The Nature of Mathematics: A Heuristic Inquiry

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This dissertation is dedicated to my loving wife, Monica. Thank you for your love and support. Our children have a wonderful mother. You do great work!

ACKNOWLEDGEMENTS

"Everything has been approved and I am ready to begin my study. What madness have I embraced, completing a self-study concerning the nature of mathematical knowledge?" This was the first thing I wrote in my dissertation journal on the night I began my study. Perhaps it was madness. Or perhaps it was grace. I believe my work has a unique character because of the influence of God in my life. Writing this dissertation provided me an opportunity to be explicit about this influence. Sometimes I was afraid to do so, but I often found the courage. In the methodology section, I do not mention all the times that I would pray that God have a hand in this work. One night, I ran up and down a hill on the Middle Tennessee State University campus, trying to motivate myself to write. As I ran, I listened to Kendrick Lamar and he sang, "I want to be the highest. Too focused. One word. Righteous." Several times that night I prayed that God would be my focus as I wrote. I believe that if God has a hand in our work then it will be great. And it is only God's approval signature that ultimately matters. My faith gives me the freedom to be creative and temporarily suspend the influence (of the abstraction that is) the standards of the field of mathematics education research. I thank God for my committee members, who reacted positively when I made my faith explicit in this dissertation.

I acknowledge my committee members and all of the other Middle Tennessee State University faculty who have believed in me and supported me in my studies. Thank you! I will single out a few of you for acknowledgements (with the caveat that I will of course neglect to mention many important persons *and* when I do express gratitude, the words I write will not do justice to the important influence you have had in my life!).

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I love my family, and this work is better because of their presence in my life. My reality checker, Monica, has contributed to this dissertation in many practical ways. During the proposal writing phase she helped me to narrow my focus so that I did not undertake a project that was too vast. When I spoke to her about my dissertation struggles, she would often have thoughtful comments that would lead me to important breakthroughs (especially regarding Chapter Four). She encouraged me to tell the whole story of the IDEA framework's creation. This dissertation is better because of her. My children, Elly, Beren, and Isa, have all influenced my thinking about the nature of mathematics, and it shows in this dissertation! For instance, it became clear early on in my study that pure mathematics is an enjoyable exploration of ideas. I wanted to find a way to inspire children to joyfully work on their own mathematical puzzles and questions. To this end I created a mathematics game called "castles" hoping that Beren and Isa would be excited about exploring mathematics on their own. Although I did not achieve my intended goal, we had fun and there were some other positive effects. In my journal I wrote, "The castle game inspired Beren to build a castle with legos."

I owe much of my academic success to my mother and father. I will always remember how my mother made sure I did my homework every day after school. I remember first grade, bringing books home in little plastic bags to read to her; and working on mathematics at the kitchen table. I must have really enjoyed those moments, because now I am planning to make a career out of school as a mathematics education professor! My father has been very supportive of my mathematical endeavors. He read my dissertation proposal; and his response was that he loved mathematics in school because it was a subject in which the teacher could not be biased against you. There were right and wrong answers, and as long as he learned the correct procedures, he got the top grades. He is a "business mathematics" expert. Perhaps over the next few years I may learn more about business mathematics, and he might get a chance to experience the

mathematics that I love, pure mathematics. I also wish to thank my brother Joseph. We had many conversations about the nature of mathematics as I completed this study. From my journal: “Phone conversation with Joseph. Elementary school mathematics should be useful for everyday life. Everything else should be optional. The credit system shouldn’t depend on algebra, geometry etc… Learning mathematics for its own sake should be optional and not forced.” I love you, and I know you will do great work in whatever field you choose! To all of my extended family, I love you. And thank you!

A few other shout outs. Thank you to my fellow MSE doctoral students. I think that I really found my identity in those “special topics” courses we took together. I would get really fired up during our discussions! In the future, MTSU will be the NOM/NOS nexus. I would also like to thank my favorite school teachers and a few other professors that have supported me throughout my schooling. Those that come to my mind immediately are Mrs. Rhonda, Mrs. Reina, Mr. Baker, Dr. Winters, Dr. Montemayor, Dr. Erenso, Dr. Hague, and Mrs. Warrenfells. Some of these thanks are long overdue! I have imagined so many people and how I would thank them in this acknowledgements section, but I have not thanked them all yet! I am surely forgetting someone! Thank you to all those who gave me feedback on this research as I made presentations across the country at job talks and conferences. The feedback I received from California State University Long Beach faculty is extra appreciated! Thank you to my friends, especially James and Tom. You both appear in my dissertation data (ask me and we can analyze it!). Thank you to Dr. Carl Langenhop, who inspired me to conduct this study. One day we had a conversation in which I realized that I would like to work on research mathematics again.

As I drove home from his apartment I thought, “How can I complete a mathematics education dissertation, and do research mathematics at the same time?” The ideas quickly fell into place, and now I am a Doctor! Thank you to my friend in Christ, Johnny Ruhl, who baptized me on the Sunday before my dissertation defense. If you were hoping for an acknowledgment, but have not received one, please let me know and I will make it up to you! Thank you all and God bless you! Love, Jeff.

ABSTRACT

What is mathematics? What does it mean to be a mathematician? What should students understand about the nature of mathematical knowledge and inquiry? Research in the field of mathematics education has found that students often have naïve views about the nature of mathematics. Some believe that mathematics is a body of unchanging knowledge, a collection of arbitrary rules and procedures that must be memorized. Mathematics is seen as an impersonal and uncreative subject. To combat the naïve view, we need a humanistic vision and explicit goals for what we hope students understand about the nature of mathematics. The goal of this dissertation was to begin a systematic inquiry into the nature of mathematics by identifying humanistic characteristics of mathematics that may serve as goals for student understanding, and to tell real-life stories to illuminate those characteristics. Using the methodological framework of heuristic inquiry, the researcher identified such characteristics by collaborating with a professional mathematician, by co-teaching an undergraduate transition-to-proof course, and being open to mathematics wherever it appeared in life. The results of this study are the IDEA Framework for the Nature of Pure Mathematics and ten corresponding stories that illuminate the characteristics of the framework. The IDEA framework consists of four foundational characteristics: Our mathematical ideas and practices are part of our ***I***dentity; mathematical ideas and knowledge are ***D***ynamic and forever refined; mathematical inquiry is an emotional ***E***xploration of ideas; and mathematical ideas and knowledge are socially vetted through ***A***rgumentation. The stories that are told to illustrate the IDEA framework capture various experiences of the researcher, from conversations with his son to emotional classroom discussions between undergraduates in a transition-to-proof course. The researcher draws several implications for teaching and research. He argues that the IDEA framework should be tested in future research for its effectiveness as an aid in designing instruction that fosters humanistic conceptions of the nature of mathematics in the minds of students. He calls for a cultural renewal of undergraduate mathematics instruction, and he questions the focus on logic and set theory within transition-to-proof courses. Some instructional alternatives are presented. The final recommendation is that nature of mathematics become a subject in its own right for both students and teachers. If students and teachers are to revise their beliefs about the nature of mathematics, then they must have the opportunities to reflect on what they believe about mathematics and be confronted with experiences that challenge those beliefs.

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Introduction

> [Benny] believes there are rules for every type of problem: “In fractions, we have 100 different kinds of rules.” He thought these rules were invented “by a man or someone who was very smart.” This was an enormous task because, “It must have took this guy a long time ... about 50 years ... because to get the rules he had to work all of the problems out like that…”. […] Benny’s view about rules and answers reveals how he learns mathematics. Mathematics consists of different rules for different types of problems. These rules have all been invented. [...] Therefore, mathematics is not a rational and logical subject in which one has to reason, analyze, seek relationships, make generalizations and verify answers. His purpose in learning mathematics is to discover the rules and to use them to solve problems. There is only one rule for each type of problem, and he does not consider the possibility that there could be other ways of solving the same problem. Since the rules have already been invented, changing a rule was wrong because the answer “would come out different” (Erlwanger, 1973, p. 54).

The case of Benny, as presented by Erlwanger (1973), provides a vivid illustration of how a student’s conception of mathematics can influence how he or she learns the subject. Because he believed that mathematics consists of arbitrary rules for solving various problems, Benny believed his role as a student of mathematics was to figure out these rules and the problems they apply to. In his dissertation, Erlwanger (1974) documented the cases of other students who developed similar conceptions of mathematics. For these students "mathematics was a system of rules or methods involving symbolic and spatial patterns. That is, their individual conception of mathematics differed in many respects from the adult conception of mathematics" (p. 205).

While Erlwanger's (1973, 1974) studies demonstrated that a student's conception of mathematics can influence mathematical learning, other scholars have claimed that a teacher's conception of mathematics can influence instruction. Thompson (1992) noted that when a teacher views mathematics as "a discipline characterized by accurate results and infallible procedures" this "can lead to instruction that places undue emphasis on the manipulation of symbols whose meanings are rarely addressed" (p. 127). Beswick (2012) found that teachers possessed conflicting views of school mathematics and mathematics as a discipline, and hypothesized that teacher education programs were not doing enough to help prospective teachers gain an informed view of the discipline of mathematics. Hersh (1997) contended that "misconception of the nature of mathematics" (p. xii) was one of the major causes of the failure of mathematics education in the United States. Similarly, Boaler (2016) wrote, "I strongly believe that if school math classrooms presented the true nature of the discipline, we would not have this nationwide dislike of math and widespread math underachievement" (pp. 22-23).

As a field, mathematics education scholars must conduct a systematic inquiry into student and teacher understandings of the nature of mathematics and come to a consensus about the kind of conceptions of mathematics we hope students develop as a result of mathematics instruction. Not only must researchers use empirical science to understand the views of mathematics that students and teachers develop in school, but we must also consider and formulate goals for students' and teachers' conceptions of the nature of mathematics. If we are to prevent future students of mathematics from viewing mathematics as arbitrary rules and procedures and teachers from emphasizing symbol manipulation in the mathematics classroom, then we must work to ensure that standards, curricula, and instruction explicitly promote an alternative view.

Alternative Conceptions of Mathematics as a Human Activity

Of course, mathematics education scholars do recognize that there is an alternative view. If mathematics was simply a collection of arbitrary rules then few of us would be in the business of mathematics education. We believe mathematics is a rewarding human activity and believe students should "enjoy the triumph of [mathematical] discovery" (Pólya, 1957, p. v). We know new mathematical discoveries are made every day, and that "Mathematics is a dynamic field that is ever changing" (NCTM, 2014, p. 72). We also acknowledge that mathematics is a fundamental aspect of human culture (Bishop, 1988), plays a role in shaping society (Skovmose, 2016), has applications in the natural sciences (Steen, 1988), and is a creative, emotional subject (Burton, 1999). Boaler (2016) noted,

> Mathematics is a cultural phenomenon; a set of ideas, connections, and relationships that we can use to make sense of the world. At its core, mathematics

> is about patterns. We can lay a mathematical lens upon the world, and when we do, we see patterns everywhere; and it is through our understanding of the patterns, developed through mathematical study, that new and powerful knowledge is created. (p. 23)

One important influence on mathematics education scholars' conceptions of the discipline of mathematics is humanistic philosophy of mathematics (Presmeg, 2007). Humanistic philosophers work from the simple assumption that mathematics is a human activity and product (Hersh, 1997). The preeminent philosopher in this tradition, Imre Lakatos, utilized the history of mathematics in order to describe the activities of mathematicians that contribute to the growth and revision of mathematical knowledge. Lakatos's book *Proofs and Refutations* (1976) is a major source of influence in the mathematics education community (Lerman, 2000). According to Lerman, the importance of *Proofs and Refutations* is "the humanistic image of mathematics it presents, as a quasiempiricist enterprise of the community of mathematicians over time rather than a monotonically increasing body of certain knowledge" (p. 22). Mathematics education scholars such as Lampert (1990) and Ball (1988) drew upon Lakatos's work and made his ideas the foundation of their approaches to teaching elementary mathematics. Consider Lampert's (1990) description of a research project in which she taught a fifth grade class.

> My research examined whether it was possible to make knowing mathematics in the classroom more like knowing mathematics in the discipline. My organizing

> ideas have been the "humility and courage" that Lakatos and Pólya take to be essential to doing mathematics. […] What has been described here thus is a new kind of practice of teaching and learning, one that engages the participants in authentic mathematical activity. (p. 58)

In her dissertation, Ball (1988) grounded what she called "mathematical pedagogy" on Lakatos's philosophy, particularly his notion of the fallibilist or revisionary nature of mathematical knowledge.

> Mathematics is not presented as a finished body of knowledge, but rather as something that changes and grows over time through a process of working from what you know to what you don't. […] Thus, when second graders think that the next number after 2 is 3, they are right—given what they know at that point. As members of a mathematical community, everyone in the class would agree. Once students encounter the set of rational numbers, perhaps through division, then the question of what the next number after 2 is becomes debatable, and the old answer—3—is clearly refuted. In this way, pupils' encounters with mathematics represent Lakatos's conception of the discipline. (pp. 200-201)

Lakatos's ideas, and the ideas of others in the humanistic tradition (e.g. Ernest, 1991; Hersh, 1997; Kitcher, 1983; Tymoczko, 1988) have influenced mathematics education scholarship and continue to be incorporated in current literature and research (e.g. Boaler, 2016; Komatsu, 2016; Larsen & Zandieh, 2008; Weber, Inglis, & Mejia-Ramos, 2014).

Mathematics education scholars' conception of the discipline of mathematics as a human activity influences research, instructional designs, and our hopes for school mathematics. And yet, students and teachers generally hold views of mathematics that are dissimilar to those of mathematics education scholars (Picker & Berry, 2000). Perhaps this is because there has not been a concerted and systematic effort within mathematics education to promote these views amongst students and teachers (Kean, 2012; Jankvist, 2015; White-Fredette, 2010). There may be a shared vision amongst many mathematics education scholars about the perceptions of mathematics we hope students and teachers develop, but if our vision is to be realized, then we need to fully articulate our goals for student and teacher understandings of the nature of mathematics. I suggest we look to science education, where significant progress has been made in articulating goals for student and teacher understanding of the nature of science.

The Nature of Science

In science education, scholars have long argued that students and teachers not only need to understand the facts of science, but also need to possess a general understanding of both science as a discipline and the nature of scientific knowledge (Lederman & Lederman, 2014; McComas, Almazroa & Clough, 1998; Peters-Burton, 2013). Proponents for teaching the nature of science (NOS) are concerned with providing students and teachers a general understanding of the scientific enterprise. As Hurd (1960) noted, "A student should learn something about the character of scientific knowledge, how it has been developed, and how it is used" (p. 34). The problem within science education is that “science continues to be primarily taught as a rigid body of knowledge rather than a way of knowing” (Peters-Burton, 2013). Scholars argue that students should appreciate science as a part of human culture, and that an understanding of NOS is important for scientific literacy (McComas et al., 1998).

Similar arguments have been made in mathematics education. Educators have recommended that students understand “mathematics as a part of cultural heritage” (NCTM, 2000, p. 4). In the National Council of Teachers of Mathematics’ (NCTM) *Principles and Standards for School Mathematics* (2000) it is written that, “Mathematics is one of the greatest cultural and intellectual achievements of humankind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects” (p. 4). As an influential element of contemporary culture, students should be aware of the cultural significance of mathematics (Bishop, 1988) and understand the role mathematics plays in shaping society (Skovmose, 2016).

Research in science education indicates that understanding NOS assists students in learning science content. Peters-Burton (2013) noted that “students who normally don’t consider themselves ‘science-minded’ may become more engaged when they are more empowered to learn how knowledge is developed and validated in the discipline of science” (p. 164). Students enjoy learning about NOS and lament when social/historical aspects of science are left out of instruction. According to McComas et al. (1998), “Incorporating the nature of science while teaching science content humanizes the sciences and conveys a great adventure rather than memorizing trivial outcomes of the process” (p. 519). Even at a very young age students can understand important NOS aspects (Lederman & Lederman, 2014). Perhaps similar positive outcomes would be seen if students had the opportunity to learn about the nature of mathematics and its cultural-historical significance, but little work has been done in this area (Jankvist, 2015).

Statement of the Problem

Researchers in mathematics education have generally not concentrated efforts on the nature of mathematics to the extent that science educators have focused on the nature of science (Kean, 2012). For example, while the Next Generation Science Standards (NGSS, 2013) contain an appendix on the nature of science that addresses the understandings that K-12 students should develop related to scientific inquiry and scientific knowledge, the Common Core State Standards (CCSSI, 2010) do not explicitly address student understandings of the nature of mathematics. The Common Core document does contain the Standards of Mathematical Practice—practices influenced by mathematics education scholars’ understanding of the discipline of mathematics and the practices of mathematicians. These standards outline the types of mathematical behavior scholars hope students engage in (e.g. persevere in problem solving), but the standards do not express specific goals about the knowledge students should have about the mathematical enterprise (e.g. mathematicians’ work on unsolved conjectures). Similarly NCTM’s (2000) process standards of problem solving, reasoning and proof, communication, connections, and representation are goals for students’ mathematical activity. These process goals were informed by scholars’ knowledge of the discipline of mathematics, but it is not made explicit that students (like scholars) should understand that these processes are fundamental to the work of practicing mathematicians. There is one exception regarding the reasoning and proof standard. According to this standard, not only are students expected to create and evaluate proofs, but also to “Recognize reasoning and proof as fundamental aspects of mathematics” (p. 402). This is a rare example of an explicit goal for student beliefs about the nature of mathematics.

The field of mathematics education has not only generally neglected to outline explicit goals for *students’* understandings of the nature of mathematics but *teachers’* understandings as well. Thompson (1992) noted, “Very few cases of teachers with an informed historical and philosophical perspective of mathematics have been documented in the literature. This observation may suggest the need to revise curricula to include courses in the history and philosophy of mathematics” (p. 141). Although courses in the history and philosophy of mathematics now exist to address this need, more work needs to be done in conceptualizing our goals for teacher understandings of the nature of mathematics before such courses can be effective. Without an explicit outline of the understandings that we hope teachers develop, how can we move forward with designing teacher education programs that do foster an informed conception of the nature of mathematics? Once we have a guiding framework(s) that outlines what students and teachers should know about the nature of mathematics, we can conduct a systematic inquiry into the teaching and learning of the nature of mathematics, and consider the instructional designs that are likely to realize our vision.

The Value of a List

In science education, scholars have done significant work conceptualizing the construct of nature of science (NOS), and have conducted research to understand how NOS can be taught to students and teachers (Irzik & Nola, 2014). Research on the teaching and learning of NOS is guided by frameworks or lists that explicitly outline an informed conception of NOS and goals for learning. According to Lederman and Lederman (2014),

> Lists serve an important function, as they help provide a concise organization of the often complex ideas and concepts they include. […] In the hands of an expert teacher, listings of desired student outcomes help guide instruction and help identify prerequisite knowledge students need to master before they can achieve a sophisticated understanding of the concept on the list. (p. 615)

For instance, the Next Generation Science Standards (NGSS, 2013) contain an appendix that lists in detail understandings of NOS that K-12 students should develop. Students should understand for example that “Scientific knowledge is open to revision in light of new evidence” (p. 4) and “Scientific models, laws, mechanisms, and theories explain natural phenomena” (p. 4). The NGSS provides detailed descriptions of how student understandings of these aspects of science should look across various grade levels.

An analogous list does not exist within mathematics education. Of course lists do exist that outline important mathematical practices (e.g. CCSSI, 2010; NCTM, 2000), mathematical habits of mind (e.g. Cuoco, Goldenberg, & Mark, 1996), and cultural features of mathematics (e.g. Boaler, 2016). However, these lists, while informed by scholars' understandings of the nature of mathematics, have not served to provide goals for student or teacher understandings of the nature of mathematics. Perhaps an implicit assumption is that if students are doing mathematics (i.e. engaging in the Standards for Mathematical Practice) then they will also come to understand the nature of mathematics. But research in science education supports the view that doing science, even engaging in laboratory apprenticeships, does not necessarily translate into an understanding of NOS (Bell, Blair, Crawford, & Lederman, 2003). Research suggests that in order for students to understand NOS, it must be explicitly addressed, and "brought to the forefront" (Bell et al., 2003, p. 504) of students' activity.

Will students be able to understand the nature of mathematics by doing mathematics in the classroom, or is an explicit discussion of the characteristics of mathematics required for such understanding to develop? As a systematic inquiry into the teaching and learning of the nature of mathematics has not been conducted, we are not in a position to answer this question. As Jankvist (2015) noted, "Only rarely is the act of providing students or teachers with certain beliefs, e.g. by changing existing ones, about mathematics or mathematics as an established and (scientifically) practiced discipline considered a goal in itself within mathematics education research" (p. 41).

A first step in conducting a systematic inquiry into the teaching and learning of the nature of mathematics is to consider the characteristics of the nature of mathematics that students or teachers should understand, and compile them into a framework/list with the purpose of conducting further research into student and teacher understandings of those characteristics. After we determine what features of the nature of mathematics can be taught and learned, we can make progress toward ensuring that students and teachers possess an informed view of the nature of mathematics.

Purpose of the Dissertation

My goal with this dissertation is to begin the work of "conceptualizing the construct" (Lederman & Lederman, 2014, p. 600) of the nature of mathematics (NOM) as has been done with the nature of science (NOS) in science education. My aim is to produce a humanistic framework for the nature of mathematics that highlights key features of the mathematics that may serve as goals for student and teacher understanding, together with corresponding narratives that illuminate these features. Through the literature review described in Chapter Two and the study outlined in Chapter Three, I have worked to produce The IDEA Framework for the Nature of Pure Mathematics. This framework and corresponding narratives are presented in Chapter Four. It is my hope that this framework may guide research, professional development, the design of curricula, and other work in mathematics education.

The first step in the creation of the framework was a thorough literature review in which I investigated humanistic philosophy of mathematics and the mathematics education research that has been influenced by this philosophy. After completing this review of the literature and compiling an initial list of characteristics of mathematics, I deepened my understanding of NOM and revised the framework by conducting heuristic self-search inquiry (Douglass & Moustakas, 1985; Moustakas, 1990; Sela-Smith, 2002). By documenting and reflecting on my own experience collaborating with a research mathematician while also teaching an undergraduate mathematics course, I had experiences highly relevant to the nature of mathematics and its teaching and learning. Drawing from these experiences, I have drafted narratives that illuminate key features of the nature of mathematics. Two research questions guided my study:

> What is the nature of mathematics? What should students understand about the nature of mathematics?[1]

To reiterate, my goal has been, through a review of the literature and heuristic self-search inquiry, to explicate key humanistic characteristics of the nature of mathematics in the form of a list, and to complement this list with supporting narratives that illuminate the characteristics.

Significance of Study

I have outlined the need for the field of mathematics education to make explicit its goals for student and teacher understandings of the nature of mathematics. Only if these goals are made explicit can we move forward with research to investigate the best ways to meet these goals. In creating a humanistic framework that explicitly highlights key characteristics of the nature of mathematics, I hope that my work will move the field of mathematics education forward in this important area.

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[1] As will be discussed in Chapter Three, the focus of these questions was narrowed for the purposes of my study.

The Value of Stories in Educational Research

During my time as a doctoral student, the mathematics education articles that have had the most influence on me have been stories. Erlwanger (1973) presented the story of Benny, a boy who constructed his own non-standard mathematical rules and beliefs while working in the Individually Prescribed Instruction (IPI) mathematics program. Benny's story illuminated the limitations of a behaviorist approach to mathematics learning. It was a particularly powerful critique in part due to the empathy a reader could feel for Benny. By telling stories about her third grade classroom, Ball (1993) demonstrated the conflicts a teacher may encounter when striving to both honor student thinking and respect the discipline of mathematics. These stories, such as the one about "Sean numbers," are enjoyable to read, and also incite significant discussion in the field of mathematics education (Bass & Ball, 2014). In another of my favorite journal articles, Lampert (1990) told the story of her fifth grade class. This story functions as an existence proof that it is possible for school students to engage in authentic mathematical practices. My purpose in recalling the stories, told by other mathematics education scholars, is to underscore the value that narratives have for educational research. My hope is that the narratives presented in Chapter Four adequately illuminate key characteristics of the nature of mathematics for future readers. I hope the narratives may be useful for fostering discussion, research, and understanding of the nature of mathematics within the field of mathematics education. These narratives are not fictional, but are true stories of what I experienced during my heuristic inquiry into the nature of mathematics.

Research Program

I do not intend for my dissertation work to be the final chapter in my own understanding of the teaching and learning of the nature of mathematics, but the beginning of a long research program. For this dissertation I created a framework that will be revised over time as needed. As seen in Figure 1, this dissertation work is the beginning of a research trajectory. The purpose of this dissertation has been to develop an initial framework and corresponding narratives. In the future, I would like to interview mathematicians and mathematics educators in order to confirm that the nature of mathematics characteristics I have outlined are consistent with others' beliefs, values, and goals. Furthermore, it will be important to consider if students and teachers can understand these characteristics of the nature of mathematics and to explore how these characteristics can be taught and understood. Subsequently, researchers may consider examining empirical studies within science education related to the nature of science and adopt similar methods to study the nature of mathematics within the field of mathematics education.

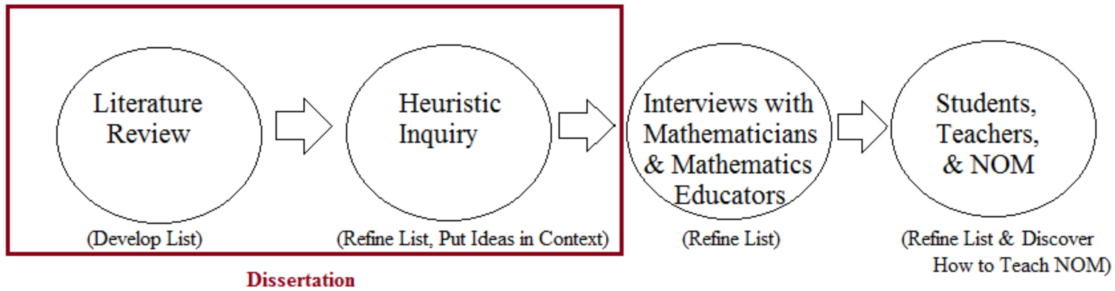

Figure 1. The Long-Term Development of a Framework for the Nature of Mathematics

Definition of Terms

Nature of Mathematics (NOM)

I conceive the nature of mathematics (NOM) to be a philosophical notion, signifying that mathematics has a nature that can be described. As I discuss in Chapter Two, this nature may vary across particular human contexts. To be clear, NOM is not simply the list of characteristics I will highlight in this dissertation. NOM is a vast philosophical concept that cannot easily be pinned down (cf. Kean, 2012). My intent has been to draw out some of the humanistic characteristics of NOM that may be important for people-in-society to know and understand in order to combat naïve views.

Several times in this chapter I have referred to NOM implicitly in terms of the discipline practiced by professional mathematicians. However there are many types of professional mathematicians. Steen (1988) noted,

> Today's mathematical sciences […] can be divided into three parts of roughly comparable size: statistical science, core [pure] mathematics, and applied

> mathematics. […] Although the boundaries between these parts overlap considerably, each province has an identifiable character that corresponds well with the three stages of the mathematical paradigm established by Newton: data, deduction, and observation. (p. 612)

However, mathematics is not only the practice of professional mathematicians. Bishop (1988) discussed fundamental mathematical practices (e.g. counting, locating, and measuring) that exist within all human cultures. According to Bishop, mathematics as practiced by professional mathematicians is only one form of mathematics. Ultimately, I envision a NOM framework that fully outlines mathematics as a broad cultural activity and delineates important characteristics of different types of mathematics. In Chapter Two, I have done some preliminary work in this broad direction, discussing ethnomathematics as well as some categories of mathematics that exist within cultures (e.g. artisan, commercial-administrative). Also the beginnings of a broad NOM framework are presented in Chapter Five with recommendations for future development.

In order to narrow the focus of my dissertation, I chose to concentrate on the mathematical practices and knowledge of pure mathematicians. Browder (1976) defined pure mathematics as “that part of mathematical activity that is done without explicit or immediate consideration of direct application to other intellectual domains or domains of human practice” (p. 542). I view pure mathematics as mathematics that is done for its own sake.

One reason that I choose to focus on pure mathematics in my dissertation study is that most of my experience as a scholar has been related to pure mathematics. It is the aspect of mathematics that I know and love. As a master's student, nearly all of the mathematics courses I enrolled in were pure mathematics courses. I also wrote a pure mathematics master's thesis in graph theory. In addition, my doctoral studies in mathematics education have primarily been focused on undergraduates' learning of mathematical proof, a key feature of pure mathematical knowledge.

Throughout this chapter I have mentioned the "discipline" of mathematics. Unless otherwise noted, I am referring to the knowledge and practices of pure mathematicians when I use this term. As I discuss in Chapter Two, my framework for the nature of mathematics as a discipline is grounded in humanistic philosophy of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1978). This philosophy is grounded upon the premise that mathematics is a human activity. Philosophers in this tradition typically focus on the discipline of pure mathematics, but acknowledge the importance of the applied (Hersh, 1997). I do not argue that pure mathematics should be valued over other forms of mathematics. I believe it is important for students to understand the complexity of the discipline of mathematics and also understand mathematics as a fundamental aspect of human cultures. I suggest an in-depth study of the nature of applied mathematics, statistics, and other forms of mathematics should be conducted by interested researchers.

The Nature of Mathematical Inquiry versus the Nature of Mathematical Knowledge

When considering pure mathematics as a discipline, I conceive of two aspects of NOM that may be fruitful to distinguish: the nature of mathematical inquiry (NOMI) and the nature of mathematical knowledge (NOMK). NOMI refers to the practices that mathematicians engage in when creating knowledge (e.g. conjecturing, proving, communicating, etc.…) and the human experience of such activity (e.g. emotional). NOMK refers to the nature of the knowledge that mathematician's produce (for instance, is mathematical knowledge absolute or subject to revision?). It should be noted that this distinction is not always clear cut. For instance, would conjecturing be categorized as NOMI or NOMK? One can make the case for NOMI—conjecturing is an important mathematical *practice* that plays a role in the creation of mathematical knowledge. On the other hand, established theorems were once conjectures. This would place conjectures in the category of NOMK. My point here is that while I believe it will be valuable to distinguish between the nature of the inquiry of mathematicians (NOMI) and the nature of the knowledge mathematicians produce (NOMK), the distinctions between knowledge and practice are not clear cut. These distinctions will be further discussed in Chapter Two.

Chapter Summary

In this chapter I have argued that the field of mathematics education needs a framework that outlines key characteristics of the nature of mathematics that students and teachers should know and understand. I agree with Hersh (1997) that "What's needed is an explication of what mathematicians do—as part of general human culture" (p. 19). We need a guiding framework to begin the work of reforming student misconceptions of the nature of mathematics so they may possess a more informed, humanistic view of the subject. My goals with this dissertation have been, through empirical means, to begin the process of designing an educational framework for the nature of mathematics, and to tell stories grounded in data that illuminate key characteristics of the nature of mathematics. My aim has been to portray mathematics as a human activity. The first draft of such a framework is presented in Chapter Two, and is based upon a review of the relevant literature in mathematics education and humanistic philosophy of mathematics.

CHAPTER TWO: REVIEW OF LITERATURE

Introduction

Students and teachers frequently have a limited view of the nature of mathematics and may believe mathematics is a static body of knowledge consisting of arbitrary rules and procedures (Beswick, 2012; Erlwanger, 1974; Muis, 2004; Presmeg, 2007; Thompson, 1992). These naïve views may negatively affect the teaching (Thompson, 1992; White-Fredette, 2010) and learning (Erlwanger, 1974) of mathematics. In contrast to these naïve views, many mathematics education scholars view and describe mathematical knowledge as a dynamic human product (Boaler, 2016), and emphasize the human aspects of mathematical work such as creativity (Burton, 1999) and fallibility (Ernest, 1991). These modern views are influenced by cultural approaches to mathematics (Bishop, 1988), theories of embodied cognition (Lakoff & Nuñez, 2000), humanistic philosophy of mathematics (Ernest, 1991), or perhaps scholars' own experiences doing mathematical work (e.g. Hersh, 1997). According to Hersh (1997), "misconception of the nature of mathematics" (p. xii) is one of the major causes of the failure of mathematics education in the United States. Thompson (1992) noted that "Very few cases of teachers with an informed historical and philosophical perspective of mathematics have been documented in the literature" (p. 141). The gap between uninformed views of mathematics (largely held by students and teachers) and the informed cultural-historic perspectives held by scholars needs to be addressed within mathematics education.

In Chapter One I made the case that the field of mathematics education needs to begin a systematic inquiry into the teaching and learning of the nature of mathematics (NOM), with a first step being the creation of an explicit framework highlighting key features of NOM that will serve as goals for student and teacher understanding. Such a framework should be useful in order to inform research, professional development, the design of curricula, and other work in mathematics education. The preliminary NOM framework in Figure 2 is informed by my study of the work of humanistic philosophers of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976), mathematics education scholars who emphasize the cultural dimensions of mathematical knowledge (e.g. Bishop, 1988, Izmirli, 2011), critical scholars (e.g. Borba & Skovmose, 1997; Harouni, 2015) and other mathematics education scholars whose work is relevant to NOM (e.g. Burton, 1995; Weber, 2010).

This first iteration of a NOM framework is clearly incomplete. I present further iterations in Chapters Four and Five based on my dissertation study, but I also believe further modifications will also need to be made post-dissertation. It is important to note that as I conducted my dissertation study, the characteristics of mathematics described in this initial framework gave me a starting place for identifying features of the nature of mathematics at work in my own mathematical research and teaching. Through completion of the study I recognized that some of these initial characteristics may not be the best goals for student understanding, but I have left this initial framework intact since it was influential to my thinking during the dissertation study.

Mathematics as a Fundamental Part of Human Cultures
• Mathematical knowledge is influenced by cultural values (Bishop, 1988; Hersh, 1997). • Mathematical knowledge is embedded within the work of artisans (Harouni, 2015). • The purpose of commercial-administrative mathematical knowledge is calculation for economic purposes. The efficiency and accuracy of mathematical procedures is valued and there is no need to understand why a procedure works (Harouni, 2015). • Western academic mathematics is one (but not the only) form of mathematics (Bishop, 1988; Izmirli, 2011).
The Nature of Pure Mathematics as a Discipline
Nature of Mathematical Knowledge (NOMK) • Mathematical knowledge is subject to revision (Hersh, 1997; Lakatos, 1976). • Mathematical knowledge is socially validated (Ernest, 1991; Lakatos, 1976). • Proofs are bearers of mathematical knowledge (Hanna & Barbeau, 2008; Weber 2010). • Informal mathematical work is the foundation of formal knowledge (Hersh, 1997; Lakatos, 1976). *Nature of Mathematical Inquiry (NOMI)* • Mathematical inquiry can be creative, emotional, and collaborative (Burton, 1995). • Each sub-discipline of mathematics has different norms, values, and standards (Burton, 1999, Weber, 2008).
Statistical and Applied Mathematics
• Mathematical knowledge is used to shape society, but cannot be considered an absolute judge (Borba &Skovmose, 1997).

Figure 2. First Iteration of a Nature of Mathematics (NOM) Framework

In this chapter, I will review the literature that is relevant to the teaching and learning of the nature of mathematics (NOM). Much of this literature has influenced the categories and characteristics of NOM that are listed in Figure 2, but the discussion will not be limited to those categories and characteristics. First we will begin by considering mathematics as a fundamental part of human culture (Bishop, 1988). This discussion will examine the way mathematics is embedded in the everyday life of various human groups, and contrast different forms of mathematical knowledge, for example artisanal, commercial-administrative, and philosophical (Harouni, 2015). Pure academic mathematics will be situated as one form of mathematics with its own unique value-laden characteristics. This discussion will lead to an examination of the philosophy of mathematics literature. Note that the question “What is the nature of mathematics?” is a philosophical one; it is thus fitting that in designing an educational framework for the nature of mathematics we draw from the philosophy of mathematics. As Ernest (1991) noted, “The philosophy of mathematics is the branch of philosophy whose task is to reflect on, and account for the nature of mathematics” (p. 3). Humanistic philosophy of mathematics has had a profound influence within mathematics education (Lerman, 2000; Toumasis, 1997), and this philosophy has had a guiding influence on my dissertation study. The philosophies of Platonism and formalism, which continue to have influence within mathematics education (Dossey, 1992, White-Fredette, 2010), will be addressed to provide a contrast to the humanistic ideas. Finally, by incorporating the work of Steen (1988), the chapter will conclude with a brief description of statistical and applied mathematics. I acknowledge that more conceptualization in these areas is needed in future research projects.

Before beginning the review of the literature, it is pertinent that I provide some insight into my personal stance as a researcher regarding the nature of mathematics and the purposes of education. Not only because this will provide the reader some insight into how I interpret the literature, but also because the inquiry framework that I am using for my dissertation study necessitates that the researcher's personal stance be made explicit.

Theoretical Orientation of the Researcher

In order to develop a useful educational framework for the nature of mathematics, I conceived that it would be beneficial to collaborate with a research mathematician and reflect on my own experience doing mathematics as the basis of my dissertation study. Because of its emphasis on the personal experience of the researcher, the methodological framework of heuristic self-search inquiry (Douglass & Moustakas, 1985; Moustakas, 1990; Sela-Smith, 2002) seemed to be a good fit for the type of work I desired to undertake. Patton (2015) noted,

> [H]euristic research epitomizes the phenomenological emphasis on meanings and knowing through personal experience; it exemplifies and places at the fore the way in which the researcher is the primary instrument in qualitative inquiry; and it challenges traditional scientific concerns about researcher objectivity and detachment. (p. 119)

I chose heuristic self-search inquiry because I thought it wise to examine deeply my own experience conducting mathematical research so that I will have some authority to speak on the characteristics of pure mathematical inquiry and knowledge for the purposes of mathematics education[1]. But heuristic inquiry is a method that not only

[1] See Chapter 1 for my description of mathematics as a discipline and the affinity I have for pure mathematics.

informs data collection but also the entire research process from the very beginnings of research to the final stages. As Moustakas (1990) noted, "The self of the researcher is present throughout the process and, while understanding the phenomenon with increasing depth, the researcher also experiences growing self-awareness and self-knowledge" (p. 9). Crucial to this process is what Moustakas called the "internal frame of reference" (p. 26). Bach (2002) noted, "The internal frame of reference is a guiding concept in heuristic research and makes possible all other processes within the model. The personal is the basic foundation, the beginning of a knowledge base" (p. 94). Thus before I provide a thorough review of the literature, I offer a glimpse into my own perceptions, thoughts, and identity, the "internal frame of reference" that influences my work.

Internal Frame of Reference

I tend to work from a humanistic-critical-theistic perspective. In describing humanistic mathematics education, Brown (1996) noted that the goal of education should be to gain an understanding of self and what it means to be human. From the critical perspective (e.g. Borba & Skovmose, 1997; Harouni, 2015; Pais, 2013) a goal of education is to empower individuals and ultimately to challenge and change an unjust social order. My humanistic-critical concerns are rooted in an underlying belief in a loving creator God. In a previous paper regarding the nature of mathematical knowledge (Pair, 2015) I wrote,

> My belief in God is a conscious commitment I have made in light of personal experiences—experiences in which the existence of God seemed undeniable. The

> commitment could be called faith. […] I believe that my accomplishments are made possible because of my trust in God. […] What needs to be transmitted to the reader is that the belief in God is central to my reality. I would be deceiving my self and others if I did not recognize that any position I take will necessarily be influenced by this belief. (p. 3)

In that paper I described how I came to the conclusion that "God and humans are co-creators of mathematics." (p. 6). I do not intend for such a statement to make it into a framework for the nature of mathematics. Rather I intend to focus on the humanistic premise that humans are creators of mathematics. I believe humanistic concerns can unite people of different perspectives.

