Developing Meaning for Algebraic Procedures: An Exploration of the Connections Undergraduate Students Make Between Algebraic Rational Expressions and Basic Number Properties

by

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With much love, I dedicate this dissertation to my father Fred McBroom.

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ABSTRACT

The attainment and retention of later algebra skills in high school has been identified as a factor significantly impacting the postsecondary success of students majoring in STEM fields. Researchers maintain that learners develop meaning for algebraic procedures by forming connections to the basic number system properties. The present study investigated the connections participants formed between algebraic procedures and basic number properties in the context of rational expressions.

An assessment, given to 107 undergraduate students in precalculus, contained three pairs of closely matched algebraic and numeric rational expressions with the operations of addition, subtraction, and division. The researcher quantitatively analyzed the distribution of scores in the numeric and algebraic context. Qualitative methods were used to analyze the strategies and errors that occurred in the participants' written work. Finally, task-based interviews were conducted with eight participants to reveal their mathematical thinking related to numeric and algebraic rational expressions.

Statistical analysis using McNemar's test indicated that the undergraduate participants' abilities related to algebraic rational expressions and rational numbers were significantly different, although serious deficiencies were noted in both cases. A small intercorrelation was found in only one of the three pairs of problems, suggesting that the participants had not formed connections between algebraic procedures and basic number properties. The analysis of the participants' written work revealed that the percent of participants who consistently applied the same procedure in the numeric and algebraic items of Problem Sets A, B, and C were 56%, 47%, and 37%, respectively. Correct

strategies led to fewer correct solutions in the algebraic context because of a diverse collection of errors. These errors exposed a lack of understanding for the distributive and multiplicative identity properties, as well as the mathematical ideas of equivalence and combining monomials. These fundamental mathematical ideas need to be better developed in primary and secondary education. At the post-secondary level, these ideas should serve as the foundation for interventions that are designed to support underprepared students. The results of the interviews were consistent with the quantitative analyses and the qualitative examination of the strategies used by the participants. The findings in all three areas of the study point to a disconnect between numeric and algebraic contexts in the participants' thinking.

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CHAPTER I: INTRODUCTION

Introduction

Producing high quality science, technology, engineering, and mathematics (STEM) graduates is necessary to drive innovation and is vital to the economy and security of the United States (Machi, 2009). On November 23, 2009, the White House announced President Obama's "Educate to Innovate" campaign – a nationwide effort made of high-powered, public-private partnerships dedicated to increasing students' interest and abilities in science and mathematics (The White House, Office of the Press Secretary, 2009). The following quote demonstrates President Obama's commitment to moving America to the top of the list in science and mathematics achievement: "Reaffirming and strengthening America's role as the world's engine of scientific discovery and technological innovation is essential to meeting the challenges of this century" (The White House, Office of the Press Secretary, 2009, para. 3). Increasing the number of workers with knowledge in STEM areas is an important factor in meeting this objective.

Supply and Demand of STEM Graduates

STEM occupations are a small part of the overall workforce in the United States, only 5% in 2008, but they drive the technological changes that shape all other occupations. It is projected that by 2018 STEM occupations will grow from 6.8 million to 8 million total jobs. Furthermore, it is estimated that 92% of these STEM jobs will be for those with at least some postsecondary education or training (Carnevale, Smith, & Strohl, 2010). In the United States, the rate of students aged 20 to 24 years old pursuing postsecondary education has significantly increased from 44% in 1980 to 61% in 2003. The completion rate, however, has not kept pace. Overall growth in higher education attainment has been slow-moving (Goldin & Katz, 2008) and STEM degrees are no exception. The number of STEM degrees conferred in 2006 was virtually the same as it was in 1995 (Snyder, Dillow, & Hoffman, 2008) and the gap between the supply and demand of talented STEM workers continues to widen.

Persistence in STEM Majors

One cost-effective method for increasing the number of STEM graduates is to focus resources on minimizing attrition from STEM fields (Ehrenberg, 2010; Soldner, Rowan-Kenyon, Kurotsuchi Inkelas, Garvey, & Robbins, 2012). A 2009 report from the National Center for Education Statistics examined longitudinal data of approximately 12,000 students who began postsecondary education in 1995 (Chen, 2009). The study found that only 37% of students who entered a STEM field during their first year of enrollment had graduated with a STEM degree six years later in 2001. The remaining students were either still working on a STEM degree (7%), had changed to a non-STEM major (27%), or left postsecondary education altogether (28%). Another recent study found that only about half of the students who entered college majoring in a STEM field eventually graduated with a STEM degree (Ehrenberg, 2010).

Because fewer students enter college as STEM majors, persistence rates need to reach much higher levels to meet the demand for more STEM graduates. Efforts to reduce attrition should be based upon an understanding of the factors that lead students to leave STEM fields (Ehrenberg, 2010), so that the benefits of these programs and interventions can be maximized.

Factors Affecting Persistence in STEM Majors

Many research studies have examined the factors that influence students' success in STEM disciplines (Ehrenberg, 2010; Griffith, 2010; Kokkelenberg & Sinha, 2010; Ost, 2010; Price, 2010; Rask, 2010; Seymour & Hewitt, 1997). Explanations for attrition in STEM fields include, but are not limited to, the students' prior academic preparation (Astin & Astin, 1993; Kokkelenberg & Sinha, 2010; Post et al., 2010), the poor quality of undergraduate STEM instruction (Seymour & Hewitt, 1997), the students' gender (Griffith, 2010; Kokkelenberg & Sinha, 2010; Price, 2010), the students' ethnicity (Griffith, 2010; Kokkelenberg & Sinha, 2010; Price, 2010), and grades students receive in introductory courses (Ost, 2010; Rask, 2010). Ehrenberg (2010) reviewed five papers in a research symposium funded by the Sloan Foundation and identified grades in introductory STEM courses and prior academic preparation as the two most important factors that influence persistence in STEM fields. Each of these factors will be described in the following sections.

Grades in introductory STEM courses. A study of approximately 5,000 students was conducted by Rask (2010) at a liberal arts college where overall completion rates were much higher than typical colleges and universities, yet the attrition in STEM departments was very high. Rask found evidence that grades received in introductory STEM courses were one reason for the high STEM attrition rate at the institution in the study. When ranking average course grades given from lowest to highest, Rask found that five of the lowest six grading departments were STEM departments and that all of the STEM departments fell below the college mean grade given. According to Rask, student sensitivity to grades received in STEM courses and how those grades compared to their non-STEM courses may have explained why so many students left STEM majors. Similarly, Ost (2010) reported that grades in introductory courses contributed to attrition in STEM fields. In his study at a large elite university, Ost found that life science and physical science students who changed to a non-STEM major were discouraged by low grades in their major courses and were enticed by higher grades in non-science courses. Rather than bring the grade distribution in STEM courses in line with non-STEM courses (Rask, 2010), educators should examine strategies that address the gaps in prior academic preparation.

Prior academic preparation. Although the results should be considered carefully because of the age of the study, a 1993 study by Astin and Astin is perhaps the most comprehensive examination of factors influencing STEM persistence ever undertaken and is often cited in the undergraduate STEM education literature (e.g., Adelman, 1998; Elliott & Strenta, 1996; Harwell, 2000; Kokkelenberg, 2010; Lewis, Menzies, Najera, & Page, 2009; Seymour, 2002; Seymour & Hewitt, 1997; Vogt, 2008). Astin and Astin examined longitudinal data for 27,065 freshmen at 388 four-year colleges and universities and found that the strongest predictor of STEM persistence after four years was the students' entering level of mathematics or academic competency as measured by scores on college entrance exams, the American College Testing (ACT) Program Assessment or Scholastic Aptitude Test (SAT), and high school grade point average

(Astin & Astin, 1993). A more recent, larger-scale study by Kokkelenberg and Sinha (2010) analyzed longitudinal data for approximately 44,000 students at a New York State University and found evidence that prior academic preparation as measured by advanced placement (AP) credits and SAT scores was a significant indicator of success in a STEM major. Therefore, educators could improve STEM persistence by identifying students with gaps in prior academic preparation and intervening early in their academic careers.

Conclusion

The individual and institutional factors that lead students to leave STEM majors are very complex, and one single solution does not exist. Special attention has been given to the discipline of mathematics, which is considered a gateway to other STEM courses (President's Council of Advisors on Science and Technology, 2012). Research has identified grades in early STEM courses and prior academic preparation as two important factors influencing success for students who major in STEM disciplines (Astin & Astin, 1993; Kokkelenberg & Sinha, 2010). Mathematics educators may contribute to solving the STEM persistence problem by seeking to understand why students struggle in early postsecondary mathematics courses such as precalculus.

Background of Study

The mathematics entry point for STEM majors is often precalculus (Post et al., 2010), and failure to succeed in this course is often a barrier for students continuing to study a STEM field (Adelman, 1998). Students' lack of algebraic manipulation skills is among several difficulties with calculus that Tall (1993) observed. Similarly, in a study by Baranchik and Cherkas (2002), success in precalculus was found to depend on the students' "later algebra skills," a term used to describe those algebra skills learned just before precalculus. Therefore, it seems logical that one potential avenue for supporting the retention of STEM majors lies within the acquisition of later algebra skills.

A review of precalculus textbooks supports the fundamental belief that the lack of algebra skills limits students' success in precalculus. In fact, in five of the seven precalculus textbooks reviewed by the researcher, the author included an algebra review section (see Table 1). The algebra topics most often covered in these review sections included real numbers, exponents, factoring polynomials, rational expressions, and radicals.

Table 1

Algebra Review Topics Found in Precalculus Textbooks

	Algebra Review Topics				
Textbook Author	Real Numbers	Exponents	Factoring Polynomials	Rational Expressions	Radicals
Young (2010)	X	X	X	X	X
Narasimhan (2009)	X	X	X	X	X
Zill and Dewer (2009)	-	-	-	-	-
Larson and Falvco (2010)	X	X	X	X	X
Cohen, Lee, and Sklar (2006)	X	X	X	X	-
Faires and DeFranza (2011)	-	-	-	-	-
Stewart, Redlin, and Watson (2011)	X	X	X	X	X

Mathematics instructor web-sites also provide documentation for the most common algebra errors made by students. Simplifying and performing operations with rational expressions was identified as a particular area of weakness for many students by Dawkins (n.d.), Schechter (2009), and Scofield (2003). Scofield (2003), a mathematics instructor for 14 years, published a comprehensive list on his course page of the most common algebra errors he witnessed students make in his precalculus and calculus classes. He observed that when faced with complicated algebraic fractions, students tended to cancel everything in sight without regard to the fact that the numerator and denominator must first be factored. Schechter (2009) called this phenomenon "undistributed cancellations" and admitted that although he saw this error fairly often, he did not have a very clear idea of why it happened.

In an essay on mathematics education, Thurston (1990) noted that many calculus students made mistakes adding fractions, particularly symbolically. He noted that students commonly made this mistake: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. Similarly, researchers who reviewed a precalculus-algebra course preceding precalculus collected anecdotal evidence from faculty members and observed that students often had trouble simplifying or performing operations with rational expressions, especially when finding least common denominators and greatest common divisors (Karim, Leisher, & Liu, 2010). These findings identify the inadequate understanding of rational expressions as a limitation for many students.

Findings from a STEM talent expansion grant with which the researcher had worked agree with the results found by Thurston (1990) and Karim, Leisher, and Liu (2010). As an evaluator for the grant whose aim was to improve retention by supporting

STEM students in early mathematics courses, the researcher had the opportunity to assess the prior academic preparation of incoming STEM majors during the grant's summer mathematics bridge program. Students in the summer bridge program had an ACT-Mathematics score between 19 and 23, inclusively, and were planning to take precalculus in the following fall semester. Data from the 2010 assessment test of precalculus readiness demonstrated that students in this program did not have many of the algebra skills necessary to succeed in precalculus. Their inability to simplify or perform operations with algebraic rational expressions was among the deficiencies noted. Only 9% of the incoming freshman STEM majors in this group correctly simplified the expression $\frac{x^2+18x+81}{x^2-81}$ on the pretest assessment, further supporting students' conceptions and misconceptions related to rational expressions as a research area that should be of interest to mathematics educators.

Contemporary research supports the idea that proficiency with algebraic processes is dependent upon conceptual understanding of basic number properties specifically related to rational numbers. A study by Brown and Quinn (2007) found a positive relationship between proficiency with fractions and success in algebra. They concluded that understanding the structure of arithmetic could have a profound effect on learning the structure of algebra. Similarly, Wu (2001) contended, "the computational aspect of numbers is essential for the learning of both higher mathematics and science as well" (p. 13). Welder (2012) and Rotman (1991) also pointed to number knowledge of fractions as a prerequisite for learning algebra.

Theoretical Framework

The theoretical framework for this study was based on the use of connected representations of knowledge to think about understanding mathematics, as described by Hiebert and Carpenter (1992). In their framework, Hiebert and Carpenter stated that mathematical understanding exists when a mathematical idea, procedure, or fact is part of the learner's internal mental network of representations. Furthermore, they asserted that learners develop meaning for algebraic procedures by forming connections to the basic number properties. This framework was used to examine the connection between students' errors and their understanding of rational expressions and basic number properties.

Statement of Purpose

The purpose of this mixed-methods study was to explore students' understanding of algebraic rational expressions. Students' conceptions and misconceptions were established using a framework of connected representations (Hiebert & Carpenter, 1992) to analyze student written work and dialogue during task-based interviews. The following research questions were addressed:

(1) What, if any, relationship exists between undergraduate students' ability to simplify and perform operations with algebraic rational expressions and their ability to do the same with rational numbers?

(2) What strategies do undergraduate students use to simplify and perform operations with algebraic and numeric rational expressions?

(3) How are undergraduate students' strategies for simplifying and performing operations with algebraic rational expressions connected to their understanding of basic number properties?

Significance of the Study

This study extended the current knowledge of undergraduate students' understanding of rational expressions and how this is linked to their understanding of basic number properties. The results of this study provided much-needed insights into students' understanding and inform the classroom practice of mathematics educators on a wide spectrum of educational levels: from the teaching and learning of fractions in elementary grades, to the teaching and learning of algebra at the secondary level, and to the teaching and learning of advanced mathematics courses at the postsecondary level. It is desirable for students to acquire algebra skills and establish an understanding of rational expressions at the earliest possible educational level, but many students will enter postsecondary education without this knowledge. It is critical that these algebra deficiencies in students be addressed so that the number of students succeeding in mathematics and persisting in STEM majors increases.

Definition of Terms

Throughout this study the term *rational number* will be used to represent the quotient $\frac{a}{b}$, where a and b are integers and b is not zero. The quotient $\frac{a}{b}$, where a and b are numeric expressions, and b is not equivalent to zero will be referred to as a *numeric rational expression*. The term *algebraic rational expression* will be used to represent the quotient $\frac{P}{Q}$, where P and Q are polynomials and the value of Q is not zero. Operations that may be performed on rational numbers and rational expressions include addition, subtraction, multiplication, and division. Additionally, the phrase "*simplified rational expression"* will be used to represent an equivalent numeric or algebraic rational expression in which the numerator and denominator are relatively prime, that is, the greatest common divisor is 1. Basic number properties include the commutative property of addition and multiplication, the associative property of addition and multiplication, and the distributive property of multiplication over addition. *Conceptual knowledge* refers to the understanding of mathematical ideas and how they are interconnected. *Procedural knowledge* refers to familiarity with procedures, recognizing when and how to use procedures, and performing procedures accurately and fluently (Kilpatrick, Swafford, & Findell, 2001).

Chapter Summary

Students who major in STEM fields will drive technology and science innovation. Increasing the retention and graduation of students who major in STEM is important for strengthening the United State's position in the global economy (Machi, 2009). Students in postsecondary education who desire to study a STEM field must also face some advanced mathematics courses. The first hurdle for many students is passing precalculus (Post et al., 2010), which requires that they be fluent in advanced algebraic procedures (Baranchik & Cherkas, 2002; Tall, 1993). Weak prior academic preparation in algebra often leads to low grades in introductory mathematics courses and discourages students from studying STEM fields. The algebraic procedures of simplifying, adding, subtracting, multiplying, and dividing rational expressions are an area perceived as a common weakness among students (Dawkins, n.d.; Schechter, 2009; Scofield, 2003).

Understanding students' conceptions and misconceptions related to rational expressions and how they connect algebraic procedures to basic number properties is an important step toward being able to promote success for all students in introductory mathematics courses, but particularly for STEM majors who might otherwise leave a STEM field of study. The next chapter will address the current body of research describing theoretical perspectives on the learning of algebra, the connected representations framework, and empirical research on the learning of algebra, specifically rational expressions.

CHAPTER II: REVIEW OF LITERATURE

Introduction

The number of students graduating each year with a STEM degree is not sufficient to meet the needs of our nation's workforce (Carnevale, Smith, & Strohl, 2010). Although many students start college as STEM majors, very few finish college with a degree in a STEM field (Snyder, Dillow, & Hoffman, 2008). One reason students do not graduate as a STEM major is their failure to succeed in the advanced mathematics courses that are required to earn a STEM degree (Rask, 2010). Students who struggle in introductory mathematics courses may perceive the STEM degree as too difficult to attain, and they often leave the STEM field for a major with less emphasis on mathematics. Mathematics educators can better understand this phenomenon by investigating the struggles that students face, and determining what students understand and do not understand about mathematics. Proficiency with algebraic procedures is one gap that educators have identified in students' academic preparation for college, particularly a lack of understanding with the procedures used to simplify or perform operations with rational expressions (Dawkins, n.d.; Schechter 2009; and Scofield, 2003).

Hiebert and Carpenter's (1992) theoretical framework suggests that understanding of algebraic procedures depends on understanding of the underlying number properties. This study explored the connections between students' understanding of algebraic rational expressions and their understanding of rational numbers. The following section begins with an examination of two other theoretical perspectives on the learning of algebra, connectionism and Gestalt theory, and establishes a theoretical framework for this study. Empirical research specifically addressing algebraic rational expressions is limited; therefore, the chapter ends with a review of research about the connection between arithmetic and algebra, a description of how researchers have historically treated the study of student errors in algebra, and research about the learning of algebra that is closely related or can be extended to simplifying and performing operations with algebraic rational expressions.

Theoretical Perspectives

Methods for measuring intelligence, aptitude, and achievement were first developed by psychologists in the early twentieth century. At that time, the mathematics domain of algebra was a commonly used medium for psychologists to test their theories of memory and acquisition of knowledge. Algebra was well-suited for this purpose because relatively few people had already mastered algebra, and the answers were easily scored (Wagner & Parker, 1993). Since that time, theories from psychology have continued to be an important influence on the teaching and learning of mathematics (Lambdin & Walcott, 2007).

Lambdin and Walcott (2007) described the psychological theories that have shaped the teaching of mathematics during the twentieth century. As psychologists learned more about how the mind works, mathematics educators applied the findings to the teaching of mathematics. The following section describes two important theoretical perspectives, connectionism and Gestalt theory, that continue to influence the current teaching and learning of mathematics (Lambdin & Walcott, 2007).

Connectionism

In the early twentieth century, the role of connections in the learning of mathematics was first explored by Thorndike, who believed that behavior could be explained by the study of bonds that form between a stimulus and a response (Lambdin & Walcott, 2007). For example, Thorndike et al. (1923) believed students formed bonds, or connections, between equivalent representations such as $a \times ab = a^2b, a(a+b) = a^2 + ab,$ and $-a \times -b = +ab$ where the left side of each equation represented a stimulus and the right side represented the appropriate response. He believed that students organized these connections into mathematical habits through practice and gradually gained a sense of what to do when faced with an algebraic expression and why the action was appropriate.

During this same time period, Thorndike's colleague Symonds (1922) studied the psychology of errors. He believed that the study of students' behavior could explain the frequency of errors they made in algebra. The behavior categories of Thorndike and others (1923) that Symonds thought were significant included multiple responses given by students to the same situation, a student's determination to get *the* answer, and a student's tendency to give partial answers. Symonds prescribed that students must "first learn to carry out procedures and manipulations, and thereby acquire the meaning and significance of what they do" (p. 104), presumably through much drill and practice. He stated, "Our analysis shows the need for drill or practice in various processes. Telling a boy what things to do does not form habits of doing them" (p. 102).

The prevailing viewpoint at that time was that repeated practice served as a means for students to form connections between the stimulus and response, thus developing procedural knowledge. As evidenced in the *Principles and Standards for School Mathematics* (NCTM, 2000), however, current opinion refutes the idea that meaning for algebraic procedures is developed through repeated practice. Regardless, this first examination of the role of connections in the learning of mathematics and inquiry into what the study of student errors could reveal about a student's knowledge is significant.

Gestalt Theory

A somewhat different view of connections was held by Gestalt psychologists (Lambdin & Walcott, 2007). Gestalt theory strongly influenced the teaching and learning of mathematics around the time of World War II. Those who supported this theory defined learning as a process of developing insights by recognizing the relationship of a part to the whole (Lambdin & Walcott, 2007). Gestalt psychologists were more interested in *how* a student knows something instead of *what* they know.

Consistent with Gestalt theory, Brownell (1947) claimed that facts and skills are tied together by meaning and thus make up the complete structure of mathematics. He advocated teaching that emphasized mathematical relationships and used real-world problems to demonstrate to students the purpose of the mathematics they learned. According to Brownell (1947), teaching mathematics in a meaningful way benefited students by building a sound foundation of knowledge that was more easily remembered, more easily transferred, and more likely to actually be used.

Fehr (1955) also believed that new evidence from Gestalt psychologists about the role of connections in learning should be incorporated into mathematics instruction. According to Fehr, learning begins by building a structure of knowledge through connected concepts, and then applying that knowledge to solve problems. He wrote a guide for teaching high school mathematics that asserted that long-lasting learning occurs when students are allowed to experience mathematics and discover laws and principles for themselves. Gestalt theory retained the idea of Thorndike and others (1923) that connections are formed in the acquisition of knowledge, but it rejected the idea that meaning can be obtained through repeated drill and practice (Lambdin & Walcott, 2007).

The Framework of Connected Representations

Hiebert and Carpenter's (1992) framework of connected representations draws from psychology research done in the early twentieth century and more recent cognitive science. The framework provides a means for explaining students' understanding that is easily communicated and understood, and can shed light on both students' successes and failures. For these reasons, the framework of connected representations was chosen to guide this study and will be discussed in detail in this section.

Hiebert and Carpenter (1992) based their framework of connected representations on both historical and more recent research in the psychology of learning. The research of Hiebert and Carpenter (1992) might be seen as an extension of Thorndike's premature ideas. As with Thorndike's research, Hiebert and Carpenter's research on the learning and teaching of mathematics was centered upon the idea that learning is equivalent to the formation of connections; however, their work included new evidence from cognitive science about the existence of internal mental representations. The idea from contemporary cognitive science that knowledge is represented internally, and that the internal representations are structured, is the primary assumption that supports the framework. Applying the cognitive science theory of internal representations to learning, Hiebert and Carpenter (1992) submitted that the construction of knowledge occurs when new information is connected to prior connections or when established connections are rearranged or abandoned. They proposed that this structure of connected representations is a useful way to describe mathematical understanding.

Hiebert and Carpenter (1992) distinguished internal representations as a way for the brain to operate when thinking about mathematical ideas and external representations as a means to facilitate the communication of mathematical ideas. They suggested that in the teaching and learning of mathematics, an external representation (pictures, symbols, and manipulatives) viewed by a student has an impact on the way that an internal representation is formed. Just as Gestalt psychologists believed that students developed insights by recognizing the relationships between a part and the whole, Hiebert and Carpenter proposed that understanding is created through the examination of the similarities or differences between different representations or through the investigation of patterns and regularities that occur within the same representation form. Hiebert and Carpenter provided some important consequences of thinking about understanding in this way – knowledge from thickly connected networks provides a strong base for the construction of new knowledge, is quickly retrieved, and is more easily preserved over time.

Other research describes the organization of knowledge by “experts” as having more pieces of conceptual knowledge in memory, more characteristics and attributes of these bits of knowledge, more links to other sections of knowledge, and more efficient means of retrieving knowledge from this network (Bransford, Brown, & Cocking, 2000). Experts’ knowledge is further described as being organized around “big ideas” and with parameters to the context in which it may be applied. In this way, an expert is able to understand and apply knowledge to new contexts rather than just retrieve facts as a novice would. The parameters, or conditions, stored with pieces of conceptual knowledge tell the expert when, where, and why to use a specific piece of information. This structure allows the expert to quickly retrieve procedures relative to a specific task.

In mathematics, and particularly algebra, there is a constant tension between the importance of conceptual knowledge and procedural knowledge (Star, 2005). Hiebert and Carpenter (1992) maintained that both kinds of knowledge are necessary for mathematical expertise and defined both in terms of connected representations. They defined conceptual knowledge as a network of internal representations rich with connections and procedural knowledge as an internal representation of a sequence of actions with connections formed between each step of the procedure. An important element in Hiebert and Carpenter’s framework is the idea that mathematical procedures always depend on conceptual knowledge of mathematical principles. Only a few connections are needed to create an internal representation of the sequence of steps in a procedure, and thus procedural knowledge does not become part of the richly connected network of conceptual knowledge. The fewer number of connections means that procedural knowledge is harder to recall or extend to new circumstances. When procedural steps are linked to conceptual knowledge, however, the procedure becomes part of a larger network and then has access to all of the knowledge in that network, extending the range of the procedure's capabilities. Furthermore, Hiebert and Carpenter (1992) explicitly claimed that meaning for algebraic procedures is created when connections are formed between the procedure and the basic number properties.

In the context of algebraic rational expressions, the theory of connected representations holds that the algebraic processes of simplifying, adding, subtracting, multiplying, and dividing algebraic rational expressions are dependent on conceptual understanding of the basic number properties including, but not limited to, the commutative property for addition and multiplication, the associative property for addition and multiplication, and the distributive property of multiplication over addition. According to Hiebert and Carpenter (1992), misconceptions and procedural errors can be understood in terms of connections. This study utilized this theory by examining students' abilities to simplify and perform operations with both numeric and algebraic rational expressions. The next section examines findings from empirical research on the learning of algebra and provides additional insights that informed the design of this study.

Research About the Learning of Algebra

The critical need for more STEM graduates has brought increased attention to issues surrounding undergraduate education. The advanced mathematics required in most STEM degrees is often an obstacle that students without sufficient previous academic preparation in algebra find difficult to overcome (President's Council of Advisors on

Science and Technology, 2012). More students could find success, however, if educators better understood the difficulties that students have with algebra, and could design post-secondary mathematics instruction that would support the needs of all students. Because of the limited research on the learning of algebra at the post-secondary level, many of the studies in the next section are set in the context of primary or secondary education. The following section describes research that explores the relationship between arithmetic and algebra, searches for root causes of the most common algebraic errors, and seeks to understand how students develop meaning of algebraic procedures.

The Relationship Between Arithmetic and Algebra

Algebra is often described as generalized arithmetic (Brown & Quinn, 2007; Kilpatrick, Swafford, & Findell, 2001; Wu, 2001). The manipulations or transformations allowed in algebra are governed by the laws of arithmetic: commutative laws of addition and multiplication, associative laws of addition and multiplication, distributive law of multiplication over addition, additive and multiplicative identities, and additive and multiplicative inverses (Carraher & Schliemann, 2007). Students who are transitioning from arithmetic to algebra need time to develop a conceptual knowledge of number properties and explicitly explore these relationships before they are taught algebraic algorithms (Brown & Quinn, 2006; Vance, 1998; Warren, 2003). Researchers have also called for instruction that gives increased attention to the structure, relationships, and reasoning of operations in arithmetic (Lee & Pang, 2012; Linchevski & Livneh, 1999; Warren, 2003).

The Common Core State Standards for Mathematics (CCSSI, 2010), adopted by 45 states and almost all territories in our nation, include language that explicitly connects arithmetic to algebra in the standards for grades six, seven, and eight, the typical period of transition from arithmetic to algebra (See Appendix A).

In the sixth grade, students develop an understanding of the distributive property in the context of algebra rather than learn meaningless rules for the manipulation of algebraic symbols. A standard from the seventh grade provides an excellent opportunity for students to compare and make connections between an algebraic and arithmetic solution. Finally, although the eighth grade standards do not specifically refer to the connection between algebra and arithmetic, language in the introduction holds the expectation that teachers should continue to use basic number properties in instruction to justify transformations of algebraic expressions and equations. These standards, however, have only recently been adopted by many states. Educators for many years to come will continue to see students face difficulties with algebra that can be attributed to deficiencies in arithmetic. With this in mind, the current research about the connection between arithmetic and algebra can be used to inform classroom instruction and to help educators understand the difficulties they see exhibited by their current students.

The goal of most contemporary research that examines the connections between arithmetic and algebra is to promote the development of algebraic thinking and operation sense in the early grades (Lee & Chang, 2012; Lee & Pang, 2012; Linchevski & Livneh, 1999; Warren, 2003). The benefits of this research to post-secondary education may not be obvious, but findings from research on algebraic thinking and reasoning in the early grades can have implications for learning algebra in the higher grades. Algebraic procedural skills learned in the higher grades require students to have developed a strong operation sense (Hiebert & Carpenter, 1992), which many studies have shown is a weakness in students transitioning from arithmetic to algebra (Linchevski & Livneh, 1999; Warren, 2003).

Near the end of students' primary education, just prior to pre-algebra or algebra, many students do not understand the order of operations (Herscovics & Linchevski, 1994; Linchevski & Livneh, 1999; Warren, 2003). In two studies, students who evaluated numerical expressions with a mixture of operations failed to demonstrate understanding of the order of operations. Students frequently performed arithmetic operations from left to right, 62% in one study (Linchevski & Livneh, 1999) and 77% in another (Herscovics & Linchevski, 1994). Researchers from both studies attributed the errors to an over-generalization of the acronyms PEMDAS (parenthesis, exponents, multiplication, division, arithmetic, and subtraction) or BOMDAS (Brackets first, or Multiplication or Division, then Addition or Subtraction).

In contrast, Linchevski and Livneh (1999) noted that when the arithmetic expression contained subtraction before multiplication, instead of addition before multiplication, only 47% of students incorrectly performed the subtraction first, a 30% decrease in the number of errors. The researchers hypothesized that the subtraction operation was not triggering recall of the order of operations rules, but seemed to serve as a signal to students to partition the expression at that point. Further into the study by Linchevski and Livneh (1999), a related phenomenon occurred when students were asked to quickly evaluate the expression $50 - 10 + 10 + 10$ without a calculator. Nearly half of the students added the 10's first, and found the answer to be $50 - 30 = 20$. Even though some students could recite the order of operation rules, they could not resist separating the expression into two pieces at the subtraction sign.

The subtraction operation also provided trouble for students in a study by Herscovics and Linchevski (1994). In this study, seventh and eighth graders without previous instruction in algebra were asked to extend their arithmetic knowledge and solve equations with a missing value. Herscovics and Linchevski observed that when some students were presented with a single term on the left-hand side of the equals sign and more than one term on the right-hand side of the equals sign, they struggled to decompose the equation and read aloud going from *right to left*. For example, in interviews students read an equation like $25 = 10 + n$ aloud as $n + 10 = 25$. When subtraction was the operation on the right side of the equals sign, this misconception led students to read an equation like $17 = 3 - n$, incorrectly as $n - 3 = 17$, which led to an incorrect answer. One explanation for this might be that this error is the result of the dominant presentation of decomposition in arithmetic as two numbers with an operation on the left side of the equal sign and an answer on the right side of the equals sign (Herscovics & Linchevski, 1994).

In summary, a significant number of students near the end of their primary education have shown an inadequate sense of the properties of operations in arithmetic. The frequent occurrence of errors in algebra may be explained by students' misconceptions about operations in arithmetic. The next section will discuss research about algebra that specifically investigated student errors to uncover student misconceptions.

The Study of Errors and Misconceptions

For at least a century, researchers have studied common errors made by students in the learning of algebra. They hoped that insights from these studies would direct the design of interventions that would eliminate these errors, or prevent them from becoming formed habits (Symonds, 1922; Thorndike et al., 1923). Psychologists in the 1920's sought to explain errors in terms of students' behavior. The arrival of computer technology in the 1980's and 1990's prompted researchers to explore the use of an intelligent computer program to systematically detect and diagnose students' algebra errors (Birenbaum, Kelly, & Tatsuoka, 1992; Payne & Squibb, 1990; Sleeman, 1984). Following this, contemporary research has focused on increasing awareness with elementary and middle school teachers on how their classroom practices may shape misconceptions that inhibit learning later in algebra (Welder, 2012). The following section will discuss how the study of students' errors has evolved over time and the merits of the current efforts to identify misconceptions.

Exploring the psychology of errors. Early in the twentieth century, Symonds (1922) associated certain behaviors with errors that students made in algebra. For example, he noted that many students demonstrated a strong desire to get *the* correct answer to a problem. Students looked to the teacher, other classmates, and the key for approval of their answers. The result was that when they perceived that some level of approval had been given, even if it had not, they stopped thinking about the problem.

Students who repeated this pattern did not develop autonomy with mathematics. Students who gave partial answers to a problem, leaving out a step or failing to complete all of the steps, were often characterized as forgetful by their teachers. Symonds presented an alternative argument – partial answers may be the result of a student who has reached his or her maximum level of attention. Today's psychologists focus on these same behaviors but refer to it in terms of cognitive load.

Diagnosing errors with mal-rules. Researchers in the 1980's were also concerned with the influence of working-memory load and attention allocation on the occurrence of errors (Payne & Squibb, 1990). The study of algebraic procedural knowledge during this time was shaped by an *Information Processing* framework that encouraged the study and classification of common errors to uncover the mental processes used by students carrying out algebraic procedures (Wagner & Parker, 1993). The goal of research during this time was to classify each and every observed student mistake as a "mal-rule" and establish a complete catalogue of errors that would allow teachers to create appropriate remediations (Birenbaum, Kelly, & Tatsuoka, 1992; Payne & Squibb, 1990; Sleeman, 1984). The identification and classification of mal-rules expended a significant amount of resources, but produced mainly inconsistent results. Payne and Squibb (1990) warned that the frequency of mal-rules was unstable, the severely skewed distributions of mal-rules provided little useful information about students' errors, and that their own mal-rule model failed to recognize the important connection between procedural and conceptual knowledge.

Researchers studying student errors also encountered a phenomenon that their studies could not explain. Birenbaum, Kelly, and Tatsuoka (1992) expected to be able to categorize and define all of the ways to solve a problem, but students continually surprised them by creating original solutions. Sleeman (1984) similarly noted that students were not consistent in the strategies they chose to solve equivalent tasks. Payne and Squibb (1990) unexpectedly found that increasing the level of a problem's difficulty did not also increase the frequency of errors for that problem. Evidence from more recent studies continues to report that students make a large variety of errors and often invent their own rules and procedures. The focus of this more recent research, however, has shifted to understanding why these errors occur and what misconceptions they reveal (Demby, 1997; Ruhl, Balatti, & Belward, 2011; Otten, Males, & Figueras, n.d.).

Uncovering students' misconceptions. Studies about students' misconceptions surrounding rational expressions typically investigate strategies students use to simplify the expressions, but do not address strategies used by students to perform operations such as addition, subtraction, multiplication, and division (Constanta, 2012; Demby, 1997; Ruhl, Balatti, & Belward, 2011; Otten, Males, & Figueras, n.d.). Perhaps this is due to the fact that the studies are typically situated in middle school or high school, and students at this level of education have not yet learned to perform operations with algebraic rational expressions. For this reason, the following section will discuss findings from the literature about simplifying algebraic rational expressions (Constanta, 2012; Ruhl, Balatti, & Belward, 2011; Otten, Males, & Figueras, n.d.) and results from one study focusing on the simplification of algebraic expressions (Demby, 1997).

The diversity of errors recorded by researchers categorizing mal-rules (Sleeman, 1984; Payne & Squibb, 1990) was also seen in more recent studies by Otten, Males, and Figueras (n.d.) and Demby (1997). In a study that examined the instruction of algebraic rational expressions in secondary school, teachers found that their prediction of the most common problems students would have simplifying algebraic rational expressions did not match the reality of the students' errors (Constanta, 2012). During instruction, attention was not given to the composition of algebraic expressions and the relationship between operations in the numerator and denominator because teachers had assumed, incorrectly, that the students knew these concepts. In a different study, researchers who examined students' reflections of their procedures used to simplify algebraic rational expression problems were surprised at the widespread confusion about the meaning of "common factor" which was assumed to be common knowledge with undergraduate students (Ruhl, Balatti, & Belward, 2011).

The schemes used to code student errors are as diverse as the errors themselves. Errors related to cancellation of factors, however, are the most prevalent errors students make when simplifying algebraic rational expressions (Constanta, 2012; Otten, Males, & Figueras, n.d.; Ruhl, Balatti, & Belward, 2011). Constanta (2012) hypothesized that cancellation errors may stem from the students' inability to perceive the numerator as a "whole" that is composed of different parts, while Otten and colleagues (n.d.) credited a misconception of the operation of division as the most likely cause of cancellation errors.

Two important implications for classroom instruction stem from the work by Demby (1997) and Constanta (2012). First, Demby observed that even when given formal instruction on rules, students frequently created their own rules, but often incorrectly. Students more easily remember rules that they have constructed themselves, so Demby suggested that classroom instruction should be designed so that students explicitly form their own rules with guidance from the teacher to ensure the rules are correct. Second, teachers in the study by Constanta (2012) realized that their assumptions about what students knew led to insufficient instruction. Constanta called for the development of a theory of learning addressing algebraic rational expressions that could guide instruction, and an increase in the variation of problem features that teachers present in class.

The most notable misconceptions identified in this research on the learning of algebraic rational expressions are related to understanding of a “common factor” (Ruhl, Balatti, & Belward, 2011) and the operation of division (Otten, Males, & Figueras, n.d.). The purpose of most of the studies examined in this section was not to identify *what* errors students made, but to understand *why* students frequently made errors simplifying algebraic rational expressions (Constanta, 2012; Ruhl, Balatti, & Belward, 2011). Researchers used interviews and written reflections in their investigations to gain insight into the thinking done by students (Constanta, 2012; Otten, Males, & Figueras, n.d.; Ruhl, Balatti, & Belward, 2011), but discounted the information that might be gleaned from studying correct solutions as well.

Chapter Summary

Although it is important for all students to have algebraic procedural knowledge, it is critical for those who desire to be scientists, physicists, or mathematicians and will study advanced mathematics. Research in the learning of mathematics from the last century combines to inform what we know today about students' conceptual knowledge and procedural knowledge of numeric and algebraic rational expressions. Research has revealed common errors, but failed to advance our understanding of why students continue to make the same errors observed by Symonds (1922), Thorndike (1923), Brownell (1947), Sleeman (1984), Payne and Squibb (1990), Warren (2003), and Constanta (2012). Although many years have passed, connectionism and Gestalt theories can still frame the problems that educators see today with the learning of algebraic procedures. Current research of algebraic procedures frequently mentions the connection between arithmetic and algebra (Herscovics & Linchevski, 1994; Linchevski & Livneh, 1999) first discussed by Thorndike in 1923. Hiebert and Carpenter's (1992) framework tells educators that algebraic procedural knowledge is connected to conceptual knowledge of number properties. Very often, a disconnect between algebra and arithmetic was seen by researchers who concluded that the student errors they observed demonstrated a lack of operation sense (Otten, Males, & Figueras, n.d.; Warren, 2003).

The Gestalt theory of learning is still relevant and could be used to explain the difficulties students in Constanta's (2012) study had with distinguishing the numerator as a "whole" to which operations could be applied, or that could be decomposed into parts. In line with Gestalt theory, current research is focused on uncovering how students acquire knowledge and not what knowledge they acquire. However, researchers often focus on students' errors, and overlook what the study of correct strategies used by students could contribute to educators' understanding of learning. In the next chapter, a description is given of how elements of the theories and research described above influenced the methodology of this study.

CHAPTER III: METHODOLOGY

Introduction

Increasing the retention and graduation of students who major in STEM disciplines is necessary to meet the future workforce demands in the United States (Carnevale, Smith, & Strohl, 2010). Mathematics has been identified as a barrier that prevents many students from continuing to study a STEM field (President's Council of Advisors on Science and Technology, 2012). Weak prior academic preparation in algebra often leads to low grades in introductory mathematics courses, such as precalculus, and discourages students from studying STEM fields (Kokkelenberg & Sinha, 2010). The algebraic procedures of simplifying, adding, subtracting, multiplying, and dividing rational expressions represent an area perceived as a common weakness among students (Dawkins, n.d.; Schechter, 2009; Scofield, 2003). The purpose of this mixed-methods study was to explore students' understanding of algebraic rational expressions and to identify the connections, if any, that they formed between algebraic procedures with rational expressions and basic number properties.

This study can best be characterized as an explanatory mixed-methods design (Gray, Mills, & Airasian, 2009). The first phase of the study began with a quantitative assessment to identify differences, if any, in students' performances with algebraic and numeric rational expressions. This assessment was also used to code and categorize the strategies found in students' work. The second phase of the study used qualitative interviews to explore students' mathematical thinking that may explain why the differences were found. This chapter will begin with a description of the context and participants of this study. Next will follow a discussion of the instruments and procedures that were used to collect the data. Finally, a description of how the data was analyzed and an explanation of how the analysis of this data answered the research questions will be provided.

Context

This study took place at a Southeastern, public university that primarily serves in-state residents. Statistics from the fall semester of 2012 indicated that of the 25,394 students who were enrolled at this university, 72% were full-time, undergraduate students (Office of Institutional Effectiveness, Planning, and Research, 2012). Students with a declared STEM major made up 21% of the undergraduate population. Although gender was almost perfectly balanced across the university, only 38% of undergraduate STEM majors were female. Minorities made up 31% of both the overall university population and undergraduate STEM majors. In the fall of 2012, the university offered fourteen sections of calculus and twenty-three sections of precalculus, indicating that the first mathematics course for a majority of STEM majors at this university is precalculus. Of the 650 students who took precalculus in the fall of 2012, approximately 44% made a grade of D or F or withdrew from the class.

Participants

This study had two phases of data collection, and thus two samples of participants. Because the purpose of the study was to examine students' abilities with algebraic rational expressions and numeric rational expressions, students taking precalculus were the most appropriate population for the study. In the next paragraphs, the sampling methods used for both phases of the study are described.

Phase One

Over 500 students take precalculus each fall at the university where this study was situated. The most appropriate method of selection from the natural groups that the different sections of precalculus form was cluster sampling (Kemper, Stringfield, & Teddlie, 2003). The university offered 23 sections of precalculus in the fall of 2012, with an average of 30 students in each section. Of the 23 sections of precalculus offered in the fall of 2012, two evening sections were excluded from this study to maintain a homogeneous sample of full-time, traditional students. Thus, to obtain a sample size adequate for statistical analysis, the researcher randomly selected five sections of precalculus and invited the students enrolled in those classes to participate in this study.

To randomly select the sections, the researcher arranged each of the sections of precalculus in a spreadsheet column and assigned each record a number. A random number generator was used to select five sections of precalculus. The researcher contacted the professors of the selected sections of precalculus by email and asked for permission to visit their classes and invite their students to participate in the study. Two professors declined to allow the researcher to visit their classes. Two additional sections of precalculus were subsequently chosen using a random number generator, and the professors of both sections agreed to allow the researcher to visit their classes and ask their students to participate in the study. The participants in the resulting sample (n = 107) had an average age of 21. The gender distribution of the sample was aligned with that of the university: 36% female, 59% male, and 5% not reported. At 26%, the minority composition for this sample was slightly less than expected, but a large number of subjects, 30%, did not disclose their race or ethnicity.

Phase Two

The second phase of the study used qualitative methods to assess the participants' mathematical thinking during task-based interview sessions. The sample size for this qualitative phase of the study was dependent, in part, on the results from the quantitative phase, which is often common in mixed-method designs (Creswell, 2007). The researcher took the written assessments of participants who consented to take part in phase two of the study and arranged them in ascending order based on the number of correct items. None of the assessments had five or more of the six items correct. Consequently, the following subgroups of interest were formed: group one - participants who answered four items correct, group two - participants who answered one, two, or three items correct, and group three - participants who answered none of the items correct. The researcher selected all three participants in group one and then used random purposive sampling, that is, taking a random sample from purposefully selected subgroups, to select five participants from groups two and three (Kemper, Stringfield, & Teddlie, 2003). The researcher used a random number generator to make the selections from groups two and three. All three of the participants in group one completed the interview. Participation was lower in groups two and three, thus two additional participants from each group were randomly selected and asked to participate. In the end, the researcher conducted eight interviews. Three of the interviews were with participants who answered four items correct, four of the interviews were with participants who answered two or three items correct, and one interview was with a participant who answered none of the items correct.

Instruments

This mixed-methods study collected data in two phases: one quantitative and one qualitative. Each phase had a unique instrument for collecting data. The first phase used an assessment instrument to collect student work, and the second phase used an interview protocol to guide task-based interviews with participants. The instruments from each phase will be described in the following paragraphs.

Phase One

In the first phase of the study, the researcher used a self-developed assessment instrument (see Appendix B) to collect data regarding participants' procedural knowledge of algebraic and numeric rational expressions. The assessment instrument was limited to six mathematics questions to control for fatigue effects (Mitchell & Jolley, 2010) and to minimize each instructor's loss of instructional time. The assessment had two parts: one with three open-ended questions that asked students to perform operations with algebraic rational expressions, and one with three open-ended questions that asked students to perform operations with numeric rational expressions. The instrument also included demographic questions, which asked for the student's gender, age, major, ACT-mathematics score, highest high school mathematics course taken, and race or ethnicity.

