A Duality between hypergraphs and cone lattices

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by

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I would like to dedicate this work to Minerva, without whose favor none of this would be possible. May I continue to garner her favor in my future endeavors.

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ABSTRACT

In this paper, we introduce and characterize the class of lattices that arise as the family of lowersets of the incidence poset for a hypergraph. In particular, we show that the following statements are logically equivalent:

1. A lattice $\mathcal{L}$ is order isomorphic to the frame of opens for a hypergraph endowed with the Classical topology.
2. A lattice $\mathcal{L}$ is bialgebraic, distributive, and its subposet of completely join-prime elements forms the incidence poset for a hypergraph.
3. A lattice $\mathcal{L}$ is a cone lattice.

We conclude the paper by extending a well-known Stone-type duality to the categories of hypergraphs coupled with finite-based HP-morphisms and cone lattices coupled with frame homomorphisms that preserve compact elements.

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CHAPTER 1

PARTIALLY ORDERED SETS AND LATTICES

In this and the succeeding chapter, our intention is to establish the basic tenants of order- and lattice-theoretic notions that are crucial in establishing the results in chapters 4 and 5. The exhibition of this material is routine, and the interested reader can find a broader introduction to this material in several sources. For instance, similar exhibitions can be found in Davey and Priestley [9], Gratzer [13], and Birkhoff [3]. While these resources were used in developing the theory used throughout, the proofs provided are the author's own unless specifically stated otherwise. A reader interested in a modified Moore-method approach to this material is encouraged to visit the Journal of Inquiry Based Learning in Mathematics, No. 44 for an appropriate set of notes.

In this section, we will introduce some definitions and concepts about posets that will play a key role in the development of the subject in subsequent sections. It stands to reason that we should begin with a few definitions; and that is primarily what this section consists of.

A *partially ordered set* (or poset for short) is a system $\mathbf{P} = (P, \leq)$ consisting of a set P and a binary relation $\leq$ on the set P satisfying the following conditions:

1. For all $x \in P$, we have $x \leq x$ (*reflexivity*).

2. If $x \leq y$ and $y \leq x$, then $x = y$ (*antisymmetry*).

3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (*transitivity*).

The binary relation $\leq$ defined above is called a *partial ordering* on the set P. Note that we are using a convention customary with binary relations — we write $x \leq y$ to mean (x, y) is a member of $\leq$.

The set $\mathtt{Su}(X)$ of all subsets of a set X is partially ordered by set inclusion. That is, the binary relation $\subseteq$ defined by

$$A \subseteq B \iff a \in B \text{ for all } a \in A$$

is a partial ordering on $\mathtt{Su}(X)$. The relation is *partial* in the sense that not all members of $\mathtt{Su}(X)$ are related under subset inclusion. For example, if $X = \{x, y, z\}$, then the subsets $A = \{x, y\}$ and $B = \{y, z\}$ are not related by subset inclusion.

Two elements x, y of a poset $\mathbf{P} = (P, \leq)$ are said to be *comparable* provided $x \leq y$ or $y \leq x$. We sometimes say that x is *below* y (or that y is *above* x). If this is not the case, we say that x and y are *incomparable* and write $x \parallel y$. (Note: The notation for incomparability is not universally used.)

In a poset $\mathbf{P}$ we write $x < y$ to mean that $x \leq y$ but $y \not\leq x$. In this case, we say the inequality is *strict*. It is acceptable to write $x \geq y$ when $y \leq x$, though we will not often have need of this convention.

A poset $\mathbf{P}$ is said to be a *chain* (or *totally ordered*) provided every element is comparable to every other element in P; that is, for all $x, y \in P$, we have $x \leq y$ or $y \leq x$. The positive integers under their natural ordering is an example of a chain. At the other extreme, we say a poset $\mathbf{P}$ is an *antichain* provided $x \leq y$ only when $x = y$. Note that the empty set and singleton sets are the only sets which are both a chain and an antichain under any partial ordering.

Definition 1.1. Let $\mathbf{P} = (P, \leq)$ be any poset. The *order dual* of $\mathbf{P}$ is defined to be the system $\mathbf{P}^{\mathrm{op}} = (P, \leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y \iff y \leq x$. We usually denote the order dual of a poset P by simply writing P^{op}.

Given any "statement" Φ about a poset $\mathbf{P}$, we can obtain its "dual" simply by replacing every occurrence of $\leq$ in the statement with $\geq$. This simple fact gives rise to an important feature of order theory known as the *duality principle*:

A statement Φ is true of all posets if and only if its dual is also true of all posets.

This simple observation will often be used to shorten proofs, particularly when the conjecture to be proved consists of two parts, one part the dual of the other. In such cases, we will prove one part and state that the other "follows by duality".

Any subset Q of a poset $\mathbf{P}$ may be regarded as a poset in its own right under the restriction to Q of the partial ordering on $\mathbf{P}$. When viewed in this manner, we say the subset Q is a *subposet* of $\mathbf{P}$. There are two particularly important examples of subposets we will be using:

Definition 1.2. Let $\mathbf{P} = (P, \leq)$ be a poset and let $L \subseteq P$. We say that L is a *lowerset* (or *order ideal*) of $\mathbf{P}$ provided, whenever $x \in P$ is such that $x \leq y$ for some $y \in L$, then $x \in L$. An *upperset* (or *order filter*) of $\mathbf{P}$ is defined to be a lowerset of $\mathbf{P}^{\mathrm{op}}$. We let $\mathtt{Low}(\mathbf{P})$ denote the set of all lowersets for $\mathbf{P}$, partially ordered by set inclusion, and let $\mathcal{U}(\mathbf{P})$ denote the set of all uppersets of $\mathbf{P}$ partially ordered by *reverse* set inclusion (that is, $A \leq B$ in $\mathcal{U}(\mathbf{P})$ if and only if $B \subseteq A$).

Definition 1.3. Let $\mathbf{P} = (P, \leq)$ be a poset and let $X \subseteq P$. The set

$$\downarrow X = \{p \in P : p \leq x \textit{ for some } x \in X\}$$

is called the *lower set generated by* X in $\mathbf{P}$. Likewise, the set

$$\uparrow X = \{p \in P : x \leq p \textit{ for some } x \in X\}$$

is called the *upperset generated by* X in $\mathbf{P}$.

A lowerset generated by a singleton is called a *principal* lowerset; it is often denoted by $\downarrow x$ instead of $\downarrow\{x\}$.

Definition 1.4. Let $\mathbf{P} = (P, \leq)$ be a poset. We say that $x \in P$ is *minimal* in $\mathbf{P}$ provided $\downarrow x = \{x\}$. A *maximal* element in $\mathbf{P}$ is a minimal element in $\mathbf{P}^{\text{op}}$.

Definition 1.5. Let $\mathbf{P} = (P, \leq)$ be a poset. We say $\mathbf{P}$ has a *least element* provided $\mathbf{P}$ has exactly one minimal element. We say that $\mathbf{P}$ has a *greatest element* provided $\mathbf{P}^{\text{op}}$ has a least element. We use $\perp$ and $\top$ to denote the least and greatest elements, respectively, of $\mathbf{P}$ (when they exist).

A poset which has a least element is said to be *lower-bounded*. A poset which has a greatest element is said to be *upper-bounded*. A *bounded* poset has both a least and a greatest element.

Definition 1.6. Let $\mathbf{P} = (P, \leq)$ be a poset and let $X \subseteq P$. We say that X is *bounded below* (or has a *lower bound* in $\mathbf{P}$ provided there exist $y \in P$ such that $y \in \downarrow x$ for all $x \in X$. We say that X is *upper-bounded* in $\mathbf{P}$ provided it is lower-bounded in $\mathbf{P}^{\text{op}}$. We let $m(X)$ and $j(X)$ denote the set of all lower-bounds and upper-bounds, respectively, for X.

1.1 Some Important Classes of Posets

In this section, we introduce a few of the fundamental classes of posets that will play a role in all of the work to follow.

Definition 1.7. Let $\mathbf{P}$ be a poset and let $X \subseteq P$. We say that X has an *infimum* (or greatest lower-bound) in $\mathbf{P}$ provided $m(X)$ has a greatest element. This element is known as the *meet* of X in $\mathbf{P}$ and is denoted by $\bigwedge X$. Likewise, we say that X has a *supremum* (or least upper bound) in $\mathbf{P}$ provided $j(X)$ has a least element. This element is known as the *join* of X in $\mathbf{P}$ and is denoted by $\bigvee X$.

When $X = \{x_1, ..., x_n\}$ has a meet in a poset $\mathbf{P}$, we often denote it by

$$\bigwedge X = x_1 \wedge ... \wedge x_n$$

and likewise denote the join of X in P by

$$\bigvee X = x_1 \vee ... \vee x_n$$

Note the use of the logic operations of conjunction and disjunction to denote the finite meets and joins.

Proposition 1.8. *If* $\mathbf{P} = (P, \leq)$ *is any poset, then* $\mathbf{P}$ *is lower-bounded if and only if* $\bigvee \emptyset$ *exists in* $\mathbf{P}$.

Proof. If we suppose $\mathbf{P}$ is lower bounded, there exists an element $\perp \in P$ such that $\perp \leq p$ for every $p \in P$. We observe that $\perp$ is an upper bound for $\emptyset$, and is therefore the least such element, hence $\bigvee \emptyset = \perp$. On the other hand, if the equation holds, we observe that such an element must be the *least* such element of $\mathbf{P}$ and therefore serves as a lower bound. □

Definition 1.9. A poset $\mathbf{J}$ is called a *join semilattice* provided every pair of elements in J has a join in $\mathbf{J}$. We say that a poset $\mathbf{P}$ is a *meet semilattice* provided $\mathbf{P}^{\mathrm{op}}$ is a join semilattice.

The notion of a join-semilattice can easily be extended to closure under finite joins using mathematical induction.

Definition 1.10. A poset $\mathbf{L}$ is said to be a *lattice* provided it is both a join and a meet semilattice.

An *identity* for a lattice is a particular equation which holds true for all elements in a given lattice. We will look at several identities, some enjoyed by all lattices;

other enjoyed only by certain lattices. These identities will prove useful in much of the work to follow.

Proposition 1.11. *Let L be a lattice, and let $x, y, z \in L$. The following identities hold.*

1. $x \vee y = y \vee x$
2. $x \wedge y = y \wedge x$
3. $(x \vee y) \vee z = x \vee (y \vee z)$
4. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
5. $x \vee x = x$
6. $x \wedge x = x$
7. $x \vee (x \wedge y) = x$
8. $x \wedge (x \vee y) = x$

Proof. We will prove claims 1,3,5, and 7. Claims 2,4,6, and 8 follow by duality.

1. One consideration we make is

$$x \vee y = \bigvee\{x, y\} = \bigvee\{y, x\} = y \vee x$$

To better illustrate, we demonstrate the uniqueness of the supremum induced by the antisymmetry of $\leq$. We observe that if $x \leq y$, then $x \vee y = y \vee x = y$. Similarly, if $y \leq x$, then $x \vee y = y \vee x = x$. If $x \mid\mid y$, then there exist $z_1, z_2 \in L$ such that $x \vee y = z_1$ and $y \vee x = z_2$. Since L is a lattice, we know there exists $w = z_1 \wedge z_2$. If w, x, y are each distinct elements of L, we must have $x, y < w$ since w is the greatest lower bound for z_1 and z_2. But this contradicts the claim that z_1 and z_2 are least upper bounds. If $w = y$, then y is the greatest lower bound for z_1 and z_2. This implies $x \leq y$, contrary to assumption. A similar contradiction occurs if $w = x$, so we must conclude $z_1 = z_2$.

3. It is sufficient to observe that $x \vee (y \vee z) = \bigvee\{x, y, z\} = (x \vee y) \vee z$.

5. We observe that x is certainly an upper bound for itself. Indeed, if x were not the *least* upper bound when compared to itself, then there must exist some $x_0 < x$ such that x_0 is an upper bound for x. This notion is absurd, so we conclude that the inequality holds. It also suffices to note that since $\leq$ is antisymmetric, it is reflexive, hence $x \vee x = x$.

7. First, we recognize that $x \leq (x \vee z)$ and $(x \wedge z) \leq x$ for any $x, z \in L$. Suppose $z = x \wedge y$. Then
$$x \leq x \vee z = x \vee (x \wedge y) \leq x \vee x = x$$
We may therefore conclude claim 7 holds.

$\square$

Viewed as binary operations on **L**, Claims 1.11 (1) - (4) tell us that $\wedge$ and $\vee$ are commutative and associative. Claim 1.11 (5)-(6) tell us that $\wedge$ and $\vee$ are *idempotent*. Identities (7) and (8) in Claim 1.11 are called the *absorption laws*.

We can think of meets and joins in a lattice **L** as defining binary operations on the underlying set:

1. Define $\vee : L \times L \longrightarrow L$ by $\vee(x, y) = x \vee y$

2. Define $\wedge : L \times L \longrightarrow L$ by $\wedge(x, y) = x \wedge y$

Let L be a set equipped with binary operations m and j which are commutative, associative, idempotent, and satisfy the absorption laws. Define binary relations on L as follows:

1. $x \leq y \iff m(x, y) = x$

2. $x \sqsupseteq y \iff j(x, y) = y$

We begin by making the following observations: since m and j are commutative, associative, idempotent, and satisfy the absorption laws, we must have

$$x = m(x, j(x, y)) = j(x, m(x, y))$$

$$y = m(y, j(x, y)) = j(y, m(x, y))$$

for any x, y in L. If $x \leq y$, then $x = m(x, y)$. Then

$$y = j(y, m(x, y)) = j(y, x) = j(x, y)$$

hence $x \sqsupseteq y$. Similarly, is $x \sqsupseteq y$, then $j(x, y) = y$. It follows that

$$x = m(x, j(x, y)) = m(x, y)$$

and $x \leq y$. From here we define $j(x, y) := x \vee y$ and $m(x, y) := x \wedge y$ and observe that

$$x \leq y \Longleftrightarrow x = x \wedge y \Longleftrightarrow y = x \vee y$$

Since L is endowed with the absorption property, we know each element in L can be expressed as the composition of the join (or meet) of that element with the meet (join) of that element with any other in L. This implies that the join (and meet) of each pair of elements exists; indeed if it failed in a single instance then the resulting absorption relation would be undefined. Therefore, it follows that L is a lattice under $\leq$.

In light of the previous paragraphs, we can think of a lattice L either as a poset in which every pair of elements has a meet and a join, *or* we may think of a lattice as a triple $\mathbf{L} = (L, \wedge, \vee)$, where L is a set, $\wedge$ and $\vee$ are binary operations which are commutative, associative, idempotent, and satisfy the absorption laws. Both viewpoints are useful.

Definition 1.12. Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice. A subset S of L is said to be a *sublattice* of $\mathbf{L}$ provided S is closed under the restrictions of $\vee$ and $\wedge$ to S.

Definition 1.13. Let $\mathbf{P}$ be a poset. A subposet D of P is *directed* provided every finite subset of D has an upper bound in D. A directed lowerset of $\mathbf{P}$ is called an *ideal* of $\mathbf{P}$.

Directed sets are sometimes called *up-directed*, but this is not standard. Note that directed sets are by definition nonempty. A directed set in $\mathbf{P}^{\mathrm{op}}$ is said to be *filtered* (or down-directed) in $\mathbf{P}$.

Let $\mathbf{P}$ be a poset and suppose $D \subseteq P$ is directed. Since D is directed, we know that any finite $F = \{f_1, f_2, \ldots, f_n\} \subseteq D$ has an upper bound $d_f \in D$. Similarly, any finite subset of D_x will be of the form $F_x = \{x \wedge f : f \in F\}$ for some finite $F \subseteq D$. Since $x \wedge f \leq x \wedge d_f$ for all f, we know every finite F_x has an upper bound in D_x, hence D_x is directed. This tells us that if $x \in P$, we observe that $D_x = \{x \wedge d : d \in D\}$ is directed, provided D_x is nonempty.

Our next result characterizes a very important class of posets:

Lemma 1.14. *Let P be a poset. The following statements are equivalent:*

1. *Every subset of P has a meet in P.*
2. *Every subset of P has a join in P.*
3. *P is both a lower-bounded join semilattice and a DCPO.*

Proof. In all that follows, let $S \subseteq P$.

$1 \Rightarrow 2$ Suppose P satisfies (1) and let S_U be the set of all upper bounds of S in P. We know that S_U is nonempty since $\bigwedge \emptyset = \top$ acts as an upper bound for P. By definition, $\bigwedge S_U \leq y$ for every y in S_U. We also observe that every $x \in S$ is a lower bound for S_U, and consequently $x \leq \bigwedge S_U$ for every $x \in S$ since $\bigwedge S_U$ is the *greatest* lower bound for S_U. This means that $\bigwedge S_U$ is an upper bound for S, hence $\bigwedge S_U \in S_U$. This means that $\bigwedge S_U$ is the least upper bound for S; that is $\bigwedge S_U = \bigvee S$, and since our choice of S was arbitrary, we have the desired implication.

$2 \Rightarrow 3$ Suppose P satisfies (2); that is, P is closed under arbitrary joins. Note that this implies that P is bounded below since $\bigvee \emptyset = \perp$ acts as a lower bound for P. Clearly, (2) guarantees that P is a join semilattice. Furthermore, (2) tells us that every directed subset of P has a join in P, hence P is a DCPO.

$3 \Rightarrow 1$ Suppose P satisfies (3) and let S_L be the set of all lower bounds for S. We know that S_L is nonempty since P is lower bounded. We claim that S_L is directed. To see this, consider $X \in \mathtt{Fin}(S_L)$. Since X is finite, $\bigvee X \in P$. Furthermore, since every element of S is an upper bound for X, $\bigvee X$ is a lower bound for S, hence $\bigvee X \in S_L$, and S_L is therefore directed.

Since S_L is directed, it follows that $\bigvee S_L \in P$. By the way we defined S_L, every member of S is an upper bound for S_L. That is $\bigvee S_L \leq x, \forall x \in S$. This shows $\bigvee S_L \in S_L$, and by definition of join, this means that S_L has a greatest element. That is, $\bigvee S_L = \bigwedge S$, which is what we intended to show.

□

Definition 1.15. A poset $\mathbf{P}$ is *complete* provided every subset of P has a meet (equivalently, a join) in $\mathbf{P}$. Complete posets are often called complete *lattices*.

Every finite lattice is necessarily complete. Note that the real numbers under their natural ordering form a chain which is not complete. Also, whenever S is an infinite set, the collection $\mathtt{Fin}(S)$ of finite subsets of S is not a complete lattice under the partial ordering of subset inclusion.

Definition 1.16. We say that a poset $\mathbf{P}$ can be *order embedded* in another poset $\mathbf{Q}$ provided there exists an isotone injection $f : P \longrightarrow Q$.

Suppose $\mathbf{P}$ is a poset. One important example of an order embedding is the mapping $p \mapsto \downarrow p$, which embeds $\mathbf{P}$ into $\mathcal{L}$; consequently every poset can be embedded into a complete lattice.

Definition 1.17. Let $\mathbf{P}$ be any poset. An *ideal* of $\mathbf{P}$ is a directed lowerset of $\mathbf{P}$, and a *filter* of $\mathbf{P}$ is a directed lowerset of $\mathbf{P}^{op}$. Let $\mathtt{Idl}(\mathbf{P})$ denote the family of all ideals of $\mathbf{P}$, partially ordered by set inclusion. Let $\mathtt{Fil}(\mathbf{P})$ denote the family of all filters of $\mathbf{P}$, partially ordered by *reverse* set inclusion.

It should be noted that a filter F in a poset $\mathbf{P}$ is an upperset that is *down directed*. That is, if $A \subseteq F$ is finite, then A has a *lower* bound in F. It should also be noted that $\mathtt{Fil}(\mathbf{P})$ is the order dual of $\mathtt{Idl}(\mathbf{P}^{op})$. (Thus, we consider $G \leq F$ in $\mathtt{Fil}(P)$ provided $F \subseteq G$.)