I have always believed that a purpose of mathematics education should be to develop in people the ability to think critically about their lives so that they may consider ways in which the nature of society can be revitalized. But I have become more aware of this desire through the dissertation process. I want to make this explicit by taking a critical stance. This stance is grounded in spiritual concerns for human welfare and manifests itself as an alignment and admiration for critical theorists in mathematics education (e.g. Ernest, Sriraman, & Ernest, 2016). I do acknowledge that there are differences in my views and those of critical theorists, particularly religious differences. But in spirit with other critical mathematics education scholars, my purpose is not to imagine how to teach the nature of mathematics within society as it is, but to teach the nature of mathematics in a way that challenges that which is contrary to the well-being of humans. My vision is that the learning of mathematics will not be an oppressive experience, but an enlightening one. I acknowledge that my increasing awareness of my personal stance is no less than the beginnings of a personal transformation which Sela-Smith (2002) claimed to be the dominant purpose of heuristic self-search inquiry. She wrote, “[A] relentless inward focus can lead to greater self-understanding, self-transformation, and reconstruction of a hindering worldview” (p. 80).

This dissertation is written from my internal frame of reference, and I am explicit about the ways in which my thinking has been transformed through this work. At the start of this project, my thoughts regarding the nature of mathematics were exclusively related to the practices and knowledge of pure mathematicians; and indeed pure mathematics has been the focus of this dissertation. But as I began to review the literature, I became increasingly aware of the fact that pure mathematics is not the only real mathematics. Mathematical ideas have always been part of human culture, and mathematical knowledge takes on a distinct character depending upon the cultural context in which it plays a role.

Mathematics as a Part of Human Culture

Many scholars in mathematics education have taken cultural approaches in describing the nature of mathematics (e.g. Bishop, 1988; D’Ambrosio, 2016; Presmeg, 2007). From this perspective, mathematics is viewed as a fundamental part of all human cultures. According to Bishop (1988), there are six cultural activities “necessary and sufficient for the development of mathematical knowledge” (p. 182): counting, locating, measuring, designing, playing, and explaining. It is from these activities that mathematical knowledge is uniquely developed in each culture.

Mathematical Knowledge is Influenced by Cultural Values

Mathematics plays a formatting role in society (Borba & Skovmose, 1997), and as a part of culture, is necessarily influenced by cultural values (Bishop, 1988). We will examine this influence by considering the differences between mathematical knowledge of people in different contexts.

Mathematical knowledge is embedded within the work of artisans.

Mathematics may be so fundamental to our lives that it is indistinguishable from our activities. Harouni (2015) noted that this is precisely the case for the artisan (e.g. a mason, shipbuilder, or a carpenter). There is no distinction between mathematical knowledge and the practice of a craft:

> The carpenter's act of measuring planks, for example, might involve operations similar to what one learns in school today, but the artisan's math is intertwined with the materials and instruments of his work (Smith, 2004). The ruler he uses defines the meaning of numbers for him. (p. 54)

The mathematical knowledge of the artisan is not distinct from his work and must be transmitted through apprenticeship. As artisanal mathematics is intertwined with the craft it is not identified as mathematics, and the artisan is not likely to claim to have mathematical knowledge. And although we can use the mathematics taught in Western schools to classify the practices of a craftsman as mathematical, the knowledge of the artisan is not generally developed in the West.

> Artisanal learning did not help shape public schooling because of the rise of industrial capitalism, which rapidly eroded the influence of the artisan class. By the late seventeenth century in Europe, apprenticeships were proving expensive and outdated (De Munck, 2007). Technological progress limited the skills needed by the majority of workers on the shop floor, and increasingly the combination of simple wage labor and machinery replaced the work of trained artisans. (Harouni, 2015, p. 63)

Thus the mathematical knowledge taught in Western schools today is shaped by a *lack* of value related to the artisan's craft.

The Economic Role of Mathematics in the Modern West

As a fundamental part of culture, mathematical knowledge is not value-free[2] (Bishop, 1988). Mathematics will take on a different character depending on the values associated with a culture. Because mathematics is embedded within and part of a culture, there is a dynamic interplay as the culture influences mathematics and vice versa. Bishop noted,

> Western culture's world-view appears to be dominated by material objects… One of the ways mathematics has gained its power is through the activity of

[2] Some scholars in science education (e.g. Lederman, Antink, & Bartos, 2014) maintain that a goal is for students to understand the value-laden nature of scientific knowledge.

> objectivising the abstractions from reality. Through its symbols (letters, numerals, figures) mathematics has taught people how to deal with abstract entities, as if they were objects. (p. 186)

The objectification of number serves economic needs, and decontextualization is necessary for determining exchange value of goods and labor within an economic system (Harouni, 2015). The high degree of certainty we are able to achieve in mathematics may be due to the historical need to create invariant objects to deal with accounting (Ernest, 2016). Ernest maintained that "at the heart of systems of numeration and measurement is the human requirement that processes of accounting should conserve the material resources being recorded and hence, by proxy, be invariant with respect to the quantities, numbers and calculations involved" (p. 382). The objectification and tendency to decontextualize number is a key feature of what Harouni (2015) termed commercial-administrative mathematics, or reckoner's mathematics.

The nature of commercial-administrative mathematical knowledge.

Reckoning schools appeared in fourteenth century Italy and were valuable to merchants and accountants who sent their children to such schools to be taught by a reckonmaster (Harouni, 2015). The curriculum of a reckoning school resembles the content that characterizes what is today sometimes called traditional elementary mathematics instruction. Harouni noted that a typical fourteenth century curriculum consisted of the following seven sections:

1. Addition, subtraction, and multiplication (including memorization of algorithms and fact tables)

2. Division by a single digit

3. Division by a two-digit number

4. Division by a three-digit number or more

5. Fractions (basic operations, used in problem situations)

6. Rule of three

7. Principles of the Florentine monetary system. (p. 55)

A key feature of the reckoner's mathematical practice was that it was not important to understand why mathematical procedures and algorithms work, but merely have confidence that they do always produce the right answer (Harouni, 2015). There was no need for justification in a reckoner's school, and the reckonmaster or textbook was the sole source of mathematical authority. The mathematics education scholar Lampert (1990) seems to be describing a reckoner's school when she outlined what it means to know mathematics in modern schools.

> Commonly, mathematics is associated with certainty: knowing it, with being able to get the right answer, quickly… These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher. Beliefs about how to do

> mathematics and what is means to know it in school are acquired through years of watching, listening, and practicing. (p. 32)

It is clear that the mathematical knowledge of the reckoner is influenced by economic values. Harouni (2015) claimed that “due to reckoning mathematics’ economic significance, it continues to dominate the lay perception of mathematics” (p. 60). Furthermore,

> In the case of mathematics education, we are dealing with an overwhelming economic fact that we hold in common with sixteenth-century Europe: commercial and administrative calculation is *still* the dominant intellectual activity of our societies. We are not merely inheritors of reckoning. We *are* reckoners—and perhaps academics, artisans, politicians, and so on—and the math we teach contains our attitudes. (p. 58)

Western Academic Mathematics is One (but not the only) Form of Mathematics

Due to political-economic reasons, artisanal mathematics is generally ignored in schools while the characteristics of commercial-administrative mathematics continue to be valued. But Harouni (2015) noted there is one other form of mathematics that is often found competing with commercial-administrative mathematics in school.

> We can think of it as *philosophical mathematics,* using *philosophy* as a blanket term to cover also priestly and academic activities. It is exemplified in the work of Euclid, in the astronomical and astrological discourses of Muslim scholars, and in the discipline of pure math that is the practice of academic mathematicians. […]

> Philosophical mathematics loves patterns. It draws them out because they hint at meaning, and meaning is the priest's and philosopher's sustenance. (p. 64)

In the past, if my students said they did not like mathematics, I may have told them that it is because they had never studied *real* mathematics. I get warm and fuzzy feelings when I read the work of scholars who make the case that students might actually enjoy mathematics if they studied the *real* thing—creating proofs, formulating conjectures, and experiencing the creativity and beauty of mathematics (Boaler, 2016; Lockhart, 2009). Clearly my favorite version of mathematics is priestly, philosophical mathematics. The mathematics I love is the mathematics of patterns. I believe that pure mathematics is valuable enough that every person should have the opportunity to know and experience it. This entails coming to understand the nature of pure mathematical practice and knowledge. But *pure* mathematics, the mathematics that I love, is not the only *real* mathematics.

Ethnomathematics.

Izmirli (2011) claimed that non-Western cultures have been "excluded from the mainstream history of mathematics, and formal, academic mathematics" (p. 33) and that "all quantitative and qualitative practices, such as counting, weighing, and measuring, comparing, sorting and classifying, which have been accumulated through generations in diverse cultures, should be encompassed as legitimate ways of doing mathematics" (p . 34). Within mathematics education, some scholars have devoted themselves to incorporating the mathematics of non-Western cultures into the Western classroom. This has largely been conducted under the research program of ethnomathematics (D'Ambrosio, 2016). According to D'Ambrosio (2004), ethnomathematics is "the mathematics which is practiced among identifiable cultural groups" (p. 196). An identifiable cultural group may be the collective of practicing mathematicians (Bishop, 1988; Mesquita & Restivo, 2013), but usually refers to non-academic groups, such as "national tribal societies […] [or] builders and welldiggers and shack-raisers in the slums" (D'Ambrosio, 2004, p. 196). D'Ambrosio (2016) claimed, "The main goal of Ethnomathematics is building up a civilization free of truculence, arrogance, intolerance, discrimination, inequity, bigotry and hatred" (p. 25).

> As our students experience multicultural mathematical activities that reflect the knowledge and behavior of people from diverse cultural environments, they may not only learn to value the mathematics but, just as important, may develop a greater respect for those that are different from themselves (D'Ambrosio, 2001, p. 303).

Other researchers in this tradition support students in connecting the mathematics of everyday life to the academic mathematics learned in school (Presmeg, 2007).

As I investigated the ethnomathematics literature I came across a criticism of ethnomathematics written by Pais (2011). Key to Pais's critique is that mathematics education must be conceived within the broader economic system of our times, i.e. global capitalism. Any pedagogical approach aimed at changing society may be compromised due to this economic reality. Instead of leading to multicultural appreciation, the incorporation of a revolutionary pedagogy into the classroom may be superficially perverted to fit inside the system of schooling. In the case of ethnomathematics, the mathematics of another culture may be used not as a tool to make students more culturally sensitive, but merely as an aid to teaching the "official curriculum" (Pais, 2011, p. 224). Any cultural insight that could be gained may be ignored for the sake of teaching Western mathematics, the mathematics which plays a significant role in the economic system. Pais noted, "[T]he process of bringing diversity and ethno-mathematical ideas into the classroom may end up conveying practices opposed to the benevolent multicultural ideas these researchers want to enforce, by promoting a desubstantialized view of Other's culture" (p. 227).

The influence of the economic system may inadvertently thwart our aims in mathematics education in other ways. Pais (2015) argued that within mathematics education, researchers repress the fact that mathematics education serves a role in the political economy. Why do students desire to learn mathematics? Pais (2013) wrote that "Mathematics allows students to accumulate credit in the school system that will allow them to continue studying and later to achieve a place in the sun" (p. 20). The ideology of the mathematics education scholar often ignores this reality, in hopes that students will desire to learn mathematics for its own sake. We must be aware that for many students, mathematical knowledge is a required commodity. Students have learned that to do mathematics means to complete exercises for the purposes of obtaining school credit. We cannot ignore the economic reality as we work to transform students' and teachers' conceptions of the nature of mathematics.

Implications and Summary of Ideas: Mathematics and Culture

I contend it may be valuable for students and teachers to understand that there are many different types of mathematics. Teachers may use this knowledge to incorporate into their instruction the mathematical practices most likely to meet their educational goals. I conjecture that even mathematics education scholars (especially those of philosophical leanings) may benefit from considering the reality of commercial-administrative mathematics. Our proposed reforms are often pitted against this cultural inheritance. In some sense, the reform movement has resulted in school being a combination of commercial-administrative mathematics and pure mathematics. Students are still expected to learn the reckoners' algorithms but also understand the meaning of the algorithms as a pure mathematician would.

> The production of contemporary math curriculum represents the outcome of a particular dialectical battle between philosophical mathematics and commercial-administrative mathematics. In a strong money economy, as soon as philosophical mathematics leaves its specialized cloisters and addresses itself to the general public, it is fated to meet commercial-administrative mathematics. While commerce and administration, which rely heavily on mathematics, want philosophical math to submit to and reinforce their agenda, the philosophical mathematician wants to reassert her independent identity and in that attempt

> brandishes her stronger, more scientific version of mathematics. This dynamic brings about a synthesis that can take a variety of forms depending on the battleground. In public education, the result tends to be disappointing to proponents both of reckoning and of philosophical math, as they do not intend to change their own practice or view of the purpose of math but only the way in which the future generation is trained in it. (Harouni, 2015, p. 66)

I have described how mathematics is a fundamental aspect of human culture, mathematical knowledge is influenced by cultural values, and academic mathematics is only one form of mathematics (Bishop, 1988). I contend it may be valuable for students and teachers to understand the cultural-embeddedness of mathematics and understand that artisanal, commercial-administrative, and philosophical mathematics are distinct forms of mathematics. I hypothesize that students may appreciate glimpses into the mathematics of other cultures (perhaps the work of artisans) and an understanding that mathematics is an intricate part of life. I strongly contend that more attention needs to be paid to the characteristics of commercial-administrative mathematics. Decontextualization reflects the values of a mathematics designed for economic purposes where the efficiency and accuracy of mathematical procedures is valued (there is no value in understanding why a procedure works). More research is needed to determine the benefits of and methods for helping students and teachers understand that mathematical knowledge is shaped by human values.

The Nature of Pure Mathematics as a Discipline

Is even *pure* mathematical knowledge subject to the values of culture? There is a perception that mathematical knowledge has a timeless objective quality (Hersh, 1997). Does the fact that 2+2 equals 4 not transcend one's humanity? In the next portion of this literature review I turn our attention to the philosophy of mathematics, particularly as it is relevant to pure, academic mathematics as a discipline. Philosophers of mathematics primarily seek to understand and describe the discipline of mathematics and the nature of its corresponding knowledge (Kitcher, 1983). As thorough expositions of the history of philosophy of mathematics have been put forth by others (e.g. Dossey, 1992; Ernest, 1991; Hersh, 1997; White-Fredette, 2010), I do not intend to put forth a complete exposition here.

Humanistic philosophy of mathematics (Ernest, 1991; Hersh, 1997; Lakatos, 1976), has informed the characteristics of the nature of pure mathematics as a discipline I outlined in Figure 2. Humanistic approaches are unique in that they take as foundational the notion that mathematical knowledge is a human product. As Hersh (1997) wrote, "To the humanist, mathematics is *ours*—our tool, our plaything" (p. 60). I will also incorporate a discussion of traditional philosophies of mathematics (Platonism and formalism), which have typically served to proliferate the idea that mathematical knowledge is absolute and value-free. These philosophical positions have had a considerable amount of influence (typically perceived as negative) on the teaching and learning of mathematics (Dossey, 1992; White-Fredette, 2010). They will provide a contrast to humanistic views.

When considering mathematics as a discipline I conceive of two aspects of NOM that may be fruitful to distinguish: the nature of mathematical knowledge (NOMK) and the nature of mathematical inquiry (NOMI). NOMK refers to the nature of the knowledge that mathematician's produce (for instance, is mathematical knowledge absolute or subject to revision?). NOMI refers to the practices that mathematicians engage in when creating knowledge (e.g. conjecturing, proving, communicating, etc.…) and the human experience of such activity (e.g. emotion). In Chapter One, I claimed that the boundaries between NOMI and NOMK are not always clear cut. Nevertheless, it may be important to make the distinction when possible as there has been confusion in science education when scholars have conflated the nature of scientific knowledge and scientific inquiry when discussing nature of science (Lederman & Lederman, 2014). The remainder of this chapter will generally follow the outline of Figure 2, first with a discussion of key features of NOMK before turning to NOMI.

Mathematical Knowledge is Subject to Revision (NOMK)

Humanistic philosophers work from the simple assumption that mathematics is a human activity and product. Thus mathematics necessarily influences and is influenced by human culture (Hersh, 1997). As a human product, mathematical knowledge is necessarily imperfect, fallible, and subject to revision (Ernest, 1991). Imre Lakatos is typically credited with the fallibilist view of mathematical knowledge (Kitcher, 1983).

Lakatos's (1976) *Proofs and Refutations* is a classroom narrative in which the instructor and students' discussions parallel the historical-conceptual development of an Eulerian conjecture as carried out by mathematicians such as Euler, Cauchy, Lhuilier, etc.… Through the story told in *Proofs and Refutations,* Lakatos demonstrates the fallible, revisionary nature of mathematical knowledge—as counterexamples are found to what are believed to be solid proofs and theorems, mathematics grows and changes. This view contrasts more traditional views such as Platonism and formalism.

Platonism.

The Platonic view of mathematical knowledge is one of the most traditional and widespread (Hersh, 1997). According to Dossey (1992), "Plato took the position that the objects of mathematics had an existence of their own, beyond the mind, in the external world" (p. 40)[3]. Brown (2008), a modern Platonist, wrote "Mathematical objects are perfectly real and exist independently of us" (p. 12), and we gain access to these objects through "the mind's eye" (p. 14). If mathematical objects are conceived to have a transcendental existence, then mathematical truth exists independently of humans and awaits human discovery. Although a standard objection against Platonism is the problem of how humans access the immaterial realm of mathematics (Ernest, 1991; Hersh, 1997), I claim the Platonic position has fallen out of favor with modern philosophers primarily because such a position "violate[s] the empiricism of modern science" (Hersh, 1997, p. 12). Modern philosophers seek to explain mathematics without appealing to a belief in

[3] Note that under the Platonic position, mathematical knowledge is necessarily free of cultural influence.

God (Pair, 2015). When we can no longer make that appeal, and we abandon Platonism, we lose our ability to assume that mathematical knowledge is timeless and objectively true.

The Euclidean (deductivist) paradigm.

The Platonic perspective formed the foundation of the Euclidean paradigm that has dominated mathematics for 2,500 years (Ernest, 1991). This view is that humans can arrive at certain mathematical truth by following the deductive process. A few self-evident truths called postulates are assumed along with some definitions of mathematical terms. Then, beginning from these postulates and definitions, one can proceed by logical deduction to arrive at other certain truths. But mathematicians eventually found they could create new bodies of useful mathematical knowledge, non-Euclidean geometries, by withholding Euclid's assumption about parallel lines (Ernest, 1991). Thus, what were considered postulates could no longer be considered obvious truths, as a different "truth" could be arrived at when one began with a different set of assumptions. According to Hersh (1997), "Geometry served from the time of Plato as proof that certainty is possible in human knowledge…. Loss of certainty in geometry threatened loss of all certainty" (p. 136). Mathematicians responded to this loss of certainty by attempting to ground the foundation of mathematics in logic, arithmetic or set theory, but these attempts led to contradictions (Hersh, 1997). Formalism, the attempt "to characterize mathematical ideas in terms of formal axiomatic systems" (Dossey, 1992, p. 41) emerged in response to these conundrums.

Formalism.

Under the formalist paradigm, mathematics begins with a collection of formal axioms and definitions. From these axioms and definitions theorems are logically deduced. Perhaps the key difference between formalism and the Euclidean paradigm is the replacement of the notion of postulate with the term axiom (although the terms are often conflated). In contrast to a postulate, which is believed to be obviously true, an axiom is not necessarily true or false, but a starting point. If we adopt *Axioms X, Y, Z* we arrive at one branch of mathematics; or we can adopt *Axioms W, X, Y* and arrive at a different branch. Under formalism, neither branch of mathematics is *truer* than the other. In describing today's formalist, Hersh (1997) wrote,

> For him, all mathematics, from arithmetic on up, is a game of logical deduction. He defines mathematics as the science of rigorous proof. […] All mathematicians can say is whether the theorem follows logically from the axioms. Mathematical theorems have no content; they're not *about* anything. On the other hand, they're absolutely free of doubt or error, because a rigorous proof has no gaps or loopholes. (p. 163)

So in a sense, formalism allows the mathematician to regain a sense of absolute certainty. But Ernest (1991) argued that it is impossible for formalism (or any deductivist paradigm) to serve as a means of arriving at absolute truth in mathematics.

> Mathematical truth ultimately depends on an irreducible set of assumptions, which are adopted without demonstration. But to qualify as true knowledge, the

> assumptions require a warrant for their assertion. There is no valid warrant for mathematical knowledge other than demonstration or proof. Therefore the assumptions are beliefs, not knowledge, and remain open to challenge, and thus to doubt. (p. 14)

Hersh (1997) noted that while formalism may be the official view, and dominant in mathematics textbooks, it is problematic in practice.

> Indeed, formalism contradicts ordinary mathematical experience. Every school teacher talks about 'facts of arithmetic' or 'facts of geometry.' In high school the Pythagorean theorem and the prime factorization theorem are learned as true statements about right triangles or about natural numbers. Yet the formalist says any talk of facts or truth is incorrect. (p. 163)

Hersh (1997) claimed that the working mathematician is caught between Platonism and formalism. "[W]hen he is doing mathematics, he is convinced that he is dealing with an objective reality… But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all" (p. 11). Perhaps Platonism and formalism continue to have influence in mathematics because people are not aware there is an alternative to these views. Hersh (1997) wrote, "To abandon both, we must abandon absolute certainty, and develop a philosophy faithful to mathematical experience" (p. 43).

Both Platonism and formalism (or a melding of the two) perpetuate the absolutist view that mathematical knowledge "consists of certain and unchallengeable truths"

(Ernest, 1991, p. 7). Humanists adopt the fallibilist view: mathematics is an imperfect human activity, and so mathematical knowledge is subject to revision. According to humanistic philosophers, we need not hold onto the idea that mathematical knowledge is absolutely certain, or that mathematics is restricted to axioms, definitions, and proof. Hersh (1997) noted, "Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them" (p.22). Lakatos's (1976) *Proofs and Refutations* highlighted the way mathematical knowledge can be revised over time as mathematicians find new counterexamples to established proofs (even proofs that were widely regarded as correct may be eventually be refuted). As Hersh (1997) noted, "*For two millennia, mathematicians and philosophers accepted reasoning that they later rejected*. Can we be sure that we, unlike our predecessors, are not overlooking big gaps? We can't. Our mathematics can't be certain" (p. 45).

Fallibilism within mathematics education.

Within mathematics education, scholars have called for a reform of school mathematics to align with the fallibilist view. Sometimes the fallibilist view is implicit in reform documents such as NCTM's (1989) standards (Toumasis, 1997). Other times the fallibilist view is passionately articulated by mathematics education scholars (Ball, 1988; Burton, 1995; Lampert, 1990). Burton (1995) noted that if we consider mathematics to consist of a body of absolute truths then "the purpose of education is to convey [the truths] into the heads of learners" (p. 276). When mathematics is presented as "information which should not be questioned" (p. 276), some learners may perceive that mathematics is a subject that only a few people have the ability to understand. Thus an absolutist conception of the nature of mathematics, in addition to being philosophically indefensible (Ernest, 1991; Hersh, 1997), also disempowers learners who do not immediately perceive the “truth” of what the teacher is trying to convey in the classroom. Burton (1995) advocated for a humanist/feminist view of mathematical knowledge in school, claiming that “Re-telling mathematics, both in terms of context and person-ness, would consequently demystify and therefore seem to offer opportunities for greater inclusivity” (p. 280).

We have some empirical evidence related to the influence of teachers’ fallibilist or absolutist views on mathematics instruction, but as Thompson (1992) noted, “Very few cases of teachers with an informed historical and philosophical perspective of mathematics have been documented in the literature” (p. 141). Lerman (1990) conducted a study with four student teachers, two who held absolutist views, and two who held fallibilist views. After watching a video extract from a secondary mathematics lesson, “the two student teachers who were the most ‘absolutist’ felt that the teacher in the extract was not directing the students enough and was too open, whereas the most ‘fallibilist’ thought she was not open enough, and was too directed” (p. 59).

Beswick (2012) conducted case studies with two mathematics teachers, incorporating Ernest’s (1989) three categories of teacher beliefs about mathematics: instrumentalist, Platonic, and problem-solving. The instrumentalist view is that mathematics consists of isolated rules; the Platonic that mathematics is discoverable body of absolute knowledge; and the problem-solving view is that mathematics is a dynamic body of knowledge created by humans (similar to the fallibilist view). Beswick (2012) sought to determine if teachers could hold disparate views of school mathematics and the discipline of mathematics, understand how these disparities might arise, and consider what implications they held for practice. Sally, an experienced secondary teacher, possessed a problem solving view of school mathematics and a Platonic view of the discipline. Her problem solving view was likely a result of the influence of the reform agenda, particularly during her three years as a Senior Curriculum Officer for her district. It appears she rarely reflected on mathematics as a discipline, since she had been a school teacher for 18 years. Sally's teaching practice was influenced by her problem solving view of school mathematics rather than her Platonic perspective of the discipline. Jennifer, a novice middle grades teacher, possessed an instrumentalist/Platonic view of mathematics. Although she tried to incorporate student-centered teaching approaches in her classroom, "[s]he appeared to be struggling to reconcile her predominantly Platonist beliefs about the nature of mathematics with a desire to teach mathematics consistently with a problem solving perspective" (p. 143). Beswick noted that Jennifer had participated in some professional learning that "appears to have been effective in inspiring her to try different approaches to teaching but not to have addressed her beliefs about what mathematics is; hence her conflict" (p. 144).

Based on the studies of Lerman (1990) and Beswick (2012) we cannot draw strong conclusions about the relationship between teachers' (absolutist/fallibilist) views of mathematics and their teaching practice. In Lerman's study, there were clear distinctions between the ways teachers with absolutist/fallibilist views interpreted teaching; the case of Jennifer (Beswick, 2012) corroborates this finding, being another instance of a teacher with absolutist views tending to prefer lecture-centered instruction. However the case of Sally (Beswick, 2012) seems to imply that it is possible for a teacher to buy into the reform movement's implicit fallibilist message for school mathematics, while still holding a Platonic view of mathematical knowledge. More research is needed to understand the interactions between a teacher's views of school mathematics, the discipline of mathematics, and teaching practice.

Critics of fallibilism.

It is in some sense revolutionary to adopt a fallibilist view of mathematical knowledge within the mathematics classroom. After all, are we not absolutely certain that $2 + 2 = 4$?[4] Is this not a truth that is beyond human experience? Although acknowledging that mathematical knowledge is subject to revision in the discipline, de Villiers (2004) critiqued fallibilism as a philosophy of *school* mathematics.

> Undeniably, our mathematical results, once proven correctly (though perhaps subject to later revision), are still the most certain of all human knowledge. I am definitely (several orders of magnitude!) more certain of the universal validity of

[4] Of course, when considering the set of integers {0, 1, 2} under the operation of addition modulo 3, we know that $2 + 2 = 1$. Ernest (1991) discussed the crucial roles language and convention plays in the development of mathematical knowledge. Hersh (1997) distinguished between number as adjective and number as noun to explain why mathematical knowledge appears absolute (but really is not).

> the Pythagorean theorem in the plane, than I am about whether the sun will rise tomorrow. Even if the earth were suddenly to be destroyed tomorrow, this would not alter the theorem's validity elsewhere in the universe. It is misleading, therefore, and not in the best educational interests of our students, to deny the existence of this extremely high level of certainty in mathematics. (p. 409)

This passage reminds me of a quote by Lakatos (1962), "Why not honestly admit mathematical fallibility, and try to defend the dignity of *fallible* knowledge from cynical skepticism[?]" (p.184). It is not harmful to teach students to have a high degree of certainty in their mathematical knowledge. The danger comes when we expect students to accept mathematical knowledge without understanding (or risk failing the course). We must treat mathematical knowledge in school as dynamic to guard against the trend that "Knowing mathematics in school [is] having a set of unexamined beliefs" (Lampert, 1990, p. 154). For the active research mathematician, her certainty may increase over time as she considers the mathematical evidence. Students must be given the opportunity to develop certainty in the same manner instead of being forced to memorize absolutely certain "knowledge."

Mathematical Knowledge is Socially Validated (NOMK)

Philosophers presenting a humanistic view of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976; Tymoczko, 1988) are frequently cited and have been influential in mathematics education (e.g. Ball, 1988; Boaler, 2016; Komatsu, 2016; Lampert, 1990; Larsen & Zandieh, 2008; Weber, Inglis, Mejia-Ramos, 2014). A key feature of humanistic philosophy that finds its way into the classroom is the notion that mathematical knowledge is socially validated.

Hersh (1997), an academic mathematician, emphasized the importance of proof in the social validation of knowledge. Hersh noted that for mathematicians, the primary purpose of proof is to convince other mathematicians that a claim is true (Hersh, 1993). But the truth of the statement is not within the proof itself, but in the refereeing. Hersh (1997) claimed, “What mathematicians at large sanction and accept *is* correct mathematics” (p. 50). Furthermore, “There are different versions of proof or rigor, depending on time, place, and other things” (p. 22). Using data from an interview study with mathematicians, Weber (2008) noted that mathematicians validated proofs differently depending if the proof was supposedly written by a student or by a professional mathematician. Thus mathematicians apply different standards depending on the community in which a proof is intended. Weber (2008) claimed that it may be valuable for students “to appreciate the social functions of proof in helping mathematical communities understand why certain theorems are true” (p. 452). How might students come to understand the social function of proof and the socially validated nature of mathematical knowledge? Although we do not have much research in this area, I propose that it may be valuable for students to be given some authority to judge what is correct through classroom discussions, and to negotiate standards of rigor and proof (cf. Ko, Yee, Bleiler-Baxter, & Boyle, 2016).

Ernest has described in detail the general process by which mathematical knowledge is socially validated in the discipline of mathematics. In his *Philosophy of Mathematics Education* (1991) Ernest presented his social constructivist philosophy of mathematics, a philosophy grounded in Lakatos's work. Ernest contended that mathematical knowledge is created through a subjective/objective cycle. Individuals subjectively create knowledge, and it is validated inter-subjectively by other mathematicians so that it becomes objective knowledge accepted by wider communities (perhaps through publication). This objective knowledge can then inspire more individual thought as it is subjectively reconstructed, and this may lead to further subjective creations, which may then in turn become objective taken-as-shared knowledge. Because of the cyclic and social nature of this process, objective knowledge is seen as tentative and subject to revision.

Mathematics students engaged in the social validation of knowledge.

We see an example of Ernest's cycle of knowledge construction in Deborah Ball's (1993) third grade classroom. Ball's teaching was directly influenced by Lakatos's philosophy (see Ball, 1988). The students in her class were often engaged in discussing, conjecturing, formulating working definitions of mathematical concepts, or justifying their reasoning to others. One day a boy named Sean said that he was thinking about the number six. He reasoned the number six is both even and odd because it can be broken up into three groups of two; two is even, but three is odd. Thus six is odd (since it is made up of an odd number of groups of two). Through the lens of Ernest's cycle of knowledge construction we perceive Sean as having constructed subjective knowledge. The other students then attempted to refute Sean's claim, barring it from objective status. For instance, Mei claimed that if six is even and odd, then there will be countless other odd numbers that are both even and odd, including ten. Ultimately, Sean's idea does contribute to the objective knowledge of the community. Ball explains to the class that Sean has discovered a new type of number. Sean provides the definition for "Sean numbers," numbers that can be formed using an odd number of groups of two. For the next several days in class students studied the properties of Sean numbers (e.g. the sum of four consecutive integers is always a Sean number.). At this stage the notion of Sean number had been accepted as objective knowledge by the classroom community. Furthermore the knowledge was subjectively reconstructed by the students and led to the refinement and growth of what was objective, taken-as-shared.

Magdalene Lampert was one of Ball's mentors, serving on her dissertation committee (Ball, 1988). She was also influenced by Lakatos's philosophy[5], and students in her fifth grade classroom were collectively responsible for the validation of mathematical knowledge. Lampert (1990) explicitly addressed how humanistic philosophy influenced her classroom practice, and she provided an existence proof that fifth grade students can engage in what she called authentic mathematical practices. Lampert described mathematics as a social process of conscious guessing, conjecturing, refuting, and generalizing where "reasoning and mathematical argument—not the teacher

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[5] Lampert (1990) also cites the influence of Póyla (1954).

or the textbook—are the primary source of an idea's legitimacy" (p. 34). Lampert showed it is possible for fifth grade students to hypothesize, argue for the legitimacy of their hypotheses, and develop acceptable norms for critiquing the reasoning of others and revising their own ideas. Lampert claimed "that the students had learned to regard themselves as a mathematical community of discourse, capable of ascertaining the legitimacy of any member's assertions using a mathematical form of argument" (p. 42). Lampert's role as a teacher was to provide a model of a mathematical expert, introduce students to tools and conventions, and "follow students' arguments as they wander around in various mathematical terrain and muster evidence as appropriate to support or challenge their assertions, and then support students as they attempt to do the same thing with one another's assertions" (p. 41). I believe Lampert's work (along with Ball's) provides insight into what classrooms might be like when a teacher possesses what Thompson (1992) called "an informed historical and philosophical perspective of mathematics" (p. 141). It is in such classrooms that students are most likely to experience and understand the socially validated nature of mathematical knowledge.

Lampert (1990) inferred that the students in her class viewed mathematics differently than students in traditional classes. Yackel and Rasmussen (2002) made a similar claim about students in an inquiry-oriented undergraduate differential equations course. They wrote that some of the students took "seriously the obligations of developing personally-meaningful solutions, of listening to and attempting to make sense of the thinking of others, and of offering explanations and justifications of their mathematical thinking" (p. 324). In other words, the students were engaged in the social validation of mathematical knowledge.

> The normative patterns of interaction serve to sustain the expectations and obligations on which they are based and thus to sustain individual participants' beliefs about their role and about what constitutes mathematical activity in this classroom. (p. 324)

Are the students in the classrooms described by Lampert (1990) and Yackel and Rasmussen (2002) developing a view of what constitutes mathematical activity in the *discipline*, or a view about "what constitutes mathematical activity in this *classroom*" (Yackel & Rasmussen, 2002, p. 324)? [emphasis added] That is, are students developing disparate notions of mathematics as a school subject versus mathematics as a discipline (cf. Beswick, 2012)?

Research in science education has found that "doing science" will not necessarily lead to students' understanding of the nature of scientific knowledge within the discipline (Bell, Blair, Crawford, & Lederman, 2003). The outcome may be different for mathematics instruction, but we need research to investigate this question. Perhaps if students are engaged in the social-validation of mathematical knowledge for a significant portion of their schooling then they may come to understand that mathematical knowledge is socially validated both in their own classrooms and within the discipline.

Proofs are Bearers of Mathematical Knowledge (NOMK)

Central to humanistic philosophy of mathematics is the distinction between descriptive and prescriptive philosophy (Ernest, 1991). Philosophy of mathematics traditionally was driven by the notion of what mathematics *ought to be*. For a Platonist or formalist, mathematics is (or should be) the most certain and absolute of human knowledge; thus the purpose of philosophy of mathematics is to justify its absolute nature. For instance, even if most published mathematical proofs contain gaps in logic (Ernest, 1991; Hersh, 1997), proof *ought* (from a prescriptive perspective) to be a rigorous deductive argument from accepted premises to conclusion.

Humanistic philosophers work from a descriptive perspective, basing their philosophy on what mathematicians actually do, rather than an ideal vision. Thus humanistic philosophy of mathematics is distinct from traditional philosophy because it incorporates sociological, anthropological, or historic studies of mathematics. Indeed the classroom discussions in Lakatos's (1976) *Proof and Refutations* paralleled an account of the historical events surrounding the development of proof of an Eulerian conjecture. Hersh (1997) wrote "A humanist sees mathematics as a social-cultural-historic activity. In that case it's clear that one can actually look, go to mathematical life and see how proof and intuition and certainty are seen or not seen there" (p. 48). The humanistic emphasis has relevance not only to philosophers, but also for mathematics education scholars who desire to understand the practices of mathematicians so that implications may be drawn to inform the teaching and learning of mathematics.

In his interviews with professional mathematicians, Weber (2010) found that mathematicians read proofs in order to learn new methods and techniques that could be used in their own work, in essence, filling their mathematical toolbox. It seems the implication for mathematics education is that it can be valuable for students to read and comprehend proofs, as they will learn new mathematical methods. Hanna & Barbeau (2008) describe a similar notion, that proofs are bearers of mathematical knowledge (Rav, 1999). Hanna and Barbeau (2008) noted that students can fill their mathematical toolbox, learning methods as they work to comprehend unfamiliar proofs. As Weber (2010) noted, mathematicians read proofs to gain insight into methods that may be valuable to use in their own work.

Pair and Bleiler (2015) reported that undergraduate students in a transition-to-proofs course found value in reading classmate's proofs in order to get new ideas to aid in their own proof construction. They described an instructional activity in the course called the "critiquing activity" during which students read several arguments for the same mathematical claim and identified the strengths and weaknesses of the arguments. One student, Krissy, wrote, "Every time we have engaged in this exercise, I have found new ideas and techniques for proof-writing that I eagerly attempted to use in my own proofs" (p. 14). This student reflection demonstrates that at least in an undergraduate transition-to-proof course, proofs can be bearers of mathematical knowledge, providing students with the opportunity to acquire new mathematical techniques. Undergraduates should be aware that a key function of proof for mathematicians is the transmission of methods

(Weber, 2010). Students should be encouraged, when reading proofs, to get ideas for methods to incorporate in their own proofs. Students may also benefit from learning about the other roles proof serves for the discipline of mathematics (de Villiers, 1990).

Informal Mathematical Work is the Foundation of Formal Knowledge (NOMK)

Within mathematics education research, researchers have identified students who view mathematics as a collection of meaningless rules and procedures rather than a logical subject that can be made sense of (Boaler, 2000; Erlwanger, 1978). This neo-formalist view is likely due to a teacher's "emphasis on the manipulation of symbols whose meanings are rarely addressed" (Thompson, 1992, p. 127). An allegiance to formalism may also result in the delegitimization and rejection of informal mathematics. For instance, Brown (1996) described a formalist mathematics instructor in a graduate course. This instructor explained that the only valuable objects in mathematics were the formalisms: axioms, definitions, and logically deduced theorems. During the writing of a proof at the board, he became temporarily stumped and resorted to draw a diagram. After using the diagram to obtain the insight needed to finish the proof, the teacher hurriedly erased his diagram and resumed the formalist presentation. According to Hersh, the "grip of formalism" (p. 186) prevents mathematicians from accepting visuals or diagrams in or as proofs. Also the work that went into the creation of axioms and theorems is hidden behind the formal presentation. Lakatos (1976) wrote,

> This [deductivist] style starts with a painstakingly stated list of *axioms*, *lemmas* and/or *definitions.* The axioms and definitions frequently look artificial and

mystifyingly complicated. One is never told how these complications arose. [...] In the deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever increasing set of eternal, immutable truths. (p.142)

Axioms may be presented to students as having a divine status that is not to be questioned (Brown, 1996). From a humanistic perspective, students should have the opportunity to understand the concepts behind axioms and why those axioms were formulated. Hersh (1991) elaborated on the importance of informal mathematical work when he claimed that mathematics has a front and a back. The front is what is typically seen, in journals, and in textbooks (e.g. axioms, definitions, theorems, proofs). The front of mathematics is the polished, finished form of mathematics. But just as important and meaningful is the behind the scenes work, the creative emotional activity that serves as the basis for formal mathematical knowledge. Hersh noted that mathematics, being a human activity, is influenced by economic and social pressures. He explained that mathematics as an institution benefits from a presentation that hides the human struggle.