In creating this instrument, the researcher took care to design items that represented the important skills related to procedural knowledge of algebraic rational expressions. The researcher considered the most common errors reported in research when designing each assessment item (Demby, 1997; Otten, Males, & Figueras, n.d.; Ruhl, Balatti, & Belward, 2011). Two mathematicians also reviewed the assessment items to ensure that the content was valid for the purpose of measuring the participants' ability to perform operations with numeric and algebraic rational expressions. The researcher designed the rational number questions to closely mirror the corresponding algebraic items. Recognizing that the order in which items are presented can affect responses (Mitchell & Jolley, 2010), the items, regardless of which part they represented, were presented in a random order on the assessment. The random order was determined by a random number generator. The directions for the assessment instructed participants to perform the given operations, show all of their work, and write their answers in simplest terms.

The first set of problems (see Figure 1) presented one numeric rational expression item and one algebraic rational expression item, each with three terms and the operations of addition and subtraction. One of the terms in each item was a whole number. The denominators in this problem set, hereafter referred to as "Problem Set A," shared no common factors.

$\frac{1+6}{2}+\frac{9}{2+3}-3$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$

Figure 1. Problem Set A.

The second set of problems (see Figure 2) presented one numeric rational expression item and one algebraic rational expression item, each with two terms and the operation of division. Before the numerator and denominator are correctly factored, this problem set, hereafter referred to as "Problem Set B," presented common terms in the numerator and denominator. The presence of common terms has been found to be a strong visual cue that leads students to inappropriately cancel terms as they would factors (Otten, Males, & Figueras, n.d.). When the numerator and denominator in Problem Set B are correctly factored, two common factors can be eliminated.

$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

Figure 2. Problem Set B.

The third set of problems (see Figure 3) presented one numeric rational expression item and one algebraic rational expression item, each with two terms and the operation of addition. The denominators in this problem set, hereafter referred to as "Problem Set C," shared one common factor. Recognizing this fact would result in more simple calculations when finding the common denominator.

$\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Figure 3. Problem Set C.

The researcher pilot tested the assessment instrument with a convenience sample of high school students who had successfully completed Algebra II, the high school course in which algebraic rational expressions are taught. Although the pilot participants did not perform well on the pilot instrument, the pilot participants' work provided valuable information about their mathematical abilities. With regard to the instrument, the pilot confirmed that the design of each item, the length of the assessment, and the assessment's level of difficulty were appropriate for use with post-secondary students in a Precalculus course.

Phase Two

In the second phase of the study, participants were asked to take part in semi-structured, task-based interviews. Although in-person interviews are time-consuming, they offer opportunities for exploration and confirmation not available with quantitative methods (Johnson & Turner, 2003). Because the researcher was unknown to the participants and not connected to their classes, the researcher served as the interviewer. An interview protocol (see Appendix C), a set of cards with mathematical problems from the assessment instrument (see Appendix D), and the assessment from phase one were the essential instruments for phase two of data collection.

The interview protocol directed the order of activities and provided questions to be asked during the interview. The first set of questions in the interview protocol was designed to elicit responses from the participants while they sorted cards that demonstrated the relationships they saw between mathematical problems from part one and part two of the assessment. The laminated cards represented each question from the phase one assessment. The researcher asked questions in the second part of the protocol while the participant reviewed problems from the assessment which they had completed in phase one of the study. When it was appropriate, the researcher also used an assessment completed by a student volunteer who was further along in her mathematics education to display alternative solutions to the participants. The researcher designed questions in this section of the instrument to uncover the participants' mathematical thinking during each step of the problem. Creating the interview protocol provided an important opportunity for the researcher to anticipate the participants' actions and what might be learned from various situations. This forethought allowed the researcher to collect rich, informative data from the interview session.

The advantage of a semi-structured interview is that the researcher can probe or ask clarifying questions as needed to uncover the mathematical thinking of the participants. In so doing, the researcher became an instrument in the study (Creswell, 2007). The qualifications of the researcher included two years of experience collecting and analyzing data as an internal evaluator for a large, externally funded grant, and two years of coursework toward a Doctor of Philosophy degree in Mathematics Education.

The coursework completed by the researcher included both qualitative and quantitative research methods.

Procedures

This section will describe the steps that were taken to collect data for this study. Prior to data collection, the researcher received approval to conduct this study from the Institutional Review Board (see Appendix E). Permission to conduct this study was also obtained from the chair of the mathematics department.

Phase One

In the first phase of data collection, the researcher contacted the professor of record for each selected precalculus section by email, described the study, and requested 20 minutes of their instructional time during the second or third week of the semester. This study sought to understand the knowledge of numeric and algebraic rational expressions with which students enter college, and so it was important that the assessments were given before related material was reached in their precalculus courses, typically in chapter two of this university's approved text. Two professors declined to participate in the study, thus two additional sections of precalculus were randomly selected, and the researcher contacted those professors. After each professor agreed to allow students in his or her class to participate in the study, the researcher scheduled a date and time when the researcher would personally give the assessment.

Phase one data collection occurred during the second and third week of the fall semester. The researcher visited each participating class on the agreed dates to administer the assessment. Each episode of data collection began with an introduction and an explanation of both phases of the study. The researcher explained to the students what their participation in the phase one assessment required and how they may volunteer for participation in phase two of the study. Next, the researcher gave students an information sheet (see Appendix F), as required by the Institutional Review Board, and asked them to take a few minutes to read over the sheet. After they had a chance to read the information sheet, the researcher gave the students the chance to ask questions. After all questions were answered, the researcher distributed the assessments, provided instructions for how to complete it, and then allowed the students 20 minutes to complete the task. Participants were instructed that calculators should not be used to complete the assessment. At the end of 20 minutes, the researcher thanked the participants for taking part in the study and collected the assessments.

Phase Two

Within one week of giving the assessment, the researcher scored the written work of participants who consented to take part in phase two and separated the assessments into three groups: group one – four items correct, group two – one, two, or three items correct, and group three – no items correct. The researcher arranged each subgroup in a separate spreadsheet and assigned each record a number. The researcher selected all three participants in group one and then used a random number generator to select five participants each from groups two and three. The researcher contacted each participant via email to schedule the task-based interviews. The researcher made three attempts to contact a participant before moving to the next randomly selected participant.

All interviews were conducted between the sixth and seventh week of the fall semester. Prior to the interviews, the researcher reviewed the assessments of each selected participant and made note of both routine and interesting solutions that might have provided insights, if explored, during the interview. The scheduled interviews were held on campus in an empty office that provided privacy. The interview room (Figure 4) was approximately ten feet in length and eight feet in width with an entrance located on the north wall. A desk was positioned against the west wall. A small rectangular table was located parallel to the desk with a space of approximately two feet in between the desk and table. The researcher positioned the video camera and tripod on the desk against the west wall, and the document camera, audio recorder, and projector were located on the rectangular table in front of the desk. The researcher placed artwork and posters in the interview room to make it more inviting to participants. The researcher also left the lights on during the interviews to ensure the comfort of the participants.

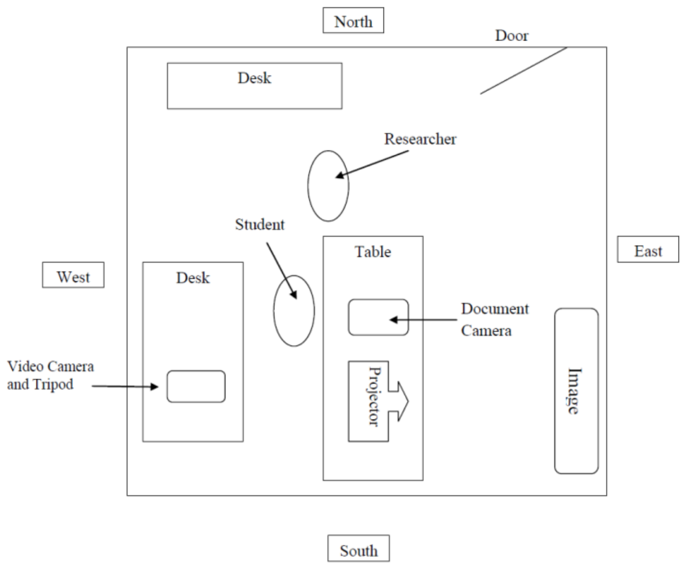

Figure 4. Diagram of interview room.

The researcher video-taped and audio-recorded the interviews to ensure collection of data. The purpose of the video was to capture gestures by the participants as they worked through the problems as well as to record how they constructed their solutions. Interview participants often perceive the level of anonymity as low (Johnson & Turner, 2003). To increase the participants' comfort level, the video-recorder was not facing the participant, but captured images that a document camera had projected onto a screen. The participants sat at the rectangular table and faced the east wall during the interview session. Participants used the document camera as they sorted cards and worked through the tasks. The video-recorder captured clear images of the subjects' work and the writing and erasures that occurred.

To establish rapport, the researcher engaged the participants in a conversation about school. When the participant appeared comfortable, the researcher started the audio and video recordings, gave a description of this phase of the study, and asked again for the interviewee's consent to take part in the interview. The researcher used questions from the interview protocol to initiate the discussion and guide the interview session. Following the interview protocol, the researcher first asked the participant if they noticed similarities in the problems that were on the assessment. The researcher then gave the participant a set of laminated cards showing one item from the assessment on each card. The researcher instructed the participant to sort the cards into meaningful groups. After the participant had formed groups, the researcher asked the participant what relationship he or she used to determine the arrangement of cards. The researcher asked the participant if it was possible to sort the cards again in a different way. The researcher continued to ask questions about the relationships perceived by the participant, and eventually ended by asking the participant directly if he or she thinks about rational numbers when solving algebraic rational expressions.

Next, the researcher presented the participant with their written assessment and asked them to look at a pre-selected pair of items. At this time, participants did not know if an item was marked correct or incorrect. Participants were asked to work through both items and explain their mathematical thinking. The researcher asked the participants to recall what they were thinking when solving the problem, to justify choices that they made, and to describe in what ways they thought the pair of problems might be related. On a few occasions, the researcher showed the participant a sample of written work completed by another student and asked for his or her thoughts about the work. After the participant had completed this task, the researcher thanked the participant for his or her contribution and stopped the video and audio recordings. Eight interviews were conducted in phase two of the study. The interview sessions ranged in length from 12 minutes to 42 minutes, but lasted 27 minutes on average.

After all interviews had been conducted, the researcher created a numbered list of all participants who volunteered to take part in the interviews. The researcher entered the names of the participants who completed interviews twice in the list. The researcher used a random number generator to randomly select the winner of a $50 gift card from Amazon.com. An impartial person observed the selection process in order to verify the random procedure and selection of the winner. The researcher notified the winner by email and arranged a time to deliver the gift card.

Data Analysis

The data analysis section will be organized by the study's research questions. Each question is listed below, followed by a description of the analysis that addressed each question.

(1) What, if any, relationship exists between undergraduate students' ability to simplify and perform operations with algebraic rational expressions and their ability to do the same with rational numbers?

The researcher used quantitative methods to address the first research question. The first analysis compared the participants' performance on the algebraic items to their performance on the numeric items. Using photocopies of each assessment collected, each item was scored as correct or incorrect. A correct item was assigned a value of one and an incorrect item was assigned a value of zero. Because the research question addresses both simplifying and performing operations with rational numbers and algebraic rational expressions, the assessment instrument specifically instructed participants to "write their answer in simplest terms" with an underline for emphasis. For that reason, responses that were not in simplest terms were coded as incorrect.

The researcher observed that participants in each of the five classes generally finished well before the 20-minute time-limit. Therefore, it was assumed that if a participant who voluntarily completed the assessment left one or two items blank, it was that they found the problem too difficult, and the item was scored as incorrect. Assessments from three subjects had three or more blank items. In this case it was assumed that the participant chose not to fully engage in the study, and the assessment was excluded from the analysis.

The researcher recorded the numbers of correct numeric and algebraic rational expression items in a spreadsheet and cross-tabulated to examine the four possible outcomes for each pair of items: 1) both numeric and algebraic items were correct, 2) both numeric and algebraic items were incorrect, 3) the numeric item was correct but the algebraic item was incorrect, and 4) the algebraic item was correct but the numeric item was incorrect. Because of to the dichotomous nature of the variables in the study, the researcher used the non-parametric McNemar's test for marginal homogeneity to determine if a difference did exist in the distribution of values across the numeric and algebraic items in each problem set. It follows that the phi coefficient is the appropriate measure for determining the intercorrelation between the responses of subjects (Sheskin, 2004). Data analysis to address the first research question included the examination of the percent of correct responses for each item, cross-tabulation tables, the McNemar's test statistic for evidence of a difference in the distribution of scores, and the inspection of the phi coefficient for evidence of a correlation between students' performance with numeric rational expressions and algebraic rational expressions.

(2) What strategies do undergraduate students use to simplify and perform operations with algebraic and numeric rational expressions?

Examining the patterns of strategies that participants use to solve problems and the errors that most frequently occur may provide teachers with insight into students' difficulties with rational numbers and algebraic rational expressions (Bejar, 1984; Radatz, 1980). To answer the second research question the researcher examined and coded the participants' written work on each assessment item and compiled detailed descriptions of the solution strategies and errors that occurred.

The researcher chose to use the hierarchical coding structure described by Ruhl, Belward, and Balatti (2011). Their perception was that the coding structures in the current literature lacked precise definitions of error categories and were unclear on the process through which categories were formed. Ruhl, Belward, and Balatti (2011) described their inductive approach to categorizing errors as similar to many other procedures found in the literature. They believed their method, however, produced a transparent procedure for the generation of error categories. The researcher coded the items in a problem set together. Following Ruhl, Belward, and Balatti's (2011) coding procedure, the first step was to identify the first process the student appeared to use in their solution. Within the categories that emerged, the researcher examined each item in detail. Every operator, term, or factor of a term that was found in the solution was coded as correct or incorrect within the context of the work by which it was preceded. When an error was encountered, the researcher made an inference about the location of the error. The researcher then reviewed the data a second time, making adjustments to the codes as necessary. The researcher recorded the location and a detailed description of the error in terms of the operation the student appeared to have used. The final step was carried out by grouping the errors but demonstrated similar operations and more broadly describing the categories that emerged. After completing the coding of all three problem sets, the researcher made another pass through the data and adjusted the category descriptions to ensure consistency throughout all three sets of problems.

In addition to identifying the error categories, the researcher analyzed the strategies and errors that occurred in the pairs of corresponding numeric and algebraic items. The researcher characterized the approach the participants used for each assessment item, and described the major errors that occurred. The researcher considered the patterns of strategies and errors that occurred in each problem, and then cross-analyzed the items within each problem set where the numeric item was correct but the algebraic item was incorrect.

(3) How are undergraduate students' strategies for simplifying and performing operations with algebraic rational expressions connected to their understanding of basic number properties?

The researcher used qualitative methods to analyze data collected from task-based interviews. Each interview was transcribed and evidence of participants' mathematical thinking was coded. The researcher, using the framework of connected representations as a guide, began with an open coding system to identify major categories of information (Creswell, 2007). The broad categories were dialogue related to errors, evidence of procedural and conceptual knowledge, and evidence of connection(s) made, or absent, between the algebraic rational expressions and rational numbers. Once the major categories were identified, the researcher used axial coding to review the data again. The researcher returned to the data and coded around the core categories looking for a description of causal conditions, situational factors, and consequences that affect the strategies identified (Creswell, 2007). The researcher reviewed each participant's data individually and developed an explanation for each case. Next, the researcher did a cross-case analysis and established a general explanation (Yin, 2009). The researcher looked for themes that emerged from the data that may identify the connections students make between algebraic procedures used to perform operations with rational expressions and the basic number properties.

Limitations of the Study

A limitation to this mixed-methods study is that the participants in the study did not receive an incentive for participation and thus may not have made a sincere effort to answer each problem fully or correctly. To mitigate this limitation, the researcher decided that students who left three or more of the six items blank had not fully engaged in the study, and these assessments were excluded from analysis. The researcher found it interesting that several students felt compelled to write notes on their assessment that indicated “it has been a while since I did fractions” or “I should have reviewed fractions like my teacher suggested.” Future replications of this study may further diminish this limitation by including a confidence or memory indicator for each question like that used in a study by Ruhl, Belward, and Balatti (2011).

Delimitations of the Study

Limitations related to the location of the study, the timing of the study, the design of the assessment instrument, and scoring of the assessments were under the control of the researcher. The Southeastern public university used as the setting for this study may not be generalizable to other populations of undergraduate students. Other institutions of higher education may find that students in STEM majors begin their study of mathematics in courses higher than precalculus. With regard to the timing of the study, although the approved textbook for this university’s precalculus course does not cover rational expressions until Chapter 2, it is possible that instructors of the precalculus sections who participated in this study may have reviewed this topic before the assessments were administered. It is also possible that the length of time between the administration of the assessment and student interviews may have impacted the students' responses. This amount of time was necessary, however, for the researcher to properly review and score the assessments, choose a sample and schedule student interviews, and accommodate the university's fall break.

It was desirable to have more replications of mathematics problems on the assessment instrument designed for this study. The assessment was limited to six items, however, to minimize the loss of instruction time to the instructors and thus increase the likelihood that they would allow the researcher access to their students. With regard to the scoring of assessments, it was necessary for the researcher to mark each item as correct or incorrect for statistical methods used in the first phase of analysis. A decision had to be made regarding the "correctness" of answers not expressed in simplest terms. The researcher and several colleagues that were informally consulted reported that answers not expressed in simplest terms are not correct, but neither are they considered incorrect. Typically they would give students partial credit for an answer that was not expressed in simplest terms. Because the research question for this study included simplification of rational expressions and the assessment instrument clearly instructed the students to write answers in simplest terms, the researcher decided to mark those items as incorrect. However, the researcher does acknowledge that this decision may have impacted the statistical analysis reported in this study.

Chapter Summary

Students' algebra abilities are an important factor affecting success in advanced college mathematics courses, and these students' persistence in degrees that require these courses (Baranchik & Cherkas, 2002). It is possible that helping students succeed in entry-level classes such as precalculus could improve the retention and graduation of STEM majors. To this end, it is important to understand the conceptual and procedural knowledge that students have when entering college. The skills of simplifying and performing operations with algebraic rational expressions are commonly recognized as an area that troubles many students (Dawkins, n.d.; Schechter, 2009; Scofield, 2003). The theory of connected representations tells us that meaning is developed for algebraic procedures by forming connections with basic number properties (Hiebert & Carpenter, 1992). This study sought to quantitatively identify if a correlation exists between students' abilities with algebraic and numeric rational expressions. The study also sought to qualitatively examine students' work and identify the most common strategies, whether correct or incorrect, that students used to simplify or perform operations with numeric and algebraic rational expressions. Furthermore, this study sought to qualitatively explore the students' mathematical thinking to reveal what connections students made between the algebraic procedures of simplifying, adding, subtracting, multiplying, and dividing and the basic properties of numbers. The next chapter will describe the data collected during the study, and discuss how the analysis and findings answer the research questions driving this study.

CHAPTER IV: RESULTS

Introduction

Algebra is considered a gateway to higher mathematics (Brown & Quinn, 2007; President's Council of Advisors on Science and Technology, 2012), and thus to many STEM degrees. Considering that even those students pursuing a technical vocation will need to know some algebra (Herscovics & Linchevski, 1994), it is important for educators to understand the difficulties that students have with this area of mathematics. Warren (2003) suggested that students' difficulties with algebra are dependent on their past experiences with arithmetic. Brown and Quinn (2007) agreed that competency with arithmetic supports student success in algebra. They specifically noted fraction concepts as the basis for many of the constructs found in the elementary algebra curriculum. Exploring students' understanding of numeric and algebraic rational expressions may help educators design interventions and instruction that will support students' learning of algebra.

This study used statistical analysis of participants' scores solving similar numeric and algebraic rational expressions to establish the degree to which participants' ability with algebraic rational expressions may be connected to their arithmetic abilities. Examination of participants' work is another means for measuring their understanding of numeric and algebraic rational expressions. In this study, the researcher examined patterns in the errors made by participants and compared their work with algebraic problems and to their work with similar arithmetic problems. Finally, task-based interviews were conducted to further investigate the participants thinking about numeric and algebraic rational expressions. The results of these analyses are presented in the following sections and organized in the following categories of analysis: Analysis of Score Distributions, Analysis of Written Work, and Analysis of Task-Based Interviews.

Analysis of Score Distributions

A contingency table was created for each pair of items showing the number of correct and incorrect numeric and algebraic items. Problem Set A required the participant to add three terms with no common factors. The percent of participants who correctly answered the numeric item was 48.6% compared to 6.5% who correctly answered the algebraic version of this item (see Table 2). Overall, only 3.7% of the participants correctly answered both items. McNemar's test for marginal homogeneity indicated that the distributions of different values across the numeric and algebraic problems were significantly different ($X^2\ (1, N = 107) = 37.961, p < .001$). The intercorrelation for this pair of problems (φ = 0.045) was less than the lower bound of 0.10 for which a small effect size should be recognized (Sheskin, 2004).

Table 2

Comparison of Participant Responses to Problem Set A

Numeric	Algebraic		Total
	Incorrect	Correct	
Incorrect	52	3	55
Correct	48	4	52
Total	100	7	107

Problem Set B required the participant to divide terms with two common factors. The percent of participants who correctly answered the numeric item was 37.4% compared to 6.5% who correctly answered the algebraic version of this item (see Table 3). Overall, 5.6% of the participants correctly answered both items. McNemar's test for marginal homogeneity indicated that the distributions of different values across the numeric and algebraic problems were significantly different ($X^2\ (1, N = 107) = 29.257, p < .001$). The intercorrelation for this problem (φ = 0.264) indicated a small effect size (Sheskin, 2004).

Table 3

Comparison of Participant Responses to Problem Set B

	Algebraic		
Numeric	Incorrect	Correct	Total
Incorrect	66	1	67
Correct	34	6	40
Total	100	7	107

Problem Set C required the participant to add two terms with only one common factor. The percent of participants who correctly answered the numeric item was 41.1% compared to 5.6% who correctly answered the algebraic version of this item (see Table 4). Overall, 2.8% of the participants correctly answered both items. McNemar's test for marginal homogeneity indicated that the distributions of different values across the numeric and algebraic problems were significantly different (X^2 $(1, N = 107) = 31.114, p < .001$). The intercorrelation for this problem (φ = .044) was less than the lower bound of 0.10 for which a small effect size should be recognized (Sheskin, 2004).

Table 4

Comparison of Participant Responses to Problem Set C

	Algebraic		
Numeric	Incorrect	Correct	Total
Incorrect	60	3	63
Correct	41	3	44
Total	101	6	107

These results indicate that the participants exhibited statistically different levels of ability with the numeric and algebraic assessment items. Furthermore, the division problem set showed a small correlation between participants' ability with the numeric and algebraic assessment items. The next section will qualitatively examine the written work provided by the participants and describe the strategies participants chose to solve the problems and the error patterns that emerged.

Analysis of Written Work

The written work was coded in terms of both specific errors and general strategies that were observed in the participants' solutions. The following sections, organized by problem sets, begin with a description of the most common errors observed in the participants' work and the frequency with which they occurred. Next, a description of the procedures used by the participants to solve the problems and a comparison of those strategies in the context of arithmetic and algebraic items is presented. Finally, each section concludes with the presentation of interesting or unusual observations. From this point forward, error categories are presented in italic text, and strategies or procedures used to perform operations with the rational numbers and expressions are enclosed in quotation marks.

Problem Set A

Problem Set A consisted of one numeric item and one algebraic item (see Figure 5). Each item had two fraction terms and one whole number term and used the operations of addition and subtraction. The denominators in each of the fraction terms did not share any common factors. Of the 107 students who participated in the study, 52 correctly answered the numeric item in Problem Set A and 7 correctly answered the algebraic item.

$\frac{1+6}{2}+\frac{9}{2+3}-3$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$

Figure 5. Problem Set A.

Error frequencies. Identifying the first process the participants appeared to use in their solutions led to the core categories of "Found Common Denominator" and "Did Not Find Common Denominator." In solutions of the numeric item of Problem Set A, 83 participants showed evidence of finding a common denominator. In solutions of the algebraic item of Problem Set A, 56 participants showed evidence of finding a common denominator.

Problem Set A numeric item. A total of 79 errors were coded in the participants' work for the numeric item of Problem Set A. The detailed log where the location and inference regarding the cause of each error was recorded is shown in Appendix G, and the grouping of the errors is shown in Appendix H. The grouping process led to the formation of four major error categories.

The *arithmetic error* occurred when participants incorrectly performed an arithmetic operation. Of the 81 errors found in the numeric item of Problem Set A, 21 were attributed to this error, most often incorrect addition or subtraction.

On 18 occasions, participants made mistakes that were categorized as *equivalent fraction errors*. Eleven of the participants either multiplied or divided by an incorrect factor to find a different form of a term. This category also described instances where in an attempt to find an equivalent fraction with a common denominator, the participant added a factor to a numerator, replaced the numerator with the new factor, or failed to change the numerator at all as in this example: $\frac{8}{5}-\frac{3}{1}$ became $\frac{8}{5}-\frac{3}{5}$. Examples of these methods of finding equivalent fractions are shown in Table 5. The errors are shown in bold print.

Table 5

Methods of Finding Equivalent Fractions

Incorrect Equivalent Fraction Methods	Example
Multiplied or divided by the wrong factor	$\frac{1+6}{2}+\frac{9}{2+3}-3=\frac{(\mathbf{1}+\mathbf{3})(1+6)}{(\mathbf{1}+\mathbf{3})(2)}+\frac{9}{2+3}-3$
Added factors to numerators	$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}=\frac{\mathbf{12}}{10}+\frac{\mathbf{11}}{10}-\frac{\mathbf{13}}{10}$
Replaced numerator with new factor	$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}=\frac{\mathbf{5}}{10}+\frac{\mathbf{2}}{10}-\frac{\mathbf{10}}{10}$
Did not change numerator	$\frac{8}{5}-\frac{3}{1}=\frac{\mathbf{8}}{5}-\frac{\mathbf{3}}{5}$

On 17 occasions, participants made *procedural errors*, of which the most common mistake was adding the numerators and denominators in place of finding a common denominator as in this example: $\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$ became $\frac{13}{8}$. On four occasions, participants used the procedure of cross-multiplication, twice with inversion of a term. For example, $\frac{1+6}{2}+\frac{9}{2+3}-3$ became $\frac{18}{35}-3$, with 18 coming from 2 multiplied by 9 and 35 coming from 7 multiplied by 5.

Nine of the 107 participants who started to work the numeric item in Problem Set A left an incomplete solution. Five of those did not continue past the simple mathematics performed in the numerator and denominator to arrive at $\frac{7}{2}+\frac{9}{5}-3$. These were classified as *persistence errors*.

The small number of errors that remained for the numeric item in Problem Set A did not warrant the creation of new error categories. Of the remaining mistakes, three participants dropped a term or denominator, two chose an incorrect common denominator, and three made errors related to cancellation of terms. Two participants committed errors that were unspecified, two participants omitted the problem, one used an incorrect decimal representation of a fraction, and one did not express the solution in simplest terms.

Problem Set A algebraic item. Altogether, the researcher coded 268 errors in the participants' work for the algebraic item of Problem Set A. Like the numeric item, the first process the participants appeared to use in their solutions led to the core categories of "Found Common Denominator" and "Did Not Find Common Denominator." The detailed log where the location and inference regarding the cause of each error was recorded is shown in Appendix I, and the grouping of the errors is shown in Appendix J. The grouping process led to the formation of ten major error categories.

On 42 occasions, participants made *procedural errors* where, similar to the numeric item, the most common mistake was adding the numerators and denominators in place of finding a common denominator. For this item, $\frac{x+2}{4}+\frac{6x}{x+1}-3$, the most common responses were $\frac{7x+2}{x+5}-\frac{3}{1}$, where the whole number was excluded from the addition operation, and $\frac{7x-1}{4x+2}$, which included the whole number. Only one participant used the procedure of cross-multiplication, as compared to four who did in the numeric item of Problem Set A. Three participants changed the expression to an equation and attempted to find a value for *x*.

The participants' solutions to this algebraic assessment item contained 31 *distribution errors*. Participants often failed to distribute a multiplier across terms inside parentheses, for example $3x + 1$ came from $3(x + 1)$ and $4x + 1$ came from $4(x + 1)$. Participants also made errors when multiplying two binomials, such as $(x + 1)(x + 2)$. Participants' responses to this particular example included $x^2 + x + 2$, $x^2 + 3x + 3$, and $x^2 + 2$, to name a few.

Participants made 30 *cancellation errors*, frequently cancelling an addend and a factor as in this example, $\frac{6x}{x+1}$ became $\frac{6}{1}$, and in another, $\frac{x+2}{4}$ became $\frac{x+1}{2}$ or $\frac{x}{2}$. The researcher observed inconsistent results of cancellation in the participants' work. The result of cancellation was sometimes a one, as in $\frac{x+2}{4}$ is equivalent to $\frac{x+1}{2}$, and sometimes a zero, as in $\frac{6x}{x+1}$ is equivalent to $\frac{6}{1}$. On 11 occasions participants left behind a zero, which the researcher called a *residual cancellation error*.

The participants mishandled the whole number term on 28 different occasions. Of these *unresolved whole number errors*, 16 participants left the whole number unchanged in the final solution and another 12 participants dropped the whole number from the solution altogether. The researcher observed 22 *equivalent fraction errors* in the participants' incorrect solutions. In nine of the solutions, participants rewrote the fraction with a common denominator but left the numerator unchanged. Participants multiplied the numerators by the wrong factor on nine other occasions. In five of these solutions, the participant multiplied the numerators by their own denominators, and on two occasions participants produced an equivalent fraction when they added a factor to the numerator.

Participants made 23 *operations with monomial errors* when they incorrectly combined like terms. Most notably, nine participants used the wrong operation, such as obtaining $2x$ from $x + 2$, and eight incorrectly added like terms. Participants made 17 *notation errors* that mainly consisted of instances where the participant incorrectly rewrote a term or operator. The incorrect solutions contained 13 *common denominator errors*. The denominators for the three terms in the expression were 4, $x + 1$, and 1 which should have produced a lowest common denominator of $4x + 4$. The incorrect common denominator used most frequently was $4x + 1$. Other examples included 4, $x + 5$, and $-4x + 8$. On nine occasions, participants began the process of solving the problem without completing the solution process. The researcher classified these errors as *persistence errors*.

The remaining 42 miscellaneous errors did not warrant the formation of additional categories. The researcher classified 11 errors as unspecified, meaning the cause of the error could not be determined and 13 participants made no attempt to solve this problem. The researcher observed eight dropped terms that did not reappear in the solution, two terms that moved from the denominator to the numerator, and one multiplication error. In four cases, the participant multiplied an expression by a number other than one, such as this example where it appeared the participant wanted to "clear" the fractions: $\frac{x+1}{2} + 6 - 3$ became $x + 1 + 12 - 6$. Only one participant committed an error described by Otten, Males, and Figueras (n.d.) as defractionalization, when they wrote $\frac{7x+2}{5+x} - 3$ as $\frac{7x}{x} + \frac{2}{5} - 3$. One participant changed the expression to an equation and made a sign error moving a term from one side of the equals sign to the other, and one participant gave a response to this algebraic item that was an answer not in simplest terms.

A very detailed analysis of the participants' written work produced the error categories above. The next section will look broadly at what strategies the participants used to approach a problem, and what was the most likely cause of failure to reach a correct solution.

Participants' strategies. The standard form solution to each of the items in Problem Set A would be a procedure to find equivalent fractions with common denominators, and then add the fractions. A variety of strategies were present in the participants' written work for the numeric and algebraic items in Problem Set A. Table 6 displays the strategies found in the participants' work for the numeric item, and indicates how often the strategy resulted in a correct or incorrect result.

Table 6

Number of Solution Strategies in Problem Set A Numeric Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Found Common Denominator	47	36
Added Across	0	5
Converted to Decimals	4	2
Cross Multiplied	0	3
Cancelled	0	1
Changed to Equation, Solved for *x*	1	0
Undetermined	0	6
Omitted	0	2
Total	52	55

Most participants recognized the need to find a common denominator before adding the fractions, although this resulted in a correct solution in only 58% of the responses that used this strategy. Within the "found common denominator" category for the numeric item, *arithmetic errors* (18) and *equivalent fraction errors* (12) accounted for most of the incorrect solutions. Four of the participants who handled the whole number term improperly either added it to the numerator of the preceding fraction, dropped it from the solution, or left it unresolved. Six of the responses did not have a distinguishable strategy. On one occasion the unusual strategy of "change to equation and solve for *x*" led to a correct solution. This participant set the original expression equal to "*x*," and thus found the correct value of "*x*," which in turn was the correct value of the expression.

Finding a common denominator was also the dominant strategy in solutions to the algebraic item of Problem Set A, although only in 55 responses as compared to 83 of the numeric solutions. The success rate of this strategy was much lower in the algebraic item – it led to a correct solution in only 13% of the solutions using this strategy. Table 7 displays the strategies found in the participants' work for the algebraic item, and indicates how often the strategy resulted in a correct or incorrect result.

Table 7

Number of Solution Strategies in Problem Set A Algebraic Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Find Common Denominator	7	48
Add Across	0	15
Multiply Across	0	1
Cancelling	0	6
Change to Equation, Solve for *x*	0	2
Clear Fractions	0	5
Invert and Cross-Multiply	0	1
Polynomial Long Division	0	1
Undetermined	0	8
Omitted	0	13
Total	7	100

Within the "find common denominator" category for the algebraic item, *equivalent fraction errors* (15), *distribution errors* (14), and *cancellation errors* (8) accounted for most of the incorrect solutions. The researcher noted that more participants chose the "add across" strategy for the algebraic item than in the numeric item. Also, an increased number of omissions and indistinguishable strategies were present in the algebraic item of Problem Set A.

Comparing Tables 6 and 7, it is clear that a different pattern of strategies emerged in the participants' solutions to the numeric and algebraic items of Problem Set A. The next section contrasts each participant's solutions to the algebraic item and numeric item. In particular, participants who answered the numeric item correctly and the algebraic item incorrectly will be examined.

Comparison of Arithmetic and Algebra. Four participants correctly answered both the numeric and algebraic items of Problem Set A. Three participants answered the algebraic item correctly, but made an arithmetic mistake on the numeric item. The strategy of "find common denominator" was used to find the correct solution in all of the seven cases.

As mentioned above, fewer participants recognized the need to find a common denominator in the algebraic item of Problem Set A. Of the 47 who correctly answered the numeric item with the "find a common denominator" strategy, 27 unsuccessfully employed the strategy in the algebra context. A dominant error pattern was not revealed in these solutions. The responses contained a mixture of *cancellation errors*, *distribution errors, equivalent fraction errors, operations with monomials errors,* and miscellaneous errors. The researcher noted that 11 participants, who used the "find common denominator" strategy in the numeric item, whether correctly or incorrectly, abandoned it in favor of the "add across" strategy in the algebraic item.

Other Observations. Four participants used the "convert to decimals" strategy to correctly answer the numeric item. The same strategy could not be applied in the algebra context, and three of the participants seemed to falter with unrecognizable strategies. The remaining participant who correctly answered the numeric item with a strategy other than "find common denominator" did deploy it in the algebra context, although with several errors. The participant who correctly solved the numeric item with the strategy "change to equation, solve for *x*" reached that correct solution by setting the numeric expression equal to "*x*." When the participant applied this strategy to the algebraic item that already included "x" terms, the expression was set equal to "0," which led to an incorrect response.

The preceding section described the errors and strategies participants used to add and subtract numeric and algebraic fractions and whole numbers. Strategies and errors found in participants' work for Problem Set B are presented in the following section. Items in this problem set used the operation of division, which required participants to utilize a completely different strategy than the one required for Problem Set A.

Problem Set B

Problem Set B consisted of one numeric item and one algebraic item (see Figure 6), each with two terms and the operation of division. The numerators and denominators in the numeric and algebraic items shared two common factors when the second term was inverted. Of the 107 participants, 40 correctly answered the numeric item and 7 correctly answered the algebraic item.

$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

Figure 6. Problem Set B.

Error Frequencies. The errors that led to incorrect solutions of items in Problem Set B are presented in this section. Distinct core categories that characterized the process the participant first appeared to use to solve the items were not suitable for this problem set. Participants used a variety of processes to solve the items in Problem Set B, such as “inverted and multiplied,” “cross-multiplied,” “divided across,” “cancelled and divided,” “found common denominator,” and “changed to decimal.”

Problem Set B numeric item. The researcher coded 127 errors in the participants’ work for the numeric item of Problem Set B. The details of the location and inference regarding the cause of each error are available in Appendix K, and documentation of the grouping process can be seen in Appendix L. The grouping process led to seven major categories of errors.

The participants made 43 *procedural errors* in their solutions to the numeric item of Problem Set B. For example, $\frac{3}{4}$ was a common response to this item when participants used the strategy “divided across.” The 3 results from 9 ÷ 3, and the 4 results from 8 ÷ 2, although if read from left to right, the correct arithmetic operations would be 3 ÷ 9 and 2 ÷ 4. Two participants inverted the first term of the expression, as opposed to the second term, and two did not find the reciprocal of the second term at all. Two participants multiplied and one added across the numerators and denominators. Two participants who used the “cross-multiplied” strategy confused the numerator and denominator in the result. Another used the strategy of “cross-multiplied” after the participant had already applied the “inverted and multiplied” strategy. After finding the reciprocal, one participant changed the sign of the second term. Although it is not necessarily incorrect, the researcher noted that nine participants found a common denominator for this item. Four of those incorrectly kept the common denominator in the final solution.

Participants made *cancellation errors* on 17 different occasions, most often cancelling the 3 in the numerator with the 3 in the denominator of the first term. This error was also frequently repeated with the 9's in the second term. When cancelling a factor with an addend, twelve participants inconsistently left behind a one in the numerator and a zero in the denominator which resulted in the expression $\frac{1}{-1} \div \frac{1}{-1}$, or 1. The researcher classified these mistakes as *residual cancellation errors*.

Fourteen participants made *arithmetic errors* and ten gave an answer that was not in lowest terms, classified as a *simplification error*. Thirteen participants made *persistence errors* when they started and did not complete the problem. The researcher observed seven *equivalent fraction errors* related to multiplying or dividing the numerator by the wrong factor.

There was not a sufficient number of remaining errors to be grouped into categories. Two of the participants who used the strategy "changed to decimal" found the incorrect decimal representation of a fraction, and another found the incorrect fraction form of a decimal. One participant made a notation error, three participants did not attempt to answer this problem, and on eight occasions, the participants' error could not be classified with certainty.

Problem Set B algebraic item. The researcher coded 252 errors in the participants' work for the algebraic item of Problem Set B. The details of the location and inference regarding the cause of each error are available in Appendix M, and documentation of the grouping process can be seen in Appendix N. The grouping process led to the classification of seven categories of errors.

Participants made 77 *cancellation errors* in the algebraic item of Problem Set B. They most frequently cancelled the x and x^2 terms and arrived at $-1 \div -1 = 1$, much the same as in the numeric item. Participants on some occasions ignored the signs and arrived at $1 \div 1 = 1$. Participants made 56 *residual cancellation errors*, leaving behind a zero as in the example given.

The participants used 32 strategies that did not produce a correct solution. These *procedural errors* included dividing or multiplying across the numerators and denominators, changing the sign of the second term, finding and keeping common denominators, and changing the expression to an equation that could be solved for "*x*." Participants also used variations of the "inverted and multiplied" strategy when they inverted the first term, changed the operation to multiplication but failed to invert, or inverted without changing the operation. Finally, one participant attempted to take the square root of the second term.

Of the 252 errors, 17 were *distribution errors*, related to the multiplication of monomials and binomials, and 16 were *simplification errors*. On eight occasions, the participants who used the incorrect strategy "divided across" confused the divisor and dividend and found $x^2 \div x$ instead of $x \div x^2$ if the expression is read from left to right. These errors were classified as *division conceptual errors*. The last major category of errors was *operations with monomials errors.* These six errors were related to combining terms using addition, subtraction, multiplication, or division. For example, a participant found $x^2 + x = 2x$ and another found $\frac{x^2}{x^2} = 0$.

The remaining 12 miscellaneous errors included four persistence errors, two equivalent fraction errors, and two notation errors. One participant made an error in an attempt at polynomial long division, one defractionalized an algebraic rational expression, one made a square root error, and one moved a number from one denominator to the other. The researcher also observed 15 errors that could not be classified and 13 participants who made no attempt to solve this problem.

This section described the errors participants made in solving the items in Problem Set B. The participants made fewer procedural errors in the algebraic item than in the numeric item. The next section will examine the ways in which participants approached the items, which strategies led to correct results, and which strategies led to incorrect solutions.

Participants' strategies. More than one procedure could lead to a correct solution to the division items in Problem Set B. The researcher observed a variety of strategies in the participants' written work. Table 8 displays the strategies found in the participants' work for the numeric item, and indicates the associated result.

Table 8

Number of Solution Strategies in Problem Set B Numeric Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Inverted and Multiplied	33	21
Found Common Denominator	2	8
Cross-Multiplied	4	4
Divided Across	0	8
Cancelled and Divided	0	5
Changed to Decimals	1	4
Multiplied Across	0	2
Multiplied Numerator by its own Denominator	0	1
Undetermined	0	11
Omitted	0	3
Total	40	67

Of the 107 participants, 77 chose a strategy that could have led to a correct solution, such as “inverted and multiplied,” “found common denominator,” “cross-multiplied,” and “changed to decimals.” These strategies resulted in a correct solution, however, in only 40 of the cases. “Inverted and multiplied” represented 21 of these cases of a correct strategy that led to an incorrect result. Within this category, five participants made arithmetic errors, two made cancellation errors, two changed the sign of the second term, and two inverted the first term. One participant made an equivalent fraction error, another followed “inverted and multiplied” with another strategy “cross-multiplied,” and one did not finish the problem. Finally, seven participants used the strategy of “inverted and multiplied,” but did not simplify the result.

Two participants who used the “cross-multiplied” strategy reversed the numerator and denominator in the solution, and two did not simplify the result. Of those participants who found a common denominator, three followed the procedure with the action of dividing across, one added across, and one multiplied across without inverting the second term. Two of these participants did not finish the problem. One participant correctly applied the "inverted and multiplied" procedure after finding a common denominator, but made a simplification error.

The strategy of "changed to decimals" was a viable choice of procedure for this numeric item, and would not have resulted in calculations too difficult to do by hand. Two participants who used this strategy made an error converting from a fraction representation to a decimal, or vice versa, and another left the answer in an unacceptable form with a decimal in the numerator.

The researcher observed that the strategies found in this numeric item were similar to those in the algebraic item, with the addition of the procedure "cancelled and divided" and the exclusion of "changed to decimals." The strategies for the algebraic item that could have led to a correct solution were "inverted and multiplied," "cross-multiplied," and "found common denominator." Table 9 shows the strategies used for the algebraic item of Problem Set B contrasted by the end result.

Table 9

Number of Solution Strategies in Problem Set B Algebraic Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Inverted and Multiplied	7	44
Cancelled and Divided	0	16
Found Common Denominator	0	5
Cross-Multiplied	0	3
Divided Across	0	6
Multiplied Across	0	3
Multiplied Numerator by its own Denominator	0	1
Took Square Root	0	1
Undetermined	0	8
Omitted	0	13
Total	7	100

Incorrect solutions that resulted from the "inverted and multiplied" strategy were attributed to cancellation errors in 21 cases. Twelve participants did not simplify the answer, two changed the sign of the second term, two committed distribution errors, and one did not finish the problem. Within this strategy, three participants found a common denominator after applying the "inverted and multiplied" procedure, and three errors could not be classified.

Two participants who "cross-multiplied" made cancellation errors, and one did not simplify the answer. Within the "found common denominator" strategy, two correctly followed this with the "inverted and multiplied" strategy, but made cancellation errors. One participant made a distribution error, and one did not finish the problem. Finally, one participant divided across after finding common denominators, which could have led to a correct solution if it had not also been followed with the “inverted and multiplied” strategy.

It can be seen in Table 9 that a large number of participants selected a correct strategy for the algebraic division item, but very few produced a correct result. The next section will contrast strategies used in the numeric and algebraic contexts. Special attention will be given to the discordant pairs of participants who had a correct numeric and an incorrect algebraic result.

Comparison of arithmetic and algebra. Six participants correctly answered the division numeric and algebraic items in Problem Set B. In each case, the participant used the “inverted and multiplied” strategy. One participant had a correct solution for the algebraic item, and an incorrect solution for the numeric item which was the result of a cancellation error. Thirty-four participants made an error in the algebraic context although they had correct solutions for the numeric item.

Twenty-eight participants correctly used the “inverted and multiplied” strategy to solve the numeric item but found an incorrect solution to the algebraic item. Of those participants, three switched to a different strategy in the algebraic context. One participant multiplied across and another incorrectly cancelled factors and addends and then divided across. One participant switched to the “found common denominator” strategy, correctly followed it with the “invert and multiply” procedure, and arrived at an incorrect solution due to a cancellation error.