Let $\mathbf{L} = (L, \leq)$ be a join semilattice and let $I \subseteq L$ be nonempty. The notion that I is a lowerset of $\mathbf{L}$ with the property that $x \vee y \in I$ whenever $x, y \in I$ is equivalent to claiming I is a directed lowerset (i.e. an ideal). Of course, the dual of this notion tells us that if $\mathbf{M} = (M, \leq)$ is a meet semilattice and $F \subseteq M$, then F is a filter of $\mathbf{M}$ if and only if F is an upperset with the property that $x \wedge y \in F$ whenever $x, y \in F$.

1.2 Zorn's Lemma and the Axiom of Choice

No discussion of basic order theory would be complete without an investigation of Zorn's Lemma (which is neither a lemma nor attributable solely to Max Zorn, one of its early defenders).

Let $\mathbf{P}$ be a nonempty poset. If every chain in $\mathbf{P}$ has an upper bound in $\mathbf{P}$, then $\mathbf{P}$ has a maximal element.

The previous, rather innocuous-looking statement is what has come to be known as *Zorn's Lemma*. It is generally taken as an axiom for order-theorists and plays a vital role in transfinite induction, as well as many existence proofs. As a quick example, recall that a *basis* for a vector space $\mathcal{V}$ over a field is a maximal, linearly independent subset. We can use Zorn's Lemma to prove that every nontrivial vector space has a basis.

To see how, let $\mathcal{V}$ be any nontrivial vector space over a field F. Since the underlying set V is not a singleton by assumption, we may select a vector $\vec{v}$ in V which is not the zero-vector. Clearly this vector is linearly independent when viewed as a singleton; hence, V contains linearly independent subsets. Now, let $P(V)$ denote the set of all linearly independent subsets of $\mathcal{V}$, partially ordered by set-inclusion, and let $C \subseteq P(V)$ be any chain. Since every member of C is a linearly independent subset of $\mathcal{V}$, it follows that $S = \bigcup C$ is also a linearly independent subset of $\mathcal{V}$ (the fact that C is a *chain* under set-inclusion is critical here). The set S clearly serves as an upper bound for C in $P(V)$; hence we know that $P(V)$ contains a maximal member by Zorn's Lemma. Any such member is the basis we seek.

Lemma 1.18. *The following statements are equivalent for any nonempty poset* $\mathbf{P}$ *:*

1. *If every chain in* $\mathbf{P}$ *has an upper bound in* $\mathbf{P}$*, then* $\mathbf{P}$ *has a maximal member.*
2. *If every chain in* $\mathbf{P}$ *has a least upper bound in* $\mathbf{P}$*, then* $\mathbf{P}$ *has a maximal member.*
3. P *contains a maximal chain.*

Proof. In order the show the equivalence of these statements, we will show $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, and $(3) \Rightarrow (1)$. $(1) \Rightarrow (2)$ is trivial, because if (1) holds and every chain in P has a least upper bound, then every chain in P is bounded and therefore has a maximal element.

To see that $(2) \Rightarrow (3)$, we proceed by contraposition. To that end, assume that no chain in P is maximal. Then for any chain $\mathcal{C} \subseteq P$, we can find a chain $\mathcal{C}' \subseteq P$ where $\mathcal{C} \subset \mathcal{C}'$. This implies that for any $c \in \mathcal{C}$, there exists $c' \in \mathcal{C}'$ where $c < c'$. Consequently P has no maximal element.

In order to see that $(3) \Rightarrow (1)$, suppose (3) holds and that every chain in P has an upper bound. Let $\mathcal{C}$ be a maximal chain in P and suppose $p \in P$ is an upper bound of $\mathcal{C}$. Then, for any $c \in \mathcal{C}$, we know $c \leq p$.

We claim that $p \in \mathcal{C}$. If not, then $\mathcal{C} \cup \{p\}$ is a chain in P where $\mathcal{C} \subset \mathcal{C} \cup \{p\}$, contradicting maximality.

Similarly, we claim that p is maximal in P. If not, then there exists $p' \in P$ where $p < p'$. But then $\mathcal{C} \cup \{p'\}$ is a chain in P where $\mathcal{C} \subset \mathcal{C} \cup \{p'\}$, again contradicting maximality. Hence p is maximal in P. $\square$

The previous lemma gives two axioms equivalent to Zorn's Lemma. The following claim gives several more.

Theorem 1.19. *Let $\mathbf{P}$ be a nonempty poset. An element $p \in P$ is proper provided $p \neq \top$. If $\mathbf{P}$ has no greatest element, then every element of P is proper. The following statements are equivalent.*

1. *If every chain in $\mathbf{P}$ has an upper bound in $\mathbf{P}$, then $\mathbf{P}$ has a maximal element.*
2. *If every chain in $\mathbf{P}$ has a proper upper bound in $\mathbf{P}$, then every chain is contained in a maximal chain.*
3. *If every chain in $\mathbf{P}$ has a proper upper bound in $\mathbf{P}$, then every chain has a maximal upper bound in $\mathbf{P}$.*
4. *If $\mathbf{F}$ is a partially ordered family of sets with the property that $\bigcup C \in F$ for every chain $C \subseteq F$, then $\mathbf{F}$ has a maximal element.*
5. *If $\mathbf{F}$ is a partially ordered family of sets with the property that a set U is a member of F if and only if every finite subset of U is a member of F, then for all $A \in F$, there exists a maximal member of $\mathbf{F}$ containing A.*

Proof. In order the show the equivalence of these statements, we will show $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, $(3) \Rightarrow (4)$, $(4) \Rightarrow (5)$, and $(5) \Rightarrow (1)$.

$(1) \Rightarrow (2)$ We will show $(1) \Rightarrow (2)$ by contraposition. Suppose we have a chain $\mathcal{C} \subseteq P$ that is not contained in a maximal chain. But this implies that $\mathcal{C}$ itself is not maximal, and since it is not contained in a maximal chain, we must have an unbounded chain, hence condition (1) does not hold.

$(2) \Rightarrow (3)$ Now suppose (2) holds, with the condition that every chain has a proper upper bound, and let $\mathcal{C} \subseteq P$ be a chain. We know we can find a maximal chain $\mathfrak{C} \subseteq P$ where $\mathcal{C} \subseteq \mathfrak{C}$. We also know we can find a proper $p \in P$ where p is an upper bound of $\mathfrak{C}$.

We claim $p \in \mathfrak{C}$. If not, then $c < p$ for all $c \in \mathfrak{C}$. But then $\mathfrak{C} \cup \{p\} \subseteq P$ is a chain where $\mathfrak{C} \subset \mathfrak{C} \cup \{p\}$, which contradicts the maximality of $\mathfrak{C}$. This also implies that p is maximal in P. To see why, suppose there is a $q \in P$ where $p < q$. We may assume q is proper, because $\mathfrak{C} \cup \{q\}$ is a chain. But then $\mathfrak{C} \subset \mathfrak{C} \cup \{q\}$, again contradicting maximality. Since p is an upper bound for $\mathfrak{C}$, it is an upper bound for $\mathcal{C}$, and (3) directly follows from here.

$(3) \Rightarrow (4)$ Let F be a partially ordered family of sets with the property that $\bigcup C \in F$ for every chain $C \subseteq F$. Suppose (3) holds. If F itself is a chain, then $\bigcup F \in F$ would be maximal, so suppose F is not itself a chain. Then we are not guaranteed that F has a top element. If $\bigcup F \notin F$, we know we have $\bigcup C \in F$ for every chain in $C \subseteq F$, and, since F is ordered by set inclusion, $\bigcup C$ is a proper upper bound for C. Since we assume that (3) holds, we know that F has a maximal element, and (4) follows from here.

$(4) \Rightarrow (5)$ Suppose (4) holds and let F be a partially ordered family of sets with the property that a set U is a member of F if and only if every finite subset of U is a member of F. Let $A \in F$ and suppose that $G \subseteq F$ is the family of all sets in F containing A. Let $\mathfrak{C} \subseteq G$ be a chain and suppose $\bigcup \mathfrak{C} \notin G$.

By construction, $A \subseteq \bigcup \mathfrak{C}$ since all elements of G contain A. Furthermore, G is the family of all sets in F containing A, so it follows $\bigcup \mathfrak{C} \notin F$. This means that there exists a finite $C \subseteq \bigcup \mathfrak{C}$ that is not in F. But $C \subseteq \bigcup \mathfrak{C}$ means that there exist some $\mathcal{C} \in \mathfrak{C}$ such that $C \subseteq \mathcal{C}$, and $C \notin F$ violates the finite set membership property of F, hence no such C exists. Therefore we must have $\bigcup \mathfrak{C} \in F$ and, since $A \subseteq \bigcup \mathfrak{C}$, $\bigcup \mathfrak{C} \in G$.

Since our choice of $\mathfrak{C} \subseteq G$ was arbitrary, we may conclude that there exists a maximal $M \in G$. We know that $A \subseteq M$, and (5) follows from here.

$(5) \Rightarrow (1)$ Suppose (5) holds and let P be a poset where every chain in P has an upper bound in P. Observe that a subset $\mathcal{C} \subseteq P$ is a chain if and only if every finite subset of $\mathcal{C}$ is a chain, and by (5) there exists a maximal $A_C \in \mathtt{Su}(\mathcal{C})$ such that $\downarrow_C c \in A_C$ (where $\downarrow_C c$ is a descending chain in $\mathcal{C}$), for all $c \in \mathcal{C}$. Since by hypothesis $\mathcal{C}$ has an upper bound, we must have $a \in A_C$ such that $\mathcal{C} = \downarrow_C a$; that is a is maximal in $\mathcal{C}$.

$\square$

Applying Zorn's Lemma is sometimes whimsically referred to as *Zornication*.

CHAPTER 2

LATTICES

2.1 Modular and Distributive Lattices

In this chapter, we will explore some of the major properties that lattices can satisfy. We begin with what is likely one of the most important properties from an historical perspective. The property takes the form of an identity and is inspired by one of the fundamental properties relating set union and set intersection, as well as a related property enjoyed by all rings.

Definition 2.20. Let $\mathbf{L} = (L, \wedge, \vee)$ be a lattice. We say that $\mathbf{P}$ is *distributive* provided joins distribute over meets and vice-versa. That is, for all $x, y, z \in L$, we have

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

The concept of distributivity is *self-dual*; that is, a lattice is distributive if and only if its order dual is distributive. Interestingly enough, we can say even more than this — the two distributive conditions are actually *equivalent*, as the following result shows.

Lemma 2.21. *Let* $\mathbf{P}$ *be a lattice and let* $x, y, z \in L$*. Then* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *if and only if* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*.*

Proof. Assume the first equation, then observe that the absorption laws give us

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\
&= x \vee [(x \vee y) \wedge z] \\
&= x \vee [(x \wedge z) \vee (y \wedge z)] \\
&= [x \vee (x \wedge z)] \vee (y \wedge z) \\
&= x \vee (y \wedge z)
\end{aligned}$$

Hence, the first equation implies the second. The fact that the second equation implies the first follows by duality.

$\square$

Under their natural ordering, the integers form a sublattice of the lattice of real numbers, as do the rational numbers. Given any set S, the set $\texttt{Fin}(S)$ of all finite subsets of S forms a (distributive) sublattice of $\texttt{Su}(S)$.

One useful observation about distributivity is that the inequality

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

holds in any lattice. First, observe for any $x, y, z \in L$ (where $\mathbf{L} = (L, \leq)$ is distributive), $x \wedge y, x \wedge z \leq x$, which implies $(x \wedge y) \vee (x \wedge z) \leq x$. On the other hand, $x \wedge y \leq y$ and $x \wedge z \leq z$, so $(x \wedge y) \vee (x \wedge z) \leq y \vee z$. We have shown that$(x \wedge y) \vee (x \wedge z)$ is a lower bound for both x and $(y \vee z)$. It is therefore a lower bound for $x \wedge (y \vee z)$, hence $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ holds in any lattice. Hence, to prove that a lattice is distributive, one need only establish the reverse inequality.

Let $\mathbf{S}_5 = \{\bot, a, b, c, \top\}$ and let $\mathbf{N}_5 = (\mathbf{S}_5, \leq)$, where $a \vee b = b \vee c = \top$, $a \vee b = b$, $a \wedge c = \bot$. We observe that every pair of elements has a meet and a join, and since $\mathbf{N}_5$ is finite, it follows that it is a complete lattice. Now consider

$$(a \wedge b) \vee (b \wedge c) = a \vee \bot = a$$

while

$$b \wedge (a \vee c) = b \wedge \top = b$$

In this case $(a \wedge b) \vee (b \wedge c) < b \wedge (a \vee c)$, which shows $\mathcal{N}_5$ is a lattice that is not distributive.

Now suppose $\mathbf{M}_5 = (\mathbf{S}_5, \leq)$, where $a \vee b = b \vee c = a \vee c = \top$ and $a \wedge b = a \wedge c = b \wedge c = \bot$. We observe this is also a lattice. Now consider

$$(a \wedge b) \vee (b \wedge c) = \bot \vee \bot = \bot$$

while

$$b \wedge (a \vee c) = b \wedge \top = b$$

In this case $(a \wedge b) \vee (b \wedge c) < b \wedge (a \vee c)$, which shows $\mathcal{M}_5$ is also a lattice that is not distributive.

The lattice $\mathbf{M}_5$ is called the *nondistributive diamond*; the lattice $\mathcal{N}_5$ is simply called the *pentagon*. These substructures play a crucial role in identifying distributive lattices. However, in order to do so, we must identify precisely when these substructures appear in relation to distributive lattices.

Definition 2.22. A lattice $\mathbf{L}$ is said to be *modular* (or weakly distributive) provided, for all $x, y, z \in L$, $z \leq x$ implies that

$$x \wedge (y \vee z) = (x \wedge y) \vee z$$

Let L be a distributive lattice and suppose, for $x, y, z \in L$ we have $z \leq x$. Then

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = (x \wedge y) \vee z$$

Hence L is modular. The lattice $\mathbf{M}_5$ is also modular, but it is not distributive. Let $x, y, z \in M_5$ such that $z \leq x$. If $z = x$, we have

$$x \wedge (y \vee x) = (x \wedge y) \vee x = x$$

by the absorption laws. We may therefore assume that $z < x$. If $x \in \{a, b, c\}$, then $z = \bot$ and

$$x \wedge (y \vee \bot) = x \wedge y = (x \wedge y) \vee \bot$$

If $x = \top$, then

$$\top \wedge (y \vee z) = (y \vee z) = (\top \wedge y) \vee z$$

In any case, the equality holds and $\mathbf{M_5}$ is indeed modular. We will verify that there exist lattices which are *not* modular. In particular, consider $\mathbf{N}_5$. Observe that

$$a \wedge (b \vee c) = a \wedge \top = a$$

while

$$(a \wedge b) \vee c = a \vee c = \top$$

and the non-distributive pentagon is not modular. Though this is a particular example, we shall now see that it can be extended generally.

Theorem 2.23. *A lattice is modular if and only if it does not contain the pentagon as a sublattice.*

Proof. Let $\mathbf{L} = (L, \wedge, \vee)$ be a lattice, and suppose that J is a sublattice of $\mathbf{L}$. If J is not modular, then there exist $x, y, z \in J$ such that $z < x$ but $x \wedge (y \vee z) \neq (x \wedge y) \vee z$. Since $x, y, z \in L$, it follows that $\mathbf{L}$ is not modular. Consequently, if $\mathbf{L}$ is modular, we see that $\mathbf{L}$ cannot contain the pentagon as a sublattice.

On the other hand, suppose that $\mathbf{L}$ is not modular. Then there exist $x, y, z \in L$ such that $z < x$ but $(x \wedge y) \vee z < x \wedge (y \vee z)$. We will use this fact to construct a pentagon in $\mathbf{L}$. Let $a = z \vee y$, $b = x \wedge y$, $c = (x \wedge y) \vee z$ and $d = x \wedge (y \vee z)$. We will prove that $\{a, b, c, d, y\}$ forms a pentagon.

By assumption, $c < d$. Also, we must have $b < y < a$. To see why, notice first that clearly $b = x \wedge y \leq y$. If $b = y$, then we have $y \leq x$. However, if this is the case, then

$$(x \wedge y) \vee z = y \vee z$$

Thus, by assumption, we have $y \vee z < x \wedge (y \vee z)$ — an impossibility. Thus, we must have $b < y$. To see that $y < a$, again first note that $y \leq y \vee z = a$. If $y = a$, then we must have $z \leq y$. Therefore,

$$x \wedge (y \vee z) = x \wedge y$$

Thus, by assumption, we have $(x \wedge y) \vee z < x \wedge y$ — an impossibility. We therefore must have $y < a$.

Now, suppose $y \leq c$. By assumption, we know

$$y < d = x \wedge (y \vee z)$$

This implies that $y \leq x$. That means $c = (x \wedge y) \vee z = y \vee z = a$, and by assumption $c = y \vee z < x \wedge (y \vee z) = d$ — an impossibility.

On the other hand, if $c = (x \wedge y) \vee z \leq y$, then $z \leq y$. It follows that $d = x \wedge (y \vee z) = x \wedge y = b$. This means $d \leq c$ — contrary to assumption.

From here, it is easy to see that $c \wedge y = b$, $d \vee y = a$, and we have the pentagon.

□

Lemma 2.24. *For a lattice* $\mathbf{L}$*, the following statements are equivalent.*

1. *The lattice* $\mathbf{L}$ *is modular.*

2. *For any* $x, y, z \in L$ *we have* $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$.

3. *For any* $x, y, z \in L$ *we have* $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge (x \vee z))$.

Proof. We will show (1) is true if and only if (2) is true; the equivalence of (3) follows by duality. Suppose L is modular, and observe that $x \wedge z \leq x$ for any $x, z \in L$. From here we have

$$(x \wedge y) \vee ((x \wedge z) = x \wedge (y \vee (x \wedge z))$$

by modularity.

Conversely, if the inequality holds and $z \leq x$, we have

$$(x \wedge y) \vee z = (x \wedge y) \vee ((x \wedge z) = x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee z)$$

and L is modular. □

Let **L** be a lattice which satisfies the following identity:

(D_1) For all $x, y, z \in L$, $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$

We can use the absorption laws to show that every lattice that satisfies this identity is modular. Indeed, if we suppose $x, y, z \in L$, we may assume $z < x$. Otherwise, if $x = z$, (D_1) reduces to $x = x$ by the absorption laws (the other cases are identical). Observe that

$$\begin{aligned}
(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z) \quad &\Leftrightarrow \\
(x \wedge y) \vee (y \wedge z) \vee z = (x \vee y) \wedge (y \vee z) \wedge x \quad &\Leftrightarrow \\
(x \wedge y) \vee [z \vee (y \wedge z)] = [x \wedge (x \vee y)] \wedge (y \vee z) \quad &\Leftrightarrow \\
(x \wedge y) \vee z = x \wedge (y \vee z) \quad &\Leftrightarrow
\end{aligned}$$

Hence L is modular. We can actually say more than that, which the next lemma will demonstrate.

Lemma 2.25. *A lattice* **L** *is distributive if and only if it satisfies Identity D_1.*

Proof. Suppose L is distributive. Observe that

$$\begin{aligned}(x \vee y) \wedge (x \vee z) \wedge (y \vee z) &= [x \vee (y \wedge z)] \wedge (y \vee z)\\ &= (y \wedge z) \vee [x \wedge (y \vee z)]\\ &= (y \wedge z) \vee (x \wedge y) \vee (x \wedge z)\\ &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)\end{aligned}$$

Hence $(D1)$ holds.

