Tail Conditional Expectations for

Extended Exponential Dispersion Models

A Thesis

Presented to the Faculty of the Department of Mathematical Sciences

Middle Tennessee State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematical Sciences

by

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August 2014

ACKNOWLEDGMENTS

I would like to express my special thanks of gratitude to Dr. Qiang Wu, who offered a lot of patience and guidance to help me in this thesis. Also, I want to thank Dr. Don Hong, who gave me the good opportunity to do this research and Dr. Rebecca Calahan, who is willing to be my defense committee member. I have learned so many new things about research.

Secondly I would also like to thank my parents and friends who helped me a lot in finishing this thesis within the limited time.

ABSTRACT

For a loss that can be incurred in a given period, the tail conditional expectation, also termed as tail value-at-risk, is the conditional average amount of loss, given that the loss exceeds a specified value. This measurement helps insurance companies to determine the amounts of capital to pay out claims resulted from catastrophic event when premium revenues are insufficient. In this paper, we extend the exponential dispersion models and derive tail conditional expectation forms of the extended models.

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CHAPTER 1

INTRODUCTION

1.1 Tail Conditional Expectation

Insurance companies often set aside amounts of capital from which they can draw in the event that premium revenues become insufficient to pay out claims. Determining these amounts needed is not an easy process. We need to be able to determine the probability distribution of the losses that it is facing. More importantly, we should determine the best risk measurement [Artzner,Delbaen,Eber and Heath 1999] to find the amount of loss to cover the claims. Tail conditional expectation (TCE) has become increasingly popular for measuring this kind of risk, especially the adequacy of existing capital and the possibility of financial ruin [Society of Actuaries, Klugman,Panjer,and Willmot 2008, Bowers,Gerber,Hickman,Jones and Nesbitt 1997]. It is one of the newest risk modeling techniques adopted by the insurance industry.

Assume for the moment that an insurance company faces the risk of losing an amount X for some fixed period of time. This generally refers to the total claims for the insurance company. We denote its distribution function by $F_X(x) = \Pr(X \leq x)$, the probability of the event $\{X \leq x\}$, and its tail function by $S_X(x) = \Pr(X > x)$. The function $S_X(x)$ is also called survival function in probability literature. Note that although the setting applies to insurance companies, it is equally applicable for any institution confronted with any risky business. It may even refer to the loss faced by an investment portfolio. The *tail conditional expectation* of X is defined as

$$TCE_X(x) = E(X \mid X > x). \tag{1}$$

It gives an average amount of the tail of the distribution. We can interpret this risk measure as the mean of worst losses beyond certain level. The formula used to evaluate TCE is

$$TCE_X(x) = \frac{1}{S_X(x)} \int_x^\infty x dF_X(x), \tag{2}$$

provided that $S_X(x) > 0$, where the integral is the Lebesgue-Stieltjes integral.

The value of x is usually set based on the potential value of the loss x_q of the distribution with the property

$$S_X(x_q) = 1 - q,$$

where $0 < q < 1$. With a small value of q, the value x_q is considered to be the amount of loss that is unlikely to happen in normal situation while it may ruin the business once it happens. It is important for a company to monitor and prepare for such an extreme situation. TCE provides a good measure for this purpose.

1.2 Exponential Dispersion Models

The class of exponential dispersion models has served as “error distributions” for generalized linear models in the sense developed by [Nelder,Wedderburn 1972]. This includes many well known discrete distributions like Poisson and Binomial as well as continuous distributions like Normal, Gamma and Inverse Gaussian. It is not surprising to find that they are becoming popular to actuaries. For example, credibility formulas for the class of exponential dispersion models preserve the property of a predictive mean [Kaas,Danneburg and Goovaerts 1997, Nelder and Verrall 1997, Landsman and Makov 1998, Landsman 2002]. A thorough and systematic investigation of exponential dispersion models was done by Jorgensen [Jorgensen 1986, Jorgensen 1987, Jorgensen 1997].