Twenty-five participants repeated the “inverted and multiplied” strategy in the algebraic context. Of those participants, nine made cancellation errors, six did not simplify the solution, and two found a common denominator. One participant changed the sign of the second term, one made a distribution error, and another did not finish the problem. Three participants who repeated the "inverted and multiplied" strategy used an unspecified procedure, and two omitted the problem although they correctly answered the numeric division item.

Other observations. It is interesting to note that 13 participants omitted the algebraic division item in Problem Set B compared to only three who omitted the numeric division problem. Ten of those participants answered at least four other questions on the assessment. The remaining three felt compelled to leave a message explaining their omission of the problem. The messages were "sorry, I don't know how to do these" followed by a sad face, "do not remember division of fractions at all," and "IDK," meaning "I don't know."

The preceding section described the errors and strategies participants used to divide numeric and algebraic rational expressions. Participants often made errors in the execution of a correct choice of strategy. In the algebraic division item of Problem Set B, more incorrect solutions were related to cancellation errors than any other category of errors. The last section describes the strategies and errors found in participants' work for Problem Set C, another problem set with the operation of addition.

Problem Set C

The final pair of problems, Problem Set C (see figure 7) consisted of one numeric and one algebraic item that again used the operation of addition. In this problem set, however, the denominators were quadratic expressions that when factored shared a common factor. Forty-four participants gave a correct response to the numeric item in Problem Set C and six gave a correct response to the algebraic item. The following section will examine the errors that led to the participants' incorrect solutions.

$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$	$\frac{1}{x^2 - x - 2} + \frac{x}{x^2 - 7x + 10}$

Figure 7. Problem Set C.

Error frequencies. As in Problem Set A, the first process the participant used in their solution could be characterized as "Found common denominator" or "Did not find common denominator." Seventy-three solutions for the numeric item of Problem Set C were characterized as using the strategy "found common denominator." For the algebraic item, 33 solutions were similarly characterized as using the procedure "found common denominator."

Problem Set C numeric item. Participants' solutions to the numeric item in Problem Set C contained 98 errors. The detailed log where the location and description of the error were recorded can be seen in Appendix O and the grouping of errors is shown in Appendix P. The numeric item for this problem set produced six categories of errors.

The largest category of errors for this item was *procedural errors*. Participants made 32 errors when they chose a strategy that would not lead to a correct solution. Twenty-eight of these errors can be attributed to the strategy of "added across." Two participants cross-multiplied, and another inverted the second term. Finally, one participant used the largest denominator as the common denominator in the solution.

Participants made 13 *arithmetic errors* and 13 *equivalent fraction errors* in this item, most often multiplying or dividing a numerator or denominator by the wrong factor. Fifteen participants made *persistence errors* when they started and did not finish this numeric item, seven finished with a *simplification error,* and six participants made *notation errors*. The remaining errors were not grouped into categories. Participants made three cancellation errors and two incorrect denominator errors. In two solutions a participant presented an answer with a decimal in the numerator, which was coded as "improper form of a fraction," and another participant defractionalized a numeric rational expression. One participant interpreted the denominators as decimals rather than factors multiplied together, and one participant gave a decimal answer that was rounded. One error was coded as "unspecified" and only one participant omitted this numeric item.

Problem Set C algebraic item. Participants made 234 errors in the solutions to the algebraic item in Problem Set C. The detailed log where the location and description of the errors were recorded can be seen in Appendix Q and the grouping of errors is shown in Appendix R. Seven major categories of errors were found in the participants' work for the algebraic item of Problem Set C.

Procedural errors was the largest category of errors for this algebraic item with 83 coded errors. Almost 80% of these occurred when participants used the strategy "added across." The small number of other strategic errors occurred when participants multiplied across, changed the expression to an equation, cross-multiplied, cross-*added*, inverted the second term, or attempted polynomial long division.

Participants made 35 *operations with monomials errors* when handling the increased number of variable terms in this item. For example, fourteen participants who used the incorrect strategy of “added across” found the sum of 1 and x in the numerator to be x. Nine participants who added across the denominators found $x^2 + x^2$ in the denominator to be $4x^2$, x^2, or x^4.

The participants made 27 *cancellation errors* in this algebraic item. Most of the errors involved cancelling a factor with an addend, but two new types of cancellation errors appeared in the participants’ work for this item. Participants cancelled the common factor shared by the denominators like this, $\frac{1}{(x+1)\cancel{(x-2)}} + \frac{x}{(x-5)\cancel{(x-2)}}$, on seven occasions. The researcher also observed five instances of a cancelled factor in the numerator and denominator after an equivalent fraction was found, and before the numerators were combined. In one unusual case, the factors reappeared in the final solution like this: $\frac{1\cancel{(x-5)}}{(x-2)(x+1)\cancel{(x-5)}} + \frac{x\cancel{(x+1)}}{(x-5)(x-2)\cancel{(x+1)}} = \frac{x}{(x-2)(x+1)(x-5)}$. Only eight *residual cancellation errors* were recorded for this algebraic item.

Participants incorrectly factored expressions or distributed multipliers, which led to 16 *distribution errors*. Nine participants made errors related to the *properties of operations*. Participants moved terms from the denominators to numerators without reason, as in this example, $\frac{2x}{2x^2-8x+8}$ became $2x - 8x + 8$, or removed a factor from the denominator without changing the numerator, as in this example, $\frac{1+x}{2x^2-8x-8}$ became

$\frac{1+x}{x^2-4x-4}$. Two participants used the property $a \div 1 = a$, incorrectly and rewrote the first term as $x^2 - x - 2$.

There were nine recorded instances of participants who started and did not finish the problem, which was classified as *persistence errors*. The remaining 24 miscellaneous errors were found in participants' solutions that dropped terms, did not simplify the answer, found incorrect equivalent fractions, and made arithmetic mistakes. Of the 107 participants, 18 omitted this problem and five made errors that could not be definitively classified.

The participants' written work was also coded in terms of the strategy with which they approached the problem, and the related outcome. The next section will examine these strategies separately in the numeric and algebraic context.

Participants' strategies. Problem Set C used the operation of addition, and so the natural choice of process in both contexts would be to find equivalent fractions with a common denominator, and then add the fractions. Some participants recognized this as the appropriate strategy and others did not. A summary of the participants' strategies and the outcomes is given in Table 10.

Table 10

Number of Solution Strategies in Problem Set C Numeric Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Found common denominator	44	29
Added across	0	14
Added numerators, multiplied denominators	0	1
Cross-multiplied	0	2
Omitted	0	1
Unspecified	0	16
Total	44	63

All of the 44 correct responses were from participants who found a common denominator. This same strategy was used incorrectly by 29 participants who made a variety of missteps that included arithmetic, equivalent fraction, simplification, common denominator, and cancellation errors. Fourteen participants added across the numerators and denominators, but made no other mistakes. In sixteen of the solutions, the researcher could not clearly identify the process that the participant followed.

In the algebraic item of Problem Set C, more participants abandoned the “found common denominator” strategy in favor of the “added across” method as shown in Table 11. Of the 33 participants who retained the “found common denominator” strategy for the algebraic item, only six had a correct result. The incorrect results were most often attributed to distribution errors and answers that were not simplified.

Table 11

Number of Solution Strategies in Problem Set C Algebraic Item Contrasted By Result

	Solution Result	
Strategy	Correct	Incorrect
Found common denominator	6	27
Added across	0	30
Cleared fractions	0	4
Cancelled as many terms as possible	0	5
Changed to an equation, solved for *x*	0	2
Multiplied across	0	2
Inverted and cross-multiplied	0	1
Cross-added	0	1
Omitted	0	18
Undetermined	0	11
Total	6	101

It is clear from Tables 10 and 11 that the participants approached the numeric and algebraic items in different ways. The next section compares the strategies the participants used in the numeric context to those used in the algebraic context. Participants who answered the numeric item correctly and the algebraic item incorrectly are of special interest.

Comparison of arithmetic and algebra. Three participants correctly answered both of the assessment items in Problem Set C. Three participants correctly answered the algebraic item and incorrectly answered the numeric item. Arithmetic and distribution errors can explain this unusual occurrence. Correct numeric solutions followed by incorrect algebraic solutions occurred in 41 cases and were further explored.

All of these 41 participants found a common denominator and correctly answered the numeric item. Seventeen of those approached the algebraic item with the same strategy, but made cancellation, distribution, or simplification errors. Ten participants used the incorrect strategy of "added across." These categories account for most of the participants who answered the numeric item correctly and the algebraic item incorrectly. In three cases, participants used strategies that could not be identified, and four participants omitted the problem.

Other observations. A common denominator could be found for the algebraic item in Problem Set C with or without factoring the denominators. The advantage of factoring was identification of the shared factor and reduced difficulty of the computations. Participants who did or did not factor before they found a common denominator were evenly split. The six participants who correctly answered this item, however, all chose to factor the denominators before finding a common denominator. The participants who did not factor made distribution errors and operations with monomials errors. Several executed their strategy correctly, but did not simplify their answer. It is not immediately obvious that $\frac{x^3-9x+10}{x^4-8x^3+15x^2+4x-20}$ can be simplified.

The preceding section described the strategies the participants used to approach these numeric and algebraic rational expressions, and the errors that they made. Examining the participants' written work, however, will not reveal why the participants made these mistakes. To answer that question, the researcher conducted task-based interviews to uncover the participants' thinking while working these problems, and what connections they made between the numeric and algebraic items, if any. The results of the interviews are presented in the next section.

Analysis of Task-Based Interviews

The researcher conducted task-based interviews to investigate the connections participants made between simplifying and performing operations with algebraic rational expressions and their understanding of basic number properties. The interview protocol instrument was divided into two distinct parts (see Appendix C). In the first part, the researcher asked questions to identify what details about the items the participants could recall from the assessment completed a few weeks earlier. Next, to discover which, if any, relationships between algebraic and numeric problems were significant to the participants, the researcher asked them to sort the problems into groups. The researcher ended part one of each interview with a question to determine if participants thought about fractions and other numeric problems when working algebraic problems.

In the second part of the task-based interviews, the researcher asked participants to review the written work from their assessments and explain their mathematical thinking. The assessments were not marked with a score, so the participants were unaware if the work was correct or incorrect. The researcher conducted interviews with eight participants and used pseudonyms in place of the participants' real names throughout the rest of the study. The sample of participants for the task-based interview was chosen from groups classified by the number of assessment items they correctly answered. Group 1 consisted of Emma, Isabella, and Sophia who each had four correct answers. Group 2 included Ethan, Liam, Mason, and Noah who each had one, two, or three correct answers. Finally, Jacob, who had no correct answers, was the only participant in Group 3. Each participant was considered an individual case, and the researcher developed case descriptions based on the relevant interview dialogue and observations. In the following paragraphs the cases are arranged by groups, although the analysis was conducted across the participants. Individual case descriptions and case analyses are given first, followed by a cross-case analysis and a general explanation.

Group 1

Emma. Emma was a white female who was 20 years of age and majored in biology. She reported that the highest level of mathematics she had completed in secondary school was calculus. In the paragraphs that follow, a detailed description of Emma's case is presented followed by an analysis of her case.

Case description. Prior to the interview, the researcher reviewed Emma's assessment (see Appendix S) and noted that she gave incorrect answers for two addition items, both in Problem Set C. In the numeric item, Emma made an error that Otten, Males, and Figueras (n.d.) called defractionalization, or breaking apart a fraction. In the algebraic item, she made an arithmetic error. The researcher selected both of these items for Emma to evaluate during her interview. The researcher observed that Emma, like many other participants, had found a common denominator for the algebraic two-term addition item by multiplying the two polynomials together, which was an inefficient method. The researcher selected an example of a more efficient solution to this problem that used factoring before finding a common denominator for Emma to review. The researcher also noted that Emma had done much of the work for the algebraic division problem in her head, so the researcher selected this problem to be reviewed during her interview.

The researcher began the interview by asking Emma what she remembered about the assessment she took in class a few weeks earlier. Emma responded, "I don't really remember that much," so the researcher moved on to the sorting activity. Emma quickly sorted the cards and arranged them in groups of numeric items and algebraic items. Emma did not immediately see another way to sort the cards, but after some thought she separated the problems by operation. The last question for Emma in the first part of the interview was "When you're working problems like this that have variables, do you think about how you would work a problem with just numbers?" Emma answered, "Yeah, alright, 'cause [*sic*] it would be similar you would just have an x in place of a number."

The researcher began the second part of the interview by asking Emma to review her solution to the algebraic division problem and describe how she had approached it.

> Emma: Ok, um, well, looks like I did $\frac{x}{x-1}$ and multiplied it by $(x-1)(x+1)$, because that is x^2-1 factored out, over x^2, and I multiplied it by the reciprocal because it's dividing. And then it looks like I factored out an x because of the x times the x^2 right there.
>
> Researcher: Ok.
>
> Emma: And then factored out an $x-1$ and an $x-1$ right here to get me $\frac{x+1}{x}$ is what it looks like.

Emma's explanation demonstrated that she understood very well how to solve the problem.

The researcher then asked Emma to describe her solution to the numeric addition problem (see Figure 8) where she had defractionalized the terms $\frac{1}{4 \cdot 5}$ into $\frac{1}{4} \cdot \frac{1}{5}$, and $\frac{3}{5 \cdot 7}$ into $\frac{3}{5} \cdot \frac{3}{7}$. Emma replied, “Um, I don’t know. I think I broke it apart, ‘cause [*sic*], I don’t know. I’m not always the best with fractions, so I did break it down more than usually [*sic*].” Emma continued to describe the remaining steps in her solution, which were correct, and then the researcher asked her to look at the corresponding algebraic problem. Emma indicated that she “cross-multiplied” to find the common denominator in the algebraic problem (see Figure 8).

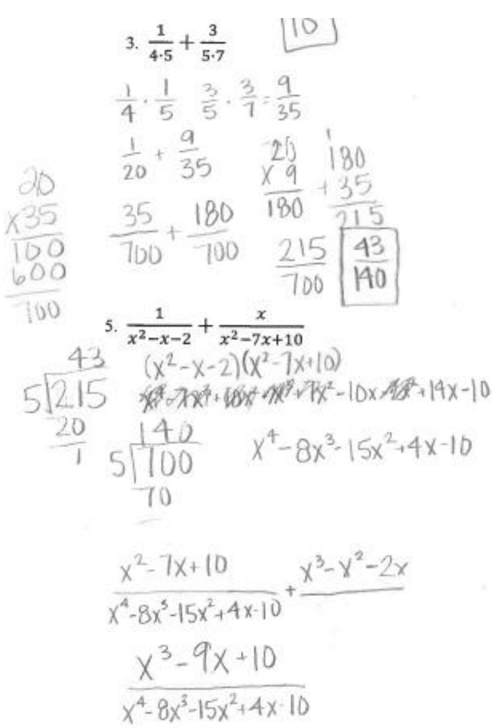

Figure 8. Emma’s solutions to Problem Set C.

The researcher drew Emma's attention back to the numeric problem and asked if she had applied the cross-multiply rule in that problem. Emma replied, "I might of," so the researcher pointed to the three-term addition problem which Emma answered correctly (see Figure 9) and asked if she had used cross-multiplication in that problem and she said, "I think so."

> Researcher: How do you use the cross multiply rule when you have three things?"
>
> Emma: Um, I don't know. I just put, I automatically put, um, 3 as 3 over (pause) or 30 over 10 in my head so that way it was already the same.
>
> Researcher: OK. What made you want to do that?
>
> Emma: Um, because when I used the cross multiplying I got, um, like 35 over 10 and 18 over 10, so if I did 3 over 10 I wouldn't have to multiply that to anything, or 30 over 10 it would already be.

Emma's response suggested that she was finding an equivalent fraction with the same denominator as the product of cross-multiplying the first two terms.

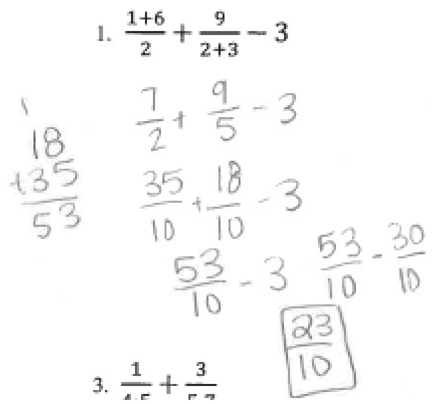

Figure 9. Emma’s solution to the numeric item in Problem Set A.

Finally, the researcher displayed a sample of a more efficient solution to the algebraic item from Problem Set C (see Figure10) and asked Emma what she thought about it. Emma recognized that the individual had multiplied each numerator by the term missing in the denominator so that each term could have a common denominator, and indicated she thought it looked right, and then paused for quite some time. After some thought, Emma said “I think I messed this one up [numeric item in Problem Set C] ‘cause I think it should be $\frac{1}{5}\cdot\frac{3}{7}$.” She recognized that she had made a mistake when she defractionalized the terms, but was confident that the rest of her steps were correct.

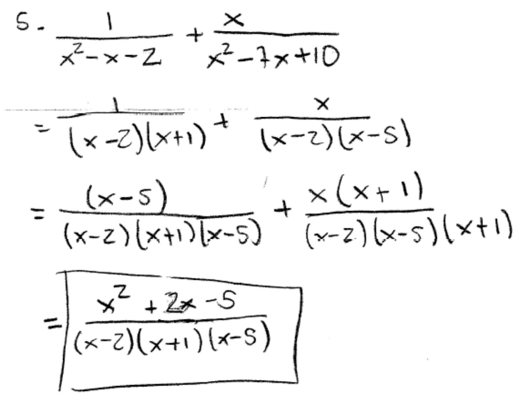

Figure 10. Efficient solution to the algebraic item in Problem Set C.

The last question in the interview was, “Do you work them [numeric problems] the same way as you do the problems with variables?” Emma answered, “Um, I feel like I do? I think I take the same steps to try to do them.” The researcher then followed with, “Do these two types of problems have all the same rules? The same procedures to follow?” Emma’s response follows.

> Emma: Yeah, I believe so. I think they have the same set of rules because, um, just *x* represents like where a number could be, or, um, a missing number, so it’s the same rules. It’s just in a variable . . .

Case analysis. Emma’s first instinct was to separate the assessment items into groups of numeric and algebraic problems, although she believed that both groups followed the same rules and procedures. At the end of the interview, Emma provided further support for this belief when she described a variable *x* as representing a missing number. It was no surprise then, that Emma consistently used the same procedures to answer the numeric and algebraic problems.

Emma gave a clear explanation of her solution to the algebraic division problem that used the invert and multiply procedure. She factored the quadratic term into two binomials and cancelled factors in the numerator and denominator before finishing with the operation of multiplication. Emma's choice of words "factored out" suggested that she understands that factors, and not addends, can be cancelled in an algebraic rational expression.

Emma had procedural knowledge of finding a common denominator through the use of equivalent fractions and cross-multiplication, but was unsure of how to apply the latter method in a problem with more than two terms. In that case, Emma combined the two procedures to solve the problem. This action could be characterized as evidence of a limit to Emma's procedural knowledge, although it also speaks to her procedural flexibility. Emma also applied cross-multiplication to the algebraic item in Problem Set C. Emma inspected a more efficient method of solving this item and easily recognized the steps that were taken, yet she did not give an indication of being familiar with the method. This seemed unusual, given that Emma completed a high level of mathematics in her secondary education.

The inspection of the more efficient method of solving the algebraic item in Problem Set C led Emma to reconsider her solution to the numeric item in the same problem set. In this item, Emma made a conceptual error described as defractionalization. She attributed this step to a desire to "break it down" more than usual because of her perceived difficulties with fractions. Emma took this step, although she was not completely confident that it should be made, to transform the problem into a format that she was more comfortable with.

Isabella. Isabella was an African American nineteen-year old female who majored in biology. She reported that calculus was the highest level of mathematics she had completed in her secondary education. Isabella came to the interview with a great attitude toward mathematics. She said she loved mathematics, and loved her precalculus class. A detailed description of Isabella's interview and an analysis of her case are presented in the paragraphs that follow.

Case description. The researcher reviewed Isabella's assessment (see Appendix T) before she arrived for the interview. Isabella had made a distribution error on the algebraic item from Problem Set A, and her solution to the algebraic item in Problem Set C was correct, other than it was not simplified. Like Emma, she had used an inefficient method for finding the common denominator to this problem. The researcher selected both of these items for Isabella's interview as well as an example of a more efficient solution to the item from Problem Set C that used factoring before finding a common as denominator. The researcher noted that in Isabella's solution to the three-term, algebraic addition item, she wrote the rational terms in a vertical arrangement, and so planned to inquire about this problem during the interview.

The researcher began by asking Isabella if she remembered any similarities between the problems on the assessment she took, and she immediately said that she did. The sorting cards were nearby, so she asked if she could use those as she described what she remembered. Isabella remembered that “For the most part, you had to find a common denominator,” so she moved the addition problems to one side, and placed the division problems together in a separate group. She also pointed out that some of the arithmetic problems required you to perform an operation before you could find a common denominator.

Isabella then arranged the cards by level of difficulty. She perceived the algebraic division problem as easiest and the algebraic item from Problem Set C as the hardest because it had “lots of multiplication.” Isabella enjoyed the sorting activity and wanted to keep going. Her third sort of the items is described in the following conversation.

> Isabella: These are the ones that have *x* variables and such, *x* variables and polynomials, and these are the ones that just have numbers.
>
> Researcher: Ok, and how are those two groups different?
>
> Isabella: This one requires you to do something with polynomials and this one, it does require operations, but it’s just numbers.
>
> Researcher: And do you solve it a different way, or are they, do they follow the same rules?
>
> Isabella: Uh, I’m gonna have to think about that one. [pause] Each of these has very similar rules, but different. They require different operations.

Isabella stared at the cards and then exclaimed, “Wait! That’s another group, too.” Isabella’s last group was arranged in terms of operations, which she did not seem to recognize as being the same as her first arrangement. When asked what makes the division problems different from the addition problems, she gave the following response.

> Isabella: Addition and subtraction, you don't have to deal with, uh, all you do is find a common denominator, that's all you do. . . This one, division, you have to do an extra step, flipping the second problem and division to multiplication, and then solve it.

The researcher gave Isabella her assessment and some time to get reacquainted with her work, and then asked her to describe her solution to the numeric item in Problem Set A. Isabella said she first found a common denominator although a lot of the work was done in her head. She said she followed a method she learned in the fourth grade that she called the "stacking method." Isabella took a piece of paper and showed the researcher how the method would be used with this problem (see Figure 11). She first wrote the fractions vertically, and then drew a fraction bar to the right of each fraction. She found $2 \cdot 5 \cdot 1 = 10$ to be her common denominator, and wrote that beneath each fraction bar on the right hand side. She described her next step for finding equivalent fractions as a "proportion system."

> Isabella: Like in this one, 1 times a number makes 10, so whatever the number is, just multiply it times this one [draws an arrow to the three]. Ok, so 1 times 10 equals 10, so I took 10 times 3 which gave me 30. And since 5 times 2 makes 10,

I took 2 times 9 and it gave me 18. And then I took 2 times 5 equals 10, and 5 times 7 equals 35.

The researcher then observed as Isabella wrote the addition and subtraction operation symbols on the left hand side, and then she combined the numerators.

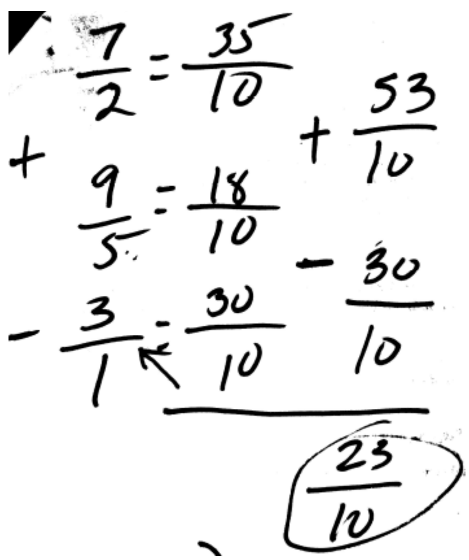

Figure 11. Isabella's stacking method.

Isabella pointed out that she had used the stacking method on the algebraic problem in the same problem set. She described the steps in her solution to this problem, and along the way became aware that she had made a distribution error. In this problem, Isabella also described the system she used to keep track of like terms (see Figure 12). Isabella used different shapes – circles, squares, and triangles – to identify terms of different degrees. She claimed that the process, which she learned in the eighth grade, helped her “not forget which terms are what.” Isabella looked at her solution of $\frac{x^2+39x+14}{4x+4}$ and said, “I felt like I wanted to do more with this, like it could have been a lot more, if I could, if I could factor this out, but it can’t because 1 times 14 or 2 times 7, I gave up on that.” The researcher asked, “But it looked like it needed that?” and she replied, “Just like one of those problems that seemed like it would.”

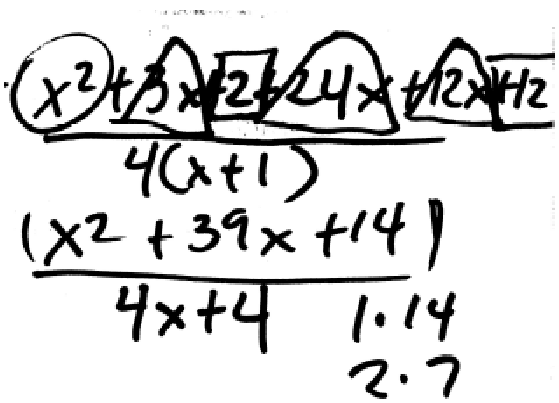

Figure 12. Isabella’s strategy for combining like terms.

The researcher then drew Isabella’s attention to the algebraic item in Problem Set C which she had earlier described as the one she perceived to be the hardest. Isabella’s solution to this problem was correct, other than the fact that it was not simplified. Isabella talked through her solution, and then the researcher displayed an example of a more efficient solution to this problem, one that used factoring to simplify the process of finding a common denominator. Isabella took time to study the work, and then responded enthusiastically, “It’s a very interesting way. . . This is a really good way. This would actually shorten the amount of work that would have to be done." She described the procedure as "very strategic." The researcher asked if she had ever seen an approach like that, and she indicated that this method was new to her.

The researcher ended the interview by asking Isabella if she thought numeric and algebraic problems were related in any way. The following is Isabella's response.

> Isabella: Yes, they follow the same operations, very similar operations, but numbers can be multiplied together and everything, but there's a limit to polynomials. Polynomials with the same, uh, same *x*, I can't remember the word, polynomials with the exact same exponent can only be added and subtracted, but any exponent, I mean any polynomial, could be multiplied and divided. It can.

Case analysis. Isabella recalled that most of the assessment items had required a common denominator. She also remembered which items she had perceived as easiest and hardest. Isabella first arranged the cards containing the assessment items based on which problems would, or would not, require a common denominator. To Isabella, this meant separating the items into groups based on the operations of addition and division. Although she later separated the cards into numeric and algebraic groups, Isabella appeared focused on the operations involved in a problem, which she connected to specific procedures. She associated the operation of addition with finding a common denominator, and the operation of division with "flipping the second problem" accompanied by changing division to multiplication. Isabella took some time to answer a question about the rules for numeric problems and algebraic problems. After some thought, she described the rules for both domains as “similar,” although she also indicated that they required different operations. Later in the interview, Isabella claimed numeric and algebraic problems required “very similar” operations, and differed in the extent to which terms could be combined.

An example of how Isabella applied similar, although slightly different, operations in a numeric and algebraic context may be seen in Isabella’s solutions to the division problems in Problem Set B. Her solution to the numeric item used the method of invert and multiply. She performed the operation of multiplication followed by the removal of common factors to simplify her answer. Isabella also used the procedure of invert and multiply in her solution to the algebraic division problem; however, she factored the quadratic expression and cancelled common factors in the numerator and denominator before completing the operation of multiplication. It is not clear why Isabella chose to approach these problems in slightly different ways.

Early in the interview, Isabella pointed out that some items required an operation to be done before a common denominator could be found. In fact, in a problem such as $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$, it is much easier to find a common denominator when the numbers are factored. This is even more true for the algebraic problem in Problem Set C. Isabella, like many participants, found the common denominator to be the product of the entire denominator of the first term and the entire denominator of the second term, not realizing that they shared a common factor. Isabella evaluated an alternative method for solving that problem, and embraced it right away. She described the procedure as being more efficient and “very strategic.” Isabella made a comment that she had not previously seen this method. Like Emma, Isabella had completed Calculus in her secondary education, and it seemed unusual that she had not been previously exposed to this method.

Isabella’s stacking method for the addition and subtraction of fractions, and her method for organizing the combination of like terms, demonstrate her methodical, procedural driven approach to mathematics. Within her stacking method, Isabella used a systematic “proportion system” for finding equivalent fractions. Isabella seemed to have internalized the procedure and was comfortable writing the equivalent fraction terms horizontally for the numeric item in Problem Set A as she visualized the stacking method and mentally performed the steps. She followed the method explicitly in the more complex, algebraic problem from the same problem set, where she wrote the terms vertically. Isabella’s stacking method was easily extended to accommodate a problem with more than two terms. That is not necessarily true for the method of cross-multiplication that is used by many students.

Isabella seemed unsure of when to “stop” a solution. She struggled with the feeling that something more could be done when the numerator had a quadratic polynomial. She attempted to factor $x^2 + 39x + 14$ using the factors of 1 and 14, then 2 and 7, and eventually she “gave up” when those did not seem to work. She gave the impression that she was not confident that this solution was completely simplified.

Sophia. Sophia was an African American female who was twenty years of age and majored in Psychology. She reported that the highest level of mathematics completed in her secondary education was precalculus. Sophia was slightly nervous at first, but quickly warmed up to the researcher. In the paragraphs that follow, a detailed description and analysis of Sophia's case are presented.

Case description. Sophia correctly answered four of the six assessment items (see Appendix U). Equivalent fraction errors were the cause of the two incorrect responses she gave to the numeric item from Problem Set C and to the algebraic item from Problem Set A. The researcher noted that for the algebraic item in Problem Set C, Sophia had factored each denominator, recognized that they shared a common factor, and then found the least common denominator. Because her written work suggested that Sophia understood the problem well, the researcher selected an incorrect solution to this problem, one that added across the numerators instead of finding equivalent fractions, for Sophia to evaluate during her interview.

The researcher started Sophia's interview with the sorting activity. Sophia separated the problems by operation, stating, "That was the very first thing I saw." The researcher asked if she saw something else and she arranged the problems in groups of polynomial and number problems. The researcher asked if she noticed anything about the problems the day she completed the assessment, and Sophia said she specifically remembered having trouble with the last step on one problem, and pointed to the algebraic problem from Problem Set C.

The researcher gave Sophia her assessment and asked her to describe the approach that she chose for the algebraic item from Problem Set C (see Figure 13). She described the process of factoring the denominators, and then multiplying each numerator by the missing factor to find equivalent fractions. Sophia then said, "And then here, was addition, ok, so what I think I did was I added across, is like, what I did. But I'm not sure." After that step, Sophia described her confusion that day in the following way.

> Sophia: I remember trying to figure out, how am I supposed to write this? And I was gonna [*sic*] do some of these over here, and then, uh, yeah, it was time after that to just say forget it. But yeah, that's when I really struggled.

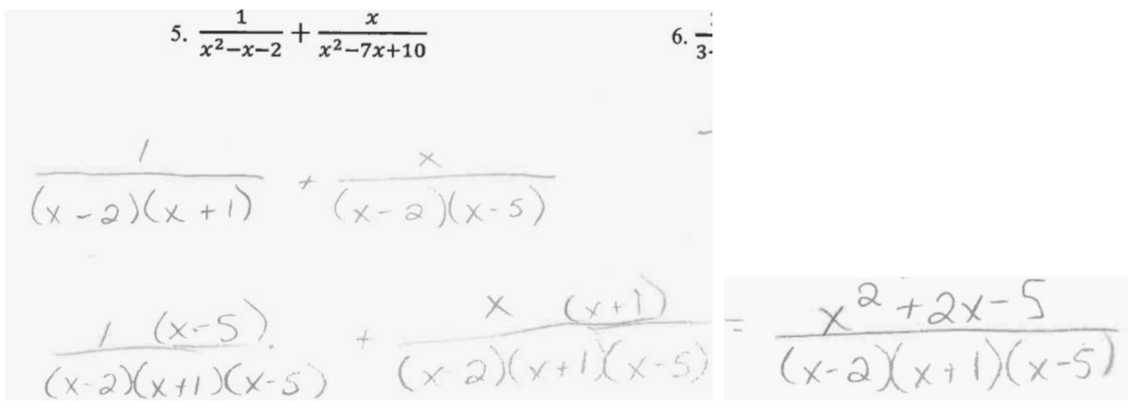

Figure 13. Sophia's solution to the algebraic item in Problem Set C.

Sophia wanted to look at a problem on the assessment that she thought would be an example of how she correctly executed the same process. She pointed to the algebraic item in Problem Set A, which the researcher had noted contained an equivalent fraction error. The researcher decided that for now she would let Sophia continue with the examination of the new problem; however, the researcher made a note to come back to the problem above.

Sophia led the researcher through her work on the algebraic item from Problem Set A. Her solution contained an equivalent fraction error, so the researcher moved her attention to the correct response she gave to the numeric item from the same problem set. Sophia described her strategy, and then the researcher asked if she saw any similarities between the two items in Problem Set A. Sophia said, “Now I do,” as she realized that each numeric item on the assessment had a corresponding algebraic item. The researcher asked, “Do you think about problems like this [numeric] when you see problems like that [algebraic]?” and Sophia exclaimed, “Nope! I wanna [*sic*] run away when I see those!” She said she did not like the algebraic problems because it was hard to remember what to do first.

The researcher then returned to the problem where Sophia struggled with the last step and asked her to look at it one more time, starting from the beginning. Sophia said the first thing she did was to factor the denominators, and then multiplied the numerators by $x + 1$ and $x + 5$. The conversation that followed is below.

Researcher: Tell me, why did you do that?

Sophia: To get a common denominator. And then I brought this one over here and multiply it top and bottom.

Researcher: And why top and bottom?

Sophia: Uh, because you’re supposed to? I just know you’re supposed to, whatever you do to the bottom you’re supposed to do to the top.

The researcher asked Sophia what she would do next, and she replied that she "really didn't know." She recognized that on the assessment she had multiplied the numerators by the factors, and then added the numerators together. Sophia had reached the correct solution of $\frac{x^2+2x-5}{(x-2)(x+1)(x-5)}$, but was not aware that this is where she should stop.

> Sophia: And then that's where I got stuck. 'Cause [*sic*] then I wasn't sure that factored . . . and I don't know, I just didn't have time to do it. 'Cause [*sic*] I was thinking about what do I do next, what do I do next? 'Cause [*sic*] at first, I had something totally different and then I was like, no that's wrong. And then I ended up getting this. And I was like, what do I do next, but now that I look at it, I think I would factor this into, uh, *x*, wait, no, multiplies to this. Yeah, I uh, yeah, I don't know.

Sophia's solution was correct, but she was not confident that she had the right answer, or if she should continue with the problem. The researcher asked Sophia to review an incorrect solution to the same problem, and describe what she thought about the method. In the work that she reviewed, the participant had combined the numerators of 1 and *x* into 1*x* without finding equivalent fractions. The researcher asked, "Do you think they did it right? Or wrong?" Sophia said, "I honestly don't know. I wouldn't have thought to do that, but I don't know." The researcher followed with this question, "How confident do you feel about yours?" Sophia replied, "Well, I don't know. With my brain, I guess confident in this. It’s what makes sense to me as far as, you know, if you [inaudible] if it was all just numbers you would do it that way.”

Case analysis. Sophia’s first instinct was to sort the assessment items by operation. When first asked if she thought about the methods used to solve numeric problems when faced with an algebraic problem, Sophia adamantly said no. She said that she wanted to run when she saw an algebraic problem, presumably because they are more difficult. Her final comment suggested, however, that she had in fact made a connection between algebra and number properties, as she relied on her knowledge of procedures with rational numbers to evaluate the incorrect written work of another participant.

Sophia’s assessment had very few errors, but her interview revealed that she was not confident that her methods or solutions were correct. On paper, Sophia’s method of finding a common denominator in the algebraic item from Problem Set C looked flawless and efficient. Sophia’s description of the process, however, suggested that she had memorized steps in a procedure to find common denominators without developing an understanding of equivalent fractions. Sophia was also unsure of how and when to end the solution to the algebraic item from Problem Set C. She was uncertain of what form to write it in, or if it could be factored and further simplified.

The researcher did not discuss the division items with Sophia during her interview. It can be seen from Sophia’s assessment, however, that like Isabella she approached the problems with slightly different methods. Sophia cancelled common factors before the operation of multiplication in the algebraic item, although she did this step after multiplication in the numeric item.

Group 2

Ethan. Ethan was an Asian, eighteen-year-old male who majored in Biology. He reported that precalculus was the highest level of mathematics he had completed in his secondary education. Ethan had a very pleasant personality and was relaxed and cooperative during his interview. The paragraphs that follow present a detailed description and analysis of Ethan's case.

Case description. The researcher reviewed Ethan's assessment (see Appendix V) before he arrived for his interview. He had incorrectly answered all three of the algebraic problems. For the division item, Ethan found a common denominator, and kept it in his final solution, instead of multiplying across the denominator. He did not do that for the numeric division item, so the researcher made a note to investigate the inconsistency. In both of the addition problems, Ethan multiplied each numerator by the factor of the common denominator that it was missing, and then wrote the common denominator in his solution as the *sum* of all of the denominators. He did not do that for the numeric items, so the researcher made a plan to ask Ethan to examine those problems. The researcher also selected a correct solution to the algebraic item in Problem Set C for Ethan to review in his interview.

The researcher began the interview by asking Ethan if he remembered thinking that there were any similarities in the problems on the assessment. He responded, "Right now, no. I can't really remember." The researcher then gave Ethan the assessment problem cards and asked him to sort them into some meaningful groups. Ethan placed the algebraic problems on the left, and the numeric problems on the right. The researcher asked how he decided on those groups, and Ethan said, "I guess its numbers and variables. Yeah, that's how I sorted them. One group with all numbers and the other group with the *x*'s." The researcher asked Ethan if he thought about the numeric problems when he was working algebraic problems, or if they require different methods to solve. Ethan responded in a way that suggested he was thinking about equations, rather than expressions.

> Ethan: Yeah, there is a different step to approach to solve the ones with variables and just the ones with the numbers, because the numbers you just have to solve for the numbers itself, just complete that question but for these variables in it, you have to solve. For me, I think about solving for the *x* instead of solving for the whole problem and stuff.

The researcher then gave Ethan his assessment and gave him time to become reacquainted with it. The researcher wanted to begin with the division problems, and instructed Ethan to look specifically at number two. Ethan had found a common denominator for the algebraic division problem, but not the numeric one. Finding a common denominator is not an incorrect strategy for dividing fractions, but it is an inefficient one. In this case, Ethan kept the common denominator in his final solution, which was incorrect. The researcher showed him the problems together and pointed out the extra step in the algebraic problem. Ethan said he "kind of froze on that step" and was unsure of his strategy. The researcher asked him if a common denominator was necessary for the numeric division problem, and he replied, “No, I don’t think it’s necessary for that problem because I guess you’re working with numbers, there’s no unknown in them. Also, I might have messed up number two.” The researcher asked, “In what way?” and Ethan responded, “Right now when I think about it, it’s kind of foggy in my head, but I believe you only find common denominators when you’re adding and subtracting, not multiplying and dividing. I think I might have messed up.”

The researcher pointed to the algebraic addition item from Problem Set A that had denominators of 4, $x + 1$, and 1 and asked, “Is that an appropriate problem to find a common denominator?” Ethan answered, “Yes. Addition or subtraction. Yeah, that’s fine, a common denominator, I believe.” The researcher asked Ethan to talk through his solution to this problem (see Figure 14).

> Ethan: Since it’s three separate parts to the problem (pause) for the first part, it’s only over, the denominator is only 4, so you have to multiply by $x + 1$ and 1 to get that section by accepting all three of the denominators. The same with the second part and the third part. The $\frac{6x}{x+1}$, it has $x + 1$, but it doesn’t have 4 or 1, so I can multiply that by 4 and 1 and then 3 with the 4 and $x + 1$ to get a common denominator so you can write all of it under the same denominator.

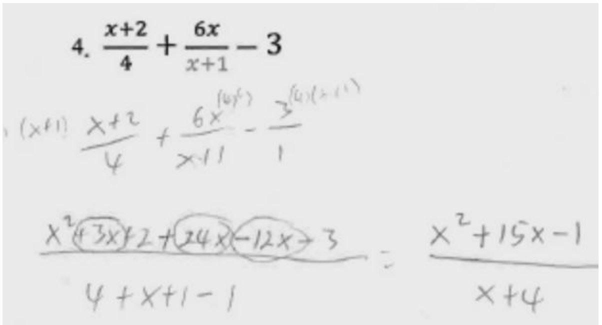

Figure 14. Ethan's solution to the algebraic item in Problem Set A.

Ethan's explanation to this point was accurate, but Ethan made an error in the common denominator. He wrote the common denominator as $4 + x + 1 - 1$ which then became $x + 4$. Ethan correctly found a common denominator when he solved the corresponding numeric item in Problem Set A (see Figure 15). The researcher pointed out the discrepancy and they discussed it in the following exchange.

> Researcher: Let me ask, in this problem you added across the top, sorry, like here at this step, added and combined your like terms and also did the same thing on the bottom. You added these different denominators, but up here it appears that you've multiplied. Should this denominator be $2 + 5 - 1$?
>
> Ethan: I guess I did (pause)
>
> Researcher: Or is it just different because they're different problems?
>
> Ethan: I mean they are different steps that you need to take. If there's a variable in the question, but now that I look at it (pause).
>
> Researcher: What are you thinking?

Ethan: I don't really know, actually. My brain's telling me what I did was right for number four, but another feeling is kind of contradicting, maybe I should have done something different.

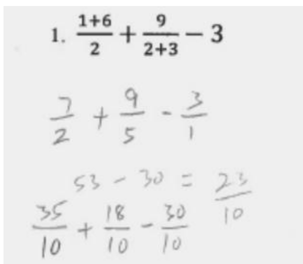

Figure 15. Ethan's solution to the numeric item in Problem Set A.

The researcher asked Ethan what his first impression was of the algebraic item from Problem Set C. He said, "Same as when I saw number four [algebraic item in Problem Set A], common denominator." The researcher pointed out that Ethan had also combined the like terms in the denominators of this problem and asked why he had done that. Ethan indicated it was to find a common denominator. The researcher asked why he found a common denominator in that way, and he replied, "I kind of have a mixed feeling about what I did." He began to question his solutions of the other problems as well, so the researcher decided to show Ethan an alternative solution to the algebraic item in Problem Set C (see Figure 10).

Ethan noticed that the denominators were factored in the sample solution, and that the numerators had been multiplied by the factors from the denominators that they were missing. He then noticed that the common denominator was the product of the factors in denominators of each term. The researcher asked, “How do you feel about that?” Ethan responded, “That’s kind of what I was thinking. Maybe I shouldn’t have combined the other terms on the denominator, just the numerator. Yeah. That looks a little more, I guess, right to me now.”

The researcher ended the interview with this question, “When you’re working these kind [*sic*] of problems, like these two were very similar but one’s numbers and one is algebraic, do you think about how you would do it with numbers when you’re working an algebraic problem?”

> Ethan: Kind of, not really. Yeah, not really because the numbers, it’s just numbers. What I see is what I do. With the variables and expressions there’s different rules to follow. The same rules doesn’t [*sic*], to me doesn’t apply to both of them.

Case analysis. Ethan did not recall any specific details of the assessment items. He first arranged the assessment items into groups of numeric problems and algebraic problems. When asked if numeric and algebraic problems required different solution methods, Ethan answered affirmatively, although his response was focused on finding a single numeric value for an expression compared to solving for “*x*.”

Ethan’s comment at the beginning of the interview suggested he thought that the presence of an unknown dictated in some way what strategy should be used to solve a problem. An example of this thinking can be found in Ethan's solutions to the division problems. He did not find a common denominator for the numeric problem, although he did for the algebraic problem. When Ethan was confronted with the inconsistencies in the division problems, he admitted that he "kind of froze" on that step. It appeared that he began to doubt his methods at this point in the interview, though he did not actually change his mind. He continued to believe that the presence of an unknown in the algebraic problem justified the difference in procedures.

Ethan approached each of the addition algebraic problems with a method that was different from his approach to the corresponding numeric problem. In the algebraic problems, Ethan used the denominators to find numerators for the equivalent fractions, and then he *added* the terms in each denominator to find a common denominator. Ethan appeared to shift his thinking about common denominators when he reviewed an alternative solution to the algebraic item from Problem Set C and noticed that the common denominator was the *product* of the denominators. His final comment, "there's different rules to follow," however, suggested that he did not change his fundamental belief that algebraic and numeric problems are not connected.

Liam. The oldest student in this phase of the study was Liam, a white male who was thirty-eight years of age. He had a Bachelor of Science degree in Business Administration and had returned to school to major in pre-dentistry. Liam appeared nervous, but was polite and cooperative during his interview. A detailed description of Liam's case and an analysis of his case are presented in the paragraphs that follow.