Conversely, suppose $(D1)$ holds. Then for any $x, y, z \in L$, there exists $\omega \in L$ such that

$$\omega = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$$

If we consider $x \vee \omega$, we observe

$$\begin{aligned}x \vee \omega &= [x \vee (x \wedge y)] \vee (x \wedge z) \vee (y \wedge z)\\ &= [x \vee (x \wedge z)] \vee (y \wedge z)\\ &= x \vee (y \wedge z)\end{aligned}$$

On the other hand, by modularity we have

$$\begin{aligned}x \vee \omega &= x \vee (x \vee y) \wedge (x \vee z) \wedge (y \vee z)\\ &= x \vee (x \vee y) \wedge (y \vee z) \wedge (x \vee z)\\ &= x \vee [x \vee [y \wedge (y \vee z)]] \wedge (x \vee z)\\ &= [x \vee x] \vee y \wedge (x \vee z)\\ &= (x \vee y) \wedge (x \vee z)\end{aligned}$$

That is, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for any $x, y, z \in L$ and L is distributive. $\square$

Let $\mathbf{L}$ be a modular lattice. Let $x, y, z \in L$ and let

- $u = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$
- $v = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$
- $a = (y \wedge z) \vee [x \wedge (y \vee z)]$
- $b = (x \wedge z) \vee [y \wedge (x \vee z)]$
- $c = (x \wedge y) \vee [z \wedge (x \vee y)]$

We claim $a \vee b = a \vee c = b \vee c = u$; will prove the case for $a \vee b$.

$$
\begin{aligned}
a \vee b &= (y \wedge z) \vee [x \wedge (y \vee z)] \vee (x \wedge z) \vee [y \wedge (x \vee z)] \\
&= [(y \wedge z) \vee x] \wedge (y \vee z) \vee [(x \wedge z) \vee y] \wedge (x \wedge z) \\
&= (y \wedge z) \vee [x \wedge (y \vee z) \vee (x \wedge z) \vee y] \wedge (x \wedge z) && \text{(By associativity)} \\
&= (y \wedge z) \vee (y \vee z) \wedge x \vee y \vee (x \wedge z) \wedge (x \vee z) && \text{(By communitivity)} \\
&= (y \vee z) \vee (y \wedge z) \wedge x \vee y \vee (x \wedge z) \wedge (x \vee z) && \text{(By communitivity)} \\
&= (y \vee z) \vee (y \wedge z) \wedge x \vee y \vee (x \wedge z) \wedge (x \vee z) && \text{(By communitivity)} \\
&= [(y \vee z) \vee (y \wedge z)] \wedge [(x \vee y) \vee (x \wedge z)] \wedge (x \vee z) && \text{(By associativity)} \\
&= (y \vee z) \wedge (x \vee y) \wedge (x \vee z) && \text{(Since } x \wedge z \leq x \leq x \vee y \text{ and } y \wedge z \leq y \vee z\text{)} \\
&= u
\end{aligned}
$$

The cases for $a \vee c$ and $b \vee c$ are similar. We observe that

$$\begin{aligned} a^{op} &= [(y \wedge z) \vee [x \wedge (y \vee z)]]^{op} \\ &= (y \vee z) \wedge [x \vee (y \wedge z)] \\ &= [(y \wedge z) \vee x] \wedge (y \vee z) && \text{(By communitivity)} \\ &= (y \wedge z) \vee [x \wedge (y \vee z) && \text{(By associativity)} \\ &= a \end{aligned}$$

This tells us that a is its own order dual. The cases for b^{op} and c^{op} are similar. We also observe that v is the order dual of u, hence $a \wedge b = a \wedge c = b \wedge c = v$ (note $u^{op} = (a \vee b)^{op} = a \wedge b$, so this claim is dual to that found above). Now, if $\chi \in \{a, b, c\}$, idempotency assures that $\chi = \chi \wedge \chi = \chi \vee \chi$. Since each χ is its order dual, we may conclude if $u = \chi \vee \chi$, then

$$v = \chi \wedge \chi = \chi = \chi \vee \chi = u$$

If L is distributive, then Lemma 2.25 tells us L satisfies $(D1)$. That is, $u = v$, for every $u, v \in L$. On the other hand, if L is not distributive, there exist $u, v \in L$ such that $v < u$. It is worth noting that in this case $a, b, c \in L$ are distinct. We are now ready to establish the main result from this section.

Theorem 2.26. *A lattice is distributive if and only if it does not contain the pentagon* $\mathbf{N}_5$ *or the non-distributive diamond* $\mathbf{M}_5$ *as a sublattice.*

Proof. Suppose L is distributive. Note this is true if and only if every sublattice of L is also distributive, hence L cannot contain the pentagon or the non-distributive diamond as a sublattice.

Conversely, if L does not contain the pentagon as a sublattice, we know that L is modular by Theorem 2.23. If L is *not* distributive, we will use the preceding paragraph to prove that L contains the non-distributive diamond as a sublattice. To that end, let $x, y, z \in L$ be distinct and let

- $u = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$
- $v = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$
- $a = (y \wedge z) \vee [x \wedge (y \vee z)]$
- $b = (x \wedge z) \vee [y \wedge (x \vee z)]$
- $c = (x \wedge y) \vee [z \wedge (x \vee y)]$

. Since L is not distributive, we know $v < u$ and a,b, and c are distinct elements.

Suppose $\chi, \gamma \in \{a, b, c\}$ such that $\chi < \gamma$. Then

$$\chi \vee \gamma = u = \gamma$$

and

$$\chi \wedge \gamma = v = \chi$$

However, there remaining element $\zeta \in \{a, b, c\}$ satisfies

$$\chi \wedge \zeta = v = \chi$$

so $\chi < \zeta$, and

$$\zeta \vee \gamma = u = \gamma$$

But then $\chi < \zeta < \gamma$ means $\zeta \wedge \gamma = \zeta$, contradicting the condition that $\chi \wedge \gamma = \zeta \wedge \gamma$, so it follows that $a, b,$ and c are incomparable. This also implies $v < \chi < u$ for each $\chi \in \{a, b, c\}$, and $\{a, b, c, u, v\} \subseteq L$ forms the non-distributive lattice. By contraposition, we conclude a modular lattice is distributive if and only if it does not contain the non-distributive lattice, which finishes the proof. $\square$

From here, we have established one of the most fundamental properties of distributive lattices. Indeed, we shall make use of the forbidden substructures of $\mathcal{M}_5$ and $\mathcal{N}_5$ to verify that certain properties are exclusive to distributive lattices. We end this section with the following lemma, which gives a kind of "cancellative" property that characterizes distributive lattices.

Lemma 2.27. *A lattice* $\mathbf{L}$ *is distributive if and only if, for all* $x, y, z \in L$, $x \vee y = x \vee z$ *and* $x \wedge y = x \wedge z$ *together imply that* $y = z$.

Proof. Suppose L is a distributive lattice and, for $x, y, z \in L$ we have $x \vee z = x \vee y$ and $x \wedge y = x \wedge z$. Observe

$$
\begin{aligned}
y &= y \vee (x \wedge y) \\
&= y \vee (x \wedge z) \\
&= (x \vee y) \wedge (x \vee z) \\
&= x \vee y && \text{(idempotency)} \\
&= x \vee z \\
&= x \vee (z \wedge (y \vee z)) \\
&= (x \vee z) \wedge (y \vee z) \\
&= z \vee (x \wedge y) \\
&= z \vee (x \wedge z) \\
&= z
\end{aligned}
$$

On the other hand, consider the case where M_5 or N_5 is a sublattice of L. Recall that in both cases, $\top = a \vee c = b \vee c$ and $\bot = a \wedge c = b \wedge c$. However, $a < b$ in N_5, and $a \mid\mid b$ in M_5, therefore if L is not distributive, there exist $x, y, z \in L$ such that we have $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ while $y \neq z$. Therefore, by contraposition, our claim holds. $\square$

2.2 Prime and Maximal Ideals

In this section, we introduce the notion of prime and maximal ideals. Although the results later in this paper makes little direct use of these objects, they lay the necessary groundwork for the results to come. As such, we will establish some important results regarding these particular classes of ideals.

Definition 2.28. Let $\mathbf{L}$ be a lattice. A ideal $I \neq L$ of $\mathbf{L}$ is said to be *prime* provided $a \wedge b \in I$ always implies that $a \in I$ or $b \in I$.

Claim 2.29. *The following statements are equivalent for a proper ideal I of a lattice $\mathbf{L}$:*

1. I *is a prime ideal of* $\mathbf{L}$.
2. $L - \mathrm{I}$ *is a filter of* $\mathbf{L}$.
3. $L - \mathrm{I}$ *is a prime filter of* $\mathbf{L}$.
4. *For all ideals* J, K *of* $\mathbf{L}$, *if* $\mathrm{J} \cap \mathrm{K} \subseteq \mathrm{I}$, *then* $\mathrm{J} \subseteq \mathrm{I}$ *or* $\mathrm{K} \subseteq \mathrm{I}$.

Proof. $(1) \Rightarrow (2)$ Suppose $I \in \mathtt{Idl}(\mathbf{L})$ is prime. This implies that $L - I$ is nonempty. If we let $x \in I$ and $y \in L - I$, we observe that since I is closed under join that we either have $x < y$ or $x \mid\mid y$. If $x < y$ then $x \vee y = y$. If $x \mid\mid y$, then $y < x \vee y$. Suppose that $x \vee y \in I$. Then

$$y = y \wedge (x \vee y) \in I$$

contradicting the fact that these sets are disjoint. So $x \vee y \in L - I$ and $L - I$ is closed under outside join.

Now, if we have $y_1, y_2 \in \bigwedge\{y_1, y_2\}$, we note that $\bigwedge\{y_1, y_2\}$ exists since $\mathbf{L}$ is a lattice. In fact $\bigwedge\{y_1, y_2\} \in L - I$ since $y_1, y_2 \notin I$ and I is prime. It follows from here that $L - I$ is closed under meet, and $L - I$ is therefore a filter of $\mathbf{L}$.

$(2) \Rightarrow (3)$ Suppose that $L - I$ is a filter of $\mathbf{L}$. Since ideals are nonempty we know that $L - I \subset L$. Suppose that $x \vee y \in L - I$ and $x \notin L - I$. This means that $x \in I$, and if $y \in I$, then $x \vee y \in I$. But this is impossible since $L - I$ and I partition L, hence $y \in L - I$. It follows from here that $L - I$ is prime.

$(3) \Rightarrow (4)$ Suppose that (3) holds and let $J, K \in \mathtt{Idl}(\mathbf{L}))$ where $J \cap K \subseteq I$ and $K \nsubseteq I$. We may assume that $J \cap K$ is nonempty. To that end, we must have $j \wedge k \in I$ for every $j \in J$ and $k \in K$.

Suppose we have $j^\star \in L - I$ for some $j^\star \in J$. Since $K \nsubseteq I$, then $K - I \subseteq L - I$ is nonempty. Since $L - I$ is a filter, it is closed under meet. That is, for every $k^- \in K - I$ and $l^- \in L - I$, we have $k^- \wedge l^- \in L - I$. This also implies that for every $k^- \in K - I$, we have $j^\star \wedge k^- \in L - I$. But $j^\star \wedge k^- \in J \cap K$ since $j^\star \in J$ and $k^- \in K$, and since $J \cap K \subseteq I$, this implies $I \cap (L - I) \neq \emptyset$, which it absurd. Hence no such $j^\star$ exists and we may conclude that $J \subseteq I$.

$(4) \Rightarrow (1)$ We assume that (4) holds and suppose the we have an ideal I. Assume that $x, y \in L$ are such that $x \wedge y \in I$. It follows that $(x] \cap (y] \subseteq I$; hence, either $(x] \subseteq I$ or $(y] \subseteq I$. It follows that $x \in I$ or $y \in I$, hence I is prime.

□

Another useful characterization of prime ideals in distributive lattices is that a proper ideal P of a lattice $\mathbf{L}$ is prime if and only if for any $I, J \in \mathtt{Idl}(\mathbf{L})$, $I \cap J = P$ implies $I = P$ or $J = P$. In order to do so, we make use of the fact that $\mathtt{Idl}(\mathbf{L})$ is distributive precisely when L is distributive (consider the order embedding $p \mapsto {\downarrow}p$; $\mathtt{Idl}(\mathbf{L})$ will contain $\mathcal{M}_5$ or $\mathcal{N}_5$ as a sublattice if and only if $\mathbf{L}$ contains one or the other). First, we assume that $I \cap J = P$ implies $I = P$ or $J = P$ and suppose I, J are such that $P^\star = I \cap J \subseteq P$ and consider $I^* = I \cup P$ and $J^* = J \cup P$. Since $\texttt{Idl}(\mathbf{L})$ is distributive, we observe

$$\begin{aligned} I^* \cap J^* &= (I \cup P) \cap (J \cup P) \\ &= (I \cap J) \cup P \\ &= P^\star \cup P \\ &= P \end{aligned}$$

By assumption we have $I^* = P$ or $J^* = P$, and by extension of the absorption property of $\cup$ we have $I = P$ or $J = P$.

To see that the converse is true, we proceed by contrapositive. That is, we suppose that $P = I \cap J$ for some $I, J \in \texttt{Idl}(\mathbf{L})$ where $P \neq I$ and $P \neq J$. This means there exist $i \in I$ and $j \in J$ where $i, j \notin P$. Since I and J are ideals, $i \in I$ and $j \in J$ implies $i \wedge j \in I$ and $i \wedge j \in J$. This means that $i \wedge j \in I \cap J = P$. But $i, j \notin P$ implies that P is not prime. We may therefore conclude that a proper ideal P of $\mathbf{L}$ is prime if and only if $I \cap J = P$ implies $I = P$ or $J = P$ for all ideals I, J of $\mathbf{L}$.

A proper ideal of a lattice $\mathbf{L}$ is *maximal* provided it is a maximal element of $\texttt{Idl}(\mathbf{L}) - \{L\}$. If $\mathbf{L}$ has a greatest element, we can use Zorn's Lemma to prove that every proper ideal of $\mathbf{L}$ is contained in a maximal ideal of $\mathbf{L}$. Let $I \in \texttt{Idl}(\mathbf{L})$ be proper and let $\mathcal{I} \subseteq \texttt{Idl}(\mathbf{L})$ be the family of all proper ideals containing I. Note that $\top \notin \bigcup \mathcal{I}$, so any member of this family is a proper ideal of $\mathbf{L}$.

Suppose we have a chain $C \subseteq \texttt{Idl}(\mathbf{L})$. Since C is ordered by set inclusion, we know that for any distinct $J_x, J_y \in C$ we have either $J_x \subset J_y$ or $J_y \subset J_x$. We observe that $\bigcup C = \{x : x \in J_i, J_i \in C\}$, so for any arbitrary $x_\alpha \in J_\alpha$ and $x_\beta \in J_\beta$ where $J_\alpha \subset J_\beta$ in C, we have $x_\alpha \vee x_\beta \in J_\beta \subseteq \bigcup C$. Furthermore, since every ideal in C is closed under outside meet, if $x \in \bigcup C$, we must have $x \wedge l \in J$, for all $l \in L$ for some $J \in C$, hence $\bigcup C$ is also closed under outside meet. This shows that $\bigcup C$ is itself an ideal. Furthermore, since $C \subseteq \mathcal{I}$, we know $\top \notin \bigcup C$, hence $\bigcup C$ is proper.

We may therefore conclude that every chain in $\mathcal{I}$ has a proper upper bound in $\mathcal{I}$. By Zorn's Lemma, this means that each chain has a maximal upper bound in $\mathcal{I}$. That is, every chain in $\mathcal{I}$ is contained in a maximal ideal, which proves our claim.

If $\mathbf{L}$ is distributive, the notion of maximal and prime ideals coincide. To see why, suppose that $I \in \mathtt{Idl}(\mathbf{L})$ is proper but not prime.

We know there exist $J, K \in \mathtt{Idl}(\mathbf{L})$ where $I = J \cap K$ while $I \neq J$ and $I \neq K$. This means $I \subset J$ and $I \subset K$, and since $J \neq K$, we know $J \subset L$ or $K \subset L$. This means we have $I \subset J \subset L$ or $I \subset K \subset L$, hence I is not maximal. Therefore, by contraposition, if I is maximal, we have I is prime.

We conclude this section with one of the most important results regarding prime ideals. The existence of prime ideals coincides directly with the existence of join- and meet-irreducible elements (see section 2.7), which will be key to establishing our main result.

Theorem 2.30 (Prime Ideal Theorem)**.** *Let L be a distributive lattice. For every ideal I and filter F of L with empty intersection, there exists a prime ideal P of L disjoint with F such that $I \subseteq P$.*

Proof. Since a filter is a directed lowerset of L^{op}, it follows that no filter is empty. Hence, we note that I must be proper if it is disjoint with a filter F. Let X denote the set of all ideals in L that contain I and are disjoint with F. A simple application of Zorn's Lemma guarantees that X has a maximal member P. We must prove that P is prime.

Suppose that $a, b \in L$ are such that $a \wedge_L b \in P$, but suppose by way of contradiction that neither a nor b is a member of P. The fact that P is maximal in X implies that $(P \vee \downarrow a) \cap F \neq \emptyset$ and $(P \vee \downarrow b) \cap F \neq \emptyset$, where the join is taken in $\mathtt{Idl}(L)$. Now, it follows that there exist $p, q \in P$ such that $a \vee_L p \in F$ and $a \vee_L b \in F$. Since F is a filter, it follows that $x = (a \vee_L p) \wedge_L (b \vee_L q) \in F$. Distributivity therefore implies that

$$x = (p \wedge_L q) \vee_L (p \wedge_L b) \vee_L (q \wedge_L a) \vee_L (a \wedge_L b) \in F$$

However, since every join above is also a member of P, it follows that $x \in P$ as well — a contradiction. □

2.3 Relatively Complemented Lattices

In this section, we will introduce one of the most important families of distributive lattices. Virtually any branch of mathematics that deals with partially ordered objects will at some point deal with structures introduced in this section.

Let $\mathbf{P}$ be a poset and let $a, b \in P$. Throughout this section, we will let $[a, b] = \uparrow a \cap \downarrow b$. This subset of $\mathbf{P}$ is called an *interval* in $\mathbf{P}$; and, of course, is nonempty if and only if $a \leq b$.

Definition 2.31. Let $\mathbf{L}$ be a lattice and let $[a, b] \subseteq L$. An element $x \in [a, b]$ has a *relative complement* in $[a, b]$ provided there exist $y \in [a, b]$ such that $x \wedge y = a$ and $x \vee y = b$. We say that $[a, b]$ is *relatively complemented* provided every element in $[a, b]$ has a relative complement in $[a, b]$. A lattice in which every interval is relatively complemented is called a *relatively complemented* lattice.

If $\mathbf{L}$ is a bounded lattice, then $L = [\bot, \top]$, and relatively complemented elements of $\mathbf{L}$ are said to be *complemented*. A complemented, distributive lattice is called a *Boolean* lattice in honor of George Boole, a prominent nineteenth century mathematician. (Notice that Boolean lattices are necessarily bounded.) Motivated by this classical definition, relatively complemented, distributive lattices are called *generalized* Boolean lattices. A generalized Boolean lattice is a Boolean lattice if and only if it is bounded.

It is easy to verify that the nondistributive diamond $\mathbf{M}_5$ and pentagon $\mathbf{N}_5$ are both complemented lattices. Furthermore, there are elements in both lattices that have multiple complements. Consequently, relative complements need not be unique.

Lemma 2.32. *Let* $\mathbf{L}$ *be a distributive lattice, and let* $a, b \in L$*. An element of* $[a, b]$ *can have at most one relative complement in* $[a, b]$*.*

Proof. Let $x \in [a, b]$ be complemented and let $y_1, y_2 \in [a, b]$ both be complements of x. This means

$$x \wedge y_1 = x \wedge y_2 = a$$

and

$$x \vee y_1 = x \vee y_2 = b$$

By Lemma 2.27 we must have $y_1 = y_2$, which is what we wanted to show. □

Let $\mathbf{L}$ be a lower bounded, distributive lattice. If $a, b \in L$ and a has a relative complement in $[\perp, a \vee b]$, then it is unique; and we denote it by $b \setminus a$. Note that a and $b \setminus a$ are orthogonal.