A random variable X is said to belong to the Exponential Dispersion Family (EDF) of distributions if its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q_λ and can be represented as follows: for some function $\kappa(\theta)$ called the cumulant:

$$dP_{\theta,\lambda} = e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x). \tag{3}$$

The parameter θ is named the canonical parameter belonging to the set

$$\Theta = \{\theta \in \mathbb{R} \mid \kappa(\theta) < \infty\}.$$

The parameter λ is called the index parameter belonging to the set of positive real numbers $\Lambda = \{\lambda \mid \lambda > 0\} = \mathbb{R}_+$. The representation in (3) is called the *reproductive* form of EDF and is denoted by $X \sim ED(\theta, \lambda)$ for a random variable belonging to this family. Another form of EDF is called the *additive* form which can be obtained by the transformation $Y = \lambda X$. Its probability measure $P^*_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q^*_λ which can be represented as

$$dP^*_{\theta,\lambda} = e^{[\theta y - \lambda\kappa(\theta)]} dQ^*_\lambda(y). \tag{4}$$

If the measure Q_λ in (3) is absolutely continuous with respect to a Lebesgue measure, then the density of X has the form

$$f_X(x) = e^{\lambda[\theta x - \kappa(\theta)]} q_\lambda(x). \tag{5}$$

The same can be said about additive model, $ED^*(\theta, \lambda)$, and Y has the density

$$f_Y(y) = e^{[\theta y - \lambda\kappa(\theta)]} q^*_\lambda(y). \tag{6}$$

Consider the loss random variable X belonging to the family of exponential dispersion models in reproductive or additive form. We will denote the tail probability function by $S(\cdot \mid \theta, \lambda)$. This simplifies the notation by dropping the subscript when no confusion can happen and emphasizes its dependence on the parameters θ and λ.

In [Landsman and Valdez 2005], the TCE is given for both reproductive form and additive form of exponential dispersion models. Suppose that the random variable X belongs to EDF. If the survival function has partial derivative with respect to θ, $\kappa(\theta)$ is a differentiable function, and one can differentiate the survival function $S(\cdot|\lambda, \theta)$ in θ under the integral sign, then

- For $X \sim ED(\mu, \lambda)$, the reproductive form of EDF,

$$TCE_X(x) = \mu + \frac{h}{\lambda}, \tag{7}$$

  where

$$h = \frac{\partial}{\partial\theta} \log S(x \mid \theta, \lambda).$$

- For $X \sim ED^*(\mu, \lambda)$, the additive form of EDF,

$$TCE_X(x) = \mu + h. \tag{8}$$

In this thesis, we will extend the formulas of exponential dispersion models and investigate their properties. The first generalization we would consider is to replace the multiplier θ in EDF by a function of θ. Such a family of models will be referred to Type I generalized exponential dispersion family (GEDF). This is motivated by the linear exponential family (LEF) [Brown 1987] which takes the form

$$f(x \mid \theta) = \frac{e^{r(\theta)x} q(x)}{p(\theta)}. \tag{9}$$

With this generalization, the GEDF includes LEF as a special case.

The second generalization is to replace the term $\lambda\kappa(\theta)$ by a general bivariate function of λ and θ. Such a family of distributions is called Type II generalized exponential dispersion family. Note that

$$\lambda\kappa(\theta) = \log\left(\int_{\mathbb{R}} e^{\lambda\theta x} dQ_\lambda(x)\right) \tag{10}$$

for the reproductive form of EDF and

$$\lambda\kappa(\theta) = \log\left(\int_{\mathbb{R}} e^{\theta y} dQ^*_\lambda(y)\right) \tag{11}$$

for the additive form of EDF. For general distributions Q_λ or Q^*_λ, the functions on the right of (10) or (11) should be a general bivariate function, not necessarily able to be written in the form of $\lambda\kappa(\theta)$. Therefore, in EDF the choice of the distributions Q_λ or Q^*_λ are restricted. The Type II generalization removes the restriction and allows to consider more general distributions.

In Chapter 2 we will study the Type I generalized exponential dispersion family, investigate its properties, and derive the formula for its tail conditional expectation. In Chapter 3, we will study the Type II generalized exponential dispersion family. Finally, we close in Chapter 4 with conclusions and discussions.