Case description. The researcher reviewed Liam's assessment (see Appendix W) before the interview session and noted that he had correctly answered two of the numeric items, and made a simple arithmetic error on the other. Liam appeared to have had difficulty with the algebraic items. He chose to take the square root of the second term in the division problem, did not finish the three-term addition problem, and omitted the two-term addition problem. The researcher decided to focus the interview on the algebraic division problem and the algebraic addition problem from Problem Set C.

The researcher asked Liam what he remembered from the assessment he took in class, and if he recalled any similarities between the problems. He said he could not remember any specific details about the problems. The researcher gave the cards with the assessment problems to Liam and asked him to arrange them in meaningful groups. Liam thought about the cards for a long time, and then arranged them with the numeric items across the top and the algebraic items across the bottom. When asked why he decided on those groups, Liam said, "These all being whole numbers at the top, whether it was adding or division, they were all whole numbers, real numbers. These down here and this one can be in a group by itself." Liam had picked up the algebraic item from Problem Set A and moved it to a third row on the bottom (see Figure 16). Liam's explanation for moving the item to the third row was that it did not have a "squared" term and should not be in the same group as the other algebraic problems. The researcher prompted Liam to sort the cards again, and he arranged the problems by operation, addition in one group, division in another. The researcher asked Liam if he were presented an algebra problem and he was struggling to solve it, would he think about how to solve it with numbers.

Liam's reply was, "Yes, you could just replace *x* with one or any other whole number and then try to solve that way."

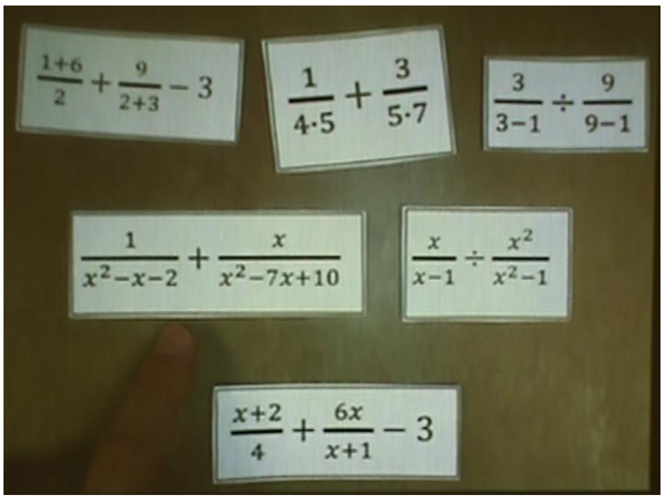

Figure 16. Liam's first arrangement of the cards.

The researcher gave Liam his assessment and he immediately pointed out the algebraic item in Problem Set C that he had omitted. He said, "This I now know can be factored into $x - 5$ and $x + 2$." Liam seemed eager to solve the problem, and the researcher let him continue to factor both denominators. The researcher asked him what he would do next, and he multiplied each numerator by its own denominator (see Figure 17). Liam then paused for a long time, and the researcher asked, "What are you thinking?" The following exchange occurred.

Liam: Well, I'm not sure if I would distribute this through [points to *x* in front of $(x - 5)(x - 2)$], if this should be like this [draws parentheses around $(x - 5)(x - 2)$], or if it's just that, if I have to do (pause). Obviously multiply this factor out [points to $(x - 5)$ $(x - 2)$ inside parentheses] and I'm just going to get this [points to original denominator].

Researcher: That's true.

Liam: For some reason I think I need to trade this to all of them, but this doesn't look right because these two are together.

Researcher: What doesn't look right?

Liam: Having this *x* multiplied or distributed through all of these. For some reason I'm wanting to pair these up first, but again I know that is just going to give me this, which I just broke out. I don't know, just taking a guess I would say it would be -5x. This is just throwing me off. I don't know if it is plus or minus, which is just why I think it's wrong. Plus $x^2 - 2x$.

Liam's misstep when finding equivalent fractions left him with a situation that he knew was incorrect; however, he was unsure of how it should be resolved.

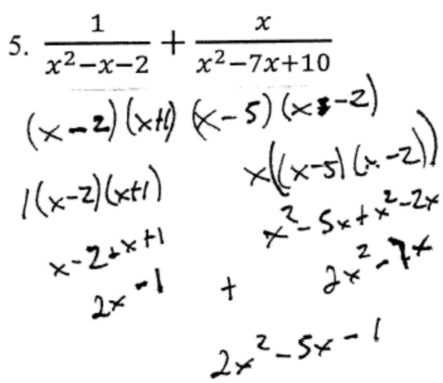

Figure 17. Liam's solution to the algebraic item in Problem Set C.

The researcher decided to shift Liam's attention to a problem that he had solved correctly and asked him to look at the numeric item from the same problem set. Liam was confident as he explained the steps he took to solve the problem. He described finding common denominators, and adding the numerators together to arrive at $\frac{19}{140}$. Liam thought his answer could possibly be simplified further, but eventually decided "I guess that's it."

The researcher wanted to discuss Liam's solution to the algebraic division problem next (see Figure 18). Liam described his approach to this problem as, "Well, it looks like I decided to square, which I'm not sure why I decided to do that, to get rid of the square, I took the square root of $x^2 - 1$." After he gave it some thought, Liam said "I think you're supposed to multiply by the reciprocal, which I don't think is the square root of $x^2 - 1$. I'm not sure." His incorrect approach brought him to the result $\frac{x}{x-1} \div \frac{x}{x-1}$. He was not sure what his next step should be, so he used this reasoning.

Liam: Well, assuming that these are both equal, you divide that into that and you just get 1. I don't think that's right, but just looking at this x over $x - 1$, if this was $\frac{1}{3}$ divided by $\frac{1}{3}$, it would be 1, you know in your head just as a factor.

Researcher: Ok, so you just reasoned through that?

Liam: Yes, these two are both the same thing.

Researcher: That makes sense.

Liam: If you replace x with a number, then this should be the same thing divided by the same thing, which would give you one. I'm still pretty sure this is not right.

Liam provided an example of using number sense to reason through an algebraic problem. He was confident that variables were merely placeholders for numbers, so if one number divided by itself was one, then so must be the algebraic expression he was working to solve.

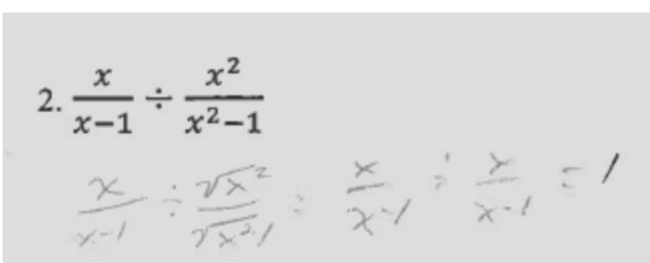

Figure 18. Liam's solution to the algebraic item in Problem Set B.

The researcher then asked Liam to review his solution to the numeric division problem. Liam described his strategy as multiplying by the reciprocal. Although it was unnecessary, Liam found a common denominator for this problem, and then arrived at the correct result. The researcher asked Liam again if he thought about solving problems with numbers when he was working algebraic problems and they had the following conversation.

> Liam: Yes. You can always try to throw in a number to see what happens if you were to replace a number with the obvious one.
>
> Researcher: Why do you do that?
>
> Liam: I guess just to see if … I don't know, I guess it's just easier to work with numbers.
>
> Researcher: Does it always work out?
>
> Liam: No, I don't think.
>
> Researcher: Are the rules different for this problem than this one [points to division problems]; the rules of algebra versus the rules of numbers?
>
> Liam: I don't think so. I think the steps should be the same. Like I said, I'm not sure why I squared this. Who knows? Just trying to simplify this. I guess I just didn't like this 2 there and wanted to get rid of it and take the square root of it.
>
> Researcher: I've heard other students say they don't like the way something looked. Are there [interrupted by Liam].
>
> Liam: Yes, it's more comfortable working with things in a certain format.

Researcher: OK. When you see a problem like that [points to algebraic division problem] it triggers the idea "I need to get rid of the x^2?"

Liam: Yes, simplify it somehow and make it look like this so we're working with the same types of numbers. They are all x's, or whatever. Having them in a similar form, if that's the right term, which I'm sure it is not.

The researcher mentioned that visual cues often prompt students to take certain actions and Liam provided this insight into why students make procedural mistakes.

Liam: Well, it's just ignorant students grasping at straws is what it is. [Laughter] When you have no idea how to solve something, you just start playing around with numbers to try and make it look like something you're comfortable with while you're traveling along. You just feel better, "Okay, that looks right," and you work it out. You're probably completely wrong.

Case analysis. Liam did not recall any specific details of the assessment items. Liam first sorted the items into groups of numeric and algebraic problems, and then further sorted the algebraic items by the degree of the variable. He confidently stated that thinking about numeric problems can help with the solution of an algebraic problem. Later into the interview, his comments indicated that he recognized that a variable x can be replaced by any number and that the same steps were required to solve numeric and algebraic problems.

Despite making this connection, Liam was unable to extend the methods of operating with rational numbers to operating with rational expressions. He chose incorrect methods to solve each of the algebraic problems that were different from the methods he applied to the same items in the numeric context. Liam could clearly explain how to find equivalent fractions in the numeric item from Problem Set C, but when he attempted to solve the algebraic item from the same set, he multiplied each numerator by its own denominator to find equivalent fractions. Liam was aware, however, that his solution did not "look right" and that this method would not lead him to a correct result. Liam was not sure how to resolve the issue, and did not look at the method he used in the numeric item for guidance.

In the algebraic division problem, Liam admitted that he made an effort to avoid the x^2 term by taking the square root of a rational expression, although he knew that this method was most likely incorrect. Liam explained that in general, when he was unsure of what steps to take, he would look for a way to adjust the problem to be one that he was more comfortable with. Despite these mistakes, Liam, who had been out of school for many years, was able to use the connection he had formed between numbers and variables to successfully reason through an unfamiliar step in the division problem.

Mason. Mason was a white, eighteen-year-old male who had not declared a major. He reported that an honors section of precalculus was the highest level of mathematics that he had completed in his secondary education. Mason was pleasant during the interview, although he was very direct and focused. His session lasted only 12 minutes, and was the shortest of the interviews. A detailed description of Mason's case and an analysis of his case are presented in the paragraphs that follow.

Case description. Mason incorrectly answered three items on the assessment (see Appendix X). He did not appear to have trouble deciding what procedure to use; his errors had all occurred during the execution of a correct procedure. He did not simplify his result for the algebraic division problem, made a simple arithmetic error in the numeric item from Problem Set C, and made a distribution error in the algebraic item from Problem Set C. The researcher made a plan to review the pair of division problems during Mason's interview. The researcher also obtained a sample of a correct solution to the algebraic problem in Problem Set C for Mason to review.

To begin the interview, the researcher asked Mason to think about the assessment he completed a few weeks prior to the interview and to describe any similarities that he recalled between the problems.

> Mason: If I remember correctly, there were a lot of non common denominators that you had to switch and do that a lot, and then simplify from the numerator to the denominator.

The researcher gave the assessment item cards to Mason and asked him to divide them into meaningful groups. Mason provided an interesting arrangement (see Figure 19). He described it in this way, "These numbers down here [pointing at the bottom half] you could get a definite answer for, up here you can't. But these two deal with addition and subtraction [pointing to left half] and these two deal with division [pointing to right half]."

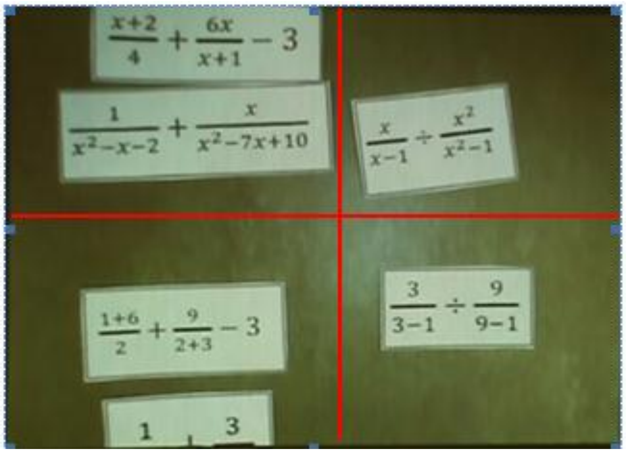

Figure 19. Mason's arrangement of the assessment items (lines added by researcher).

The researcher selected the two division items and asked Mason if he approached the problems differently because one was numeric and one was algebraic. Mason considered the procedures needed to answer the numeric and algebraic problems and then decided that he used the same approach in both cases.

The researcher gave Mason his assessment and asked him to look over it and try to remember what he was thinking on that day. When Mason appeared ready, the researcher asked him to describe how he solved the algebraic division item.

> Mason: Ok, well, number two was pretty easy. It seemed like, uh, instead of cross-multiplying, I just flipped the numerator and the denominator and then

> multiplied. And then I got that equation, the $\frac{x^3-x}{x^3-x^2}$ . . . Then I just factored out an *x* since those were common terms. Multiplied and you could just cancel those out for the answer right there.

The researcher then asked Mason to look at the corresponding numeric item and identify how it was related to the algebraic problem. Mason said, "You pretty much use the same concept. It's just numerical, not alphabetical. So, I pretty much did the exact same thing with flipping the second fraction, then multiplying."

Next, the researcher asked Mason to review a numeric addition problem and his immediate reaction was, "Well, this, you have to find a common denominator." The researcher asked, "Why was that?" and Mason responded, "Because you are adding and subtracting." Mason demonstrated that he understood "finding a common denominator" as he explained his work in the problem (see Figure 20).

> Mason: I really couldn't think of a good common denominator off the top of my head, so I just multiplied 20 by 35 to get 700 and then you multiply by 20 over 20 and 35 over 35 'cause [*sic*] then it would be just like multiplying by one. . . And once again, you just get an answer and simplify to the least amount.

The researcher showed Mason how someone else had solved the same numeric problem. The researcher pointed out that in this solution they used 140 as the common denominator. The researcher said, "Yes, well, I'm trying to think, how did they come up with 140? Any idea?" Mason replied, "Not really."

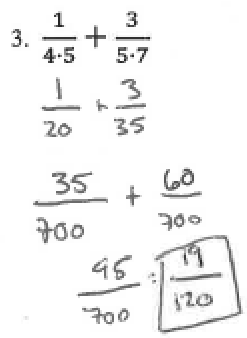

Figure 20. Mason's solution to the numeric item in Problem Set C.

The researcher moved next to the algebraic item from Problem Set C. Mason said his first impression of that problem was "Oh man . . . That is gonna [*sic*] be a long one." The researcher asked Mason to describe his steps, and he began by saying that it was "the same thing as in number 3, just find the least common denominator, but there's really not one between the two, so you just have to multiply them together." Mason finished describing his steps, and the researcher asked if $\frac{2x^2-8x+8}{x^4-8x^3+15x^2+4x-2}$ was his final answer. Mason said, "Yes. You could probably simplify that a little bit more, but I really don't know how you would."

The researcher then selected a sample solution to the problem that had used a different method to find the common denominator, and had an answer that was simplified further than Mason's (see Figure 21). Mason looked over the work and then said, "Wow, that is a lot simpler." The researcher asked, "But do you think it's right?" and Mason indicated that he thought it was correct. The researcher asked Mason to describe the steps in this solution and the following conversation occurred.

> Mason: They factored out the bottom terms.
>
> Researcher: Ok, and then what did they do?
>
> Mason: They realized there was only one factor that wasn't common between them and so they multiplied by that.
>
> Researcher: Um hum, so that brings us to here?
>
> Mason: Yes, and then they distributed the *x* in the second half right over here.
>
> Researcher: OK.
>
> Mason: And then just added right there [inaudible].
>
> Researcher: So, it's interesting that they broke this down into its parts and then found a common denominator.
>
> Mason: Right.
>
> Researcher: Look at number three. Um, they didn't do that here because it already starts with the parts already broken down.
>
> Mason: Oh.
>
> Researcher: What do you think the common denominator would be just by looking at that?
>
> Mason: Four times five times seven. There we go!
>
> Researcher: Ok.

Mason: That's actually really smart. I've never seen that before. In either of those. That's (pause) that's really helpful.

5. $\dfrac{1}{x^2-x-2}+\dfrac{x}{x^2-7x+10}$

$=\dfrac{1}{(x-2)(x+1)}+\dfrac{x}{(x-2)(x-5)}$

$=\dfrac{(x-5)}{(x-2)(x+1)(x-5)}+\dfrac{x(x+1)}{(x-2)(x-5)(x+1)}$

$=\boxed{\dfrac{x^2+2x-5}{(x-2)(x+1)(x-5)}}$

Figure 21. Efficient solution to the algebraic item in Problem Set C.

The researcher finished by asking Mason how much he thought about what he can or cannot do with numbers when working algebraic problems. Mason replied, "Very little, actually. . . I like to keep the two kind of separated and then when I go back and check, I'll go use numbers and put them in for *x*." The researcher asked, "Ok, but when you're actually working it out, you don't think about that?" and he responded that, no, he did not. He added, "Like, until I sat down and looked at them I didn't really put two and six together at all when we were doing it. . . Now that I had a chance to look at it I can see that they are the exact same, but, yeah, that's interesting."

Case analysis. Mason remembered "a lot of non common denominators" from the problems on the assessment. When asked to sort the assessment items into groups, Mason formed a unique arrangement. He thought of the items both in terms of the expected result – a definite answer for numeric problems – and in terms of the operations that were required. At the end of the interview, Mason stated that he did not think of numeric and algebraic problems together. In fact, he indicated that he had not made that connection between these problems until he sat down for the interview.

Mason demonstrated a general knowledge of what procedures to use and when they should be applied. He chose correct procedures for all of the problems, and made only minor errors in the execution of these methods. Mason's comments also suggested that he understood that "finding a common denominator" involved finding equivalent fractions by multiplying the numerator and denominator of a fraction by the same number, or by "one."

Mason easily found common denominators in his work, although he was not finding *least* common denominators. Mason used the product of 20 and 35 as the common denominator for the numeric item in Problem Set C. The large common denominator that resulted, 700, may have contributed the arithmetic error that led him to an incorrect solution for this problem. Mason examined a solution to this problem that found a common denominator of 140, but was unable to describe how to arrive at that number.

The algebraic item in Problem Set C also required Mason to find a common denominator. He claimed that the least common denominator for the algebraic item in Problem Set C was the product of both of the denominators, which made the algebraic item in Problem Set C seem long and tedious. This method also made it difficult for

Mason to recognize that his answer could be further simplified. He inspected a more efficient method for this problem, and this time was able to easily follow the steps and explain that individual's thinking. He described the solution as "really smart" and "helpful." He was also able to immediately apply the concept to the numeric item in Problem Set C and quickly identified the least common denominator as the product of 4, 5, and 7, solely through inspection. Like Isabella, although he had completed an honors precalculus class in his secondary education, Mason claimed that he had not seen this more efficient method of finding equivalent fractions with common denominators, but embraced it right away.

Noah. Noah was a white male who was eighteen years of age and majoring in Chemistry. He reported that the highest level of mathematics completed in his secondary education was Trigonometry. Noah enjoyed discussing and working the problems and at 41 minutes, his was the longest interview. The paragraphs that follow present a detailed description of Noah's case followed by an analysis of his case.

Case description. Noah answered the two numeric addition problems correct on his assessment (see Appendix Y). In the other addition problems, he found common denominators, but made two distribution errors. Noah's result for the numeric division problem was not simplified, and he made a cancellation error in the algebraic division problem. The researcher selected Problem Sets A and B to review during Noah's interview. The researcher also selected a correct solution to the algebraic division problem to share with Noah.

The researcher began the interview by asking Noah if he recalled any similarities between the problems from the assessment that he completed a few weeks ago. He remembered "finding the common denominator." The researcher then engaged Noah in the sorting activity. The denominator in each item was the first thing that caught the attention of Noah. He separated the problems into groups with numeric denominators and algebraic denominators. The researcher invited him to identify other similarities in the assessment items and he pointed out that the two division problems were similar because they did not use addition or subtraction, thus dividing the problems into groups based on the required operations. The researcher pointed to the numeric and algebraic division items and asked, "Did you do anything different when you were working those?" Noah responded, "Yes, I just solved them completely different." Noah then immediately contradicted himself when he said that he inverted and multiplied each problem.

The researcher gave Noah his assessment and asked him to review his solution to the algebraic division problem, and describe what he was thinking as he solved the problem. Noah described inverting the second term, and then using the distributive property as he multiplied across to get this result, $\frac{x^3-1x}{x-x^2}$. Noah then made a cancellation error and arrived at $\frac{x^3-1}{-x^2}$. Noah and the researcher had the following conversation.

Noah: The *x*'s cancelled out. I just cancelled. (pause) Yes, I just cancelled out the *x*'s.

Researcher: OK. How did you know to do that?

Noah: Since they were the same. Anything on the numerator that's exactly the same as something in the denominator gets knocked out.

Researcher: OK

At this point, Noah became aware of the distribution error he made. The researcher then brought his attention back to the cancellation of the *x*'s.

Researcher: And, so when you cancelled that, what was left behind?

Noah: Just the $x^3 - 1$, just the $-x^2$.

Researcher: OK. So when you cancelled the *x* in the numerator, it left behind a one. In the denominator it didn't leave anything behind. Tell me why, why it works that way.

Noah: Because of the minus, because you'll have to take minus one [pause].

Researcher: OK, and so the *x* stays on the bottom because?

Noah: It was already minus. It was a plus *x*, so [pause].

Researcher: OK.

Noah: I'm pretty sure that's what I thought.

It appeared that Noah believed that when a cancelled term was in the leading position, the result of the cancellation was a zero, and if the term followed an operator in the expression, like an addition or subtraction sign, the result of the cancellation was a one. The researcher asked Noah to describe his solution to the corresponding numeric item.

He described the steps he took to invert, multiply, and then simplify his answer. When asked if he saw any similarities between the two problems, Noah began to ponder the cancellation of the three and nine before multiplying the numerators and denominators. He wrote $\frac{8}{2\cdot 9} = \frac{8}{18}$, which was incorrect, but it convinced him that cancellation would not work in this situation. He said, “The nine and the three, but, I, wait, that doesn’t work. Never mind. Yeah, you can’t cancel the denominator with the numerator. You cancel the numerator with the denominator. So that doesn’t work.” This statement showed that Noah had a misconception about cancellation, which became the focus of the interview. The researcher asked Noah, “What’s the difference?” and he began to explain his thinking as he wrote examples for the researcher to see (see Figure 22).

> Noah: Because I like to think of an example, like if you have eight over four. Yeah, now that I think about it, that isn’t right at all. [smiles]
>
> Researcher: Ok.
>
> Noah: But eight over four, for example, the four is the smaller number, and the denominator on the eight is the largest one of the numerator, so four goes into eight and makes two. However, if it was four over eight, that, that wouldn’t work at all. Or (pause) actually it still would. That would still work!

Noah made the comment that simplifying after he had an answer was much easier for him. The researcher asked him to try the problem again with cancelling before multiplying. Noah realized that he could cancel two factors and arrived at the solution $\frac{4}{3}$. He understood then that his answer on the assessment could have been further simplified.

Figure 22. Noah’s written work during interview related to reducing fractions.

The researcher noticed that many participants would cancel before multiplication in the algebraic division problem, but not in the numeric problem, and asked Noah why he used different methods for those two problems.

> Noah: Whole numbers, it’s a lot easier to simplify afterwards with whole numbers as where when working with variables, it’s much easier while you’re dividing.
>
> Researcher: OK, why is that so?
>
> Noah: Because it’s already pre-simplified. If you plug something into the *x*’s, it’s pre-simplified, and you don’t have to cancel out anywhere what you’d already do if it was whole numbers. But here since you have the whole numbers, it’s easier to just do it afterwards from my perspective.

The researcher showed Noah an alternative solution to the algebraic division problem. He noticed that the expressions were factored and cancelled before they were multiplied and then he wrote $\frac{x}{\cancel{x-1}} \cdot \frac{\cancel{(x-1)}(x+1)}{x^2}$ on his paper. Noah finished his solution by multiplying across the numerators and denominators to arrive at $\frac{x^2+x}{x^2}$. The researcher pointed out that this was different from the sample they were looking at, and asked Noah to identify which one he thought was correct. Noah chose his solution, saying that "Yes, because those *x*'s could cancel even there, so" and he wrote $\frac{\cancel{x^2}+x}{\cancel{x^2}} = x$. The researcher compared this result to Noah's solution on the assessment and asked him which one was correct. He said, "Honestly, I'm not sure," and considered them for a moment before choosing his result of *x* as "probably" the correct answer because the factoring "made sense" to him.

The researcher asked Noah to look at the numeric item from Problem Set A, and Noah asked if he could just solve it again and see if he arrived at the same answer. The researcher allowed him to continue and he worked the problem in the exact same way, and arrived at a correct solution. The researcher then turned his attention to the corresponding algebraic problem. He worked it on a separate piece of paper, and then the researcher asked him to compare it to his original solution (see Figure 23). The solutions were different, so the researcher asked Noah what he noticed about the two. He identified the place in his original solution where the lack of parentheses had led him to make a distribution error. Noah was confident that he had reached a final solution, $\frac{x^2+15x-10}{4x+4}$, to the algebraic item in problem set A. The researcher engaged him in a conversation about knowing when a solution was complete.

Researcher: You seem very confident in that.

Noah: Yes. Also because it looks like a math answer.

Researcher: What does that mean?

Noah: It has a quadratic in the numerator. This one does not [his original solution].

Researcher: OK, it looked very "mathy?"

Noah: It makes me assume that it looks, this looks more correct. Just like the factoring in this one [algebraic division problem]. When the person factored it, that definitely looked a lot more correct than not factoring it. Test taking skills . . . Just using like test-taking logic, it would make me want to factor that, and I didn't, which makes me think.

Researcher: Makes you want to factor?

Noah: Makes me, makes me, (pause) when they factored that one, it makes me think that the, that they would want you to factor it much more. It makes more sense to factor, because it does factor, and it makes sense (pause); It seemed like a more practical answer. Like this one [original solution] doesn't have a quadratic in the, in the numerator, so (pause).

Researcher: So you don't feel good about it?

Noah: I don't feel good about that one, but that one [work he just completed] looks, that one looks quite a bit more correct.

Researcher: Ok, but you said earlier when you see a quadratic that it makes you want to factor it.

Noah: Yeah, but.

Researcher: How do you know right now whether to factor it or whether to stop?

Noah: I would stop because that doesn't, it doesn't factor. Not easily. I could use the quadratic formula, but I'm not going to. But it's, as far as simplifying, that's as simple as it gets since that doesn't factor.

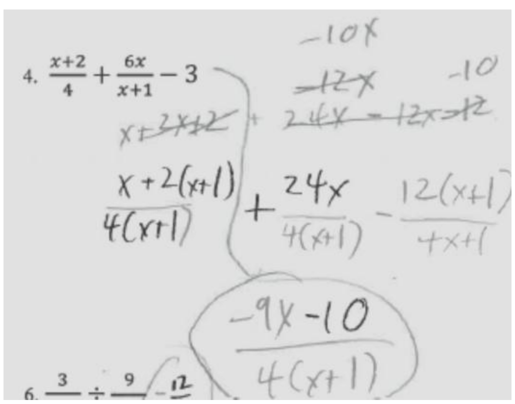

Figure 23. Noah's solution to the algebraic item in Problem Set A.

The researcher then returned to the work Noah had just completed and asked if he would cancel anything in $\frac{x^2+15x-10}{4x+4}$. Noah replied, "Once again, it's not simple enough, yeah" and he cancelled 15*x* and 4*x* using subtraction, which gave him 11*x*, and he divided a factor of 2 from 4 in the denominator and 10 in the numerator so that his answer was now $\frac{x^2+11x-5}{2}$. The researcher pointed out that in one case he subtracted, and in the other he divided and asked Noah if those were both ways to cancel. Noah answered, "Uh, yes, because (pause) yes, because this has the exponent [points to 15*x*]. The exponents, I'm pretty sure the exponent changes how you do it."

The conversation that followed lasted approximately 20 minutes as Noah used numbers and variables in different configurations to test the rules of cancelling. He started with $\frac{10}{5} = 2$ and compared that to $\frac{10x}{5x} = \frac{2x}{x}$, which he tested with $x = 2$. Noah came to the conclusion that he must keep the *x* in the denominator, and the numbers would need to divide evenly for cancelling to occur. Noah then simplified $\frac{2x}{x}$ to $1x$, which seemed to work for him when $x = 2$. Noah then tried $x = 3$ with $\frac{10x}{5x}$ and found $\frac{30}{15} = 2$, which satisfied him. Noah continued testing other arrangements and inserted "larger" test numbers until the researcher asked him to describe the conclusion he had reached. Noah said, "No matter what number I use to cancel, the *x*'s can still cancel out. If they're both in the numerator and denominator." The researcher then posed this problem, $\frac{5+x}{x}$, and asked Noah if the *x*'s in this problem would cancel because they met his description of both being in the numerator and denominator.

> Noah: Yes, but with the addition of a five, that may, it may mess things up to simplify. Because now you're using different (pause), when I was doing this, $\frac{x}{2x}$, it

didn't matter what x I put in, I'd always get one-half. This one, like if I put in a one, I get six. If I put in a two, I get seven-halves.

Researcher: And why do you think that is?

Noah: Because you still need the x's on the numerator and the denominator when added. When just straight dividing that trick seems to work, but with the, when you add another x in, another number in there, it doesn't seem to work.

Researcher: Ok.

Noah: So, I would say they don't cancel out at that point.

Noah's interview lasted 44 minutes which was longer than any other. He seemed to enjoy himself, and at the end remarked "That was fun! That was nice!"

Case analysis. Noah initially sorted the assessment item cards into groups of numeric and algebraic problems. He stated that they would be solved in different ways, although he had applied exactly the same correct procedures to each pair of numeric and algebraic items on the assessment. His only mistakes were execution errors related to the distribution property and cancellation.

Cancellation errors were common in the participants' written work. The interview revealed that Noah had several misconceptions related to the cancellation of factors. The first was discovered when he explained his work for the algebraic division problem. His comments suggested that he believed the position of a term would determine if a cancellation would result in a one or zero. Using Noah's reasoning, the result of cancelling a leading term was zero, and the result of cancelling a term that followed an operator like addition or subtraction, was one. Noah preferred to cancel common factors after multiplication in the numeric problems, and before multiplication in algebraic problems. In the latter case, he called this "pre-simplification." Noah presented another interpretation of cancellation in the numeric division problem. Noah considered how cancellation could be carried out before the operation of multiplication, and was unsure if the 3 in the numerator could cancel with the 9 in the denominator. He first claimed that it was not possible because "you cancel numerators with denominators," although the opposite was not true. Noah attempted to demonstrate this fact using numbers, and found that he had contradicted himself. He eventually saw that the numbers would in fact cancel and that his answer on the assessment had not been completely simplified.

In a different problem, Noah indicated that he thought "exponents" and numbers had different cancellation rules, subtraction and division, respectively. He also attempted to test this rule using numbers, although he was unsatisfied with the inconsistent results that he saw in the examples he created. Some of Noah's actions, such as substituting numbers for variables to test theories, suggested that he made a connection between number properties and rules of algebra. He also drew the correct conclusion when asked if $\frac{x+5}{x}$ could be further simplified.

Finally, Noah described his perception of when a solution is complete or correct. He said some solutions "looked" like a mathematics answer. The presence of a quadratic in a solution gave him more confidence that the answer was correct. He was aware, however, that if the quadratic did not factor, the solution could not be further simplified, and that was his cue to stop. The researcher also noted that it was interesting that Noah described an answer that resulted from factoring as being “more practical.”

Group 3

Jacob. Jacob was a thirty-four year-old, white, male. He reported that Algebra I was the highest level of mathematics he had completed in his secondary education. Jacob was comfortable discussing his weaknesses in mathematics, and the degree to which he pursued resources outside of the classroom demonstrated that he took personal responsibility for his education. He also had a wife who was a full-time student, and a two-year-old son. The details and analysis of Jacob’s case are presented in the paragraphs that follow.

Case description. Jacob did not correctly answer any of the six assessment items, although he attempted all of them except the algebraic division problem. He wrote a note on the assessment (see Appendix Z) that said, “Fun note while I’m wasting time. Maybe I should have reviewed fractions like my professor said to do.” The researcher later learned that Jacob was a non-traditional student returning to school. The researcher reviewed Jacob’s strategies and errors to prepare for the interview. Jacob’s strategy for the numeric addition items was to find a common denominator and equivalent fractions, and then to add the numerators. His approach for the algebraic item in Problem Set A, however, revealed a different strategy. Jacob added the terms across the numerators and denominators to obtain his result. Jacob’s incorrect thinking for the numeric division problem was not obvious from his written work. The researcher noted that any of the problems would be suitable for review in Jacob's interview, but decided to focus on Problem Set A.

The researcher asked Jacob if he remembered the problems from the assessment, and if he had noticed any similarities between them. Jacob's response focused on the operations that he remembered. He said, "There were two or three division and multiplication, but I can't remember if there were any addition, but the more I remember it was mostly multiplication, division, and fractions."

The researcher indicated that the assessment problems were printed on individual cards, and asked Jacob to sort them into groups that had some meaning to him. Jacob took some time and then arranged the cards in two columns (see Figure 24). When he finished, the researcher said "Tell me why you chose that arrangement." Jacob's response was "Difficulty. Perceived difficulty." He indicated the group of algebraic items and said "It's actually one of these three potentially be [*sic*] the most difficult." When asked what makes them more difficult, he replied, "There's just more going on. You have an exponent. *X* is raised to an exponent."

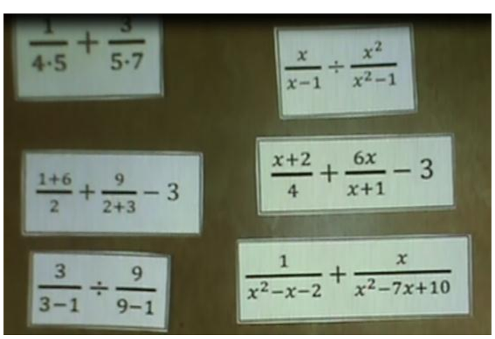

Figure 24. Jacob's arrangement of the assessment items.

The researcher asked Jacob to look at the items from Problem Set A and identify any similarities that he noticed. Jacob said, “They look like they’re pretty close, just one uses *x* instead of numeric values.” The researcher followed with this question, “When you see a problem like this, do you think about how you would work a problem that just has numbers, or are they completely separate in your mind?” Jacob replied, “Probably completely separate in my mind.” The researcher asked Jacob to describe his solution to the numeric item, and then presented him with the algebraic item where he modified the procedure. The following is Jacob’s reaction.

> Jacob: It looks like I just added for the denominators and put the denominator and numerator completely different. I did a completely different way to try to work it.
>
> Researcher: Ok.
>
> Jacob: I didn’t try to find a common denominator on that one.
>
> Researcher: Do you feel like you should have?
>
> Jacob: I don’t know.
>
> Researcher: Or are you pretty confident that what you have here is good?
>
> Jacob: No, I don’t feel confident. I feel like it could be worked differently, but I don’t know. It just looked different, when I see them, even though they, there’s a pattern the way I reason in my mind, it looked different.

Jacob thought about the discrepancy some more, and finally concluded that the problems "should be solved differently." Jacob was troubled by the inconsistency in his strategy for adding fractions, but was not convinced that it was wrong. Jacob contemplated if a numeric and algebraic problem with the same operation should be solved in different ways, and had the following to say about the concept of a variable.

> Researcher: Visually, what looks out of place or unusual about them? What's triggering you to say, "That doesn't look right."
>
> Jacob: It's just that *x* is not a numeric value. It's just *x*, a variable to be determined. That's really the difference. That'd be the difference in all of them. *X* is just a variable. It's that concept that *x* is just a number, but unable to really realize that *x* is a number. That struck an idea, but *x*, I feel it wasn't, I never really grasped what *x* represents and that's been my struggle.
>
> Researcher: What does *x* represent?
>
> Jacob: Any value that, it could be any value really. It's different if this [algebraic division problem] were to equal something. *X* has a definite value because it could change, it can change, see that could change from one of these [algebraic division problem].
>
> Researcher: If *x* can be any number, then would the method to solve this one [three-term algebraic addition item] be different from the method to solve this one [three-term numeric addition item]? To perform these operations?

Jacob: Yes, I would think, because *x* isn't defined even though this [three-term algebraic addition problem] may be if this were six, this one nine, if this were two and this was $x + 3$, the answer would be different I guess.

Later in the interview, Jacob was asked again to compare numeric and algebraic problems.

Jacob: I would say, for me, and probably for others who struggle with fractions, is the fact that this is algebraic and this is just numbers, again it's not *x*, it's just a number, and ignoring the fact that the letter, it's just like we did, never mind, I don't remember what they call it, polynomial or exponential, whatever, I just, variable *x, y*, it's just a number and that's what the teacher kept saying, *don't worry, it's just a number*.

Jacob mentioned that he had been working to correct his deficiencies, and wanted to show the researcher something he had been trying to learn. Jacob told the researcher that he had memorized a rule for adding fractions, $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$,that he found in a text book, although he was not confident that he had accurately recalled the formula. Jacob demonstrated how to use the rule and correctly added the first two terms in the algebraic addition item from Problem Set A. When the researcher asked what would happen to the third term, he said "I don't know (pause). I would separate it and then put it over (pause) it needs to have the same denominator (pause) I don't know, I would ignore it. Pretend it doesn't exist."

The researcher decided to show Jacob an alternative solution to the algebraic item in Problem Set A (see Figure 25) and ask Jacob what he thought of this work. Jacob said the method looked familiar to him, and described it in this way.

> Jacob: You're keeping it uniform. Everything's getting multiplied by the same thing instead of the way I did it up here, but I don't, this way it looks more right (pause) they did a good job on making it look more uniform, giving it some legitimacy, maybe, whether it's right or its wrong.

4. $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

$$= \frac{x+2}{4}\left(\frac{x+1}{x+1}\right) + \frac{6x}{x+1}\left(\frac{4}{4}\right) - 3\left(\frac{x+1}{x+1}\right)\left(\frac{4}{4}\right)$$

$$= \frac{x^2+3x+2 \;+24x \;-12x-12}{4(x+1)}$$

$$= \boxed{\frac{x^2+15x-10}{4(x+1)}}$$

Figure 25. Alternative solution to the algebraic item in Problem Set A.

Jacob told the researcher some details about his status in school. He had returned to school to pursue a degree in engineering. He was keenly aware of his deficiencies in algebra, and was working hard to overcome these difficulties. Jacob described the many textbooks that he had purchased based on recommendations from different mathematics teachers and web-sites. The following is Jacob's description of how he used these textbooks and other resources.

> Jacob: Usually what I do is, I've actually bought a lot of math books, old math books that had been suggested on different websites. People that are math teachers said that they like this, so I bought some of those, and I've bought textbooks, and what I'll do is jump from a text or one textbook, to the one I use in class, to the next. If it deals with something like fractions, I'll go back to one of the more basic math books that I have and check that out and then I'll check websites like Kahn Academy, Purplemath.com. There are a plethora of different websites.
>
> Researcher: There's one called Paul's Online Math.
>
> Jacob: Yes, that one's really good.
>
> Researcher: It is, isn't it?
>
> Jacob: Lamar.edu has some really good resources. There's quite a few I have bookmarked or tabbed, and it's just online resources. What I try to do – and I use YouTube a lot – what I try to do is just get a bunch of different perspectives on how someone else may teach it.
>
> Researcher: Ok, I was just going to ask, even from textbook to textbook different authors do some things different ways. There's more than one way to do some things. Do you ever find that to be troubling, or helpful?

> Jacob: No, I actually prefer that. I prefer talking to a peer from class about something because they're going to be closer to my level so they may be able to use the language that may not be over my head and they tend to be a little more patient because they know they'd just learned this. I really like different opinions and like you said, different authors, completely different style.

The researcher continued to talk with Jacob about the resources he had found, and asked him if he used the tutoring lab on campus. He said he had used the lab, but found it to be "hit or miss," depending on which tutor he would get. The researcher asked Jacob a final question about the applicability of rules in numeric and algebraic contexts. Jacob said, "Again, I would say math being logical, yes, but it would be the same rule for both."

Case analysis. Jacob reported that the operations of the problems was the aspect that he recalled from the assessment. When asked to sort the problems, he arranged them in what he perceived as levels of difficulty, with the algebraic items being more difficult than the numeric problems. Jacob saw that the pair of addition items in Problem Set A were essentially the same, only one had variables and the other had numbers. Despite that, Jacob indicated that he kept numeric and algebraic problems separate in his thinking. When he compared his solutions to this pair of problems, Jacob questioned his choice of different methods. At the same time, he could not imagine that the problems should be worked in the same way when they looked so different.

Jacob recalled the memorized steps of a procedure for adding fractions, although it was not useful when he was presented with a problem that required him to extend the rule to three terms. When asked how he would handle the third term, Jacob said he would ignore it, or pretend it was not there. Jacob reviewed an alternative solution for adding fractions, and indicated that it seemed familiar. He was impressed by the visual "uniformity" of the solution, which gave it "legitimacy."

Jacob admitted that he struggled to internalize the idea that a variable is a placeholder for a number, even though he had heard this fact several times, and he continued to think of variables as separate from numbers. Jacob's statements that "*x* is not a numeric value" and later "it could be any value, really" demonstrated how conflicted he was about the meaning of a variable. Jacob seemed to understand that when a variable was used in an equation, only certain values of the variable would make the equation true. The concept of a variable in an expression, however, confused him. He failed to understand the idea that in that context, a variable could take on any value. However, Jacob's last comment that arithmetic and algebra have the same rules suggested that he had formed a procedural connection between the two domains. Jacob's description of the lengths to which he had gone to improve his mathematics abilities suggested that he was a highly motivated student with a strong desire to succeed.

Cross-case Analysis

In addition to analyzing each individual case, the researcher conducted a cross-case analysis of the eight participants as one unit, and also made comparisons across the established groups. The researcher looked for commonalities and differences among the participants. This cross-case analysis revealed five themes but are presented in the paragraphs that follow.

Correct choice of procedure. A correct solution generally follows the choice of a correct procedure. In the case of the addition and subtraction items in Problem Sets A and C, one correct procedure could have been to find equivalent fractions with common denominators and add the numerators. A second approach to addition and subtraction problems would be to “cross-multiply” using the rule $\frac{\mathrm{A}}{\mathrm{B}} + \frac{\mathrm{C}}{\mathrm{D}} = \frac{\mathrm{AD+BC}}{\mathrm{BD}}$, where A, B, C, and D are integers or polynomials.

In Problem Set A, all eight participants chose a correct strategy to solve the numeric item. Two participants used incorrect strategies to solve the algebraic items. Liam’s procedure was unspecified, and Jacob incorrectly added across the numerators and denominators. In Problem Set C, all eight participants chose a correct procedure to solve the numeric item. Liam omitted the algebraic item in this problem set. All of the remaining participants chose a correct procedure for the algebraic item in Problem Set C.

Problem Set B contained a numeric and algebraic division problem for which there were several correct strategies. One possibility was to invert the second term and multiply across the numerators and denominators. Another strategy would be to cross-multiply, and a third procedure was to find equivalent fractions with common denominators, and then to divide the numerators. Seven of the eight participants chose a correct strategy for the numeric item. One participant omitted the algebraic division item, and two participants did not choose a correct strategy for the algebraic division item. Ethan found a common denominator and then multiplied the numerators and Liam chose the strategy of taking the square root of the second term, and then divided across the numerators and denominators.

Consistent application of procedure. If a participant has formed a connection between numeric and algebraic rational numbers, then it would be reasonable to expect them to use the same procedure in both contexts. Five of the eight participants were consistent in their application of procedures in the numeric and algebraic problems. Three participants, however, failed to provide evidence of connecting fraction procedures to algebraic procedures. In the case of Ethan, he chose the strategy of cross-multiplication for the numeric item, but found a common denominator and multiplied across the numerators in the algebraic context. Jacob found equivalent fractions with common denominators for the numeric item in Problem Set A, although he chose to add across the numerators and denominators for the algebraic item in this set of problems. Liam chose correct procedures for each of the three numeric items, although he did not apply any of these strategies to the three algebraic items. In problem Set A he added across the numerators and denominators, in Problem Set B he took the square root of the second term and divided across numerators and denominators, and finally, he omitted the algebraic item in Problem Set C.

Efficient nature of procedure. Efficiency is a desirable characteristic for procedures. In the case of addition and subtraction of fractions, finding a least common denominator is considered an efficient strategy. A least common denominator reduces the size of the numbers in a numeric context, and the amount of work required to multiply and combine like terms in the algebraic context. Although correct solutions can be found without a least common denominator, one would expect to see at least some efficient strategies used in college mathematics courses such as precalculus. Four of the eight participants found a least common denominator for the numeric item in Problem Set C by recognizing that the two denominators shared one common factor. Only two participants found a least common denominator for the algebraic item in this same problem set by factoring the denominators and recognizing that they shared a common factor. Isabella and Mason responded favorably when the researcher displayed a sample of an efficient method for solving the algebraic item in Problem Set C. Isabella called it “very strategic” and Mason said it was “very smart.” Both Isabella and Mason indicated this was the first time they had seen this strategy, despite the fact that they had completed precalculus and calculus, respectively, in their secondary education.