Proposition 2.33. *Let* $\mathbf{L}$ *be a lower bounded, distributive lattice. The following statements are equivalent.*

1. *The lattice* $\mathbf{L}$ *is a generalized Boolean lattice.*

2. *The element* $b \setminus a$ *exists for all* $a, b \in L$*.*

Proof. We observe that $(1) \implies (2)$ by the way we define generalized Boolean lattices. To see the converse, let $y = (b \setminus x) \vee (a \setminus x) \vee a$ for $x \in [a, b]$ and observe

$$\begin{aligned} x \vee y &= [x \vee (b \setminus x)] \vee [(a \setminus x) \vee a] & x \wedge y &= x \wedge [(b \setminus x) \vee (a \setminus x) \vee a] \\ &= (x \vee b) \vee (x \vee a) & &= (x \wedge (b \setminus x)) \vee (x \wedge (a \setminus x)) \vee (x \wedge a) \\ &= x \vee b & &= \perp \vee \perp \vee a \\ &= b & &= a \end{aligned}$$

This means L is a generalized Boolean lattice, and the two statements are equivalent. □

If $\mathbf{B}$ is a Boolean lattice, we observe that since B is bounded below, there exists $\perp \in B$ that serves as a least element. Note $\{\perp\} \in \bigcap \mathtt{Idl}(B)$. Suppose $I \in \mathtt{Idl}(B)$ and let $a, b \in I$. By the property of ideals, we know $a \vee b \in I$, and $[\perp, a \vee b] \subseteq I$ since I is a lowerset. Since B is a Boolean lattice, we know there exists a unique $b \setminus a \in [\perp, a \vee b]$, and since our choice of a and b were arbitrary, we may conclude I is a generalized Boolean lattice because of Proposition 2.33.

Lemma 2.34. *Every prime ideal of a relatively complemented lattice* $\mathbf{L}$ *is maximal.*

Proof. Let P be a prime ideal of $\mathbf{L}$, let $x \in L - P$, and consider $I = (P \cup \{x\}]$. We will prove that $I = L$. To this end, suppose that $y \in L - P$ is distinct from x. Let $z \in {\downarrow}x \cap P$. Since $\mathbf{L}$ is relatively complemented, there exist $d \in [z, x \vee y]$ such that $x \wedge d = z$ and $x \vee d = x \vee y$. Now, since P is a prime ideal and $x \notin P$, we must conclude that $d \in P$. However, this implies that $x \vee d \in I$; consequently, we may conclude that $y \in I$, as desired.

□

Proposition 2.35. *Let* $\mathbf{L}$ *be a distributive lattice and let* $a, b \in L$.

1. *The set* $F = \{x \in L : b \leq x \vee c\}$ *is a nonempty filter of* L.

2. *Let $G = [\uparrow c \cup F)$. If $a \in G$, there exist $u \in F$ such that $u \wedge c \leq a$. The element $d = a \vee (u \wedge b)$ is the relative complement of c in $[a, b]$.*

Proof. First, we recognize that $b \leq b \vee l$ for any $l \in L$, hence $b \in F$. To verify that F is a filter, suppose $x \in F$ and $x \leq y$ for some $y \in L$. Since $b \leq x \vee c$, and $x \leq y$ implies $x \vee c \leq y \vee c$, it follows that $b \leq y \vee c$, hence $y \in F$, and F is indeed a filter of L.

For the second claim, we recognize that $a \leq c, d \leq b$ and that L is distributive. By assumption, $u \wedge c \leq a$, and since $u \in F$ we have $b \leq u \vee c$. From here we observe the following:

$$
\begin{aligned}
c \wedge d &= c \wedge [a \vee (u \wedge b)] & \qquad c \vee d &= c \vee [a \vee (u \wedge b)] \\
&= (c \wedge a) \vee [c \wedge (u \wedge b)] & &= (c \vee a) \vee (u \wedge b) \\
&= (c \wedge a) \vee (c \wedge b \wedge u) & &= c \vee (u \wedge b) \\
&= (c \wedge a) \vee (c \wedge u) & &= (c \vee u) \wedge (c \vee b) \\
&= a \vee (c \wedge u) & &= (c \vee u) \wedge b \\
&= a & &= b
\end{aligned}
$$

We may therefore conclude that c and d are relative complements in $[a, b]$. □

Let $\mathbf{L}$ be a distributive lattice that is not a generalized Boolean lattice. There exist $a, b \in L$ and $c \in [a, b]$ such that c has no relative complement, and we know $a \leq c \leq b$. In fact, we may assume that $a < c < b$, since if we must have distinct a and b, and since both are relative complements in $[a, b]$, c cannot be either. With that in mind, we observe that $\uparrow c$ is a filter containing c but not a. Since $(a] \cap \uparrow c = \emptyset$, we know there exists a prime $P \in \mathtt{Idl}(L)$ where $(a] \subseteq P$ and $\uparrow c \cap P = \emptyset$ by the prime ideal theorem. Hence $a \in P$ while $c \notin P$. In what follows, let F and G be the same filters defined in Proposition 2.35. We observe that if $a \in G$, then $[a, b]$ is relatively complemented, contrary to our initial assumption. We may therefore conclude $(a] \cap G = \emptyset$, hence there exists a prime $P \in \texttt{Idl}(L)$ disjoint with G that contains $(a]$. Without loss of generality, we may assume this is the same P from part (2). We observe that since $b \in G$, we have $b \notin P$, and since $c < b$ it follows that $b \notin\downarrow c$. Since P is disjoint with G, and therefore F, we know that for every $p \in P$ we have either $p \vee c < b$ or $p \vee c \mid\mid b$. More to the point, there is no p in P such that $p \vee c = b$. We may therefore conclude that $b \notin I$, which is what we wanted to show. Since $b \notin I$, it follows that $\uparrow b \cap I = \emptyset$. Again, by the prime ideal theorem, we must have a prime $Q \in \texttt{Idl}(L)$ where $I \subseteq Q$ not containing $\uparrow b$. We observe that since $c \in Q$ and $P \subseteq Q$, we must have $P \subset Q$.

Notice that Lemma 2.34 tells us that the prime ideals of a relatively complemented lattice form an antichain. The previous paragraph, along with Proposition 2.35 and Lemma 2.34, tell us that among distributive lattices, the generalized Boolean lattices are *precisely* the ones whose prime ideals form antichains.

Let $\mathbf{L}$ be a bounded lattice. Whenever $x \in L$ has a unique complement, it is customary to let x^c denote that element. Other common symbols for this element include $\overline{x}$ and $\neg x$.

Theorem 2.36. *Let* $\mathbf{B}$ *be a Boolean lattice and let* $x, y, z \in B$*. The following statements hold:*

1. $x \wedge y = \bot \iff x \leq y^c$
2. $x = (x^c)^c$
3. $x \wedge y \leq z \iff y \leq x^c \vee z$
4. $(x \vee y)^c = x^c \wedge y^c$
5. $(x \wedge y)^c = x^c \vee y^c$

The last two properties in the previous claim are known as *De Morgan's Laws*.

The proof of Theorem 2.36 is established via routine calculations and is left to the reader.

Corollary 2.37. *Let L be a Boolean lattice and $x, y, z \in L$. Then*

$$x \vee y = \top \iff y^c \leq x$$

$$z \leq x \vee y \iff x^c \wedge z \leq y$$

Proof. We recognize these equations are simply the order dual of items (1) and (3) of Theorem 2.36. □

Definition 2.38. A lattice $\mathbf{L}$ is said to be *join continuous* if for all $y \in L$ and $X \subseteq L$ such that $\bigvee X$ exists, then $\bigvee\{y \wedge x : x \in X\}$ exists also; and we have

$$y \wedge \bigvee X = \bigvee\{y \wedge x : x \in X\}$$

If we suppose L is a Boolean lattice, there is a top element $\top \in L$. Then for every $x \in X$ we have $x = \top \wedge x \leq z$. That is, z is an upper bound for X. Since x_0 is the *least* upper bound for X, we must have $x_0 \leq z$. It therefore follows that $y \wedge x_0 \leq z$ for any y. We may therefore conclude that every Boolean lattice is join continuous.

Definition 2.39. Let $\mathbf{P}$ be a lower-bounded poset. We say that an element $a \in P$ is an *atom* of $\mathbf{P}$ provided $\perp \prec a$. Likewise, an element c of an upper-bounded poset $\mathbf{P}$ is a *co-atom* of P provided c is an atom of $\mathbf{P}^{\mathrm{op}}$. We say that a lower-bounded poset $\mathbf{P}$ is *atomic* provided $\downarrow x$ contains an atom for all $\perp < x \in P$.

Lemma 2.40. *Let $\mathbf{L}$ be a uniquely complemented lattice and let $a \in L$ be an atom. The element a^c is a co-atom in $\mathbf{L}$ (that is, an atom in $\mathbf{L}^{op}$).*

Proof. Observe that for any $\bot < x \in L$, we either have $a \mid\mid x$ or $a \leq x$. If $a \mid\mid x$, then $a \wedge x = \bot$, and by Theorem 2.36 part (1) we have $x \leq a^c$. On the other hand, if $a \leq x$, then by Corollary 2.37 we have $a^c \vee x = \top$. In either case, this precludes the existence of an x^* such that $a^c < x^* < \top$, hence $a^c \prec \top$. □

Let **L** be a Boolean lattice. By definition, every Boolean lattice is a relatively complemented distributive lattice. We observe that Lemma 2.32 guarantees that complements are unique. Furthermore, we note that Lemma 2.40 guarantees that atoms and co-atom complements occur in pairs. We may therefore conclude **L** is atomic if and only if its order dual $\mathbf{L}^{\mathrm{op}}$ is atomic.

We also observe that $\uparrow x$ contains at least one co-atom; call this element a^c. Since $x \leq a^c$, we have $x \wedge a = \bot$ where $a \in L$ is the atomic complement of a^c by Theorem 2.36 part (1). This means $a \mid\mid x$, hence $x < x \vee a$. We can also show that **L** is atomic if and only if the top element is a join of atoms. Let $A = \{a \in L : \bot \prec a\}$. That is, A is the atomic set of L. If $\top = \bigvee A$, then for any $x \in L$ we have

$$x = x \wedge \top = x \wedge \bigvee A = \bigvee \{x \wedge a : a \in A\}$$

Therefore, for every $\bot < x \in L$, there exists $a \in A$ such that $a \leq x$. Hence L is atomic.

Conversely, suppose L is atomic. Clearly, A is bounded since L is bounded. That is, $\top$ is an upper bound for A. However, for any $x < \top \in L$, we observe that there exists $a \in A$ such that $x < a \vee x$. Consequently, any such x cannot be an upper bound for A, hence $\top$ is the *least* upper bound for A. Hence $\bigvee A = \top$, which verifies our claim.

Theorem 2.41. *Let* **L** *be a complete, atomic Boolean lattice. Every element of* **L** *is the join of a set of atoms.*

Proof. In all that follows, let $A = \{a \in L : \bot \prec a\}$. That is, A is the atomic set of L. We know that $\top = \bigvee A$. Furthermore, we observe that $\bigvee \emptyset = \bot$, and $\emptyset \in \mathtt{Su}(A)$. Let $\bot < x < \top \in L$; we may assume that $x \notin A$ since there would be nothing to show otherwise. We know that for every $a \in A$, either $a \mid\mid x$ or $a < x$. We may therefore partition A into $A = A_< \cup A_{||}$ where $A_< = \{a \in A : a < x\}$ and $A_{||} = \{a \in A : a \mid\mid x\}$. Observe that

$$\begin{aligned}
x &= x \wedge \top \\
&= x \wedge \bigvee A \\
&= \bigvee \{x \wedge a : a \in A\} \\
&= \bigvee \big\{\{x \wedge a_{||} : a_{||} \in A_{||}\} \cup \{x \wedge a_< : a_< \in A_<\}\big\} \\
&= \bigvee \big\{\{\bot\} \cup \{x \wedge a_< : a_< \in A_<\}\big\} && \text{(If } a \mid\mid x\text{, then } x \wedge a = \bot\text{)} \\
&= \bigvee \{x \wedge a_< : a_< \in A_<\} && \text{(If } a < x\text{, then } x \wedge a = a\text{)} \\
&= \bigvee \{a_< : a_< \in A_<\}
\end{aligned}$$

Hence x may be expressed as the join of a set of atoms. □

We shall end this section with a classic exhibition. It is well known that a complete Boolean lattice is atomic if and only if it is order isomorphic to the powerset of some set. To demonstrate this, first let L be a complete atomic Boolean lattice and let $A = \{a \in L : \bot \prec a\}$ be its atomic set. Define $f : \mathtt{Su}(A) \longrightarrow L$ by the map $f(S) = \bigvee S$ for $S \in \mathtt{Su}(A)$. We have that f is surjective from Theorem 2.41. Suppose we have $x \in L$ where $x = f(S_1) = f(S_2)$. From the proof of Theorem 2.41 we must have $S_1 = S_2 = \{a_< : a_< \in A_<\}$ where $A_< = \{a \in A : a < x\}$, hence f is bijective. Furthermore, if we have $S_a \subseteq S_b$ in $\mathtt{Su}(A)$, it follows that $f(S_a) \leq f(S_b)$ in L, hence f is an isomorphism.

On the other hand, if L is isomorphic to the powerset of some set; we may call this set X. We observe that every powerset is bounded and also a field of sets, and is therefore uniquely complemented and distributive under set inclusion. Since L is isomorphic to $\mathtt{Su}(X)$, there must exist some map $g : L \longrightarrow \mathtt{Su}(X)$; and there must be $L_A \subseteq L$ such that $L_A = \big\{l \in L : g(l) = \{x\}, \forall x \in X\big\}$. That is, L_A is the preimage set of all the singletons in $\mathtt{Su}(X)$. From here we observe that $\bigcup\big\{\{x\} : x \in X\big\} = X$, which by hypothesis is order isomorphic to the top element of L. That is, $\bigvee L_A = \top_L$ and we may conclude that L is an atomic Boolean lattice.

2.4 Adjunctions and Heyting Lattices

In this section, we will explore another important class of distributive lattice — the *Heyting* lattices. These lattices are inspired by a branch of mathematics known as *intuitionistic* logic. We begin with a concept which will have far-reaching implications in other sections.

Definition 2.42. Let $\mathbf{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ be posets, and let $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ be functions. We say that f and g form an *adjunction* provided

$$f(p) \sqsubseteq q \iff p \leq g(q)$$

for all $p \in P$ and $q \in Q$. We will use the symbol $(f, g) : P \leftrightharpoons Q$ to indicate that an adjunction exists between the posets $\mathbf{P}$ and $\mathbf{Q}$ and to label the component functions.

When working with adjunctions, we will usually dispense with distinct symbols for the partial orders on the posets $\mathbf{P}$ and $\mathbf{Q}$ unless special care is needed. Adjunctions are often called *residuations*. If $(f, g) : P \leftrightharpoons Q$, then we call f a *left adjoint* for g and call g a *right adjoint* for f. The functions f and g are often called left and right *residuals*, but we shall stick to the term “adjoint” .

Lemma 2.43. *Suppose $(f, g) : P \leftrightharpoons Q$ is an adjunction between posets $\mathbf{P} = (P, \leq)$ and $\mathbf{Q} = (Q, \preceq)$. The following statements are true:*

1. *$f(g(q)) \preceq q$ and $p \leq g(f(p))$ for all $p \in P$ and $q \in Q$.*

2. *Both f and g are order homomorphisms.*

Proof. Observe that $f(p) \leq f(p)$ and $g(q) \leq g(q)$ is always true, hence $f(g(q)) \preceq q$ and $p \leq g(f(p))$ for all $p \in P$ and $q \in Q$.

Suppose $x, y \in Q$ such that $x \preceq y$. Since g is the right adjoint of f, we know by (1) we have $f(g(x)) \preceq y$ and by the definition of adjunction we have $g(x) \leq g(y)$. We may therefore conclude that g is isotone. The proof that f is isotone mirrors this argument. □

If $(f, g) : P \leftrightharpoons Q$ is an adjunction between posets $\mathbf{P}$ and $\mathbf{Q}$, then it is well known that f preserves all existing joins and g preserves all existing meets. To see why this is true, suppose $\mathcal{F} \subseteq P$ and $\mathcal{G} \subseteq Q$ such that $p_F = \bigvee_P \mathcal{F}$ and $q_g = \bigwedge_Q \mathcal{G}$ exist. Observe that since both f and g are isotone, $f(p_F)$ and $g(q_G)$ are upper and lower bounds for $f(F)$ and $g(G)$, respectively. The fact that these maps are isotone also ensures that if p_l and q_u are any lower and upper bounds for $g(G)$ and $f(F)$ we must have $f(p_l) \leq_Q q_g(q_g \in G)$ since $p \leq_P g(q_g)(q_g \in G)$. That is, $f(p_l)$ is a lower bound for G whenever p_l is a lower bound for $g(G)$. Since g is isotone, we may therefore conclude $f(p_l) \leq_Q q_G$, hence $\bigwedge g(G) = g(q_G)$. Similarly, if $g(q_u)$ is an upper bound for F, we have $p_f(p_f \in F) \leq_P g(q_u)$, hence $g(p_f)(p_f \in F) \leq_Q q_u$ and so $\bigvee f(F) = f(p_F)$.

It is also well known that $g \circ f \circ g = g$ and $f \circ g \circ f = f$. We will prove that $f \circ g \circ f = f$; the proof that $g \circ f \circ g = g$ will mirror this one. Let $p \in P$ and observe that since f is a function, there exists a $q \in Q$ such that $f(p) = q$. By Lemma 2.43

we have $f(g(q)) \leq_Q q$. That is $f(g(f(p))) \leq f(p)$. Since f is isotone and $p \leq g(q)$, we must have $f(g(f(p))) = f(p)$, hence $f \circ g \circ f = f$.

Theorem 2.44. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be posets and let* $f : P \longrightarrow Q$ *and* $g : Q \longrightarrow P$ *be functions. The following statements are equivalent:*

1. *The mappings* f *and* g *form an adjunction.*

2. *The mapping* g *is isotone, and for all* $p \in P$, $g^{-1}(\uparrow p) = \uparrow f(p)$

3. *The mapping* f *is isotone, and for all* $q \in Q$, $f^{-1}(\downarrow q) = \downarrow g(q)$.

Proof. First, suppose $(f, g) : P \leftrightharpoons Q$ is an adjunction. We know f is isotone, and for all $q \in Q$ we know $f \circ g(q) \leq q$, hence $g(q) \in f^{-1}(\downarrow q)$. Conversely, if $x \in f^{-1}(\downarrow q)$, it follows that $f(x) \leq q$, hence $x \leq g(q)$ and $\bigvee f^{-1}(\downarrow q) = g(q)$. This tells us (1) implies (3).

On the other hand, if for all $q \in Q$ we have $f^{-1}(\downarrow q) = \downarrow g(q)$, then suppose $x \in P$ and $y \in Q$ such that $f(x) \leq y$. It follows that $x \in f^{-1}(\downarrow y)$ which implies $x \leq g(y)$. If we had initially assumed $x \leq g(y)$, then again we find $\bigvee f^{-1}(\downarrow y) = g(y)$ and $f(x) \leq y$. This proves (3) implies (1). The equivalence of (1) and (2) is similar and left to the reader as an exercise. $\square$

Let $(f, g) : P \leftrightharpoons Q$ be an adjunction. Claim 2.44 tells us that f uniquely determines g and vice-versa. Indeed, we know that, for all $q \in Q$ and all $p \in P$,

- $g(q) = \bigvee_P f^{-1}(\downarrow q)$, and
- $f(p) = \bigwedge_Q g^{-1}(\uparrow p)$.

Hence, we are justified in referring to g as *the* right adjoint of f and to f as *the* left adjoint of g.