CHAPTER 2

TYPE I GENERALIZED EXPONENTIAL DISPERSION FAMILY

We consider the Type I generalization of the reproductive form of the exponential dispersion model which takes the form

$$dP_{\theta,\lambda} = e^{\lambda[r(\theta)x-\kappa(\theta)]}dQ_\lambda(x). \tag{12}$$

If the measure Q_λ is absolutely continuous with respect to the Lebesgue measure, then the density of X has the form

$$f_X(x) = e^{\lambda[r(\theta)x-\kappa(\theta)]}q_\lambda(x). \tag{13}$$

2.1 Mean and Variance of Type I GEDF

For the reproductive form of Type I GEDF, we can deduce the formulae for its mean and variance.

Theorem 2.1. *Suppose that a random variable X belongs to the Type I GEDF whose reproductive form is given by* (12)*. If its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to the measure Q_λ, both $r(\theta)$ and $\kappa(\theta)$ have the second derivatives and $r(\theta)$ is invertible, then the mean value of X is*

$$\mu = \mu(\theta) = \frac{\kappa'(\theta)}{r'(\theta)} \tag{14}$$

and the variance of X is

$$\mathbf{Var}(X) = \frac{\mu'(\theta)}{\lambda r'(\theta)}. \tag{15}$$

Proof. We compute the cumulant generating function as follows:

$$\begin{aligned} K_X(t) = \log E(e^{Xt}) &= \log \left\{ \int_{\mathbb{R}} e^{xt} e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \right\} \\ &= \log \left\{ \int_{\mathbb{R}} e^{xt + \lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \right\} \\ &= \log \left\{ \int_{\mathbb{R}} e^{\lambda[(r(\theta)+t/\lambda)x] - \kappa(\theta)} dQ_\lambda(x) \right\}. \end{aligned}$$

Let ξ be the number such that $r(\theta) + t/\lambda = r(\xi)$. Since $r(\theta)$ is invertible, we have $\xi = r^{-1}(r(\theta) + t/\lambda)$. Then

$$\begin{aligned} K_X(t) &= \log \left\{ e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]} \int_{\mathbb{R}} e^{\lambda\left[(r(\theta)+t/\lambda)x - \kappa(r^{-1}(r(\theta)+t/\lambda))\right]} dQ_\lambda(x) \right\} \\ &= \log \left\{ e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]} \int_{\mathbb{R}} e^{\lambda[r(\xi)x - \kappa(\xi)]} dQ_\lambda(x) \right\} \\ &= \log \left\{ e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]} \right\} \\ &= \lambda \left[\kappa(r^{-1}(r(\theta) + t/\lambda)) - \kappa(\theta) \right]. \end{aligned}$$

It follows that its moment generating function is

$$M_X(t) = E(e^{Xt}) = e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]}. \tag{16}$$

Knowing that $(f^{-1}(x))' = 1/f'(f^{-1}(x)))$ for an invertible differentiable function, we can find the first order derivative of the moment generating function:

$$\begin{aligned} M_X'(t) &= e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]} \lambda \kappa' \left(r^{-1}(r(\theta) + t/\lambda)\right) \left(r^{-1}(r(\theta) + t/\lambda)\right)' \\ &= e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda)) - \kappa(\theta)\right]} \frac{\kappa' \left(r^{-1}(r(\theta) + t/\lambda)\right)}{r' \left(r^{-1}(r(\theta) + t/\lambda)\right)}. \end{aligned}$$

When $t = 0$, we obtain

$$\mu = M_X'(0) = e^{\lambda\left[\kappa(r^{-1}(r(\theta)))-\kappa(\theta)\right]} \frac{\kappa'\left(r^{-1}(r(\theta))\right)}{r'\left(r^{-1}(r(\theta))\right)} = \frac{\kappa'(\theta)}{r'(\theta)}.$$

To derive the formula for the variance, we compute the second order derivative of the moment generating function:

$$\begin{aligned}
M_X''(t) &= (M_X'(t))' \\
&= \left\{e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]}\right\}' \frac{\kappa'\left(r^{-1}(r(\theta)+t/\lambda)\right)}{r'\left(r^{-1}(r(\theta)+t/\lambda)\right)} \\
&\quad + e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \frac{\kappa''\left(r^{-1}(r(\theta)+t/\lambda)\right)}{r'\left(r^{-1}(r(\theta)+t/\lambda)\right)} \\
&\quad + e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \kappa\left(r^{-1}(r(\theta)+t/\lambda)\right) \left[r'\left(r^{-1}(r(\theta)+t/\lambda)\right)^{-1}\right]' \\
&= e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \left[\frac{\kappa'\left(r^{-1}(r(\theta)+t/\lambda)\right)}{r'\left(r^{-1}(r(\theta)+t/\lambda)\right)}\right]^2 \\
&\quad + e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \frac{\kappa''\left(r^{-1}(r(\theta)+t/\lambda)\right)}{\lambda\left[r'\left(r^{-1}(r(\theta)+t/\lambda)\right)\right]^2} \\
&\quad + e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda))-\kappa(\theta)\right]} \kappa'\left(r^{-1}(r(\theta)+t/\lambda)\right) \\
&\qquad \times \frac{-r''\left(r^{-1}(r(\theta)+t/\lambda)\right)}{\left[r'\left(r^{-1}(r(\theta)+t/\lambda)\right)\right]^2} (r^{-1}(r(\theta)+t/\lambda))' \\
&= e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \left[\frac{\kappa'\left(r^{-1}(r(\theta)+t/\lambda)\right)}{r'(r^{-1}(r(\theta)+t/\lambda))}\right]^2 \\
&\quad + e^{\lambda\left[\kappa\left(r^{-1}(r(\theta)+t/\lambda)\right)-\kappa(\theta)\right]} \frac{\kappa''\left(r^{-1}(r(\theta)+t/\lambda)\right)}{\lambda\left[r'\left(r^{-1}(r(\theta)+t/\lambda)\right)\right]^2} \\
&\quad + e^{\lambda\left[\kappa(r^{-1}(r(\theta)+t/\lambda))\right]} \kappa'\left(r^{-1}(r(\theta)+t/\lambda)\right) \frac{-r''(r^{-1}(r(\theta)+t/\lambda))}{\lambda\left[r'\left(r^{-1}(r(\theta)+t/\lambda)\right)\right]^3}.
\end{aligned}$$

When $t = 0$, we have

$$E(X^2) = M''_X(0) = \left[\frac{\kappa'(\theta)}{r'(\theta)}\right]^2 + \frac{\kappa''(\theta)}{\lambda[r'(\theta)]^2} - \frac{\kappa'(\theta)r''(\theta)}{\lambda[r'(\theta)]^3}.$$

Then we can get the variance as

$$\begin{aligned}
\mathbf{Var}(X) &= E(X^2) - \mu^2 \\
&= \frac{\kappa''(\theta)}{\lambda[r'(\theta)]^2} - \frac{\kappa'(\theta)r''(\theta)}{\lambda[r'(\theta)]^3} \\
&= \frac{1}{\lambda[r'(\theta)]^2}\left[\kappa''(\theta) - \frac{\kappa'(\theta)r''(\theta)}{r'(\theta)}\right] \\
&= \frac{1}{\lambda\,[r'(\theta)]}\left(\frac{\kappa'(\theta)}{r'(\theta)}\right)' \\
&= \frac{\mu'(\theta)}{\lambda r'(\theta)}.
\end{aligned}$$

where we have used the fact $\mu = \mu(\theta) = \frac{\kappa'(\theta)}{r'(\theta)}$ obtained above. This complete the proof of the theorem. □

When $\lambda = 1$ and $\kappa(\theta) = \log(p(\theta))$, the Type I GEDF reduces to the LEF (9). Theorem 2.1 gives

$$\mu = \mu(\theta) = \frac{\kappa'(\theta)}{r'(\theta)} = \frac{p'(\theta)}{r'(\theta)p'(\theta)}.$$

and $\mathbf{Var}(X) = \frac{\mu'(\theta)}{r'(\theta)}$. These coincide the results for linear exponential family in [Klugman,Panjer,and Willmot 2008].