> The standard exposition purges mathematics of the personal, the controversial, the tentative, leaving little trace of humanity in the creator or the consumer. […] If mathematics were presented in the style in which it’s created, few would believe its universality, unity, certainty, or objectivity. These myths support the institution of mathematics. For mathematics is not only an art and a science, but also an institution, with budgets, administrations, rank, status, awards, and grants. (p. 38)

Students within a mathematics classroom, especially an undergraduate setting should understand the nature of the informal work that ultimately leads to polished theorems. Lakatos (1976) claimed mathematics is a quasi-empirical discipline. De Villiers (2004) summarized the notion of quasi-empiricism:

> [T]he objects in mathematics, though largely abstract and imaginary, can be subjected to empirical testing much as scientific theories are. Quasi-empiricism will, therefore, refer here to all non-deductive methods involving experimental, intuitive, inductive, or analogical reasoning. (p. 398)

Lakatos (1976) showed how quasi-empirical methods (e.g. using a counterexample to refute a theorem statement), are implemented in practice and contribute to the development of mathematical knowledge. Mathematicians often use examples to look for patterns and make conjectures (de Villiers, 2004); it is this informal work that ultimately leads to the formal theorem. In interviews in which mathematicians were asked to determine if a proof was valid, Weber (2008) found that mathematicians used inductive examples to make sense of deductive inferences within a proof. Students should understand that mathematics is not only axioms, definitions and the following of deductive steps. Inductive methods also play a crucial role in creating (Lakatos, 1976) and validating (Weber, 2008) mathematical knowledge.

Mathematical Inquiry can be Creative, Emotional, and Collaborative (NOMI)

Note that in discussing informal knowledge, I have described both the nature of mathematical knowledge (formal knowledge is dependent upon informal work) but have also touched on aspects of the informal practices of mathematical inquiry (e.g. inductive methods). The rest of this section will focus on characteristics of the nature of mathematical inquiry (NOMI).

The mathematics education scholar Leone Burton (1999) wrote, “you cannot separate the mathematics from the people who produce it” (p. 134). Burton (1999) interviewed 35 female and 35 male research mathematicians. She focused on the participants’ “‘life history’ as mathematicians, especially their feelings about the nature of knowing mathematics, and how they come to know” (p. 122). Burton found that knowing mathematics was related not only to rationality and logic, but also emotion.

> So, coming to know, for my participants, was represented by feelings, the powerful sense of Aha! which is what holds them in mathematics […] Whether your knowing is robust, or not, for the moment that you know that you know the power of that knowledge lies in the feelings it evokes not externally in the mathematics [...] There is a chasm between this perspective on coming to know, and the transmission pedagogy of the classroom dependent as it is on acquiring the knowledge of the expert. (p. 135)

Burton noted that the feelings described by the mathematicians (e.g. euphoria, excitement) contrast sharply with the descriptions often given by students (e.g. boring, frustrating, anxiety-causing). She wrote that mathematicians “gain pleasure and satisfaction from the feelings which are associated with knowing” (p. 134). Emotion is interconnected with conviction and can drive mathematical work. As Pólya (1954) noted, one is often convinced that a mathematical claim is true before one begins a proof. Burton claimed that this conviction is a feeling that leads a mathematician to persevere: "because of your feelings you remain *convinced* that a path is there" (p. 134). Certainly a goal is for every mathematics student to have the opportunity to experience these human aspects of mathematics and understand that the work of the pure mathematician involves creativity, collaboration, and intuition.

Each Sub-discipline of Mathematics has Different Norms, Values, and Standards (NOMI)

Burton (1999), in her interviews with mathematicians from diverse sub-disciplines, found that while there are similarities (e.g. collaboration, pressure to publish) across disciplines, each sub-discipline had its own culture. In his interviews with mathematicians, Weber (2008) found that proof techniques are context-dependent, i.e. proof techniques are valuable within certain domains of mathematics (e.g. algebra, graph theory, real analysis). He recommended that instead of viewing proof techniques as context-free, instructors should recognize that certain proof validation strategies "must be learned in the context of studying particular mathematical domains" (p. 452).

Summary of Ideas Related to the Nature of Pure Mathematics

In terms of the nature of mathematical knowledge (NOMK) I have discussed that mathematical knowledge is subject to revision (Hersh, 1997; Lakatos, 1976), mathematical knowledge is socially validated (Ernest, 1991; Lakatos, 1976), proofs are bearers of mathematical knowledge (Hanna & Barbeau, 2008, Weber 2010), and informal mathematical work is the foundation of formal knowledge (Hersh, 1991; Lakatos, 1976). In terms of the nature of mathematical inquiry (NOMI) I discussed that mathematical inquiry can involve collaboration and be creative and emotional (Burton, 1995, 1999, 2002), and that there is diversity of mathematical practice across sub-disciplines of academic mathematics (Burton, 1998; Weber, 2008).

Statistical and Applied Mathematics

Steen (1988) broke the mathematical sciences into three component parts: statistical science, core mathematics, and applied mathematics. I believe students should understand the difference between these three types of mathematics. We have already discussed core or pure mathematics, and will here briefly describe statistical and applied mathematics. It is beyond my level of expertise to provide a full description of the nature of these types of mathematics. More work must be done beyond this dissertation study to adequately articulate goals for student and teacher understanding related to these forms of mathematics.

Steen (1988) noted that “Statistical science investigates problems associated with uncertainty in the collection, analysis, and interpretation of data” (p. 612). And “Applied mathematics fits mathematical methods to the observations and theories of science. It is a principal conduit for scientific ideas to stimulate mathematical innovation and for mathematical tools to solve scientific problems” (p. 612). Statistics and mathematical modeling are becoming increasingly important in society, and citizens should understand the nature of each in order to make informed decisions on matters in which these types of mathematics are applied.

Mathematical Knowledge is Used to Shape Society, but cannot be Considered an Absolute Judge

Mathematics is often applied in society but this application sometimes portrays mathematics in an absolutist way. Borba and Skovmose (1997) noted that when mathematics is seen as above humans and free of human influence that it can be used to shape society in the guise of a neutral mechanism. When we problematize a human situation the mathematical outcome is not the end of the story. Mathematics is only one of many features that should be considered when making societal decisions. As Borba and Skovmose (1997) put it, "the problem arises when one believes that by applying 'a perfect body of knowledge' to a problem one will have *'the* solution'" (p. 18). In my view, it is critical that we not make decisions contrary to human interests simply because it is mathematically optimal.

Borba and Skovmose (1997) argued that pedagogy focused on correct answers and procedures will reinforce the absolutist ideology of mathematics. Teachers should not only be aware of the influence of this ideology, but challenge it with alternative pedagogies. Alrø & Skovmose (1996) claimed that one way to combat the ideology of certainty is to focus on students' good reasons for their mathematical actions rather than on correcting their mistakes. Such pedagogies must emphasize discussion and context rather than consistently present students with situations for which there is one right answer. Students should be aware that when applied, mathematics is not an impartial judge, and cannot be a final arbiter in societal decisions.

Chapter Summary

In this chapter I have described mathematics as a fundamental cultural activity. I have situated pure mathematics as one type of mathematics among many, with its own unique practices and form of knowledge. I have articulated some characteristics of mathematics as a cultural activity, especially pure mathematics in particular, to form an initial draft of a NOM framework. This initial draft was foundational to my subsequent dissertation study in which I immersed myself in doing mathematics in order to shed light on NOM and its teaching and learning. This initial framework was also refined through my dissertation study, the details of which are described subsequently in Chapter Three.

CHAPTER THREE: METHODOLOGY

Purpose and Research Questions

The purpose of my dissertation is to begin the work of "conceptualizing the construct" (Lederman & Lederman, 2014, p. 600) of the nature of mathematics as has been done with the nature of science in science education scholarship. I have begun the creation of a humanistic educational framework for the nature of mathematics (NOM), and drafted narratives that help to clarify and illuminate key features of NOM. Two broad questions, "What is the nature of mathematics?" and "What should students understand about the nature of mathematics?" were explored in the previous chapters. These broad questions are not limited to one type of mathematics, say pure or applied, and they are not limited to a specific student demographic. But for my dissertation study, I narrowed my focus to one type of mathematics, pure mathematics, and I focused on undergraduate students' understanding the nature of pure mathematics within a transition-to-proof course. I sought to understand, "What is the nature of pure mathematics?" and "What should undergraduate students in a transition-to-proof course understand about the nature of pure mathematics?" To answer these questions I worked together with a research mathematician on an unsolved conjecture in graph theory, and I co-taught an undergraduate transition-to-proof course with another mathematics education scholar. In regard to the framework presented in Chapter Two, my aim with the dissertation was to further conceptualize and revise the second major category: the nature of pure mathematics as a discipline.

A Focus on Pure Mathematics and Undergraduates

In this dissertation, I have done some work in reflecting on various types of mathematics and goals for students' understanding of NOM at all grade levels. But during my dissertation study, I made efforts to narrow my focus to pure mathematics and undergraduates' understanding of NOM. In Chapter One, I provided a personal rationale for why I focused on pure mathematics based on my academic background. I now provide additional rationale for this decision and also provide rationale for my decision to narrow my focus to undergraduates.

It is widely known in mathematics education that the public has little idea what pure mathematicians do (Picker & Berry, 2000). Picker and Berry found that for school students, "mathematicians are essentially invisible" (p. 88) and that "Pupils believe that mathematicians do applications similar to those they have seen in their own mathematics classes, including arithmetic computation, area and perimeter, and measurement" (p. 88). Certainly, students do not come to have a realistic image of the practicing mathematician by completing school; I believe school portrays mathematics as something wholly different from the work of mathematicians. Jo Boaler (2016) wrote,

> This wide gulf between real mathematics and school mathematics is at the heart of the math problems we face in education. I strongly believe that if school math classrooms presented the true nature of the discipline, we would not have this nationwide dislike of math and widespread math underachievement. (pp. 22-23)

But, in order for school mathematics to present the "true nature of the discipline," we need an explication of the discipline in a form that educators can understand.

If we hope, as Rock and Shaw (2000) did, that children see "mathematics as a modern-day career choice" (p. 554) then much more work must be done so that people in general, but especially mathematics teachers, understand the work that mathematicians do. I believe the university must take the lead in teaching NOM if we are to see effects in primary and secondary schools. The university is where mathematics teachers are educated; I agree with Fried (2014) that those who are "mathematically educated must feel at home with mathematics, appreciate its power, and know it as a part of one's culture" (p. 30). The conceptions of NOM that future teachers develop in the university will likely stay with them as they begin to teach, and will likely have an influence on their students' understanding[1]. If NOM cannot be taught at the university, the home of disciplinary mathematics, then where can it be taught?

Methodological Framework: Heuristic Inquiry

The methodological framework that I chose for this study is called heuristic inquiry, which was coined by the humanistic psychologist Clark Moustakas. Heuristic is an adjective, defined by google as "enabling a person to discover or learn something for themselves." I sought to understand what is the nature of pure mathematics (NOM[2])? But

[1] Thompson (1984) noted that more research is needed to understand how and if a teacher's conception of mathematics influences the conceptions' of the students. In Thompson (1990) she again reiterated that virtually no research has been conducted in this area. Although I presented a few related studies in Chapter Two (e.g. Beswick, 2012; Lerman, 1990), to my knowledge scholars in mathematics education still have yet to thoroughly investigate this issue.
[2] In subsequent sections of this chapter, NOM refers almost exclusively to the nature of pure mathematics.

of course, pure mathematics is what mathematicians do (Ernest, 1991; Hersh, 1997). As Courant and Robbins (1941) claimed in the introduction to their book *What is Mathematics?*: “For scholars and laymen alike it is not philosophy but active experience in mathematics itself that can alone answer the question: What is mathematics? (p. xix)” I reasoned that if I wanted to understand the nature of pure mathematics, then I must have experience doing pure mathematics.

I decided that the core of my dissertation would be the documentation of and reflection on my own experience doing pure mathematics research in collaboration with an established research mathematician. Patton (2015) wrote that the core question of heuristic inquiry is “What is my experience of this phenomenon and the essential experience of others who also experience this phenomenon intensely?” (p. 118). In this light, heuristic inquiry seemed to be a perfect fit to study my experience doing pure mathematics.

Context and Data Collection

Heuristic inquiry is a self-study, and the inquiry “brings to the fore the personal experience and insights of the researcher” (Patton, 2015, p. 118). A key feature of heuristic inquiry is the concept of immersion:

> The researcher is alert to all possibilities for meaning and enters fully into life with others wherever the theme is being expressed or talked about—in public settings, in social contexts, or in professional meetings. Virtually anything connected with the research question becomes raw material for immersion, for

> staying with, and for maintaining a sustained focus and concentration. (Moustakas, 1990, p. 28)

In order to honor the immersive spirit of heuristic inquiry I decided to collect a large amount of qualitative data in order to capture my experience of mathematics wherever it emerged over the course of one university semester. All the sources of data that were collected and analyzed in this study are listed in Figure 3 and then subsequently described.

Mathematics Collaboration Data
• Audio-recordings of discussions with mathematician • Hard copies of mathematical work (whiteboard photos and personal notebooks)
Mathematics Course Data
• Class materials (e.g. handouts, PowerPoint slides) • Audio recordings of whole class discussions • Audio recordings of discussions with co-instructor • Student homework, classwork, and exit tickets
Journal Data
• Journal in which the researcher reflected on his experiences doing mathematics, teaching mathematics, discussing NOM, and reading NOM literature
Other Data
• Informal Interviews • Personal Audio / Other Photos / Documents / Notes

Figure 3: Data Sources

Mathematics Collaboration Data

To deeply understand the nature of mathematics, I knew it was necessary for me to engage deeply in mathematical inquiry. To that end I sought collaboration with a graph theorist and we worked together on an unsolved conjecture. The graph theorist, whom I will refer to as Dr. Combinatorial, is a full professor and active research mathematician. I recorded all of our conversations in which we discussed the conjecture. I also kept hard copies or photos of all of our mathematical work. For instance, Figure 4 is a picture from the whiteboard in Dr. Combinatorial's office. It shows many of the diagrams we drew for ourselves in order to communicate mathematical ideas to each other.

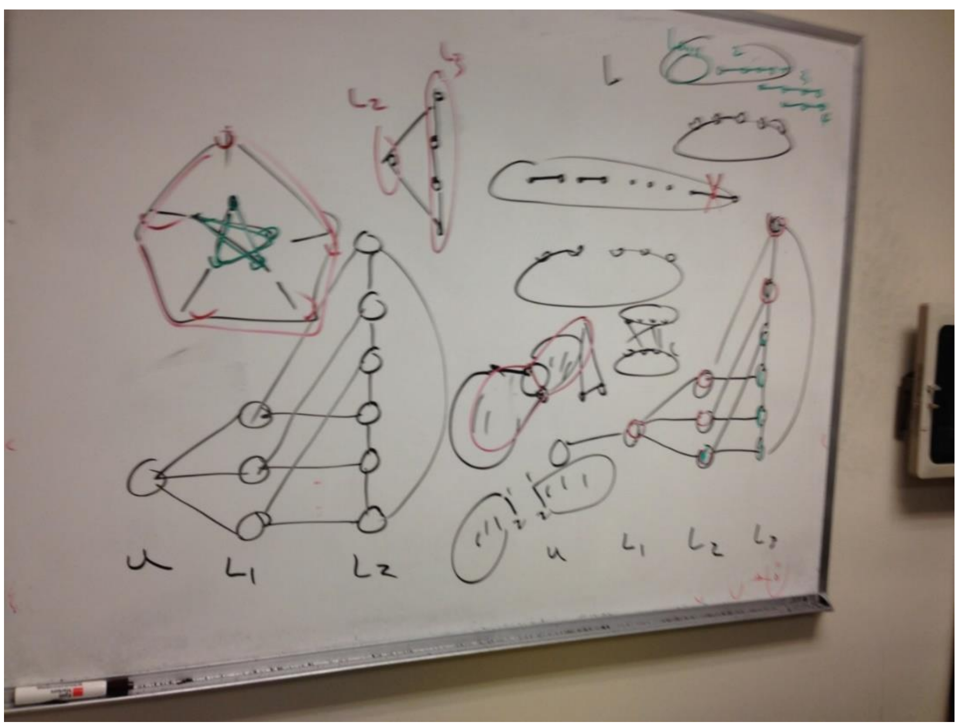


Figure 4. Mathematical Work on Dr. Combinatorial's White Board

Much of my mathematical work on the unsolved conjecture was done on my own time, and I kept a few mathematical notebooks in which I worked on the conjecture. These notebooks document my mathematical work and reflections. Throughout the process of working on the conjecture, I was not only doing mathematics, but I was constantly making an effort to reflect on my own experience as it was related to my research questions. The mathematics notebooks are filled with notes documenting my reflections on NOM in which I consider what students may benefit from understanding about NOMI and NOMK. For example, in the excerpt from my mathematics notebook shown below in Figure 5, there is a note, “I remembered Weber et al. 2014, sometimes trust an authority to discover or put forth effort into understanding it… If there are odd cycles, we must have a cycle of length 5?”

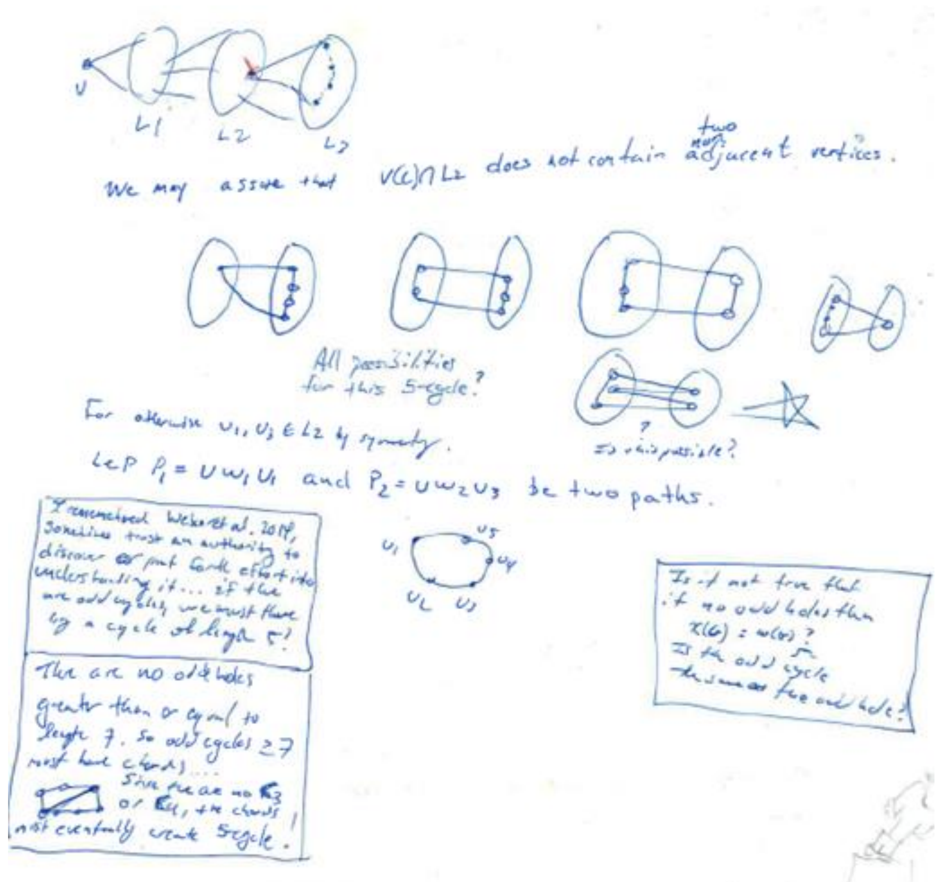


Figure 5. An Excerpt from a Mathematics Notebook

That note references a mathematics education paper by Weber et al. (2014) in which the authors, drawing from interview studies with mathematicians, noted that sometimes mathematicians accept mathematical claims made by respected authorities without justifying those claims themselves or even reading a proof. As an educational implication, the authors claimed that undergraduate students may benefit from accepting the validity of an argument put forth by an authority and push themselves to make sense of non-intuitive deductions within the argument. In my own case, I was unsure of a claim in Dr. Combinatorial's unpublished paper. I reminded myself of the advice of Weber and colleagues, and I proceeded to try to make sense of the claim under the assumption that it was correct. The note is illustrative of my efforts to document my experience doing and reflecting on the nature of my own mathematical inquiry while also considering what may be valuable for undergraduate mathematics students to understand about the nature of mathematical inquiry and knowledge.

Mathematics Course Data

During this dissertation study I also co-taught a course required of undergraduate mathematics majors. The course is called "Foundations of Higher Mathematics" and is meant to serve as a transition course as students proceed from lower-level to upper-level mathematics coursework. The transition represents a shift from the traditional procedurally-based school mathematics to the work that more closely resembles that of pure mathematicians. Thus a key element of the course is teaching students about the role of conjectures, theorems, and proofs in the discipline of pure mathematics (i.e. teaching NOMI and NOMK). I co-taught this course with another mathematics education scholar, Dr. Amicable, who had designed the course and taught it for seven prior semesters. I fully took over teaching the last month of the semester as she had to take a leave of absence. The course was inquiry-based in nature and students were constantly working together to draft arguments, critique arguments, and discuss and debate mathematical ideas and proof writing techniques.

The data I gathered included audio recordings of discussions I had with the co-instructor, audio of whole-class discussions, student homework, classwork, exit tickets, and all other class materials. Twenty three students from the course agreed to participate in my study. Dr. Amicable asked all of the students to decide which number type best captured their own personalities. See Appendix A for a description of Dr. Amicable and the undergraduate students along with their chosen number types, which I have chosen to be their pseudonyms in this study. I chose the number type Surreal as my own pseudonym. The students in the course had a variety of majors (e.g. mathematics education, industrial mathematics, aerospace, etc.…). These are also listed in the Appendix A if known. As an example of course data, Figure 6 is a photo of a poster that a small-group of students had produced and presented during one class session.

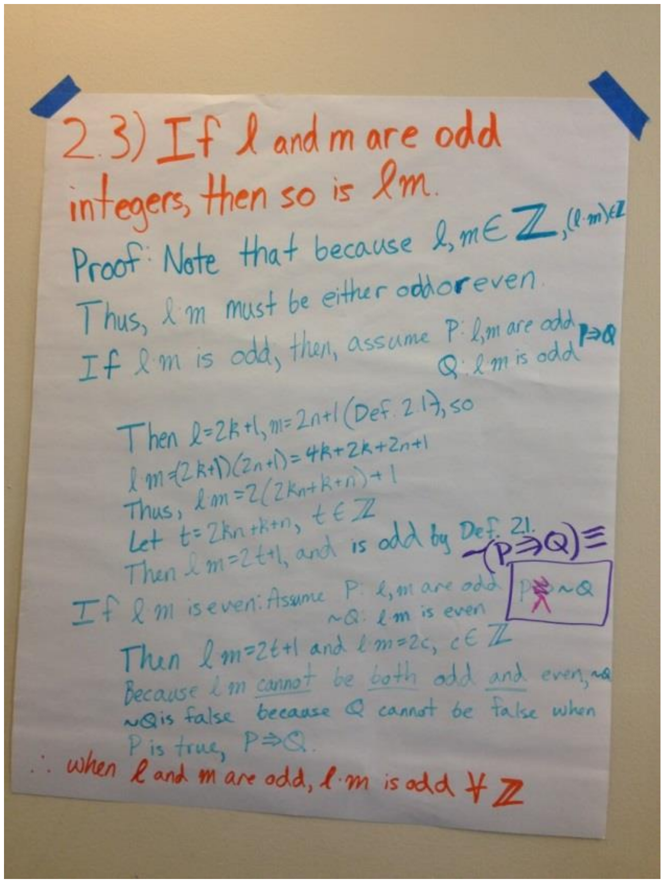


Figure 6: A Small Group's Proof

Journal Data

A crucial feature of my data for this self-study is a personal journal that I kept in order to write and reflect about my daily experiences doing and teaching mathematics. My writings were particularly focused on documenting and reflecting on my experiences relevant to NOM and its teaching and learning. Sometimes these reflections were directly related to the data I was collecting in regard to the mathematics collaboration or the mathematics course. For instance, consider the journal entry below which opens with a student's exit ticket from class.

> ---Has the math community changed over time when it comes to what counts as proof and what doesn't? (Composite).
>
> Composite wrote that as her question today. I feel a mixture of helplessness and excitement. Excitement that I could design instruction so that Composite could understand that indeed the math community has evolved in this regard. But I am discouraged because I don't see a way to disrupt the flow of Foundations of Higher Mathematics to make this possible. I imagine a future time when I am a faculty and I can design a lesson in which students critique proofs from mathematicians of different eras.

Notice that in this journal entry I am excited a student is asking a question about NOM, and I am thinking about a possible instructional design so that I could teach the student that, yes, indeed standards of proof have evolved within the discipline of mathematics.

The journal provided me a means to catalog meaningful events during data collection, and was used post data collection to identify critical NOM characteristics and corresponding narratives. It also served as an outlet for engaging in the concepts and processes of heuristic inquiry (Moustakas, 1990). For instance, one process of heuristic inquiry is self-dialogue:

> One may enter into dialogue with the phenomenon, allowing the phenomenon to speak directly to one's own experience, to be questioned by it. In this way, one is

> able to encounter and examine it, to engage in a rhythmic flow with it—back and forth, again and again—until one has uncovered its multiple meanings" (p. 16).

Here is an example of self-dialogue from my dissertation journal that was written after I worked through a proof of a well-known graph theory result after initially making a mistake:

> Skeptical Mathematician: What is this young mathematics education scholar think he is doing? He has just proven one of the most elementary results of graph theory. Clearly he does not have the knowledge or skill needed to conduct mathematical research or speak on the nature of mathematical knowledge.
>
> Me: Oh, I overheard you speaking about my work. Indeed you are justified in questioning my qualifications. There are many others more qualified than I to conduct mathematical research. But consider the mathematics student, learning mathematics for the first time. I wish to discover what aspects of the nature of mathematical knowledge they would consider fruitful. And I have something here in this excerpt. Mathematicians make mistakes. Mistakes are common in calculation. When we make a mistake, we may feel uneasy or puzzled. Such uneasiness can be a sign that we need to reconsider our mathematical work, and determine if we have made an error. It seems this might be valuable for the young elementary student as well as the undergraduate mathematics major just learning proof.

Other concepts central to heuristic inquiry which I sought to engage in were tacit knowing and intuition. During my study, I realized that the overall purpose of heuristic inquiry was what Sela-Smith (2002) called “a relentless inward focus” (p. 80). From my journal: “In essence I have decided that the unifying theme of heuristic inquiry is the relentless inward focus. Thus I try to articulate what is just barely perceptible within my consciousness.” When writing in my journal I always sought to bring to the surface and make explicit those thoughts and emotions what otherwise may have gone unnoticed.

Other Data

Informal Interviews.

Another important source of data came from audio recordings of informal coffee-shop style interviews that I conducted with persons whom I was interested in speaking to about NOM. These interviews generally consisted of conversations about NOM and interviewees’ opinions about what students should understand about NOM. Six people agreed to such interviews, and in some cases multiple interviews were conducted. Most notably are two mathematicians, Dr. Algebraic and Dr. Differential. When speaking to these mathematicians I was able to get feedback on my ideas about possible goals for students’ understanding of the nature of mathematics. Here is an example from an interview with Dr. Algebraic:

> Researcher: Do you think that it matters, say, for undergraduates to understand that mathematical knowledge is subject to revision?

Dr. Algebraic: I think it is. I wouldn't say that isn't. I wouldn't say that the fate of nations hangs on them knowing that. But it would be a good thing for them to know that. I think it should be part of their undergraduate exposure to have that knowledge. I have had [graduate] students tell me that there has been no new mathematics in the last 300 years. In all seriousness tell me that.

Researcher: Wow.

Dr. Algebraic: Yeah it was kind of scary.

Personal data.

I collected other personal data to capture and document my experiences and reflections relevant to the nature of mathematics. Sometimes I made audio recordings with my iPhone to capture the ideas that were bouncing in my head while driving. Other times I would take notes on my iPhone, perhaps laying in my bed at night or at a social function, and wanting to capture emotions I was feeling or what I thought might be a significant NOM idea came to mind. Here are some examples:

November 7, 2016 at 8:37pm

I am sad. I thought I had made progress on the problem, but I still am at the same place… I have to make sense of Gallai's theorem.

November 14, 2016 at 7:51pm

How is that for statistics? Trump for the win. Chainsaw videos. Email leaks at the last minute. "Trump protests sixth day in a row" is the headline on CNN. Children understand this mathematics. … Mathematics plays a part in the culture wars.

These quotations provide some insight into how I approached heuristic inquiry. Douglass and Moustakas (1985) wrote, “It is the focus on the human person in experience and that person’s reflective search, awareness, and discovery that constitutes the essential core of heuristic investigation” (p. 42). I sought to capture my experience of mathematics in whatever ways it entered into my consciousness. I wished to document my reflective search and discoveries. Consider one more example, the photo in Figure 7 that I took while on an airplane on the way to a mathematics education workshop. Looking down I was fascinated with the geometric patterns I saw on the ground below. I could not help but reflect on the idea that mathematics shapes our world (Borba & Skovmose, 1997). As I was not sure which of my experiences would prove to provide me deep insight into the nature of mathematics, I documented as much of my experience as possible.

Figure 7: Mathematics Shapes Our World

Data Analysis

I collected a very large amount of data; and it was a challenge to organize and make sense of it in achieving my dissertation goals. I will provide a brief overview of my data analysis before going into greater detail in subsequent sections. Douglass & Moustakas (1985) noted that heuristic inquiry is a process but does not lend itself to a particular methodology:

> As a conceptual framework of human science, heuristics offers an attitude with which to approach research, but does not prescribe a methodology. […] It is the focus on the human person in experience and that person's reflective search,

> awareness, and discovery that constitutes the essential core of heuristic investigation. (p. 42)

Although heuristic research does not prescribe a methodology, Moustakas (1990) did outline six phases of heuristic inquiry that he believed were crucial to the process: initial engagement, immersion, incubation, illumination, explication, and creative synthesis. I provide a brief overview of those phases as they were related to my work.

Overview of the Analysis Process in Terms of the Phases of Heuristic Inquiry

Initial engagement.

Initial engagement (Moustakas, 1990) refers to the beginning stages of the research process during which the researcher identifies a personally important problem or research question. For my inquiry, this included the initial review of the literature and also the early stages of data collection. Moustakas wrote,

> The question lingers within the researcher and awaits the disciplined commitment that will reveal its underlying meanings. The engagement or encountering of a question that holds personal powers is a process that requires inner receptiveness, a willingness to enter fully into the theme, and to discover from within the spectrum of life experiences that will clarify and expand knowledge of the topic and illuminate the terms of the question. (p. 27)

Immersion and illumination.

The process of initial engagement (Moustakas, 1990) continued into the early stages of my data collection. Gradually I became *immersed* in mathematics as I collaborated with a research mathematician, taught an undergraduate mathematics course, and reflected deeply on these experiences in my journal. Moustakas (1990) wrote, "the researcher lives the question in waking, sleeping, and even dream states. Everything in his or her life becomes crystallized around the question" (p. 28). Consider this excerpt from my research journal:

> Pretty sure I had a relevant dream about the nature of mathematics. Someone was presenting to me the solution to a mathematics problem. The problem involved a diagram. This person had partitioned off the diagram, with one important section. This important section was key to their argument. But it only became apparent why this was so important when they explained that there was a unit square such that this subsection could be partitioned into exactly a whole number of unit squares. Once that was explained I could visualize it perfectly… [continued below]

By immersing myself in my research questions I was able to achieve insight as key features of the nature of mathematics were *illuminated.* Moustakas (1990) wrote that when "the researcher is open and receptive to tacit knowledge and intuition […] in a receptive mind without conscious striving or concentration, the insight or modification occurs" (p. 29). Consider the journal entry continued below after recounting my dream:

> [continued from above]… In mathematics the same problem likely exists. Mathematicians come up with ideas for proofs, but the final proof does not explain the insight. Why did you decide to partition the graph in this manner?

> Was there some reason you thought it would be valuable to approach the proof this way? Proof is viewed just as a convincing argument; hence the intuition that led to the proof is believed to be not worth mentioning. Sometimes it is difficult to comprehend arguments or see their significance because of this. … [continued below]

Heuristic inquiry is a very emotional process aimed at bringing to consciousness the tacit and repressed: "[T]he researcher may discover refinements of meaning and understanding or may penetrate to the core of the phenomenon until it suddenly yields glimmerings that lead to a unifying picture" (Douglass & Moustakas, 1985, p. 50). During data collection I put myself into the habit of expressing what was surfacing in my consciousness. No matter how left-field the ideas seemed, I let sub-conscious thoughts come to the surface. I often described this in my journal as recognizing thoughts “bubbling up”:

> …[continued from above] Another thing that is bubbling up from within my consciousness is pure mathematics is in the same class as football as regard to human activities that are pursued as ends in themselves and do little to bring about the kingdom of god on earth (or end injustice, etc.…). Of course, we can still play football, and football should be taught to those who have a passion for it. But football should not be forced on all members of the population. The same goes for pure mathematics. By the time a student is a senior in high school, the state has a responsibility to educate its citizens in the mathematics necessary to function

within society. This is not done. Common core mathematics is pure/applied mathematics. It is not societal mathematics. What should the curriculum of maths be?

Incubation.

Incubation is the phase of heuristic inquiry during which the researcher takes a rest period from the inquiry. I purposefully set aside time for incubation immediately following data collection. The goal of incubation is to allow the unconscious mind to process one's experience so that new, tacit understandings emerge. Moustakas wrote (1990), "The period of incubation allows the inner workings of the tacit dimension and intuition to continue to clarify and extend understanding on levels outside the immediate awareness" (p. 29).

Explication.

Moustakas (1990) wrote that "The purpose of the explication phase is to fully examine what has awakened in consciousness, in order to understand its various layers of meaning" (p. 31). This occurred as I analyzed the data (described in more detail subsequently) and worked to create the final version of a humanistic framework for the nature of mathematics and related narratives.

Creative Synthesis.

Through data collection I documented my experience doing mathematics, teaching mathematics, and reflecting deeply on the nature of mathematical knowledge for the purposes of education. A goal of my dissertation was to draft narrative stories that highlight key events or themes related to NOM. I see these stories as the creative synthesis that Moustakas (1990) described:

> Finally, the heuristic researcher develops a *creative synthesis*, an original integration of the material that reflects the researcher's intuition, imagination, and personal knowledge of meanings and essences of the experience. The creative synthesis may take the form of a lyric poem, a song, a narrative description, a story, or a metaphoric tale. In this way the experience as a whole is presented, and, unlike most research studies, the individual persons remain intact. (p. 51)

As Moustakas (1990) wrote, "Transcriptions, notes, and personal documents are gathered together and organized by the investigator into a sequence that tells the story of each research participant" (p. 49). These narratives put the person at the center of mathematics and will be useful for understanding the NOM aspects I identified through this inquiry.

A Thorough Description of the Analysis Procedure

Analysis during data collection.

Patton (2015) describes heuristic inquiry as a *method of analysis*. Analysis is on-going from the very beginning stages of the inquiry. I was engaged in analysis every time I wrote in my journal to reflect on what had happened in the undergraduate class or what I was experiencing through my collaboration with Dr. Combinatorial. I was constantly debating with myself in an effort to determine what undergraduate students should understand about the nature of mathematics and why. Two main ideas became very prominent during the data collection phase: mathematics is an enjoyable exploration of ideas, and our mathematical ideas are part of our identity. I became very excited about these ideas during data collection, and I would talk about them to various people. Here is an example from a conversation I (Surreal) had with my co-instructor (Dr. Amicable) in the undergraduate course:

Surreal: This math stuff has been really exciting lately.

Dr. Amicable: Oh really? Have you made a breakthrough?

Surreal: I think so.

Dr. Amicable: Okay good!

Surreal: ::laughing::

Dr. Amicable: It looks exciting. It looks like my undergrad research experiences on that paper [“that paper” referring to my mathematics notebook].

Surreal: Mmm yep. *Math is this enjoyable experience of ideas. It is really exciting!*

Throughout the data collection process I had in mind the research question, “What should students understand about the nature of mathematics?” Whenever I had an idea for a possible NOM goal I wrote it out (often in my journal) and then saved it into a single word document. At the end of data collection I had the following list of fifteen possible candidates for a NOM framework (in addition to the initial characteristics identified in the literature review):

- NOMI - Mathematicians make mistakes. When a mistake is made, a mathematician may feel uneasiness or puzzlement that may be a signal that one should look for errors in their own work. A contrasting fact is that one may also feel emotion (perhaps related to the certainty one has) when one finds a correct solution. [inspired by my discovery that the number of edges in a complete graph of n vertices is n(n-1)/2]
- NOM? - Mathematician's practice is funded because mathematicians have proved valuable in wartime (Barany).
- NOMK - The teaching of mathematics affects the knowledge of mathematics (standardization).
- NOMI - Some mathematicians are able to achieve insights into problems by changing their focus or perception within a mathematical situation. The essence of research.
- NOMI - It may be fruitful to suspend one's desire for absolute conviction. Accept on fact that what an established mathematician has put forth is true. Use the mathematician's method in one's own proofs. Do not be worried about the details. Come up with a broad idea of a proof. Be critical later. Let it go.
- NOMI - Mathematics is a dialectic of criticism and justification… refutations and proofs… they require different mindsets. .. humility and courage…

- NOMK - Mathematical concepts of previously invented objects are sometimes applied to different mathematical objects in order to generate new concepts! (radii of graphs). In other words, some advanced concepts are analogous to well-known concepts. (diameter of a graph).
- NOMI - Mathematicians work on problems in their mind often, perhaps when participating in other activities (like mowing the grass). (10-5-16 and earlier)
- NOM - Mathematics plays a major role in the credit system which students have to move through within education. (Pais).
- NOMI/K - Symbols and definitions need to be standardized for ease in communication. But oftentimes there are many different ways that a concept is defined in the literature. It is important to be clear how you define it. Some definitions are equivalent.
- NOMI/K - Proof is viewed just as a convincing argument, hence the intuition that led to the proof is believed to be not worth mentioning. Sometimes it is difficult to comprehend arguments or see their significance because of this. Also some modern day concepts (function composition) seem totally pointless when introduced without also mentioning the motivation for their creation. The implication is we need to teach more history of math.

- NOM - School mathematics is also a phenomenon. It is influenced by conceptions of mathematics and other types of mathematics. Students learn school mathematics to earn credit. NOSM -School mathematics is malleable. But social, political, and cultural factors make it resistant to change. Of course the same could be said for pure mathematics I suppose.
- NOMI - Mathematicians enjoy doing mathematics.
- NOM - Mathematics involves criticism of people's ideas and argumentation.
- NOM - Ideas are the basis of mathematics. Numerals are distinct from numbers. (Lakoff & Nunez) What are numbers? Adjectives (Hersh), Embodied Mathematics (Lakoff & Nunez, Kitcher).

Throughout data collection I frequently reflected on the NOM characteristics from this list. Often these ideas were the topics of conversation during the informal interviews, as I asked mathematicians and others if they considered these characteristics to be worthy goals for student understanding of the nature of mathematics. This NOM characteristics played an important role in the later stages of analysis.

Post data collection analysis: early stage.

After a semester of data collection, I took incubation (Moustakas, 1990) seriously. I put away my dissertation work for several weeks during the 2016-2017 winter break. This incubation period came to an end in early January when I received my first invitation for an on-campus job interview for a university faculty position. Thus I went back to data analysis so I could present preliminary results at a job talk.