Lemma 2.45. *Whenever* **P** *is a complete lattice and* **Q** *is a poset, then a mapping $f : P \to Q$ has a right adjoint if and only if f preserves arbitrary joins.*

Proof. First, suppose $g : Q \longrightarrow P$ is the right adjoint of f. This implies $(f, g) : P \leftrightarrows Q$ forms an adjunction. Since P is complete, we know P is closed under arbitrary joins, therefore f must preserve arbitrary joins.

Conversely, suppose f preserves arbitrary joins. Note that this is sufficient to claim that f is isotone. Define a map $g : q \mapsto \bigvee_P f^{-1}(\downarrow q)$ for all $q \in Q$. Since f is isotone, we note $\bot_P \in f^{-1}(\downarrow q)$ for every $q \in Q$. If $f^{-1}(\downarrow q) = \{\bot_P\}$, we have $\bot_P = \bigvee_P f^{-1}(\downarrow q)$. That is, $g(q) = \bot_P$. This means $\downarrow g(q) = \{\bot_P\} = f^{-1}(\downarrow q)$.

Note that if $p_y \in f^{-1}(\downarrow q_x)$ for some $q_x \in Q$, then $p_y \in \downarrow \bigvee_P f^{-1}(\downarrow q_x) = \downarrow g(q_x)$, hence $f^{-1}(\downarrow q_x) \subseteq \downarrow g(q_x)$. On the other hand, suppose $p_y \in \downarrow g(q_x)$. Then we observe

$$\begin{aligned} f(p_y) &\leq f(g(q_x)) \\ &= f(\bigvee_P f^{-1}(\downarrow q_x)) \\ &= \bigvee_Q f(f^{-1}(\downarrow q_x)) \\ &= q_x \end{aligned}$$

This means $f(p_y) \in \downarrow q_x$, hence $p_y \in f^{-1}(\downarrow q_x)$. This means $\downarrow g(q_x) \subseteq f^{-1}(\downarrow q_x)$, hence $\downarrow g(q_x) = f^{-1}(\downarrow q_x)$. Therefore, by Theorem 2.44, $(f, g) : P \leftrightarrows Q$ forms an adjunction. $\square$

Likewise, whenever **Q** is a complete lattice and **P** is a poset, a mapping $g : Q \to P$ has a left adjoint if and only if g preserves arbitrary meets. Both of these statements require completeness.

However, if $\mathbf{P}$ is not a complete lattice, the preservation of all existing joins is not sufficient to guarantee that f has a right adjoint. Consider $\mathcal{N} = \mathbb{N} \cup \{0\}$ and consider $f : \mathcal{N} \longrightarrow \{0\}$. We observe that since $\mathcal{N}$ is a chain, the join of any two elements is simply the maximum function. That is $\bigvee\{x, y\} = \mathtt{max}(x, y)$. It therefore follows that for any finite subset $S \subseteq \mathcal{N}$, we have $\mathtt{max}(S) = n$ for some $n \in S$ (namely the largest member), and since *every* element maps to 0, f trivially preserves these joins. However, we observe $f^{-1}(\downarrow 0) = f^{-1}(0) = \mathcal{N}$, and $\bigvee \mathcal{N}$ does not exist, hence there is no function $g : \{0\} \longrightarrow \mathcal{N}$ such that $g(0) = \bigvee f^{-1}(\downarrow 0)$; that is, f has no right adjoint.

One very useful example of an adjunction that which exists when two posets are order isomorphic. If $f : P \longrightarrow Q$ is a bijection between posets and both f and f^{-1} are order preserving, then we have

$$f(p) \leq_Q q \Longleftrightarrow p \leq_P f^{-1}(q)$$

Hence $(f, f^{-1}) : P \leftrightharpoons Q$ is an adjunction. However, if either f or f^{-1} is *not* order preserving, then there exists

$$f(p) \leq_Q q \;\text{ where }\; p \nleq_P f^{-1}(q) \;\text{ or }\; f(p) \nleq_Q P \;\text{ where }\; p \leq_P f^{-1}(q)$$

and an adjunction does not exist between f and f^{-1}. This shows that order isomorphisms are adjunctions, hence they preserve arbitrary meets and joins.

Definition 2.46. Let $\mathbf{L}$ be a bounded lattice. We say that $\mathbf{L}$ is a *Heyting* lattice (or a *Brouwerian lattice*) if for all $a, b \in L$, there exists an element $c \in L$ such that, for all $x \in L$,

$$a \wedge x \leq b \iff x \leq c.$$

In a Heyting lattice, it is easy to see that the element c is uniquely determined by a and b. The element c is usually denoted by $a \to b$ and is called the *relative* *pseudocomplement of a with respect to b.* In this context, the arrow is known as the *Heyting arrow*, or *implication*.

Let $\mathbf{L}$ be a Heyting lattice and let $a, b \in L$. We note that if $a \wedge x \leq b \iff x \leq c$, we have $a \wedge c \leq b$. Idempotency tells us that c is the largest value that satisfies this claim, so c is an upper bound for $\{x \in L : a \wedge x \leq b\}$. Observe that if y is any upper bound for $\{x \in L : a \wedge x \leq b\}$, we must have $c \leq y$, hence $c = \bigvee\{x \in L : a \wedge x \leq b\}$. This tells us that we can characterize the Heyting arrow in the following manner: for any $x \in L$, $a \to b = \bigvee\{x \in L : a \wedge x \leq b\}$.

Heyting lattices are very common structures. For example, all Boolean lattices are Heyting lattices — take $a \to b = a^c \vee b$, where a^c denotes the complement of a.

Let $\mathbf{L}$ be a Heyting lattice and let $a \in L$ be fixed. We can define maps

$$m(x) = a \wedge x \qquad i(x) = a \to x$$

For any $x, y \in L$ we observe that $a \wedge x \leq y \iff x \leq a \to y$. In other words, $m(x) \leq y \iff x \leq i(y)$. Therefore $(m, i) : L \leftrightarrows L$ forms an adjunction by definition. Suppose $S, T \subseteq L$ such that $\bigvee S, \bigwedge T \in L$. Since $(m, i) : L \leftrightarrows L$ forms an adjunction for a fixed $a \in L$, we know m preserves existing joins and i preserves existing meets. This means

$$a \wedge \bigvee S = m(\bigvee S) = \bigvee m(S) = \bigvee\{a \wedge s : s \in S\}$$

and

$$a \to \bigwedge T = i(\bigwedge T) = \bigwedge i(T) = \bigwedge\{a \to t : t \in T\}$$

Therefore the following properties hold:

1. The lattice $\mathbf{L}$ is join continuous (and in particular is distributive).

2. Whenever $X \subseteq L$ is such that $\bigwedge X$ exists, then

$$a \to \bigwedge X = \bigwedge\{a \to x : x \in X\}.$$

Definition 2.47. A complete, join continuous lattice is called a *frame*.

Note that every complete Heyting lattice is automatically a frame. The converse is also true. To see why, let F be a frame and let $a, b, x \in F$. Let $\mathcal{C} = \{c \in F : a \wedge c \leq b\}$; from here we observe $a \to b = \bigvee \mathcal{C}$. If $a \wedge x \leq b$, then $x \in \mathcal{C}$. This implies $x \leq \bigvee \mathcal{C}$. On the other hand, if $x \leq \bigvee \mathcal{C}$, then

$$x = x \wedge \bigvee \mathcal{C} = \bigvee \{x \wedge c : a \wedge c \leq b\}$$

hence

$$a \wedge x = (a \wedge x) \wedge \bigvee \mathcal{C} = \bigvee \{(a \wedge x) \wedge c : a \wedge c \leq b\}$$

We observe that b is an upper bound for $a \wedge x \wedge c$ since $a \wedge c \leq b$, hence $a \wedge x = \bigvee \{(a \wedge x) \wedge c : a \wedge c \leq b\} \leq b$. This shows that $a \wedge x \leq b$ if and only if $x \leq a \to b$ and F is a Heyting algebra.

Let **L** be a Heyting lattice and observe that we have

$$c \leq x \to b \iff c \wedge x \leq b \iff x \leq c \to b.$$

Define two maps $\lambda_b : L \to L^{\text{op}}$ and $\rho_b : L^{\text{op}} \to L$ by

$$\lambda_b(x) = x \to b = \rho_b(x),$$

Observe that if $x, y, b \in L$ we have

$$y \leq x \to b \Longleftrightarrow x \wedge y \leq b \Longleftrightarrow x \leq y \to b$$

This is equivalent to the statement

$$\lambda_b(x) \leq_{op} y \Longleftrightarrow x \leq \rho_b(y)$$

Hence $(\lambda_b, \rho_b) L \leftrightharpoons L^{\text{op}}$ forms an adjunction.

1. We recall that

$$x \to b = \bigvee \{c \in L : c \wedge x \leq b\} \qquad y \to b = \bigvee \{c \in L : c \wedge y \leq b\}$$

Suppose $x \leq y$. Let $\mathcal{F} = \{c \in L : c \wedge x \leq b\}$ and $\mathcal{G} = \{c \in L : c \wedge y \leq b\}$ and suppose $z \in \mathcal{G}$. This means that $z \wedge y \leq b$, therefore $z \wedge x \leq b$ and $z \in \mathcal{F}$. This shows us that $\mathcal{G} \subseteq \mathcal{F}$, therefore we must have $\bigvee \mathcal{G} \leq \bigvee \mathcal{F}$. That is, if $x \leq y$, then $y \to b \leq x \to b$. $\square$

2. Since $(\lambda_b, \rho_b) L \leftrightharpoons L^{\mathrm{op}}$ forms an adjunction, we know λ_b preserves existing joins. This means

$$\begin{aligned}
\lambda_b(\bigvee X) &= \bigvee_{op} \lambda(X) \\
&= \bigvee_{op} \{\lambda(x) : x \in X\} \\
&= \bigvee_{op} \{x \to b : x \in X\} \\
&= \bigwedge \{x \to b : x \in X\}
\end{aligned}$$

We have therefore established the following result:

1. The Heyting arrow is order-reversing in its left argument; that is, if $x \leq y$, then $y \to b \leq x \to b$.

2. Whenever $X \subseteq L$ is such that $\bigvee X$ exists, then

$$\bigvee X \to b = \bigwedge \{x \to b : x \in X\}.$$

Hence we often say that the Heyting arrow is *self-adjoint*.

2.5 Closure Operators

In the last section, we briefly introduced adjunctions between posets and looked at one important application — Heyting lattices. In this section, we continue our look at adjunctions by taking advantage of another important property they possess.

Definition 2.48. Let $\mathbf{P}$ be a poset. A function $\varphi : P \longrightarrow P$ is called a *closure operator* on $\mathbf{P}$ provided φ is isotone, idempotent, and enlarging. That is, provided, for all $x, y \in P$, we have the following:

- $x \leq y \implies \varphi(x) \leq \varphi(y)$ (The mapping is isotone.)
- $\varphi(\varphi(x)) = \varphi(x)$ (The mapping is idempotent.)
- $x \leq \varphi(x)$ (The mapping is enlarging.)

Definition 2.49. Let $\mathbf{P}$ be a poset. A function $\psi : P \longrightarrow P$ is called a *kernel operator* on $\mathbf{P}$ provided ψ is isotone, idempotent, and reducing. That is, provided, for all $x, y \in P$, we have the following:

- $x \leq y \implies \psi(x) \leq \psi(y)$
- $\psi(\psi(x)) = \psi(x)$
- $\psi(x) \leq x$

Lemma 2.50. *Let $\mathbf{P}$ and $\mathbf{Q}$ be posets and let $(f, g) : P \leftrightharpoons Q$. The map $\varphi = g \circ f$ is a closure operator on $\mathbf{P}$ and $\psi = f \circ g$ is a kernel operator on $\mathbf{Q}$.*

Proof. We observe that both φ and ψ are isotone by the properties of adjunctions. Furthermore we have

$$\varphi \circ \varphi = g \circ f \circ g \circ f = g \circ f = \varphi$$

and

$$\psi \circ \psi = f \circ g \circ f \circ g = f \circ g = \psi$$

and both are idempotent. By Lemma 1.4.2, we know that for $p \in P$ and $q \in Q$, we have $\psi(q) \leq q$ and $p \leq \varphi(p)$ for every $p \in P$ and $q \in Q$, hence ψ is reducing and φ is enlarging. Therefore, φ is a closure operator and ψ is a kernel operator. $\square$

Lemma 2.51. *Let* $\mathbf{P}$ *be a poset and let* $\theta : P \longrightarrow P$ *be a function. Let* Q *be the image of* P *under* θ*. Let* $g : Q \longrightarrow P$ *be the inclusion map* $g(q) = q$*.*

1. *If* θ *is a closure operator, then* $(\theta, g) : P \leftrightharpoons Q$*.*

2. *If* θ *is a kernel operator, then* $(g, \theta) : Q \leftrightharpoons P$*.*

Proof. Suppose θ is a closure operator. Since θ is enlarging, we have $p \leq \theta(p)$. Suppose $\theta(p) \leq q$ for some $q \in P$. Since θ is isotone, we know this is true if and only if $p \leq q = g(q)$, hence $(\theta, g) : P \leftrightharpoons Q$ forms an adjunction. A dual argument of the above argument proves this claim. $\square$

Let $\mathbf{P}$ be a poset and let $\varphi : P \longrightarrow P$ be a closure operator. By Lemma 2.51 we know that $(\varphi, g) : P \leftrightharpoons Q$ forms an adjunction; in our case where $Q = P$ is the image set under φ and $g(q) = p$. This is sufficient to claim that φ preserves existing joins.

Lemmas 2.50 and 2.51 tell us that there is an intimate connection between adjunctions and closure (or kernel) operators. This connection tells us some important facts about closure and kernel operators. For example, as in the previous claim, suppose that $\theta : P \longrightarrow P$ is a closure operator and let $Q = \theta(P)$. The fact that the inclusion map $g : Q \longrightarrow P$ is the right adjoint to θ tells us that whenever $X \subseteq Q$ is such that $\bigwedge_Q X$ exists, then

$$g(\bigwedge_Q X) = \bigwedge_P \{g(x) : x \in X\} = \bigwedge_P X$$

The converse of this statement is also true; that is, whenever $X \subseteq Q$ is such that $\bigwedge_P X$ exists, then $\bigwedge_Q X$ also exists and $\bigwedge_Q X = \bigwedge_P X$.

To see why this is so, let $x_0 = \bigwedge_P X$. Suppose $z \in Q$ is such that $z \leq \theta(x)$ for all $x \in X$. (Note that such z exist by assumption, since $\theta(x_0)$ has this property.) Then, the fact that $(\theta, g) : P \leftrightharpoons Q$ tells us that

$$\begin{aligned} z \leq \theta(x) \quad &\Longleftrightarrow \quad z = g(z) \leq x (\forall x \in X) \\ &\Longleftrightarrow \quad z = g(z) \leq x_0 \\ &\Longleftrightarrow \quad z \leq \theta(x_0) \end{aligned}$$

Thus, $\theta(x_0)$ is the greatest lower bound of the set $\theta(X)$ in Q. However, since $X \subseteq Q$ by assumption, we know that $\theta(X) = X$. Consequently, we know

$$\bigwedge_Q \theta(X) = \bigwedge_Q X = \theta(x_0)$$

It only remains to prove that $\theta(x_0) = x_0$. Since θ is a closure operator, we know at once that $x_0 \leq_P \theta(x_0)$. However, since $\theta(x_0)$ is itself a lower bound in P for the set X, it follows that $\theta(x_0) \leq_P x_0$ as well.

We can summarize the previous discussion in the following way: If $\mathbf{P}$ is a poset and $\theta : P \longrightarrow P$ is a closure operator, then the set $Q = \theta(P)$ is *completely meet faithful* in P; that is, for all $X \subseteq Q$, we have

1. $\bigwedge_P X$ exists $\iff$ $\bigwedge_Q X$ exists, and
2. $\bigwedge_P X = \bigwedge_Q X$

Let $\mathbf{P}$ be a poset and suppose $\theta : P \longrightarrow P$ is a kernel operator. We can claim that $Q = \theta(P)$ is *completely join faithful* in P; that is, for all $X \subseteq Q$, we have

1. $\bigvee_P X$ exists $\iff$ $\bigvee_Q X$ exists, and

2. $\bigvee_P X = \bigvee_Q X$

Suppose that $\theta : P \longrightarrow P$ is a kernel operator and let $Q = \theta(P)$. The fact that the inclusion map $g : Q \longrightarrow P$ is the left adjoint to θ tells us that whenever $X \subseteq Q$ is such that $\bigvee_Q X$ exists, then

$$g(\bigvee_Q X) = \bigvee_P \{g(x) : x \in X\} = \bigvee_P X$$

The converse of this statement is also true; that is, whenever $X \subseteq Q$ is such that $\bigvee_P X$ exists, then $\bigvee_Q X$ also exists and $\bigvee_Q X = \bigvee_P X$.

To see why this is so, let $x_0 = \bigvee_P X$. Suppose $z \in Q$ is such that $\theta(x) \leq z$ for all $x \in X$. (Note that such z exist by assumption, since $\theta(x_0)$ has this property.) Then, the fact that $(g, \theta) : Q \leftrightharpoons P$ tells us that

$$\begin{aligned}
\theta(x) \leq z \quad &\Longleftrightarrow \quad x \leq z = g(z)(\forall x \in X) \\
&\Longleftrightarrow \quad x \leq z = g(z) \\
&\Longleftrightarrow \quad \theta(x_0) \leq z
\end{aligned}$$

Thus, $\theta(x_0)$ is the greatest lower bound of the set $\theta(X)$ in Q. However, since $X \subseteq Q$ by assumption, we know that $\theta(X) = X$. Consequently, we know

$$\bigvee_Q \theta(X) = \bigvee_Q X = \theta(x_0)$$

It only remains to prove that $\theta(x_0) = x_0$. Since θ is a kernel operator, we know at once that $\theta(x_0) \leq_P x_0$. However, since $\theta(x_0)$ is itself an upper bound in P for the set X, it follows that $x_0 \leq_P \theta(x_0)$ as well.

Definition 2.52. Let $\mathbf{P}$ be a poset and let $X \subseteq P$ be a subposet of $\mathbf{P}$. We say that X is a *closure retract* of $\mathbf{P}$ if, for all $p \in P$, the set $U(p) = \{x \in X : p \leq x\}$ has a least element. We say that X is a *kernel retract* of $\mathbf{P}$ if X^{op} is a closure retract in $\mathbf{P}^{\mathrm{op}}$. Members of a closure retract are often called *closed* elements; those of kernel retracts are sometimes called *open* elements.

Let $\mathbf{P}$ be a poset. The lattice $\mathtt{Low}(\mathbf{P})$ of lowersets of $\mathbf{P}$ is a closure retract of the powerset of P. To see why, suppose $S \in \mathtt{Su}(P)$ and consider the set $U(S) = \{\downarrow T \in \mathtt{Low}(P) : S \subseteq \downarrow T\}$. We observe that $\downarrow S \in U(S)$, and that it is the *least* such element of $\mathtt{Low}(P)$ with this property for any $S \in \mathtt{Su}(P)$. It follows from here that $\mathtt{Low}(P)$ is a closure retract of $\mathtt{Su}(P)$.

Of course, we are left with the question, "How are closure operators and closure retracts related?" First, we make a proposition:

Proposition 2.53. *Let* $\mathbf{P}$ *be a poset and suppose that* $\varphi : P \longrightarrow P$ *is a closure operator of* $\mathbf{P}$. *The set* $\varphi(P)$ *is a closure retract of* $\mathbf{P}$.

Proof. Since φ is a closure operator, it is enlarging. That is, $p \leq \varphi(p)$ for all $p \in P$. We know φ is also isotone, so for any p, the set $U(p) = \{\varphi(x) \in \varphi(P) : p \leq \varphi(x)\}$ has a least element; namely $\varphi(p)$. It is worth noting that this implies $\varphi(p) = \bigwedge U(p)$. Therefore $\varphi(P)$ is a closure retract on P. $\square$

Suppose $\mathbf{P}$ be a poset and suppose that $X \subseteq P$ is a closure retract. Define a mapping $\varphi_X : P \longrightarrow P$ by $\varphi_X(p) = \bigwedge U(p)$ and let $\mathtt{Ran}(\varphi)$ denote the range of φ. By definition, $\mathtt{Ran}(\varphi) \subseteq X$, and for any $p \in X$, we have $\varphi(p) = \bigwedge\{x \in X : p \leq x\} = p$, hence $p \in \mathtt{Ran}(\varphi)$ and equality follows from here.