2.2 TCE of Type I GEDF

Theorem 2.2. *Under the assumptions of Theorem 2.1, if one can also differentiate the tail function $S(\cdot|\theta, \lambda)$ in θ under the integral sign, then the tail conditional* *expectation is given by*

$$TCE_X(x) = \mu + \frac{h}{\lambda r'(\theta)}, \tag{17}$$

where

$$h = \frac{\partial}{\partial \theta} \log S(x \mid \theta, \lambda).$$

Proof. Recall the tail function is

$$S(x \mid \theta, \lambda) = \int_x^\infty e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x).$$

We have

$$\begin{aligned} h = \frac{\partial}{\partial \theta} \log S(x \mid \theta, \lambda) &= \frac{1}{S(x \mid \theta, \lambda)} \int_x^\infty \frac{\partial}{\partial \theta} \left\{ e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \right\} \\ &= \frac{1}{S(x \mid \theta, \lambda)} \int_x^\infty \lambda \left[r'(\theta)x - \kappa'(\theta) \right] e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \\ &= \frac{\lambda}{S(x \mid \theta, \lambda)} \left\{ \int_x^\infty r'(\theta) x e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \right. \\ &\qquad \left. - \int_x^\infty \kappa'(\theta) e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x) \right\} \\ &= \frac{\lambda}{S(x \mid \theta, \lambda)} \left[r'(\theta) \int_x^\infty x dP_{\theta,\lambda} - \kappa'(\theta) S(x \mid \theta, \lambda) \right] \\ &= \lambda \left[r'(\theta) TCE_X(x) - \kappa'(\theta) \right], \end{aligned}$$

with $dP_{\theta,\lambda} = e^{\lambda[r(\theta)x - \kappa(\theta)]} dQ_\lambda(x)$. This gives

$$TCE_X(x) = \frac{\kappa'(\theta)}{r'(\theta)} + \frac{h}{\lambda r'(\theta)}.$$

The conclusion follows by noticing that $\mu = \frac{\kappa'(\theta)}{r'(\theta)}$. □

CHAPTER 3

TYPE II GENERALIZED EXPONENTIAL DISPERSION FAMILY

In this chapter we consider the Type II generalization of EDF. This time we focus on the additive form. The additive form of Type II GEDF is given by

$$dP^*_{\theta,\lambda} = e^{[\theta y - \kappa(\lambda,\theta)]} dQ^*_\lambda(y) \tag{18}$$

or

$$f_Y(y) = e^{[\theta y - \kappa(\lambda,\theta)]} q^*_\lambda(y) \tag{19}$$

3.1 Mean and Variance of Type II GEDF

Theorem 3.1. *Suppose that a random variable Y belongs to the Type II GEDF whose distribution is given by* (18)*. If its probability measure $P^*_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q^*_λ and $\kappa(\lambda,\theta)$ has the second partial derivative to θ, then the mean value of Y is*

$$\mu = \frac{\partial \kappa(\lambda,\theta)}{\partial \theta}, \tag{20}$$

and the variance of Y is

$$\mathbf{Var}(Y) = \frac{\partial^2 \kappa(\lambda,\theta)}{\partial \theta^2}. \tag{21}$$

Proof. The generating function can be derived as follows:

$$\begin{aligned}
K_Y(t) = \log E(e^{Yt}) &= \log\left\{\int_{\mathbb{R}} e^{yt} e^{\theta y - \kappa(\lambda,\theta)} dQ^*_\lambda(y)\right\} \\
&= \log\left\{\int_{\mathbb{R}} e^{(\theta+t)y - \kappa(\lambda,\theta)} dQ^*_\lambda(y)\right\} \\
&= \log\left\{e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)} \int_{\mathbb{R}} e^{(\theta+t)y - \kappa(\lambda,\theta+t)} dQ^*_\lambda(y)\right\}
\end{aligned}$$

$$= \kappa(\lambda, \theta + t) - \kappa(\lambda, \theta).$$

Then the moment generating function is $M_Y(t) = e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)}$. Thus,

$$M'_Y(t) = e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)} \frac{\partial \kappa(\lambda, \theta + t)}{\partial t} = e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)} \frac{\partial \kappa(\lambda, \theta + t)}{\partial \theta}.$$

When $t = 0$, we have the mean value

$$\mu = M'_Y(0) = \frac{\partial \kappa(\lambda, \theta)}{\partial \theta}.$$

The second order derivative of $M_X(t)$ is

$$M''_Y(t) \quad = e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)} \left(\frac{\partial \kappa(\lambda, \theta + t)}{\partial \theta} \right)^2 + e^{\kappa(\lambda,\theta+t)-\kappa(\lambda,\theta)} \frac{\partial^2 \kappa(\lambda, \theta + t)}{\partial \theta^2}.$$