In the division items, the participants could choose to simplify before or after performing the operation of multiplication. Simplifying before the operation is considered to be a more efficient procedure. Emma was the only participant who simplified before multiplication in the numeric item. Emma, Isabella, and Sophia simplified before multiplication in the algebraic item. The fact that the participants applied an efficient algorithm for simplification more often in the algebraic context was an unexpected result.

Effect of challenge. The researcher used two strategies to challenge the participants’ mathematical thinking. The researcher displayed alternative strategies for the participants’ to review during the interview, and in some cases the researcher asked participants to compare their own numeric and algebraic solutions. Of the four occasions when participants were asked to compare one of their own correct solutions to an incorrect solution, two recognized a mistake they had made, and two did not. The participants were asked to compare their own incorrect solution to a correct alternative solution on four different occasions. Each time, the participants changed their thinking. Jacob reported that a strategy “looked familiar” in one solution and began to recall a method for finding an equivalent fraction after viewing another. Noah said the factoring in the alternative solution “made sense” and looked “more correct.” Ethan examined a correct alternative solution and realized that he should have multiplied the denominators instead of adding the terms to find a common denominator.

Connection between numeric and algebraic contexts. The theoretical framework that guided this study asserted that meaning for procedures is developed when the procedures are connected to number properties. The researcher asked questions during the interviews to draw out the participants’ thinking about the relationship between numeric and algebraic contexts. The three participants in group one had the most correct items on the assessment, and also gave evidence of the strongest connections between numeric and algebraic contexts. Emma recognized that problems in both domains “take the same steps” and follow the same rules because a variable represents a number. Sophia was asked if a certain algebraic procedure was correct, and she reasoned that it would make sense, because that was the same procedure to use if it was a numeric problem. Isabella agreed that they follow similar rules, but noted that they require different operations because there was a limit to the extent to which variables can be combined.

In groups two and three, only Liam provided consistent evidence that he had formed a connection between numeric and algebraic properties. The remaining participants made at least one or more statements that indicated their connection was weak. Ethan made contradictory statements with regard to the division problems in Problem Set B. Ethan thought that the presence of an unknown determined the necessity of a common denominator in the division problems, but also recognized that the problems required the same steps because “it’s [numeric division item] kind of like the same steps as number two [algebraic division item], except it’s just with numbers.”

Mason made this statement, “You pretty much use the same concept, it’s just numerical, not alphabetical” which indicated he was aware of the connection between operations on numeric and algebraic problems. Later, Mason remarked that he preferred to keep them separate in his thinking, and that in fact he had not made the connection between the assessment items until the researcher brought it to his attention. Jacob thought the items in Problem Set A should be solved differently, and was unsure if the same rules applied to both numeric and algebraic problems. He also reported that he kept the two domains separate in his mind. Noah’s comments were strong evidence that he did not make a connection between the operations on numeric and algebraic problems. He said they would be solved completely differently, and like Ethan, he believed that “exponents” dictated which procedure should be used to simplify through cancellation.

Summary of Task-Based Interview Results

The preceding section described the qualitative data collected during task-based interviews with eight participants. These results are not generalizable to the entire population, but they do provide insights into the participants’ mathematical thinking while performing operations on numeric and algebraic rational expressions. In particular, these results identified possible causes for errors made by the participants, and described the extent to which participants connected the properties of numbers with algebraic procedures.

Chapter Summary

The purpose of this study was to explore students' understanding of algebraic rational expressions and the connections they make, if any, to rational numbers. The researcher used statistical analysis of the participants' scores on the assessment of rational expressions to determine to what extent the scores for numeric items and algebraic items followed the same distribution. The researcher examined the participants' written work and coded hundreds of mathematical errors along with the strategies used by the participants to solve each problem. The researcher compared the patterns of errors and strategies found in the algebraic context to those found in the numeric context. Finally, the researcher conducted task-based interviews with a sample of participants to gain insight into their mathematical thinking.

The results of this study illuminated the deficiencies that the participants had with algebraic rational expressions and rational numbers. Many students enter college with insufficient knowledge of these concepts, and algebra in general, which hinders their success in higher mathematics courses. When these students face challenges in the introductory mathematics courses required by their major, many are motivated to choose a different field of study that does not require such an extensive knowledge of mathematics. This may explain, in part, why the nation has a shortage of students who major in a STEM field. The purpose of this study was to investigate one area of algebra, rational expressions, but many algebra skills are encompassed in this one mathematical concept. A discussion of the results of the study and the implications for the learning of mathematics are discussed in the next chapter.

CHAPTER V: DISCUSSION

Introduction

This study investigated three different aspects of the difficulties undergraduate students have with rational expressions. One purpose of the study was to investigate and compare participants' abilities to simplify and perform operations with rational numbers and algebraic rational expressions. Another goal of this study was to determine the strategies participants used and the errors they made when solving numeric rational expressions and algebraic rational expressions. Finally, this study aimed to examine the extent to which students connected algebraic procedures to the methods of simplifying and performing operations with numbers. A restatement of the research problem and a review of the methods used in this study are presented first in this final chapter. This chapter ends with a discussion of the results and the implications for mathematics education.

The Research Problem

As previously mentioned, this study was conducted to address an area of mathematics that many undergraduate students find difficult. Rational expressions have been identified as an area of mathematics where students consistently make errors (Dawkins, n.d.; Schechter, 2009; Scofield, 2003). Competency with rational expressions, as well as other areas of algebra, is essential for students to succeed in advanced college mathematics courses (Baranchik & Cherkas, 2002; Tall, 1993). Success in these advanced college mathematics courses is critical for students who major in a STEM field. Students without the necessary prerequisite knowledge of algebra often struggle through the introductory college courses required for a major in STEM fields, and many leave their field of study for a major outside of STEM (Astin & Astin, 1993; Kokkelenberg & Sinha, 2010). It follows, then, that educators may contribute to the retention of students who major in a STEM field by investigating students' difficulties with areas of algebra such as rational expressions.

Review of Methodology

This mixed-methods study used quantitative and qualitative methods to analyze the data. Statistical analyses quantitatively compared the participants' abilities to simplify and perform operations on matched pairs of numeric and algebraic rational expressions. The study also used qualitative methods to further understand the participants' knowledge of rational numbers and algebraic rational expressions. The researcher examined each assessment item and coded the errors and strategies that were found. The researcher looked for patterns in the categories of errors and the strategies the participants used in the numeric and algebraic context. Finally, this study used task-based interviews to reveal participants' mathematical thinking related to rational numbers and algebraic rational expressions, and the degree to which they connected the two concepts. A discussion of the quantitative and qualitative results is presented in the following sections.

Discussion of the Results

Quantitative Results

The results of the quantitative analysis in this study suggest that undergraduate students have serious deficiencies with algebraic procedures in the context of rational expressions. Less than 14% of participants correctly answered one or more of the algebraic rational expressions. In each of the three problem pairs, a significant difference was found in the distribution of scores, meaning the subjects had different abilities with algebraic and numeric problems. Although 69.2% of participants correctly answered one or more numeric items, the percentage of correct answers for each individual numeric item never exceeded 50%. Research has shown that proficiency with rational numbers is related to success in algebra (Brown & Quinn, 2007; Welder, 2012). The small percent of correct responses demonstrated the degree to which these participants were competent with rational numbers and algebraic rational expressions. Therefore, this result indicates that it is likely that deficiencies with rational numbers may also contribute to students' difficulties with college-level mathematics.

A correlation between participants' abilities with algebraic and numeric rational expressions was found only in the division problem set, and then it could only be categorized as a small effect. The small correlation seen between numeric and algebraic division operations in this study may be related to the consistency with which participants applied procedures in both contexts. The absence of medium or strong correlations between the algebraic and numeric items would suggest that although the participants were more likely to get a numeric item correct and the corresponding algebraic item incorrect, there was no relationship between their abilities in both contexts. Since Hiebert and Carpenter's (1992) framework tells us that algebraic procedural knowledge is connected to conceptual knowledge of number properties, these results suggest that very few participants in this study had made the connection between the two contexts.

Qualitative Results

This study utilized qualitative methods to examine the strategies participants chose to simplify and perform operations in the different contexts of numeric and algebraic rational expressions. Additionally, the researcher identified and categorized the errors made by the participants in each setting. Also, task-based interviews revealed the participants' mathematical thinking that may provide insight into the connections students make between algebraic procedures and properties of numbers. These strategies, errors, and connections will be discussed in the paragraphs that follow.

Strategies. Research of algebraic procedures frequently mentions the connection between arithmetic and algebra (Herscovics & Linchevski, 1994; Linchevski & Livneh, 1999). If the participants in this study connected algebraic procedural knowledge to conceptual knowledge of number properties as described by Hiebert and Carpenter's (1992) framework, then it would be logical to expect that they would apply the same procedures in the numeric and algebraic contexts. The percent of participants who consistently applied the same procedure in the numeric and algebraic items of Problem Sets A, B, and C were 56%, 47%, and 37%, respectively. This result indicated that less than half of the participants in this study connected algebra operations with number operations.

Several different methods could be used to solve the division items in Problem Set B. In the numeric item, 54 participants chose the procedure of "inverted and multiplied," and 51 participants chose the same procedure for the algebraic item in this problem set. There was more variation in the choice of procedure between the numeric and algebraic contexts for the addition items. In Problem Set A, 83 participants chose the strategy “found common denominator” while only 55 chose this same strategy for the algebraic item in that problem set. Similarly, 73 participants chose the strategy “found common denominator” in the numeric item of Problem Set C, although 33 chose that strategy for the algebraic item in the same problem set. The interview results revealed that many of those participants often chose inefficient strategies of finding a common denominator or simplifying a numeric or algebraic expression. Brown and Quinn (2006) reported a similar phenomenon related to the simplification of numeric fractions.

The success rate of correct strategies was vastly different between the numeric and algebraic contexts. In Problem Set A, 58% of the correct strategies led to a correct result for the numeric items and only 13% of the algebraic items. In Problem Set B, the success rate for correct strategies for the numeric and algebraic items was 52% and 12%, respectively. Similarly, the success rate for correct strategies in Problem Set C was 60% in the numeric context and 18% in the algebraic context. The missteps that prevented participants with a correct choice of strategy from reaching a correct solution included *equivalent fraction errors*, *distribution errors*, *cancellation errors*, and several others.

Errors. The participants made more errors in the algebraic items than the numeric items. Mistakes such as *distribution errors* and *operations with monomial errors* that were not possible, or less likely, to occur in the numeric context were one factor that contributed to the increased number of errors in the algebraic context. *Procedural errors* and *cancellation errors*, however, could have occurred in both contexts although they were found more often in the algebraic items. One of the most common procedural errors in the algebraic context was the strategy of “add across” which Wu (2001) believed is a natural extension of students’ knowledge of addition with numbers.

Cancellation errors have previously been reported as the most prevalent mistakes in studies of simplifying algebraic rational expressions (Constanta, 2012; Otten, Males, & Figueras, n.d.; Ruhl, Balatti, & Belward, 2011). In the study by Constanta (2012), he found that students could not discern the difference between a term and a factor. The cancellation of terms, and not factors, was also the cause of most of the cancellation errors in this study.

Ruhl, Balatti, & Belward (2011) reported finding misconceptions related to common factors in their study. In this study, the common factors in the denominators of the numeric and algebraic items Problem Set C were often ignored and participants found a common denominator that was not a *least* common denominator. *Equivalent fraction errors* were consistently found in both the numeric and algebraic items in Problem Set A. Many of these errors occurred when the participant multiplied or divided part of a fraction by the wrong factor.

Connections. The qualitative data gathered from the interviews provided an opportunity for the researcher to investigate the degree to which the participants connected rational number and algebraic rational expression concepts. The results of the interviews were consistent with the quantitative analyses and the qualitative examination of the strategies used by the participants. The findings in all three areas of the study point to a disconnect between numeric and algebraic contexts in the participants’ thinking.

Emma, Sophia, and Isabella provided evidence of connections between operations with numbers and operations with algebraic expressions. Isabella easily recalled algorithms she had learned in the fourth grade, suggesting that she had developed a thickly connected network of knowledge like that described by Heibert and Carpenter (1992). Mason and Jacob made their disconnect explicitly clear when they said they preferred to think separately about operations and properties of numeric and algebraic problems. Despite this lack of a connection between number and algebraic procedures, Mason had correct solutions on several of the assessment items suggesting that he may have applied algorithms that he had memorized but for which he had not developed meaning.

Liam and Jacob, the non-traditional students, struggled to recall facts and procedures related to rational numbers and expressions that they had learned many years ago. This suggested that their earlier mathematics learning experiences had led to the development of sparse, frail networks of knowledge. When students are engaged in the development of algorithms, they develop meaning for procedures (Brown & Quinn, 2006; IES, 2010; NCTM, 2000). Procedures learned in a meaningful way have more connections than memorized algorithms and form stronger, more durable networks of knowledge (Hiebert & Carpenter, 1992).

Ethan and Noah's comments demonstrated that they had formed separate networks of knowledge related to numeric and algebraic operations. Ethan believed that an expression with an unknown required a different procedure than the same expression with numeric terms. Noah believed that different rules for simplification existed for numeric and algebraic expressions. According to Hiebert and Carpenter (1992), the development of new knowledge may involve forming new connections, rearranging connections, or breaking false connections.

Implications for Mathematics Education

The results of this study have implications for mathematics education on several educational levels. The learning of rational numbers in the primary grades lays the foundation for the learning of algebraic rational expressions in secondary school, and the learning of more advanced mathematics at the post-secondary level. The following paragraphs will address implications for the initial instruction of algebraic rational expressions and for remediation of the topic.

Instruction of Algebraic Rational Expressions

According to Skemp (1976), students are able to develop an instrumental understanding when we want them to develop relational understanding. If students have not developed a conceptual understanding of rational numbers, merely memorizing algorithms for adding, subtracting, multiplying, and dividing, they will not be able to make the connections between basic number properties and algebraic procedures described by Hiebert and Carpenter (1992). Instructors teaching algebraic rational expressions as a new concept should design learning experiences for students that engage them in the development of algorithms and support the development of the important connections between arithmetic and algebra.

Remediation of Algebraic Rational Expressions

Secondary and post-secondary educators hope that students will have developed the critical understanding of rational numbers in prior grades, but as this study shows, it is very likely that they do not have this knowledge. Educators of under-prepared students cannot assume, as the teachers did in Constanta's (2012) study, that students have developed an understanding of fundamental concepts such as equivalent fractions and common factors. It would also be unwise to assume that students will form connections and develop new knowledge when they are simply presented with new material. Remediation for students like Ethan and Noah may be harder as it would require breaking the false connections that were formed between non-existent rules and algebraic procedures before new connections could be made.

Instructors of courses for which performing operations with algebraic rational expression is prerequisite knowledge should design learning experiences for students that explicitly address the categories of errors identified in this study. Findings from this study, similar to those of Guzman, Kieran, and Martinez (2010), suggest that instructors may use the evaluation of alternative strategies as a tool to encourage correct mathematical thinking in their students with misconceptions. The interaction between the instructor and students at these levels should include explanations and justifications of procedures so that all students may understand why procedures work and when they should be applied.

Future Areas of Research

This study focused on the abilities that post-secondary, undergraduate students had with numeric and algebraic rational expressions and the connections they made between arithmetic and algebra. As mentioned before, the foundation for this knowledge is obtained in the learning of arithmetic in the primary grades, and in the transition between arithmetic and algebra in the middle grades. Research is needed to determine if rational numbers are taught at the elementary level in a way that allows this knowledge to be extended to the algebraic context in the later grades. It is also desirable to learn more about the transition students make from arithmetic to algebra in the middle grades. Educators at this level need research that would identify the learning experiences that facilitate the students' connections between arithmetic and algebra during this transition. Finally, at the post-secondary level, research that evaluates different means of remediation is needed to determine the most effective methods of increasing students' understanding of rational numbers and algebraic rational expressions, thus opening their access to higher mathematics.

Chapter Summary

Technology and science innovations will be driven by students who major in STEM fields. Increasing the retention and graduation rates for students who major in STEM fields will impact the strength of the position of the United States in the global economy (Machi, 2009). Weak prior academic preparation in algebra often leads to low grades in introductory mathematics courses and discourages students from studying STEM fields. It is possible that helping students succeed in entry-level classes such as precalculus could improve the retention and graduation of students who major in a STEM field. To this end, it is important to understand the conceptual and procedural knowledge that students have when entering college. This study established the existence and the extent of students' algebraic deficiencies with rational expressions, and provided insights into the factors that may influence students' difficulties with rational numbers and algebraic expressions.

While it is important for all students to have algebraic procedural knowledge, it is critical for those who desire to be scientists, physicists, or mathematicians and will study advanced mathematics. It is possible that if the algebra deficiencies in students are identified and addressed early, the number of students succeeding in mathematics and persisting in STEM majors will increase. Understanding students' conceptions and misconceptions related to rational expressions and how they connect algebraic procedures to basic number properties is an important step towards being able to promote success for all students in introductory mathematics courses, but particularly for students who major in STEM that might otherwise leave a STEM field of study.

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APPENDICES

Appendix A

Common Core State Standards – Mathematics

<u>*Mathematics, Grade 6, Expressions and Equations*</u>

CCSS.Math.Content.6.EE.A.3 Apply properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression $3(2 + x)$ *to produce the equivalent expression* $6 + 3x$*; apply the distributive property to the expression* $24x + 18y$ *to produce the equivalent expression* $6(4x + 3y)$*; apply properties of operations to* $y + y + y$ *to produce the equivalent expression* $3y$*.*

<u>*Mathematics, Grade 7, Expressions and Equations*</u>

CCSS.Math.Content.7.EE.B.4a Solve word problems leading to equations of the form $px + q = r$*, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of operations used in each approach.*

<u>*Mathematics, Grade 8, Introduction*</u>

Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equations.

Appendix B

Rational Numbers and Expressions Assessment Instrument

Rational Numbers and Expressions Study
Assessment

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, <u>show all of your work</u>, and <u>write your answer in simplest terms.</u>

1. $\frac{1+6}{2} + \frac{9}{2+3} - 3$

2. $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

3. $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

4. $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

5. $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

6. $\frac{3}{3-1} \div \frac{9}{9-1}$

Rational Numbers and Expressions Study
Assessment

Please provide the following information about your background.		
Age: ______	**Ethnicity and Race:**	_____American Indian or Alaska Native
Gender: _____Male ______Female		_____Asian
ACT-Math score ___________		_____ Black or African American
Major: ____		_____ Hispanic or Latino
Highest Math Course		____ Native Hawaiian or Pacific Islander
Taken in High School:____________________________		_____ White

If you consent to participate in an individual interview, please provide the information requested below.

Name____

Email ___________________ @mtmail.mtsu.edu Phone______________________

Each person who volunteers will be entered into a drawing for a $50 gift card from Amazon.com. Those who are randomly selected, *and who complete the interview*, will be entered twice into the drawing. The drawing will take place when all interviews have been completed. The winner of the gift card will be notified by email and phone.

Thank you for participating!

Appendix C

Interview Protocol

Rational Numbers and Expressions Study
Interview Protocol

Interviewer: *Thank you for meeting with me today. I appreciate that you volunteered to do this interview. I will ask you a few questions and we will talk about some problems from the assessment you took a few weeks ago. I don't expect it to take longer than 30 minutes, unless you want to talk longer. I am video-recording and audio-recording today's interview so that I do not have to write notes, but I am only recording the images projected onto the screen. Even so, the video-tape and any other information gathered will be kept confidential. I will not share anything you say or do during the interview with the teacher of your class. Your participation is voluntary, and you may stop at any time. Do you agree to participate? Great, let's get started.*

<u>Part 1:</u>

Interviewer: *Think about the assessment you took a few weeks ago. Did you notice any similarities between the problems?*

Give the laminated cards to the student.

Interviewer: *Each of these cards has a problem from the assessment. Can you sort these into two or more groups?*

Probes: *What relationship do the cards in each group have with one another?*

Can you arrange the cards in a different way?

Interviewer: *When you work a problem like this (show algebraic problem) do you think about fractions that just have numbers, like this (show numeric problem)?*

<u>Part 2:</u>

Give the student their assessment form, and point out the pre-selected pair of problems.

Interviewer: *Do you remember these problems? Take a minute to look at the first problem, and walk me through your solution. Explain to me what you did at each step.*

Probes: *Can you choose to do that?*

Can you justify why you did that?

Here is an example where someone did it differently. What do you think about this?

Repeat this with the second problem.

Interviewer: *What, if any, relationship do you see between the two problems?*

Interviewer: *That's all, we are done! I really appreciate that you participated in my study. I will enter your name twice in the drawing which will be held when all of the interviews are done. Thank you for coming, and I hope the rest of the semester is great!*

Appendix D

Rational Expression Cards

$$\frac{x+2}{4}+\frac{6x}{x+1}-3$$

$$\frac{1+6}{2}+\frac{9}{5}-3$$

$$\frac{1}{2x^2-3x-2}+\frac{x}{x^2-7x+10}$$

$$\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$$

$$\frac{x}{x-1}\div\frac{x^2}{x^2-1}$$

$$\frac{3}{3-1}\div\frac{9}{9-1}$$

Appendix E

Institutional Review Board Approval

August 29, 2012

Jennifer Yantz, Dr. Angela Barlow
Mathematics and Science Education
Jly2d@mtmail.mtsu.edu, Angela.Barlow@mtsu.edu

Protocol Title: "Developing Meaning for Algebraic Procedures: An Exploration of the Connections Undergraduate Students Make Between Algebraic Rational Expressions and Basic Number Properties"
Protocol Number: 13-037

Dear Investigator(s),

The exemption is pursuant to 45 CFR 46.101(b) (2). This is because your research involves the use of educational tests, survey procedures, interview procedures or observation of public behavior.

You will need to submit an end-of-project report to the Office of Compliance upon completion of your research. Complete research means that you have finished collecting data and you are ready to submit your thesis and/or publish your findings. Should you not finish your research within the three (3) year period, you must submit a Progress Report and request a continuation prior to the expiration date. Please allow time for review and requested revisions. Your study expires on **August 29, 2015.**

Any change to the protocol must be submitted to the IRB before implementing this change. According to MTSU Policy, a researcher is defined as anyone who works with data or has contact with participants. Anyone meeting this definition needs to be listed on the protocol and needs to provide a certificate of training to the Office of Compliance. **If you add researchers to an approved project, please forward an updated list of researchers and their certificates of training to the Office of Compliance before they begin to work on the project.** Once your research is completed, please send us a copy of the final report questionnaire to the Office of Compliance. This form can be located at www.mtsu.edu/irb on the forms page.

Also, all research materials must be retained by the PI or **faculty advisor (if the PI is a student)** for at least three (3) years after study completion. Should you have any questions or need additional information, please do not hesitate to contact me.

Sincerely,
Andrew W. Jones
Graduate Assistant to:
Emily Born
Compliance Officer
615-494-8918
Emily.Born@mtsu.edu

Appendix F

Information Sheet

Rational Numbers and Expressions Study
Information Sheet

Principal Investigator: Jennifer Yantz
Study Title: Developing Meaning for Algebraic Procedures: An Exploration of the Connections Undergraduate Students Make Between Algebraic Rational Expressions and Basic Number Properties
Institution: Middle Tennessee State University

The following information is provided to inform you about the research project and your participation in it. Please read this form carefully and feel free to ask any questions you may have about this study and the information given below. You will be given an opportunity to ask questions, and your questions will be answered. Also, you will be given a copy of this consent form.

Your participation in this research study is voluntary. You are also free to withdraw from this study at any time. For additional information about giving consent or your rights as a participant in this study, please feel free to contact the MTSU Office of Compliance at (615) 494-8918.

1. **Purpose of the study:**
   You are being asked to participate in a research study because the Department of Mathematics is committed to improving undergraduate instruction and providing you with the best education possible. The specific purpose of the study is to learn more about what students know of rational expressions.

2. **Description of procedures to be followed and approximate duration of the study:**
   In a moment, I will give you an assessment with six mathematics problems and four demographic questions to answer. You will have 20 minutes to complete the assessment. I will collect the assessments when you are finished. This assessment will not have any impact on your grade for this course.

3. **Expected costs:**
   There are no expected costs for this study.

4. **Description of the discomforts, inconveniences, and/or risks that can be reasonably expected as a result of participation in this study:**
   Participation in this study will require you to give up about 20 minutes of your instruction time.

5. **Compensation in case of study-related injury:**
   N/A

6. **Anticipated benefits from this study:**
   a) The potential benefits to science and humankind that may result from this study are an increased understanding of the mathematical abilities of undergraduate students in Precalculus courses.
   b) There are no potential personal benefits to you from this study.

7. **Alternative treatments available:**
   N/A

8. **Compensation for participation:**
   There is no compensation for participating in today's assessment. Students who volunteer to participate in the next phase of this study, personal interviews, will be entered into a drawing to win a $50 gift card from Amazon.com. Volunteers who are selected and complete the interview will be entered twice in the drawing.

9. **Circumstances under which the Principal Investigator may withdraw you from study participation:**
   N/A

10. **What happens if you choose to withdraw from study participation:**
    There will be no repercussions from withdrawing to participate in this study.

Contact Information. If you should have any questions about this research study or possible injury, please feel free to contact Jennifer Yantz at 615-479-7050 or my Faculty Advisor, Dr. Angela Barlow at 615-898-5353.

Confidentiality. All efforts, within reason, will be made to keep the personal information in your research record private but total privacy cannot be promised. Your information may be shared with MTSU or the government, such as the Middle Tennessee State University Institutional Review Board, Federal Government Office for Human Research Protections

Appendix G

Problem Set A Numeric Item Errors

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Did not find common denominator"				
Total No.	8				
Location of Error	$\frac{1+6}{2}+\frac{9}{2+3}-3$			$\frac{7}{2}+\frac{9}{5}$	Omitted
Errors	$\frac{16}{7}-3$	$\frac{16}{7}-3$	$\frac{7}{2}+\frac{9}{5}$	$\frac{18}{35}$	N/A
Description of Errors	Added numerator without finding common denominator	Added denominators instead of finding common denominator	Dropped a term	Cross multiplied	N/A
Error Count	2	2	1	1	2

Figure G1. Description and frequency of errors in Problem Set A numeric item core category "Did not find common denominator."

Problem: $\frac{1+6}{2} + \frac{9}{2+3} - 3$

	Errors in core category "Did not find common denominator"				
Total No.	11				
Location of Error	$\frac{\overbrace{1+6}^{7}}{2} + \frac{9}{\underbrace{2+3}_{5}} - 3$	$\left(\frac{7}{2} + \frac{9}{5}\right) - 3$		$\frac{7}{2} + \frac{9}{5} - \frac{3}{1}$	$\frac{7}{2} + \frac{9}{5} - \frac{3}{1}$
Error	Did not continue	$(3.5 - 5.8) - 3$	Did not continue	$\frac{13}{8}$	$\frac{13}{8}$
Description of Error	Did not continue	Incorrect equivalent decimal	Did not continue	Added numerators without finding common denominator	Added denominators (with an error) instead of finding common denominator
Error Count	1	1	5	2	2

Figure G2. Description and frequency of errors in Problem Set A numeric item core category "Did not find common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Did not find common denominator"				
Total No.	5				
Location of Error	$\frac{1+6}{2}+\frac{9}{2+3}-3$	$\frac{4+6}{9+54}-\frac{3}{1}$	$\frac{1+\cancel{6}^{3}}{\cancel{2}}+\frac{9}{2+3}-3$		$\frac{1+6}{2}+\frac{\cancel{9}^{3}}{2+\cancel{3}}-3$
Error	$\frac{1+6}{2}\times\frac{2+3}{9}=\frac{4+6}{9+54}$	$\frac{4+6}{9+54}\times\frac{1}{-3}=\frac{-12\pm 18}{9+54}$	$1+3+\frac{3}{2}-3$		$1+3+\frac{3}{2}-3$
Description of Error	Inverted 2nd term and cross multiplied	Inverted 2nd term and cross multiplied	Cancelled addend in numerator with factor in denominator	Result of cancellation was zero	Cancelled addend in denominator with factor in numerator
Error Count	1	1	1	1	1

Figure G3. Description and frequency of errors in Problem Set A numeric item core category "Did not find common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Did not find common denominator"					
Total No.	6					
Location of Error	$\frac{-12 \pm 18}{9+54}$	$\frac{-30}{65}$	$3.5+\frac{9}{5}-3$	$\frac{7}{2}+\frac{9}{5}-3$		
Error	$\frac{-30}{65}$	$\frac{-6}{12}$	$\frac{1}{2}+\frac{9}{5}$	$\frac{16}{7}-3$	$\frac{14}{7}-3$	$\frac{14}{7}-3$
Description of Error	Arithmetic error - addition	Equivalent Fraction error – Division error in denominator	Did not continue	Did not continue	Added numerators (with an error) without finding common denominator	Added denominators instead of finding common denominator
Error Count	1	1	1	1	1	1

Figure G4. Description and frequency of errors in Problem Set A numeric item core category "Did not find common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Did not find common denominator"	
Total No.	2	
Location of Error	$\frac{1+6}{2}+\frac{9}{2+3}-3$	$\frac{18}{35}-3$
Error	$\frac{18}{35}-3$	$-2\frac{17}{1}$
Description of Error	Cross multiplied (18 = 2 x 9, 35 = 7 x 5)	Unspecified
Error Count	1	1

Figure G5. Description and frequency of errors in Problem Set A numeric item core category "Did not find common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"		
Total No.	6		
Location of Errors	$\frac{1+6}{2}+\frac{9}{2+3}-3$		$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$
Errors	$\frac{5}{2}+\frac{9}{5}-\frac{3}{1}$	$\frac{5}{2}+\frac{9}{6}-\frac{3}{1}$	$\frac{7}{10}+\frac{9}{10}-\frac{3}{1}$
Description of Errors	Arithmetic error - addition	Arithmetic error - addition	Equivalent Fraction Error - did not multiply numerator by factor
Error Count	1	4	1

Figure G6. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2} + \frac{9}{2+3} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{7}{2} + \frac{9}{5} - \frac{3}{1}$			$\frac{1+6}{2} + \frac{9}{2+3} - 3$
Errors	$\frac{7}{10} + \frac{9}{10} - \frac{3}{10}$	$\frac{5}{10} + \frac{2}{10} - \frac{10}{10}$	$\frac{7}{10} + \frac{9}{10} - \frac{3}{1}$	$\frac{1+6+3}{2+3} + \frac{9}{2+3} + \frac{-6-9}{2+3}$
Description of Errors	Equivalent Fraction Error - did not multiply numerator by factor	Equivalent Fraction Error multiplied numerator by wrong factor	Equivalent Fraction Error - did not multiply numerator by factor	Equivalent Fraction error – added a term to both numerator and denominator
Error Count	1	1	1	1

Figure G7. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2} + \frac{9}{2+3} - 3$

	Errors in core category "Found common denominator"					
Total No.	7					
Location of Errors	$\frac{1+6}{2} + \frac{9}{2+3} - 3$		$\frac{7}{2} + \frac{9}{5} - \frac{3}{1}$		$\frac{35}{10} + \frac{18}{10} - 3$	
Errors	$\frac{1+6+3}{2+3} + \frac{9}{2+3} + \frac{-6-9}{2+3}$	$\frac{7}{2} + \frac{9}{5}$	$\frac{5}{10} + \frac{2}{10} - \frac{10}{10}$	$\frac{5}{10} + \frac{2}{10} - \frac{10}{10}$	$\frac{55}{10} - 3$	$\frac{43}{10} - \frac{30}{10}$
Description of Errors	Incorrect common denominator – chose common addends instead of factors	Dropped a term	Equivalent Fraction Error replaced numerator with factor needed for equivalent fraction	Equivalent Fraction Error replaced numerator with factor needed for equivalent fraction	Arithmetic error - addition	Arithmetic error - addition
Error Count	1	1	1	1	1	2

Figure G8. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"				
Total No.	7				
Location of Errors	$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}$		$\frac{(2+3)1+6}{2(2+3)}+\frac{9\,(2)}{2(2+3)}-\frac{30}{10}$	$\frac{12}{6}$	$3\frac{5}{10}+1\frac{8}{10}-3$
Errors	$\frac{83}{10}$	$\frac{13}{10}$	$\frac{42}{10}+\frac{18}{10}-\frac{30}{10}$	$\frac{1}{2}$	$3\frac{13}{10}-3$
Description of Errors	Arithmetic error – addition	Arithmetic error – addition	Arithmetic error - multiplication	Equivalent Fraction Error – inverted simplified fraction	Made error combining mixed fractions
Error Count	3	1	1	1	1

Figure G9. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"			
Total No.	5			
Location of Errors	$\frac{43}{10}-\frac{30}{10}$	$\frac{8}{5}-\frac{3}{1}$	$\frac{35}{10}+\frac{18}{10}-3$	$\frac{53}{10}-\frac{30}{10}$
Errors	$\frac{3}{10}$	$\frac{8}{5}-\frac{3}{5}$	$\frac{53}{10}-3$	$\frac{156}{30}-\frac{30}{30}$
Description of Errors	Arithmetic error - addition	Equivalent Fraction Error did not multiply numerator by a factor	Answer not simplified	Equivalent Fraction Error did not multiply numerator by a factor
Error Count	2	1	1	1

Figure G10. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}$	$\frac{53}{10}-3$		$\frac{7}{2}+\frac{9}{5}$
Errors	$\frac{73}{10}$	$\frac{51}{10}$	$\frac{56}{10}$	$\frac{35}{10}+\frac{19}{10}$
Description of Errors	Arithmetic error – addition	Combined a whole number and the numerator of a fraction without finding common denominator	Combined a whole number and the numerator of a fraction without finding common denominator	Equivalent Fraction Error added factor to numerator instead of multiplying
Error Count	1	1	1	1

Figure G11. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"				
Total No.	5				
Location of Errors	$\frac{35+18-30}{10}$	$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$			
Errors	Did not continue	$\frac{12}{10}+\frac{11}{10}-\frac{13}{10}$	$\frac{12}{10}+\frac{11}{10}-\frac{13}{10}$	$\frac{12}{10}+\frac{11}{10}-\frac{13}{10}$	$\frac{70}{10}+\frac{18}{10}-\frac{30}{10}$
Description of Errors	Did not continue	Equivalent Fraction Error – added factor to numerator	Equivalent Fraction Error – added factor to numerator	Equivalent Fraction Error – added factor to numerator	Equivalent Fraction Error – multiplied by common denominator instead of missing factor
Error Count	1	1	1	1	1

Figure G12. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{1+6}{2}+\frac{9}{2+3}-3$	$\frac{1+6}{2}+\frac{9}{2+3}-3$		
Errors	$\frac{9}{2+3}+\frac{9}{2+3}-\frac{3}{2+3}$	$\frac{9}{2+3}+\frac{9}{2+3}-\frac{3}{2+3}$	$\frac{9}{2+3}+\frac{9}{2+3}-\frac{3}{2+3}$	$\frac{7}{2}+\frac{9}{6}-\frac{3}{1}$
Description of Errors	Unspecified error	Incorrect common denominator – Chose common addends instead of factors	Equivalent Fraction Error – did not multiply numerator by a factor	Arithmetic error – addition
Error Count	1	1	1	1

Figure G13. Description and frequency of errors in Problem Set A numeric item core category "Found common denominator."

Problem: $\frac{1+6}{2}+\frac{9}{2+3}-3$

<table>
<tr><td></td><td colspan="5">Errors in core category “Found common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="5">5</td></tr>
<tr><td>Location of Errors</td><td>$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$</td><td>$\frac{1+6}{2}+\frac{9}{2+3}-3$</td><td>$\frac{43}{10}-\frac{30}{10}$</td><td colspan="2">$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$</td></tr>
<tr><td>Errors</td><td>$\frac{35}{10}+\frac{18}{10}+\frac{30}{10}$</td><td>$\frac{(1+3)(1+6)}{(1+3)(2)}$</td><td>13</td><td>$\frac{49}{10}+\frac{18}{10}-\frac{30}{10}$</td><td>$\frac{16}{10}-3$</td></tr>
<tr><td>Description of Errors</td><td>Changed a sign</td><td>Equivalent Fraction Error – multiplied by wrong factor</td><td>Dropped denominator</td><td>Arithmetic error - multiplication</td><td>Added numerators before finding equivalent fractions</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure G14. Description and frequency of errors in Problem Set A numeric item core category “Found common denominator.”

Appendix H

Error Categories in Problem Set A Numeric Item

Description of Errors in Problem 1 $\frac{1+6}{2}+\frac{9}{2+3}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Arithmetic error - addition	1	Arithmetic Error	21
Arithmetic error - addition	1		
Arithmetic error - addition	4		
Arithmetic error - addition	1		
Arithmetic error - addition	2		
Arithmetic error - addition	2		
Arithmetic error – addition	3		
Arithmetic error – addition	1		
Arithmetic error – addition	1		
Arithmetic error – addition	1		
Arithmetic error - multiplication	1		
Arithmetic error - multiplication	1		
Changed a sign	1		
Made error combining mixed fractions	1		

Figure H1. Error Categories in Problem Set A Numeric Item.

Description of Errors in Problem 1 $\frac{1+6}{2}+\frac{9}{2+3}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Equivalent Fraction Error - did not multiply numerator by factor	1	Equivalent Fraction Error	18
Equivalent Fraction Error - did not multiply numerator by factor	1		
Equivalent Fraction Error - did not multiply numerator by factor	1		
Equivalent Fraction error – added a term to both numerator and denominator	1		
Equivalent Fraction Error – added factor to numerator	1		
Equivalent Fraction Error – added factor to numerator	1		
Equivalent Fraction Error – added factor to numerator	1		
Equivalent Fraction Error – did not multiply numerator by a factor	1		
Equivalent Fraction error – division error in denominator	1		
Equivalent Fraction Error – inverted simplified fraction	1		
Equivalent Fraction Error – multiplied by common denominator instead of missing factor	1		
Equivalent Fraction Error – multiplied by wrong factor	1		
Equivalent Fraction Error -added factor to numerator instead of multiplying	1		
Equivalent Fraction Error -did not multiply numerator by a factor	1		
Equivalent Fraction Error -did not multiply numerator by a factor	1		
Equivalent Fraction Error -multiplied numerator by wrong factor	1		
Equivalent Fraction Error -replaced numerator with factor needed for equivalent fraction	1		
Equivalent Fraction Error - replaced numerator with factor needed for equivalent fraction	1		

Figure H2. Error Categories in Problem Set A Numeric Item.

Description of Errors in Problem 1 $\frac{1+6}{2}+\frac{9}{2+3}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added numerator without finding common denominator	2	Procedural Error	17
Added denominators (with an error) instead of finding common denominator	2		
Added denominators instead of finding common denominator	2		
Added denominators instead of finding common denominator	1		
Added numerators (with an error) without finding common denominator	1		
Added numerators before finding equivalent fractions	1		
Added numerators without finding common denominator	2		
Cross multiplied (18 = 2 x 9, 35 = 7 x 5)	1		
Cross multiplied	1		
Inverted 2nd term and cross multiplied	1		
Inverted 2nd term and cross multiplied	1		
Combined a whole number and the numerator of a fraction without finding common denominator	1		
Combined a whole number and the numerator of a fraction without finding common denominator	1		
Did not continue	1	Persistence Error	9
Did not continue	5		
Did not continue	1		
Did not continue	1		
Did not continue	1		

Figure H3. Error Categories in Problem Set A Numeric Item.

Description of Errors in Problem 1 $\frac{1+6}{2}+\frac{9}{2+3}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Dropped a term	1	Miscellaneous Errors	10
Dropped a term	1		
Dropped denominator	1		
Cancelled addend in denominator with factor in numerator	1		
Cancelled addend in numerator with factor in denominator	1		
Result of cancellation was zero	1		
Incorrect common denominator – Chose common addends instead of factors	1		
Incorrect common denominator – Chose common addends instead of factors	1		
Answer not simplified	1		
Incorrect decimal form of a fraction	1		
Unspecified error	1	Unspecified Error	2
Unspecified error	1		
Omitted	2	Omitted	2
		Total	79

Figure H4. Error Categories in Problem Set A Numeric Item.

Appendix I

Problem Set A Algebraic Item Errors

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"				
Total No.	18				
Location of Error	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		$\frac{7x+2}{x+5} - \frac{3}{1}$		$\frac{x+2}{4}$
Error	$\frac{7x+2}{x+5} - \frac{3}{1}$	$\frac{7x+2}{x+5} - \frac{3}{1}$	$\frac{7x-1}{x-4}$	$\frac{7x-1}{x-4}$	$\frac{2x}{4}$
Description of Error	Added numerators without finding common denominator	Added denominators	Added numerators without finding common denominator	Added denominators	Combined terms by changing operation from addition to multiplication
Error Count	6	7	1	1	3

Figure I1. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"					
Total No.	8					
Location of Error	$\frac{x+2}{4}+\frac{6x}{x+1}-3$			$\frac{6x}{x+1}$	$\frac{2\cancel{x}}{4}+\frac{6x}{\cancel{x}}$	
Error	$\frac{x+2+6x-3}{4+x+1-1}$	$\frac{6x+x+2}{x+5}-\frac{3}{1}$	$\frac{x+2+6x-3}{4+x+1-1}$	$\frac{6x}{x}$	$\frac{1}{2}+6$	
Description of Error	Added numerators without finding common denominator	Added numerators without finding common denominator	Added denominators	Combined terms by changing operation from addition to multiplication	Cancelled parts of terms	Result of cancellatio n was zero
Error Count	1	1	1	3	1	1

Figure I2. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"			
Total No.	16			
Location of Error	$\frac{6x}{x+1}$	-3		
Error	$6x^2 + 6x$	-3	-3	-3
Description of Error	Multiplied twice by denominator. Once to cancel with denominator and again to change numerator	Dropped term from solution but reappears later	Dropped term from solution, never reappears	Left whole number unchanged
Error Count	1	4	2	9

Figure I3. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"					
Total No.	6					
Location of Error	$\frac{x+2}{4}$				$\frac{7\cancel{x}+2}{\cancel{x}+5}$	
Error	$4x + \frac{1}{2}$	$4x + \frac{1}{2}$	$4x + \frac{1}{2}$	$\frac{x}{2} + \frac{6}{1}$	$\frac{9}{5}$	
Description of Error	Multiplied numerator by its own denominator	Distribution Error – multiplied factor times first term only	Cancelled a factor of 2 from a term in the numerator with a factor in the denominator	Cancelled term in numerator with factor in denominator	Cancelled factors of terms	Result of cancellation was zero
Error Count	1	1	1	1	1	1

Figure I4. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"					
Total No.	6					
Location of Error	$\frac{x+2}{4}+\frac{6x}{x+1}-3$	$4\frac{(x+2)}{4}$		$\frac{6x}{x+1}$	$\frac{6x}{x+1}$	$9x+6$
Error	$4\frac{(x+2)}{4}$	$4x+8$		$5x+1$	$5x+1$	$3x+2$
Description of Error	Equivalent Fraction Error multiplied numerator by its own denominator	Cancelled factor with denominator and distributed it to numerator	Result of cancellation was zero	Cancellation error subtraction	Moved term from denominator to numerator	Divided an expression by 3 inappropriate simplification
Error Count	1	1	1	1	1	1

Figure I5. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"		
Total No.	4		
Location of Error	$\frac{2+x}{4} + \frac{6x}{x+1} - 3$	$\left(\frac{x+2}{4} + \frac{6x}{x+1} - 3\right) 4x + 1$	$\frac{x+2}{4} + \frac{6x}{x+1} - \frac{3}{1}$
Error	Did not finish	$\frac{4x^2 + 7x + 2}{16x}$	$\frac{7x - 1}{4x + 2}$
Description of Error	Did not finish	Unspecified	Added terms in the numerator
Error Count	2	1	1

Figure I6. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"			
Total No.	4			
Location of Error	$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{3}{1}$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$	$\frac{x+2}{4}+6x+1-3$	
Error	$\frac{7x-1}{4x+2}$	$\frac{x+2}{4}+6x+1-3$	$7x$	
Description of Error	Like Term Error mixed addition and multiplication	Moved term from denominator to numerator	Added numerators without finding common denominator	Dropped a denominator, never reappeared
Error Count	1	1	1	1

Figure I7. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

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Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"				
Total No.	7				
Location of Error	$\frac{6x+x+2}{x+5} - \frac{3}{1}$		$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		$\frac{7x+2}{5+x}$
Error	$\frac{6x+x-3}{x+4}$	$\frac{6x+x-3}{x+4}$	$\frac{x+2+6x}{4+x+1}$	$\frac{x+2+6x}{4+x+1}$	$\frac{7x}{x} + \frac{2}{5} - 3$
Description of Error	Did not combine all like terms	Unspecified	Added terms in numerator without finding common denominator	Added terms in denominator	Broke apart expression
Error Count	1	1	2	2	1

Figure I8. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Did not find common denominator"				
Total No.	5				
Location of Error	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$			$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	
Error	$x^2 + 3 + 6x + 4 - 3$			$\frac{1}{2} + \quad -3$	$\frac{1}{2} + \quad -3$
Description of Error	Took common denominator factor missing from denominator and added it to numerator (with a mistake)	Like term error addition	Dropped denominators	Cancelled factor from term in numerator with factor in denominator ½ came from (x + 2)/4	Dropped a term (x)
Error Count	1	1	1	1	1

Figure I9. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"					
Total No.	6					
Location of Error	$\frac{x+2}{4}+\frac{6x}{x+1}-3$		$\frac{x+2}{4}+\frac{6x}{x+1}-3$			
Error	$\frac{1}{2}+\emptyset-3$	$\frac{1}{2}+\emptyset-3$	$\frac{x}{2}+\frac{6}{1}$		$\frac{x}{2}+\frac{6}{1}$	$\frac{x}{2}+\frac{6}{1}$
Description of Error	Left out a part Scratch work indicated an attempt at long division with 6*x* divided by $x+1$	Did not finish	Cancelled a factor in numerator with term in denominator	Result of cancellation was zero	Dropped the -3 term	Did not finish
Error Count	1	1	1	1	1	1

Figure I10. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

<table>
<tr><td></td><td colspan="3">Errors in core category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="3">3</td></tr>
<tr><td>Location of Error</td><td colspan="2">$\frac{x+2}{4}+\frac{6x}{x+1}-3$</td><td>$\frac{2x}{4}+\frac{6x}{x}-3$</td></tr>
<tr><td>Error</td><td>$\frac{x+2}{4}+\frac{x+1}{6x}=\frac{4x+1}{6x^2+2}-3$</td><td>$\frac{x+2}{4}+\frac{x+1}{6x}=\frac{4x+1}{6x^2+2}-3$</td><td>$.5x+6x-3$</td></tr>
<tr><td>Description of Error</td><td>Inverted second fraction and cross multiplied</td><td>Distribution error – monomial x binomial</td><td>Unspecified</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure I11. Description and frequency of errors in Problem Set A algebraic item core category “Did not find common denominator.”