It is worth noting that the previous sentence demonstrates that if $p \in X$, then $\varphi(p) = p$. This means that if $q \in P$ such that $\varphi(q) = p$, we have $\varphi(\varphi(q)) = \varphi(p) = p$. Furthermore, if $q_1 \leq q_2 \in P$, we must have $\bigwedge U(q_1) \leq \bigwedge U(q_2) \in X$, hence φ is isotone. We observe the definition of $U(p)$ guarantees that φ is enlarging, hence

φ is a closure operator. Proposition 2.53 demonstrated that $\varphi(p) = \varphi_X(p)$. This implies $\varphi(P) = X$, hence $\varphi = \varphi_X$. This tells us that the closure retracts of a poset $\mathbf{P}$ are precisely the images of $\mathbf{P}$ under its closure operators; moreover, the closure retracts of $\mathbf{P}$ uniquely determine the closure operators on P and vice-versa.

2.6 Compact Generation

Definition 2.54. Let $\mathbf{P}$ be a DCPO. An element $c \in P$ is *compact* provided, whenever $D \subseteq P$ is directed and such that $x \leq \bigvee D$, then $c \leq d$ for some $d \in D$. Compact elements are sometimes called *isolated* or *finite* elements. We will let $\mathtt{Com}(\mathbf{P})$ represent the subposet of compact elements from $\mathbf{P}$.

Observe that we can characterize compact elements by the notion of being inaccessible by directed joins. For each $x \in P$ for a poset $\mathcal{P} = (P, \leq)$, we shall denote the set $K_x = {\downarrow}x \cap \mathtt{Com}(P)$ as the set of all compact elements of P less than or equal to x; note that K_x is necessarily directed.

It is worth noting that if $\mathbf{L}$ is a complete lattice, then $\mathtt{Com}(\mathbf{L})$ is a join subsemilattice of L. Suppose $x, y \in \mathtt{Com}(L)$ such that $x \mid\mid y$ and suppose D is a directed subset of L such that $\bigvee D = x \vee y$. If we consider the sets

$$D_x = \{x \wedge d : d \in D\}$$

and

$$D_y = \{y \wedge d : d \in D\}$$

we observe that D_x and D_y are directed. Since L is complete, $\bigvee D_x = l \in L$ exists. Since x is an upper bound for L, we have $l \leq x$. If $l < x$, then $x \wedge d < x$ for every $d \in D$. But then $\bigvee D \leq x$, contrary to the hypothesis that $\bigvee D = x \vee y$. Since x is compact, we have $x \in D_x$. A similar argument shows us that $y \in D_y$. This implies that there exist $d_x, d_y \in D$ such that $x \leq d_x$ and $y \leq d_y$. It follows that $x \vee y \leq d_x \vee d_y$, and since $\bigvee D = x \vee y$, we must have $d_x \vee d_y = x \vee y$. Since D is directed, we know that $\{d_x, d_y\}$ has an upper bound in D, which implies $x \vee y \in D$, hence $x \vee y$ is compact.

Since the main result of this paper pertains to characterizing certain classes of complete lattices, we make the following observations. Suppose L is a join-continuous lattice. If element $c \in L$ is compact and D is directed, then $c \leq \bigvee D$ always implies $c \leq d$ for some $d \in D$. Furthermore, if $c \leq \bigvee S$ for any $S \subseteq L$, then $c \leq \bigvee T$ for some finite $T \subseteq S$. To see why, let $S \subseteq L$ such that $c \leq \bigvee S$. We observe that $\texttt{Fin}(S)$ is always directed. This means that we can extract a finite $F_s \in \texttt{Fin}(S)$ such that $c \leq \bigvee F_s$, which implies

$$
\begin{aligned}
c &= c \wedge \bigvee F_s \\
&= \bigvee \{c \wedge s_i : s_i \in F_s\}
\end{aligned}
$$

Since $c = (c \wedge s_1) \vee (c \wedge s_2) \vee \cdots \vee (c \wedge s_n)$, for some finite $F_s = \{s_1, s_2, \ldots, s_n\}$, it follows that $c \leq s_i$ for some $s_i \in F_s \subseteq S$. If we restrict our focus to join-continuous lattice we find an element c is compact in a lattice L if and only if whenever $X \subseteq L$ is such that $c \leq \bigvee X$, then there exist finite $F \subseteq X$ such that $c \leq \bigvee F$.

Observe that the converse is not necessarily true if P is not join-continuous. Consider $\mathcal{Q} = \mathbb{N} \cup \{\omega\}$ where $z < \omega$ for every $n \in \mathbb{N}$. Let B be the four-element Boolean lattice with top element $\top$, bottom element 0. Construct P by letting

- $0 < 1$
- $\top < \omega$
- $\top \mid\mid n$ for every $n \in \mathbb{N}$

We observe that $\top$ is compact, but $\top$ fails the given conditions.

Definition 2.55. A poset $\mathbf{P}$ is said to be *compactly generated* provided it is a DCPO and every element of $\mathbf{P}$ is the join of a directed family of compact elements in $\mathbf{P}$. Compactly generated posets are often called *algebraic* posets.

To illustrate compact generation, let X be any set. It is routine to show that $(\mathtt{Su}(X), \subseteq)$ is compactly generated with $\mathtt{Com}(\mathtt{Su}(X)) = \mathtt{Fin}(X)$. Indeed, we observe that any finite set is compact since they are inaccessable by directed joins. That is, $\mathtt{Fin}(X) \subseteq \mathtt{Com}(\mathtt{Su}(X))$. On the other hand, we observe that any infinite set *is* accessible by directed joins (i.e. the union of its finite subsets), hence $\mathtt{Fin}(X) = \mathtt{Com}(\mathtt{Su}(X))$. For any $Y \in \mathtt{Su}(X)$, we let $\mathcal{F} = \{F \in \mathtt{Fin}(X) : F \subseteq Y\}$ and observe that $\mathcal{F}$ is directed with $\bigvee \mathcal{F} = \bigcup \mathcal{F} = Y$, hence $\mathtt{Su}(X)$ is compactly generated.

Similarly, if $\mathbf{P}$ is any poset, we can let $\mathcal{F} = \{\downarrow F : F \in \mathtt{Fin}(P)\}$. We note that any finitely generated lowerset is inaccessible by directed joins in $\mathtt{Low}(P)$, hence $\mathcal{F} \subseteq \mathtt{Com}(\mathtt{Low}(P))$. On the other hand, if $S \in \mathtt{Su}(P)$ is infinite, we note that $\downarrow S = \bigcup\{\downarrow F : F \in \mathtt{Fin}(P) \cap S\}$. This tells us that $\mathcal{F} = \mathtt{Com}(\mathtt{Low}(P))$, and demonstrates that $\mathtt{Low}(P)$ is compactly generated. We have thus proved the following result:

Lemma 2.56. *Let $\mathbf{P}$ be any poset. The lattice $(\mathtt{Low}(\mathbf{P}), \subseteq)$ is compactly generated with $\mathtt{Com}(\mathtt{Low}(\mathbf{P})) = \{\downarrow F : F \in \mathtt{Fin}(P)\}$.*

Before we leave this section, we make the following observation regarding algebraic distributive lattices and frames:

Lemma 2.57. *Every lower-bounded, algebraic distributive lattice is join continuous (and thus a frame).*

Proof. Let L be a compactly generated distributive lattice and let $x, y \in L$. We know there exists a directed $F_y \subseteq \mathtt{Com}(L)$ such that $y = \bigvee F_y$, and since L is complete, we know $\bigvee\{x \wedge f : f \in F_y\} = z$ exists. We have $z \leq x \wedge y$ since $\bigvee\{x \wedge f : f \in F_y\} \leq x \wedge \bigvee F_y$ is always true.

Let $c \in F_{x\wedge y}$ where $F_{x\wedge y} \subseteq \texttt{Com}(P)$ is such that $\bigvee F_{x\wedge y} = x \wedge y$. This implies that $c \in F_x \cap F_y$ (where F_x is similarly defined), which in turn implies $c \leq f_0$ for some $f_0 \in \{x \wedge f : f \in F_y\}$. This means that z is an upper bound for $F_{x\wedge y}$, hence

$$x \wedge y = x \wedge \bigvee F_y = \bigvee\{x \wedge f : f \in F_y\} = z$$

and L is join continuous. $\square$

In light of Lemma 2.57, we have the following characterization of compact elements as a corollary:

Corollary 2.58. *If* $\mathbf{L}$ *is a lower-bounded, algebraic distributive lattice, and* $c \in L$*, then the following statements are equivalent:*

1. *The element* c *is compact in* $\mathbf{L}$*.*
2. *Whenever* $X \subseteq L$ *is such that* $c \leq \bigvee X$*, then there exist finite* $F \subseteq X$ *such that* $c \leq \bigvee F$*.*

Before we leave this section, we provide a remark regarding *meets* of compact elements. We cannot assume that the meet of two compact elements is compact, even in a complete lattice. Consider the lattice $\mathcal{L} = (\mathbb{N} \cup \{\omega, u, v, \top\}, \leq)$, where

1. We have $u, v \notin \mathbb{N}$.
2. The partial order $\leq$ is the natural order on $\mathbb{N}$.
3. The element ω is the first transfinite ordinal.
4. We have $\omega < u$ and $\omega < v$ and $\omega = u \wedge v$.
5. The element $\top$ satisfies $l \leq \top$ for all $l \in L$.

First, we note that $\bigvee \mathbb{N} = \omega$ while $\omega \notin \mathbb{N}$, so ω is not compact. We also observe that both u and v are both maximal in $\mathcal{P}$, and we have $\omega \prec u$ and $\omega \prec v$. Consequently, for any $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}$ such that $\bigvee \mathcal{S} = u$ and $\bigvee \mathcal{T} = v$, we must have $\omega, u \in \mathcal{S}$ and $\omega, v \in \mathcal{T}$, hence both u and v are compact.

2.7 Irreducible Elements in Lattices

Representing a given structure (such a a group, ring, lattice, etc.) in terms of a "canonical" set of elements under a specific operation is a natural problem that arises in the study of algebra. Usually this canonical set consists of those elements which are "irreducible" with regard to the specified operation. An elementary example would be the representation of positive integers as products of primes (which are irreducible with regard to multiplication).

Definition 2.59. Let $\mathbf{L}$ be a meet semilattice. An element $p \in L$ is *meet irreducible* if, for all $F \in \mathtt{Fin}(L)$, $p = \bigwedge F$ always implies $p = f$ for some $f \in F$. An element j of a join semilattice $\mathbf{L}$ is *join irreducible* provided it is meet irreducible in $\mathbf{L}^{\mathrm{op}}$.

Note that the greatest element of a meet semilattice (if it exists) cannot be meet irreducible. The concepts of *completely* meet irreducible and *completely* join irreducible elements can be defined in a (complete) meet or join semilattice by removing the restriction that the set F be finite.

Let L be a lower-bounded, compactly generated lattice and let $a, b \in L$ be such that $a \not\leq b$. We will first verify that $K_a - K_b$ is nonempty. Since L is compactly generated, there exist $F_a, F_b \subseteq \mathtt{Com}(P)$ such that $\bigvee F_a = a$ and $\bigvee F_b = b$. If $b < a$, it follows that $F_b \subset F_a$, and if $a \mid\mid b$, there exist $c_a \in F_a - F_b$ and $c_b \in F_b - F_a$. In either case, there exists $c_a \in F_a - F_b$, hence $K_a - K_b$ is nonempty.

Let $c \in K_a - K_b$ and let $X = \{x \in L : b \in {\downarrow}x$ and $c \notin {\downarrow}x\}$. We may assume $\bigvee X = q$ and let $F_q \subseteq \mathtt{Com}(X)$ be a directed subset such that $\bigvee X = \bigvee F_q$. Clearly, $c \notin \mathtt{Com}(X)$, hence $q \in X$, and q is maximal. Suppose $q = \bigvee X = \bigwedge S$ for some $S \subseteq L$. By the definition of X and the maximality of q, we have $q < l \in L$ implies $c, b \in\downarrow l$. Clearly, this means $b \in\downarrow s$ for every $s \in S$. Since $\bigwedge = q$ and $c \notin K_q$, there must be some $s_q \in S$ such that $c \notin K_s$. This would imply $s_q \in X$; this means $s_q = q$ and q is completely meet irreducible. We have proved the following claim:

Claim 2.60. *Let L be a compactly generated lattice and let $a, b \in L$ be such that $a \not\leq b$.*

1. *We have $K_a - K_b$ is nonempty.*
2. *Let $c \in K_a - K_b$ and let $X = \{x \in L : b \in\downarrow x$ and $c \notin\downarrow x\}$. Then X has a maximal member.*
3. *If q is a maximal member of X then q must be completely meet irreducible.*

We now prove a well known result:

Theorem 2.61. *Let $\mathbf{L}$ be a complete lattice, let $a \in L$, and let $Q \subseteq L$.*

1. *Suppose that, for all $b \in L$ such that $a \not\leq b$, there exist $q \in Q$ such that $a \leq q$ and $q \not\leq b$. We have $a = \bigwedge Q$.*
2. *Every element of a lower-bounded, compactly generated lattice is the meet of a set of completely meet irreducible elements.*

Proof. To prove (1), we let $Q = \{q \in L : a \leq q$ and $q \not\leq b\}$. From here we observe that, by the way we define Q, we have a is a lower bound of Q and $a \in Q$, hence $\bigwedge Q = a$. For (2), suppose $l \in L$ and let $M \subseteq L$ such that

$$M = \{m \in L : l \leq m \text{ and } m \text{ is completely meet irreducible}\}$$

Since L is complete, $\bigwedge M = m_l$ exists and $l \leq m_l$. If we suppose $l < m_l$, we know there exist $F_l \subset F_{m_l} \subseteq \texttt{Com}(L)$ such that $l = \bigvee F_l < \bigvee F_{m_l} = m_l$. This implies that there exists $c \in F_{m_l} - F_l$, and we therefore know the set $X_l = \{x \in L : l \in \downarrow x \text{ and } c \notin \downarrow x\}$ is nonempty and $\bigvee X_l = q_l$ is completely meet irreducible. By the way we defined M, we have $q_l \in M$, hence $m_l \leq q_l$. But this implies $c \in \downarrow q_l$; a contradiction. We may therefore conclude $l = \bigwedge M = m_l$ as desired. $\square$

Part (2) of the last theorem is a famous result due to Garrett Birkhoff. It will have important implications in much of our later work.

Definition 2.62. Let $\mathbf{L}$ be a meet semilattice. An element $p \in L$ is *meet prime* if, for all $F \in \texttt{Fin}(L)$, $p \geq \bigwedge F$ always implies $p \geq f$ for some $f \in F$. An element j of a join semilattice $\mathbf{L}$ is *join prime* provided it is meet prime in $\mathbf{L}^{\text{op}}$.

The concepts of *completely* meet prime and *completely* join prime elements can be defined in a (complete) meet or join semilattice by removing the restriction that the set F be finite. It should be noted that every meet prime element is meet irreducible. To see why, let L be a meet semilattice and suppose $m \in L$ is meet prime. For any $F \in \texttt{Fin}(L)$, we know that there exists $f \in F$ such that $f \leq m$ whenever $\bigwedge F \leq m$. This means that if $m = \bigwedge F$ we have $m = f$; otherwise m fails to be a lower bound. Of course, if we remove the condition that F must be finite we can extend this notion to completely meet irreducible and completely meet prime elements.

This of course raises the question whether meet irreducible and meet prime elements are equivalent. It turns out that a meet irreducible element in a lattice need not be meet prime and that a meet irreducible element in a complete lattice need not be completely meet irreducible or meet prime. In the finite case, consider $\mathcal{M}_5 = \{\bot, a, b, c, \top\}$, i.e. non-distributive diamond. We observe that $x \in \{a, b, c\}$ is meet irreducible, since they are each covered only by $\top$ and therefore must be in any $F \subseteq \mathcal{M}_5$ such that $\bigwedge F = x$. However, we note that for any such x, we have $y, z \in F = \{a, b, c\} - \{x\}$ where $\bigwedge F = y \wedge z = \bot < x$, hence any such x is not meet prime.

We can extend this argument to an infinite complete example as follows: let $\mathcal{M}_{5^\infty} = \{\bot, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \top\}$ where $\mathcal{Q}_i = \{\mathbb{N}_i \cup \{\omega_i\}\}$. Further suppose that $\mathbb{N}_i$ is ordered under the usual ordering, with ω_i the appropriate first transfinite element for each copy of $\mathbb{N}$. Finally, suppose that $\omega_i \prec \top$ and $\bot \prec 1_i$ for each i and $q_i \mid\mid q_j$ when $i \neq j$ for every appropriate $q \in \mathcal{Q}$. We observe that this and its dual form a complete, non-distributive modular lattice. We observe that any finite $F \subseteq \mathcal{M}_{5^\infty}$ such that $\bigvee F = \omega_i$, we must have $\omega_i \in F$, hence ω_i is join irreducible. However, we know that $\bigvee \mathbb{N}_i = \omega_i$, and $\omega_i \notin \mathbb{N}_i$, hence ω_i is not completely join irreducible.

We also observe that for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, we have $\omega_i \vee \omega_j = \omega_i \vee \omega_k = \omega_j \vee \omega_k = \top$, and since each of these elements are incomparable we note that each are not join prime. We have shown that each ω is join irreducible, but not completely join irreducible nor join prime. Our claim comes from observing these elements in $\mathcal{M}_{5^\infty}^{op}$.

Observe that the examples given above are both non-distributive. Since our work in upcoming chapters deals with meet-irreducible elements in distributive lattices, it will be useful to understand precisely when these elements coincide with meet prime elements.

Lemma 2.63. *In a distributive lattice, every meet irreducible element is meet prime.*

Proof. Let L be any lattice and let $x \in L$ be meet irreducible but not meet prime. This means that there exists $y, z \in L$ such that $y \wedge z = \alpha < x$, but $y \nleq x$ and $z \nleq x$ (this also implies $y \mid\mid z$). Since L is a lattice, there exists $\bigvee\{x, y, z\} = \omega \in L$, which is strictly larger than each of the three elements. We know that x is not strictly less than both, so x is incomparable to at least one of the two elements. If $x < y$ and $x \mid\mid z$, we observe $\{\alpha, x, y, z, \omega\} \cong \mathcal{N}_5$. Similarly, if $x \mid\mid y$ and $x \mid\mid z$, then $\{\alpha, x, y, z, \omega\} \cong \mathcal{M}_5$. This shows that if an element is meet irreducible but not meet prime in a lattice L, then L either contains the nondistributive diamond or pentagon as a sublattice. Our claim follows by contraposition. $\square$

It is worth noting that even in a complete distributive lattice, a completely meet irreducible element need not be completely meet prime. We observe that the open set lattice of the usual space on the real line $(\mathbb{R}, \Omega_U)$ is a complete, distributive lattice (indeed, a frame); ordered under set inclusion. Observe that $A = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$ is completely meet irreducible. If we let $\mathcal{F} = \left\{(-\frac{1}{i}, \frac{1}{i}) : i \in I\right\}$ where the index I is infinite, we observe that $\bigcap \mathcal{F} = \{0\}$. Since singletons aren't open under Ω_U, it follows $\bigwedge \mathcal{F} = \emptyset$. This means $\bigwedge \mathcal{F} \subseteq A$. However, we have $0 \in f_i$ for every $f_i \in \mathcal{F}$ hence A is *not* completely meet prime.