When $t = 0$, we have

$$E(Y^2) = M''_Y(0) = \left(\frac{\partial \kappa(\lambda, \theta)}{\partial \theta} \right)^2 + \frac{\partial^2 \kappa(\lambda, \theta)}{\partial \theta^2}.$$

Then we get the variance of extended addtive model

$$\mathbf{Var}(Y) = E(Y^2) - \mu^2 = \frac{\partial^2 \kappa(\lambda, \theta)}{\partial \theta^2}.$$

This completes the proof. □

3.2 TCE of Type II GEDF

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, if one can differentiate $S(\cdot|\theta, \lambda)$ in θ under the integral sign, then*

$$TCE_Y(y) = \mu + h. \tag{22}$$

Proof. The proof is similar to that of Theorem 2.2. The tail function is

$$S(y \mid \theta, \lambda) = \int_y^\infty e^{\theta y - \kappa(\lambda,\theta)} dQ_\lambda^*(y).$$

Then

$$\begin{aligned}
h = \frac{\partial}{\partial\theta} \log S(y \mid \theta, \lambda) &= \frac{1}{S(y \mid \theta, \lambda)} \int_y^\infty \frac{\partial}{\partial\theta} \left\{ e^{\theta y - \kappa(\lambda,\theta)]} dQ_\lambda^*(y) \right\} \\
&= \frac{1}{S(y \mid \theta, \lambda)} \left[\int_y^\infty y e^{\theta y - \kappa(\lambda,\theta)} dQ_\lambda^*(y) \right. \\
&\qquad \left. - \frac{\partial \kappa(\lambda,\theta)}{\partial\theta} \int_y^\infty e^{\theta y - \kappa(\lambda,\theta)} dQ_\lambda^*(y) \right] \\
&= \frac{1}{S(y \mid \theta, \lambda)} \left[\int_y^\infty y dP_{\theta,\lambda}^* - \frac{\partial \kappa(\lambda,\theta)}{\partial\theta} \int_y^\infty e^{\theta y - \kappa(\lambda,\theta)]} dQ_\lambda^*(y) \right] \\
&= TCE_Y(y) - \frac{\partial \kappa(\lambda,\theta)}{\partial\theta}.
\end{aligned}$$

This together with (20) proves the conclusion. □

CHAPTER 4

CONCLUSION AND DISCUSSION

This thesis examines tail conditional expectations for loss random variables that belong to the class of generalized exponential dispersion models. For the exponential dispersion models in [Landsman and Valdez 2005], it has both reproductive form and additive form. We extended the exponential dispersion families based on these two forms. Two types of generalization are considered. The representations for their means and variances are derived. The tail conditional expectations are characterized. By comparing our results with those in [Landsman and Valdez 2005] we see they share great similarity.

In Chapter 2 and Chapter 3 we have considered the two types of generalization separately. One can also consider them simultaneously to derive even more general and complicated distributions. In this case the reproductive form of the distribution is

$$dP_{\theta,\lambda} = e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} dQ_\lambda(x),$$

and the additive form is

$$dP^*_{\theta,\lambda} = e^{r(\theta)y - \kappa(\lambda,\theta)} dQ^*_\lambda(y).$$

Similar results can be derived using the same techniques and more complicated computations.

- For the reproductive form, there are

$$\mu = \mu(\theta) = \frac{\partial_2 \kappa(\lambda,\theta)}{\lambda r'(\theta)},\ \mathbf{Var}(X) = \frac{\mu'(\theta)}{\lambda r'(\theta)},\ \text{and}\ TCE_X(x) = \mu + \frac{h}{\lambda r'(\theta)}.$$

- For additive form, there are

$$\mu = \mu(\theta) = \frac{\partial_2 \kappa(\lambda,\theta)}{r'(\theta)},\ \mathbf{Var}(Y) = \frac{\mu'(\theta)}{r'(\theta)},\ \text{and}\ TCE_Y(y) = \mu + \frac{h}{r'(\theta)}.$$

Finally, we remark that the EDF contains a couple of special distributions that are commonly seen in probability theory and used in loss modeling. These special distributions are clearly also examples of generalized exponential dispersion models. Although it seems that no commonly seen special distributions belong to GEDF but not to EDF, it is not difficult to theoretically construct such distributions. The generalization allows more freedom to model loss variables which may be beneficial when the losses cannot be well modeled using commonly seen special distributions.

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