I began to review my reflective journal as well as the journals in which I conducted mathematical research. I sought to identify features of the nature of mathematics for which I could tell clear and compelling stories. I used the qualitative data analysis software ATLAS.ti to open code my reflective journal. The codes I assigned were designed to summarize or capture the meaning expressed in interesting quotations within the data (Saldaña, 2012). This initial coding process primarily helped me to re-familiarize myself with the data from my study post-incubation and identify possible NOM narratives. Based on this initial review, I identified three NOM characteristics and associated narratives:

- Proofs are bearers of mathematical knowledge. (NOMK)
- Mathematical knowledge is subject to revision. (NOMK)
- Pure mathematics research is an exploration of mathematical ideas. (NOMI)

I then worked to collect relevant pieces of data to support and tell these stories. I transcribed audio recordings of some whole-class discussions in the transition-to-proof course as well as some conversations that I had with Dr. Combinatorial. I made digital copies of my mathematics notebooks and identified important diagrams and text. I accessed relevant photos of the mathematical work Dr. Combinatorial and I had used to communicate on his whiteboard.

Through this process, I identified several other important NOM features that were related to the three I had originally identified. The narratives I drafted to present at the job talk ultimately highlighted the following elaborated NOM characteristics:

- Proofs are bearers of mathematical knowledge. / By reading proofs a mathematician may learn about a method of proof that may be valuable in their own research.
- Mathematical knowledge is subject to revision. / Mathematicians make mistakes.
- Pure mathematics involves (an often enjoyable) exploration of mathematical ideas. / Mathematical ideas are part of our personal identity. / Mathematics involves argumentation and criticism of ideas.

At this point the narratives were not in complete written form, but in the form of a PowerPoint presentation and verbal stories. I made similar presentations at a couple more universities, and I also gave a related presentation at the Twentieth Annual Conference on Research in Undergraduate Mathematics Education (RUME). At that talk (Pair, 2017) I discussed the four NOMK features that were in the initial Chapter Two framework:

- Mathematical knowledge is subject to revision.
- Mathematical knowledge is socially validated.
- Proofs are bearers of mathematical knowledge.
- Informal mathematical work is foundational to formal knowledge.

Although this talk was originally intended to be a purely theoretical presentation based on my literature review, I incorporated data from my dissertation to illuminate the four characteristics. As I gave all of these presentations, and I received helpful feedback that deepened my understanding of NOM and the related characteristics. One way in which this feedback influenced the NOM framework is in regard to the idea that "mathematical knowledge is subject to revision." At RUME some mathematicians in the audience were critical of this idea, perhaps because of a conflation with knowledge and truth. I would contend that while pure mathematical truth may not be subject to revision, our knowledge is. Nevertheless, I reworded this statement so it would be less abrasive: "Mathematical knowledge is dynamic and refined over time." It now appears in the final version of the framework as "Mathematical knowledge is dynamic and forever changing." Although I am now considering revising it to, "Mathematical knowledge is dynamic and forever refined."

Post data collection analysis: intermediate stage.

After the job presentations and RUME talk I began the process of transcribing all of the audio data. Originally, I only planned to transcribe critical moments that I had identified in my journal during data collection. However, sometimes as I examined the data, what at first seemed to me to be irrelevant to NOM later took on significance. In my journal I wrote, "I realize that I may go back and listen to these recordings, and discover something new… some new illuminations that I did not think were important at the time." Thus, I decided to personally transcribe all of the audio data.

During this transcription process, I continued to engage in analysis, often pausing to reflect on the data and draw implications for the nature of mathematics. Consider the following transcription from a discussion in the undergraduate course. Notice that part of the text is bracketed []. I used brackets to block off my analytic notes within the transcriptions.

> Surreal (Me): So I want to pull your attention back to the definition, "all possible truth values of the constituent statements." [This is so stale taking all this time to study truth tables and logical equivalence. It doesn't capture this spirit that mathematics is ours, our tool, our plaything, to be used how we wish. Milos, Brian and the Seldens say they use a just in time approach with logic. Introduce it to the students if they need it—but they will invent proof by contradiction on their own. Why take several weeks doing these truth tables? I know Odd said something like this in his mid-semester reviews, that we spent too much time on logic. Why not just move past this myth that you have to have formal logic to do mathematics, that mathematics is essentially logic? Why not take an idea-based approach? Mathematics is an exploration of pure ideas. Logic can be a tool to help us construct arguments, but logic is not necessary to do mathematics. I need to study if it is possible to teach mathematics this way. Ideas. Not strictly logic or sets; those are not the foundations of higher mathematics. If visuals are the best way to portray mathematical ideas, then allow the students to use visuals or any other form of reasoning that best helps them construct ideas.] Both statements are

> true or both false. Have we looked at all possible values of the constituent statements here? No. We've looked at one possibility in the case where they are all true. That's a great learning opportunity, thank you all for that.

These blocked sections of analysis served as signposts in the data that I could go back to later for further analysis.

While still in the transcription process, I decided to draft several narratives in order to get feedback from my dissertation chairs. Written drafts of narratives were constructed for the following NOM characteristics:

- Pure mathematics involves an exploration of ideas. (NOMI)
- Our mathematical ideas are part of our identity. (NOMI/NOMK)
- Mathematics involves argumentation and criticism of ideas.(NOMI)
- Proofs are bearers of mathematical knowledge. (NOMK)
- Mathematical knowledge is dynamic and refined over time. (NOMK)

After drafting those narratives, I continued transcription and analysis, and I began making note of other possible narratives and NOM characteristics. When I was nearly finished with the transcriptions, I created the NOM framework shown in Figure 8 and included it in a presentation I gave as a guest speaker in a doctoral course in which students were studying the nature of mathematics and science.

HUMANISTIC NOM FRAMEWORK

Pure Mathematics is an Exploration of Ideas (NOMI)

-Mathematics Involves Argumentation and Criticism of Ideas (NOMI)

-Our Ideas/Practices Are Part of Our Identity (NOMK/I)

-Mathematics can be Emotional (NOMI)

-Enjoyable Exploration of Ideas (NOMI)

-Proofs are bearers of mathematical knowledge (NOMK)

-Mathematical knowledge is dynamic and refined over time (NOMK)

-Mathematical Knowledge is Socially Validated (NOMK)

-Informal Mathematical Work is Foundational to Formal Knowledge (NOMI/K)

Figure 8: Early Draft of Revised NOM Framework

Post data collection analysis: nearing the final stage.

After finishing the transcriptions, I decided to review and code *all* of the data one last time. I had several motivations for pursuing a final coding. I sought supporting evidence for the NOM characteristics I had already identified. I wanted to make sure I did not overlook any crucial events related to NOM. I also wished to code in such a way that I could organize my data so that it would aid in my construction of compelling stories for each NOM characteristic. But, I was not sure which method of coding would best help me achieve these goals.

Recall that during the early stages of post-data collection analysis, I coded my research journal as well as a few other pieces of data. I decided to review the open codes I had produced during that early stage and determine if any were useful. There were 127 codes, most of which I judged would not be useful to recode my data. Some of the codes that were used most often were [credit system], [election], [ideas], [limitations], [methodology], and [mistakes].

I decided to code the data based on the possible NOM characteristics I had identified up to that point of analysis. These selected NOM characteristics came from the initial framework in Chapter Two and also those ideas identified in subsequent analysis stages. I also created codes for methodology and implications.[3] My plan, which I carried out, was to categorize all of the data according to these NOM characteristics and determine if there was sufficient data to draft a compelling narrative to illuminate the features of mathematics or to identify data that could supplement the narratives I had already created. I also reasoned this categorization of the data would be helpful in considering implications and summarizing methodological decisions I had made throughout the dissertation process. The codes and the corresponding descriptions that I began using for the final coding are listed below:

- [culture] Mathematical knowledge is influenced by cultural values.
- [artisan] Mathematical knowledge is embedded within the work of artisans.
- [reckoner] The purpose of commercial-administrative (reckoners') mathematical knowledge is calculation for economic purposes.

[3] I also created the code [religion], but I did not include it in my final analysis.

- [ethno] Western academic mathematics is one (but not the only) form of mathematics.
- [dynamic] Mathematical knowledge is subject to revision, dynamic, and refined over time. (NOMK) (Mathematicians make mistakes.)
- [socially validated] Mathematical knowledge is socially validated (NOMK)
- [proofs as bearers] Proofs are bearers of mathematical knowledge. (NOMK)
- [informal] Informal work is foundational to formal knowledge. (NOMK/I)
- [sub-disciplines] Each sub-discipline of mathematics has different norms, values, and standards. (NOMI)
- [emotional] Mathematical inquiry can be emotional. (NOMI) (enjoy, happy, sad, frustrated, excited, etc.…)
- [creative] Mathematical inquiry can be creative. (NOMI)
- [collaborative] Mathematical inquiry is collaborative.
- [statistics]
- [applied]
- [shape society] Mathematical knowledge is used to shape society, but cannot be considered an absolute judge.
- [ideas] Pure Mathematics is an Exploration of Ideas. (NOMI) -ideas are the basis of mathematics.

- [criticism] Mathematics Involves Argumentation and Criticism of Ideas. (NOMI)
- [identity] Our Ideas/Practices Are Part of Our Identity. (NOMK/I)
- [change perspective] Some mathematicians are able to achieve insights into problems by changing their focus or perception within a mathematical situation (the essence of research). NOMI
- [suspend conviction] It may be fruitful to suspend one's desire for absolute conviction. Accept what an established mathematician has put forth as true. Be critical later. Let it go. NOMI
- [conventional vs reasonable] There is mathematical knowledge related to convention and mathematical knowledge grounded in reasoning.
- [style] style is important in mathematics (nomk/nomi?)
- [credit system] Mathematics plays a major role in the credit system which students have to move through within education. NOSM (nature of school mathematics).
- [implications]
- [methodology]

As I began to code the entire set of data using ATLAS.ti, I began to encounter some quotes that seemed relevant to NOM but did not fall under the codes I had created. So I created a few new codes:

- [elite] – Pure mathematics is perceived as an elite practice. (NOMI/K)
- [communication] – Mathematicians communicate through proof. Symbols are important for mathematical communication. (NOMI/K)
- [deduction] – A mathematical proof is a deductive argument—it starts from what we know to be true and demonstrates something new. (examples and diagrams play a role in the proving process). (NOMK)
- [other] – Anything else that does not fit into the preexisting categories.

During the coding process I also created analytic memos in ATLAS.ti and a separate WORD file to document further methodological decisions and reflect on implications. For instance, during this coding/analysis I noticed that mathematical knowledge being socially validated is closely related to the notion that mathematics involves argumentation and criticism of ideas. Ultimately, I decided to consolidate the two codes [criticism] and [socially validated] into one code, [argumentation]. In the final framework I capture both these ideas with the statement "Mathematical ideas and knowledge are socially vetted through argumentation."

After reviewing and coding all of the data, I used ATLAS.ti to produce an output of the pieces of data (quotations) associated with each code. I then reviewed each of these codes and the corresponding sets of data quotations and made notes on possible narratives or described my decision not to go further with the characteristic in regard to my NOM framework. For instance:

For the code [collaborative] Mathematical inquiry is collaborative. I had only one quotation from a transcript of Foundations along with some initial analysis within the blocked text: Dr. Amicable: In your groups think about that. … [Dr. Amicable always puts it on the groups. Cooperative/collaborative learning.]

Collaboration was a constant in the course. But there was not a story that jumped out to me as I coded the data. Of course, within the mathematical collaboration between Dr. Combinatorial and I, collaboration was prominent by nature of the relationship. It is certainly not a stretch to argue that mathematical inquiry is often collaborative, and I believe students should understand this aspect of the nature of mathematics. I believe we can take this is as a given and teach students the value of collaborative work in mathematics. Nevertheless, it does not occupy a place in the final framework due to the lack of compelling narratives.

Post data collection analysis: final stage.

After the final run-through of the data, I continued to draft narratives, and evaluate, reflect on, and refine the NOM framework. I struggled to get the framework into a final form that I was comfortable with. I wanted to rephrase all the NOM characteristics in my own words. I thought the list was too long. I recalled a talk given by Schoenfeld (2016) in which he recommended that any good framework have five or fewer categories because "If you have too many things to work on, it's difficult to keep all of them in your mind.[4]" I was not sure if some characteristics should be sub-features

[4] http://ats.berkeley.edu/publications/AHS_MAA_Talk_without_video.pdf

of others. For example, I wondered if the notion that "pure mathematical inquiry is an exploration of ideas" subsumed the idea that "informal mathematical knowledge is foundational to formal knowledge." I made a PowerPoint document in which I reworded characteristics, removed characteristics, and generally tried to put the framework into a satisfactory form. I sometimes printed these iterations and revised them (See Figure 9 for an example).

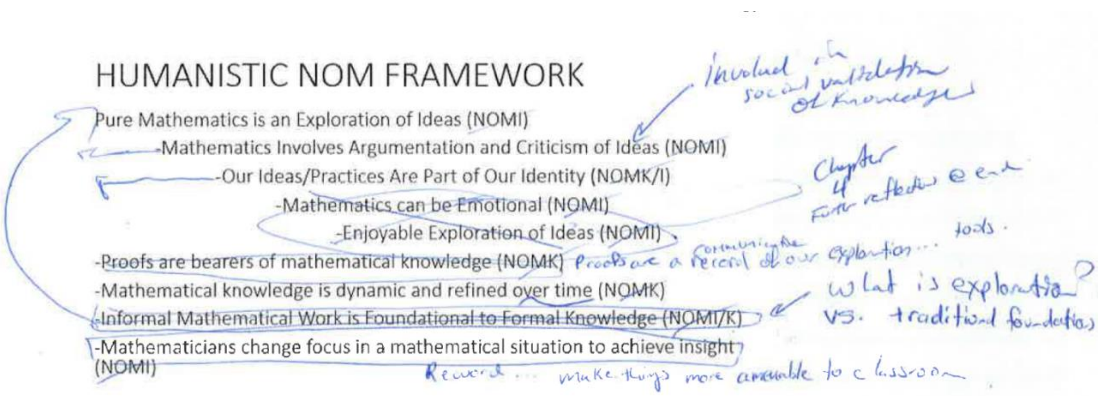


Figure 9. Revising the Framework

At times, I was frustrated. Instead of writing my dissertation I would pace in my home or take walks. Perhaps I was further engaged in incubation: "The period of incubation allows the inner workings of the tacit dimension and intuition to continue to clarify and extend understanding on levels outside the immediate awareness" (Moustakas, 1990, p. 29). Ultimately, I began to reflect on my entire dissertation study, and I decided I needed to produce a final framework that I was proud of and captured the essence of my experience. I recalled the phase of creative synthesis within heuristic inquiry, and I realized that in addition to the narratives that I was creating, the final NOM framework was also part of my creative synthesis. I read again what Moustakas had written: “Finally, the heuristic researcher develops a *creative synthesis*, an original integration of the material that reflects the researcher’s intuition, imagination, and personal knowledge of meanings and essences of the experience” (p. 51). I underlined those words: intuition, imagination, personal knowledge, meanings and essences. I gave myself creative license to craft a NOM framework that was not only grounded in my data but also captured the core themes I experienced during heuristic inquiry.

I went back to my NOM lists and thought more about which categories could be subsumed under the others. I wished to find the smallest number of characteristics that could provide a framework for teaching students about the humanistic nature of mathematics. At one point I had narrowed my list down to four categories: 1) Pure mathematical inquiry is an exploration of ideas; 2) Our mathematical ideas are part of our identity; 3) Mathematical knowledge is dynamic and ever changing; and 4) Mathematics involves social validation, argumentation, and criticism of ideas. After further reflection I decided (temporarily) to eliminate the fourth characteristic. I reasoned that one of the reasons mathematical knowledge is dynamic and forever changing is because of the process of social validation and argumentation. I decided to narrow the list down to the first three categories and eliminate the fourth. But after getting down to three categories, I was still not satisfied.

I again felt stuck. I knew I needed to write, but I was struggling. I sat down one night and opened a new word document titled "reflective analysis and discussion." I wrote the following:

> Mathematical inquiry is an **exploration** of ideas.
>
> Mathematical ideas are part of our **identity**.
>
> Mathematical knowledge is **dynamic** and ever changing.
>
> DIE dynamic identity exploration. … Woe is me. <u>Suffering</u> through the final stages of a dissertation product. Completing a project unlike any other I have ever completed, for the first time. …

The next day I put off going into my office to work until late in the afternoon. I decided to talk to my wife about the NOM framework. I told her I had narrowed my list down to three categories. I explained that mathematical ideas are part of our identity. She wanted evidence. I told her some of my stories (presented in Chapter Four), and she understood. I went on to explain that mathematical knowledge is dynamic, and that mathematics involves an exploration of ideas. Overall, she thought the categories were good.

I felt better after talking to her, but still uneasy. I was about to go to my office to write more, but I stopped on my way out of the house and stood in the doorway. I told my wife how if you take the first letters from the key words in the three characteristics, it spells DIE (dynamic, identity, exploration). My wife recalls "I remember you were standing at the door. And you were talking about your framework or whatever you call it.

You were naming your keywords. You were like, 'It spells DIE.'" She thought that was funny. And she said, "If only you had an A." If only I had an A? I was not sure what she meant. She said, "Because then you could spell IDEA." I immediately blurted out "Argumentation!" I now had my framework. I went to my office and excitedly shared my findings with a colleague. I then wrote the following in my "reflective analysis and discussion" document:

> Excited. I have been thinking about the framework for days. Can I condense categories? A selected few so that they are easy to remember? I was considering three. Mathematical ideas and practices are part of our (human) identity (both individual and collective). Mathematical inquiry is an exploration of ideas. Mathematical knowledge is dynamic and forever changing (both individually and collectively). I discussed these with my wife (she asked me what my evidence was.) I told her that there was an acronym: DIE. Then she said, if only you had an A. So then you could have IDEA. !! Argumentation was the fourth on my list. Thank God for this.

Here are the four characteristics that I ultimately decided upon: 1) Our mathematical ideas and practices are part of our *identity*; 2) Mathematical knowledge is *dynamic* and forever refined; 3) Pure mathematical inquiry is an *exploration* of ideas; and 4) Mathematical ideas and knowledge are socially vetted through *argumentation*. These characteristics are the foundational categories of the "IDEA Framework for the Nature of Pure Mathematics" presented in Chapter Four.

After sharing this framework with my dissertation chairs, they suggested I go back to data analysis, and see if I could provide further justification for these four characteristics. Returning to ATLAS.ti, I produced a code frequency chart, shown in Figure 10, for all of the codes related to a NOM characteristic for which there were at least five coded quotations in the data. Notice that the four characteristics that I ultimately chose to be the basis of a NOM framework correspond to 4 out of 5 of the most used codes during data analysis ([ideas (exploration of)], [dynamic], [identity], [argumentation][5]). By examining the frequency chart, I was able to verify that I had arrived at four NOM characteristics that I not only felt were crucial to my experience during this study, but were also grounded in the data.

[5] Recall that the [argumentation] code was created after merging the [socially validated] and [criticism] codes.

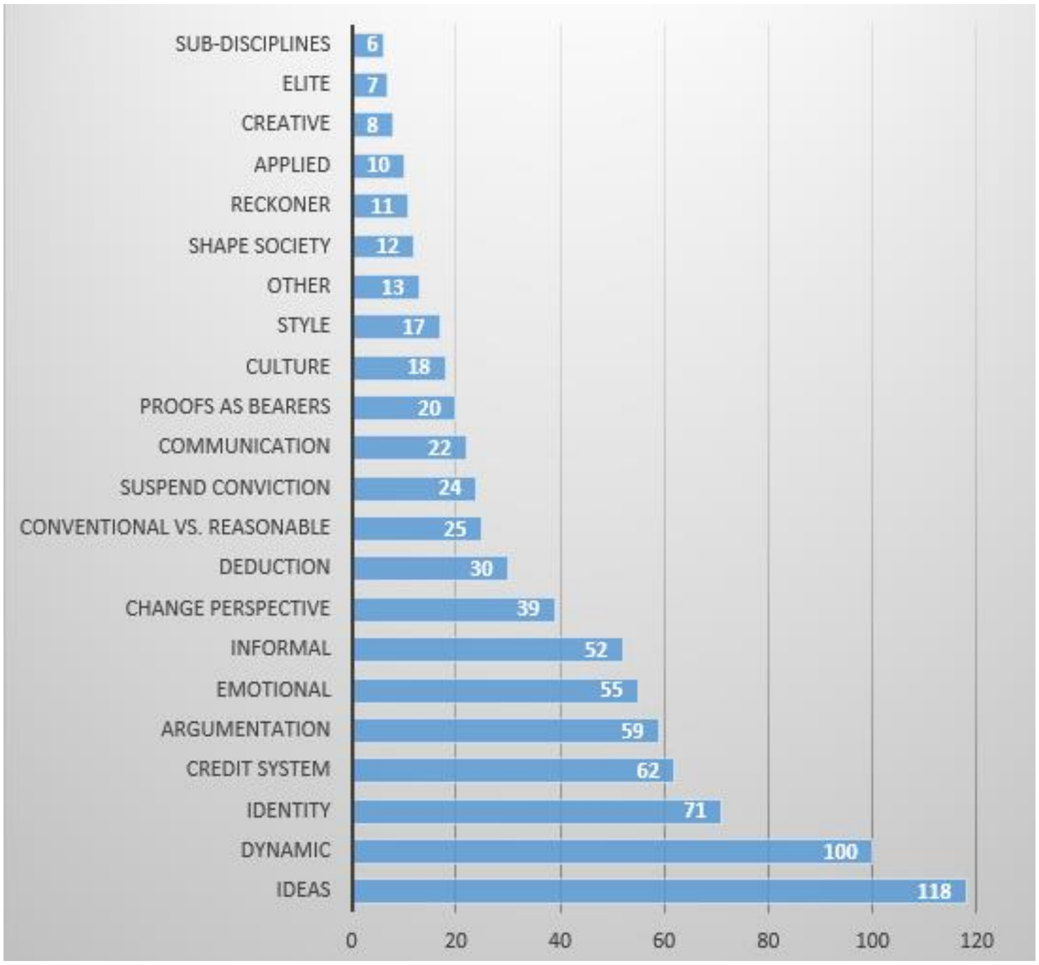


Figure 10. Code Frequencies

To further illustrate the encompassing nature of the four characteristics, I made a list of characteristics and sub-characteristics shown in Figure 11. This list demonstrates that many of the other NOM characteristics that I identified during my study are related to the four foundational categories. Some of these characteristics are further discussed in Chapters Four and Five.

Our Mathematical Ideas and Practices are Part of Our **Identity**.

- Mathematical inquiry can be emotional and creative.
- Mathematical knowledge is influenced by cultural values.
- Western academic mathematics is one (but not the only) form of mathematics.

Mathematical Knowledge is **Dynamic** and Forever Changing.

- There is mathematical knowledge related to convention and mathematical knowledge grounded in reasoning.
- Pure mathematics is perceived as an elite practice.
- Style is important in mathematics.

Pure Mathematical Inquiry is an **Exploration** of Ideas.

- Informal mathematical work is foundational to formal knowledge.
- Mathematicians change focus in a mathematical situation to achieve insight.
- Proofs are bearers of mathematical knowledge (they bear information about how the exploration was conducted).
- It may be fruitful to suspend one's desire for absolute conviction during inquiry.

Mathematical Ideas and Knowledge are Socially Vetted through **Argumentation**.

- Mathematicians communicate through proof.
- A mathematical proof is a deductive argument.

Figure 11. An Association of NOM Characteristics

Note that I did not choose to incorporate the characteristic "Mathematics plays a major role in the credit system which students have to move through within education" into my NOM framework even though it corresponded to the fourth highest number of data quotations. This notion, that there is a conflict between teaching the nature of pure mathematics and our current educational system, was on my mind throughout the dissertation process. But this is related more to implications regarding teaching NOM than the humanistic characteristics of mathematics I experienced during my dissertation study. I discuss the conflict between the educational credit system and the promotion of a humanistic NOM vision in Chapter Five.

Limitations and Delimitations

Subjectivity and Storytelling

I view this dissertation as being to the far-left end of a subjectivist/objectivist continuum. Critics may claim that any conclusions drawn from such a personal study may not be useful for informing mathematics education. I realize that when I choose a topic to write about, I do so because of some personal interest usually grounded in my experience. If I am able to successfully tell my story through writing, then it becomes clearer why the topic is important to me, and how it may be relevant to other persons. As a young scholar, I saw this dissertation as a personal opportunity for self-transformation, the main purpose of heuristic self-search inquiry (Sela-Smith, 2002). My thinking on NOM has been transformed in many ways, and in Chapter Four these transformations are sometimes discussed.

Brown, Cooney, & Jones (1990) noted that, “The question of the value of research from a humanistic perspective rests largely on whether one sees science as telling a story” (p. 650). Indeed, one of the criteria I considered essential for a NOM characteristic to be part of my framework was for there to exist compelling stories that illuminated that characteristic. I have already received very positive feedback on these stories from colleagues. People have told me that they identify with my work. Others have said they see the nature of mathematics in a new light. Patton (2015) wrote that stories help “us learn about specific individuals and society and culture more generally" (p. 128). However, the results I present in Chapter Four arose out of a particular context. If this study was conducted by a different scholar, with a different mathematician, in a different field of mathematics, in a different course, and at a different university, then the framework and narratives would have been different. Recall (from Chapter One) that I conceive of this dissertation as being the first step in a long research progression. One important next step is to get feedback from others regarding the value of the NOM characteristics I have identified.

Some Notes on my Qualifications to Engage in a Mathematics Collaboration

Because this was a personal self-study, it is important that the reader gets a sense of who I am as a person and my relevant qualifications. I have already written about my theoretical orientation in Chapter Two and my affinity for pure mathematics in Chapter One, but I also note here that I am uniquely qualified for philosophical work and for pure mathematical research. My undergraduate studies were not in mathematics, but philosophy (I earned a Bachelor of Arts Degree). I did study a bit of undergraduate mathematics as a physics minor. This mathematical study, along with my experience tutoring mathematics professionally for a few years, was enough for me to conditionally meet the requirements to enter a master's program in mathematics. I completed the master's degree, taking predominantly courses in pure mathematics, and I defended a thesis in the field of graph theory. I have had prior experience conducting mathematical research, and this experience provided me the confidence and experience I needed to work with a professional mathematician for this dissertation project.

One limitation to this study is that the mathematical collaboration lasted for only one semester. The research mathematician and I have still not accomplished the goal of the mathematics collaboration, which was to prove an open conjecture in the field of graph theory. If we had succeeded in this task, and gone through the steps of writing a communicative research paper for publication, then my conclusions about the nature of mathematics may have been different.

Comments on Data Collection

This was an immersive study for which I collected different types of data from a variety of sources. Even so, I recognize that other data sources and methods of data collection could have provided me more insight into the nature of mathematics and its teaching and learning.

In regard to the mathematics collaboration, video data may have been helpful. I made audio recordings of all of the conversations between Dr. Combinatorial and I. And I took pictures of all our mathematical work. But sometimes I found it difficult to reconcile these two sources and determine which feature of a diagram from a photo was being referred to in an audio-recording. While it is possible video data may have been useful, it is also possible that collecting video data may have created an awkward situation in which Dr. Combinatorial or I may have acted in a reserved manner.

In regard to the undergraduate transition-to-proof course, I believe the audio recording of whole class discussions worked well. But it would have been nice to also have recordings of small-group conversations. This would have required significantly more resources and time commitment to collect and analyze this data, but I am sure students said things in small groups that would have provided insight into their understanding of the nature of mathematics that did not come out in whole group discussions.

Chapter Summary

In this chapter I have outlined the methodology for my dissertation study. By doing mathematics with an active research mathematician, and by teaching undergraduates in a transition to higher mathematics course, I have had experiences highly relevant to the nature of mathematics (NOM). I have reflected deeply on these experiences and considered the aspects of NOM that undergraduate mathematics students and instructors should know and understand. Heuristic self-search inquiry has provided the important conceptual tools (e.g. immersion, illumination) that have enabled me to be in touch with my emotions and inner awareness, so that I was able to create narratives that inform a humanistic educational NOM framework.

CHAPTER FOUR: RESULTS

Introduction

I now present a humanistic educational framework for the nature of pure mathematics along with corresponding narratives that illuminate the features of the framework. The chapter begins with the framework, followed by some mathematical preliminaries before the illuminating narratives are presented. Through heuristic inquiry, I have documented my work doing mathematics with an active research mathematician, teaching undergraduates in a transition to higher mathematics course, and engaging in other activities related to my work as a mathematics education scholar. I have reflected deeply on these experiences and considered humanistic aspects of the nature of pure mathematics that undergraduate mathematics students in a transition-to-proof course should understand. I have sought to articulate these aspects so that the field of mathematics education has a framework that can guide both research and teaching related to the nature of mathematics.

The IDEA Framework for the Nature of Pure Mathematics

I have chosen four characteristics of the nature of mathematics (NOM) to serve as the basis of a humanistic educational framework: 1) Our mathematical ideas and practices are part of our *identity*; 2) Mathematical knowledge is *dynamic* and forever changing; 3) Pure mathematical inquiry is an *exploration* of ideas; and 4) Mathematical ideas and knowledge are socially vetted through *argumentation*. I call the framework of which these characteristics are foundational as “The IDEA Framework for the Nature of Pure

Mathematics" and it is shown in Figure 12. Note that IDEA corresponds to the key concepts of each of the four characteristics: I-Identity, D-Dynamic, E-Exploration, and A-Argumentation.

The IDEA Framework for the Nature of Pure Mathematics

Our Mathematical Ideas and Practices are Part of Our **Identity**.

Mathematical Knowledge is **Dynamic** and Forever Changing.

Pure Mathematical Inquiry is an **Exploration** of Ideas.

Mathematical Ideas and Knowledge are Socially Vetted through **Argumentation**.

Figure 12. The IDEA Framework for the Nature of Pure Mathematics.

I believe it is important to have a modest list of NOM categories to guide our general efforts teaching NOM, but no list will ever exhaustively outline all that students should understand about NOM. In addition to the four primary categories of the IDEA framework, the narratives in this chapter also feature four secondary categories that were significant to my study: 1) Mathematical inquiry and argumentation can be emotional. 2) Informal mathematical work is foundational to formal knowledge. 3) Mathematicians change focus in a mathematical situation to achieve insight. 4) Proofs are bearers of mathematical knowledge. These secondary categories are related to, but not necessarily subcomponents of, the four foundational categories in the IDEA framework. All of these characteristics of NOM will be illuminated in the subsequent narratives. See Figure 13 for a list of all the NOM characteristics together with the names of the narratives for which each NOM characteristic plays a role.

Nature of Mathematics Characteristics	Corresponding Stories
Foundational IDEA Characteristics	
I - Our mathematical ideas and practices are part of our *identity*.	• If No One Agrees With You • Toothpicks, etc. • Odd, Even, Odd, Even • Coloring
D - Mathematical knowledge is *dynamic* and forever changing.	• Mistakes • If No One Agrees With You • Odd, Even, Odd, Even • Levels • Cases • Coloring • We Are the Future
E - Pure mathematical inquiry is an *exploration* of ideas.	• Tension • Toothpicks, etc. • Odd, Even, Odd, Even • Levels • Mistakes • Coloring • The Essence of Research
A - Mathematical ideas and knowledge are socially vetted through *argumentation*.	• If No One Agrees With You • Cases • Mistakes • We Are the Future
Secondary Characteristics	
Mathematical inquiry and argumentation can be emotional.	• Tension • Toothpicks, etc. • Odd, Even, Odd, Even • Coloring
Informal mathematical work is foundational to formal knowledge.	• Tension. Toothpicks, etc. • Coloring
Mathematicians change focus in a mathematical situation to achieve insight.	• Coloring • The Essence of Research
Proofs are bearers of mathematical knowledge.	• Levels • Cases

Figure 13. List of NOM Characteristics and Corresponding Narratives

Some Notes on the Narratives

In creating the following narratives, I have strived to capture the humanistic side of pure mathematics. Mathematics is a human activity—an activity for which I have much affection. But mathematics is often presented to students as a body of absolute knowledge, separate and distinct from humanity (Burton, 1995; Ernest, 1991). Scholars have argued instead for a humanistic view of mathematics, putting the person at the center of mathematics (Burton, 1995). Ernest (1991) noted that "Anything else [than a human view of mathematics] alienates and disempowers learners" (p. xii). Similarly, Burton (1995) argued, from a feminist perspective, that "Re-telling mathematics, both in terms of context and person-ness, would consequently demystify and therefore seem to offer opportunities for greater inclusivity" (p. 280). As Hersh (1997) wrote, "To the humanist, mathematics is *ours*—our tool, our plaything" (p. 60). I have tried to capture this humanist spirit in both the IDEA framework as well as the corresponding narratives.

The narratives in this chapter capture my experience doing mathematics, teaching mathematics, and reflecting deeply on the nature of mathematical inquiry and knowledge for the purposes of education. I see these stories as the creative synthesis that Moustakas (1990) described:

> Finally, the heuristic researcher develops a *creative synthesis*, an original integration of the material that reflects the researcher's intuition, imagination, and personal knowledge of meanings and essences of the experience. The creative synthesis may take the form of a lyric poem, a song, a narrative description, a

> story, or a metaphoric tale. In this way, the experience as a whole is presented, and, unlike most research studies, the individual persons remain intact. (p. 51)

The narratives I have drafted feature direct quotations and excerpts from the data. As Moustakas (1990) wrote, “Transcriptions, notes, and personal documents are gathered together and organized by the investigator into a sequence that tells the story of each research participant” (p. 49). With these narratives, I have put people at the center of mathematics.

Context, Setting, and Pseudonyms

Before presenting the narratives, I will remind the reader of the context for this study and also discuss some mathematical preliminaries. Many of the experiences described took place at a state university within a city in the southeastern United States. At that university, data was collected from an undergraduate transition-to-proof course, which I co-taught. For an introductory assignment, the students were asked to consider a number type that best represented their selves. Students chose number types such as binary, permutation, whole, natural, real, positive, infinitely repeating decimal, etc.… I have chosen to use their personally-selected number types as pseudonyms for the students. In the event a participant did not choose a number type, I have chosen one for them. I have chosen the number type Surreal for myself. The co-instructor of the transition-to-proof course is Dr. Amicable. The research mathematician with whom I worked on an unsolved conjecture in graph theory will be referred to as Dr. Combinatorial. In Appendix A you will find a detailed list of each of the participants of this study, as well as some descriptions of the participants and some demographic information.

Mathematical Preliminaries

In order to best convey my results, I need you to step into the world of graph theory, at least get your feet wet a bit, in order to understand the context of my work. I will present an intuitive overview of some of the concepts that were involved in my mathematics research that served as a backdrop for my reflection on the nature of pure mathematics.

A (Very) Brief Introduction to Graph Theory

A graph is a collection of vertices and edges. The graph in Figure 14 has three vertices and three edges. It is called a 3-cycle, or a triangle. The graph in Figure 15 is a 4-cycle. I will call it a square; although we could draw it so it looks nothing like a square as in Figure 16.

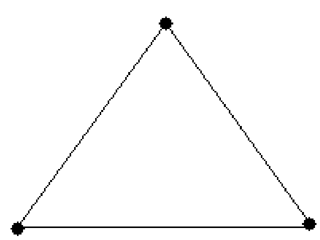


Figure 14. 3-cycle (Triangle)

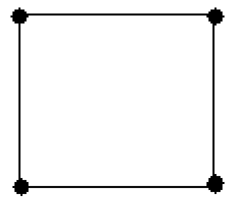


Figure 15. 4-cycle (Square)

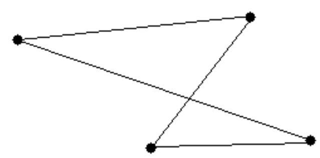


Figure 16. 4-cycle (That Looks Nothing Like a Square)

We say that these two graphs in Figures 15 and 16 are isomorphic as the connectivity relationship between the vertices is the same for each graph. Please take a moment to answer this question: If we were to color the vertices of the triangle and square so that no two adjacent[1] vertices shared the same color, then what is the minimum number of colors we would need? (Turn to the next page and see Figure 17 for the answer).

[1] If an edge connects two vertices, then we say the vertices are adjacent.

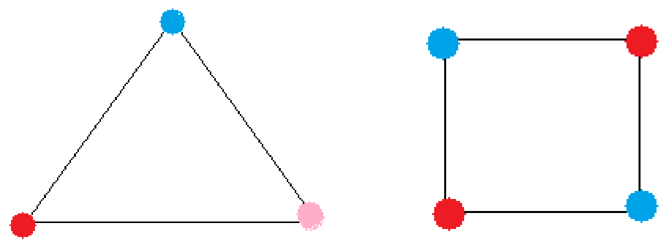


Figure 17: Colorings of the Triangle and Square

For the triangle we need three colors, but for the square we can get away with two! The minimum number of colors one can use to color the vertices of a graph so that no two adjacent vertices share the same color is called the chromatic number. What is the chromatic number of the pentagon shown in Figure 18? Figure this out yourself before turning to the next page.

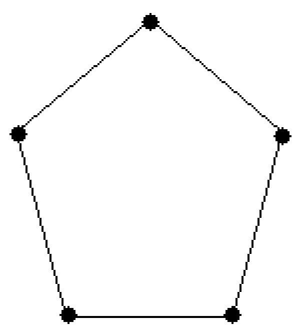


Figure 18: 5-cycle (Pentagon)

The chromatic number is three! Good, you are quick! Here is one possible 3-coloring in Figure 19. How about the chromatic number of the Petersen graph (see Figure 20)?

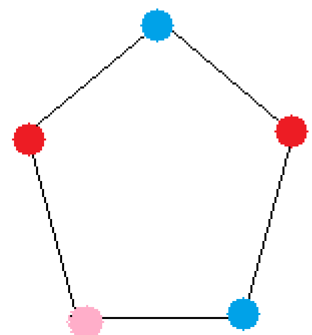


Figure 19. 3-coloring of the Pentagon

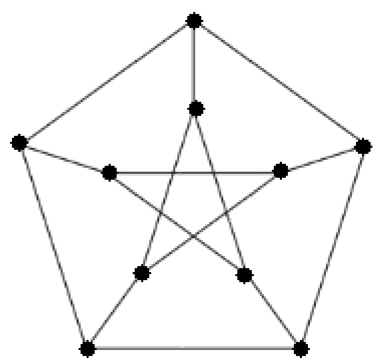


Figure 20. The Petersen Graph

I will not give the chromatic number of the Petersen graph away just yet (take some time to think about it!). The Petersen graph has several interesting properties and has been the subject of many research articles and even books (e.g. Holton & Sheehan, 1993). If you take some time to study it, you may discover some of these properties. For instance, you might observe that the Petersen graph does not have any triangles or squares, i.e. no 3-cycles or 4-cycles. But you can find several 5-cycles. Since the size of the smallest cycle is five, a graph theorist would say that the girth (size of the smallest cycle) of this graph is five. Another interesting fact is that every odd cycle greater than a length of five has a chord, an edge that is not part of the cycle that joins two vertices of the cycle. For instance, consider Figure 21.

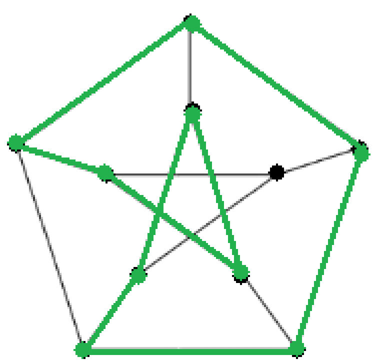


Figure 21. A 9-cycle Within the Petersen Graph

In green I have traced a 9-cycle within the Petersen Graph. Notice that there are a few *chords*: edges that are not part of the cycle but connect two cycle vertices (these chords are not green but connect two green vertices). If you examine the picture you can find three such edges (shown in red in Figure 22).