Lemma 2.64. *Let* $\mathbf{L}$ *be a complete lattice. If* $j \in L$ *is completely join prime, then* $\downarrow j$ *is co-atomic; in particular,* j *is compact in* $\mathbf{L}$*.*

Proof. Let $\mathcal{F} = \downarrow j - \{j\}$. We observe that if there exists $\top_{\mathcal{F}} \in (\mathcal{F}, \leq)$, then $\mathcal{F} = \downarrow \top_{\mathcal{F}}$ and $\top_{\mathcal{F}} \prec j$. We proceed under the assumption that no such element exists. Suppose $\mathcal{C} \subseteq \mathcal{F}$ is a chain. Since j is join-prime, we observe that since $f < j$ for every $f \in \mathcal{F}$, we know j is an upper bound for every chain $\mathcal{C}$. This means $\bigvee C < j$, hence $\bigvee \mathcal{C} \in \mathcal{F}$. That is, every chain $\mathcal{F}$ has a proper upper bound in $\mathcal{F}$, which means every chain has a maximal upper bound in $\mathcal{F}$ by Zorn's Lemma. If $\mathcal{M} \subseteq \mathcal{F}$ is the set of all such maximal elements, it follows by the way $\mathcal{F}$ is defined that $m \prec j$ for all $m \in \mathcal{M}$. This implies j is inaccessible by directed joins; that is, j is compact. $\square$

Theorem 2.65. *Let* $\mathbf{L}$ *be a lower-bounded, compactly generated, distributive lattice. The following statements are equivalent:*

1. *The lattice* $\mathbf{L}^{op}$ *is algebraic (lower-bounded and compactly generated);*

2. *Every completely meet irreducible element of* $\mathbf{L}$ *is completely meet prime;*

3. *Every element of* $\mathbf{L}$ *is the join of a set of completely join prime elements;*

4. *The lattice* $\mathbf{L}$ *is order isomorphic to* $\mathtt{Low}(\mathbf{P})$ *for some poset* $\mathbf{P}$.

Proof. $1 \Longrightarrow 2$ Suppose L is bialgebraic and let $x \in L$ be completely meet irreducible such that $x = \bigwedge K_x$ for $K_x \subseteq \mathtt{Com}(L^{op})$. This means $x = k_x$ for some $k_x \in K_x$. Suppose $\bigwedge S = s \leq x$ for some $S \subseteq L$. It follows that $s = \bigwedge K_s$ for some $K_s \subseteq \mathtt{Com}(L^{op})$. It is worth noting that this means $K_x \subseteq K_s$, which implies there exists $k_s \in K_s$ such that $k_s \leq x$, which in turn implies that there exists $s_k \in S$ such that $k_s \in K_{s_k}$. This means $s_k \leq x$, hence x is meet prime.

Observe that by duality we have every join irreducible element in L is meet irreducible and therefore meet prime in L^{op}. This implies that every join irreducible element is also join prime in L, and vice-versa.

$2 \Longrightarrow 1$ Suppose L is an algebraic, distributive lattice that satisfies (2). By Birkhoff's Theorem, for every $x \in L$ we have $x = \bigwedge M$ where M is the set of all meet prime elements satisfying $M \subseteq \uparrow x$. It is worth observing that M is directed in L^{op}. Since every m_x is completely meet prime in L, it is completely join prime in L^{op}. Consequently, if D is directed in L^{op} where $\bigwedge D = \bigvee_{op} D = m_x$, we must have $m_x = d$ for some $d \in D$. This means that every m_x is compact with respect to L^{op}, hence L^{op} is algebraic.

$1 \Longleftrightarrow 2 \Longleftrightarrow 3$ Since L^{op} is algebraic, Birkhoff's Theorem guarantees every $x \in L^{op}$ we have $x = \bigwedge_{op} M$ where M is the set of all meet prime elements satisfying $M \subseteq \uparrow_{op} x$. Our claim follows by observing the order duals of these sets in L. Consequently, if there exists an element of L that is *not* the join of a set of completely join prime elements, L^{op} is not algebraic, so (1), (2), and (3) are equivalent.

$3 \iff 4$ Let P be the set of all completely join prime elements in L and define $\varphi : L \longrightarrow \texttt{Low}(P)$ as the map $\varphi(x) = \downarrow x \cap P$. Since $x_1 \leq x_2$ if and only if $\downarrow x_1 \cap P \subseteq \downarrow x_2 \cap P$, we know φ is isotone. If we define $\psi : \texttt{Low}(P) \longrightarrow L$ as the map $\psi(\downarrow X) = \bigvee \downarrow X$, we note that $x = \bigvee \downarrow X \leq \bigvee \downarrow Y = y$ if and only if $\downarrow X \subseteq \downarrow Y$, hence ψ is isotone. If we let $S \subseteq P$ such that $\bigvee S = x$, we observe

$$\psi(\varphi(x)) = \psi(\downarrow x \cap P) = \psi(\downarrow S) = \bigvee S = x$$

and

$$\varphi(\psi(\downarrow S)) = \varphi(\bigvee S) = \varphi(x) = \downarrow x \cap P = \downarrow S$$

This tells us that φ and ψ are mutually inverse, and therefore $L \cong \texttt{Low}(P)$.

Incidentally, if (3) does *not* hold, then φ and ψ are not mutually inverse, so we may therefore conclude that (3) and (4) are equivalent.

□

We will say that a lattice $\mathbf{L}$ is *bicompactly generated* provided both $\mathbf{L}$ and $\mathbf{L}^{\text{op}}$ are lower-bounded, compactly generated posets. (In keeping with the fact that compactly generated posets are often called algebraic posets, such lattices are often said to be *bialgebraic*.) We observe that Theorem 2.65 guarantees that bialgebraic lattices are *precisely* the ones where completely meet-irreducible elements and completely meet prime elements coincide.

CHAPTER 3

BASIC CATEGORY THEORY AND DUAL EQUIVALENCE

This chapter is intended to be a very brief primer on the basics of category theory. Our focus is to introduce the basics concepts necessary to establish our intended duality. Since we do not need much more than the basics, and the remainder is not germane to this paper, this chapter will be relatively brief. The material from this chapter can also be found in Mac Lane [20], as this text remains the authority in all things categorical.

3.1 Category Theory

Category theory is a relatively new branch of mathematics that has far reaching consequences. It arose in the early- to mid-twentieth century, largely out of trends in algebraic topology and the study of homology and homotopy group theory. One broad notion that emerged is that seemingly unrelated mathematical objects (such as topological surfaces and group theory) have surprising relationships, largely through the morphisms defined on those objects. Category theory has developed as a foundational tool to establish these relationships.

Definition 3.66. Suppose we have two collections (not necessarily sets or functions) $\mathtt{Ob} = \{a, b, c, \dots\}$ and $\mathtt{Ar} = \{f, g, h, \dots\}$. We say $\mathcal{M} = (\mathtt{Ob}, \mathtt{Ar}, \mathtt{dom}(), \mathtt{cod}())$ is a *metagraph* provided

- For every $a \in \mathtt{Ar}$, there exists $o_d, o_c \in \mathtt{Ob}$ such that $\mathtt{dom}(a) = o_d$ and $\mathtt{cod}(a) = o_c$. We shall say $\mathtt{dom}(a)$ is the *domain* of a and $\mathtt{cod}(a)$ is the *codomain* of a.
- We may denote the above property as $a : o_d \longrightarrow o_c$ or as the following diagram:

$$o_d \xrightarrow{a} o_c$$

We say $\mathcal{M}$ is the metagraph on the collection of *objects* $\mathtt{Ob}$ and the collection of *arrows* (or *morphisms*) $\mathtt{Ar}$.

If we suppose that $\mathcal{M} = (\mathtt{Ob}, \mathtt{Ar}, \mathtt{dom}(), \mathtt{cod}())$ is a metagraph and, for every object $o \in \mathtt{Ob}$, there exists an arrow $a_o \in \mathtt{Ar}$ such that $\mathtt{dom}(a_o) = o = \mathtt{cod}(a_o)$, we say o has identity a_o and designate $a_o = \mathtt{Id}_o$. If we suppose that for every $f, g \in \mathtt{Ar}$ such that $\mathtt{dom}(g) = \mathtt{cod}(g)$, we have a map $h \in \mathtt{Ar}$ where

- We have $\mathtt{dom}(h) = \mathtt{dom}(f)$.
- We have $\mathtt{cod}(h) = \mathtt{cod}(g)$.
- The diagram below commutes:

$$\begin{tikzcd} \mathtt{dom}(f) \arrow[r, "h"] \arrow[d, "f"'] & \mathtt{cod}(g) \\ \mathtt{dom}(g) = \mathtt{cod}(f) \arrow[ur, "g"'] & \end{tikzcd}$$

We say h is the *composition* of g and f and denote this map $h = g \circ f$. We say a metagraph $\mathcal{M}$ is a *metacategory* provided $\mathtt{Ar}$ is closed under composition, for every $o \in \mathtt{Ob}$, there exists $\mathtt{Id}_o \in \mathtt{Ar}$, and the following axioms are satisfied:

Associativity of Composition If $A, B, C, D \in \mathtt{Ob}$ with arrows $f, g, h \in \mathtt{Ar}$ such that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we also have $h \circ (g \circ f) = (h \circ g) \circ f$.

Existence of Unit If $f : A \longrightarrow B$ and $g : B \longrightarrow C$, and $\mathtt{Id}_B : B \longrightarrow B$ is the identity map on B, then $\mathtt{Id}_B \circ f = f$ and $g \circ \mathtt{Id}_B = g$. This relationship can be illustrated as the following diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
B \arrow[ur, "\mathtt{Id}_B"] \arrow[r, "g"] & C
\end{tikzcd}$$

To demonstrate the utility of the notion of a metacategory, an interested reader could easily verify that if we define $\mathtt{Ob}_S$ as the collection of all sets and $\mathtt{Ar}_S$ as the collection of all functions between sets, the quadruple $\mathcal{S} = (\mathtt{Ob}_S, \mathtt{Ar}_S, \mathtt{dom}(), \mathtt{cod}())$ forms a metacategory. We recall that the notion of a "set of all sets" leads to a logical paradox (namely Cantor's paradox). This tells us that $\mathtt{Ob}_S$ and $\mathtt{Ar}_S$, while collections of objects, are not sets and cannot be examined via the axioms of set theory. However, our focus will be directed on constructions that are consistent with our notions of set theory (these constructions are sometimes referred to as being *concrete*).

We say a *directed graph* is a set $\mathcal{G} = (\mathtt{Ob} \cup \mathtt{Ar})$ where $\mathtt{Ob}$ is a set of objects, and $\mathtt{Ar}$ is a set of morphisms, with two functions $\mathtt{dom} : \mathtt{Ar} \longrightarrow \mathtt{Ob}$ and $\mathtt{cod} : \mathtt{Ar} \longrightarrow \mathtt{Ob}$ that assigns each arrow a domain and a codomain. If $A, B \in \mathtt{Ob}$, we denote the class of all morphisms from A to B to be the collection

$$\hom(A, B) = \{\varphi : A \longrightarrow B\} \text{ where } \hom(A, B) \text{ is a subclass of } \mathtt{Ar}$$

In this graph, the vertices are the objects and the edges are the arrows.

In a directed graph $\mathtt{G} = (\mathtt{Ob} \cup \mathtt{Ar})$, we say the *product over* $\mathtt{Ob}$ is the set $\Pi_{\mathtt{Ob}} \subseteq \mathtt{Ar} \times \mathtt{Ar}$ such that

$$\Pi_{\mathtt{Ob}} = \{(g, f) : \mathtt{cod}(f) = \mathtt{dom}(g)\}$$

We say a directed graph is a *category* provided there exists an embedding $\mathtt{Id} : \mathtt{Ob} \longrightarrow \mathtt{Ar}$ where $\mathtt{Id}(A) = \mathtt{Id}_A$ and an embedding $\circ : \Pi_{\mathtt{Ob}} \longrightarrow \mathtt{Ar}$ where $\circ((g, f)) =$

$g \circ f$, provided $\mathtt{Id}_A$ and $g \circ f$ define the identity and composition arrows (respectively) and $\mathtt{G}$ satisfies the axioms of metacategories previously defined.

3.2 Functors, Natural Transformations, and Categorical Equivalence versus Dual Equivalence

We will switch now our focus to comparing categories. Recall from algebra that the notion of homomorphism in either a group or a ring is a way to compare the structural relation between two of these objects. In categories, we would like a similar way of comparing two categories; this immediately implies preserving relationships between objects and arrows.

Definition 3.67. Let $\mathtt{C} = (\mathtt{Ob}_C \cup \mathtt{Ar}_C)$ and $\mathtt{D} = (\mathtt{Ob}_D \cup \mathtt{Ar}_D)$ be categories. A *functor* $\mathtt{F} \subseteq \mathtt{C} \times \mathtt{D}$ is a map

$$\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$$

such that $\mathtt{F}(\mathtt{Ob}_C) \subseteq \mathtt{Ob}_D$ and $F(\mathtt{Ar}_C) \subseteq \mathtt{Ar}_D$ that preserves arrows in the following manner:

- When $A, B \in \mathtt{Ob}_C$ and

  $$A \xrightarrow{f} B$$

  then we have

  $$\mathtt{F}(A) \xrightarrow{\mathtt{F}(f)} \mathtt{F}(B)$$

- When $A \in \mathtt{Ob}_C$ we have $\mathtt{F}(\mathtt{Id}_A) = \mathtt{Id}_{\mathtt{F}(A)}$.

- If $f, g \in \mathtt{Ar}$ we have $\mathtt{F}(g \circ f) = \mathtt{F}(g) \circ \mathtt{F}(f)$.

$$\begin{tikzcd} A \arrow[r, "g \circ f"] \arrow[d, "f"'] & C & & & \mathtt{F}(A) \arrow[r, "\mathtt{F}(g \circ f)"] \arrow[d, "\mathtt{F}(f)"'] & \mathtt{F}(C) \\ B \arrow[ur, "g"] & & \mathtt{C} \arrow[r, "\mathtt{F}"] & \mathtt{D} & \mathtt{F}(B) \arrow[ur, "\mathtt{F}(g)"] & \end{tikzcd}$$

Figure 1: Commuting diagrams for functors.

We shall now illustrate the previous definition. Suppose $\mathtt{C}$ is a category with three objects and three nonidentity arrows, and let $\mathtt{D}$ be another category. Then a functor $\mathtt{F} : \mathtt{C} \to \mathtt{D}$ could be represented by the following diagrams:

Functors also compose in the expected manner; that is if $\mathtt{C}, \mathtt{D}$, and $\mathtt{E}$ are categories and $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ and $\mathtt{G} : \mathtt{D} \longrightarrow \mathtt{E}$ are functors, then for every $A \in \mathtt{Ob}_C$ and $f \in \mathtt{Ar}_C$, we have $\mathtt{G}(\mathtt{F}(A)) \in \mathtt{Ob}_E$ and $\mathtt{G}(\mathtt{F}(f)) \in \mathtt{Ob}_E$. The diagram for composition of functors is identical (save for the names of objects and arrows) to that described for metacategories.

We will now focus on a few definitions describing certain classes of functors. These will mirror similar notions from algebra, such as injective and surjective homomorphims. Suppose $\mathtt{C}$ and $\mathtt{D}$ are categories and $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ is a functor between them. We say $\mathcal{F}$ is *full* provided for every $c_1, c_2 \in \mathtt{Ob}_C$ and $g : \mathtt{F}(c_1) \longrightarrow \mathtt{F}(c_2) \in \mathtt{Ar}_D$, there exists $f : c_1 \longrightarrow c_2 \in \mathtt{Ar}_C$ such that $\mathtt{F}(f) = g$. On the other hand, if for every parallel arrows $f_1 : c_1 \longrightarrow c_2$ and $f_2 : c_1 \longrightarrow c_2$ in $\mathtt{Ar}_C$ where $\mathtt{F}(f_1) = \mathtt{F}(f_2)$ we have $f_1 = f_2$, we say $\mathtt{F}$ is *faithful*. We say a full and faithful functor $\mathtt{F}$ is an *isomorphism* provided $\mathtt{F}$ is a bijection on both objects and arrows. It is worth noting that a categorical isomorphism exists if and only if there exist functors $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ and $\mathtt{G} : \mathtt{D} \longrightarrow \mathtt{C}$ where $\mathtt{G} \circ \mathtt{F} = \mathtt{Id}_\mathtt{C}$ and $\mathtt{F} \circ \mathtt{G} = \mathtt{Id}_\mathtt{D}$. In this case, we have an *equivalence of categories*.

$$
\begin{tikzcd}
A \arrow[r, "g \circ f"] \arrow[dd, "f"'] & C & & & \mathtt{F}(A) & \mathtt{F}(C) \arrow[l, "\mathtt{F}(g \circ f)"'] \arrow[ddl, "\mathtt{F}(g)"'] \\
& & \mathtt{C} \arrow[r, "\mathtt{F}"] & \mathtt{D} & & \\
B \arrow[uur, "g"] & & & & \mathtt{F}(B) \arrow[uu, "\mathtt{F}(f)"] &
\end{tikzcd}
$$

Figure 2: Commuting diagram for a contravariant functor. Note how the arrows are reversed in the codomain. Compare this with the commuting diagram for functors (Figure 1).

In category theory, there is another type of equivalence: that of the dual equivalence. The main result of this paper seeks to establish just such a relationship. Unlike an equivalence of categories, a dual equivalence establishes an equivalence between a category and the *dual* of another (that is, a category where all arrows are reversed). An interested reader would do well to consult Clark and Davey [6]; this is one of the most comprehensive texts on the subject of categorical dualities and where they are found.

Definition 3.68. Suppose $\mathtt{C}$ and $\mathtt{D}$ are categories and $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ is a functor. We say $\mathtt{F}$ is *contravariant* provided the following properties hold:

1. For any $A, B \in \mathtt{Ob}_C$ and any $f : A \longrightarrow B \in \mathtt{Ar}_C$, we have $\mathtt{F}(f) : \mathtt{F}(B) \longrightarrow \mathtt{F}(A) \in \mathtt{Ar}_D$.
2. We have $\mathtt{F}(\mathtt{Id}_A) = \mathtt{Id}_{\mathtt{F}(A)}$ for every $A \in \mathtt{Ob}_C$.
3. If $A, B, C \in \mathtt{Ob}_C$ and $f : A \longrightarrow B, g : B \longrightarrow C \in \mathtt{Ar}_C$, then $\mathtt{F}(g \circ f) = \mathtt{F}(f) \circ \mathtt{F}(g) : \mathtt{F}(C) \longrightarrow \mathtt{F}(A) \in \mathtt{Ar}_D$.

A contravariant functor, as defined above, satisfies the following diagrams:

$$\begin{tikzcd}
\mathtt{C} \arrow[dd, shift right, "\mathtt{F}"'] \arrow[dd, shift left, "\mathtt{G}"] & A \arrow[r, "f"] & B \\
 & \mathtt{F}(A) \arrow[r, "\mathtt{F}(f)"] \arrow[d, "\tau_A"'] & \mathtt{F}(B) \arrow[d, "\tau_B"'] \\
\mathtt{D} & \mathtt{G}(A) \arrow[r, "\mathtt{G}(f)"] & \mathtt{G}(B)
\end{tikzcd}$$

Figure 3: Commuting diagram for a natural transformation.

In a certain sense, a dual equivalence is almost like finding a "mirror-image" of a category rather than finding a (mathematically) identical construction. As such, we shall be interested in a finding a pair of contravariant functors $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ and $\mathtt{G} : \mathtt{D} \longrightarrow \mathtt{C}$ which are in a very general sense inverse to one another. However, in order to define precisely when this happens, we need a way to compare functors.

It is frequently said that category theory is all about the arrows. As such, we have seen that in a category, the arrows between objects determine the structure of a category. Similarly, we have seen that functors act as arrows between categories themselves. It should come as no surprise that there exist arrows between functors as well; these arrows are known as natural transformations.