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"					
Total No.	9					
Location of Error	$\frac{x+2}{4}+\frac{6x}{x+1}-3$			$\frac{x+1}{2}+\frac{6}{1}$		$\frac{x+2}{4}+\frac{6x}{x+1}-3$
Error	$\frac{x+1}{2}+\frac{6}{1}$	$\frac{x+1}{2}+\frac{6}{1}$		$\frac{x+7}{3}-3$	$\frac{x+7}{3}-3$	$(x+2)+6x+1-3$
Description of Error	Cancellation error – cancelled factor from term in numerator with factor of denominator	Cancellation error – cancelled factor from numerator with factor of term in denominator	Result of cancellation was zero	Added numerators without finding common denominator	Added denominators	Unspecified
Error Count	2	2	2	1	1	1

Figure I12. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"						
Total No.	9						
Location of Error	$\frac{x+2+6x}{4+x+1}-3$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$			$\frac{7\cancel{x}+\cancel{2}}{\cancel{x}+\cancel{4}}-3$		
Error	$\frac{7x+2x-3}{x+5}$	$\frac{7x+2}{4x+1}-3$	$\frac{7x+2}{4x+1}-3$	$\frac{7x+2}{x+4}-3$	$\frac{7+1}{2}-3$	$\frac{7+1}{2}-3$	$\frac{7+1}{2}-3$
Description of Error	Notation Error 2 became 2*x*	Added numerators without finding common denominator	Incorrect denominator 4*x* + 1	Added denominators	Cancelled factor in terms	Result of cancellation was zero	Cancelled factor in terms
Error Count	1	2	2	1	1	1	1

Figure I13. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category “Did not find common denominator”					
Total No.	6					
Location of Error	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		$(x+1)\frac{x+2}{4} - 3 = \frac{6x}{\cancel{x+1}}\cancel{(x+1)}$			$\frac{x+1}{2} + 6 - 3$
Error	$\frac{x+2}{4} - 3 = \frac{6x}{x+1}$	$\frac{x+2}{4} - 3 = \frac{6x}{x+1}$	$\frac{x^2+3x+20}{10} - 3 = \frac{6x}{6}$			$x + 1 + 12 - 6$
Description of Error	Changed to an equation	Sign Error. Since this was set equal to zero, the signs of the right hand side were wrong	Distribution Error multiplied only the first term on the left side by $(x + 1)$	Unspecified error	Distribution Error divided only one term by factor	Multiplied an expression by a factor other than one
Error Count	1	1	1	1	1	1

Figure I14. Description and frequency of errors in Problem Set A algebraic item core category “Did not find common denominator.”

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

<table>
<tr><td></td><td colspan="5">Errors in core category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="5">8</td></tr>
<tr><td>Location of Error</td><td colspan="3">$\frac{x+2}{4} + \frac{6x}{x+1} - 3$</td><td>$\frac{1}{\cancel{2}} + \frac{\cancel{6}x}{1} - \frac{3}{1}$</td><td>$\frac{x+2}{4} + \frac{6x}{x+1} - 3$</td></tr>
<tr><td>Error</td><td colspan="2">$\frac{1}{2} + \frac{6x}{1} - \frac{3}{1}$</td><td>$\frac{1}{2} + \frac{6x}{1} - \frac{3}{1}$</td><td>$\frac{1}{1\cancel{2}} + \frac{3\cancel{6}x}{1} - \frac{3}{1}$</td><td>$4(x+2) + 6x(x+1) - 3$</td></tr>
<tr><td>Description of Error</td><td>Cancelled addend from numerator of one term with addend in denominator of another term</td><td>Result of cancellation was zero</td><td>Cancelled term (2) in numerator with factor in denominator.</td><td>Cancelled factor in numerator of one term with factor in denominator of another</td><td>Multiplied all numerators by their denominators</td></tr>
<tr><td>Error Count</td><td>2</td><td>2</td><td>1</td><td>1</td><td>2</td></tr>
</table>

Figure I15. Description and frequency of errors in Problem Set A algebraic item core category “Did not find common denominator.”

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"		
Total No.	3		
Location of Error	$4(x+2)+6x(x+1)-3$	$6x^2+18x-3$	$\frac{2}{4}+\frac{6x}{1}-\frac{3}{1}$
Error	$4x+8x+6x^2+6x-3$	$2x^2+6x-1$	$6x\frac{1}{4}$
Description of Error	Distribution Error an extra "*x*" appeared in second term	Divided entire expression by a number other than one	Unspecified error
Error Count	1	1	1

Figure I16. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Did not find common denominator"	
Total No.	2	
Location of Error	$6x^2+10x+8-3$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$
Error	$6x^2+10x+5=0$	$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{3}{1}$
Description of Error	Changed to an equation equal to zero, solved for *x*	Rewrote problem but did not finish
Error Count	1	1

Figure I17. Description and frequency of errors in Problem Set A algebraic item core category "Did not find common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category “Found common denominator”						
Total No.	36						
Location of Errors	$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{12x+12}{4(x+1)}$	$\frac{x+2}{4}$	$\frac{6x}{x+1}$	-3		$\frac{x+2}{4}+\frac{6x}{x+1}-3$	Omitted
Errors	$\frac{-5x+14}{4x+4}$	$\frac{x+2}{4(x+1)}$	$\frac{6x}{4(x+1)}$	-3	-3	$\frac{a}{b}-3$ Where a/b is an algebraic expression	N/A
Description of Errors	Didn’t distribute “-“ sign +14 came from 2 + 12	Equivalent Fraction Error Didn’t multiply numerator by factor	Equivalent Fraction Error Didn’t multiply numerator by factor	Dropped term from solution but reappears later	Dropped term from solution, never reappears	Left whole number unchanged	N/A
Error Count	1	4	2	1	9	6	13

Figure I18. Description and frequency of errors in Problem Set A algebraic item core category “Found common denominator.”

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Error	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		$\frac{2x}{4} + \frac{6x}{1x} - 3$
Error	$\frac{6x^2+2}{4x+1}$	$\frac{6x^2+2}{4x+1}$	$\frac{3}{4}x - 3$
Description of Error	Combined numerators without finding common denominator	Like Term Error operation unspecified	Unspecified error
Error Count	1	1	1

Figure I19. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"				
Total No.	7				
Location of Errors	$\frac{(x+2)(x+1)}{4(x+1)}$	$-\frac{3(x+1)}{4(x+1)}\left(\frac{4}{4}\right)$	$(x+1)(x+2)$	$-3(x+1)$	$\frac{(x+1)(x+2)}{4(x+1)}$
Errors	$\frac{x^2+3x+2}{4x+1}$	$-\frac{3(x+1)}{4(x+1)}$	x^2+3x+3	$-3x+1$	$\frac{x^2+3x+3}{4x+4}$
Description of Errors	Notation Error – dropped parentheses (regained later)	Notation Error – dropped a term (4/4)	Distribution Error binomial x binomial	Distribution Error Scalar x binomial	Distribution Error binomial x binomial
Error Count	1	1	2	1	2

Figure I20. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{x^2+3x+2+24x-12\cancel{(x+1)}}{4\cancel{(x+1)}}$	$\frac{6x}{x+1}$	$\frac{(x+1)(x+2)}{4(x+1)}+\frac{4(6x)}{4(x+1)}-3$	$\frac{x+2}{4}$
Errors	$\frac{x^2+27x-10}{4}$	$\frac{x+2}{4}+\frac{20x}{4}-3$	$\frac{x^2+x+2}{4(x+1)}+\frac{4(6x)}{4(x+1)}-3$	$\frac{x^2+4x+4}{4x+4}$
Description of Errors	Cancelled a factor of a term in numerator with a factor in denominator	Unspecified equivalent fraction error	Distribution Error binomial x binomial	Equivalent Fraction Error Multiplied numerator by wrong factor
Error Count	1	1	1	1

Figure I21. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x+2}{4}+\frac{6x}{x+1}-3$		
Errors	$\frac{x+2}{4}+\frac{20x}{4}-3$	$\frac{x^2+2x}{4}+\frac{24x^2}{4}-\frac{12x}{4}$	$\frac{x^2+2x}{4}+\frac{24x^2}{4}-\frac{12x}{4}$
Description of Errors	Incorrect common denominator (4)	Incorrect common denominator (4)	Equivalent Fraction Error – used factor of $4x$ although common denominator is written as 4
Error Count	1	1	1

Figure I22. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"			
Total No.	5			
Location of Errors	$\frac{21x+2}{4}-\frac{12}{4}$	$\frac{x+2}{4}+\frac{6x}{x+1}-3$	$\frac{x^2+28x+4}{4x+4}$	$4x+4$
Errors	$\frac{2x-10}{4}$	$\frac{x^2+2x}{4}+\frac{6x^2}{1}-3x$	$\frac{x^2+7x+1}{x+1}$	$4x+1$
Description of Errors	Notation Error dropped a "1" from 21	Multiplied all numerators by *x*, but *x* is not a factor in the common denominator chosen	Unspecified	Notation Error carried it through rest of solution
Error Count	1	1	1	2

Figure I23. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x}{1}\left(\frac{x+2}{4}+\frac{6x}{x+1}-\frac{3}{1}\right)$		$\left(\frac{7x+2}{4x+4}\right)-3$
Errors	$\left(\frac{x^2+2}{4}+\frac{6x}{1}-\frac{3x}{1}\right)$	$\left(\frac{x^2+2}{4}+\frac{6x}{1}-\frac{3x}{1}\right)$	$\left(\frac{3x+2}{3x+4}\right)-3$
Description of Errors	Distribution Error monomial x binomial	Cancelled a scalar with addend in denominator	Unspecified
Error Count	1	1	1

Figure I24. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x^2+3x+2}{4(x+1)} + \frac{24x}{4(x+1)} + \frac{-(12x+12)}{4(x+1)}$	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	$\frac{6x}{x+1}$
Errors	$\frac{x^2+15x+10}{4(x+1)}$	$\frac{x+2}{4} + \frac{6x}{x+1} = 3$	$\frac{32x}{4x+4}$
Description of Errors	Sign Error – distribution of -1	Changed expression to an equation	Arithmetic error - multiplication
Error Count	1	1	1

Figure I25. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	-3	$x\left(\frac{(x+2)}{4}\right)$	$\frac{(x+1)(x+2)}{4(x+1}$	$4x+4$
Errors	$-\frac{4(3x-3)}{4(x+1}$	$\frac{x^2+2x}{4x}$	$\frac{x^2+x+x+2}{4x+4}$	$4x+1$
Description of Errors	Distribution Error – sign error	Like Term Error multiplication	Distribution Error binomial x binomial	Notation Error caused incorrect equivalent fraction, but did not carry it through solution
Error Count	1	1	1	1

Figure I26. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{6x}{x+1}$	$\frac{(x+1)6x}{x+1}$	$\frac{(x+1)(x+2)}{4}$	$\frac{(x+1)6x}{x+1}$
Errors	$\frac{(x+1)6x}{x+1}$	$\frac{6x^2}{(x+1)(x+1)}$	$\frac{x^2+2}{4(x+1)}$	$\frac{6x^2}{(x+1)(x+1)}$
Description of Errors	Equivalent Fraction Error – multiplied numerator by its own denominator	Distribution error – binomial x monomial	Distribution error – binomial x binomial	Incorrect denominator
Error Count	1	1	1	1

Figure I27. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$-(x+1)\frac{3}{1}$		$\frac{x^2}{4(x+1)}$ $\frac{num}{x^2+2x+1}$ $\frac{3x+1}{x+1}$	$\frac{(x+1)(x+2)}{4}$
Errors	$-\frac{3x+1}{x+1}$	$-\frac{3x+1}{x+1}$	Did not finish	$\frac{x^2+2}{4(x+1)}$
Description of Errors	Distribution Error – scalar x binomial	Incorrect denominator	Did not finish	Distribution Error – binomial x binomial
Error Count	1	1	1	1

Figure I28. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"	
Total No.	2	
Location of Errors	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	$\frac{(4x+4)(x+2) + 4x+4(6x) - 4x+4(3)}{4x+4}$
Errors	$\frac{(4x+4)(x+2) + 4x+4(6x) - 4x+4(3)}{4x+4}$	$4x^2 + 8x + 42x + 8 + 24x^2 + 24 - 12x + 12$
Description of Errors	Equivalent Fraction Error multiplied each numerator by entire common denominator	Dropped denominator
Error Count	1	1

Figure I29. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"	
Total No.	2	
Location of Errors	$\frac{x+2}{4}$	$\frac{(4x+4)(x+2) + 4x+4(6x) - \ 4x+4(3)}{4x+4}$
Errors	$\frac{2x+8}{4x+4}$	$4x^2 + 8x + 42x + 8 + 24x^2 + 24 - 12x + 12$
Description of Errors	Unspecified	Like Term Error - multiplication
Error Count	1	1

Figure I30. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"	
Total No.	3	
Location of Errors	$\frac{(4x+4)(x+2)+4x+4(6x)-\ 4x+4(3)}{4x+4}$	$\frac{x^2+x(x+2)}{4}$
Errors	$4x^2+8x+42x+8+24x^2+24-12x+12$	$\frac{x^3+2x}{4x^2+4x}$
Description of Errors	Distribution Error – binomial x scalar caused by lack of parentheses	Distribution Error – failed to distribute a factor
Error Count	2	1

Figure I31. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"	
Total No.	4	
Location of Errors	$\frac{x+2}{4}+\frac{6x}{x+1}-3$	$\frac{x^2+3x+2+24x-12x+-3}{4x+1}$
Errors	$\frac{x+2}{(x+1)4}+\frac{6x}{(4)x+1}-\frac{3(4x+1)}{1}$	$\frac{x^2+5x-1}{4x+1}$
Description of Errors	Incorrect common denominator – $4x$ + 1	Notation Error 5 in place of 15
Error Count	3	1

Figure I32. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	$\frac{(4x)6x}{(4x)x+1}$	$\frac{x^3+2x}{4x^2+4x} + \frac{10x^2}{4x^2+4x} - \frac{12x^2+12x}{4x^2+4x}$
Errors	$\frac{x^3+2x}{4x^2+4x} + \frac{10x^2}{4x^2+4x} - \frac{12x^2+12x}{4x^2+4x}$	$\frac{10x^2}{4x^2+4x}$	$\frac{x^3-2x^2+14x}{4x^2+4x}$
Description of Errors	Incorrect common denominator $4x^2 + 4x$	Like Term Error multiplication	Sign error – distribution of "-"
Error Count	1	1	1

Figure I33. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{2x+8}{4x+4} + \frac{24x}{4x+4} - 3$	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	$\frac{(x+2)(x+1)\left(\frac{1}{4}\right)}{x+1}$	
Errors	$\frac{26x^2+8}{4x+4} - 3$	$\frac{x^2+8x}{5x+1}$	$\left(\frac{1}{4}x + \frac{1}{2}\right)\left(\frac{1}{4}x + \frac{1}{4}\right)$	$\left(\frac{1}{4}x + \frac{1}{2}\right)\left(\frac{1}{4}x + \frac{1}{4}\right)$
Description of Errors	Like Term Error multiplication	Incorrect common denominator 5*x* + 1	Multiplied each set of parentheses by 1/4	Dropped denominator of *x* + 1
Error Count	1	1	1	1

Figure I34. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors				
Total No.	5				
Location of Errors	$\frac{6x}{x+1}-\frac{3(x+1)}{x+1}$		$\frac{1}{16}x^2+\frac{1}{16}x+\frac{1}{8}x+\frac{1}{8}+6x-3x+1$	$\frac{6x}{x+1}$	
Errors	$6x-3x+1$	$6x-3x+1$	$\frac{1}{16}x^2+\frac{1}{16}x+\frac{2}{16}x+\frac{1}{8}$	$\frac{6x}{(x+1)^2}$	$\frac{6x}{(x+1)^2}$
Description of Errors	Distribution Error – multiplication over addition	Dropped denominator of $x+1$	Dropped terms at end	Incorrect denominator $(x+1)^2$	Equivalent Fraction Error – didn't multiply numerator by a factor
Error Count	1	1	1	1	1

Figure I35. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"		
Total No.	4		
Location of Errors	$\frac{x+2\ (x+1)}{4x+4}$	$\frac{x^2+2x+2}{4x+4}+\frac{6x\cdot 4}{4x+4}-\frac{12x+12}{4x+4}$	
Errors	$\frac{x^2+2x+2}{4x+4}$	$\frac{x^2+20x+20}{4x+4}$	$\frac{x^2+20x+20}{4x+4}$
Description of Errors	Distribution Error – binomial x binomial due to lack of parentheses	Like Term Error addition of monomials	Like Term Error addition of monomials
Error Count	2	1	1

Figure I36. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		$\frac{x+2}{4} + \frac{6x}{x+1} - 3$
Errors	$\frac{x+2+6x-12x-24}{4x+8}$	$\frac{x+2+6x-12x-24}{4x+8}$	$\frac{x+2}{4} + \frac{6x}{x+1} - \frac{3}{1}$
Description of Errors	Incorrect denominator $4x + 8$	Equivalent Fraction Error didn't multiply numerator by factor	Did not finish
Error Count	1	1	1

Figure I37. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$-\dfrac{3(4)(x+1)}{1}$	$\dfrac{x+2}{4} + \dfrac{6x}{x+1} - 3$	$\dfrac{x+2}{4} + \dfrac{6x}{x+1} - 3$
Errors	$\dfrac{-12x + 3}{4 + x + 1 - 1}$	$\dfrac{x^2 + 3x + 2 + 24x - 12x + 3}{4 + x + 1 - 1}$	$\dfrac{x+2}{4(x+1)} + \dfrac{6x}{x+1} - \dfrac{3}{x+1}$
Description of Errors	Sign error – distribution of "-" over addition	Added terms in denominator	Equivalent Fraction Error Added a factor of $x + 1$ to denominator of 1rst and 3rd terms, but did not multiply factor times numerator
Error Count	1	1	1

Figure I38. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"				
Total No.	6				
Location of Errors	$\frac{x+2}{4(x+1)} + \frac{6x}{x+1} - \frac{3}{x+1}$				
Errors	$\frac{(x+1)(x+2)(4)}{4x+1} + \frac{6x^2+1}{x+1} + \frac{3x+1}{x+1}$				
Description of Errors	Equivalent Fraction Error multiplied numerator by wrong factor	Multiplication error – monomial x binomial	Multiplication error – monomial x binomial	Distribution Error – scalar x binomial	Notation Error – changed sign for no apparent reason
Error Count	1	1	1	1	2

Figure I39. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	$\frac{x-10}{4} + \frac{6x}{x-1}$	$\frac{(x-10)\cancel{(4)}(x-1)}{4} + \frac{6x(4)\cancel{(x-1)}}{\cancel{x-1}}$
Errors	$\frac{x+2}{4} + \frac{6x}{x-1} - 3$	$\left(\frac{x-10}{4} + \frac{6x}{x-1}\right)(4x-4)$	$(x-10)(x-1)6x(4)$
Description of Errors	Notation Error. When re-copying the problem, changed operation in denominator of second term	Equivalent Fraction Error multiplied by factor other than one	Changed operation of addition between terms to multiplication
Error Count	1	1	1

Figure I40. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	6			
Location of Errors	$(x-10)(x-1)6x(4)$	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$		
Errors	$(x^2 - 11x + 10)(24x)$	$\frac{7x+2}{x+5} - 3$	$\frac{7x+2}{x+5} - 3$	$\frac{7x+2}{x+5} - 3$
Description of Errors	Did not finish	Added terms in numerator without common denominator	Added terms in denominator	Left whole number unchanged
Error Count	1	2	2	1

Figure I41. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{(x+2)(x+1)+(4)(6x)-(4)(3)(x+1)}{(4)(x+1)(1)}$	$\frac{(x+2)\cancel{(x+1)}+\cancel{(4)}(6x)-\cancel{(4)}(3)\cancel{(x+1)}}{(4)(x+1)(1)}$	
Errors	$\frac{(x+2)\cancel{(x+1)}+(4)(6x)-(4)(3)\cancel{(x+1)}}{(4)(x+1)(1)}$	$\frac{(x+2)(6x)}{(4)(x+1)(1)}$	$\frac{(x+2)(6x)}{(4)(x+1)(1)}$
Description of Errors	Cancelled a factor of two terms in the numerator	Changed operation from addition to multiplication	Did not finish
Error Count	1	1	1

Figure I42. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"		
Total No.	3		
Location of Errors	$\frac{x+2}{4}+\frac{\cancel{6x}}{\cancel{x}+1}-3$	$\frac{(x+2)(x+1)+(4)(6x)-(4)(3)(x+1)}{(4)(x+1)(1)}$	$(x+1)\frac{(x+2)}{4}+\frac{6x}{x+1}(4)$
Errors	$\frac{x+2}{4}+5x-3$	$\frac{(x+2)(x+1)+\cancel{(4)}(6x)-\cancel{(4)}(3)(x+1)}{(4)(x+1)(1)}$	$\frac{4x+8}{4x+4}+\frac{24x}{4x+4}$
Description of Errors	Cancelled a factor of numerator with term in denominator	Cancelled a factor of two terms in numerator	Equivalent Fraction Error. Multiplied numerator by wrong factor
Error Count	1	1	1

Figure I43. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{28x+8}{4x+4}$		$\frac{25x+2}{4x+4}$	$\frac{x^2+3x+2+24x-12x-12}{4(x+1)}$
Errors	$7x+2$	$7x+2$	$\frac{25x+1}{4x+2}$	$\frac{x^2+17x-10}{4(x+1)}$
Description of Errors	Cancelled factor of a term with factor of another term	Cancelled factor of a term with factor of another term	Cancelled factor of a term in numerator with factor of term in denominator	Like Term Error addition
Error Count	1	1	1	1

Figure I44. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"					
Total No.	6					
Location of Errors	$\frac{x^2+2x+3}{4x+4}$	$\frac{24x}{4x+4}$		$\frac{12x+12}{4x+4}$	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$	
Errors	$\frac{x^2+x+3}{2x+4}$	$\frac{6x}{4}$		$3x+3$	$\frac{2x}{4} + \frac{6x}{1x}$	$\frac{2x}{4} + \frac{6x}{1x}$
Description of Errors	Cancelled factors of terms	Cancelled factors of terms	Result of cancellation was zero	Cancelled factors of terms	Combined terms by changing operation from + to x	Combined terms by changing operation from + to x
Error Count	1	1	1	1	1	1

Figure I45. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{2x^2}{4x\cancel{4}} + \frac{24\cancel{6}x}{\cancel{1x}\,4x}$		$\frac{x^2+2x+x+2}{4x+4} + \frac{24x}{4x+4} - \frac{12x+12}{4x+4}$	
Errors	$\frac{26x}{4x}$		$(4x+4)\left(\frac{x^2+2x+x+2}{4x+4} + \frac{24x}{4x+4} - \frac{12x+12}{4x+4}\right)$	$x^2+14x+x+14$
Description of Errors	Like Term Error addition	Answer not simplified	Equivalent Fraction Error multiplied expression by number other than one	Like Term Error addition
Error Count	1	1	1	1

Figure I46. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Problem: $\frac{x+2}{4}+\frac{6x}{x+1}-3$

	Errors in core category "Found common denominator"			
Total No.	4			
Location of Errors	$\frac{x+2}{4}+\frac{6x}{x+1}-3$	$\frac{x^2+27x+2}{4x+4}$		$\frac{x+2}{4}+\frac{6x}{x+1}-3$
Errors	$\frac{x+2}{4(x+1)}+\frac{6x}{(x+1)(4)}$	$\frac{x^2+27x}{4x+\ 2}$		$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{4}{4}$
Description of Errors	Notation Error wrote terms with new denominators but numerators were unchanged. Later wrote numerators correctly	Cancelled terms, not factors	Result of cancellation was zero	Equivalent Fraction Error – 3 became 4/4
Error Count	1	1	1	1

Figure I47. Description and frequency of errors in Problem Set A algebraic item core category "Found common denominator."

Appendix J

Error Categories for Problem Set A Algebraic Item

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added denominators	7	Procedural Error	42
Added denominators	1		
Added denominators	1		
Added denominators	1		
Added denominators	1		
Added numerators without finding common denom.	6		
Added numerators without finding common denom.	1		
Added numerators without finding common denom.	1		
Added numerators without finding common denom.	1		
Added numerators without finding common denom.	1		
Added numerators without finding common denom.	1		
Added numerators without finding common denom.	2		
Added terms in denominator	2		
Added terms in denominator	1		
Added terms in denominator	2		
Added terms in numerator without common denom.	2		
Added terms in numerator without finding common denom.	2		
Added terms in the numerator	1		
Changed expression to an equation	1		
Changed to an equation	1		
Changed to an equation equal to zero, solved for *x*	1		
Combined numerators without finding common denominator	1		
Inverted second fraction and cross multiplied	1		
Multiplied all numerators by their denominators	2		
Multiplied all numerators by *x*, but *x* is not a factor in the common denominator chosen	1		

Figure J1. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Didn't distribute "-" sign +14 came from 2 + 12	1	Distribution Error	31
Distribution Error An extra "*x*" appeared in second term	1		
Distribution Error – binomial x binomial	1		
Distribution Error – binomial x binomial	1		
Distribution Error – binomial x binomial due to lack of parentheses	2		
Distribution Error – binomial x monomial	1		
Distribution Error – binomial x scalar caused by lack of parentheses	2		
Distribution Error – failed to distribute a factor	1		
Distribution Error – monomial x binomial	1		
Distribution Error – multiplication over addition	1		
Distribution Error – multiplied factor times first term only	1		
Distribution Error – scalar x binomial	1		
Distribution Error – scalar x binomial	1		
Distribution Error – sign error	1		
Distribution Error binomial x binomial	2		
Distribution Error binomial x binomial	2		
Distribution Error binomial x binomial	1		
Distribution Error binomial x binomial	1		
Distribution Error divided only one term by factor	1		
Distribution Error monomial x binomial	1		
Distribution Error scalar x binomial	1		
Distribution Error multiplied only the first term on the left side by (x + 1)	1		
Multiplication Error – monomial x binomial	1		
Multiplication Error – monomial x binomial	1		
Sign Error – distribution of "-"	1		
Sign Error – distribution of "-" over addition	1		
Sign Error – distribution of -1	1		

Figure J2. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	Number of errors	Error Category	Total Number of Errors
Cancellation error – cancelled factor from numerator with factor of term in denominator	2	Cancellation Error	30
Cancellation error – cancelled factor from term in numerator with factor of denominator	2		
Cancellation Error subtraction	1		
Cancelled a factor in numerator with term in denom.	1		
Cancelled a factor of a term in the numerator with a factor in the denominator	1		
Cancelled a factor of a term in numerator with a factor in denominator	1		
Cancelled a factor of numerator with term in denom.	1		
Cancelled a factor of two terms in numerator	1		
Cancelled a factor of two terms in the numerator	1		
Cancelled a scalar with addend in denominator	1		
Cancelled addend from numerator of one term with addend in denominator of another term	2		
Cancelled factor from term in numerator with factor in denominator ½ came from $(x + 2)/4$	1		
Cancelled factor in numerator of one term with factor in denominator of another	1		
Cancelled factor in terms	1		
Cancelled factor in terms	1		
Cancelled factors of terms in numerator and denom.	1		
Cancelled factor of a term with factor of another term	1		
Cancelled factor of a term with factor of another term	1		
Cancelled factor with denominator <u>and</u> distributed it to numerator	1		
Cancelled factors of terms	1		
Cancelled factors of terms	1		
Cancelled factors of terms	1		
Cancelled factors of terms	1		
Cancelled parts of terms	1		
Cancelled term in numerator with factor in denom.	1		
Cancelled term in numerator with factor in denom.	1		
Cancelled terms, not factors	1		

Figure J3. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	Number of errors	Error Category	Total Number of Errors
Result of cancellation was zero	1	Residual Cancellation Error	11
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Dropped term from solution, never reappears (-3)	2	Unresolved whole number error	28
Dropped term from solution, never reappears (-3)	9		
Dropped the -3 term	1		
Left whole number unchanged	9		
Left whole number unchanged	6		
Left whole number unchanged	1		
Like Term Error - multiplication	1	Operations with Monomials Error	23
Like term error addition	1		
Like term error addition	1		
Like term error addition	1		
Like term error addition	1		
Combined terms by changing operation from + to x	1		
Combined terms by changing operation from + to x	1		
Like Term Error addition of monomials	1		
Like Term Error addition of monomials	1		
Like Term Error mixed addition and multiplication	1		
Like Term Error multiplication	1		
Like Term Error multiplication	1		
Like Term Error multiplication	1		
Like Term Error operation unspecified	1		
Did not combine all like terms	1		
Combined terms by changing operation from + to x	3		
Combined terms by changing operation from + to x	3		
Changed operation from addition to multiplication	1		
Changed operation of + between terms to x	1		

Figure J4. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	Number of errors	Error Category	Total Number of Errors
Equivalent Fraction Error – 3 became 4/4	1	Equivalent Fraction Error	22
Equivalent Fraction Error added a factor of x + 1 to denominator of 1rst and 3rd terms, but did not multiply factor times numerator	1		
Equivalent Fraction Error – didn't multiply numerator by a factor	1		
Equivalent Fraction Error didn't multiply numerator by factor	1		
Equivalent Fraction Error multiplied by factor other than one	1		
Equivalent Fraction Error – multiplied numerator by its own denominator	1		
Equivalent Fraction Error multiplied numerator by wrong factor	1		
Equivalent Fraction Error multiplied numerator by wrong factor	1		
Equivalent Fraction Error – used factor of 4*x* although common denominator is written as 4	1		
Equivalent Fraction Error didn't multiply numerator by factor	4		
Equivalent Fraction Error didn't multiply numerator by factor	2		
Equivalent Fraction Error multiplied each numerator by entire common denominator	1		
Equivalent Fraction Error multiplied expression by number other than one	1		
Equivalent Fraction Error multiplied numerator by its own denominator	1		
Equivalent Fraction Error multiplied numerator by wrong factor	1		
Multiplied numerator by its own denominator	1		
Took common denominator factor missing from denominator and added it to numerator (with a mistake)	1		
Multiplied twice by denominator. Once to cancel with denominator and again to change numerator	1		

Figure J5. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Notation Error carried it through rest of solution	2	Notation Error	17
Notation Error – changed sign for no apparent reason	2		
Notation Error dropped a "1" from 21	1		
Notation Error – dropped a term (4/4)	1		
Notation Error – dropped parentheses	1		
Notation Error 2 became 2x	1		
Notation Error 5 in place of 15	1		
Notation Error Caused incorrect equivalent fraction, but did not carry it through solution	1		
Notation Error Wrote terms with new denominators but numerators were unchanged. (Corrected later)	1		
Notation Error. When re-copying the problem, changed operation in denominator of second term	1		
Dropped term from solution but reappears later (-3)	4		
Dropped term from solution but reappears later (-3)	1		
Incorrect common denominator – $4x + 1$	3	Common Denominator Error	13
Incorrect common denominator (4)	1		
Incorrect common denominator (4)	1		
Incorrect common denominator $4x^2 + 4x$	1		
Incorrect common denominator $5x + 1$	1		
Incorrect denominator $(x + 1)(x + 1)$	1		
Incorrect denominator $(x + 1)$	1		
Incorrect denominator $4x + 8$	1		
Incorrect denominator $(x + 1)^2$	1		
Incorrect denominator $4x + 1$	2		
Did not finish	2	Persistence Error	9
Did not finish	1		
Did not finish	1		
Did not finish	1		
Did not finish	1		
Did not finish	1		
Did not finish	1		
Rewrote problem but did not finish	1		

Figure J6. Error categories in Problem Set A algebraic item.

Description of Errors in Problem 4 $\frac{x+2}{4}+\frac{6x}{x+1}-3$	**Number of errors**	**Error Category**	**Total Number of Errors**
Dropped a denominator, never reappeared	1	Miscellaneous Errors	18
Dropped a term (x)	1		
Dropped denominator	1		
Dropped denominator of $x + 1$	1		
Dropped denominator of $x + 1$	1		
Dropped terms at end	1		
Dropped denominators	1		
Left out a part (present in scratch work)	1		
Divided an expression by 3 inappropriate simplified	1		
Divided expression by a number other than one	1		
Multiplied an expression by a factor other than one	1		
Multiplied each set of parentheses by 1/4	1		
Split apart a fraction	1		
Moved term from denominator to numerator	1		
Moved term from denominator to numerator	1		
Answer not simplified	1		
Sign error. Since the expression was set equal to zero, the sign of the right hand side was wrong	1		
Arithmetic error - multiplication	1		
Unspecified	1	Unspecified Errors	11
Unspecified	1		
Unspecified	1		
Unspecified	1		
Unspecified	1		
Unspecified	1		
Unspecified	1		
Unspecified equivalent fraction error	1		
Unspecified error	1		
Unspecified error	1		
Unspecified error	1		
Omitted	13	Omitted	13
		Total	268

Figure J7. Error categories in Problem Set A algebraic item.

Appendix K

Problem Set B Numeric Item Errors

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors				
Total No.	16				
Location of Error	Omitted	Unspecified	$\frac{3}{2} / \frac{9}{8}$		$\frac{3}{2} \cdot \frac{8}{9}$
Error	N/A	N/A	$\frac{1.1}{1.1}$	$\frac{1.1}{1.1}$	$\frac{24}{18}$
Description of Error	N/A	N/A	Incorrect decimal form	Incorrect decimal form	Answer Not Simplified
Error Count	3	8	1	1	3

Figure K1. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors			
Total No.	10			
Location of Error	$\frac{\cancel{3}}{\cancel{3}-1} \div \frac{\cancel{9}}{\cancel{9}-1}$			
Error	$-1 \cdot -1$		$-1 \cdot -1$	
Description of Error	Cancelled Addends	Result of cancellation was zero	Cancelled Addends	Result of cancellation was zero
Error Count	3	2	3	2

Figure K2. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors		
Total No.	3		
Location of Error	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{8}{8}\frac{3}{2} \div \frac{9}{8}\frac{\cdot 2}{\cdot 2}$
Error	$\frac{3}{2} \times \frac{8}{9}$	$\frac{8}{8}\frac{3}{2} \div \frac{9}{8}\frac{\cdot 2}{\cdot 2}$	$\frac{24}{16} \div \frac{7}{16}$
Description of Error	Did not finish	Found Common Denominator	Arithmetic error - multiplication
Error Count	1	1	1

Figure K3. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors				
Total No.	6				
Location of Error	$\frac{24}{16} \div \frac{7}{16}$			$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{3}{3-1} \div \frac{9}{9-1}$
Error	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{3}{2} \div \frac{9}{8} \left(\frac{2}{3}\right)$	$\frac{3}{2} \cdot \frac{9}{8}$
Description of Error	Divided across numerators	Arithmetic error – division	Kept common denominator (as in addition or subtraction)	Found reciprocal of first term	Did not find reciprocal
Error Count	1	1	1	1	2

Figure K4. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors					
Total No.	16					
Location of Error	$\frac{\cancel{3}}{\cancel{3}-\cancel{1}} \div \frac{3\cancel{9}-\cancel{1}}{\cancel{9}3}$					$\frac{1\ \cancel{3}}{3-1} \cdot \frac{9-1}{\cancel{9}\ 3}$
Error	1	1	1	1	1	$\frac{9-1}{9-3}$
Description of Error	Cancelled addends (3)	Result of cancellation was zero	Cancelled addends (-1)	Result of cancellation was zero	Cancelled addends (9)	Answer Not Simplified
Error Count	3	3	3	3	3	1

Figure K5. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors			
Total No.	14			
Location of Error	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{3}{2} \div \frac{9}{8}$	$\frac{12}{8} \div \frac{9}{8}$	
Error	$\frac{3}{2} \div \frac{9}{8}$	$\frac{12}{8} \div \frac{9}{8}$	$\frac{108}{8}$	$\frac{108}{8}$
Description of Error	Did not finish	Found common denominator	Multiplied across numerator without changing operation	Kept common denominator as if adding or subtracting
Error Count	7	4	1	1

Figure K6. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors			
Total No.	4			
Location of Error	$\frac{3}{3-1} \div \frac{9}{9-1}$		$\left(\frac{2}{2}\right)\frac{3}{2}\cdot\frac{8}{9}\left(\frac{9}{9}\right)$	$\frac{3}{3-1} \div \frac{9}{9-1}$
Error	$\left(\frac{2}{2}\right)\frac{3}{2}\cdot\frac{8}{9}\left(\frac{9}{9}\right)$	$\left(\frac{2}{2}\right)\frac{3}{2}\cdot\frac{8}{9}\left(\frac{9}{9}\right)$	$\frac{2(3)\cdot 8(9)}{2(9)}$	$\frac{3}{2}\cdot -\frac{8}{9}$
Description of Error	Found common denominator	Equivalent fraction error – multiplied by wrong factor	Equivalent fraction error – multiplied only numerator by factor	Changed sign of second term
Error Count	1	1	1	1

Figure K7. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors					
Total No.	21					
Location of Error	$\frac{3}{2} \div \frac{9}{8}$		$\frac{3}{3-1} \div \frac{9}{9-1}$		$\frac{3}{3-1} \div \frac{9-1}{9}$	$\frac{3 \cdot 8}{2 \cdot 9}$
Error	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{2} \div \frac{9}{10}$		$\frac{27-3}{29-9}$	$= \frac{24}{18} = \frac{12}{9}$
Description of Error	Divided across numerators	Divided across denominators	Arithmetic error – subtraction	Did not finish	Arithmetic error – multiplication	Answer Not Simplified
Error Count	6	6	1	1	1	6

Figure K8. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors						
Total No.	9						
Location of Error	$\frac{3}{3-1} \div \frac{9}{9-1}$			$\frac{2}{4}$	$\frac{12}{8} \div \frac{9}{8}$		$\frac{3}{3-1} \div \frac{9}{9-1}$
Error	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1.03		$\frac{3}{4} \cdot \frac{8}{9}$
Description of Error	Divided across numerators	Arithmetic error – division	Divided across denominators	Equivalent fraction error – only divided numerator by factor	Divided across numerators	Arithmetic error - division	Arithmetic error – subtraction
Error Count	1	1	1	1	1	1	3

Figure K9. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors				
Total No.	5				
Location of Error	$8 \cdot \frac{3}{2} \div \frac{9}{8} \cdot 2$			$\frac{3}{18} \div \frac{9}{18}$	
Error	$\frac{3}{18} \div \frac{9}{18}$	$\frac{3}{18} \div \frac{9}{18}$	$\frac{3}{18} \div \frac{9}{18}$	$= \frac{}{18}$	
Description of Error	Found common denominator	Arithmetic error – multiplication	Equivalent fraction error – did not multiply numerator by factor	Kept common denominator	Did not finish
Error Count	1	1	1	1	1

Figure K10. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors			
Total No.	5			
Location of Error	$\frac{3}{2} \div \frac{9}{8}$	$\frac{3}{2} \div \frac{9}{8}$		
Error	$= \frac{18}{24}$	$\frac{6}{8} / \frac{9}{8}$		
Description of Error	Reversed numerator and denominator when cross-multiplying	Found common denominator	Equivalent fraction error – multiplied numerator by wrong factor	Did not finish
Error Count	2	1	1	1

Figure K11. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors						
Total No.	7						
Location of Error	$\frac{\cancel{3}}{\cancel{3}-1} \div \frac{\cancel{9}}{\cancel{9}-1}$				$\frac{0}{-1} \div \frac{0}{-1}$	$\frac{3}{2} \div \frac{9}{8}$	$\frac{-1}{-1}$
Error	$\frac{0}{-1} \div \frac{0}{-1}$	$\frac{0}{-1} \div \frac{0}{-1}$	$\frac{0}{-1} \div \frac{0}{-1}$	$\frac{0}{-1} \div \frac{0}{-1}$	= 1	$\frac{1.5}{1.125}$	= -1
Description of Error	Cancelled addends	Result of cancellation was zero	Cancelled addends	Result of cancellation was zero	Arithmetic error – division by zero	Did not finish	Arithmetic error - sign
Error Count	1	1	1	1	1	1	1

Figure K12. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors					
Total No.	6					
Location of Error	$1.125\ \overline{)1.50000}$	$= 1.3$	$\left(\frac{3}{2}\right)\left(\frac{8}{9}\right)$	$\frac{3}{2} \div \frac{9}{8}$		$1\frac{1}{2} \div 1\frac{1}{8}$
Error	$= 1.3$	$\frac{1}{3}$	$\frac{21}{18}$	$\frac{\frac{1}{3}}{\frac{2}{8}}$	$\frac{\frac{1}{3}}{\frac{2}{8}}$	$= 1\frac{1}{16}$
Description of Error	Arithmetic error – division	Incorrect fraction form of decimal	Arithmetic error – multiplication	Divided across numerators	Divided across denominators	Multiplied across denominators
Error Count	1	1	1	1	1	1

Figure K13. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors					
Total No.	6					
Location of Error	$\frac{3}{2} \div \frac{9}{8}$	$\frac{24}{36}$	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{24}{16} \div \frac{18}{16}$
Error	$\frac{2}{3} \cdot \frac{9}{8}$	$= \frac{6}{9}$	$\frac{3}{3-1} \times \frac{9+1}{9}$	$\frac{3(3-1)}{9(9-1)}$	$\frac{24}{16} \div \frac{18}{16}$	$\frac{6}{8} \div \frac{9}{8}$
Description of Error	Found reciprocal of first term	Not simplified	Notation error - Changed sign	Procedure error – multiplied each numerator by its denominator	Found Common Denominator	Equivalent fraction error – divided numerator by wrong factor
Error Count	1	1	1	1	1	1

Figure K14. Description and frequency of errors in Problem Set B numeric item.

Problem: $\frac{3}{3-1} \div \frac{9}{9-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{6}{8} \div \frac{9}{8}$		$\frac{3}{2} \cdot \frac{8}{9}$	$\frac{16}{27}$	$\frac{3}{2} \div \frac{9}{8}$
Error	$= \frac{15}{8}$	$= \frac{15}{8}$	$\frac{16}{27}$	$\frac{3}{9}$	$\frac{12}{8} \div \frac{9}{8}$
Description of Error	Added across numerators	Kept common denominator	Cross multiplied after already applying invert and multiply procedure	Equivalent fraction error – divided numerator by wrong factor	Did not finish
Error Count	1	1	1	1	1

Figure K15. Description and frequency of errors in Problem Set B numeric item.

Appendix L

Error Categories for Problem Set B Numeric Item

Description of Error in Problem 6 $\frac{3}{3-1} \div \frac{9}{9-1}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added across numerators	1	Procedural Error	43
Divided across denominators	6		
Divided across denominators	1		
Divided across denominators	1		
Divided across numerators	1		
Divided across numerators	6		
Divided across numerators	1		
Divided across numerators	1		
Divided across numerators	1		
Changed sign of second term	1		
Cross multiplied after already applying invert and multiply procedure	1		
Did not find reciprocal	2		
Found Common Denominator	1		
Found common denominator	4		
Found common denominator	1		
Found common denominator	1		
Found common denominator	1		
Found Common Denominator	1		
Found reciprocal of first term	1		
Found reciprocal of first term	1		
Kept common denominator	1		
Kept common denominator	1		
Kept common denominator as in addition or subtraction	1		
Kept common denominator as if adding or subtracting	1		
Multiplied across denominators	1		
Multiplied across numerator without changing operation	1		
Reversed numerator and denominator when cross-multiplying	2		
Procedure error – multiplied each numerator by its denom.	1		

Figure L1. Error categories in Problem Set B numeric item.