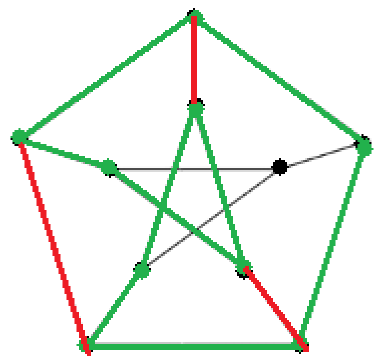


Figure 22: Three Chords of the 9-cycle

Let us take these interesting properties of the Petersen graph and generalize to a broader class of graphs. Imagine the class of all graphs with no 3-cycles, no 4-cycles, at least one 5-cycle, and such that any cycles of odd length greater than 5 must have chords.[2] The graph theorist, Dr. Combinatorial, whom I collaborated with for this dissertation project, did a lot of work on connectivity with this class of graphs, and has conjectured that the chromatic number for the class is 3. Indeed, the Petersen graph's chromatic number is 3 (see the 3-coloring in Figure 23).

---

[2] In math speak, we are referring to graphs with girth 5 and no induced odd cycles (no odd holes) of length greater than 5.

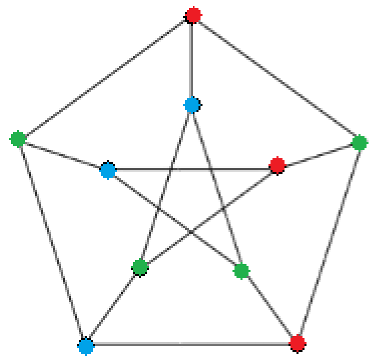


Figure 23. A 3-coloring of the Petersen Graph

The purpose of Dr. Combinatorial and my collaboration was ultimately aimed at proving his conjecture (the chromatic number is three for the class of graphs with no triangles, no squares, at least one pentagon, and such that any other odd cycles have a chord). It was in this context and through our work on this conjecture that I reflected on the nature of pure mathematical inquiry and knowledge.

I think that should be sufficient for you to follow the mathematics discussed in the narratives that follow. Some new concepts will be introduced as needed. But before moving on to the narratives, and to ensure that you understand the class of graphs the conjecture refers to, try to prove the following: Any graph in the class described above does not contain a 7-cycle. Really, try it! No pressure! If you are unable to find a proof, do not worry! I will discuss this more in the first narrative.

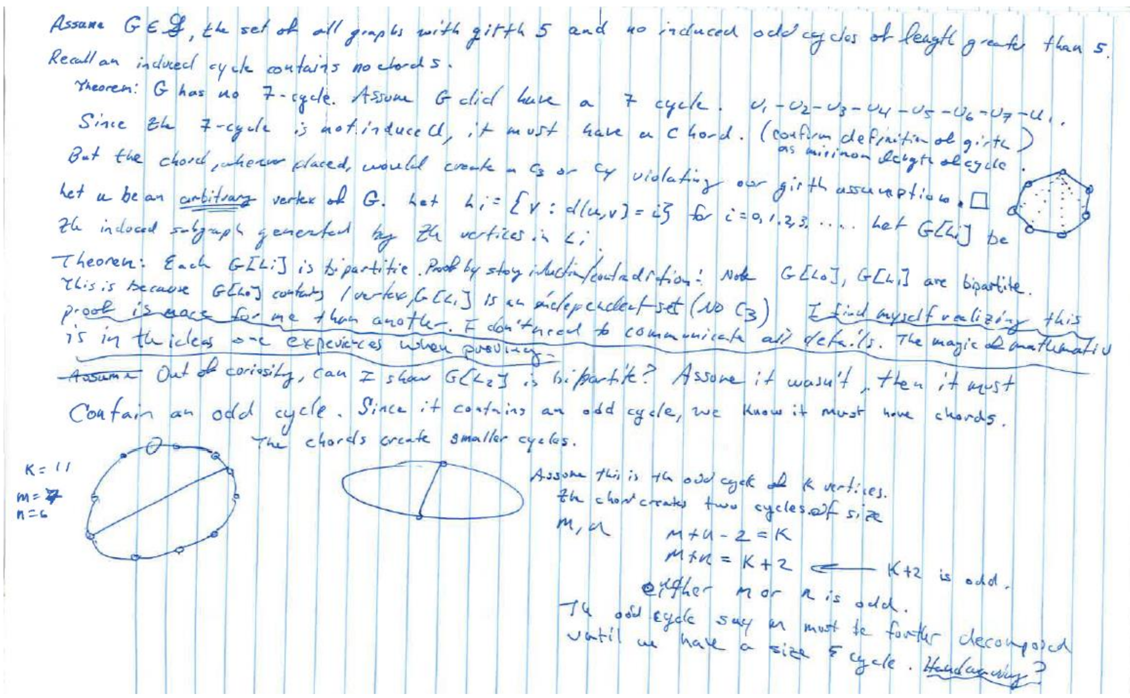
Introduction to the Nature of Mathematics Narratives

As the original intention of these narratives was to convey essential features of the nature of mathematics, I will briefly introduce each narrative and point the reader to the NOM characteristics they might see at work in the narratives. Not only are the four characteristics of the IDEA framework highlighted, but I also discuss some additional NOM characteristics. I believe the four characteristics of the IDEA framework are foundational to a humanistic understanding of the nature of mathematics; but with any subject, one can always go deeper. The structure of the chapter is as follows. The title of the narrative will be in bold, followed by a brief introduction in *italics*, and the remainder of the narrative in regular text.

Tension

The first narrative, Tension, introduces the notion that pure mathematical inquiry is an exploration of ideas. Furthermore, this exploration of ideas can be very enjoyable—an example of the emotional nature of mathematical inquiry. The reader may also observe the importance of informal mathematical work (e.g. diagrams) to the development of mathematical understanding.

One of the first significant realizations I had during my inquiry into pure mathematics was that to engage with pure mathematics involves an exploration of ideas. One night I began to work on Dr. Combinatorial's conjecture, and I wanted to summarize the important theorems I had just begun to understand. I wished to solidify them in my own mind so that I could make progress on the conjecture. I sat on my bed at home, and in my notebook, I wrote the text and diagrams shown in Figure 24. I now explain this text in detail.


Assume $G \in \mathcal{G}$, the set of all graphs with girth 5 and no induced odd cycles of length greater than 5. Recall an induced cycle contains no chords.

Theorem: G has no 7-cycle. Assume G did have a 7 cycle. $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_1$. Since the 7-cycle is not induced, it must have a chord. (confirm definition of girth as minimum length of cycle). But the chord, wherever placed, would create a C_3 or C_4 violating our girth assumption. □

Let u be an arbitrary vertex of G. Let $L_i := \{v : d(u,v) = i\}$ for $i = 0, 1, 2, 3, \ldots$. Let $G[L_i]$ be the induced subgraph generated by the vertices in L_i.

Theorem: Each $G[L_i]$ is bipartite. Proof by strong induction/contradiction: Note $G[L_0]$, $G[L_1]$ are bipartite. This is because $G[L_0]$ contains 1 vertex, $G[L_1]$ is an independent set (no C_3). I find myself realizing this proof is more for me than another. I don't need to communicate all details. The magic of mathematics is in the ideas one experiences when proving.

~~Assume~~ Out of curiosity, can I show $G[L_2]$ is bipartite? Assume it wasn't, then it must contain an odd cycle. Since it contains an odd cycle, we know it must have chords. The chords create smaller cycles.

$K = 11$
$m = 7$
$n = 6$

Assume this is the odd cycle of k vertices. The chord creates two cycles of size m, n

$m + n - 2 = K$
$m + n = K + 2$ ⟵ $K+2$ is odd.
either m or n is odd.
The odd cycle say m must be further decomposed until we have a size 5 cycle. Hand-wavy?

Figure 24. Excerpt from Mathematics Notebook

The first thing I did was to imagine an arbitrary graph from the class of graphs pertaining to the conjecture and remind myself of some related meanings: “Assume $G \in \mathcal{G}$, [G is a graph in our class] the set of all graphs with girth 5 and no induced odd cycles of length greater than 5. Recall an induced cycle contains no chords.” You should also recall girth is the size of the smallest cycle in a graph; and since our graph contains no 3-cycles and no 4-cycles, but it does contain a 5-cycle, the girth is 5.

Next, I proved the theorem mentioned previously (have you tried yourself yet?) that our graph G has no 7-cycles. To prove this I worked by contradiction. I assumed that G did have a 7-cycle. I called the cycle $u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_7 - u_1$. Recall that for our class of graphs any odd cycle of length greater than five must have a chord[3]. Thus the 7-cycle must have a chord, but where is it? Notice how I drew a diagram in the upper right of the page. This diagram is of a 7-cycle and I have reproduced it below in Figure 25.

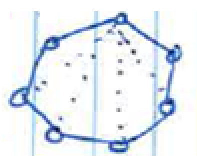


Figure 25: 7-Cycle with Possible chords

Also notice how I drew some dashed lines. These represent possible chords (we said the 7-cycle must have at least one chord, so it has to go somewhere!). Since there is some symmetry with the 7-cycle, I arbitrarily picked the uppermost vertex on the paper and drew all possible chords from there. What did I notice? Consider the right-most chord. Do you see the little triangular 3-cycle? Good! Move on to the next chord to the left. If you start at the top vertex, move straight down this chord, and then to the right (counterclockwise) you will trace a total of four edges before you get back to the original

[3] We call cycles without chords *induced cycles*, another name that is used for the same concept is *hole*.

vertex, a 4-cycle. You can check that the other two chords will also create either a 4-cycle or a 3-cycle. Since we have assumed our graph does not have 3-cycles or 4-cycles, we have reached a contradiction. I wrote, "But the chord, wherever placed, would create a C_3 [3-cycle] or a C_4 [4-cycle] violating our girth assumption."[4] So we have proved that any graph within the class does not have a 7-cycle. It is a nice little baby theorem, that proved very helpful in subsequent work.

What does my experience proving this say about the nature of mathematical inquiry being an exploration of ideas? What I now notice is that there is a tension in my written proof. That night when I sat down to write, my purpose was to solidify in my mind what I knew could be proven, and move on to try to prove new things. The tension is between this purpose and a conflicting purpose of writing to satisfy some norms of proof writing that I learned in school. In at least one line, it is clear that I was writing the proof as I would write them in my graduate mathematics courses as if I expected it to be read and graded. Notice that I labeled the 7-cycle as $u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_7 - u_1$, but I did not use this symbolization elsewhere in the proof. The diagram and my counting of chords was sufficient to convince me that the theorem was true, and I understood why it was true. The last line is a bit of a hand-wave. "But the chord wherever placed would create a C_3 [3-cycle] or a C_4 [4-cycle]." I could have opted to write some sort of argument: Arbitrarily choose vertex u_1; if we form the chord $u_1 - u_3$ then a 3-

[4] Note that the notation C_k is sometimes used to denote a cycle of size k (k is a natural number).

cycle $u_1 - u_2 - u_3 - u_1$ is formed. etc… Can you see the tension? On the one hand, working for personal understanding and on the other writing with the standards of rigor I believe are expected in mathematical writing. The conflict is between a personal exploration and understanding of ideas versus the crafting of a communicative proof that satisfies perceived norms of rigor and symbolization.

After proving that theorem I moved onto another one, which needed a proof by induction. I will spare you the details of that proof for now, and draw your attention to the underlined words on the page, reproduced in Figure 26.

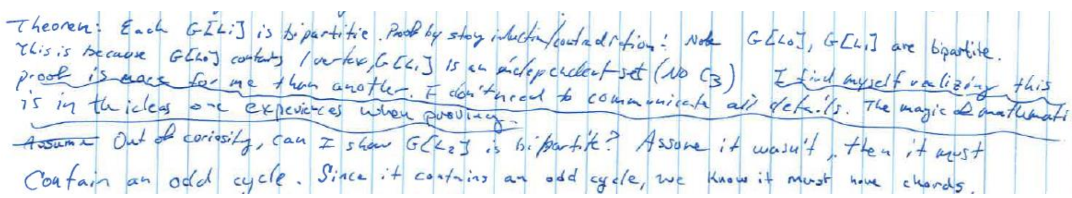

Theorem: Each $G[L_i]$ is bipartite. Proof by strong induction/contradiction: Note $G[L_0]$, $G[L_1]$ are bipartite. This is because $G[L_0]$ contains 1 vertex, $G[L_1]$ is an independent set (No C_3) I find myself realizing this proof is more for me than another. I don't need to communicate all details. The magic of mathematics is in the ideas one experiences when proving.
~~Assume~~ Out of curiosity, can I show $G[L_2]$ is bipartite? Assume it wasn't, then it must Contain an odd cycle. Since it contains an odd cycle, we know it must have chords.

Figure 26. Underlined Text

These words come after I wrote out minute details of the basis step for the $n = 0$ and $n = 1$ cases that were already clear in my own mind (but may not have been clear to a reader). The underlined text reads, “I find myself realizing this proof is more for me than another. I don’t need to communicate all the details. The magic of mathematics is in the ideas one experiences when proving.” Essentially I was giving myself permission, with those words, to drop any unnecessary symbolism and tedious explication, and just explore the mathematical ideas (and document that exploration). The very next thing I wrote was, “Out of curiosity, can I show [the $n = 2$ case]?” I already knew a proof by induction could prove for all cases, but I decided to look at a specific case so I could better understand the general argument. My subsequent work is shown below in Figure 27.

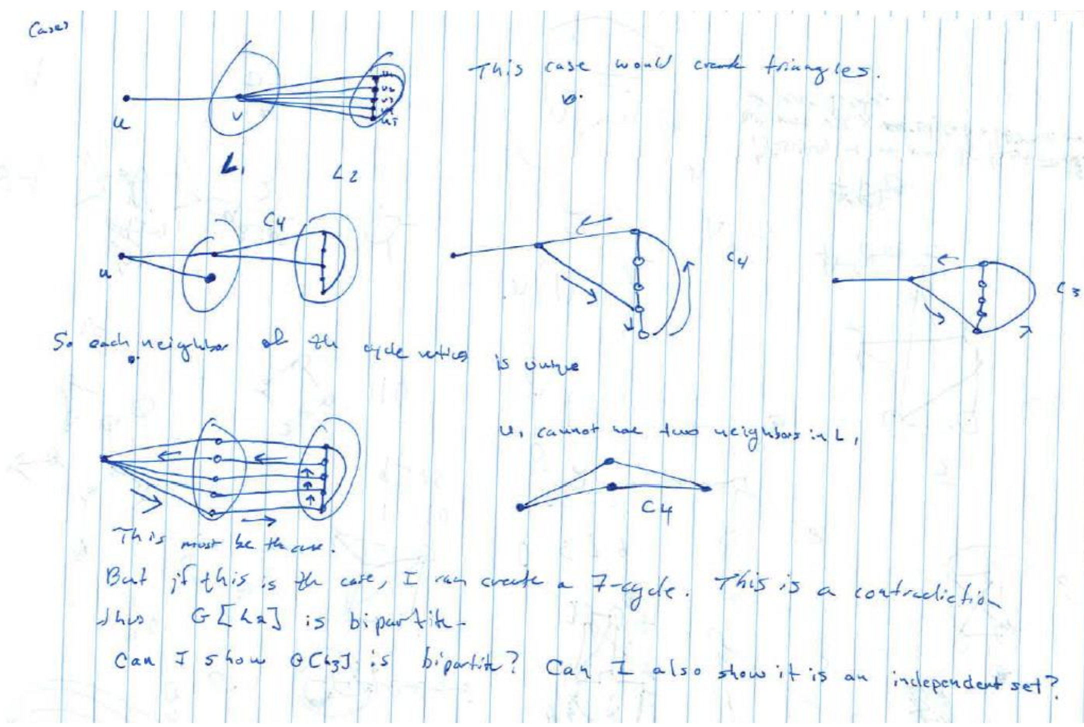


Figure 27. Exploring the Second Case

I worked through this case myself, drawing the very interesting figures shown above as an aid. Then I wanted to keep going. I moved on to the $n = 3$ case even though in the past I had worked through an induction proof by Dr. Combinatorial and his colleagues that covers all the cases. I was enjoying looking at the individual cases, and gaining insight through my work on them. I found the ideas involved in these types of proofs intellectually stimulating. What I really want to communicate to the reader is that as I began exploring the mathematical ideas related to this conjecture, I found deep satisfaction. Pure mathematics is an *enjoyable* exploration of ideas. The mathematics came alive through the proving process. Consider this journal entry.

> It is interesting how I see the problem forming. The proof of the problem is different in nature than the class of graphs the proof refers to. The proof has its own concept imagery in my mind—different mathematical processes and procedures disjoint from the class of graphs itself. … The mathematics is alive within the proof. When I imagine the truth of the conjecture, it is some sad lonely objective reality. But the proof is where the magic is. It is where my mind is. It is where the structure can be seen.

Toothpicks, Popsicle Sticks, Coffee Stirrers, and Pick-Up Sticks

This narrative continues with the themes of mathematical inquiry being an exploration of ideas, mathematics being emotional, and informal work being essential to mathematical understanding. Although more subtle, one can see two additional NOM characteristics from the IDEA framework. This story takes place just after I had a breakthrough in my understanding and I desired to push my understanding to an even higher level—mathematical knowledge is dynamic and forever refined. Also observe that *the Petersen graph and related ideas were not only part of Petersen's identity, but are now part of my own.*

Late one night, close to midnight, I had a breakthrough in my work. While working in my home office I realized there was a crucial structure involving even cycles that kept showing up in the class of graphs I had been studying (with girth five and no odd holes greater than length five). The key insight came when I found an alternate representation of the Petersen graph. I drew it in my notebook and wrote, "Who was Petersen?" See Figure 28.

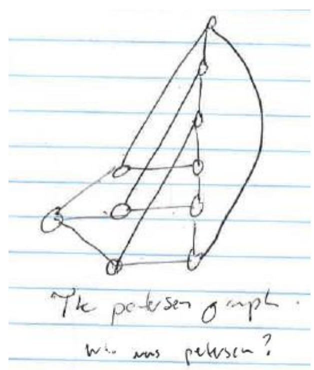


Figure 28. An Alternative Drawing of the Petersen Graph

In Figure 29 below are the typical drawing of the Petersen graph in the upper left corner and the isomorphic drawing that I discovered in the lower right corner.

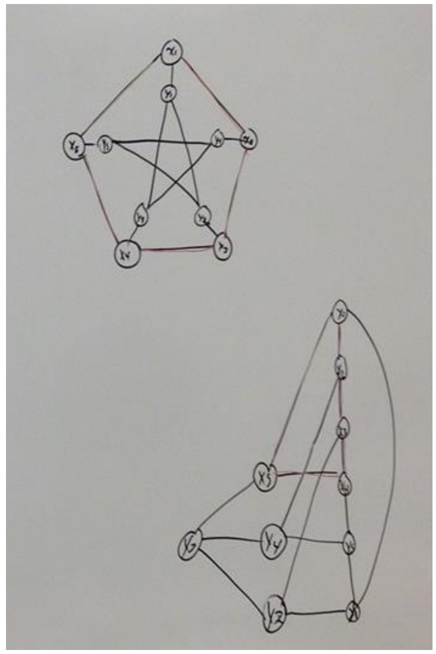


Figure 29: Two Drawings of the Petersen Graph

After discovering this alternate representation, I wanted to better understand its structure. I wanted to visualize it in a different way, so I decided to create a 3-dimensional model. I brainstormed for building supplies. I went to my children’s room for playdough. The room was dark and the children were sleeping so I was as quiet as possible. Next I made my way to the kitchen for toothpicks. Shortly thereafter I had created the (sad) graph shown in Figure 30.

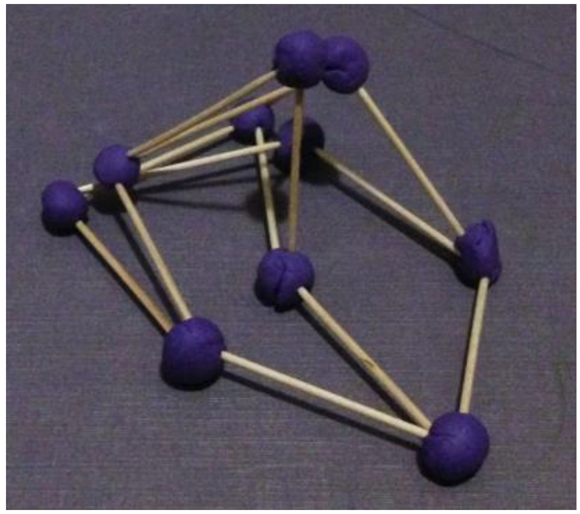


Figure 30. My First Attempt at a 3-Dimensional Petersen Graph

Even though the graph I had created was not perfect, I was still excited and proud. I found my wife (in bed but still awake) and showed her my creation. I expressed my dissatisfaction with the model, and showed her the original drawings that inspired me. She said the alternate drawing for the Petersen graph looked like “a funky boat.” She very excitedly took over the process of building a better model. We were no longer only studying graph theory, but also engineering and geometry. The toothpicks were not long enough. Our efforts kept collapsing. My wife decided to get some Popsicle sticks and coffee stirrers. Eventually we were somewhat satisfied with the 3-dimensional model shown in Figure 31.

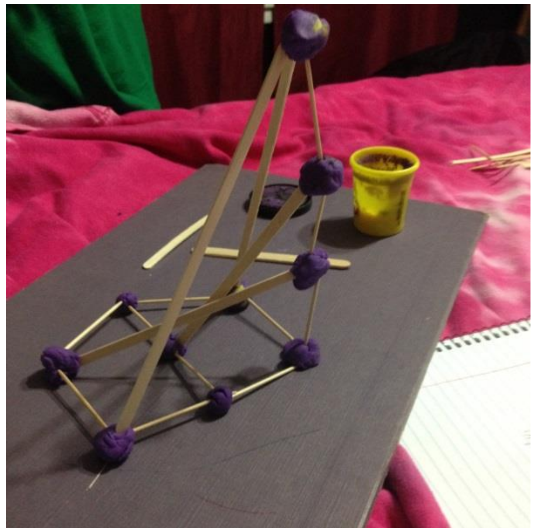


Figure 31. 3-Dimensional Petersen Graph Made with a Variety of Sticks and Playdough

After we created this model, I briefly pursued another question while lying in bed. Who was Petersen? It was late at night so I did not pursue this question long (I only looked on Wikipedia an a few other websites). I found out that Julius Petersen was Danish, and had written a famous paper in graph theory called "Die theorie der regularen graphs" which was published in December of 1891. The Petersen graph was created as a counterexample to a mathematical statement attributed to Peter Guthrie Tait.

A couple of weeks later I visited another mathematician's office, Dr. Algebraic, and I found that he had similar 3-D structures that he created for his dissertation. I have placed a photo of my coffee stirrer/popsicle/toothpick/play-dough model alongside his modeling-clay/pick-up stick lattices for comparison in Figure 32.

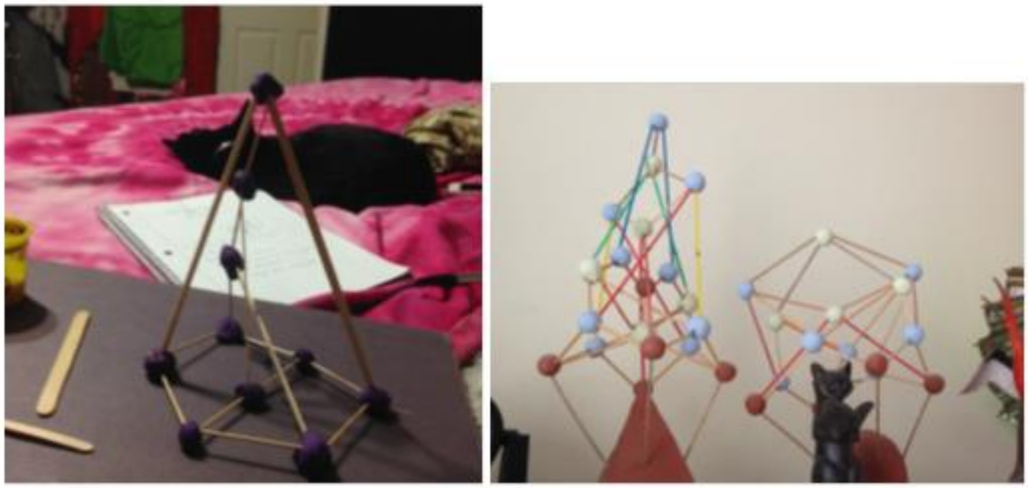


Figure 32. 3-Dimensional Structures (Both Photos Coincidentally with Black Cats)

I have found that pure mathematics involves an enjoyable exploration of interesting structure. The mathematics was so enjoyable and interesting that my wife and I stayed up late at night making 3-D mathematical structures. It was also evident from my collaboration with Dr. Combinatorial, that he also found great enjoyment from doing mathematics. Here is an excerpt from one of our conversations

> Surreal: It's fun. I'm sort of starting to see what you mean. The good thing about mathematics is you can think about it all the time.
>
> Dr. Combinatorial: ::laughing::
>
> Surreal: It is very enjoyable work.
>
> Dr. Combinatorial: If you have something in your mind you just cannot get rid of it.

Dr. Combinatorial is a full professor, and he continues to be an active researcher. He genuinely *enjoys* working on mathematics problems. Mathematics is what Dr. Combinatorial does when he has spare time. “If you have something in your mind, you just cannot get rid of it.” What we have in our minds are mathematical ideas and unsolved puzzles. To work on these puzzles and explore these ideas is fascinating and enjoyable. We need to find a way to help students realize that this is what pure mathematics is all about.

If No One Agrees With You

I now tell the story of Binary, a first generation college student, whose idea was subject to the scrutiny of the classroom community. In this narrative, we can see at least three characteristics of the IDEA framework at play. We see a clear example of how mathematical ideas are socially vetted through argumentation in an inquiry-oriented classroom. The reader will also observe how I first came to understand that mathematical ideas are part of our personal identity. Also note the dynamic nature of mathematical knowledge for this particular classroom community and its individual members. Most of this story is told in present tense.

One day in Foundations of Higher Mathematics some small groups are working to create group proofs for different theorems and presenting their proofs of those theorems to the class. The Yellow Team, consisting of team members Positive, Natural, and Odd create the poster shown in Figure 33 and present their proof to the class.

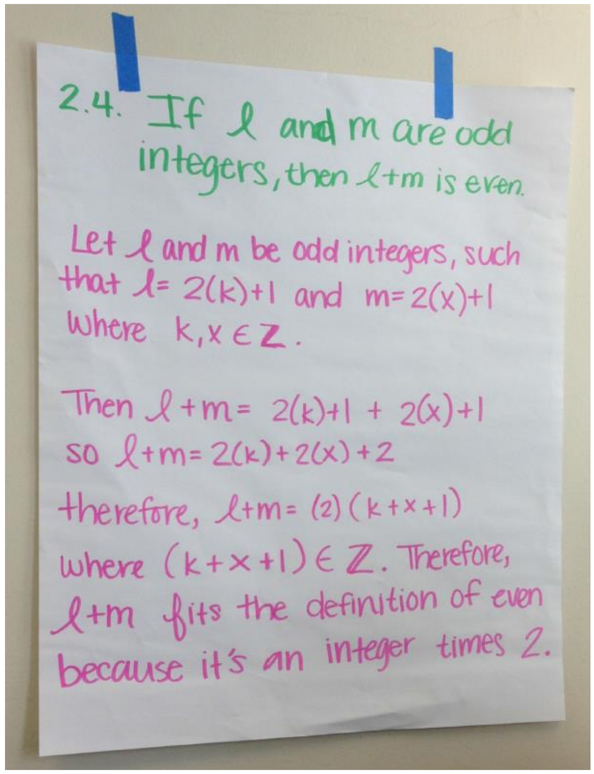


Figure 33. Yellow Team's Poster

In general, the class as a whole likes the group's argument and does not have many questions. Infinitely Repeating Decimal's comments are representative of the class: "It is pretty straightforward, it doesn't get more concise than that." However, Binary does have an important question. He asks, "If instead of having the *x,* if you did put *k*, would it make the argument less strong?" Looking at the poster, we see Binary is referring to the original delineation of the odd numbers *l* and *m* as $l = 2k + 1$ and $m = 2x + 1$ where *k* and *x* are integers. Would the argument be less strong if we simply defined $l = 2k + 1$ and $m = 2k + 1$*?* One of the presenters, Odd, responds "Oh so like *k* here and *k* here?"

(pointing to the *k* and *x* in the original delineations). “Yeah,” Binary replies. Odd explains, “Yeah, because basically what you did was, you didn’t say that it’s any two odd integers, you said the same integers. So you basically just said $l + l$.” Binary attempts to explain that he thinks *k* is sufficient by appealing to the definition of odd number. But he is interrupted as another student, Whole, interjects “If you want to stick with *k* you could be like *k* subscript 1 and *k* subscript 2.” The dialogue continues:

> Binary: I just feel like if they both were *k* it would still make a strong, same, strong argument.
>
> Odd: Cause you’ve got basically, If *l* equals *2k+1* and *m* equals *2k+1*, as *k* increases the other *k* is increasing so *l* and *m* would always be equal.
>
> Binary: But it’s still going to be odd. That’s what I am saying.
>
> Odd: It’s an odd number but it’s the same odd number. So it doesn’t cover any combination of two odd numbers. So basically you would only be able to say like 3+3 or 6 err, not 6. Or 7+7. But with this, because these are different, we could say like, let *l* be equal to 3 and *m* be equal to 7.
>
> Real: So it hits the rubric on, to be more general. Make it more universal. Cover all cases instead of something specific.
>
> Infinitely Repeating Decimal: If you really wanted them to be k’s you could use subscripts like k_1 and k_2.
>
> Odd: Yeah.

Infinitely Repeating Decimal: You just need to show that they are not the same integer.

Binary: Yeah. I am just trying to figure out if I just left that as k, and he had that. Would I get less points?

At this point several members of the class laugh and some chime in that indeed the argument would receive less points. Dr. Amicable brings the class's attention back to the mathematics. She asks for a show of hands to see how many students understand Odd's explanation, and asks Infinity to explain in her own words.

Infinity: Yeah if you made both the variables, the *k* and the *x* the same. If you did make them both *k* then they would be the same number. So you would get the same outcome of *l* and *m*. And so your numbers wouldn't vary. So like he was talking about how the answer would be consistently the same throughout. … If you make both of those k's in the equation where x is. Then you are going to get the same exact number.

Dr. Amicable: Hmm. So is it right, if I were to summarize what you are saying Infinity; would it be right to say that essentially we are changing this condition to if *l* and *m* are the same odd integers? Then $l + m$…

Infinity: Yea. Because you would be plugging in…. You would be using the same variables.

Dr. Amicable: Okay. Binary what do you think?

At this point I had expected Binary to jump on board with the class consensus. But he stands firm.

Binary: I am just saying. In general if you were to use examples, then yeah. But just in general I feel like no matter what you put in, by the definition of an odd number, it's going to come back to the exact same thing.

Whole (interrupting): But you have to have some kind of variance in it when you are adding two.

Binary: But not in the way he did, maybe in another proof, yes. But…

Whole: It's the same principle though.

Composite(2): Okay but if you do use k in both l and m… If you say $2k + 1$ is equal to l and $2k + 1$ is equal to m. You are going to go to the next step where it adds, and you will have $2k + 1$ plus $2k + 1$. And then instead of being $2k + 2x$ it's going to be $2k + 2k$ which equals $4k$. And you are going to have $4k + 2$ which is not going to be something that looks like the definition of an even number.

Binary: [emotional] Yes you will. You pull out 2.

Odd: It would still be even.

Binary: It would still be even.

Binary stated "It would still be even" with conviction and several people in the class begin to talk. Of course the result $4k + 2 = 2(2k + 1)$ is in the form of an even number and several members of the class agree. But they also stress that by using $l = 2k + 1$ and $m = 2k + 1$ the result is "more narrow" because "you've got to cover the spectrum."

Subsequently the students and instructors begin to consider examples that may serve to change Binary's mind. Binary still does not. "I understand what ya'll are saying. It makes perfect sense to me what ya'll are saying. Don't get me wrong. What ya'll are saying is 100% correct. But I'm saying that this way still satisfies everything to me for an odd number."

Time runs out for the class and the students are required to write one big idea for the day and one question they have for their exit tickets. Whole wrote "Clarity is key to success. You need to be able to differentiate variables." Integer's exit slip reads, "Differentiating variables in a proof is a basic, but really important issue." Binary's big idea was "If no one agrees with you, you're wrong."

After class Surreal and Dr. Amicable were very happy about the discussion that students had engaged in regarding Binary's question. Like any good mathematics education scholar should be, they were very pleased with the level of mathematical discourse in the classroom. As students left the room, Even asked if she could stop by Dr. Amicable's office. Later on, Dr. Amicable and Surreal had this e-mail exchange:

> Dr. Amicable: … we will need to chat about Even's concern. In brief, she felt that Binary was under attack today in class and it made her feel very uncomfortable. She recognizes that Binary may not have felt under attack, but she felt that for him. I appreciated her coming to share her feelings. Perhaps we can address the best ways to critique and also remain professional in our classroom

setting (at the beginning of next class), but I'd like to chat with you about your thoughts.

Surreal: Okay we can chat. I did not think Binary was under attack. Only his idea was under attack! But I also recognize that students have never experienced mathematical argumentation and so it may be hard for some of the students to deal with it. Although I do not have immediate thoughts about what we would tell students, I think engaging in a dialogue with students may be productive.

Dr. Amicable: I agree. I did not see it as an attack on Binary either, but it wouldn't hurt to talk with the students about critiquing an idea rather than a person.

Notice that Surreal's initial thought was that Binary was not under attack. "Only his idea was under attack!" But are our ideas not also our selves? When we criticize another person's ideas, are we not criticizing the person as well?

The next day in class Surreal began by asking students to talk about their big ideas and questions from the previous class. Many of the students said they had conversations outside of class about Binary's ideas. Others said they were trying to think of new ways to convince Binary of their point of view. Infinitely Repeating Decimal's big idea was "I'm really struggling to figure out a different way to represent that $l = 2k + 1$ and $m = 2k + 1$ only satisfies $l = m$. There has got to be a way though!" He expanded upon this idea during class.

Infinitely Repeating Decimal: You know, to me, the discussion we had on Tuesday, it was very clear that set of restrictions only satisfies $l = m$, but to someone else if it is not clear—like they think that can be interpreted differently. I think you have an obligation to make sure that everyone is on the same page. Whether one person or another changes their position, I think it is very important that everybody agrees on a given definition or theorem, etcetera. But I couldn't figure out any other way to represent that, to possibly represent it in another way that might make it more clear.

Similarly, Real's big idea was "How can we convince/persuade Binary to differentiate his k's." The classroom dialogue continues:

Real: Yeah I think clearly we spent a lot of time in class on it the other day, and it's an important point. Generality, or proving that something is universally true, is more valuable than obviously proving specific cases. I think it is important that we help Binary get to that point. But I just wasn't sure exactly how to persuade him that we needed to differentiate the k's in that specific example to ensure that we have a general case that our proof covers all the bases.

Surreal: So it seems like Infinitely Repeating Decimal and Real are thinking, "We've got this idea and we want to convince Binary of it." And Binary felt, I think; how did you feel Binary? [Recall Binary's big idea: "If no one agrees with you, you're wrong."]

Binary: The big idea is like, yeah I agree with ya'll 100%. But ya'll are not listening to me when I say that. With Infinitely Repeating Decimal, what he's trying to say, I completely agree. But I'm not looking at it as just "k." When I see that definition of odd—For me, I feel like two times any number in the world plus one would be odd. So I'm feeling like, when I see a definition I am taking that definition plussing that definition to get this new definition. So when I see that I just take k and I make it like z or like $2z + 1$ equals odd, and that's how I'm seeing it. So even though the k only satisfies $l = m$. To me, I feel like just the definition alone, no matter the variable, is enough to prove the theorem.

Surreal: Sure. I think that you're right. That's a good idea. I think that we are not used to mathematical argumentation. So we've got some ideas, we're trying to convince each other of our claims and it's important that we are criticizing each other's ideas. Right? Nobody, I do not think, was trying to attack Binary. Like, "You're wrong!" I don't think anybody was trying to do that. We are just trying to come to an agreement on these ideas and sometimes mathematical communication isn't easy. But has anybody ever had to argue mathematically in a math class before this one?

::class is generally silent::

Odd: Not to this extent.

Surreal: Odd? Maybe a little bit? So this is something different, we have to be sure we argue against people's ideas. We don't argue against any people. I don't think we are doing that. Dr. Amicable, do you want to add anything about that?

Dr. Amicable: Yeah. I think this is an important idea and I want to make sure that we pause in this moment to really think about what it means to argue in mathematics. Right? What it means to critique. So it's important that we just be professional in how we do that and emphasize that what we are critiquing is an idea, a mathematical idea, we are not pointing out a particular person. And Binary, like I said last time, I was really happy that you were standing your ground. And you were like "You know what? I still believe this, and I am going to stand firm in that." And that was really awesome to us. Surreal and I were reflecting on that. So let's just keep that in mind. Okay. So that we want to be kind to one another and respectful of one another's ideas. But also strong in our convictions of what is mathematically true.

…

Whole: This goes to what you said [Dr. Amicable]. And Binary I am not pointing at you. I am just talking about what she said. When you are talking about fighting for your convictions and stuff. At a certain point, if you are stuck on one idea, but that idea is wrong. But you haven't seen it as wrong yet. You are convinced that it is true. But the idea is wrong. When should you reach that point of coming to the idea of everyone else instead of sticking to your convictions?

Dr. Amicable: Yeah that's a great question. I do not have the answer to that. I wish I did, I'd be a lot richer. But it's a great question right? Should Binary agree with everyone else just because everyone else says that?

Odd: No because I say it. ::class laughing::

Surreal: No. ::laughing::

Odd: Oh. I mean not that.

Surreal: Even sometimes, I think, mathematicians are attacked; their ideas are attacked and everybody in the whole field thinks "this mathematician, his math isn't good." And then 2000 years later, this guy's math who was rejected is what we use now. So I know that we are not all mathematicians here, but maybe we want to be mathematicians right? We are apprentice mathematicians, some of us. And so I think it's important that we stand behind our convictions but also consider the other point of view. I think that is what is important. The ability to step back and say, "Okay what is everybody trying to say here? How is it different from my opinion? Is it really different?" Or maybe it is just a new idea under the surface that we haven't talked about that can generate a new concept.

Dr. Amicable: I know when I have been to math conferences and mathematicians are presenting their work, sometimes it gets fairly heated in the room, right? Surreal, you've been to such sessions?

Surreal: I don't know if I have been to those.

::class laughing::

Dr. Amicable: No? Well I have seen things that were similar to what we saw in class on Tuesday where it's back and forth like "I'm not sure I understand why you can say that because I see it this way." And what is uncomfortable for some of us in that situation is that we have sort of grown up in this system where we see math as black or white. It's like one or the other. It's right or it's wrong. And what we are learning, I hope what we are learning in this class, and as we move into upper level mathematics is that there is a lot more grey area in mathematics than we're used to, than we're accustomed perhaps, than we're comfortable. Actually there is a lot of emotion involved in mathematics. And there is a lot of grey area involved. So Whole, I wish I could answer your question as in, this is when you should cater to the crowd, and this is when you should hold your convictions but it's really a personal thing that you need to continue to work through. And Binary's going to keep thinking about these ideas and we're going to keep thinking about Binary's idea. About this idea of variable and what it is representing in general. Okay? So let's just keep thinking. I like what Surreal said about we need to keep thinking about one another's ideas, and really try to understand the other idea. The more we can understand someone else's idea, the deeper our own understanding will become. Alright?

Binary: I think no one really understands my idea. I think people are just so focused on 2k+1 that they are not looking at the bigger picture here.

Whole (interrupting): Well earlier before class today, Complex brought it up and we were talking about this ... And he brought it up that what you were saying break it down by the definition of variable—what the actual definition of variable is …

Over the course of two days, this was at least the fourth time that Whole interrupted Binary. Complex shares an idea, and Surreal, wanting to move on to other course content, asks Binary to explain his idea one last time.