Description of Error in Problem 6 $\frac{3}{3-1} \div \frac{9}{9-1}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Cancelled Addends	3	Cancellation Error	17
Cancelled Addends	3		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends (-1)	3		
Cancelled addends (3)	3		
Cancelled addends (9)	3		
Result of cancellation was zero	2	Residual Cancellation Error	12
Result of cancellation was zero	2		
Result of cancellation was zero	3		
Result of cancellation was zero	3		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Arithmetic error - division	1	Arithmetic Error	14
Arithmetic error – division	1		
Arithmetic error – division	1		
Arithmetic error – division	1		
Arithmetic error – division by zero	1		
Arithmetic error - multiplication	1		
Arithmetic error – multiplication	1		
Arithmetic error – multiplication	1		
Arithmetic error – multiplication	1		
Arithmetic error - sign	1		
Arithmetic error – subtraction	1		
Arithmetic error – subtraction	3		
Did not finish	1	Persistence Error	13
Did not finish	7		
Did not finish	1		
Did not finish	1		
Did not finish	1		
Did not finish	1		
Did not finish	1		

Figure L2. Error categories in Problem Set B numeric item.

Description of Error in Problem 6 $\frac{3}{3-1} \div \frac{9}{9-1}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Answer Not Simplified	3	Simplification Error	11
Answer Not Simplified	1		
Answer Not Simplified	6		
Not simplified	1		
Equivalent fraction error – did not multiply numerator by factor	1	Equivalent Fraction Error	7
Equivalent fraction error – divided numerator by wrong factor	1		
Equivalent fraction error – divided numerator by wrong factor	1		
Equivalent fraction error – multiplied by wrong factor	1		
Equivalent fraction error – multiplied numerator by wrong factor	1		
Equivalent fraction error – only divided numerator by factor	1		
Equivalent fraction error –multiplied only numerator by factor	1		
Incorrect decimal form	1	Miscellaneous Errors	4
Incorrect decimal form	1		
Incorrect fraction form of decimal	1		
Notation error - Changed sign	1		
Unspecified	8	Unspecified	8
Omitted	3	Omitted	3
		Total	127

Figure L3. Error categories in Problem Set B numeric item.

Appendix M

Problem Set B Algebraic Item Errors

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	34					
Location of Error	$\frac{x(x^2-1)}{x^2(x-1)}$	$\frac{\cancel{x^3} - x}{\cancel{x^3} - x^2}$	$\frac{\cancel{x^3} - x}{\cancel{x^3} - x^2}$	$\frac{x^3 \cancel{- x}}{x^3 - \cancel{x^2}}$	Omitted	Unspecified
Error	1	$\frac{1}{x}$	$\frac{1}{x}$	$\frac{1}{x}$	N/A	N/A
Description of Error	Cancellation error unspecified	Cancelled addends	Result of cancellation was zero	Cancelled addends	N/A	N/A
Error Count	1	3	3	2	13	12

Figure M1. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors						
Total No.	12						
Location of Error	$\frac{x}{(x-1)} * \frac{x^2-1}{x^2}$			$\frac{\cancel{x^3}-1}{\cancel{x^3}}$	$\frac{x^2-1}{x^2-x}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{\cancel{x^3}-1}{\cancel{x^3}}$
Error	$\frac{x^3-1}{x^3-1}$	$\frac{x^3-1}{x^3-1}$	$\frac{x^3-1}{x^3}$	-1	$\frac{x^2-1}{x^2-x}$	$\frac{x}{x-1} \div \frac{x^2-1}{x^2}$	-1
Description of Error	Distribution Error – monomial x binomial	Distribution Error – binomial x monomial	Distribution Error binomial x monomial	Result of cancellation was zero	Answer not simplified	Notation Error (because next step used multiplication operation)	Cancelled addend in numerator with factor in denominator
Error Count	2	1	1	1	5	1	1

Figure M2. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	18					
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{\cancel{x}}{\cancel{x}-1} / \frac{\cancel{x^2}}{\cancel{x^2}-1}$		$\frac{\cancel{x^3}-x}{\cancel{x^3}-x^2}$	$\frac{\cancel{x^3}-x}{\cancel{x^3}-x^2}$	$\frac{\cancel{x^3}-x}{\cancel{x^3}-x^2}$
Error	$\frac{x}{x-1} / \frac{x^2}{x^2-1}$	$\frac{-1}{-1}$	$\frac{-1}{-1}$	$\frac{x}{x^2}$	$\frac{x}{x^2}$	$\frac{x}{x^2}$
Description of Error	Did not change to multiplication and take reciprocal of 2nd fraction	Cancelled addends	Cancelled addends	Cancelled addends	Result of cancellation was zero	Answer not simplified
Error Count	1	5	4	3	3	2

Figure M3. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	18					
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$		$\frac{\cancel{x}}{\cancel{x}-1} \cdot \frac{\cancel{x^2}-1}{\cancel{x^2}}$			
Error	$\frac{x^2}{x^2-1} \div \frac{x^2}{x^2-1}$		-1 • -1	-1 • -1	-1 • -1	-1 • -1
Description of Error	Found common denominator	Distribution error – monomial x binomial *x (x*-1*)*	Cancelled addends	Result of cancellation was zero	Cancelled addends	Result of cancellation was zero
Error Count	1	1	4	4	3	5

Figure M4. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	4			
Location of Error	$\frac{x^2-1}{x-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$		$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	Not Simplified	$\frac{x^2}{x^2-x} / \frac{x^3}{x^3-x}$		$\frac{x^2}{x^2-1} \bullet \frac{x-1}{x}$
Description of Error	Answer not simplified	Did not change to multiplication and take reciprocal of 2nd fraction	Multiplied all terms by “x”	Found reciprocal of first term
Error Count	1	1	1	1

Figure M5. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

<table>
<tr><td></td><td colspan="6">Errors</td></tr>
<tr><td>Total No.</td><td colspan="6">8</td></tr>
<tr><td>Location of Error</td><td colspan="2">$\frac{x}{x-1} / \frac{x^2}{x^2-1}$</td><td colspan="2">$\frac{\cancel{x^3} - x^2}{\cancel{x^3} - x}$</td><td colspan="2">$\frac{\cancel{x}}{x-1} \cdot \frac{x^2-1}{\cancel{x^2}-x}$</td></tr>
<tr><td>Error</td><td>$\frac{-1}{-1}$</td><td>$\frac{-1}{-1}$</td><td>$\frac{-x^2}{-x}$</td><td>$\frac{-x^2}{-x}$</td><td colspan="2">$\frac{x^2-1}{x^2-1}$</td></tr>
<tr><td>Description of Error</td><td>Result of cancellation was zero</td><td>Result of cancellation was zero</td><td>Cancelled addends</td><td>Result of cancellation was zero</td><td>Cancelled addends</td><td>Distribution Error – monomial x binomial
$x(x-1) \neq x^2-1$</td></tr>
<tr><td>Error Count</td><td>2</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure M6. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	10					
Location of Error	$\frac{x^2-1}{x^2-x}$	$-1 \div \frac{x^{\cancel{2}}}{x^{\cancel{2}}-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$			
Error	$\frac{x^2-1}{x(x-1)}$	1	$x\overline{)x^2}$	$x-1\overline{)x^2-1}$	$x\overline{)x^2}$	$x-1\overline{)x^2-1}$
Description of Error	Answer not simplified	Cancelled factor in numerator with addend in denominator	Divided numerators	Divided denominators	Confused divisor and dividend	Confused divisor and dividend
Error Count	1	1	2	2	2	2

Figure M7. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	7				
Location of Error	$x-1\,\overline{)x^2-1}$	$\frac{\cancel{x}}{\cancel{x}-1}$	$\frac{\cancel{x}}{\cancel{x}-1}$	$\frac{x}{x\cancel{-1}} \cdot \frac{x^2\cancel{-1}}{x^2}$	
Error	$x-1$	-1	-1	1	1
Description of Error	Division of polynomials error	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Cancelled addends	Result of cancellation was zero
Error Count	1	1	1	2	2

Figure M8. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	7				
Location of Error	$\frac{\cancel{x}}{\cancel{x}-1} \cdot \frac{x^2-1}{x^2}$	$\frac{x}{x-1} \cdot \frac{\cancel{x^2}-1}{\cancel{x^2}}$	$\frac{x^2+x}{x^2}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	1	1	$\frac{x^2+x}{x^2}$	$\frac{x \cdot x}{x-1} \cdot \frac{x-1}{x \cdot x}$	$\frac{x^3}{x^3-x-x^2+1}$
Description of Error	Cancelled factor in numerator with addend in denominator	Cancelled addend in numerator with factor in denominator	Answer not simplified	Unspecified	Multiplied Across
Error Count	2	2	1	1	1

Figure M9. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	5			
Location of Error	$\left(\frac{x^2}{x^2}\right)\frac{x}{x-1} \cdot \frac{x^2-1}{x^2}\left(\frac{x-1}{x-1}\right)$	$\frac{x^3 \cdot x^2 - 1}{x - 1(x^2)}$	$\frac{x^5-1}{x^2(x-1)}$ $x^3 - x^2$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	$\frac{x^3 \cdot x^2 - 1}{x - 1(x^2)}$	$\frac{x^5 - 1}{x^2(x-1)}$	$\frac{x^5-1}{x}$	$\frac{x}{x-1} \div - \frac{x^2-1}{x^2}$
Description of Error	Found common denominator and Equivalent Fractions	Distribution Error – monomial x binomial (attributed to dropped parentheses)	Like Term Error - subtraction	Changed sign of 2nd term
Error Count	2	1	1	1

Figure M10. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	6					
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$			$\frac{x}{-x} \div \frac{\cancel{x^2}}{\cancel{x^2}-2}$	$-1 \div \frac{\cancel{x^2}}{\cancel{x^2}-1}$
Error	$\frac{x}{x-1} \div -\frac{x^2-1}{x^2}$	$\frac{x}{-x} \div \frac{\cancel{x^2}}{\cancel{x^2}-2}$			-2	1
Description of Error	Did not finish	Moved (-1) from denominator of first term to second term	Cancelled factor in numerator with addend in denominator	Cancelled factor in numerator with addend in denominator	Unspecified	Result of cancellation was zero
Error Count	1	1	1	1	1	1

Figure M11. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	8					
Location of Error	$-1 \div \frac{\cancel{x}}{\cancel{x} - 1}$		$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$		$\frac{x}{x-1} \div \frac{(x)(x)}{(x-1)(x+1)}$	
Error	1		$\frac{x}{x-1}$		$\dfrac{\frac{x}{(x)(x)}}{\frac{x-1}{(x-1)(x+1)}}$	
Description of Error	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Divided across numerators	Divided across denominators
Error Count	1	1	2	2	1	1

Figure M12. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

<table>
<tr><td></td><td colspan="6">Errors</td></tr>
<tr><td>Total No.</td><td colspan="6">6</td></tr>
<tr><td>Location of Error</td><td colspan="2">$\frac{\frac{x}{(x)(x)}}{\frac{x-1}{(x-1)(x+1)}}$</td><td colspan="2">$\frac{\frac{\cancel{1}}{\cancel{x}}\left(\frac{\cancel{x}}{\cancel{1}}\right)}{\frac{1}{x+1}\left(\frac{\cancel{x}}{\cancel{x}(x+1)}\right)}$
x^2+x</td><td colspan="2">$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$</td></tr>
<tr><td>Error</td><td colspan="2">$\frac{\frac{1}{x}\left(\frac{x}{1}\right)}{\frac{1}{x+1}\left(\frac{x}{x(x+1)}\right)}$</td><td colspan="2">$\frac{1}{2x}$</td><td colspan="2">$\frac{x^3-x}{x^3-x-x^2+1}$</td></tr>
<tr><td>Description of Error</td><td>Equivalent Fraction Error Multiplied numerator by factor other than one</td><td>Equivalent Fraction Error Multiplied denominator by factor other than one</td><td>Distribution Error binomial x binomial $(x+1)(x+1) \neq x^2+x$</td><td>Like Term Error addition $x^2+x \neq 2x$</td><td>Cross-multiplied first term numerator with second term denominator</td><td>Multiplied across denominators</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure M13. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	4			
Location of Error	$\frac{\cancel{x^3} - \cancel{x}}{\cancel{x^3} - \cancel{x} - x^2 + 1}$	$\frac{\cancel{x^3} - \cancel{x}}{\cancel{x^3} - \cancel{x} - x^2 + 1}$	$\frac{\cancel{x^3} - \cancel{x}}{\cancel{x^3} - \cancel{x} - x^2 + 1}$	$\frac{x(x^2-1)}{x^2(x-1)}$
Error	$\frac{1}{-x^2+1}$	$\frac{1}{-x^2+1}$	$\frac{1}{-x^2+1}$	$\frac{x(x^2-1)}{x^2(x-1)}$
Description of Error	Cancelled addends	Result of cancellation was zero	Cancelled addends	Answer not simplified
Error Count	1	1	1	1

Figure M14. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{x}{x-1} \bullet \frac{x^2-1}{x^2}$	$\frac{x^3-1x}{x-x^2}$		$\frac{x(x^2-1)}{x^2(x-1)}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	$\frac{x^3-1x}{x-x^2}$	$\frac{x^3-1}{x^2}$		$\frac{x^3-x}{x^3-x}$	$x^2 \, \frac{x}{x-1} \cdot \frac{x^2-1}{x^2} \, x-1$
Description of Error	Distribution Error binomial x monomial	Cancelled addends	Result of cancellation was zero	Distribution Error – monomial x binomial	Found common denominator (did not multiply across denominators)
Error Count	1	1	1	1	1

Figure M15. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{x}{x-1} \div \frac{x^2-1}{x^2}$			
Error	$\frac{x}{x-1} \div \frac{x^2-1}{x^2}$	$= x$			
Description of Error	Inverted 2nd term but did not change operation to multiplication	Cancelled addends	Result of cancellation was zero	Cancelled addends	Result of cancellation was zero
Error Count	1	1	1	1	1

Figure M16. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors					
Total No.	26					
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$				$\frac{\cancel{x}}{\cancel{x}-1} \div \frac{\cancel{x^2}}{\cancel{x^2}-1}$	$\frac{\cancel{x}}{\cancel{x}-1} \div \frac{x^{\cancel{2}}}{x^{\cancel{2}}-1}$
Error	$-1 \div -1$	$-1 \div -1$	$-1 \div -1$	$-1 \div -1$	$\frac{x}{x-1}$	$\frac{x}{x-1}$
Description of Error	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Cancelled factor in numerator with addend in denominator	Did not invert and multiply
Error Count	6	6	6	6	1	1

Figure M17. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	9			
Location of Error	$\frac{x}{x^2}$	$\frac{x^{\cancel{6}4} - x^{\cancel{5}3} - x^{\cancel{4}2} - x^{\cancel{3}}}{x^{\cancel{3}} - x^{\cancel{2}}}$	$\frac{x^3 - x}{x^3 - x^2}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	0	$\frac{x^4 - x^3 - x^2 + x}{x}$	$\frac{x^3 - x}{x^3 - x^2}$	$(x^2 - 1)\frac{x}{x-1} \div \frac{x^2}{x^2-1}(x-1)$
Description of Error	Like Term Error division	Like Term Error division (x^2 / x^2 = 0)	Answer not simplified	Found equivalent fractions for common denominator
Error Count	1	1	5	2

Figure M18. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors		
Total No.	4		
Location of Error	$(x^2-1)\dfrac{x}{x-1} \div \dfrac{x^2}{x^2-1}(x-1)$	$\dfrac{\cancel{(x^2-1)}(x) \div x^2\cancel{(x-1)}}{\cancel{(x-1)}\cancel{(x^2-1)}}$	$\dfrac{x}{1} \cdot \dfrac{1}{x^2}$
Error	$\dfrac{(x^2-1)(x) \div x^2(x-1)}{(x-1)(x^2-1)}$	$\dfrac{x}{1} \div \dfrac{x^2}{1}$	$\dfrac{x}{x^2}$
Description of Error	Kept common denominator and divided numerator	Cancelled factors in an expression that had an operation of division	Answer not simplified
Error Count	1	1	2

Figure M19. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{\cancel{x^3} - x}{\cancel{x^3} - x^2}$		$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{x+1}{x}$	
Error	$\frac{x}{x^2}$		$\frac{x}{x-1} \cdot \frac{x^2-1}{x^2}$	1	
Description of Error	Cancelled Addends	Result of cancellation was zero	Did not finish	Cancelled addend in numerator with factor in denominator	Result of cancellation was zero
Error Count	1	1	1	1	1

Figure M20. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	4			
Location of Error	$\frac{x(x^2-1)}{(x-1)(x^2-1)} \div \frac{x^2(x-1)}{(x^2-1)(x-1)}$	$\frac{\cancel{x^3} - x}{\cancel{x^3} - x^2}$		
Error	$\frac{x(x^2-1)}{x^3-x^2-x+1} \div \frac{x^2(x-1)}{x^3-x^2-x+1}$	$\frac{x}{x^2}$	$\frac{x}{x^2}$	$\frac{x}{x^2}$
Description of Error	Did not finish	Cancelled addends	Result of cancellation was zero	Answer not simplified
Error Count	1	1	1	1

Figure M21. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors		
Total No.	6		
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{(x+1)(x-1)}$	$\frac{\cancel{x^2}+x}{\cancel{(x+1)(x-1)}} \cdot \frac{\cancel{(x+1)(x-1)}}{\cancel{x^2}}$	
Error	$\frac{x(x+1)}{(x+1)(x-1)} \div \frac{x^2}{(x+1)(x-1)}$	x	
Description of Error	Found equivalent fraction for common denominator	Cancelled addend in numerator with factor in denominator	Result of cancellation was zero
Error Count	2	2	2

Figure M22. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	9				
Location of Error	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$				$\frac{x}{x-1} \div \frac{x^2}{(x-1)(x+1)}$
Error	$\frac{x}{x+1}$		$\frac{x}{x+1}$		$\frac{x}{x-1} \div$
Description of Error	Divided across numerators 4-16	Confused divisor and dividend	Divided across denominators	Confused divisor and dividend	Did not finish
Error Count	2	2	2	2	1

Figure M23. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

<table>
<tr><td></td><td colspan="5">Errors</td></tr>
<tr><td>Total No.</td><td colspan="5">6</td></tr>
<tr><td>Location of Error</td><td colspan="2">$\dfrac{\frac{x}{x-1}}{\frac{\cancel{x^2}}{\cancel{x^2}-1}}$</td><td colspan="2">$\dfrac{\frac{x}{x\cancel{-1}}}{\cancel{-1}}$</td><td>$\frac{x}{x}$</td></tr>
<tr><td>Error</td><td colspan="2">$\dfrac{\frac{x}{x-1}}{-1}$</td><td colspan="2">$\frac{x}{x}$</td><td>$x$</td></tr>
<tr><td>Description of Error</td><td>Cancelled factor in numerator with addend in denominator</td><td>Result of cancellation was zero</td><td>Cancelled addend in numerator with factor in denominator</td><td>Result of cancellation was zero</td><td>Like Term Error division</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td></tr>
</table>

Figure M24. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors		
Total No.	3		
Location of Error	$\frac{1\cancel{x}}{\cancel{x-1}} \cdot \frac{x\,\cancel{x^2-1}}{x^{\cancel{2}}}$	$\frac{1\cancel{x}}{\cancel{x-1}} \cdot \frac{x\,\cancel{x^2-1}}{x^{\cancel{2}}}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
Error	$\frac{x}{x}$	$\frac{x}{x}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1} = 0$
Description of Error	Cancelled addends	Cancelled addends	Changed to an equation equal to zero
Error Count	1	1	1

Figure M25. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{\cancel{x}}{\cancel{x}-1} / \frac{\cancel{x^2}}{\cancel{x^2}-1}$				
Error	$\frac{+1}{-1}$			$\frac{+1}{-1}$	
Description of Error	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero	Sign error	Cancelled factor in numerator with addend in denominator	Result of cancellation was zero
Error Count	1	1	1	1	1

Figure M26. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\dfrac{x}{x-1} \div \dfrac{x^2}{x^2-1}$	$\dfrac{x(x-1)}{x^2(x^2-1)}$	$\dfrac{x}{x-1} \div \dfrac{x^2}{x^2-1}$	$\dfrac{x}{x-1} \cdot \dfrac{-x^2-1}{x^2}$	
Error	$\dfrac{x(x-1)}{x^2(x^2-1)}$	$\dfrac{x^2-1x}{x^4-1x}$	$\dfrac{x}{x-1} \cdot \dfrac{-x^2-1}{x^2}$	$\dfrac{x^3+x}{-x^3+x^2}$	$\dfrac{x^3+x}{-x^3+x^2}$
Description of Error	Multiplied numerators times denominators	Distribution Error monomial x binomial	Sign Error	Sign Error	Sign Error
Error Count	1	1	1	1	1

Figure M27. Description and frequency of errors in Problem Set B algebraic item.

Problem : $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors			
Total No.	4			
Location of Error	$\frac{x^3+x}{-x^3+x^2}$		$\frac{\cancel{x}}{\cancel{x-1}} \bullet \frac{\cancel{x^2-1}}{\cancel{x^2}}$	$\frac{x}{x-1} \div \frac{x^2}{(x-1)(x+1)}$
Error	$-1 + \frac{x}{x^2}$	$-1 + \frac{x}{x^2}$	x	$\frac{x+(x+1)}{(x-1)(x+1)} \div \frac{x^2}{(x-1)(x+1)}$
Description of Error	Cancelled addends	Broke apart a rational expression	Unspecified	Notation Error (later multiplied)
Error Count	1	1	1	1

Figure M28. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

<table>
<tr><td></td><td colspan="6">Errors</td></tr>
<tr><td>Total No.</td><td colspan="6">7</td></tr>
<tr><td>Location of Error</td><td colspan="3">$\frac{\cancel{x}}{\cancel{x}-1} \div \frac{x^{\cancel{2}}}{x^{\cancel{2}}-1}$</td><td colspan="3">$-1/\frac{x^1}{x^1-1}$</td></tr>
<tr><td>Error</td><td colspan="2">$-1/\frac{x^1}{x^1-1}$</td><td>$-1/\frac{x^1}{x^1-1}$</td><td>-1</td><td>-1</td><td>-1</td></tr>
<tr><td>Description of Error</td><td>Cancelled factor in numerator with addend in denominator</td><td>Result of cancellation was zero</td><td>Cancelled factor in numerator with addend in denominator</td><td>Cancelled factor in numerator with addend in denominators</td><td>Result of cancellation was zero</td><td>Arithmetic Error division (Sign Error)</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>2</td></tr>
</table>

Figure M29. Description and frequency of errors in Problem Set B algebraic item.

Problem: $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

	Errors				
Total No.	5				
Location of Error	$\frac{\cancel{x^2}+x}{\cancel{x^2}}$		$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	$\frac{x}{x-1} \div \frac{\sqrt{x^2}}{\sqrt{x^2-1}}$	$\frac{\cancel{x}}{\cancel{x}-1} \div \frac{x^{\cancel{2}}}{x^{\cancel{2}}-1}$
Error	x		$\frac{x}{x-1} \div \frac{\sqrt{x^2}}{\sqrt{x^2-1}}$	$\frac{x}{x-1} \div \frac{x}{x-1}$	$\frac{x}{x}$
Description of Error	Cancelled addend in numerator with factor in denominator	Result of cancellation was zero.	Took square root of second term	Square Root Error	Cancelled factor in numerator with addend in denominator
Error Count	1	1	1	1	1

Figure M30. Description and frequency of errors in Problem Set B algebraic item.

Appendix N

Error Categories for Problem Set B Algebraic Item

Description of Errors in Problem 2 $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Cancellation Error unspecified	1	Cancellation Error	77
Cancelled addend in numerator with factor in denominator	1		
Cancelled addend in numerator with factor in denominator	2		
Cancelled addend in numerator with factor in denominator	1		
Cancelled addend in numerator with factor in denominator	2		
Cancelled addend in numerator with factor in denominator	1		
Cancelled addend in numerator with factor in denominator	1		
Cancelled addends	3		
Cancelled addends	2		
Cancelled addends	5		
Cancelled addends	4		
Cancelled addends	3		
Cancelled addends	4		
Cancelled addends	3		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	2		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		

Figure N1. Error categories in Problem Set B algebraic item.

Description of Errors in Problem 2 $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	Number of errors	Error Category	Total Number of Errors
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	2		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	2		
Cancelled factor in numerator with addend in denominator	6		
Cancelled factor in numerator with addend in denominator	6	Cancellation Error (continued)	
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominator	1		
Cancelled factor in numerator with addend in denominators	1		
Cancelled factors in an expression that had an operation of division	1		

Figure N2. Error categories in Problem Set B algebraic item.

Description of Errors in Problem 2 $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	Number of errors	Error Category	Total Number of Errors
Result of cancellation was zero	1	Residual Cancellation Error	56
Result of cancellation was zero	3		
Result of cancellation was zero	3		
Result of cancellation was zero	4		
Result of cancellation was zero	5		
Result of cancellation was zero	2		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	6		
Result of cancellation was zero	6		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	2		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		

Figure N3. Error categories in Problem Set B algebraic item.

Description of Errors in Problem 2 $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	Number of errors	Error Category	Total Number of Errors
Changed sign of 2nd term	1	Procedural Error	32
Changed to an equation equal to zero	1		
Cross-multiplied first term numerator with second term denominator	1		
Did not change to multiplication and take reciprocal of 2nd fraction	1		
Did not change to multiplication and take reciprocal of 2nd fraction	1		
Did not invert and multiply	1		
Divided across denominators	1		
Divided across denominators	2		
Divided across numerators	1		
Divided across numerators 4-16	2		
Divided denominators	2		
Divided numerators	2		
Found common denominator	1		
Found common denominator (did not multiply across denominators)	1		
Found common denominator and Equivalent Fractions	2		
Found equivalent fraction for common denominator	2		
Found equivalent fractions for common denominator	2		
Found reciprocal of first term	1		
Inverted 2nd term but did not change operation to multiplication	1		
Kept common denominator and divided numerator	1		
Multiplied across	1		
Multiplied across denominators	1		
Multiplied all terms by "x"	1		
Multiplied numerators times denominators	1		
Took square root of second term	1		

Figure N4. Error categories in Problem Set B algebraic item.

Description of Errors in Problem 2 $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Distribution Error – binomial x monomial	1	Distribution Error	17
Distribution Error – monomial x binomial	2		
Distribution Error – monomial x binomial	1		
Distribution Error – monomial x binomial	1		
Distribution Error – monomial x binomial (attributed to dropped parentheses)	1		
Distribution error – monomial x binomial $x\,(x - 1)$	1		
Distribution Error binomial x binomial $(x + 1)(x + 1) \neq x^2 + x$	1		
Distribution Error binomial x monomial	1		
Distribution Error binomial x monomial	1		
Distribution Error monomial x binomial	1		
Sign error	1		
Sign Error	1		
Sign Error	1		
Sign Error	1		
Arithmetic Error division (Sign Error)	2		
Answer not simplified	2	Simplification Error	16
Answer not simplified	1		
Answer not simplified	1		
Answer not simplified	1		
Answer not simplified	1		
Answer not simplified	5		
Answer not simplified	2		
Answer not simplified	5		
Answer not simplified	1		
Confused divisor and dividend	2	Division Conceptual Error	8
Confused divisor and dividend	2		
Confused divisor and dividend	2		
Confused divisor and dividend	2		

Figure N5. Error categories in Problem Set B algebraic item.

<table>
<tr><th>Description of Errors in Problem 2
$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$</th><th>Number of errors</th><th>Error Category</th><th>Total Number of Errors</th></tr>
<tr><td>Like Term Error - subtraction</td><td>1</td><td rowspan="5">Operations with Monomials Errors</td><td rowspan="5">6</td></tr>
<tr><td>Like Term Error addition $x^2 + x \neq 2x$</td><td>1</td></tr>
<tr><td>Like Term Error division</td><td>1</td></tr>
<tr><td>Like Term Error division</td><td>2</td></tr>
<tr><td>Like Term Error division ($x^2 / x^2 = 0$)</td><td>1</td></tr>
<tr><td>Did not finish</td><td>1</td><td rowspan="15"></td><td rowspan="15"></td></tr>
<tr><td>Did not finish</td><td>1</td></tr>
<tr><td>Did not finish</td><td>1</td></tr>
<tr><td>Did not finish</td><td>1</td></tr>
<tr><td>Equivalent Fraction Error Multiplied denominator by factor other than one</td><td>1</td></tr>
<tr><td>Equivalent Fraction Error Multiplied numerator by factor other than one</td><td>1</td></tr>
<tr><td>Notation Error (because next step used multiplication operation)</td><td>1</td></tr>
<tr><td>Notation Error (later multiplied)</td><td>1</td></tr>
<tr><td>Division of polynomials error</td><td>1</td></tr>
<tr><td>Square Root Error</td><td>1</td></tr>
<tr><td>Broke apart a rational expression</td><td>1</td></tr>
<tr><td>Moved (-1) from denominator of first term to second term</td><td>1</td></tr>
<tr><td>Unspecified</td><td>12</td><td rowspan="4">Unspecified Errors</td><td rowspan="4">15</td></tr>
<tr><td>Unspecified</td><td>1</td></tr>
<tr><td>Unspecified</td><td>1</td></tr>
<tr><td>Unspecified</td><td>1</td></tr>
<tr><td>Omitted</td><td>13</td><td>Omitted</td><td>13</td></tr>
<tr><td></td><td></td><td>Total</td><td>240</td></tr>
</table>

Figure N6. Error categories in Problem Set B algebraic item.

Appendix O

Problem Set C Numeric Item Errors

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

	Errors in Core Category "Found Common Denominator"					
Total No.	8					
Location of Errors	$\frac{35}{700} + \frac{60}{700}$	$\frac{95}{700}$	$\frac{7}{140} + \frac{12}{140}$	$\frac{12}{70}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$
Errors	$\frac{7}{90} + \frac{12}{90}$	$\frac{95}{700}$	$\frac{24}{140}$	$\frac{12}{70}$	$\frac{7}{140} + \frac{9}{140}$	$\frac{1}{4} \cdot \frac{1}{5} + \frac{3}{5} \cdot \frac{3}{7}$
Description of Errors	Arithmetic Error – division	Answer not simplified	Arithmetic Error – addition	Answer not simplified	Equivalent Fraction Error – multiplied numerator by wrong factor	Broke apart fraction
Error Count	1	3	1	1	1	1

Figure O1. Description and frequency of errors in Problem Set C numeric item core category "Found common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

	Errors in Core Category “Found Common Denominator”							
Total No.	8							
Location of Errors	$\frac{1.75}{35} + \frac{3}{35}$	$\frac{1}{4 \cdot \cancel{5}} + \frac{3}{\cancel{5} \cdot 7}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{35}{700} + \frac{60}{700}$		$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$	$\frac{10}{45} + \frac{30}{57}$
Errors	$\frac{4.75}{35}$	$\frac{1}{4} + \frac{3}{7}$	$\frac{3}{60} + \frac{6}{60}$	$\frac{90}{700}$	$\frac{90}{700}$	$\frac{10}{45} + \frac{30}{57}$	$\frac{570}{2565} + \frac{1710}{2565}$
Description of Errors	Improper form of a fraction	Cancellation Error – cancelled two factors in denominators	Arithmetic Error – multiplication	Arithmetic Error – addition	Answer not simplified	Mistook factors in denominator for decimals, multiplied each term by 10/10	Equivalent Fraction Error – multiplied numerator by wrong factor
Error Count	1	1	2	1	1	1	1

Figure O2. Description and frequency of errors in Problem Set C numeric item core category “Found common denominator.”

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

Errors in Core Category "Found Common Denominator"						
Total No.	6					
Location of Errors	$\frac{2280}{2565}$	$\left(\frac{7}{7}\right)\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}\left(\frac{4}{4}\right)$	$\frac{7}{140} + \frac{12}{140}$	$\frac{9}{60}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1 \cdot 35 - 20 \cdot 3}{20 \cdot 35}$
Errors	$\frac{2280}{2565}$	$\frac{8}{140} + \frac{12}{140}$	$\frac{9}{140}$	$\frac{9}{60}$	$\frac{1 \cdot 35 - 20 \cdot 3}{20 \cdot 35}$	$\frac{35 - 60}{600}$
Description of Errors	Answer not simplified	Arithmetic Error – added instead of multiplied	Notation Error "9" in place of "19"	Answer not simplified	Notation Error changed addition to subtraction	Arithmetic Error – multiplication
Error Count	1	1	1	1	1	1

Figure O3. Description and frequency of errors in Problem Set C numeric item core category "Found common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

Errors in Core Category "Found Common Denominator"						
Total No.	6					
Location of Errors	$\frac{35-60}{600}$	$\frac{25}{600}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{7}{140} + \frac{12}{140}$
Errors	$\frac{25}{600}$	$\frac{1}{150}$	$\frac{7}{140} + \frac{4}{140}$	$\frac{4}{700}$	$\frac{35}{750} + \frac{60}{750}$	$\frac{17}{140}$
Description of Errors	Arithmetic Error – addition	Equivalent Fraction Error – divided denominator by wrong factor	Arithmetic Error – multiplication	Added numerators without finding equivalent fractions	Arithmetic Error – multiplication	Arithmetic Error – addition
Error Count	1	1	1	1	1	1

Figure O4. Description and frequency of errors in Problem Set C numeric item core category "Found common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

Errors in Core Category "Found Common Denominator"						
Total No.	6					
Location of Errors	$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$		$\frac{256.5}{4.5} + \frac{769.5}{5.7}$	$\frac{95}{700}$	$\frac{1}{20} + \frac{3}{35}$	
Errors	$\frac{256.5}{4.5} + \frac{769.5}{5.7}$	$\frac{256.5}{4.5} + \frac{769.5}{5.7}$	$\frac{1026.0}{256.5}$	$\frac{19}{120}$	$\frac{1}{140} + \frac{3}{140}$	$\frac{1}{140} + \frac{3}{140}$
Description of Errors	Notation Error – wrote equivalent fractions with previous denominators	Equivalent Fraction Error – multiplied numerator by wrong factor	Improper form of fraction	Arithmetic Error – division	Equivalent Fraction Error – changed denominator but not numerator	Equivalent Fraction Error – changed denominator but not numerator
Error Count	1	1	1	1	1	1

Figure O5. Description and frequency of errors in Problem Set C numeric item core category "Found common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

Errors in Core Category "Found Common Denominator"						
Total No.	6					
Location of Errors	$\frac{4}{140}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$			$\left(\frac{70}{70}\right)\frac{1}{20} + \frac{3}{35}\left(\frac{4}{4}\right)$
Errors	$\frac{2}{2}$	$\frac{7}{140} + \frac{20}{140}$	$\frac{1}{5} + \frac{3}{5}$	$\frac{1}{5} + \frac{3}{5}$	$\frac{1}{5} + \frac{3}{5}$	$\frac{70 + 12}{140}$
Description of Errors	Equivalent Fraction Error – divided denominator by wrong factor	Equivalent Fraction Error – confused numerator with factor in denominator	Incorrect Common Denominator – used the common factor as the common denominator	Equivalent Fraction Error – removed a factor from denominator only	Equivalent Fraction Error – removed a factor from denominator only	Equivalent Fraction Error – multiplied numerator by wrong factor
Error Count	1	1	1	1	1	1

Figure O6. Description and frequency of errors in Problem Set C numeric item core category "Found common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

<table>
<tr><th colspan="4">Errors in Core Category “Found Common Denominator”</th></tr>
<tr><td>Total No.</td><td colspan="3">3</td></tr>
<tr><td>Location of Errors</td><td>$\frac{1}{20} + \frac{3}{35}$
$35 = 3 \cdot 7$
$20 = 4 \cdot 5$</td><td colspan="2">$\frac{1}{4 \cdot \cancel{5}} + \frac{3}{\cancel{5} \cdot 7}$</td></tr>
<tr><td>Errors</td><td>$\frac{35}{420} + \frac{60}{420}$</td><td>$\frac{7}{28} + \frac{21}{28}$</td><td>$\frac{7}{28} + \frac{21}{28}$</td></tr>
<tr><td>Description of Errors</td><td>Incorrect common denominator – factored 35 into 3 x 7, finds denominator 420 = 3 x 7 x 4 x 5</td><td>Cancellation Error – cancelled common factors in denominators</td><td>Equivalent Fraction Error – multiplied numerator by wrong factor</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure O7. Description and frequency of errors in Problem Set C numeric item core category “Found common denominator.”

Problem: $\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$

	Errors in Core Category “Did not find common denominator”						
Total No.	43						
Location of Errors	Omitted	Unspecified	$\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$	$\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$		$\frac{1}{4\cdot \cancel{5}}+\frac{3}{\cancel{5}\cdot 7}$	$\frac{1}{4}+\frac{3}{7}$
Errors	N/A	N/A	$\frac{1}{20}+\frac{3}{35}$	$\frac{4}{55}$	$\frac{4}{55}$	$\frac{1}{4}+\frac{3}{7}$	$\frac{4}{7}$
Description of Errors	N/A	N/A	Did not finish	Added across numerators	Added across denominators	Cancellation Error – cancelled two factors in denominators	Added numerators without finding equivalent fractions with common denominator
Error Count	1	1	15	14	10	1	1

Figure O8. Description and frequency of errors in Problem Set C numeric item core category “Did not find common denominator.”

Problem: $\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$

	Errors in Core Category "Did not find common denominator"					
Total No.	8					
Location of Errors	$\frac{1}{4}+\frac{3}{7}$	$\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$	$\frac{1}{20}+\frac{35}{3}$	$\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$		$\frac{95}{700}$
Errors	$\frac{4}{7}$	$\frac{1}{20}+\frac{35}{3}$	$\frac{710}{3}$	$\frac{4}{10.2}$	$\frac{4}{10.2}$	$.135714$
Description of Errors	Took largest denominator for use as solution denominator	Inverted 2nd term	Cross-multiplied	Notation Error – mistook factor in denominator for a decimal	Added across denominators	Rounded decimal answer was not correct
Error Count	1	1	1	2	2	1

Figure O9. Description and frequency of errors in Problem Set C numeric item core category "Did not find common denominator."

Problem: $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

	Errors in Core Category "Did not find common denominator"			
Total No.	4			
Location of Errors	$\frac{1}{20} + \frac{3}{35}$	$\frac{4}{55}$	$\frac{1}{20} + \frac{3}{35}$	$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$
Errors	$\frac{1.5}{35} + \frac{3}{35}$	$\frac{1}{14}$	$\frac{60}{35}$	$\frac{1}{20} + \frac{4}{35}$
Description of Errors	Equivalent Fraction Error – multiplied numerator by wrong factor	Arithmetic Error – division	Cross-multiplied	Notation Error – incorrectly copied number down
Error Count	1	1	1	1

Figure O10. Description and frequency of errors in Problem Set C numeric item core category "Did not find common denominator."

Appendix P

Error Categories for Problem Set C Numeric Item

Description of Errors in Problem 3 $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added across denominators	10	Procedural Error	32
Added across denominators	2		
Added across denominators	14		
Added numerators without finding equivalent fractions with common denominator	1		
Cross-multiplied	1		
Cross-multiplied	1		
Inverted 2nd term	1		
Added numerators without finding equivalent fractions	1		
Took largest denominator for use as solution denominator	1		
Arithmetic Error – added instead of multiplied	1	Arithmetic Error	13
Arithmetic Error – addition	1		
Arithmetic Error – addition	1		
Arithmetic Error – addition	1		
Arithmetic Error – addition	1		
Arithmetic Error – division	1		
Arithmetic Error – division	1		
Arithmetic Error – division	1		
Arithmetic Error – multiplication	2		
Arithmetic Error – multiplication	1		
Arithmetic Error – multiplication	1		
Arithmetic Error – multiplication	1		
Did Not Finish	15	Persistence Error	15

Figure P1. Error categories in Problem Set C numeric item.

Description of Errors in Problem 3 $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$	Number of errors	Error Category	Total Number of Errors
Equivalent Fraction Error – changed denominator but not numerator	1	Equivalent Fraction Error	13
Equivalent Fraction Error – changed denominator but not numerator	1		
Equivalent Fraction Error – confused numerator with factor in denominator	1		
Equivalent Fraction Error – divided denominator by wrong factor	1		
Equivalent Fraction Error – divided denominator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error – removed a factor from denom. only	1		
Equivalent Fraction Error – removed a factor from denom. only	1		
Answer not simplified	3	Simplification Error	7
Answer not simplified	1		
Answer not simplified	1		
Answer not simplified	1		
Answer not simplified	1		
Notation Error – incorrectly copied number down	1	Notation Error	6
Notation Error – mistook denom. factor for a decimal	2		
Notation Error – wrote equivalent fractions with previous denominators	1		
Notation Error "9" in place of "19"	1		
Notation Error changed addition to subtraction	1		

Figure P2. Error categories in Problem Set C numeric item.

Description of Errors in Problem 3 $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Cancellation Error – Cancelled common factors in denominators	1	Miscellaneous Errors	10
Cancellation Error – cancelled two factors in denominators	1		
Cancellation Error – cancelled two factors in denominators	1		
Incorrect common denominator – factored 35 into 3 x 7, finds denominator 420 = 3 x 7 x 4 x 5	1		
Incorrect common denominator – used the common factor as the common denominator	1		
Broke apart fraction	1		
Improper form of a fraction	1		
Improper form of fraction	1		
Mistook factors in denominator for decimals, multiplied each term by 10/10	1		
Rounded decimal answer was not correct	1		
Unspecified	1	Unspecified	1
Omitted	1	Omitted	1
		Total	98

Figure P3. Error categories in Problem Set C numeric item.

Appendix Q

Problem Set C Algebraic Item Errors

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Found Common Denominator"			
Total No.	4			
Location of Errors	$\frac{1}{(x+1)(x-2)}+\frac{x}{(x-5)(x-2)}$	$\frac{x^2+2x-5}{x^2-4x-5}$	$\frac{x^2+2x-5}{x^2-4x-5}$	$(x^2-x-2)(x^2-7x+10)$
Errors	$\frac{1}{(x+1)\cancel{(x-2)}}+\frac{x}{(x-5)\cancel{(x-2)}}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$x^4-8x^3-15x^2+4x-10$
Description of Errors	Cancelled common factor in denominators	Cancelled addends	Cancelled addends	Arithmetic Error – Multiplication
Error Count	1	1	1	1

Figure Q1. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Found Common Denominator"		
Total No.	5		
Location of Errors	$\frac{x^2+2x-5}{x^2-4x-5}$	$\frac{x^2+2x-5}{x^2-4x-5}$	$\frac{(x^2-7x+10)}{(x^2-x-2)(x^2-7x+10)}+\frac{x(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$
Errors	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{x^3-9x+10}{x^4-8x^3+15x^2+4x-20}$
Description of Errors	Result of cancellation was zero	Result of cancellation was zero	Answer Not Simplified
Error Count	1	1	3

Figure Q2. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$:

	Errors in Core Category "Found Common Denominator"	
Total No.	4	
Location of Errors	$\frac{(x^2-7x+10)}{(x^2-x-2)(x^2-7x+10)}+\frac{x(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$
Errors	$\frac{x^3-9x+10}{(x^2-x-2)(x^2-7x+10)}$	$x^2-2\ +x^2-7x+11$
Description of Errors	Answer Not Simplified	Cross Added
Error Count	3	1

Figure Q3. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$\frac{x^2-7x+10+x^3-x^2-2x}{(x^2-x-2)(x^2-7x+10)}$	$\frac{x^3-9x}{x^4-8x^3+15x^2+4x-20}$
Errors	$\frac{x^3-9x}{(x^2-x-2)(x^2-7x+10)}$	$\frac{x^3-9x}{x^4-8x^3+15x^2+4x-20}$
Description of Errors	Dropped a term	Answer not simplified
Error Count	1	1

Figure Q4. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"			
Total No.	4		
Location of Errors	$\frac{1\cancel{(x-5)}}{(x-2)(x+1)\cancel{(x-5)}}+\frac{x\cancel{(x+1)}}{(x-5)(x-2)\cancel{(x+1)}}$		
Errors	$\frac{x}{(x-2)(x+1)(x-5)}$		
Description of Errors	Cancelled factor in numerator and denominator after finding common denominator	Cancelled factor in numerator and denominated after finding common denominator	Cancelled factor reappeared in denominator
Error Count	2	1	1

Figure Q5. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$x^4-8x^3-3x^2-2x^2+4x-20$	
Errors	$x^4-8x^3-3x^2-2x^2+4x-20+x^5-8x^4+x^3+4x^2-20x$	
Description of Errors	Multiplied common denominator expression by x then adds to itself	Dropped denominators
Error Count	1	1

Figure Q6. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$$	
Errors	$\frac{(x^2-7x+10)+(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$	$\frac{(x^2-7x+10)+(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$
Description of Errors	Distribution Error – Did not distribute multiplier to all terms	Distribution Error – Did not distribute multiplier to all terms
Error Count	1	1

Figure Q7. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

<table>
<tr><td colspan="5">Errors in Core Category “Found Common Denominator”</td></tr>
<tr><td>Total No.</td><td colspan="4">4</td></tr>
<tr><td>Location of Errors</td><td>$\frac{(x^2-7x+10)+(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$</td><td colspan="3">$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$</td></tr>
<tr><td>Errors</td><td>$\frac{2x^2-8x+8}{x^4-8x^3+15x^2+4x-2}$</td><td>$\frac{2x^2+7x-5}{x^4-7x^2-10x}$</td><td>$\frac{2x^2+7x-5}{x^4-7x^2-10x}$</td><td>$\frac{2x^2+7x-5}{x^4-7x^2-10x}$</td></tr>
<tr><td>Description of Errors</td><td>Distribution Error – polynomial x polynomial</td><td>Multiplied like terms in (x^2 - x- 2) and (x^2+ 7x - 5) to obtain common denominator</td><td>Like Term Error – Multiplication (-2)(-5) $\neq$ -10x</td><td>Found expressions that when added to numerators produce denominator, added these together for numerators</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q8. Description and frequency of errors in Problem Set C algebraic item core category “Found common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"			
Total No.	3		
Location of Errors	$\frac{2x^2+7x-5}{x^4-7x^2-10x}$	$\frac{2x+2}{x^3-7x-10}$	
Errors	$\frac{2x+2}{x^3-7x-10}$	$\frac{4}{x^2-3}$	$\frac{4}{x^2-3}$
Description of Errors	Equivalent Fraction Error – Divided all but one term by *x* (then 7-5=2)	Equivalent Fraction Error – divided only terms with variable *x* by *x*	Arithmetic Error – addition
Error Count	1	1	1

Figure Q9. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

<table>
<tr><th colspan="4">Errors in Core Category “Found Common Denominator”</th></tr>
<tr><td>Total No.</td><td colspan="3">4</td></tr>
<tr><td>Location of Errors</td><td>$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-5)(x-2)}$</td><td colspan="2">$(x^2-x-2)(x^2-7x+10)$</td></tr>
<tr><td>Errors</td><td>$\frac{(x+1)}{(x-2)}+\frac{(x-5)x}{(x-2)}$</td><td colspan="2">$x^4-x^3-2x^2-7x^3+7x^2+7x+10x^2-10x-10$</td></tr>
<tr><td>Description of Errors</td><td>Moved all but common factor in denominator into numerators to create common denominator</td><td>Distribution Error – polynomial x polynomial</td><td>Distribution Error – polynomial x polynomial</td></tr>
<tr><td>Error Count</td><td>1</td><td>2</td><td>1</td></tr>
</table>

Figure Q10. Description and frequency of errors in Problem Set C algebraic item core category “Found common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$x^4 - x^3 - 2x^2 - 7x^3 + 7x^2 + 7x + 10x^2 - 10x - 10$	$\frac{x^2-4x+1}{x-2}$
Errors	$x^4 - 10x^3 + 15x^2 - 3x$	$\frac{(x-3)(x-1)}{x-2}$
Description of Errors	Like Term Error Addition: $-x^3 - 7x^3 \neq -10x^3$	Distribution Error – incorrectly factored expression
Error Count	1	1

Figure Q11. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

<table>
<tr><td colspan="5">Errors in Core Category “Found Common Denominator”</td></tr>
<tr><td>Total No.</td><td colspan="4">4</td></tr>
<tr><td>Location of Errors</td><td>$\frac{x^3-9x+10}{(x^2-x-2)(x^2-7x+10)}$</td><td colspan="2">$\frac{\cancel{(x-5)}+(x^2+x)}{(x-2)(x+1)\cancel{(x-5)}}$</td><td>$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-5)(x-2)}$</td></tr>
<tr><td>Errors</td><td>$\frac{x(x+10)(x-1)}{(x^2-x-2)(x^2-7x+10)}$</td><td colspan="2">$\frac{x(x+1)}{(x-2)(x+1)}$</td><td>$\frac{(x-5)+x^2+1}{(x-2)(x+1)(x-5)}$</td></tr>
<tr><td>Description of Errors</td><td>Distribution Error – incorrectly factored expression</td><td>Cancelled addend in numerator with factor in denominator</td><td>Result of cancellation was zero</td><td>Distribution Error – monomial x binomial</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q12. Description and frequency of errors in Problem Set C algebraic item core category “Found common denominator.”