> Binary: For me when I see k, I pretty much in my head, I put an odd number times 2, and I put the z which means any real integer, plus 1. So I know if I see $2z + 1$ that represents any possible odd number. So I put $(2z + 1) + (2z + 1)$ equals an even number. That's what's in my head. So when I see z I know I can put in any number imaginable and get an even number. And for this example I feel like that was enough proof. You didn't need an example. You didn't need any other variables, and that is the idea that I had.

Surreal tries to get the class moving on to another task. "Can we move on now?" But the rest of the class wants to continue to argue and discuss. At one point Odd says "You can only fit one in there at one time. … Let's separate $2k$ to be $k + k$. Then suddenly it is not odd anymore because k could be both 2 and 3 at the same time. Does that make sense?" Binary replies solemnly "You said it all wrong bro. You said it all wrong." Later on, Dr. Amicable reiterates that the instructors value classroom discourse. She also recognizes that Binary has an advanced perspective of variable.

> Dr. Amicable: Yeah. I think that's a really important idea…. You know this conversation; we're allowing it to go so long and we are digging into it so deep because I value this type of thought. We value this type of thought. And the really cool thing is that, Binary, you have such an advanced perspective on this. Your

perspective is really advanced and really deep. You are seeing the generality of this situation in ways that I wish many of the students in my other classes could see. So we don't mean to downplay your idea at all. Sometimes we need to communicate to people that don't have that same depth of generality.

…

Dr. Amicable: What do you say we move on?

Students: Awwwww.

Dr. Amicable: Yeah we could probably talk for another hour about this.

A couple weeks later, in my journal I wrote:

Emotions today I remember Binary hasn't talked the last two class periods. … We said we were not criticizing Binary, just his ideas. But Even took it as criticisms of him. Our ideas are our selves. NOM - mathematics involves criticism of people's ideas and argumentation. Students are not ready for a class in which their ideas (and hence their selves) are criticized against the "objective" standard. Even in school math… a bad grade signifies bad ideas. The teacher is criticizing the ideas and evaluating them based on perceived objective standards. Filtering and homogenizing thought. I guess what typically happens is that students are told the "right" ideas. Take away the creative act. When we allow students to take part in the creation of the process, we can judge their ideas relative to objective standards. … I have a vision of pre-service teachers afraid to speak in class.

> Mistakes are okay! Push our thinking forward as a community. Courage and humility.

I was referring to Lampert (1990) when I wrote of courage and humility. She wrote,

> My argument about what is entailed in teaching students about the nature of mathematical knowledge draws on work in the history and philosophy of mathematics. This work supports a vision of knowing mathematics in the discipline that differs from knowing mathematics in conventional classrooms. My research examined whether it was possible to make knowing mathematics in the classroom more like knowing mathematics in the discipline. My organizing ideas have been the "humility and courage" that Lakatos and Pólya take to be essential to doing mathematics. I have treated these as social virtues, and I have explored whether and how they can be deliberately taught, nurtured, and acquired in a school mathematics class. I concluded that these virtues can be taught and learned. What has been described here thus is a new kind of practice of teaching and learning, one that engages the participants in authentic mathematical activity. (p. 59)

If we are to open the classroom so that student ideas drive the discussion, we must recognize that these ideas are part of students' personal identities. It will take courage for students to put their ideas forth to the classroom community for criticism, and humility for them to realize that their ideas may need to be refined in light of new evidence and ideas.

Odd, Even, Odd, Even

This narrative is a transcription of a moment of conversation between Dr. Amicable and Surreal during one of their course planning meetings. Surreal tells Dr. Amicable a story about his then six-year old son. In this story we see at least three elements from the IDEA framework: pure mathematical inquiry is an exploration of ideas, mathematical ideas are part of our identity, and our knowledge is dynamic. My son is excited to share the results of his mathematical explorations, personally taking ownership of an idea after he learns that there are some mathematical ideas that no one has ever had before.

> Surreal: That's what I want people to know about math. Mathematicians, at least some of them, really enjoy this work because it is exciting and it is interesting. And if that is never valued in the classroom then how will students ever learn that is something that can happen? You will never believe what my son said to me two days ago. So I told him two weeks ago I am working on an unsolved problem. Nobody has solved it before. And he was like "Wow not even your teacher?" And I was like "No. Nobody has solved it. Never before." ::laughing:: So he thought that was really cool I think. So then the other night in bed he was like, "3, 6, 9, 12, 15, 18, 21, I notice that if you count by threes then it goes odd even odd even odd even odd even." And he was really excited about that idea. And then I said "Oh yeah, that's really cool." And then the next day I mentioned his idea again and he was like "I can't believe that I was the first person who ever thought of that."

::laughing::

Dr. Amicable: That's awesome!

Levels

The following two narratives, Levels and Cases, are companions that were originally drafted together to highlight the NOM characteristic that proofs are bearers of mathematical knowledge. In terms of the IDEA framework, Levels can be viewed as a story about the exploration of mathematical ideas and of the dynamic nature of mathematical knowledge. The narrative begins with a lengthy quote from the mathematician and philosopher of mathematics, Yehuda Rav. As Rav explains the idea that proofs are bearers of mathematical knowledge, he also alludes to the fact that mathematics is an exploration of ideas, mathematical knowledge is socially vetted through argumentation, and the mathematician's knowledge is dynamic.

> [P]roofs rather than the statement-form of theorems are the bearers of mathematical knowledge. Theorems are in a sense just tags, labels for proofs, summaries of information, headlines of news, editorial devices. The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systematisation of results—the entire mathematical know-how is embedded in proofs. When mathematicians pick up a paper for study, they turn their attention to the proofs, since proofs are the centre of gravity of a research paper. Theorems indicate the subject matter, resume major points, and as every research

> mathematician knows, they are usually formulated after a proof-strategy was developed, after innovative ideas were elaborated in the process of 'tossing ideas around'. Proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies—devices which can be assimilated for one's own research and be further developed. Needless to stress that in studying proofs—or experimental procedures—we are also engaged in a cumulative collective verification process. (Rav, 1999, p. 20).

Keith Weber, a mathematics education scholar from Rutgers, collected evidence that corroborates Rav's idea that proofs are bearers of mathematical knowledge in his interviews with professional mathematicians. Weber (2010) found that one of the reasons mathematicians read proofs is in order to learn new methods and new techniques that could be used in their own work, in essence, filling their mathematical toolbox (cf. Hanna & Barbeau, 2008).

When I spoke to Dr. Combinatorial about my desire to begin mathematical research for the purposes of my dissertation, he e-mailed me some PowerPoint slides and told me to make sense of them and come back and talk to him after I had done so[5]. This was no easy task! I found myself reading chapters from graph theory textbooks (e.g. Harris, Hirst, and Mossinghoff, 2008; Dietstel, 2010) related to graph coloring and

[5] After we agreed to work on his conjecture, he later sent me a paper that I also was tasked with making sense of and explaining to him. This paper was written by Dr. Combinatorial and two of his colleagues, and was under review for publication.

watching videos of mathematicians proving theorems related to the chromatic number to get acquainted with the ideas. Based on the work I had done in my review of the literature, I was well aware of the notion that proofs are bearers of mathematical knowledge. In an early journal entry I wrote, “I am excited reading this graph theory text with the knowledge that the proof techniques in this textbook may be valuable in my future work. I feel I did not understand this as I worked on my master’s thesis.” A couple days later, I came to a realization: “In everything I am reading, including the graph theory textbook, Dr. Combinatorial’s PowerPoint slides, this video by Seymour, the same proof method is being applied. Seymour called it a levelling…”

For instance, here in Figure 34 is an excerpt from one of Dr. Combinatorial’s PowerPoint slides:

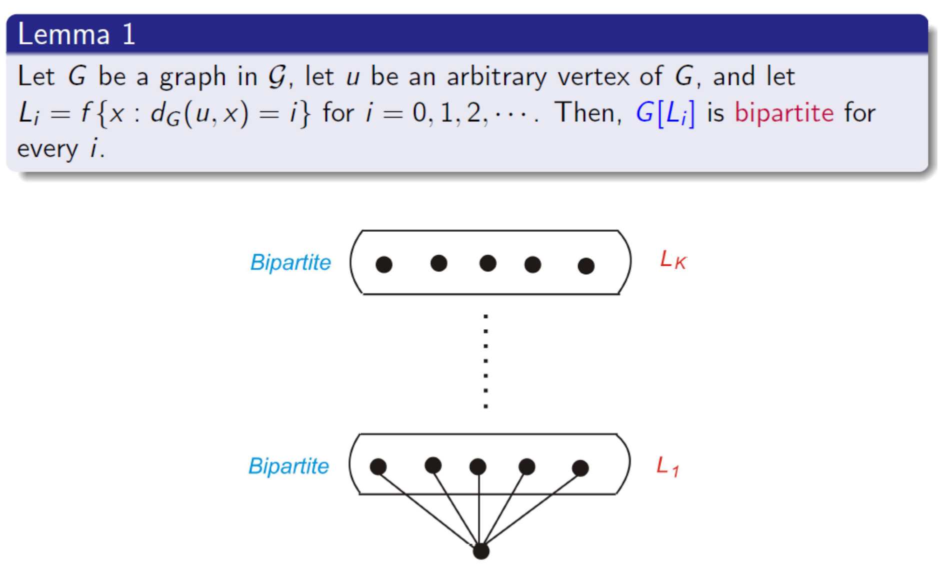


Figure 34. Dr. Combinatorial’s PowerPoint Slide

I was unsure about the meaning of the notation, $L_i = f\{x: d_G(u,x) = i\}$ for $i = 0,1,2, \ldots$ (One reason that I perhaps misunderstood is that there is a typo in the text! That f should not be there.) I was also struggling to understand the proof technique that Dr. Combinatorial was using. But finally it all clicked when I watched a YouTube video of the Princeton mathematician Paul Seymour[6]. What Seymour explained is that every graph can be partitioned into levels. For instance, let's take a look at the Petersen graph shown in Figure 35.

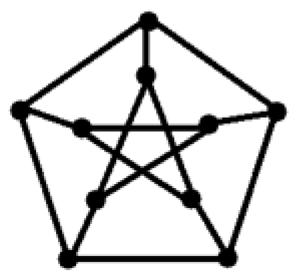


Figure 35. The Petersen Graph

We will construct levels by first choosing any arbitrary vertex from the Petersen graph. For instance, choose the red vertex in the upper left corner, labeled u in Figure 36.

[6] https://www.youtube.com/watch?v=mlf8yhq9tJ4

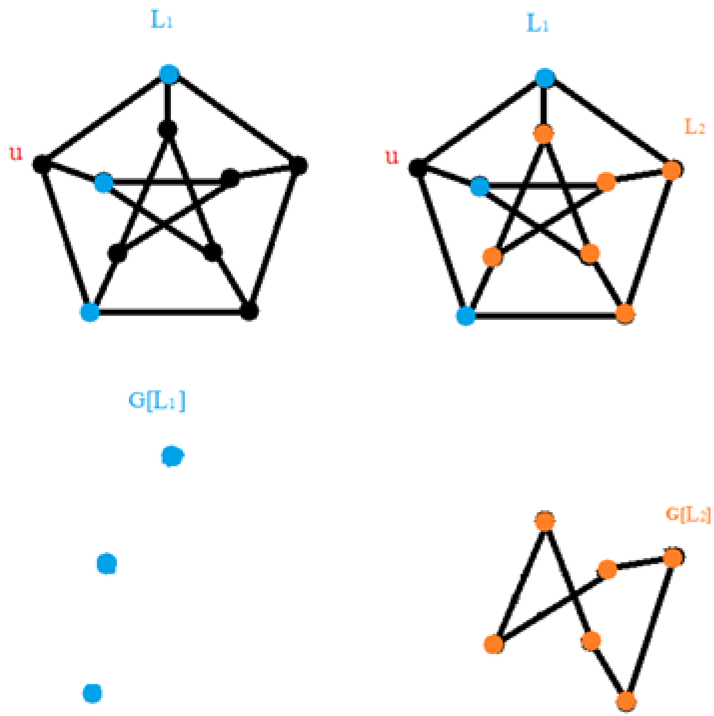


Figure 36. Partitioning the Petersen Graph into Levels

Notice that all the blue vertices are a distance of one edge away form u. We say that those vertices are in Level 1, denoted L_1. Then all the orange vertices shown in Figure 36 are a distance (shortest path) of two edges away from u. We say those vertices are in Level 2 (L_2). We can then consider the *induced subgraphs* of these collections of vertices, which are the graphs formed by all the vertices and all the edges that connect those vertices. We denote these induced subgraphs by $G[L_1]$ and $G[L_2]$ respectively and they are shown in Figure 36. Try to make sense of these definitions and the diagrams above before continuing.

This method of partitioning a graph into levels is crucial to the approach Dr. Combinatorial and I used to prove lemmas related to our conjecture. It was our standard method of attack. In Figure 37 is an excerpt from one of my mathematics notebooks. At the time I drafted the work on this page I had a robust conception of the levelling process. I can reflect on the dynamic nature of my own knowledge as I gradually began to understand the technique. I understood that Dr. Combinatorial's notation $L_i = \{x: d_G(u,x) = i\}$ referred to the set of all vertices a distance of *i* edges away from some arbitrary vertex *u*. Each oval shape in Figure 37 represents the induced subgraph of a different level (again an induced subgraph consists of all the vertices a given distance from *u* along with all the edges that connect those vertices).

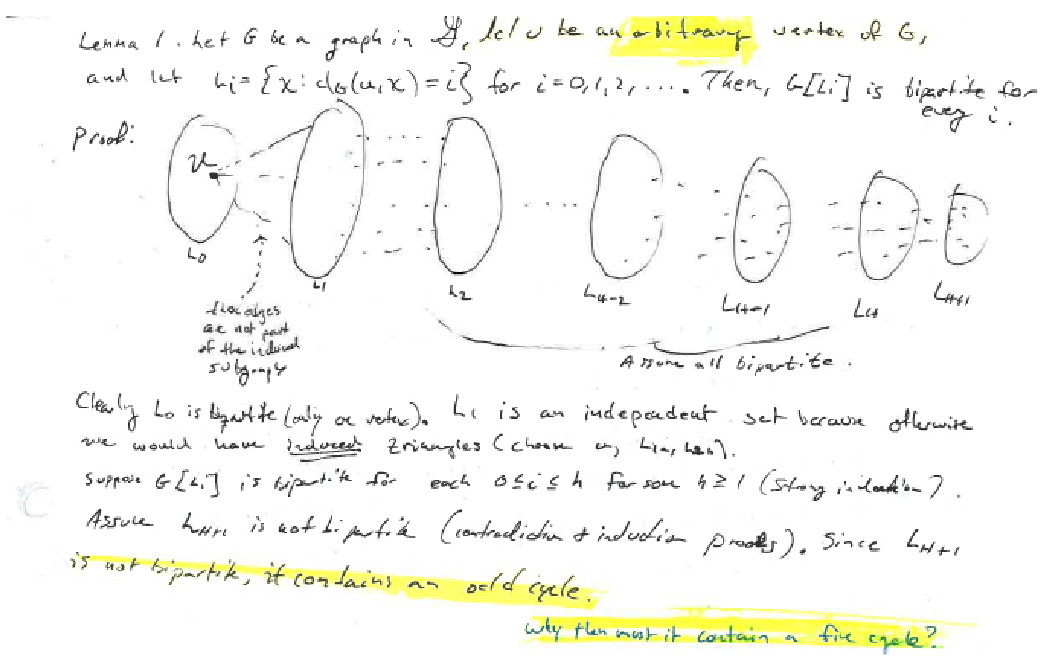


Figure 37. Level after Level

Again, I want to emphasize that I did not understand this technique until I saw it in several places (Dr. Combinatorial's PowerPoint slide, a math textbook, and finally a YouTube video). Proofs are bearers of mathematical knowledge, and mathematicians can read other people's proofs to gain ideas about techniques to use in their own work. In my case, I had already inherited a technique from Dr. Combinatorial, but it was not until I read (or watched) other proofs that I fully understood it. Below in Figure 38 is a photo from Dr. Combinatorial's whiteboard to further illustrate this concept.

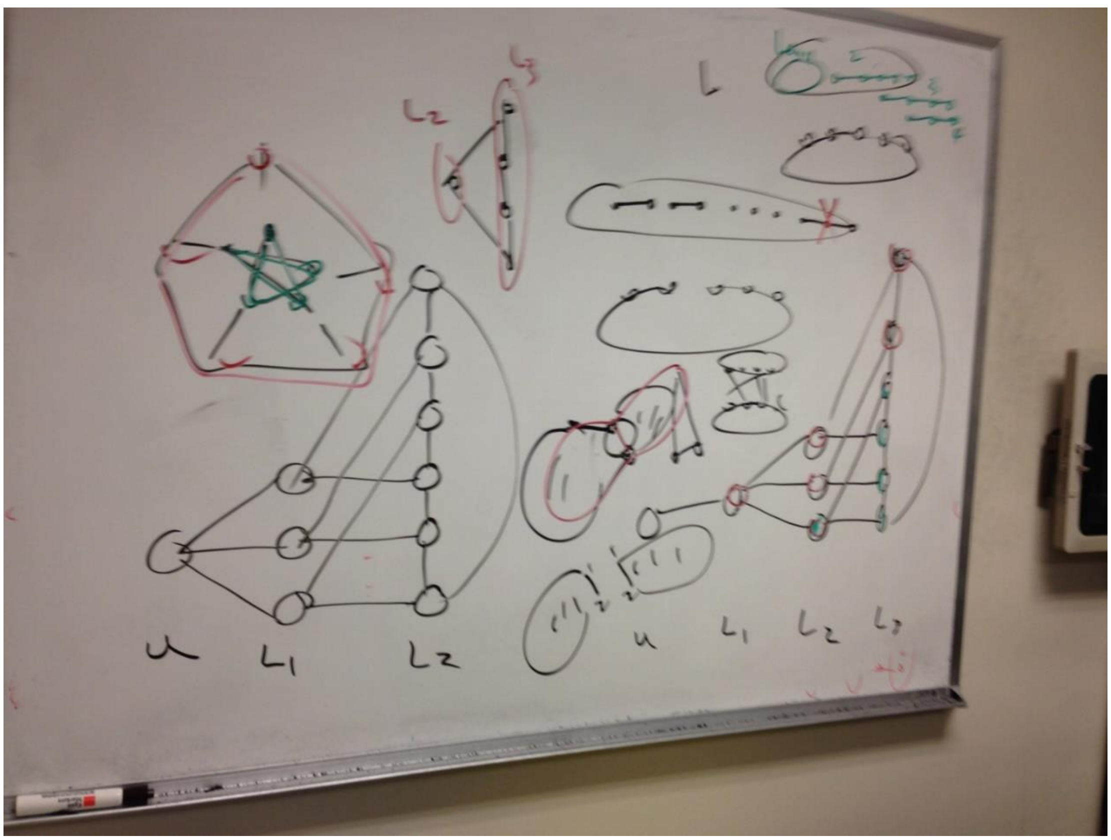


Figure 38. Dr. Combinatorial's Whiteboard

Notice that in the lower left hand corner of the whiteboard there is the alternate representation of the Petersen graph I mentioned in a previous story. Notice how it is labeled according to the levels: first with the vertex u and then levels L_1 and L_2. This levelling was a crucial technique that we used in our work.

Cases

It may be beneficial for students learning to construct proofs to know and understand that they may find new techniques and methods to use in their own work from other people's proofs. The companion piece to Levels, Cases demonstrates how students may come to understand that proofs are bearers of mathematical knowledge in the classroom. In terms of the IDEA framework, it is another instance of the social vetting of mathematics through discourse and argumentation. In this case it was the validity of a proof technique that needed to be debated.

One day in class, a group of three students, Whole, Permutation, and Binary presented their argument for the claim that "if p is an integer, then $p^2 + 3p + 2$ is even" to the class. These students used a proof by cases. It was not clear to all the other students that this approach was valid. Consider the transcript below, beginning with a summary of the group's proof.

Proof: If p is even, then $p = 2k$ for some integer k. Then

$p^2 + 3p + 2 = (2k)^2 + 3(2k) + 2 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$.

If p is odd, then $p = 2j + 1$ for some integer j. Then

$p^2 + 3p + 2 = (2j + 1)^2 + 3(2j + 1) + 2 = 4j^2 + 10j + 6 = 2(2j^2 + 5j + 3)$.

Even: I was slightly confused because it says "p is any integer." That is the only thing that it says, and "$p^2 + 3p + 2$ is even." So why did you go from p is any integer to p is even and p is odd?

Whole: So we could split it up and see how it applies to an even and odd.

Permutation: Technically there are two subsets of the integers. Technically p is an even integer or p is an odd integer so we had to prove for both.

Dr. Amicable: Even, do you buy that that covers all cases if they check for evens and they check for odds?

Even: Ummm. I need a second to think.

Dr. Amicable: Finite, what do you think?

Finite: On the top part, the top proof, one problem I have is; so when p is even, shouldn't it be when p is an integer? Because integers can be both even or odd.

Prime: They did a proof for even numbers and then a proof for odd.

Whole: But that proof for that one up there is for when p is even. So it says when p is even, of course this is true for when p is even. Then $p^2 + 3p + 2$ is even, and then you look at the bottom. It says when p is an odd integer, then that is going to be even as well. So it's specifically for when p is even, and then specifically for when p is odd, showing for when it's even or odd. It's like what Permutation was saying earlier, you technically do have two subsets when it comes to integers you have the evens or odds.

> Dr. Amicable: This is a unique strategy that we can put into our toolbox, right? That if we are trying to prove something in general for any integer. Then we could try to split it in two, this is what would be called a proof by cases. So we could split it into two cases where we test it for evens and we test it for odds, and since we have shown for both of those that it works, then it would work for all integers.

I think it is great that Dr. Amicable legitimizes proof by cases as a method students can use in their future proofs, and in their future toolbox. What I think might also be good is being explicit, telling students that this is one reason mathematicians read proofs, and that they may also find it valuable in their own work. A couple of students mentioned that they may have learned something new on this day of class in their exit tickets: Finite wrote “Sometimes proving by parts is helpful.” Composite reflected, “Consider I can learn so much from seeing other people’s work! Really enjoyable too!”

Mistakes

This narrative primarily highlights the fact that the mathematician’s knowledge is dynamic and forever changing. One reason that our knowledge is dynamic is that humans (even mathematicians) make mistakes. The process of social vetting may even fail as several mathematicians are convinced that a proof is correct. Also the reader should observe the collaborative exploration of ideas between Dr. Combinatorial and I. Note how the social vetting process is crucial in helping Dr. Combinatorial and I revise our knowledge. The section begins with a discussion of Ernest’s distinction between subjective and objective knowledge.

Paul Ernest (1991) described his social constructivist philosophy of mathematics, and he argued that mathematical knowledge is generated through cyclical process. There is both objective knowledge, ratified and generally agreed upon by the mathematics community, and there is subjective knowledge of individuals. As members of the community become enculturated into this objective knowledge, they interpret it in their own way, perhaps creating new mathematical ideas and transforming what is objective. But also an individual mathematician's subjective knowledge is sometimes transformed or revised (perhaps through discussions with other mathematicians). When we critique each other's ideas and find flaws in each other's reasoning, our knowledge is refined. See Figure 39 for my representation of Ernest's cycle of knowledge construction in mathematics.

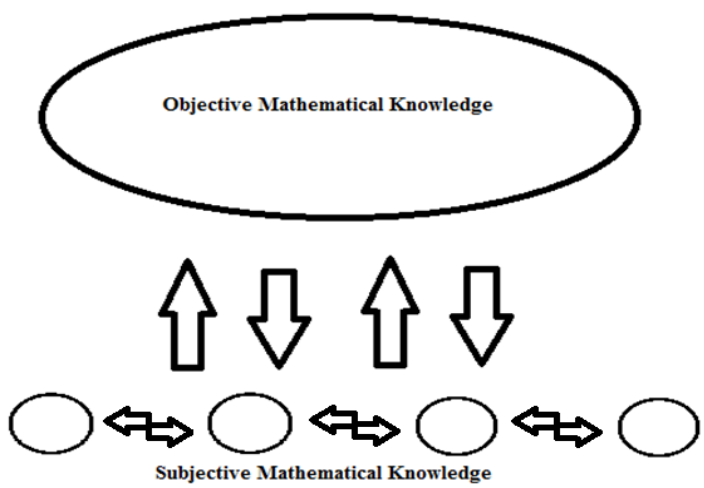


Figure 39. Paul Ernest's Cycle of Mathematical Knowledge Creation

It is not controversial that our subjective knowledge is revised. Another way of putting this is that *mathematicians make mistakes.* In my work with Dr. Combinatorial, I found that we often made mistakes. One day Dr. Combinatorial provided a verbal proof of a claim about any graph in the class of graphs we were interested in (graphs with no 3-cycles, no 4-cycles, at least one 5-cycle, and all other odd cycles having a chord). I was very excited about his proof.

Dr. Combinatorial claimed that when we partitioned the vertices of the graph into levels based on the distance each vertex was from some arbitrary vertex (as discussed in the *Levels* story), the induced subgraph of the vertices at the third level will form an independent set of vertices. In other words, Dr. Combinatorial claimed that none of the vertices in Level 3 are adjacent (there is no edge connecting any vertex of L_3). His reasoning was by contradiction. He assumed that we did have an edge in Level 3. Figure 40 is a picture of Dr. Combinatorial drawing this hypothetical edge.

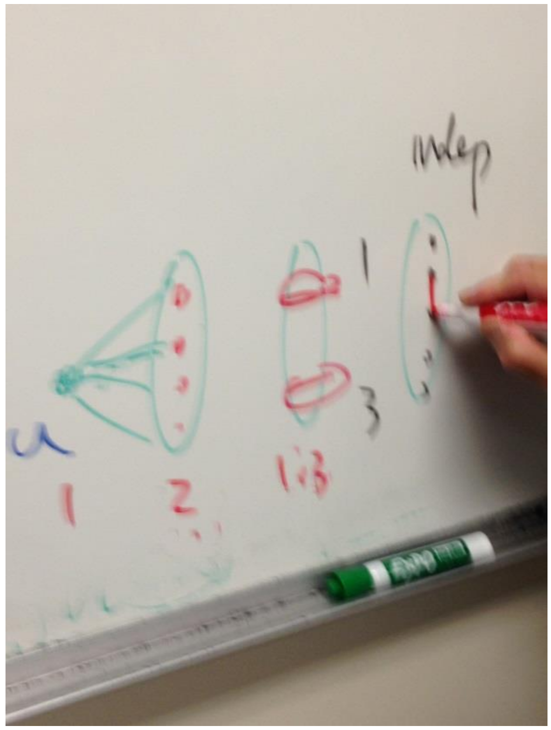


Figure 40. Hypothetical Edge in Level Three

So we have assumed there is an edge in Level 3. Now consider the two vertices that form the endpoints of that edge. Each of those must have a neighbor in Level 2. (Think about why this is true if it is not clear!) Furthermore, those neighbors must be distinct, or otherwise we would have a triangle! This is shown below in Figure 41.

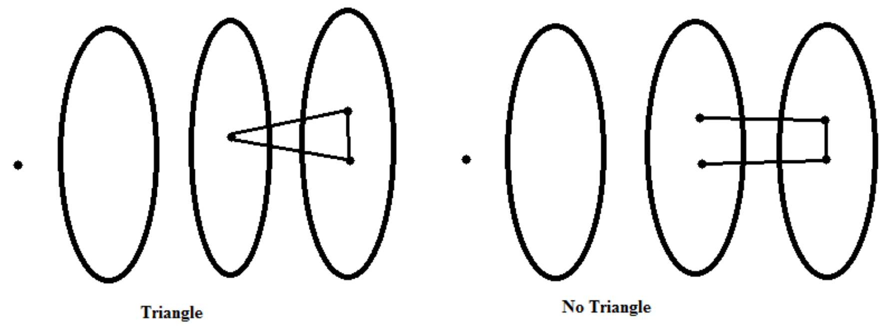


Figure 41. The Neighbors in Level Two Must be Distinct

Dr. Combinatorial used similar reasoning and explained that the vertices in Level 2 must also have distinct neighbors in Level 1. Those vertices connect back to our arbitrary initial vertex *u*. Then if you count the number of edges all the way around (see Figure 42), you will find that you have a 7-cycle. Do you recall from the *Tension* narrative that our class of graphs has no 7-cycles? We have arrived at a contradiction, and thus we have proven our claim that Level 3 is independent.

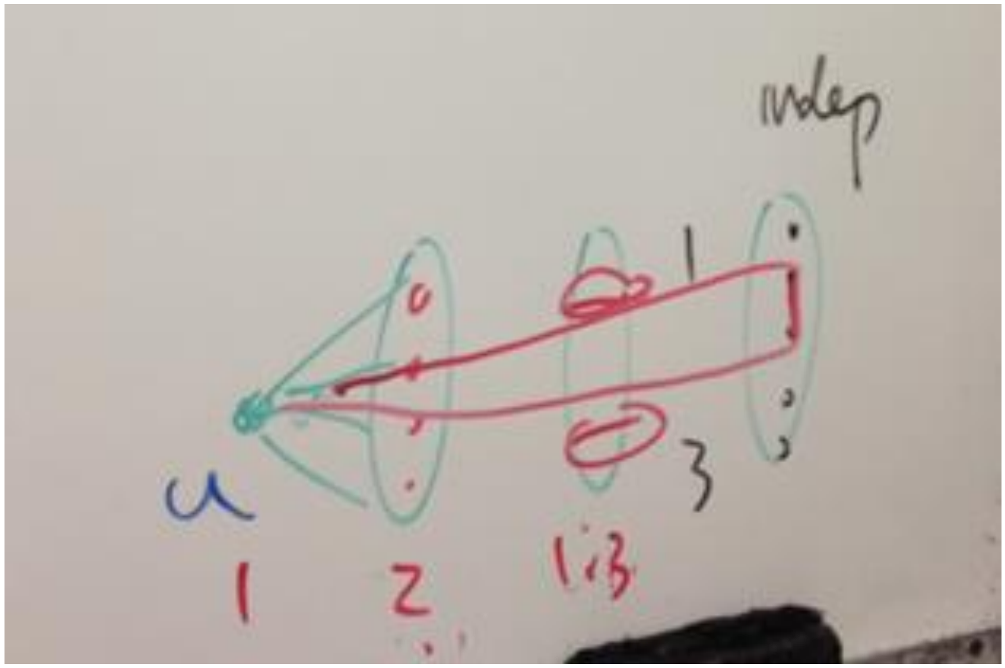


Figure 42. The Diagram Dr. Combinatorial Used in his “Proof” That Level Three is Independent

I was very excited about Dr. Combinatorial’s claim because, if it was true, then we could prove that as long as our graph only contained three levels, the chromatic number is 3. But over the course of the next week I began trying to reprove Dr. Combinatorial’s claim on my own, and I found that it was not true. I found some counterexamples. There must be a hole in his proof.

In Figure 43 there is a counterexample that I drew on Dr. Combinatorial’s whiteboard a week later (The 6/5 circles are related to a different idea).

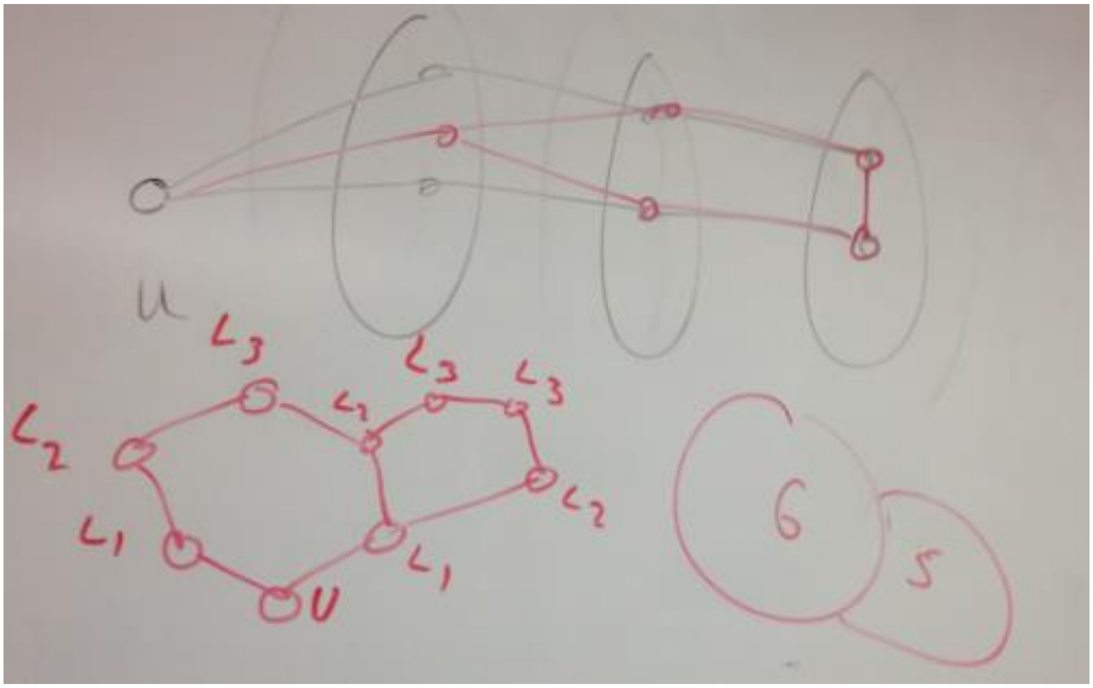


Figure 43. Counterexample on the Whiteboard

Notice that in the lower left hand corner of the whiteboard I have drawn a graph (with girth 5 and no odd holes of greater length) and labeled the vertices according to levels after picking an arbitrary initial vertex u. Also notice that there are two vertices from Level 3 that are connected with an edge. This contradicts Dr. Combinatorial's claim that the third level is independent. What is the problem with Dr. Combinatorial's proof? Well let's again assume we have an edge in Level 3 (shown in the uppermost diagram in Figure 43). Then the vertices that comprise that edge must have distinct neighbors in Level 2 (or else a triangle would form). But, those parent vertices need not have distinct neighbors in Level 1 (as Dr. Combinatorial assumed). Rather they can connect to the same vertex, and we will have the 5-cycle shown in the top-most diagram in Figure 43.

So I showed these counterexamples to Dr. Combinatorial, and here is a transcript of our conversation afterwards.

Dr. Combinatorial: So that's a problem with my claim?

Surreal: Yeah, just the claim you made last week.

Dr. Combinatorial: It's not good, right? I have to clear my, some thoughts in my mind.

::speaks softly while thinking and looking at the whiteboard for several seconds::

Dr. C: This is not necessarily a 7-cycle?

Surreal: No.

Dr. Combinatorial: Okay.

::long silence::

Dr. Combinatorial: We cannot uh, salvage this.

But of course we could salvage it a bit. We were wrong, but we just needed to refine our knowledge. We now know that whenever there is an edge in Level 3, it must always be an edge of a five cycle that has a single vertex in Level 1.

We made plenty of other mistakes during our collaboration. In the original PowerPoint Dr. Combinatorial had sent me through which I acquainted myself with his problem was the slide shown in Figure 44.

Lemma 2

If G has radius at most 5, then $G[L_2 \cup L_3]$ is bipartite.

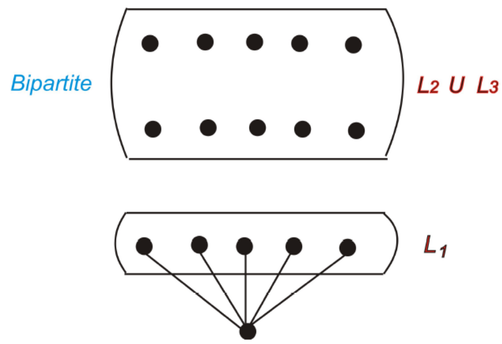


Figure 44. Claim that the Induced Subgraph of Level 2 Union Level 3 is Bipartite

The slide contains the claim that the graph induced by the vertices in Level 2 together with the vertices from Level 3 will form a bipartite graph (if the radius[7] is five or less). A bipartite graph is a graph that can be partitioned into two sets of vertices, each independent of the other (i.e. there are no connections across the two sets). But in my work trying to prove this claim, I found the counterexample shown below in Figure 45. Notice there is a 5-cycle across the two levels.

[7] The radius of a graph is analogous to the radius of a circle. To say the radius of the graph is 5 essentially says you can partition it into five or fewer levels using the techniques described in the *Levels* narrative.

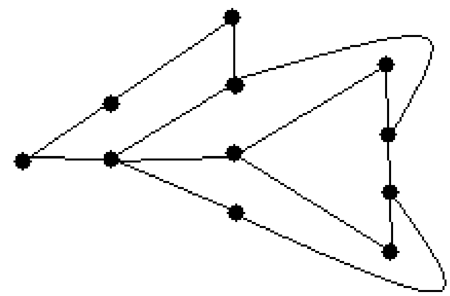


Figure 45. Counterexample to the Claim that L2 Union L3 is Bipartite

Dr. Combinatorial and I made mistakes. But the thing is, even though we made mistakes, they did not bring us down. When I discussed the mistake with Dr. Combinatorial, he said, “I am glad you see this.” When Dr. Combinatorial or I made mistakes it was a good thing. We are not going to prove the conjecture if we do not identify the flaws in our own knowledge and remedy them. Do our students also understand that mistakes are a major part of mathematical work? And that it is important we identify our mistakes so we can refine our knowledge and push our thinking forward?

Coloring

This collection of stories has an overarching mathematical theme of identity. The problems we choose to work on, and the ways in which we work on them (our practices) are part of our identity. One can also still see the concurrent themes of emotional exploration through informal ideas, and the dynamic nature of subjective knowledge.

One evening I was at home working on Dr. Combinatorial's conjecture in my bedroom. My wife walked in on me and said "What are you doing in here? Are you coloring?" Although I was coloring, I replied, "No. I'm doing math." To which my wife replied, "You can't trick me. I see you coloring with your academic ways." See Figure 46.

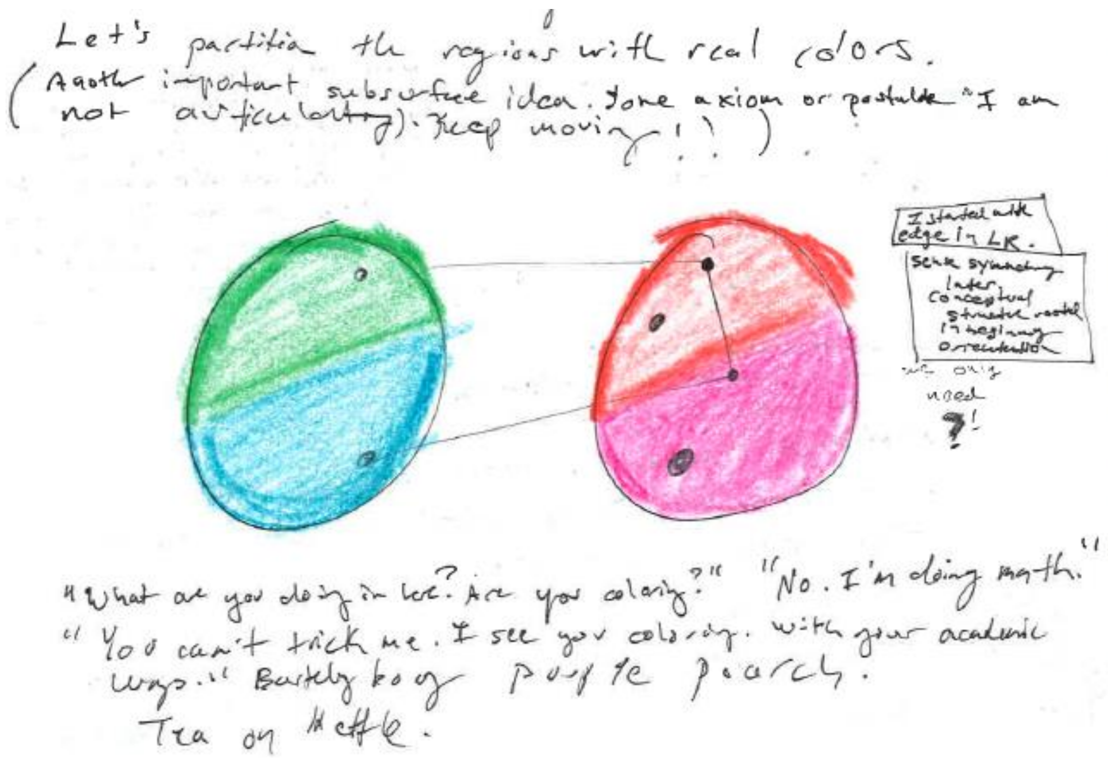


Figure 46. Partitioning with Real Colors

Indeed I was doing mathematics, but I was also coloring. Reflecting back on the time during which I worked on the conjecture, I recall I was very pleased with the fact that I could do both coloring and mathematics at the same time. The coloring seemed to make the mathematics more accessible to other people so that I could discuss my work. I could quickly explain what a graph was as well as the rules of the chromatic game (e.g. Color the vertices so that if two vertices are connected by an edge, they cannot have the same color. Then what is the smallest number of colors you need to color all the vertices?). One day I told Dr. Amicable how "yesterday I spent ten extra minutes at a tutoring session telling my student about this work. I am just really excited." Coloring was also an avenue to get my family members interested in my research. I began to regularly ask my children to color different graphs. My youngest son would often color with me. See Figure 47 for a coloring he did of a 5-cycle.