Problem: $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$\frac{1x-5}{(x-2)(x+1)(x-5)} + \frac{x^2+x}{(x-2)(x+1)(x-5)}$	$\frac{x-5+x^2+x}{(x-5)(x+10)(x-2)}$
Errors	$\frac{x^2+2x-5}{x^3-6x^2-3x+10}$	$\frac{x^2+2x-5}{}$
Description of Errors	Distribution Error – Sign Error	Did not finish
Error Count	1	1

Figure Q13. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$\frac{1}{(x-1)(x-2)}+\frac{x}{(x-5)(x-2)}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$
Errors	$\frac{1x}{(x-2)^2(x-5)(x-1)}$	$\frac{x^2-7x+10+x^3-x^2-2x}{x^2-x-2+x^2-7x+10}$
Description of Errors	Multiplied across numerators	Added across denominators
Error Count	1	1

Figure Q14. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$\frac{1}{(x+1)(x-2)}+\frac{x}{(x-5)(x-2)}$	$\frac{1}{(x+1)(x-2)}+\frac{x}{(x-5)(x-2)}$
Errors	$\frac{(x+1)(x-5)}{(x+1)(x-2)}+\frac{x(x+1)(x-5)}{(x-5)(x-2)}$	$\frac{(x+1)(x-5)}{(x+1)(x-2)}+\frac{x(x+1)(x-5)}{(x-5)(x-2)}$
Description of Errors	Equivalent Fraction Error Multiplied numerator by wrong factor	Equivalent Fraction Error – Multiplied numerator by wrong factor
Error Count	1	1

Figure Q15. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"			
Total No.	3		
Location of Errors	$\frac{\cancel{(x+1)}(x-5)}{\cancel{(x+1)}(x-2)}+\frac{x(x+1)\cancel{(x-5)}}{\cancel{(x-5)}(x-2)}$		$\frac{(1)(x^2-7x+10)+(x)(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$
Errors	$\frac{(x-5)+x^2+x}{(x-2)}$		$\frac{2x^2-8x+8}{(x^2-x-2)(x^2-7x+10)}$
Description of Errors	Cancelled factor in equivalent fraction before adding	Cancelled factor in equivalent fraction before adding	Distribution Error – failed to distribute a factor
Error Count	1	1	1

Figure Q16. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

Errors in Core Category "Found Common Denominator"		
Total No.	2	
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	
Errors	$\frac{1+x}{(x-1)(x+1)(x-2)(x-5)}$	$\frac{1+x}{(x-1)(x+1)(x-2)(x-5)}$
Description of Errors	Added numerators before finding equivalent fractions	Distribution Error – incorrectly factored an expression
Error Count	1	1

Figure Q17. Description and frequency of errors in Problem Set C algebraic item core category "Found common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"			
Total No.	23			
Location of Errors	Omitted	Unspecified	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	
Errors	N/A	N/A	$\frac{1+x}{x^2-x-2+x^2-7x+10}$	$\frac{1+x}{x^2-x-2+x^2-7x+10}$
Description of Errors	N/A	N/A	Added across numerators	Added across denominators
Error Count	18	3	1	1

Figure Q18. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category “Did not find common denominator”			
Total No.	4			
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$		$1x^2-1x-2+2x^3-7x^2+10x$	
Errors	$\frac{1}{x^2-2}+\frac{x}{-8^2x+10}$		$2x^3+6x^2+11x-2$	$2x^3+6x^2+11x-2$
Description of Errors	Moved a term from one denominator to the other	Unspecified	Like Term Error - Addition	Like Term Error - Addition
Error Count	1	1	1	1

Figure Q19. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	5				
Location of Errors	$\frac{1}{\cancel{x^2}-x-2}+\frac{x}{\cancel{x^2}-7x+10}$				$\frac{x}{-8x+8}$
Errors	$\frac{x}{-8x+8}$	$\frac{x}{-8x+8}$	$\frac{x}{-8x+8}$	$\frac{x}{-8x+8}$	$\frac{1}{-7+8}$
Description of Errors	Cancelled common terms in denominators	Added across numerators	Like Term Error - Addition	Added across denominators	Cancelled addends – subtraction
Error Count	1	1	1	1	1

Figure Q20. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

<table>
<tr><td></td><td colspan="5">Errors in Core Category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="5">5</td></tr>
<tr><td>Location of Errors</td><td>$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$</td><td colspan="3">$\frac{1}{\cancel{(x-2)}(x+1)}+\frac{x}{(x-5)\cancel{(x-2)}}$</td><td>$(x-5)+(x+1)$</td></tr>
<tr><td>Errors</td><td>$1x^2-1x-2+2x^3-7x^2+10x$</td><td colspan="3">$(x-5)+(x+1)$</td><td>x^2-4</td></tr>
<tr><td>Description of Errors</td><td>Multiplied each numerator by its own denominator</td><td>Cancelled common factor in denominators</td><td>Added across denominators</td><td>Dropped numerators</td><td>Like Term Error – Addition $x+x\neq x^2$</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q21. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"			
Total No.	4			
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$			$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-2)(x-5)}$
Errors	$\frac{x}{(x-2)(x+1)+(x+2)(x-5)}$		$\frac{x}{(x-2)(x+1)+(x+2)(x-5)}$	$\frac{1}{(x-2)(x+1)}+\frac{1}{(x-2)(-5)}$
Description of Errors	Added across numerators	Like Term Error ($1+x\neq x$)	Added across denominators	Cancellation Error – Cancelled factor in numerator with addend in denominator
Error Count	1	1	1	1

Figure Q22. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"		
Total No.	6		
Location of Errors	$\frac{1}{(x-2)(x+1)}+\frac{1}{5x+10}$	$\frac{1}{(x-2)(x+1)}+\frac{1}{5x+10}$	$1\cdot\left(\frac{1}{x^2-x-2}\right)+\frac{x}{x^2-7x+10}$
Errors	$\frac{1}{(x-2)(x+1)}+\frac{1}{5x+10}$	$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-5)(x-2)}$	$x^2-x-2+\frac{x}{x^2-7x+10}$
Description of Errors	Did not finish	Did not finish	Confused 1 ÷ a with a ÷ 1 = a
Error Count	1	3	2

Figure Q23. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"					
Total No.	7					
Location of Errors	$x^2-x-2+\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$				
Errors	$x^2-x-2+\frac{x}{x^2-7x+10}$	$\frac{1x}{x^4+7x+8}$		$\frac{1x}{x^4+7x+8}$	$\frac{1x}{x^4+7x+8}$	$\frac{1x}{x^4+7x+8}$
Description of Errors	Did not finish	Added across numerators	Like Term Error – addition $1+x \neq x$	Added across denominators	Like Term Error – Addition $x^2+x^2 \neq x^4$	Like Term Error – Addition $-x$ - $7x \neq +7x$
Error Count	2	1	1	1	1	1

Figure Q24. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	5				
Location of Errors	$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-5)(x-2)}$	$(x-2)\left(\frac{1}{x+1}+\frac{x}{x-5}\right)$		$\frac{x^2-7x+10+x^3-x^2-2x}{(x^2-x-2)(x^2-7x+10)}$	
Errors	$(x-2)\left(\frac{1}{x+1}+\frac{x}{x-5}\right)$	$(x-2)[(x+1)^{-1}+(x-5)(x-2)^{-1}]$		$\frac{x^3-2x^2-5x}{x^4-8x^3-15x^2+4x-20}$	
Description of Errors	Multiplied expression by common factor removed from denominator	Dropped a term (x)	Distribution Error – factor outside parentheses was also included inside parentheses	Like Term Error – addition $7x$ + (-2x) $\neq$ -5x	Dropped a term
Error Count	1	1	1	1	1

Figure Q25. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

<table>
<tr><td></td><td colspan="5">Errors in Core Category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="5">23</td></tr>
<tr><td>Location of Errors</td><td>$(x-2)[(x+1)^{-1}+(x-5)(x-2)^{-1}]$</td><td colspan="2">$\dfrac{1}{x^2-x-2}+\dfrac{x}{x^2-7x+10}$</td><td colspan="2">$\dfrac{1+x}{2x^2-8x-8}$</td></tr>
<tr><td>Errors</td><td>$(x-2)[(x+1)^{-1}+(x-5)(x-2)^{-1}]$</td><td>$\dfrac{1+x}{2x^2-8x+8}$</td><td>$\dfrac{1+x}{2x^2-8x+8}$</td><td colspan="2">$x+1\,\big\vert\overline{2x^2-8x+8}$</td></tr>
<tr><td>Description of Errors</td><td>Did not finish</td><td>Added across numerators</td><td>Added across denominators</td><td>Used long division</td><td>Confused divisor with dividend</td></tr>
<tr><td>Error Count</td><td>1</td><td>10</td><td>10</td><td>1</td><td>1</td></tr>
</table>

Figure Q26. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"			
Total No.	8			
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$			
Errors	$\frac{x}{x^2-8x+8}$	$\frac{x}{x^2-8x+8}$	$\frac{x}{x^2-8x+8}$	$\frac{x}{x^2-8x+8}$
Description of Errors	Added across numerators	Like Term Error – addition $1+x\neq x$	Added across denominator	Like Term Error – addition $x^2+x^2\neq x^2$
Error Count	2	2	2	2

Figure Q27. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"		
Total No.	15		
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	
Errors	$\frac{x^2-7x+10}{x^4-8x^3+15x^2+4x-20}+\frac{x^3-7x^2+10x}{x^4-8x^3+15x^2+4x-20}$	$\frac{x}{2x^2-8x+8}$	$\frac{x}{2x^2-8x+8}$
Description of Errors	Equivalent Fraction Error – multiplied numerator by wrong factor	Added across numerators	Like Term Error – addition $1+x\neq x$
Error Count	1	8	6

Figure Q28. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	16				
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$			
Errors	$\frac{x}{2x^2-8x-8}$	$\frac{x}{x^4-8x+8}$	$\frac{x}{x^4-8x+8}$	$\frac{x}{x^4-8x+8}$	$\frac{x}{x^4-8x+8}$
Description of Errors	Added across denominators	Added across numerators	Like Term Error – addition $1 + x \neq x$	Added across denominators	Like Term Error – addition $x^2 + x^2 \neq x^4$
Error Count	8	2	2	2	2

Figure Q29. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"			
Total No.	4			
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x^2-7x+10}{x}$	$\frac{1}{\cancel{x^2}-x\cancel{-2}}+\frac{x}{\cancel{x^2}-7x\cancel{+10}}$	
Errors	$\frac{1}{x^2-x-2}+\frac{x^2-7x+10}{x}$	$\frac{x^4-8x^3+15x^2+4x-20}{x}$	$\frac{1}{-x+1}+\frac{x}{-7x-5}$	
Description of Errors	Inverted 2nd term	Cross-multiplied	Cancelled common term in denominators	Cancelled factor from two terms in denominator
Error Count	1	1	1	1

Figure Q30. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category “Did not find common denominator”				
Total No.	5				
Location of Errors	$\frac{1}{-x+1}+\frac{x}{-7x-5}$	$\frac{\cancel{x}}{-7\cancel{x}-5}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$		
Errors	$\frac{x}{-7x-5}$	$-\frac{1}{2}$	$\frac{1+x}{4x^2-8x+8}$	$\frac{1+x}{4x^2-8x+8}$	$\frac{1+x}{4x^2-8x+8}$
Description of Errors	Dropped a term	Cancelled addends	Added across numerators	Added across denominators	Like Term Error – addition $x^2+x^2\neq 4x^2$
Error Count	1	1	1	1	1

Figure Q31. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category “Did not find common denominator”					
Total No.	6					
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$			$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$		
Errors	$\frac{1+x}{x^4-8x+8}$	$\frac{1+x}{x^4-8x+8}$	$\frac{1+x}{x^4-8x+8}$	$\frac{2x}{2x^2-8x+8}$	$\frac{2x}{2x^2-8x+8}$	$\frac{2x}{2x^2-8x+8}$
Description of Errors	Added across numerators	Added across denominators	Like Term Error – addition $x^2+x^2 \neq x^4$	Added across numerators	Like Term Error – addition $1+x \neq 2x$	Added across denominators
Error Count	1	1	1	1	1	1

Figure Q32. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

<table>
<tr><td></td><td colspan="7">Errors in Core Category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="7">7</td></tr>
<tr><td>Location of Errors</td><td colspan="3">$\frac{2x}{2x^2-8x+8}$</td><td>$2x-8x+8$</td><td colspan="3">$\frac{1}{\cancel{(x-2)}(x+1)}+\frac{x}{(x-5)\cancel{(x-2)}}$</td></tr>
<tr><td>Errors</td><td colspan="2">$2x \div 2x^2 = 2x$</td><td>$2x-8x+8$</td><td>$6x+8$</td><td colspan="3">$\frac{x}{(x+1)(x-5)}$</td></tr>
<tr><td>Description of Errors</td><td>Equivalent Fraction Error Divided numerator by one term from denominator</td><td>Like Term Error – division $2x \div 2x^2 \neq 2x$</td><td>Moved all terms to numerator</td><td>Like Term Error - addition $2x - 8x \neq 6x$</td><td>Cancelled common factor in denominators</td><td>Multiplied across numerators</td><td>Multiplied across denominators</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q33. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	5				
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$				
Errors	$\frac{1x}{x^4-8x^2+8}$	$\frac{1x}{x^4-8x^2+8}$	$\frac{1x}{x^4-8x^2+8}$	$\frac{1x}{x^4-8x^2+8}$	$\frac{1x}{x^4-8x^2+8}$
Description of Errors	Added across numerators	Like Term Error – addition $1 + x \neq 1x$	Added across denominators	Like Term Error - addition $x^2 + x^2 \neq x^4$	Like Term Error –addition $x - 7x \neq -8x^2$
Error Count	1	1	1	1	1

Figure Q34. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"	
Total No.	2	
Location of Errors	$\frac{1+x}{2x^2-8x+8}$	$\frac{1}{\cancel{(x-2)}(x+1)}+\frac{x}{(x-5)\cancel{(x-2)}}$
Errors	$2\left(\frac{1+x}{x^2-4x+4}\right)$	$\frac{1}{x+1}+\frac{x}{x-5}$
Description of Errors	Moved a factor from the denominator outside of parentheses (essentially to numerator) and left rational expression inside	Cancelled common factor in denominators
Error Count	1	1

Figure Q35. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

<table>
<tr><td></td><td colspan="3">Errors in Core Category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="3">3</td></tr>
<tr><td>Location of Errors</td><td>$\frac{1}{x+1}+\frac{x}{x-5}$</td><td colspan="2">$(x+1)\frac{1}{x+1}+\frac{x}{x-5}=0(x+1)$</td></tr>
<tr><td>Errors</td><td>$\frac{1}{x+1}+\frac{x}{x-5}=0$</td><td>$1+\frac{x}{x-5}=x+1$</td><td>$1+\frac{x}{x-5}=x+1$</td></tr>
<tr><td>Description of Errors</td><td>Changed to equation equal to zero</td><td>Distribution Error – Failed to distribute multiplier to all terms</td><td>Arithmetic Error – Multiplication by zero</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q36. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"					
Total No.	6					
Location of Errors	$1+\frac{\cancel{x}}{\cancel{x}-5}=x+1$		$1+\frac{1}{-5}=x+1$	$\frac{1+x}{2x^2-8x-8}$	$\frac{1}{x^2-x-2}+\frac{\cancel{x}}{\cancel{x^2}-7x+10}$	
Errors	$1+\frac{1}{-5}=x+1$		$\frac{5}{5}+\frac{1}{-5}=\frac{6}{5}=x+1$	$\frac{1+x}{x^2-4x-4}$	$\frac{1}{x^2-x-x}+x-7x+10$	
Description of Errors	Cancelled addends	Result of cancellation was zero	Arithmetic Error addition	Removed a factor of two from denominator	Cancelled addend	Dropped numerator
Error Count	1	1	1	1	1	1

Figure Q37. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

<table>
<tr><td></td><td colspan="5">Errors in Core Category “Did not find common denominator”</td></tr>
<tr><td>Total No.</td><td colspan="5">5</td></tr>
<tr><td>Location of Errors</td><td>$\frac{1}{x^2-x-2} + x - 7x + 10$</td><td colspan="4">$\frac{x-7x+10}{x^2-x-2}$</td></tr>
<tr><td>Errors</td><td>$\frac{x-7x+10}{x^2-x-2}$</td><td colspan="2">$x - 6x - 5$</td><td colspan="2">$x - 6x - 5$</td></tr>
<tr><td>Description of Errors</td><td>Unspecified</td><td>Cancelled addends</td><td>Result of cancellation was zero</td><td>Cancelled addends</td><td>Result of cancellation was zero</td></tr>
<tr><td>Error Count</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Figure Q38. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"			
Total No.	4			
Location of Errors	$\frac{x-7x+10}{x^2-x-2}$		$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$
Errors	$x-6x-5$	$x-6x-5$	$\frac{1}{x^2-x-2}=-\frac{x}{x^2-7x+10}$	$1(x^2-x-2)+x(x^2-7x+10)$
Description of Errors	Moved term to numerator	Cancelled Addends – Subtraction	Changed to equation equal to zero	Multiplied each numerator by its own denominator
Error Count	1	1	1	1

Figure Q39. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"	
Total No.	2	
Location of Errors	$\frac{1}{x^2-x-2}=-\frac{x}{x^2-7x+10}$	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$
Errors	$(x^2-7x+10)\frac{1}{x^2-x-2}=-\frac{x}{x^2-7x+10}(x^2-7x+10)$	$\frac{1}{x^2-x-2}+\frac{1}{x^2-6x+10}$
Description of Errors	Did not finish	Cancelled addend – subtraction
Error Count	1	1

Figure Q40. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	5				
Location of Errors	$\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$		$1(x^2-x-2)+x(x^2-7x+10)$	$\frac{1}{x^2-\cancel{x}-2}+\frac{\cancel{x}}{x^2-7x+10}$	
Errors	$\frac{1}{2x^2-7x+8}$		$x^2-x-2+x^3-7x+10x$	$\frac{1}{x^2-2}+\frac{1}{x^2-7x+10}$	
Description of Errors	Multiplied across numerators	Added across denominators	Like Term Error Multiplication	Cancelled addends	Result of cancellation was zero
Error Count	1	1	1	1	1

Figure Q41. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"				
Total No.	5				
Location of Errors	$\frac{1}{x^2-2}+\frac{1}{x^2-7x+10}$	$\frac{1}{\cancel{x^2}-2}+\frac{\cancel{x^2}-7x+10}{1}$			
Errors	$\frac{1}{x^2-2}+\frac{x^2-7x+10}{1}$	$\frac{-7x+11}{-2}$			
Description of Errors	Inverted 2nd term	Cancelled addends	Result of cancellation was zero	Added across numerators	Multiplied across denominators
Error Count	1	1	1	1	1

Figure Q42. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category "Did not find common denominator"	
Total No.	2	
Location of Errors	$\frac{1}{(x-2)(x+1)}+\frac{x}{(x-5)(x-2)}$	$1(x-2)(x+1) \quad x(x-5)(x-2)$
Errors	$1(x-2)(x+1) \quad x(x-5)(x-2)$	$x-2+x+1 \quad + \quad x^2-5x+x^2-2x$
Description of Errors	Multiplied each numerator by its own denominator	Added two binomials when operation should have been multiplication
Error Count	1	1

Figure Q43. Description and frequency of errors in Problem Set C algebraic item core category "Did not find common denominator."

Problem: $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

	Errors in Core Category “Did not find common denominator”
Total No.	1
Location of Errors	$1(x-2)(x+1) \quad x(x-5)(x-2)$
Errors	$x-2+x+1 \;+\; x^2-5x+x^2-2x$
Description of Errors	Distribution Error – Multiplication of three terms
Error Count	1

Figure Q44. Description and frequency of errors in Problem Set C algebraic item core category “Did not find common denominator.”

Appendix R

Error Categories for Problem Set C Algebraic Item

Description of Error in Problem 5 $\frac{1}{x^2 - x - 2} + \frac{x}{x^2 - 7x + 10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added across denominator	2	Procedural Error	83
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	10		
Added across denominators	8		
Added across denominators	2		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across denominators	1		
Added across numerators	1		
Added across numerators	1		
Added across numerators	1		
Added across numerators	1		
Added across numerators	10		
Added across numerators	2		
Added across numerators	8		
Added across numerators	2		

Figure R1. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Added across numerators	1	Procedural Error (continued)	
Added across numerators	1		
Added across numerators	1		
Added across numerators	1		
Added across numerators	1		
Added numerators before finding equivalent fractions	1		
Changed to equation equal to zero	1		
Changed to equation equal to zero	1		
Cross added	1		
Cross-multiplied	1		
Found expressions that when added to numerators produced denominator, added these together for numerators	1		
Inverted 2^{nd} term	1		
Inverted 2^{nd} term	1		
Multiplied across denominators	1		
Multiplied across denominators	1		
Multiplied across numerators	1		
Multiplied across numerators	1		
Multiplied across numerators	1		
Multiplied common denominator expression by *x* then adds to itself	1		
Multiplied each numerator by its own denominator	1		
Multiplied each numerator by its own denominator	1		
Multiplied each numerator by its own denominator	1		
Multiplied expression by common factor removed from denominator	1		
Used synthetic division	1		

Figure R2. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2 - x - 2} + \frac{x}{x^2 - 7x + 10}$	Number of errors	Error Category	Total Number of Errors
Like Term Error Addition: $-x^3 - 7x^3 \neq -10x^3$	1	Operations with Monomials Error	35
Like Term Error - Addition	1		
Like Term Error - Addition	1		
Like Term Error - Addition	1		
Like Term Error – addition $1 + x \neq x$	6		
Like Term Error - addition $2x – 8x \neq 6x$	1		
Like Term Error – addition $x^2 + x^2 \neq 4x^2$	1		
Like Term Error – addition $x^2 + x^2 \neq x^4$	1		
Like Term Error – addition $1 + x \neq x$	2		
Like Term Error – addition $7x + (-2x) \neq -5x$	1		
Like Term Error – Addition $x + x \neq x^2$	1		
Like Term Error – addition x2 + x2 ≠ x2	2		
Like Term Error – addition $x^2 + x^2 \neq x^4$	2		
Like Term Error – Addition $x^2+x^2 \neq x^4$	1		
Like Term Error – addition $1 + x$	1		
Like Term Error – addition $1 + x \neq 2x$	1		
Like Term Error – addition $1 + x \neq x$	2		
Like Term Error – addition $1+x \neq x$	1		
Like Term Error – Addition $–x - 7x \neq +7x$	1		
Like Term Error - addition $x^2 + x^2 \neq x^4$	1		
Like Term Error – division $2x \div 2x^2 \neq 2x$	1		
Like Term Error – Multiplication $(-2)(-5) \neq -10x$	1		
Like Term Error ($1 + x \neq x$)	1		
Like Term Error –addition $x - 7x \neq -8x^2$	1		
Like Term Error Multiplication	1		
Added two binomials when operation should have been multiplication	1		

Figure R3. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Cancellation Error – Cancelled factor in numerator with addend in denominator	1	Cancellation Error	27
Cancelled addend	1		
Cancelled addend – subtraction	1		
Cancelled addend in numerator with factor in denominator	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends	1		
Cancelled addends – subtraction	1		
Cancelled Addends – Subtraction	1		
Cancelled common factor in denominators	1		
Cancelled common factor in denominators	1		
Cancelled common factor in denominators	1		
Cancelled common factor in denominators	1		
Cancelled common term in denominators	1		
Cancelled common terms in denominators	1		
Cancelled factor from two terms in denominator	1		
Cancelled factor in equivalent fraction before adding	1		
Cancelled factor in equivalent fraction before adding	1		
Cancelled factor in numerator and denominated after finding common denominator	1		
Cancelled factor in numerator and denominator after finding common denominator	2		
Cancelled factor reappeared in denominator	1		

Figure R4. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Result of cancellation was zero	1	Residual Cancellation Error	8
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Result of cancellation was zero	1		
Distribution Error – Did not distribute multiplier to all terms	1	Distribution Error	16
Distribution Error – Did not distribute multiplier to all terms	1		
Distribution Error – factor outside parentheses was also included inside parentheses	1		
Distribution Error – failed to distribute a factor	1		
Distribution Error – Failed to distribute multiplier to all terms	1		
Distribution Error – incorrectly factored an expression	1		
Distribution Error – incorrectly factored expression	1		
Distribution Error – incorrectly factored expression	1		
Distribution Error – monomial *x* binomial	1		
Distribution Error – Multiplication of three terms	1		
Distribution Error – polynomial x polynomial	1		
Distribution Error – polynomial x polynomial	2		
Distribution Error – polynomial x polynomial	1		
Distribution Error – Sign Error	1		
Multiplied like terms in $(x^2 - x - 2)$ and $(x^2 + 7x - 5)$ to obtain common denominator	1		

Figure R5. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2 - x - 2} + \frac{x}{x^2 - 7x + 10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Moved a factor from the denominator to outside of the parentheses (essentially to numerator) and left rational expression inside	1	Properties of Operations Error	9
Moved a term from one denominator to the other	1		
Moved all but common factor in denominator into numerators to create common denominator	1		
Moved all terms to numerator	1		
Moved term to numerator	1		
Removed a factor of two from denominator	1		
Confused 1 ÷ a with a ÷ 1 = a	2		
Confused divisor with dividend	1		
Did not finish	1	Persistence Error	9
Did not finish	1		
Did not finish	3		
Did not finish	2		
Did not finish	1		
Did not finish	1		
Dropped a term	1	Miscellaneous Errors	24
Dropped a term	1		
Dropped a term	1		
Dropped a term (x)	1		
Dropped denominators	1		
Dropped numerator	1		
Dropped numerators	1		

Figure R6. Error categories in Problem Set C algebraic item.

Description of Error in Problem 5 $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$	**Number of errors**	**Error Category**	**Total Number of Errors**
Answer Not Simplified	3	Miscellaneous Errors (Continued)	
Answer Not Simplified	3		
Answer not simplified	1		
Equivalent Fraction Error – Divided all but one term by *x* (then 7-5=2)	1		
Equivalent Fraction Error – divided only terms with variable *x* by *x*	1		
Equivalent Fraction Error – Multiplied numerator by wrong factor	1		
Equivalent Fraction Error – multiplied numerator by wrong factor	1		
Equivalent Fraction Error Divided numerator by one term from denominator	1		
Equivalent Fraction Error Multiplied numerator by wrong factor	1		
Arithmetic Error – addition	1		
Arithmetic Error – Multiplication	1		
Arithmetic Error addition	1		
Arithmetic Error – Multiplication by zero	1		
Unspecified	3	Unspecified	5
Unspecified	1		
Unspecified	1		
Omitted	18	Omitted	18
		Total	234

Figure R7. Error categories in Problem Set C algebraic item.

Appendix S

Emma's Assessment

Rational Numbers and Expressions Study
Assessment

3-6

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

1. $\frac{1+6}{2} + \frac{9}{2+3} - 3$

$\frac{7}{2} + \frac{9}{5} - 3$

$\frac{35}{10} + \frac{18}{10} - 3$

$\frac{53}{10} - 3 \quad \frac{53}{10} - \frac{30}{10}$

$\boxed{\frac{23}{10}}$

$18 + 35 = 53$

2. $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

$\frac{-x-1}{x} \quad \frac{x+1}{x}$

3. $\frac{1}{4\cdot 5} + \frac{3}{5\cdot 7}$

$\frac{1}{4}\cdot\frac{1}{5} \quad \frac{3}{5}\cdot\frac{3}{7} = \frac{9}{35}$

$\frac{1}{20} + \frac{9}{35}$

$\frac{35}{700} + \frac{180}{700} \quad \frac{215}{700} \quad \boxed{\frac{43}{140}}$

$20 \times 9 = 180 \quad 180 + 35 = 215$

20×35: 100, 600, 700

4. $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

$\frac{x+2}{4} + \frac{6x}{x+1} - \frac{3}{1}$

$(x+2)(x+1)$

$4(x+1)$

$\frac{x^2+3x+2}{4x+4} + \frac{24x}{4x+4} - \frac{12x+12}{4x+4}$

$x^2+27x+2-12x-12$

$\frac{x^2+15x-10}{4x+4}$ (x+

5. $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

$5\overline{)215} = 43$, 20, 1

$5\overline{)700} = 140$, 70

$(x^2-x-2)(x^2-7x+10)$

$-10x + 14x - 10$

$x^4 - 8x^3 - 15x^2 + 4x - 10$

$\frac{x^2-7x+10}{x^4-8x^3-15x^2+4x-10} + \frac{x^3-x^2-2x}{}$

$\frac{x^3-9x+10}{x^4-8x^3-15x^2+4x-10}$

6. $\frac{3}{3-1} \div \frac{9}{9-1}$

$\frac{\cancel{3}}{\cancel{2}} \cdot \frac{-8}{9} = \frac{-12}{9} = \frac{4}{3}$

Appendix T

Isabella's Assessment

Rational Numbers and Expressions Study
Assessment

23-2

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

Answers

1. $\frac{23}{10}$

2. $\frac{x+1}{x}$

3. $\frac{19}{140}$

4. $\frac{x^2+39x+14}{4x+4}$

5. (at bottom right)

6. $\frac{4}{3}$

1. $\frac{1+6}{2}+\frac{9}{2+3}-3$

$\frac{7}{2}+\frac{9}{5}-\frac{30}{10}$

$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}$

$=\frac{23}{10}$

2. $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

$\frac{x}{x-1}\cdot\frac{x^2-1}{x^2}=\frac{\cancel{x}}{\cancel{x-1}}\cdot\frac{(x+1)\cancel{(x-1)}}{\cancel{x^2}}$

$=\frac{x+1}{x}$

3. $\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$

$\frac{1}{20}+\frac{3}{35}$

$\frac{35}{700}+\frac{60}{700}$

$=\frac{95}{700}=\frac{19}{140}$

140
$5\overline{)700}$
5
20
20

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$\frac{x+2}{4}\quad\frac{(x+2)(x+1)}{4(x+1)}=\frac{x^2+3x+2}{4(x+1)}$

$\frac{6x}{x+1}\quad\frac{6x(4)}{4(x+1)}=\frac{24x}{4(x+1)}$

$\frac{3}{1}\quad\frac{3(4)(x+1)}{4(x+1)}=\frac{12x+12}{4(x+1)}$

$\frac{x^2+3x+2+24x+12x+12}{4(x+1)}$

$\frac{x^2+39x+14}{4(x+1)}$

$=\frac{x^2+39x+14}{4x+4}$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

$\frac{1}{x^2-x-2}\quad\frac{x^2-7x+10}{(x^2-x-2)(x^2-7x+10)}$

$\frac{x}{x^2-7x+10}\quad\frac{x(x^2-x-2)}{(x^2-7x+10)(x^2-x-2)}$

$x^4-\boxed{x^3}-2x^2-\boxed{7x^3}+7x^2+14x+10x^2-10x-20$

$x^4-8x^3+15x^2+4x-20$

$\frac{x^2-7x+10+x^3-x^2-2x}{x^4-8x^3+15x^2+4x-20}=\boxed{\frac{x^3-9x+10}{x^4-8x^3+15x^2+4x-20}}$

6. $\frac{3}{3-1}\div\frac{9}{9-1}$

$\frac{3}{2}\cdot\frac{8}{9}$

$\frac{24}{18}=\frac{4}{3}$

Appendix U

Sophia's Assessment

3-5

Rational Numbers and Expressions Study
Assessment

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

1. $\frac{1+6}{2} + \frac{9}{2+3} - 3$

$\frac{7 \cdot 5}{2 \cdot 5} + \frac{9 \cdot 2}{5 \cdot 2}$

$\frac{35}{10} + \frac{18}{10} - \frac{30}{10} = \frac{23}{10}$

2. $\frac{x}{x-1} \div \frac{x^2}{x^2-1}$

$(x+1)(x-1)$

$\frac{x}{x-1} \cdot \frac{x^2-1}{x^2}$

$\frac{\cancel{x}}{\cancel{x-1}} \cdot \frac{(x+1)\cancel{(x-1)}}{\cancel{x^2}}$

$= \boxed{\frac{x+1}{x}}$

3. $\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$

$\frac{1 \cdot 7}{20 \cdot 7} + \frac{3 \cdot 3}{35}$

$\frac{7}{140} + \frac{9}{140} = \frac{16}{140} = \frac{8}{70} = \boxed{\frac{4}{35}}$

4. $\frac{x+2}{4} + \frac{6x}{x+1} - 3$

$\frac{x+2}{4(x+1)} + \frac{6x}{4(x+1)} - \frac{12x+12}{4(x+1)}$

$\boxed{\frac{-5x+14}{4x+4}}$

5. $\frac{1}{x^2-x-2} + \frac{x}{x^2-7x+10}$

6. $\frac{3}{3-1} \div \frac{9}{9-1}$

$\frac{3}{2} \cdot \frac{8}{9} = \frac{24}{18} = \frac{8}{6} = \boxed{\frac{4}{3}}$

$\frac{1}{(x-2)(x+1)} + \frac{x}{(x-2)(x-5)}$

$\frac{1\ (x-5)}{(x-2)(x+1)(x-5)} + \frac{x\ (x+1)}{(x-2)(x+1)(x-5)} = \frac{x^2+2x-5}{(x-2)(x+1)(x-5)}$

$x-5+x^2+x$

$= x^2+2x-5$

Appendix V

Ethan's Assessment

Rational Numbers and Expressions Study
Assessment

20-5

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

1. $\frac{1+6}{2}+\frac{9}{2+3}-3$

$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$

$53-30=\frac{23}{10}$

$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}$

2. $\frac{x}{x-1}\div\frac{x^2}{x^2-1}$

$\frac{\frac{x}{x-1}}{\frac{x^2}{x^2-1}}$ $\quad x^2 \frac{x}{x-1}\ \frac{x^2-1}{x^2}\ x-1$

$\frac{x^2(x)\ (x^2-1)(x-1)}{x^2(x-1)}$

$x^3(x^3-x^2-x+1)$ $\quad \frac{x^6-x^5-x^4+x^3}{x^2(x-1)}$

3. $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$

$\frac{1}{20}+\frac{3}{35}$

$\frac{7}{140}+\frac{12}{140}=\frac{19}{140}$

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$1(x+1)\ \frac{x+2}{4}+\frac{6x^{(4)(1)}}{x+1}-\frac{3^{(4)(x+1)}}{1}$

$\frac{x^2+3x+2+24x-12x-3}{4+x+1-1}=\frac{x^2+15x-1}{x+4}$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

$\frac{1^{\,x^2-7x+10}}{x^2-x-2}+\frac{x^{\,(x^2-x-2)}}{x^2-7x+10}$

$\frac{x^3-7x+10+x^3-x^2-2x}{x^2-x-2+x^2-7x+10}$

$\frac{x^3-9x+10}{2x^2-8x+8}$

6. $\frac{3}{3-1}\div\frac{9}{9-1}$

$\frac{\frac{3}{2}}{\frac{9}{8}}$ $\quad \frac{3}{2}\,\frac{8}{9}=\frac{24}{18}=\frac{4}{3}$

Appendix W

Liam's Assessment

Rational Numbers and Expressions Study Assessment

23 - 27

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

1. $\frac{1+6}{2}+\frac{9}{2+3}-3$

$\left(\frac{7}{2}\right)+\frac{9}{6}-\left(\frac{3}{1}\right)$

$\frac{21}{6}+\frac{9}{6}-\frac{18}{6}$

$\frac{30}{6}-\frac{18}{6}=\frac{12}{6}=2$

2. $\frac{x}{x-1}\div\frac{x^2}{x^2-1}$

$\frac{x}{x-1}\div\frac{\sqrt{x^2}}{\sqrt{x^2-1}}=\frac{x}{x-1}\div\frac{x}{x-1}=1$

3. $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}$

$\frac{1}{20}+\frac{3}{35}$

$\frac{7}{140}+\frac{12}{140}=\frac{19}{140}$

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{3}{1}$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

6. $\frac{3}{3-1}\div\frac{9}{9-1}$

$\frac{3}{2}\div\frac{9}{8}$

$\frac{12}{8}\times\frac{8}{9}=\frac{12}{9}=\frac{4}{3}$

Appendix X

Mason's Assessment

Rational Numbers and Expressions Study Assessment

7-5

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, <u>show all of your work</u>, and <u>write your answer in simplest terms.</u>

1. $\frac{1+6}{2}+\frac{9}{2+3}-3$

$\frac{7}{2}+\frac{9}{\cancel{2+3}\ 5}-\frac{30}{10}$

$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}$

$\boxed{\frac{23}{10}}$

2. $\frac{x}{x-1}\div\frac{x^2}{x^2-1}$

$\frac{x}{x-1}\cdot\left(\frac{x^2-1}{x^2}\right)$

$\frac{x(x^2-1)}{x^2(x-1)}=\frac{x^3-x}{x^3-x^2}=\frac{\cancel{(x)}(x^2-1)}{\cancel{(x)}(x^2-x)}$

$\boxed{\frac{x^2-1}{x^2-x}}$

3. $\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$

$\frac{1}{20}+\frac{3}{35}$

$\frac{35}{700}+\frac{60}{700}$

$\frac{95}{700}=\boxed{\frac{19}{120}}$

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$\frac{x+2}{4}+\frac{6x}{x+1}-\frac{3}{1}=\frac{(x+2)(x+1)}{4x+4}+\frac{24x}{4x+4}-\frac{3(4x+4)}{4x+4}$

$\frac{(x^2+3x+2)+24x-12x-12}{4x+4}=\boxed{\frac{x^2+15x-10}{4x+4}}$

if $x=1$
$\frac{1}{2}+\frac{1}{4}=-\frac{1}{4}$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

$\frac{(x^2-7x+10)+(x^2-x-2)}{(x^2-x-2)(x^2-7x+10)}$

$x^4-x^3-2x^2-7x^3+7x^2+14x+10x^2-10x-2$

$x^4-8x^3+15x^2+4x-2$

$\frac{2x^2-8x+8}{x^4-8x^3+15x^2+4x-2}$

6. $\frac{3}{3-1}\div\frac{9}{9-1}$

$\frac{3}{2}\cdot\frac{8}{9}$

$\frac{24}{18}=\frac{12}{9}=\boxed{\frac{4}{3}}$

Assessment Y

Noah's Assessment

Rational Numbers and Expressions Study
Assessment

20-2

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

1. $\frac{1+6}{2}+\frac{9}{2+3}-3 = \frac{23}{10}$

$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$

53

4

$\frac{35}{10}+\frac{18}{10}-\frac{30}{10}=\frac{23}{10}$

2. $\frac{x}{x-1}\div\frac{x^2}{x^2-1}$

$\frac{x^3-\cancel{x}}{x-x^2}$

$\frac{x}{x-1}\cdot\frac{x^2-1}{x^2}$

$\frac{x^3-1}{x^2}$

3. $\frac{1}{4\cdot 5}+\frac{3}{5\cdot 7}=\frac{19}{140}$

1

35

2

70

4

140

$\frac{1}{20}+\frac{3}{35}$

$\frac{7}{140}+\frac{12}{140}=\frac{19}{140}$

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$-10x$

$\cancel{-12x}$ $\quad -10$

$x\cancel{+2x+2}$ $+$ $24x \cancel{-12x-12}$

$\frac{x+2(x+1)}{4(x+1)}+\frac{24x}{4(x+1)}-\frac{12(x+1)}{4x+4}$

$\frac{-9x-10}{4(x+1)}$

$(x^2-x-2)(x-5)$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$ $\frac{x^2+2x-5}{x^3-6x^2-3x+10}$

6. $\frac{3}{3-1}\div\frac{9}{9-1}=\frac{12}{9}$

$\frac{3\cdot 8}{2\cdot 9}=\frac{24}{18}=\frac{12}{9}$

$x^3-5x^2-x^2+5x-2x+10$

$x^3-6x^2-3x+10$

$(x-2)(x+1)$

$\frac{1x-5}{(x-2)(x+1)(x-5)}+\frac{x^2+x}{(x-2)(x+1)(x-5)}$

x^2+2x-5

$x^2+x-2x-2$

$(x-2)(x+1)(x-5)$

$(x-5)(x-2)$

$(x\quad)(x\quad)$

Assessment Z

Jacob's Assessment

Rational Numbers and Expressions Study
Assessment

4-11

Please answer each of the six problems below. You may not use a calculator.

Perform the given operation, show all of your work, and write your answer in simplest terms.

fun motor I wrote wasting time.

maybe I should have reversed fractions like my professor said to do.

1. $\frac{1+6}{2}+\frac{9}{2+3}-3$

$\frac{7}{2}+\frac{9}{5}-\frac{3}{1}$

$\frac{5}{10}+\frac{2}{10}-\frac{10}{10}\quad -\frac{3}{10}$

2. $\frac{x}{x-1}\div\frac{x^2}{x^2-1}$

do not remember division of fractions at all.

3. $\frac{1}{4\cdot5}+\frac{3}{5\cdot7}$

$\frac{1}{20}+\frac{3}{35}$

$\frac{1\cdot35-20\cdot3}{20\cdot35}\quad\frac{35-60}{600}$

$\frac{25}{600}\quad\frac{1}{150}$

4. $\frac{x+2}{4}+\frac{6x}{x+1}-3$

$\frac{6x+x+2}{x+5}-\frac{3}{?}$

$\frac{6x+x-3}{x+4}$

5. $\frac{1}{x^2-x-2}+\frac{x}{x^2-7x+10}$

$1\cdot x$

$x^2-7x+10\cdot x^3-x^2-2x$

$\frac{x^3-2x^2-5x}{x^4-8x^3-15x^2+4x-20}$

$x^2-x-2(x^2-7x+10)$

$x^4-7x^3+10x^2-x^3+7x^2-10x-2x^2+14x-20$

$-8x^3\quad 17x^3\quad 15x^2$

6. $\frac{3}{321}\div\frac{9}{981}$

$\frac{18}{4}\div\frac{24}{4}$