Figure 47. My Son's Coloring

One day I was working with Dr. Combinatorial in his office, and we were discussing some details of the coloring conjecture. Below is the brief dialogue along with the related drawing from Dr. Combinatorial's Whiteboard in Figure 48. Notice what Dr. Combinatorial says about using colors.

> Dr. Combinatorial: As long as these two are different we are fine.
>
> Surreal: Let me make sure I agree with you. So if this one is red. ::coloring:: And this one is green. ::coloring:: And then say this one is brown. ::coloring:: And this one is red or something. ::coloring:: Then I could just say well, I will color this one green and color all those green. That would be okay. Or color this one red and that one green? Yeah it would be okay. We could do that. So I guess the problem is. I am thinking these have to both be the same color. And these, one's green, and one's red. Is this the problem perhaps? Yeah that is a problem. Yep okay.

Figure 48. Colors Versus Numerals

Dr. Combinatorial: Okay. Comments. If you are an English major, you use green red. If you are a mathematics major you use 1,2,3.

Surreal: Why is that?

Dr. Combinatorial: Because you are a mathematics major.

Surreal: So? ::laughing::

Dr. Combinatorial: Because you may confuse color.

Surreal: Ok.

Dr. Combinatorial: To distinguish color is not as easy as to distinguish number.

Surreal: But they are just labels though to me.

Dr. Combinatorial: Yeah. Just labels. Different colors. Just labels. But if you label, using 1,2 is easier.

Surreal: Okay.

I could sort of see Dr. Combinatorial's point. One may confuse the brown and the red. Certainly numerals would be a more efficient, quicker means of labeling. But what did he mean? An English major uses colors? But a mathematician uses numerals? I used to be an English major, and ultimately my undergraduate degree was in philosophy rather than mathematics. I prefer coloring to using numerals. Does this mean I am not a mathematics person?

Of course I rejected Dr. Combinatorial's advice. I took pride in being colorful with my work. I remember telling my wife that I was rebelling. A solitary, perhaps narcissistic rebellion. In one of my notebooks I wrote, “I can rebel, but only in this privileged white male sort of way that means nothing.”

In the coming weeks I became frustrated working on the conjecture. On the advice of Dr. Combinatorial I had begun to try different approaches attacking the problem. None of these approaches were fruitful. I could not see the big picture for how we would ultimately prove that any graph in our class would need only three colors. I remember feeling slightly defeated. Sitting in my home office I happened to look at a picture on my wall of some work I did as a master’s student. See Figure 49.

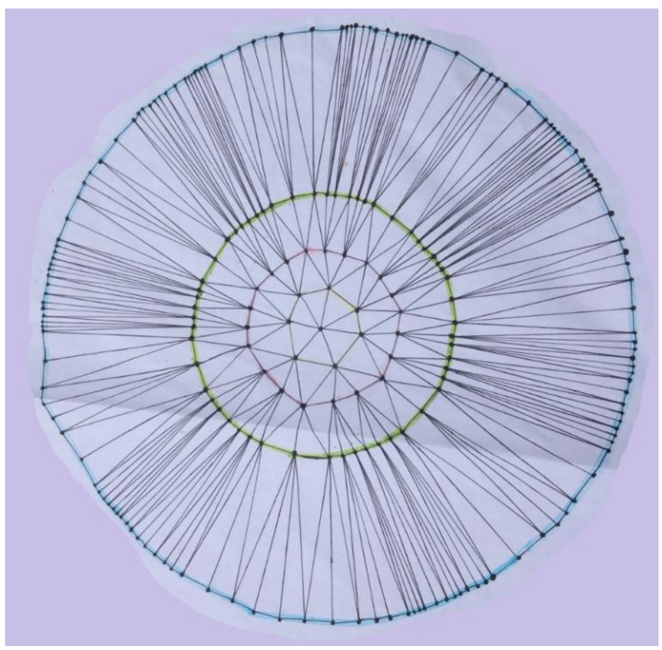


Figure 49. The Beginnings of an Infinite-Triangle Composed Graph

I was inspired to use 5-cycles instead of triangles and do something similar. I began constructing an infinite graph composed of pentagons using the method shown in Figure 50.

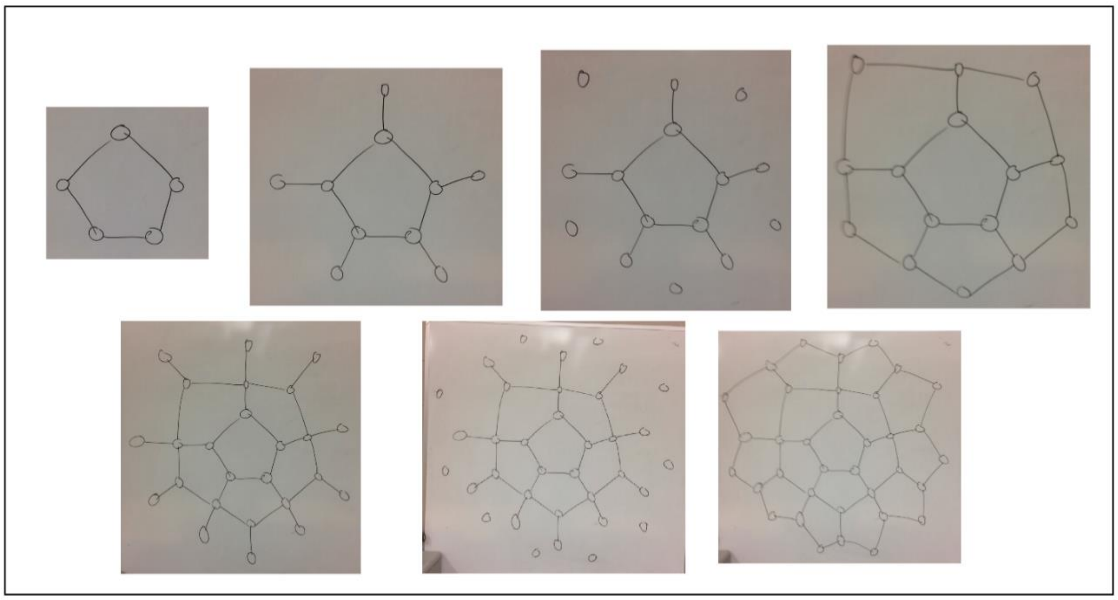


Figure 50. The Start of an Infinite Pentagon-Composed Graph

My daughter and I did some work on the problem together. We did a few iterations and started coloring the outer "rings" first and then moved toward the center pentagon. Using this approach we needed four colors the first time we tried. Later we were able to achieve three. From my journal: "I remember we discovered 5, 10, 20, 40 is the number of vertices/edges in new rings (cycle circles). Also my daughter found the pattern 1, 5, 10, 20 if we count pentagons instead of ring vertices." She also told me to be sure and give her credit if this idea ever made it into a book."

It took me a few days to convince myself that we only needed three colors for the infinite pentagon-composed graph. Eventually I solved it by changing my focus. Instead of starting with an outer ring and coloring the vertices moving ring by ring to the inner pentagon, why not start with the inner pentagon and move out? When I changed my focus, I soon solved the problem. I found an algorithm for coloring each ring in which the red color served as an insulator between distinct blue/green paths. Figure 51 is the coloring that convinced me.

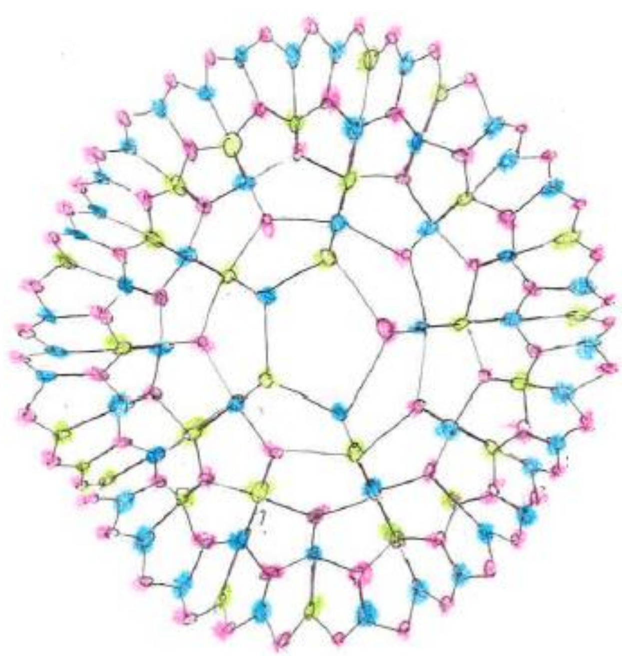


Figure 51. Coloring the Five Rings of the Infinite Pentagon Graph

During the time I was working on the infinite pentagon graph, I was sharing this graph and a few other graphs (that could be extended to infinity) with my children. Figure 52 has a triangle structure, and Figure 53 has a square structure. My wife called the square structure crystalline. My youngest son quickly figured out it only needed two colors, and we colored it together. I hung it on the wall in my home office.

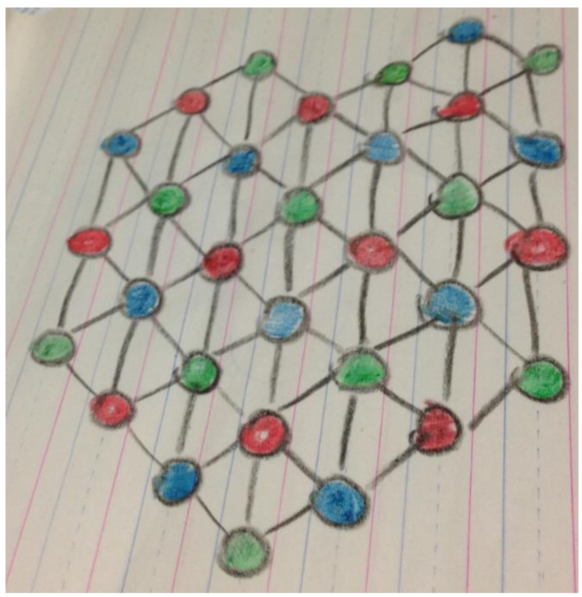


Figure 52. Triangulation

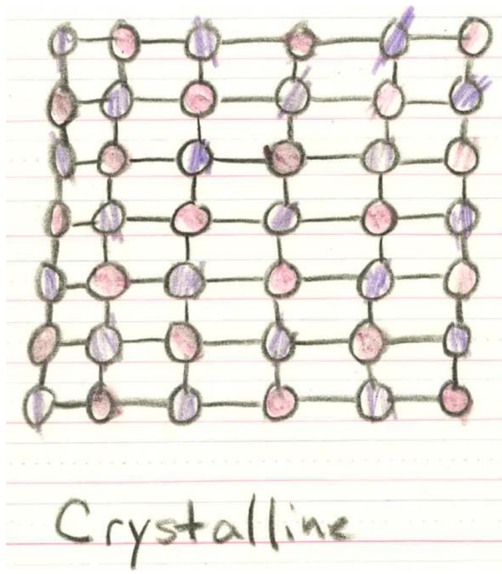


Figure 53. Crystalline

A few nights later I wrote this in one of my mathematics notebooks: “Excited about the crystalline structure on my wall. Affront to identity when Dr. Combinatorial said an English major will use colors. I embrace that identity. A vision where mathematics can work for more than a solitary few. Inspired a bit by Piper’s ‘hire me’ blog.[8] Do what you love.”

The Essence of Research

In the Colors narrative, I described how a change in focus helped me to solve the problem related to the infinite pentagonal graph. Rather than focus on an arbitrary ring and move towards the center, I started from the center and moved outward. Sometimes

[8] http://www.theliberatedmathematician.com/2016/05/hire-me/

we can change our perspective, focus on mathematical ideas in a different way, or use a new method in order to gain insight into a problem. This narrative continues with that idea, which is also related to the notions that mathematics is an exploration of our ideas, and our knowledge is dynamic.

Early on in my work I was working through a graph theory textbook, proving some basic results in coloring theory in order to familiarize myself with the content so that I would be prepared to do productive work on the unsolved conjecture. From my journal:

> I have been studying a graph theory textbook by Harris, Hirst, and Mossinghoff (2008). There is a Chapter on "Bounds of Chromatic Number." Since I am researching this topic, I thought the textbook would be a nice way to expose myself to some of the standard results in the field as well as standard methods for proving the results. I was struggling to understand the theorem that for any graph, the chromatic number will be less than or equal to the maximum degree of the graph plus one, that is for any graph G, $\chi(G) \leq \Delta G + 1$ where ΔG is the maximum degree of G.

The degree of a vertex is the number of edges that emanate from it. For example, consider the triangle graph. All the vertices of the triangle graph have a degree of two, thus the maximum degree of the triangle graph is two. We know the chromatic number is three, which is one more than the maximum degree (two). This is a simple example of the theorem. I tried to work out my own proof for a general graph. I started the proof by focusing on a vertex of max degree which I called v1. I reasoned I could give it a color, but then I needed to show we could color the rest of the graph with at most ΔG (the maximum degree of G) colors. I had trouble showing this so I took a break. From my journal:

> Later after a nap and food and several other interventions, I began to think about the problem again while I was playing Frisbee with my son. I realized that if I consider all the vertices around v1 to have already been assigned a color, then the worst case is they have all been assigned different colors, thus we need one more color, so the total number of colors is equal to the max degree plus 1. … The theorem seems clear to me now. I believe that I achieved an insight in this problem by slightly altering my focus within the graph. Rather than thinking I would color v1 and then be forced to color all the adjacent vertices, I imagined that all the adjacent vertices were already colored and I would then color v1. That change of focus was enough for me to see the truth of the theorem. Not sure what changing focus has to do with the NOMI. I suppose that (since there is at least one) some mathematicians are able to achieve insights into problems by changing their focus or perception within a mathematical situation.

Dr. Combinatorial and I also changed our focus several times during our collaboration in our efforts to prove the conjecture. I learned that we were not the only mathematicians who tried to achieve insights into problems by changing perspective.

Consider the following conversation I had with Dr. Algebraic who called a shift in focus "the essence of research":

> Dr. Algebraic: [A mathematician] invited me to come to a talk, and there was this conversation that came up about embedding graphs on the torus and how very very complicated it was, and how everybody around the world was kind of ground to a halt with this research because it was so complicated. And I was looking at it and thinking, I don't understand this, but I don't understand why they are doing it that way either. So I went back and started thinking about it. And I am trying to come up with a different way of approaching the whole problem now. Not that it was wrong. But it was just too hard.
>
> Surreal: So I think that is what I was trying to say earlier about changing focus. There is this way you are attacking a problem. You study the same problem but you just shift your focus slightly in some mathematical way.
>
> Dr. Algebraic: That is the essence of research really, is a shift in your focus.

We Are the Future

This final story features a classroom discussion that arose after a disagreement amongst students regarding best practices for proof writing. The class discussion leads to a commentary on acceptable mathematical practices both within the classroom and in the discipline of mathematics. The standards of the discipline are negotiated amongst mathematicians, and the discipline of mathematics has a dynamic nature due to the fact that its members are constantly changing.

Near the end of class, students have just debated how much detail they need to put into their proofs. If k is an integer and j is an integer, do you have to write $k + j$ is an integer if you use it in your proof? Do you have to go even further and justify this by mentioning the closure property of the integers under addition? Some of the students say yes. Others say no. Others want to know if they will be “docked for points.”

Dr. Amicable says that the students should do whatever the classroom community agrees is best for communication. She asks me what I am thinking and I say, “You probably don’t want to know.” Nevertheless, I mention that in professional mathematics papers, there will often be gaps. I say, “It is assumed the mathematician audience knows these things. This sometimes makes the papers difficult for me to read—for someone like me who is not a super mathematician. So I would maybe appreciate some clarity sometimes.”

After further discussion, Dr. Amicable says she believes the students should always write their proofs with their classmates as the intended readers in mind, always knowing that they could be called upon to present their work to their peers in class. She also mentions the practice of “handwaving.”

> Dr. Amicable: I was talking to Infinitely Repeating Decimal the other day about handwaving. You can kind of like wave your hands and say, “Oh yeah this is true and I am sure Dr. Amicable would agree with that.”
>
> Odd: It can be shown that, dot, dot, dot.

Dr. Amicable: Yeah! *Clearly*. You will see that in textbooks. *Clearly*, dot, dot, dot. But it is not so clear to me. So let's try to push ourselves. This is all about learning. Let's push ourselves go the extra mile to clarify and communicate as best we can. To our peers. If you are communicating well to your peers then it is likely you are communicating to Surreal and I as well.

Infinitely Repeating Decimal asks if he, or any other member of the class, were going to write up something for publication, "Would it be viewed in a negative light if it was too expository in areas in which it over explains?" Surreal explains that it is a difference of opinion.

Surreal: Well when I did my thesis my professor was like, if we are going to publish this you will have to cut a bunch of stuff. But to me the papers are so hard to read. I would welcome someone coming into the mathematics community who was very explanatory. I just wish more mathematicians could really clearly convey their ideas. But it is just a difference of opinion. There is another mathematician I know who says, that is the fun of it. You have to go check everything yourself and make sure you do all the side work.

::class laughing::

Odd recommends footnotes as a "happy medium" and Infinitely Repeating Decimal agrees. Then Dr. Amicable poses an interesting question taking the discussion to a new level: "You know who the next generation of mathematicians are right?" Someone hesitantly says "us."

Dr. Amicable: Yes! Right? So *you* are the community. And you will be able to determine those things. What counts as proof is really determined by who is in the community. So that's what's really neat. So if you all go out there and say I'm going to become a mathematician, and I'm going to change this. Just like Surreal. He is going to be right along with you. I want to change it so that it is a little bit easier to understand these arguments. Right?

Infinitely Repeating Decimal: We are going to change the world. I am going to change the entire mathematics community just for you.

Chapter Summary

In this chapter I have presented the IDEA Framework for the Nature of Pure Mathematics. The framework consists of four foundational characteristics that may serve as goals for student understanding of the nature of mathematics: 1) Our mathematical ideas and practices are part of our *identity*; 2) Mathematical knowledge is *dynamic* and forever refined; 3) Pure mathematical inquiry is an *exploration* of ideas; and 4) Mathematical ideas and knowledge are socially vetted through *argumentation*. These characteristics, along with some secondary NOM characteristics were illuminated in ten complementary narratives.

CHAPTER FIVE: DISCUSSION

Introduction

In this final chapter, I synthesize the results from the previous chapters and discuss implications for teaching and research. I revisit the broad nature of mathematics (NOM) framework from Chapter Two, and merge it with the IDEA framework to produce an encompassing list of NOM characteristics that may be further explored and refined in future studies. I then argue that the IDEA Framework for the Nature of Pure Mathematics can be used to inform IDEA-based mathematics instruction, the goal of which is to open up a space for both the development of a humanistic conception of mathematics within the minds of students and the development of their mathematical knowledge through engagement in mathematical inquiry. Throughout this discussion, I draw implications for university instruction, and I encourage mathematics instructors to make instructional choices that position mathematical ideas as the focus of pure mathematics courses. I take issue with the foundationalist picture of mathematics that is implicitly promoted in many transition-to-proof courses, and I offer some alternatives based on the IDEA framework. I then discuss implications for school mathematics: more emphasis needs to be placed on ideas rather than symbols in school; and teachers need to be conscious whether they are teaching pure or other forms of mathematics (e.g. commercial-administrative). The chapter concludes with future directions and implications for research.

Revising the Broad Framework

The broad framework presented in Chapter Two had three categories: 1) Mathematics as a fundamental part of human cultures; 2) Pure mathematics as a discipline; and 3) Statistical and applied mathematics. An aim of my dissertation study was to revise the second category, and I have replaced it with an expanded version of the IDEA framework (see Figure 54).

Mathematics as a Fundamental Part of Human Cultures • Western academic mathematics is one (but not the only) form of mathematics • Mathematical knowledge is influenced by cultural values. • Mathematical knowledge is embedded within the work of artisans. • The purpose of commercial-administrative mathematical knowledge is calculation for economic purposes; the efficiency and accuracy of mathematical procedures is valued.
The IDEA Framework for the Nature of Pure Mathematics • Our mathematical ideas and practices are part of our *identity*. • Mathematical knowledge is *dynamic* and forever changing. • Pure mathematical inquiry is an *exploration* of ideas. • Mathematical ideas and knowledge are socially vetted through *argumentation*. *Secondary Characteristics* • Proofs are bearers of mathematical knowledge. • Mathematics can be emotional and creative. • Informal mathematical work is foundational to formal knowledge. • Mathematicians change focus in a mathematical situation to achieve insight.
Statistical and Applied Mathematics • Mathematical knowledge is used to shape society, but cannot be considered an absolute judge.

Figure 54. Broad Framework for Continuing Research

The expanded version of the IDEA framework includes the four foundational NOM characteristics as well as the secondary characteristics discussed in Chapter Four. I believe each of these characteristics may be considered valuable for students, teachers, and scholars to understand; but research is needed to investigate the teaching and learning of these characteristics. I believe the field could benefit from in-depth studies of the nature of mathematics as a fundamental cultural activity, or the nature of statistical and applied mathematics. Heuristic inquiry could be fruitful for such endeavors.

While the original framework divided the nature of pure mathematics as a discipline into the nature of mathematical knowledge (NOMK) and the nature of mathematical inquiry (NOMI), I do not make this distinction in the broad framework presented here. I do think researchers should understand these distinctions and be wary of possible difficulties that may occur from conflating these as has apparently has been done with the nature of scientific inquiry and knowledge within science education (Lederman & Lederman, 2014). However, I have yet to experience such difficulties. Furthermore, I believe the IDEA framework has pedagogical value for promoting a humanistic conception of the nature of mathematics. The distinction between NOMI and NOMK may be less important for practitioners than it is for researchers. While researchers interested in either cognition or practice may wish to make distinctions when conducting research, what mathematics teachers need is a tool to promote a new vision of the nature of mathematics. I believe the IDEA framework may serve this purpose.

A Discussion of the IDEA Framework for the Nature of Pure Mathematics

During my study, I became passionate about pure mathematical ideas. Ideas are the heart of pure mathematics. Indeed, the unifying theme of ideas runs through the four characteristics of the IDEA framework. I believe students may benefit if we structure pure mathematics classrooms so that mathematical ideas are at the heart of students' work and class discussions. Students should understand mathematics is an exploration of ideas. Students should develop confidence in creating and sharing their own personal ideas (even though their ideas will be subject to criticism). Students should understand that their ideas will be forever refined as long as they continue to study mathematics.

Research is needed to investigate the potential of the IDEA framework to be a productive tool both for providing instructors with achievable goals for students' understanding of the nature of mathematics and as a guide for structuring mathematical inquiry in the classroom. The promise of the IDEA framework is that by following it, the mathematics classroom will revolve around meaningful ideas, and students will develop a positive conception of mathematics. Ideas should be the focus of instruction. But, are ideas not already the focus of study within mathematics classrooms? I contend that ideas are not predominately the subject of instruction, but rather the subjects are conventions, symbolic manipulations, memorizations, and deductions. Students are often asked to prove theorems without regard to the underlying ideas. By following a deductive process students successfully write credit-worthy proofs. Consider a reflection from my journal:

> Grading the problem set on mathematical induction. I realize that I can grade the proofs without even knowing what is being proved exactly. $n! \geq 2^{n-1}$. I just follow the students' induction steps and see if they have a valid deductive argument. Not a lot of idea exploration.

While Dr. Amicable and I encouraged students to make meaning of statements before proving, perhaps by constructing examples, what was ultimately deemed credit-worthy was a valid deductive proof. I believe students frequently engaged in a syntactical proof production process like that defined by Weber and Alcock (2004):

> We define a syntactic proof production as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. […] In the mathematics community, a syntactic proof production can be colloquially defined as a proof in which all one does is 'unwrap the definitions' and 'push symbols'. (p. 210)

Regardless if the students' proof production was syntactic or not, I was able to engage in a syntactic grading process (at least within the induction problem set). Students' scores were based on my ability to follow their symbolic deductions; I paid no attention to the underlying ideas behind a theorem.

The Fields Medalist William Thurston (1998) wrote, "We mathematicians need to put far greater effort into communicating mathematical ideas. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking" (p. 346). In Foundations of Higher

Mathematics, students' ability to write deductive proofs was prioritized over the ability to explore and communicate ideas. Perhaps it is a sign of the times. A result of the culture. According to Hersh (1997),

> Mathematics as an abstract deductive system is associated with our culture. But people created mathematical ideas long before there were abstract deductive systems. Perhaps mathematical ideas will be here after abstract deductive systems have had their day and passed on. (p. 232)

Are we satisfied to be part of a culture in which instructors and students pay less attention to the ideas behind proofs and more attention on producing valid deductions? I believe that instructors of mathematics and scholars of mathematics education must put serious thought into how we structure pure mathematics courses. What is needed is a renewal of culture. To renew the culture of pure mathematics instruction will require a commitment from instructors and scholars to make choices that promote the values and vision that are expressed by humanistic philosophers of mathematics and represented in the IDEA framework. In the following sections, I continue this call for cultural renewal by focusing on each of the four characteristics of the IDEA framework. After making a few points for each characteristic, I suggest some IDEA-based instructional alternatives in transition-to-proof courses that depart from the standard foundationalist approach in which logic and set theory are privileged.

Our Mathematical Ideas and Practices are Part of Our Identity

Mathematical ideas are part of our personal identity. Consider the language we use when we talk about mathematics. “I think no one really understands *my* idea” (Binary). “That is *my* conjecture… If you really want to work on this one, then I will work hard with you on it. I hope we can settle this” (Dr. Combinatorial). “I can’t believe that *I* was the first person who ever thought of that” (My Son). Boaler (2015) wrote, “All children start their lives motivated to come up with their own ideas—about mathematics and other things” (p. 172). Just last night[1] I reminded my son of his original idea and he said, “Yeah and it works multiplying by fives too. 5, 10, 15, 20. It goes odd, even, odd, even…” How often do we give our undergraduate students the opportunity to create and be proud of their own ideas?

I do not think many mathematicians would object to the notion that our ideas are part of our identity. But many students experience mathematical ideas differently. The instructor tells students up front what the right ideas are. It is up for the students to conform. When we teach pure mathematics in an authoritarian manner, we take away the creative act. If students are to see mathematical ideas as part of their identity, then the pure mathematics classroom needs to be a site of idea-generation rather than a site of indoctrination into what is “right.”

The “right” ideas are those that have been socially vetted within the mathematics community—those concepts and methods at the crest of the wave of human mathematical

[1] The date was Sunday May 28th, 2017.

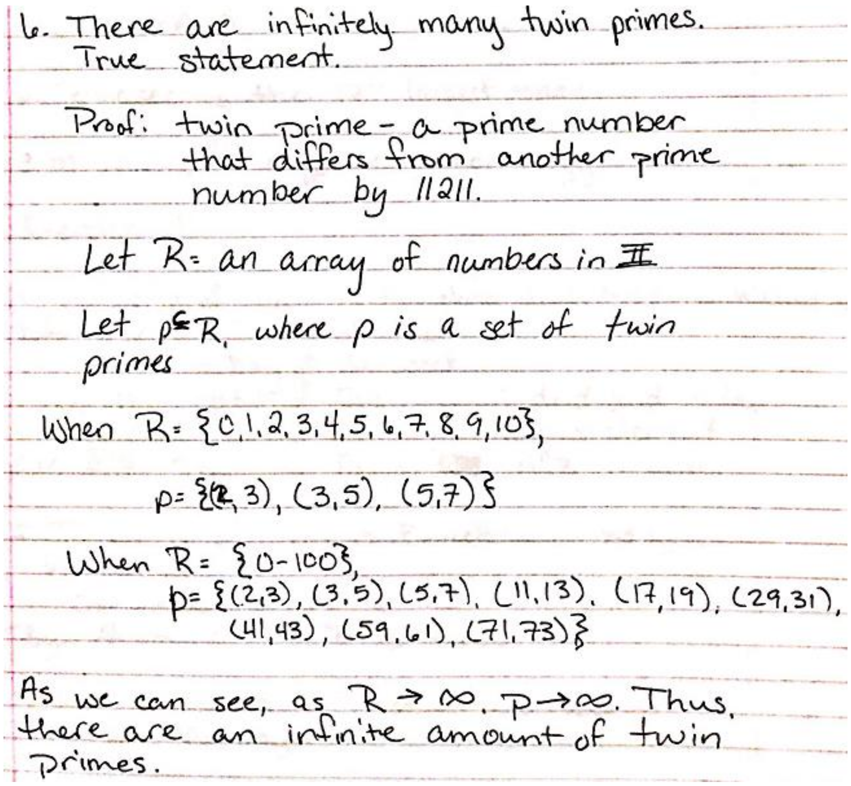
achievement. Our human legacy, the power of the mind in full force. This objective mathematical knowledge is part of our collective identity. It is ours to appreciate and admire. Certainly students should have the opportunity to learn about these ideas if they desire. But when the focus of instruction is predominately on conveyance of the “right” ideas, then we will fail to open up a space for the personal. Boaler (2015) claimed that, “children are wrongly led to believe that all of the ideas already have been had and their job is simply to receive them” (p. 172). Do we teach undergraduates the same? The wave of mathematical knowledge is dynamic. It is fluid. It gains power and takes shape because of new human minds, eager to contribute to its evolving structure.

Mathematical Knowledge is Dynamic and Forever Changing

It was the awareness that mathematical knowledge is dynamic that provided my son with the impetus for exploring ideas and taking ownership of them. “Not even your teacher? Not even dead people?” he asked when I told him I was working on a problem that no one had solved. Up till that point in his life, his conception of mathematics was similar to that of Benny’s (Erlwanger, 1973). Benny and my son believed that all mathematics had already been invented long ago. Only after learning that there was still mathematics left to discover did my son have his original idea. Unless we treat knowledge as dynamic for our students, then they will not be able to understand the other three characteristics in the IDEA framework. If the instructor always tells students what is correct, then there is no room for argumentation. If all the ideas have already been discovered, then there is neither need for exploration.

In order to teach students that mathematical knowledge is dynamic, we must ensure they understand there are unresolved mathematical questions. Mathematicians make new discoveries every day. Dr. Algebraic remarked, “I have had [graduate] students tell me that there has been no new mathematics in the last 300 years. In all seriousness tell me that.” To what degree do our undergraduate mathematics majors understand the dynamic nature of mathematical knowledge?

Many of the students in Foundations of Higher Mathematics were not familiar with what I consider to be the most famous unsolved problem, the twin primes conjecture. On an early assignment students were provided the statement “There are infinitely many twin primes.” They were asked to decide if it was a statement, if it was true or false, and to provide a rationale for their decisions. Consider some of the student responses: “Yes, it is a statement and it is true.” “This is a statement because it has been written and proven as a conjecture by multiple mathematicians.” “True because no matter how many you find there can always be more the higher/lower you count.” Infinitely Repeating Decimal created the “proof” shown in Figure 55.



6. There are infinitely many twin primes.
True statement.

Proof: twin prime – a prime number that differs from another prime number by 11211.

Let R = an array of numbers in $\mathbb{Z}$

Let $p \subseteq R$, where p is a set of twin primes

When $R = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$p = \{(2, 3), (3, 5), (5, 7)\}$

When $R = \{0\text{-}100\}$,
$p = \{(2,3), (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73)\}$

As we can see, as $R \to \infty$, $p \to \infty$. Thus, there are an infinite amount of twin primes.

Figure 55. Proof of the Twin Prime Conjecture

We must do a better job teaching students that mathematical knowledge is dynamic. Undergraduates should be aware of outstanding conjectures and understand their epistemological status. Dr. Amicable and I tried to paint a dynamic picture of mathematics for our students. We told them they were the future of the discipline. We taught them that what counts as a proof is negotiated amongst mathematicians, and gave them the opportunity to debate what makes a good proof themselves. We encouraged them to see the value of mistakes in revising their knowledge.

To understand the dynamic nature of mathematical knowledge includes understanding how one's personal knowledge and the knowledge of one's community can change and grow. We have not reached the end of our knowledge of mathematics. Neither have we reached the *end of mathematics instruction*. We play a part in creating the world in which students will experience mathematical ideas. Let us, instructors and scholars, create classrooms in which original ideas are encouraged and students see the dynamic nature of knowledge.

Pure Mathematical Inquiry is an Emotional Exploration of Ideas

Through the narratives told in Chapter Four, I highlighted the enjoyable exploration of ideas that I experienced while working on Dr. Combinatorial's conjecture. In the pure mathematics classroom, it is more important for student to experience the joy of mathematical exploration than to be forced to be familiar with the "right" ideas. How often do undergraduates have the opportunity for mathematical exploration?

A senior mathematics major agreed to informal interviews as part of my dissertation study. I spoke to him one day during final exam week, about three hours before his exam in Number Theory. I was excited for him (who would not be excited about writing number theory proofs?). But he was discouraged. "Three hours of memorizing proofs," he told me. "And then two hours of writing proofs." I tried to encourage him. I said, "There are probably some really cool ideas underneath all these proofs!" Then he excitedly told me about how he had "accidentally started discovering partition theory halfway through the semester" and "there were some cool ideas." He said, “I accidentally studied that. I ended up doing a bunch of stuff that we didn’t cover, unintentionally without thinking about it. And this book made me realize people write their own stuff. Like, Oh!” The student told me about the Dirichlet divisor problem, and how he had attempted to map the Dirichlet divisor function to polynomial functions such that the roots would be prime. He was very excited about exploring ideas, but he considered this a distraction from his coursework. I replied, “To me the whole thing about undergraduate mathematics is, why don’t we just have students do what you did? Pursue mathematical ideas because they want to, rather than because they have to.” “Yeah. Honestly it is killing me this week,” he said. “I am just like ahhh!”

How about we start supporting students in pursuing the topics that interest them? Let us give students the option of working on instructor selected problems or choosing their own path. Would it be that difficult to restructure our courses so that idea exploration is the expectation? From my journal:

> If students are persevering day after day to solve a problem, trying different approaches, thinking about possibilities, then they are mathematicians. Such students are at the pinnacle, pushing boundaries. If grades are given, they should receive the highest mark.

Let us reward, rather than discourage, our students when they explore their own questions. Let us make idea exploration standard in pure mathematics courses.

Mathematical Ideas and Knowledge are Socially Vetted through Argumentation

In the traditional environment, one is incorrect if one's ideas differ from the teacher's ideas. But how should mathematical ideas be treated in an inquiry-oriented classroom in which student ideas are the basis of instruction? It is widely accepted within the mathematics education community that the classroom should be a site of discourse in which students conjecture and debate mathematical ideas (NCTM, 2014). It is the responsibility of the classroom community to accept or reject ideas using mathematical justifications.

One thing I learned from this study, especially from the story of Binary, is that it is not a simple task to open up the classroom to argumentation. Instructors must consider students' prior experiences and feelings. Some of the undergraduates in the Foundations of Higher Mathematics course expressed apprehension when it came to sharing their ideas. On a self-evaluation, Even wrote, "I could do better on speaking up in class. I often feel like my ideas are wrong or weird… I feel I could say more things in class, but I'm nervous." Exponential, who dropped the course, wrote,

> I don't think I can really improve [on whole class-discussions] very much. Large class discussions are intimidating, and my opinion is usually not worth extra discussion. Plus, sometimes in class discussing may get a little out of hand and I don't like to participate in arguments about something I'm not very familiar.

We should not ignore such issues. Students need more low-stakes experiences with argumentation before they reach their upper-division courses. When I asked the class if they had engaged in mathematical argumentation previously, only Odd said "not to this extent." Binary did not speak for two class periods following the criticism of his idea. He was a first-generation college student, and he graduated the semester he completed Foundations of Higher Mathematics. What kind of stress do we put on students when it is not until their final year of college that they are asked to argue about mathematics? We need to teach argumentation earlier. We need more opportunities for proof and critique in lower-level undergraduate courses. Students should learn to argue in College Algebra, Pre-calculus, and Calculus. We must teach them that it takes courage to put forth a mathematical argument, and humility to know their ideas may need to be revised (Lampert, 1990). It is not the sole responsibility of K-12 mathematics teachers to facilitate argumentation. The onus is on the university to change the culture of mathematics education.

Alternatives to the Foundationalist Picture of Mathematics

If we hope students see mathematical ideas as part of their own identity, then the pure mathematics classroom cannot be a place where all the mathematical ideas students are to have are chosen up front before the semester begins. If we wish for students to experience the enjoyable exploration of mathematical ideas, then we must provide them with idea-rich mathematical contexts and conjectures to work on or support them in coming up with their own. If we wish that students see mathematical knowledge as dynamic, then we must allow the class as a whole to work for an extended duration on a problem, so they can see their knowledge grow. If they are to understand mathematical knowledge is socially vetted through argumentation, they must have the opportunity to critique their peers' ideas.

Two of the goals of the Foundations of Higher Mathematics course were for students to 1) learn how to write deductive proofs and 2) understand the foundations of mathematics: logic and set theory. Figure 56 is a PowerPoint slide presented to students on the first day set theory was studied in class. It presents the foundationalist view of the nature of mathematics.

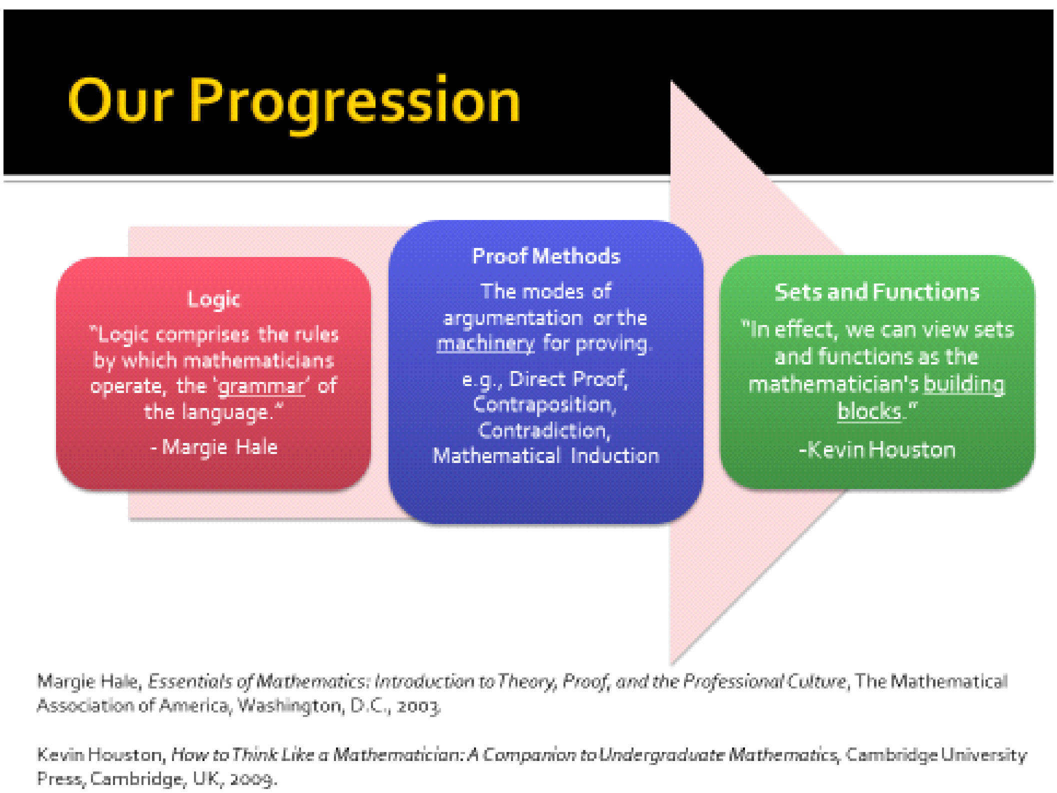


Figure 56. The Foundationalist Picture of Mathematics

I question whether this foundationalist picture of mathematics is true or useful. Is an understanding of formal logic necessary to do mathematics? Are sets and functions the building blocks of mathematics? As a house cannot stand without its foundations, do we say that mathematics cannot stand without the logic and set theory students become acquainted with in transition-to-proof courses? I worry that the emphasis we place on these topics are a result of (failed) efforts to find indubitable foundations of mathematics. According to Ernest (1991),

> Logicism is the school of thought that regards pure mathematics as a part of logic. […] If all mathematics can be expressed in purely logical terms and proved from logical principles alone, then the certainty of mathematical knowledge can be reduced to that of logic. Logic was considered to provide a certain foundation for truth. (pp. 8-9)

In efforts to find a foundation of truth, mathematics was reduced to logic and set theory. In 1984, Goodman wrote, "Twenty years ago there was a firm consensus that mathematics is set theory" (p. 21). Mathematics is set theory? Do we need truth tables or set theory to understand that $1 + 1 = 2$? Or to make a conjecture about the chromatic number of the Petersen graph? Or does explicit instruction in logic and set theory, and a focus on valid deductions, inhibit students from experiencing mathematics as an exploration of ideas?

Recently I was engaged in an e-mail exchange with a mathematician, and I attempted to argue that we can do mathematics without set theory. I wrote, "I conjecture that it was only fairly recently that mathematicians formalized comparisons and counting with sets and elements." His reply:

> Significant progress WAS made consequent to the formalization of the theory of sets. But contrary to what Poincare (may not have been he, but somebody said it) prophesied, "set theory is a disease from which mathematics will eventually recover," it has rather consolidated a foundation on which all of the rest of mathematics can be built and its language can be used universally within the discipline.

We have a choice to make. On the one hand, if we believe that logic and set theory do provide the foundation of mathematical truth, then we should make this explicit to students. We could structure Foundations of Higher Mathematics in a way that students understand what they are attempting to do is to place mathematics on a foundation of indubitable truth. Students could learn logic and set theory in this context. The other option is to drop the "foundations" façade and teach transition-to-proof courses with a focus on mathematical ideas and the humanistic characteristics such as those outlined in the IDEA framework. Let students get their feet wet in research mathematics. Instructors may probe the mathematicians in their department and find some idea-rich, high-level problems for students (perhaps even unsolved conjectures!). We could allow the students the choice to work on the problems they are interested in, and encourage them to communicate their results to the class. Students could be supported in engaging in an exploration of mathematics for an entire semester. I hypothesize that students would then naturally develop and refine their intuitive notions of logic and sets as needed. Either we teach foundations for what it is, an attempt to find an absolute indubitable foundation for mathematical truth, or we can take an IDEA-based humanistic approach, allowing students to explore mathematical ideas through their work proving rich conjectures.

The university needs to take the lead with IDEA-based instruction, not only in traditional proof courses, but in other courses in which pure mathematics is studied. Whether the course is College Algebra, Pre-calculus, Calculus, or Linear Algebra, our students need to experience the humanistic side of mathematics. They need to know mathematics is a subject about ideas. Mathematics education scholars have been pushing a humanistic vision for school mathematics for at least thirty years. Perhaps it is time the university did its part. This will involve changing instructor beliefs about the purpose of mathematics instruction. Henderson, Beach, and Finkelstein, (2011) wrote,

> First, effective change strategies must be aligned with or seek to change the beliefs of the individuals involved. Second, change strategies need to involve long-term interventions, lasting a semester, a year, and longer. Third, colleges and universities are complex systems. Developing a successful change strategy means first understanding the system and then designing a strategy that is compatible with this system. (p. 978).

Each university is a complex system, and will face different challenges to reforming undergraduate pure mathematics instruction. One challenge that I see as common to all universities concerns a possible conflict between teaching a humanistic

IDEA-based mathematics course, and the function of the university to provide students with economic credit.

The Credit System

Just as I was beginning the dissertation, I read several articles by Pais (2011, 2013, 2015) in which he argued that the influence of the capitalist economic system may inadvertently thwart our aims in mathematics education. Students learn mathematics to earn credit. Every student in Foundations of Higher Mathematics needed course credit to complete their university degree. Grades were often on the students' minds.

> Even: My question is how will the proofs be graded, point system or otherwise?
>
> Whole: Would that be docked for points?
>
> Composite*2*: So that's a proof. That's a 100 right there? Good work?
>
> Binary: Yeah. I am just trying to figure out if I just left that as *k*, and he had that. Would I get less points?

We must be aware that for many students, mathematical knowledge is a required commodity. How do we support students in learning pure mathematics (which I described in Chapter One as mathematics that is done for its own sake), when there is always course credit at stake? How do we, as instructors, promote idea exploration when we are also part of the system? An excerpt from my journal:

> I just graded Even's problem set 8, and I realize I am just playing the credit game. I took off 7 points from example 4.2 because she did not read the instructions carefully. I am not assessing mathematical understanding. I am assessing the

ability to follow instructions. I decided to only take off 5 points after writing this reflection.

I do not have simple answers, but I believe we should be more aware of how the credit system affects our behavior as instructors and scholars. If we take IDEA-based mathematics instruction seriously, we will have to carefully consider how grades are assigned. How do we measure the degree to which a student has explored a mathematical idea? What can we do to ensure we value more than the students' ability to write deductive proofs?

One suggestion I have is to change the nature of assignments in an introduction-to-proofs course. Rather than ask students to write proof after proof of what are often disjoint ideas, ask them to write an exposition, or a mini-paper, on a mathematical topic. Dr. Amicable has taught foundations of mathematics for several semesters, and one of her favorite days of instruction is when she assigns each group of students a different set theory concept (e.g. subset, empty set, power set) and asks them to become the class experts on that concept. The students become familiar with the concept definitions, and provide examples and non-examples in their presentations to the class. Students would not only be tasked with proving theorems related to an idea, but also demonstrate their understanding by providing examples or perhaps acknowledging some of the patterns they noticed during the proving process. Proof should not be treated as an isolated topic (for an entire course). It should be a mathematical tool that serves our purposes in understanding and exploring mathematical ideas. Mini-papers would give confidence to some students who know how to generate examples but are still timid in writing proofs. Such assignments may eliminate the confusion students have regarding examples in a proof (examples do provide conviction, they just do not meet the deductive standard). Students could present the results of their idea explorations. Of course, part of the challenge is finding an idea-rich concept, such as the chromatic number of graphs. But I think we are up for it!

Again, I wish to emphasize that university instructors have choices to make. We can continue with the current culture, doing what pays the bills without making a meaningful impact on students' mathematical lives. Or we can make a commitment to cultural renewal, and do our best to reform our classrooms so that ideas become the focus. The choices we make extend beyond the university. The university has a huge impact on the general population's perceptions of mathematics. Our teachers are educated at the university. What they experience in their undergraduate mathematics courses, they will carry with them into the schools.

Implications for School Mathematics

I now discuss some limited implications for school mathematics. What should school students and teachers understand about the nature of mathematics? For the dissertation study, I narrowed my focus to undergraduates in a pure mathematics course. Are the characteristics of the IDEA framework worthy goals for students' understanding in school? My six year old son benefitted from understanding the dynamic nature of mathematical knowledge. The understanding that mathematical knowledge is still in the making inspired him to explore his own ideas.

More research is needed to understand the degree to which the IDEA framework may be applicable to other contexts beyond an undergraduate pure mathematics course. But the framework does capture the same humanistic spirit that we have seen in other mathematics education texts that push for reform in school mathematics. The beliefs mathematics education scholars collectively hold dear regarding mathematics and its teaching and learning have certainly influenced my creation of the IDEA framework, perhaps subconsciously. The *Dynamic* characteristic of the IDEA framework is closely related to one of the productive beliefs that NCTM (2014) presented in *Principles to Actions*: “Mathematics is a dynamic field that is ever changing” (p. 72). This idea, that the field is changing, is captured in the *We Are the Future* narrative. Indeed, the IDEA framework shares much in common with the ideas that the mathematics education community has taken as foundational for the last 30 years. Consider how the IDEA framework aligns with Sfard’s (2003) description of the NCTM standards documents in terms of the dynamic nature of knowledge:

> The *Standards* documents of the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 1995) are the result of a serious and comprehensive attempt to teach ‘mathematics with a human face.’ This means much care for both mathematics and the student. What is being taught is ‘mathematics in the making’ rather than mathematics as a static body of knowledge. (p. 353)

Also consider how Boaler's (2016) description of what makes a mathematical mindset emphasizes idea exploration and dynamic knowledge:

> Children need to see math as a conceptual, growth subject that they should think about and make sense of. When students see math as a series of short questions, they cannot see the role for their own inner growth and learning. They think that math is a fixed set of methods that either they get or they don't. When students see math as a broad landscape of unexplored puzzles in which they can wander around, asking questions and thinking about relationships, they understand that their role is thinking, sense making, and growing. When students see mathematics as a set of ideas and relationships and their role as one of thinking about the ideas, and making sense of them, they have a mathematical mindset. (p. 34)

Historically, mathematics education scholars value student exploration of ideas and instruction that promotes a dynamic perspective of mathematical knowledge. The IDEA framework captures the spirit of mathematical pedagogy that guided Ball (1993) and Lampert's (1990) instruction. Both scholars' classrooms were environments where students were encouraged not only to create their own ideas and take ownership of them, but also engage in critique of their classmates' ideas. Lampert (1990) wrote, "My organizing ideas have been the 'humility and courage' that Lakatos and Pólya take to be essential to doing mathematics" (p. 58). If we are to open the classroom so that student ideas drive the discussion, we must recognize that these ideas are part of students' personal identity. It will take courage for students to put their ideas forth to the classroom community for criticism, and humility for them to realize that their ideas may need to be refined in light of new evidence and ideas.

The Symbolic Standard

Certainly more efforts need to be made so that ideas are the focus of instruction when students learn pure mathematics in school. I opened Chapter One with a quote from Erlwanger (1973) in which he described 6^{th} grade student Benny's conception of mathematics. Benny did not understand that mathematics is about an exploration of ideas. He did not have the opportunity to have his own mathematical ideas. He thought that mathematical rules were invented years ago by one man, and he had no conception of the dynamic nature of mathematical knowledge. He had no experience in deepening his understanding through mathematical argumentation. To Benny, the mathematics was the symbols on the page and the manipulations of those symbols. Erlwanger wrote,

> But fractions, to Benny, are mostly just symbols of the form $\frac{a}{b}$ added according to certain rules. This concept of fractions and rules leads to errors such as $\frac{2}{1}+\frac{1}{2}=\frac{3}{3}=1$. Further, $\frac{2}{1}+\frac{1}{2}$ is "just like saying $\frac{1}{2}+\frac{1}{2}$ because $\frac{2}{1}$, reverse that, $\frac{1}{2}$. So it will come out one whole no matter which way. 1 is 1." (p. 92)

If we fail to teach that mathematics is about ideas, then students tend to focus on the manipulation of symbols. Thompson (1992) noted that when a teacher views mathematics as "a discipline characterized by accurate results and infallible procedures" this "can lead to instruction that places undue emphasis on the manipulation of symbols whose meanings are rarely addressed" (p. 127). All too often in school we focus on the symbols rather than the ideas and meanings behind them.

We can teach students that mathematics is about ideas beginning in pre-school. Children should learn that the numeral 2 is a way to represent an idea, the number two. The numeral should not be equated with the number. If we fail to distinguish between ideas and the symbols representing them, then the root of the problem begins from day one. Students will see mathematics as the symbols. Mathematics is not in the symbols, it is in the ideas of which the symbols are created for various purposes. Taking the humanistic stance means to be critical of the symbolic standard, the dehumanizing mechanics of tending to symbols on a page.

How do we overcome the symbolic standard, and promote a humanistic vision of mathematics in school? Working within the school credit system, many teachers believe they must train students to answer test questions. It will be difficult to implement IDEA-based instruction in any classroom that focuses on right answers and correcting mistakes. Of course, some students claim to like mathematics because there are right and wrong answers. When we provide students with the right ideas up front, we save them from having to deal with the struggle of creativity. Students learn to memorize and reproduce actions repeatedly and as efficiently as possible. What sort of world are we creating, when we teach people that to attain the most prestigious knowledge (mathematical) is to engage in a creative-less act?

How do we Teach Other Forms of Mathematics?

I believe school students would benefit from having the opportunity to explore and create mathematical ideas. Many students do not have this opportunity, and the IDEA framework may be useful for supporting teachers in creating classroom spaces in which students come to understand and experience the nature of pure mathematics. As Boaler (2016) noted,

> This wide gulf between real mathematics and school mathematics is at the heart of the math problems we face in education. I strongly believe that if school math classrooms presented the true nature of the discipline, we would not have this nationwide dislike of math and widespread math underachievement. (pp. 22-23)

But let us think more carefully about Boaler's claim. What is "real" mathematics? She wrote, "When we ask mathematicians what math is, they will say it is the study of patterns" (p. 22). Recall Harouni (2015) distinguished between three types of mathematics: artisanal, commercial-administrative, and philosophical. He wrote that, "Philosophical mathematics loves patterns," (p. 64). Furthermore,

> The more we look at philosophical mathematics, the more we understand the regular complaint of academic mathematicians (e.g., Lockhart, 2009) that the subject taught in elementary and secondary classrooms is far removed from the mathematics that they know. (p. 65).

In *The Mathematician's Lament*, Lockhart (2009) wrote, "I'm not complaining about the presence of facts and formulas in our mathematics classes, I'm complaining about the lack of *mathematics* in our mathematics classes" (p. 29). Lockhart, like Boaler, is implying that pure mathematics is the only real mathematics. But when we say that pure mathematics is the only real mathematics, we do injustice to other forms, perhaps implying that they are not real or even *impure*. If we continue with this analogy and consider Paul's warning that "love of money is the root of all evil," [2] then impure mathematics is commercial-administrative mathematics. One may view it not as a mathematics of exploration, but rather a mathematics of exploitation. I do not think it is the best idea to demonize other forms of mathematics, or consider them unreal. If the mathematics classroom is purged of commercial-administrative mathematics, then how will our citizens be productive in a society in which money matters?

Teachers need to be conscious of the different types of mathematics because each are likely to require a different type of instruction. Historically, the pedagogy of teaching commercial-administrative mathematics, which was done in reckoning schools, emphasized efficiency and correctness in procedures rather than meaning making through exploration (Harouni, 2015). Teaching by demonstration and replication may be an effective way to teach the algorithms of commercial-administrative mathematics. My recommendation is that while school students would benefit from experiencing the joy of pure mathematical exploration, we cannot deceive ourselves and believe that pure mathematics is the only "real" mathematics. We must acknowledge that students need to learn about business mathematics, applied mathematics, and statistics. More thought is

[2] 1 Timothy 6:10

needed on the part of teachers and researchers regarding how we teach each of these forms of mathematics. It is naïve to think they can all be taught in the same manner.

Future Directions for Research

I do not intend for my dissertation work to be the final chapter in my own understanding of the teaching and learning of the nature of mathematics, but the beginning of a long research program. In the future, I would like to interview mathematicians and mathematics educators in order to confirm that the characteristics I have outlined in the IDEA framework are consistent with others' values and goals. I wish to share my narratives with them and engage in a discussion about the value of teaching using the IDEA framework. Furthermore, it will be important to consider how these characteristics can be taught. In the future, the characteristics of the IDEA framework may need to be modified. Consider the research trajectory shown in Figure 57, which was originally presented in Chapter One.

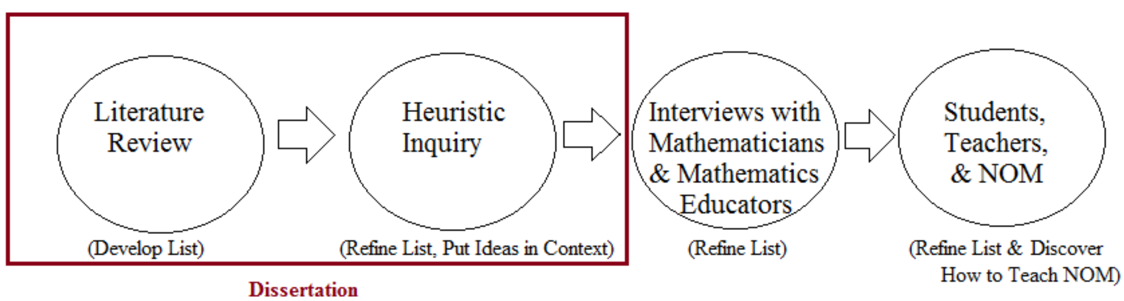


Figure 57. The Long-Term Development of a Framework for the Nature of Mathematics

In addition, researchers may consider examining empirical studies that have been completed within science education on the nature of science (NOS), and adopt similar methods to study NOM within the field of mathematics education. For instance, we may investigate explicit and reflective teaching in which NOM is “brought to the forefront” (Bell et al., 2003, p. 504) of students’ activity as has been done to teach NOS. Science education scholars have also developed research instruments designed to measure students’ understanding of NOS (Lederman, Abd-El-Khalick, Bell, & Schwartz, 2002). Similar open-ended measures may be useful for research into the effectiveness of instructional strategies for teaching students NOM.

Implications for Research

Heuristic inquiry may be a fruitful methodology for other scholars in mathematics education, and here I make some recommendations regarding this methodology for interested scholars. I have learned that heuristic inquiry is a deeply personal process. The essence of heuristic inquiry is a “relentless inward focus” (Sela-Smith, 2002, p. 80). It requires researcher creativity, not only because of the ultimate creative synthesis required, but also because the method “does not prescribe a methodology” (Douglass & Moustakas, 1985, p. 42). Heuristic inquiry is not the framework for a researcher who desires a straightforward project with a clear delineation of steps. Rather it is for brave researchers who have a desire to engage personally with a topic and are open to a transforming their understanding. The researcher must be prepared to tell his or her own story and the stories of those close to them.

One area in which I believe heuristic inquiry may be fruitful is in teaching undergraduate and graduate students about the nature of mathematics through thesis projects. As Courant and Robbins (1941) claimed, “For scholars and laymen alike it is not philosophy but active experience in mathematics itself that can alone answer the question: What is mathematics? (p. xix)” Undergraduate students are sometimes viewed as being inadequately prepared for mathematical research. But why not support them in research, not with the expectation that they will necessarily produce a result, but with the expectation that they will learn more about the nature of mathematics? In the future, I plan to continue working on Dr. Combinatorial’s conjecture. I envision working together with undergraduates, supporting them in their own heuristic inquiries into the nature of pure mathematics. One of my goals would be to teach my students about the four characteristics of the IDEA framework; but I would also support them in coming to their own conclusions about the nature of mathematics. Similar projects may also be valuable for master’s students in the field of mathematics education. Mathematics education majors are not usually expected to engage in mathematics research, but it may be valuable for them to collaborate with research mathematicians and reflect on the nature of mathematics. As teachers or future teachers, it is important that such students reflect on their own view of mathematics and consider what their current or future students should understand about the nature of mathematics.

Moustakas (1990) wrote that “The heuristic process is autobiographic, yet with virtually every question that matters personally there is also a social—and perhaps universal—significance” (p. 15). The IDEA Framework for the Nature of Pure Mathematics was born out of my personal experience doing and teaching mathematics. This list was a product of deep and long-term reflection. I believe the characteristics of the IDEA framework that I have identified do have social significance, and they will be valuable goals for students’ understanding of the nature of mathematics. However, we have to be careful how we use the list to research and teach NOM.

Lederman and Lederman (2014) noted, “Each item on a list is just a label or symbol for a much more in-depth and detailed elaboration” (p. 615). It is crucial that educators must not use a list trivially as a checklist requirement. In reference to NOS lists, Matthews (2015) wrote, “The negative side is that the list can, despite the wishes of its creators, function as a mantra, as a catechism, as yet another something to be learned” (p. 393). It would be counter-productive for students to be given the statement “mathematical inquiry involves an exploration of ideas” and asked to respond true or false. The narratives in Chapter Four may be used as tools for providing instructors and students with a deeper understanding of the characteristics of the IDEA framework. Of course, individuals will not be able to fully understand that “mathematical inquiry involves an exploration of ideas” unless they have a chance to experience such an exploration themselves.

In Chapter One, I argued that the field of mathematics education needs to conduct a systematic inquiry into the teaching and learning of NOM and ultimately arrive at a consensus NOM view as has been done in science education. But we must also remember that NOM is a philosophical subject for which there are not absolute answers. We will never arrive at a definitive NOM list that represents the true nature of mathematics. Our lists, like our philosophy, are value-laden. The IDEA framework embodies a humanistic spirit and a vision for what I hope students, teachers, and scholars may come to understand about the nature of mathematics. I would be thrilled if scholars and teachers adopted the IDEA framework in research and teaching. However, I believe the real need is to open up a sustained dialogue about NOM both within mathematics education scholarship and in mathematics classrooms.

How Do We Teach the Nature of Mathematics?

How do we teach the nature of mathematics? This is not a simple question. One approach may be to choose an aspect of the IDEA framework and conduct research to understand the teaching and learning of that aspect. For instance, what instructional activities support students in coming to view mathematical knowledge as dynamic? Dr. Amicable and I often attempted to teach our students about the nature of mathematics by telling them about what mathematicians do. It may be valuable to have mathematicians as guest speakers so that they may provide a picture to the students of what mathematical work is like (e.g. telling them about the refereeing process or their own mathematical explorations). But if our students are really going to come to have robust conceptions of the nature of mathematics, then they need more than telling. I even contend that they need more than IDEA-based instruction.

I believe that NOM should be treated as a philosophical subject, open to debate within classroom communities. Students need opportunities to reflect and discuss NOM and come to their own conclusions. In Figure 58, there is a poster some doctoral students made after reading Lakatos and discussing the nature of mathematics (NOM) for a few weeks in one of their courses. I really like it!

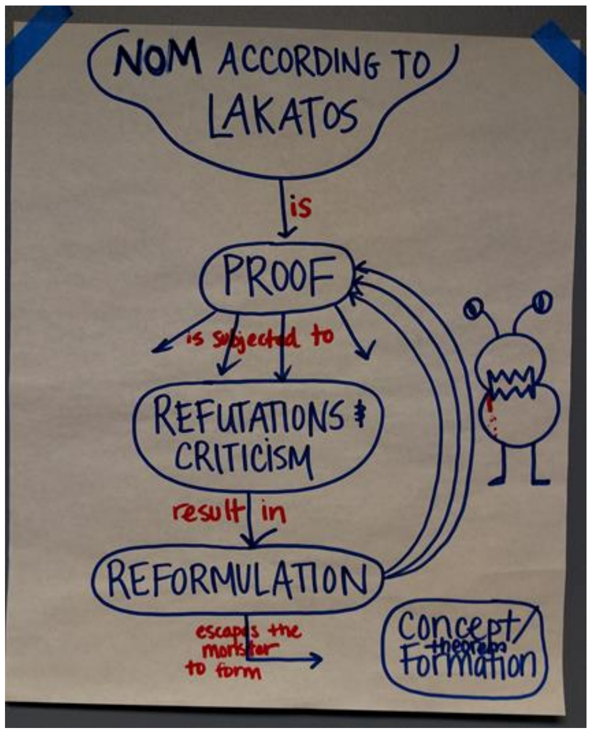


Figure 58. Doctoral Poster

The poster in Figure 58 inspired me. I believe we need explicit instruction in which NOM is treated as a subject for study and debate within classrooms (like it was for the doctoral students who made the poster). For instance, students could create their own

NOM frameworks, perhaps both at the beginning and end of a course. With knowledge of the NOM conceptions that students bring to a course, an instructor could design instruction to challenge naïve views. While I do think it will be valuable to design instruction aimed at helping students come to understand the characteristics of the IDEA framework, I also believe we should encourage students to come to their own conclusions. A lesson I take away from this dissertation project is that we all can come to understand NOM by collectively learning through inquiry, rather than under strict authority. NOM is ours. It is what we make it.

Chapter Summary

In this chapter, I have made strong recommendations for pure mathematics instruction at the university level. I contend that if students are to understand the characteristics of mathematics presented in the IDEA framework, then a cultural renewal of university mathematics instruction is necessary. Ideas must be the focus of mathematics classrooms, and students must have the opportunity to create their own ideas through personal exploration. If we are to implement IDEA-based instruction, then we have to consider alternatives to the deductivist approach in which logic and set theory are the dominant topics of transition-to-proof courses. I propose students be provided the opportunity to explore rich conjectures and create expositions on mathematical topics. We also have to do better teaching our students about unsolved conjectures in mathematics and supporting them in productive mathematical argumentation.

I discussed implications for school mathematics, and recommended that scholars and teachers pay more attention to different types of mathematics. School mathematics includes the study of pure mathematics, statistics, applied mathematics, and commercial-administrative mathematics; and different pedagogical methods may be needed to teach each type. Lastly, I discussed implications for future research. I recommend that the IDEA framework not be treated authoritatively as mathematical knowledge has been treated in school. Rather the framework and the corresponding narratives can be used as tools to foster discussion and reflection on the nature of mathematics. What is truly needed in order to bring about changes in students' conceptions of mathematics is to provide them opportunities to explicitly reflect on their own beliefs while also being confronted with positions that challenge those beliefs.

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APPENDICES

APPENDIX A – DESCRIPTION OF CHARACTERS

Major Participants

Surreal: I (the researcher) have chosen to refer to myself as Surreal. Surreal is a privileged white male, mathematics education scholar, father, musician, and aspiring Christian. The Surreal Numbers were first "created" by John Conway in 1969 (http://mathworld.wolfram.com/SurrealNumber.html). I first heard of the surreal numbers after the 2015 Conference on Research in Undergraduate Mathematics Education. It was a snowy night in Pittsburgh and I and several other graduate students were returning home from the bar. One young mathematician whom I had met was very fascinated with the surreal numbers, and excitedly tried to explain their significance to me. There is a novelette describing the Surreal Numbers by Knuth, "How two ex-students turned onto pure mathematics and found total happiness." Why do the surreal numbers best represent me? I am not sure, but I can pick whichever number type I would like!

Dr. Combinatorial: Dr. Combinatorial is the research mathematician that I collaborated with on an unsolved problem in graph theory. He formulated the conjecture that we are working on after having studied a particular class of graphs for several years. Most of his other work in the area is related to connectivity. He is a full professor, and active research mathematician in the field of combinatorics and graph theory. Dr. Combinatorial was born and raised in China before coming to the United States. Dr. Combinatorial is a Christian.

Dr. Amicable: Dr. Amicable is an assistant professor of mathematics, mathematics education scholar, and my co-instructor of the transition-to-proof course that was part of this study. Amicable numbers are two different numbers so related that the sum of the proper divisors of each is equal to the other number. Dr. Amicable explained that as a mathematics education scholar, her research informs her teaching and vice versa. All of the factors involved in her research sum to influence her teaching, and all the factors involved in her teaching sum to influence her research. Thus she is Dr. Amicable. She had taught the transition-to-proof course for six semesters prior to this study. She is a white American, and a Christian.

Undergraduate Students

Some of the students in this study provided their own descriptions for why they chose the number type they did as part of a course assignment at the beginning of the course. I present those here verbatim while also adding some demographic details. In the case where students did not complete this assignment, I have chosen a number type for them. Not all of the students listed here are referenced in the body of the dissertation.

Integers (Psychology Major, Asian, Female, English Language Learner) - Integers can be positive and negative can be neutral & extreme – only exact (perfect) whole numbers. Me-sometimes be positive, sometimes not.-pretty lots of emotional ups and downs tend to expect extreme situations (both happiest situations and worst situations) – want to be a kind of perfectionist (a little obsession… ?)

Infinity (MathEd Major, White, Female) - Infinity is an abstract concept describing something without any bound or larger than any number. The number type that describes me best is infinity. Just like myself it doesn't have a bound and it's larger than any number. I perceive my life to have no boundaries, believing anything is possible if you have the right mindset. Although I'm not larger than "numbers" I am larger than anything else. You can overcome anything as long as you have the right method.

Complex (MathEd, White, Male) – Complex numbers: real, imaginary, and sums and differences of real and imaginary numbers. Me: I am rooted in real life: reasoning, analyzing, and socializing. However, I am often found with my head in the clouds, thinking outside the box, and daydreaming. These come together in creativity and puns, but are separated by not focusing, and easily getting sidetracked.

Permutation (Professional Mathematics, White, Female) – I feel that permutation (arrangement) numbers define my personality best. Like permutation numbers I logically look at every situation in my life and can see all the different pathways it can take. This often causes me to over-analyze all my decisions, but I feel more secure in making decisions after having explored every option available to me. Moreover, permutation numbers are most often used in probability problems which also play a large role in my decision making processes in life. I see each decision I make as having a certain probability set of outcomes and then choose the one most favorable to me. While nothing is for sure in life, I go each day analyzing permutations of different events and using estimated probabilities to make decisions that give me a more secure, confident approach to my actions.

Rational (Business Mathematics, White, Female). - I would be a rational number. I think Rational numbers best describe me because a rational number is one that can be expressed as a ratio. In the same way, I can be expressed as a ratio of the different aspects of my personality or of my life.

Fibonacci (Aerospace Technology, White, Male) - Fibonacci Number Sequence. I find this relatable to myself because it seems to me that certain events in my life take shape because of two prior choices I made before that event. The sequence of events that follow that event are directly related by adding the previous two events. For example, when I was 18 I decided to take a year off before starting school. Taking a year off and making my own money led me to unproductive spending and a careless lifestyle. A careless lifestyle and unproductive spending led me to speeding tickets and going to traffic court which led me assessing where I was headed and where I want to be. This allowed me to recognize the choices I had to make in order to be in a place that I desire most.

Composite: (Math Education, Asian American, Female) – The definition of a composite number is a “natural number which has at least three distinct natural number divisors or in simpler terms, a natural number that has more than two factors.” My number type is composite number because many important people and factors have influenced and made me who I am today. I have lived half of my life in Korea and half of my life here, so ended up being influenced by Korean culture, American culture, which really is a mix of lots of different cultures already, Hispanic, and Korean American, which really ended up being really different than just Korean culture that I was used to. My dream of being a math teacher has been influenced by my dad, who loves math like I do, my favorite math teacher, Dr. Gardener, and the language barrier that I had to face when I moved from South Korea to United States. My most important life decision has always somehow included the input of my family, my friends, and some very close teachers. My religion of Christianity has been influenced by my parents, my friends, and social media, which may seem very shallow, but I had my reasons. I have also been told that my personality is really weird but I like to claim that it is just unique which is pretty much a mix of nice, mean, quiet, talkative, and just straight up crazy.

Composite2 (Math Education, White, Female) - I am composite because I am made of many different “parts.” I am well-rounded. I speak Spanish. I’m a good cook. I love baking. I’m a wife. I’m a student. I’m a bank teller, etc… I play many ‘roles’ in life!

Odd. (Industrial Mathematics, Physics Minor, White Male) - I chose this pseudonym.

Rational2 (Computer Science, White Male) - Rational Numbers multiple representations – I can take complex ideas and replace terms to make the ideas simpler to understand.

Infinitely Repeating Decimal (Professional Mathematics, White Male) - I feel that an infinitely repeating decimal is a good representation of me because outside of this class I work as a session drummer and as a captain head of a high school drum line. Like the repeating decimal, the percussion section of a group keeps the constant pulse; unwavering and unchanging.

Exponential Number: (Computer Science, White, Male, Dropped after a couple of months) - Exponential Number – Rising or Expanding at a steady and usually rapid rate. – I feel like this fits me very well because I'm about to graduate with my bachelors in computer science. I believe that my life will grow at an exponential rate.

Binary: (Computer Science, Black, Male, First Generation College Student) - Binary Numbers. Very simple once you get to know me. But can be very confusing if you don't.

Positive: (Mathematics(statistics)&Sociology, White, Female) - Positive numbers best describe me. I work my hardest to maintain a positive attitude about anything I do, and give off a positive vibe to anyone who is around me. Positive numbers are greater than 0, and generally seen with (+) sign in front of the number.

Whole: (Math Education, white, male) - A number without fractions; whole number. A whole number is not complex at all. It doesn't have any tricks up its sleeves. It doesn't throw you through a loop or try to tag along any decimal friends. A whole number is simple, and it is non-chaotic. The reason I most associate with a whole number is because as a person, I am simple and non-chaotic. I do not have much complexity to me and I do not make things harder than they have to be. I am simple, I am organized, and what you see with me is what you get. I don't add complexity when added to situations. When I am involved, things are just easy. That is just like whole numbers. You start your mathematical career learning how to count with them. They are simple, they are noncomplex, and they are the numbers people flock to.

Even (Advanced Mathematics, white, female). I chose this pseudonym.

Natural (General Mathematics, Male, English Language Learner, I am not sure where Natural immigrated from) - My choice of the set of number from the list of set of numbers is natural number. Natural numbers is appealing to me because I love nature in its very nature. Natural numbers are beautiful and it helps me visualize in things better. It is easy to list them. It is easy to study. Natural numeral is used globally for mathematical operations. It reflects my natural way of growing up in a small village to a bigger city in the world Houston, TX. I was attached to nature. I love nature, I miss nature that is why I pick natural number.

Real (Mathematics, White, Male) - Upon initial contemplation of the assignment to compare myself to a type of number, I thought it to be a bit of a challenge. Yes, we are all complex, but are we that similar to complex numbers? Making the case for 'imaginary' might have been entertaining, but ultimately I decided to compare myself to the set of Real Numbers. While I do regard myself to be a genuine, straightforward, 'what you see is what you get' type of individual, my argument goes deeper than that. The reals are somewhat like an onion, or an ogre, with many layers, or subsets. My children occasionally accuse me of being grumpy. I do not recall being labeled an ogre. However many layers, or subsets of myself integrate to complete me. Naturally, at the core of my relationship with my wife we are best friends. As husband and wife, we are whole. Parenting together certainly adds a layer, or three. Wrapped around all that (even though it may seem Irrational at first glance) we have been business partners for nearly eighteen years! Being a father and a dad to my three children is closely related, and yet separate set of the roles and relationships that make me who I am.

Finite: (Math Education, Phillipino, Male). I chose this pseudonym.

Prime: (Unknown major, White, Male). I chose this pseudonym.

Quaternion: (Statistics, Black, Female). I chose this pseudonym.

Cardinal: (Unknown, Unknown, Male). I chose this pseudonym.

Finite2: (Unknown, White, Male). I chose this pseudonym.

Other Participants

Dr. Algebraic: Dr. Algebraic is an associate professor of mathematics with research interests in algebra and number theory. I met with Dr. Algebraic a few times over the semester to discuss my ideas related to the nature of mathematics. Dr. Algebraic is a white male, and a Christian.

Dr. Differential: Dr. Differential is a retired mathematician. During his career his work was primarily in linear algebra and differential equations. He does not consider himself to be exclusively a pure or applied mathematician, but believes the distinction between the two types of mathematicians is somewhat arbitrary. However, he recognizes that he is a bit more applied than the typical mathematican. Dr. Differential is white and

92 years old. Although I did not quote him in the main text, he gave me important feedback on the NOM characteristics I was considering for my frameworks.

APPENDIX B: INSTITUTIONAL REVIEW BOARD APPROVAL

IRB

INSTITUTIONAL REVIEW BOARD

Office of Research
Compliance, 010A
Sam Ingram Building,
2269 Middle
Tennessee Blvd
Murfreesboro, TN
37129

MIDDLE TENNESSEE STATE UNIVERSITY

IRBN001 - EXPEDITED PROTOCOL APPROVAL NOTICE

Friday, August 12, 2016

Investigator(s): Jeffrey Pair (Student PI), Sarah Bleiler-Baxter (FA) and Jeremy Strayer (FA)
Investigator(s') Email(s): *jeffrey.pair@mtsu.edu; sarah.bleiler@mtsu.edu; jeremy.strayer@mtsu.edu*
Department: Mathematics and Science Education

Study Title: ***The nature of mathematics: a heuristic inquiry***
Protocol ID: **16-2320**

Dear Investigator(s),
The above identified research proposal has been reviewed by the MTSU Institutional Review Board (IRB) through the **EXPEDITED** mechanism under 45 CFR 46.110 and 21 CFR 56.110 within the category *(7) Research on individual or group characteristics or behavior* A summary of the IRB action and other particulars in regard to this protocol application is tabulated as shown below:

IRB Action	APPROVED for one year from the date of this notification	
Date of expiration	**8/12/2017**	
Sample Size	100 (ONE HUNDRED)	
Participant Pool	**Adult (mix of several types of individuals)**	
Exceptions	Collection of voice recording and hand writing samples is permitted	
Restrictions	**Collection of signed informed consent is mandatory**	
Comments	NONE	
Amendments	**Date**	**Post-approval Amendments**
		NONE

This protocol can be continued for up to THREE years (**8/12/2019**) by obtaining a continuation approval prior to **8/12/2017**. Refer to the following schedule to plan your annual project reports and be aware that you may not receive a separate reminder to complete your continuing reviews. Failure in obtaining an approval for continuation will automatically result in cancellation of this protocol. Moreover, the completion of this study MUST be notified to the Office of Compliance by filing a final report in order to close-out the protocol.

Continuing Review Schedule:

Reporting Period	Requisition Deadline	IRB Comments
First year report	7/12/2017	INCOMPLETE
Second year report	7/12/2018	INCOMPLETE
Final report	7/12/2019	INCOMPLETE

The investigator(s) indicated in this notification should read and abide by all of the post-approval conditions imposed with this approval. Refer to the post-approval guidelines posted in the MTSU IRB's website. Any unanticipated harms to participants or adverse events must be reported to the Office of Compliance at (615) 494-8918 within 48 hours of the incident. Amendments to this protocol must be approved by the IRB. Inclusion of new researchers must also be approved by the Office of Compliance before they begin to work on the project.

All of the research-related records, which include signed consent forms, investigator information and other documents related to the study, must be retained by the PI or the faculty advisor (if the PI is a student) at the secure location mentioned in the protocol application. The data storage must be maintained for at least three (3) years after study completion. Subsequently, the researcher may destroy the data in a manner that maintains confidentiality and anonymity. IRB reserves the right to modify, change or cancel the terms of this letter without prior notice. Be advised that IRB also reserves the right to inspect or audit your records if needed.

Sincerely,

Institutional Review Board
Middle Tennessee State University
Email: irb_information@mtsu.edu (for questions)
irb_submissions@mtsu.edu (for documents)

Quick Links:
Click here for a detailed list of the post-approval responsibilities. More information on expedited procedures can be found here.