Definition 3.69. Suppose $\mathtt{C}$ and $\mathtt{D}$ are categories and $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ and $\mathtt{G} : \mathtt{D} \longrightarrow \mathtt{C}$ are functors. A *natural transformation* is a function $\tau : \mathtt{F} \longrightarrow \mathtt{G}$ such that for every $\mathtt{Ob}_C$, there exists an arrow $\tau_A = \tau(A) : \mathtt{F}(A) \longrightarrow \mathtt{G}(A) \in \mathtt{Ar}_D$ where every arrow $f : A \longrightarrow B \in \mathtt{Ar}_C$ yields the following commuting diagrams:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "e_A"'] & B \arrow[d, "e_B"'] \\ \mathtt{G} \circ \mathtt{F}(A) \arrow[r, "\mathtt{G} \circ \mathtt{F}(\varphi)"] & \mathtt{G} \circ \mathtt{F}(B) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\epsilon_X"'] & Y \arrow[d, "\epsilon_Y"'] \\ \mathtt{F} \circ \mathtt{G}(X) \arrow[r, "\mathtt{F} \circ \mathtt{G}(\psi)"] & \mathtt{F} \circ \mathtt{G}(Y) \end{tikzcd}$$

Figure 4: Commuting diagram for Condition (1) of Definition 3.70. Observe that $e : \mathtt{Ob}_C \longrightarrow \mathtt{G} \circ \mathtt{F}(\mathtt{Ob}_C)$ and $\epsilon : \mathtt{Ob}_D \longrightarrow \mathtt{F} \circ \mathtt{G}(\mathtt{Ob}_D)$ form natural transformations.

The arrows $\tau_A, \tau_B \in \mathtt{Ar}_D$ are called the *components* the natural transformation τ; whenever this relationship holds we say the arrow τ_A is *natural* in A.

If every component of a natural transformation has a definable inverse $\tau_A^{-1} : \mathtt{G}(A) \longrightarrow \mathtt{F}(A)$, then τ is a *natural isomorphism*. A natural isomorphism is sometimes denoted $\tau : \mathtt{F} \cong \mathtt{G}$.

Definition 3.70. Suppose $\mathtt{C}$ and $\mathtt{D}$ are categories and $\mathtt{F} : \mathtt{C} \longrightarrow \mathtt{D}$ and $\mathtt{G} : \mathtt{D} \longrightarrow \mathtt{C}$ are a pair of contravariant functors with the property that, for $A \in \mathtt{Ob}_C$ and $B \in \mathtt{Ob}_C$ there exist morphisms

$$e_A : A \longrightarrow \mathtt{G} \circ \mathtt{F}(A) \text{ and } \epsilon_B : B \longrightarrow \mathtt{F} \circ \mathtt{G}(B)$$

We say $\langle \mathtt{F}, \mathtt{G}, e, \epsilon \rangle$ forms a *dual adjunction* between $\mathtt{C}$ and $\mathtt{D}$, and $\mathtt{F}$ and $\mathtt{G}$ are *dually adjoint* if the following conditions hold:

1. For every $\varphi : A \longrightarrow B \in \mathtt{Ar}_C$ and $\psi : X \longrightarrow Y \in \mathtt{Ar}_D$, the diagrams in Figure 4 commute; that is $e_B \circ \varphi = \mathtt{G} \circ \mathtt{F}(\varphi) \circ e_A$ and $\epsilon_Y \circ \psi = \mathtt{F} \circ \mathtt{G}(\psi) \circ \epsilon_X$.

2. For every $A \in \mathtt{Ob}_C$ and $X \in \mathtt{Ob}_D$, there exists a bijection between $\hom(A, \mathtt{G}(X)) \subseteq \mathtt{Ar}_C$ and $\hom(X, \mathtt{F}(A)) \subseteq \mathtt{Ar}_D$ associating φ and ψ where we have $\varphi = \mathtt{G}(\mathtt{F}(\varphi \circ \epsilon_X)) \circ e_A$ and $\psi = \mathtt{F}(\mathtt{G}(\psi \circ e_A)) \circ \epsilon_X$. That is, the diagrams in Figure 5 commute.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "e_A"'] & \mathtt{G}(X) \\ \mathtt{G} \circ \mathtt{F}(A) \arrow[ur, "\mathtt{G}(\varphi)"] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\epsilon_X"'] & \mathtt{F}(A) \\ \mathtt{F} \circ \mathtt{G}(X) \arrow[ur, "\mathtt{F}(\psi)"] & \end{tikzcd}$$

Figure 5: Commuting Diagram for Condition (2) of Definition 3.70

When a dual adjunction as described above exists, we say $\langle \mathtt{F}, \mathtt{G}, e, \epsilon \rangle$ is a *dual representation* provided $e : A \longrightarrow \mathtt{G} \circ \mathtt{F}(A)$ is an isomorphism for every $A \in \mathtt{Ob}_C$. Similarly, we say a dual representation $\langle \mathtt{F}, \mathtt{G}, e, \epsilon \rangle$ constitutes a *dual equivalence* if $\epsilon_X : \mathtt{Ob}_D \longrightarrow \mathtt{F} \circ \mathtt{G}(X)$ for every $X \in \mathtt{Ob}_D$ is an isomorphism as well. Indeed, in this case $\langle \mathtt{G}, \mathtt{F}, \epsilon, e \rangle$ constitutes a dual representation as well. We will end this section with a theorem which characterizes dual equivalences, demonstrating how the two categories are related when such a relation exists.

Theorem 3.71 (Clark and Davey)**.** *Suppose $\langle \mathtt{F}, \mathtt{G}, e, \epsilon \rangle$ is a dual equivalence between categories $\mathtt{C}$ and $\mathtt{D}$. Let $A, B \in \mathtt{Ob}_C$ and $X, Y \in \mathtt{Ob}_D$, and suppose $\varphi : A \longrightarrow B \in \mathtt{Ar}_C$ and $\psi : X \longrightarrow Y$. The following statements are true:*

1. *There is an object $X \in \mathtt{Ob}_D$, namely $X = \mathtt{F}(A)$, such that $A \cong \mathtt{G}(X)$.*

2. *There is an object $A \in \mathtt{Ob}_C$, namely $A = \mathtt{G}(X)$, such that $X \cong \mathtt{F}(A)$.*

3. *Both $\mathtt{F}$ and $\mathtt{G}$ are full and faithful.*

4. *The arrow $\varphi \in \mathtt{Ar}_C$ is an isomorphism if and only if $\mathtt{F}(\varphi)$ is an isomorphism, and $A \cong B$ if and only if $\mathtt{F}(A) \cong \mathtt{F}(B)$.*

5. *The arrow $\psi \in \mathtt{Ar}_D$ is and isomorphism if and only if $\mathtt{G}(\psi)$ is an isomorphism, and $X \cong Y$ if and only if $\mathtt{G}(X) \cong \mathtt{G}(Y)$.*

Proof. Observe that (1) and (2) follow directly from Definition 3.70. To prove (3), we will prove that $\mathtt{F}$ is full and faithful in ; the fact that $\langle \mathtt{G}, \mathtt{F}, \epsilon, e\rangle$ is a dual representation proves the case for $\mathtt{G}$. If we suppose $u, v \in \mathtt{Ar}_C$ such that $\mathtt{dom}(c) = A = \mathtt{dom}(d)$ and $\mathtt{cod}(c) = B = \mathtt{cod}(d)$ for $A, B \in \mathtt{Ob}_C$ where $\mathtt{F}(u) = \mathtt{F}(v)$, we use the hypothesis that e_B is an isomorphism to obtain

$$u = e_B^{-1} \circ \mathtt{G} \circ \mathtt{F}(u) \circ e_A = e_B^{-1} \circ \mathtt{G} \circ \mathtt{F}(v) \circ e_A = v$$

We may therefore conclude $\mathtt{F}$ is faithful. To see that $\mathtt{F}$ is full, suppose $\theta : \mathtt{F}(B) \longrightarrow \mathtt{F}(A) \in \mathtt{Ar}_D$. By part (2) of Definition 3.70 we set $X = \mathtt{F}(B)$, and observe that $\mathtt{F}(e_B) \circ \epsilon_{\mathtt{F}(B)} = \mathtt{Id}_{\mathtt{F}(B)}$ since $\mathtt{G}(\mathtt{Id}_{\mathtt{F}(B)}) = \mathtt{Id}_{\mathtt{G}\circ\mathtt{F}(B)}$ and therefore $\epsilon_{\mathtt{F}(B)} = \mathtt{F}^{-1}(e_B)$. From here we observe that

$$\begin{aligned}
\theta &= \mathtt{F}(\mathtt{G}(\theta) \circ e_A) \circ \epsilon_{\mathtt{F}(B)} \\
&= \mathtt{F}(\mathtt{G}(\theta) \circ e_A) \circ \mathtt{F}^{-1}(e_B) \\
&= \mathtt{F}(\mathtt{G}(\theta) \circ e_A) \circ \mathtt{F}(e_B^{-1}) \\
&= \mathtt{F}(e_B^{-1} \circ \mathtt{G}(\theta) \circ e_A) \\
&= \mathtt{F}(v) \text{ for some } v \in \mathtt{Ar}_C
\end{aligned}$$

From here we see that $\mathtt{F}$ is full as well. Observe that the fact that (3) holds forces us to conclude that (4) and (5) hold as well since both functors are full. □

CHAPTER 4

HYPERGRAPHS AND CONE LATTICES

4.1 The Graph Topology

For the most part, the concepts and methods used in this study are standard to the arenas of graph theory and order theory; however, because these are somewhat disparate realms, we take care to provide careful definitions and background information for those notions specific to one area or the other. We begin the the concept of "hypergraph."

There are many constructs that come under the general heading "graph", and these constructs fall into a loose hierarchy. It seems fitting therefore to open this exploration with a brief explanation as to what we will mean when we speak of a graph. Perusing the literature, one soon learns that all definitions of the term "graph" strive to capture the notion of a set of "vertices" coupled with "edges" that convey "adjacency" between vertices. The results appearing in the sections below apply to a number of these constructs; therefore, it is useful introduce a single definition that encompasses them all.

In all that follows, we will let $\mathtt{Su}(X)$ denote the powerset of any set X; and we will let $\mathtt{Su}(X)' = \mathtt{Su}(X) - \{\emptyset\}$.

Definition 4.72. Let G and I be disjoint, nonempty sets and consider the set $I \times \mathtt{Su}(G)'$. Let $\pi_G : I \times \mathtt{Su}(G)' \longrightarrow \mathtt{Su}(G)'$ denote the projection map. A *hypergraph* on the set G is a pair $\mathcal{G} = (G, E)$, where $E \subseteq I \times \mathtt{Su}(G)'$ is nonempty. The members of G are called *vertices* or *nodes*, and the members of E are called *edges*. Two vertices x and y are *adjacent* provided the following condition is met.

- There exist $e \in E$ such that $x, y \in \pi_G(e)$.

Observe that given the previous definition we have a vertex $v \in V$ is adjacent to itself if and only if $(i, \{x\}) \in E$. An edge e is *incident* to a vertex x provided $x \in \pi_G(e)$. We will say that $e, f \in E$ are *coincident* provided $e \neq f$ but $\pi_G(e) = \pi_G(f)$.

Readers familiar with hypergraphs will notice that our definition appears somewhat nonstandard, since the edge-set of a hypergraph is usually defined to be a nonempty collection of *multisets* of G (See Berge [2] for example.) However, our definition of "edge" is merely a formal approach to describing multisets that makes coincident edges easily distinguishable as objects. (Hypergraphs with coincident edges are sometimes called *multi-hypergraphs*.) A hypergraph is *simple* provided it contains no coincident edges.

In a hypergraph $\mathcal{G} = (G, E)$, we say that $e \in E$ is a *loop* provided $\pi_G(e)$ is a singleton. It is worth noting that a vertex $x \in G$ is adjacent to itself if and only if there is a loop incident to x.

Some authors also require that every vertex be adjacent to a distinct vertex so that no vertex is "isolated." (See for example Berge [2].) We will not make that restriction in general. Indeed, we will refer to a hypergraph $\mathcal{G} = (G, E)$ as *social* when no vertex is isolated.

A hypergraph $\mathcal{G} = (G, E)$ is called a *graph* provided provided $\pi_G(e)$ contains at most two elements for every edge e. Note that graphs may contain loops as well as coincident edges.

Some authors reserve the term "graph" for simple hypergraphs in which $\pi_G(e)$ always contains *exactly* two elements and use the term "multigraph" to allow the possibility of loops and coincident edges. However, use of the term "multigraph" is far from uniform in graph theory circles. Some authors require that at least one pair of vertices in a multigraph be adjacent via multiple edges — see Skiena [25] for example.) Others use the term "pseudograph" when discussing multigraphs with loops (for example, see Chartrand and Zhang [5]) while others use the term "multigraph" to refer to graph-constructs containing loops *or* containing coincident edges (see Tutte [32] for example). Some authors recommend that the term be abandoned (see West [34] for example); following this recommendation, we will avoid using the term altogether.

Definition 4.73. Let $\mathcal{G} = (G, E)$ by a hypergraph. For $x \in G$, let

$$E(x) = \{e \in E : x \in \pi_G(e)\} \quad \textit{and} \quad B(x) = \{x\} \cup E(x)$$

We call $E(x)$ the *edge-neighborhood* of the vertex x, and we call $B(x)$ the *edge-ball* generated by x.

Suppose that $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are graphs. In the literature, a *graph* homomorphism from $\mathcal{G}$ to $\mathcal{H}$ is defined to be a mapping $f : G \longrightarrow H$ that preserves adjacency and does not identify adjacent vertices unless the image vertex contains a loop. The following definition translates this concept into our context.

Definition 4.74. Suppose $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are hypergraphs. A function $f : G \cup E_G \longrightarrow H \cup E_H$ is an *HG-morphism* provided

1. We have $f(G) \subseteq H$ and $f(E_G) \subseteq E_H$.

2. If $x \in G$, then $f(B(x)) \subseteq B(f(x))$.

Lemma 4.75. *Suppose $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are hypergraphs, and suppose $f : G \cup E_G \longrightarrow H \cup E_H$ satisfies conditions (1) and (3) of Definition 4.74. The function f is an HG-morphism if and only if, for all $e \in E_G$, we have $f(\pi_G(e)) \subseteq \pi_G(f(e))$.*

Proof. Let $f : G \cup E_G \longrightarrow H \cup E_H$ be such that $f(G) \subseteq H$ and $f(E_G) \subseteq E_H$ and first suppose that for all $e \in E_G$, we have $f(\pi_G(e)) \subseteq \pi_H(f(e))$. If $e_x \in E_G$ such that $e_x \in B(x)$, observe that $x \in \pi_G(e_x)$, and by hypothesis we have $f(\pi_G(e_x)) \subseteq \pi_H(f(x))$. Since $f(x) \in f(\pi_G(e_x))$, it follows that $f(e_x) \in B(f(x))$. which proves that the hypothesis is sufficient to show that f is an HG-morphism.

□

In his 2005 Ph.D. dissertation, Antoine Vella described a novel topology on hypergraphs based on edge-balls and identified a number of its properties (see Vella [33]).

Definition 4.76. Let $\mathcal{G} = (G, E)$ be a hypergraph. A subset X of $G \cup E$ is *graph-open* provided one of the following conditions is met.

- *We have* $X \subseteq E$.
- *If* $x \in X \cap G$, *then* $E(x) \subseteq X$.

We include the openness of edge-only sets as Condition (1) purely for emphasis since it is actually implied by Condition (2). It is easy to see that the collection $\Omega(\mathcal{G})$ of graph-open sets forms a topology on $G \cup E$. This topology is called the *graph* or *classical* topology on $\mathcal{G}$.

It is worth noting that the family

$$\mathtt{B}(\mathcal{G}) = \{\{e\} : e \in E\} \cup \{B(x) : x \in G\}$$

forms a compact-open basis for the graph topology on any hypergraph $\mathcal{G} = (G, E)$. In the work to follow, we will let

$$\mathtt{JS}(\mathtt{B}(\mathcal{G})) = \{\bigcup F : F \in \mathtt{Fin}(\mathtt{B}(\mathcal{G}))\}$$

where $\mathtt{Fin}(\mathtt{B}(\mathcal{G}))$ denotes the set of all finite subsets of $\mathtt{B}(\mathcal{G})$. It is clear that $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$ is a lower-bounded join-semilattice under set inclusion. It is also clear that $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$ serves as another compact-open basis for the graph topology on $\mathcal{G}$. We will refer to $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$ as the join-semilattice *generated* by the basis $\mathtt{B}(\mathcal{G})$.

It is also worth noting that the graph topology is an example of an *Alexandroff topology*; that is, one in which the intersection of *any* family of open sets is open. In light of this fact, the poset $(\Omega(\mathcal{G}), \subseteq)$ is a complete, distributive lattice. The join (supremum) for any subset of $\Omega(\mathcal{G})$ is simply its union ; likewise, the meet (infimum) of the set is simply its intersection. We will have much more to say about this lattice in Section 4.3

Theorem 4.77. *Every HG-morphism is continuous with respect to the graph topology.*

Proof. Suppose $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are hypergraphs, and suppose $f : G \cup E_G \longrightarrow H \cup E_H$ is an HG-morphism. Let $y \in H$ and suppose $x \in f^{-1}(B(y))$. Observe that if $x \in G$, we necessarily have $f(x) = y$ and $f(B(x)) \subseteq B(y)$ since f is an HG-morphism. This immediately implies $B(x) \subseteq f^{-1}(B(y))$. Furthermore, if $x \in E_G$, we know $\{x\} \in \Omega(\mathcal{G})$. In either case, every element in $f^{-1}(B(y))$ is contained in an open subset of itself, hence $f^{-1}(B(y)) \in \Omega(\mathcal{G})$ and f is continuous with respect to the graph topology. $\square$

Theorem 4.78. *Let $\mathcal{G} = (G, E)$ be a hypergraph, and let $G \cup E$ be endowed with the graph topology. The following statements are true.*

1. *The set $G \cup E$ has the T_0 separation property under the graph topology.*

2. *The set $G \cup E$ is sober under this topology; that is, if X is a join-prime member of the lattice $\Gamma(\mathcal{G})$ of graph-closed subsets of $G \cup E$, then X is the closure of a singleton.*

3. *The set $G \cup E$ has a basis of compact open sets that forms a lower-bounded join sub semilattice of $\Omega(\mathcal{G})$.*

4. *If $B(x) \cap B(y)$ is finite for all distinct $x, y \in G$, then the set $G \cup E$ has a basis of compact open sets that forms a lower-bounded sublattice of $\Omega(\mathcal{G})$.*

Proof. To prove (1), it is sufficient to observe that all singleton edges are open, and if $x, y \in G$ are distinct, then $y \notin B(x)$ (the same is true of x and $B(y)$).

In order to prove (2), it will be useful to characterize the closure of singletons. Observe that since every singleton edge is open under the graph topology and every edge-ball contains a single vertex, no vertices are limit points to any open set. That is, if $x \in G$ we have $\overline{\{x\}} = \{x\}$. On the other hand, since an open set contains a vertex always means it contains its adjacent edges, the closure of any edge will contain *all* of its adjacent vertices. That is, for any $e \in E$ we have $\overline{\{e\}} = \{e\} \cup \pi_G(e)$. From here it is evident that (up to equivalence) a closed set will *not* be the closure of singleton when either it contains two (or more) vertices or it contains an edge and a non-adjacent vertex.

To see why the closure of multiple vertices fails to be join-prime, observe that if $x, y \in G$ are distinct, we clearly have $\{x\} \in \bigcup\{\{x, y\}\}$ where $\{\{x, y\}\} \subseteq \Gamma(\mathcal{G})$, but $\{x\} \notin \{\{x, y\}\}$, and therefore fails to be join-prime (see Definition 2.62 and consider its dual). Similarly, to see why the second case fails to be join-prime, consider $e \in E$ and $x \in G - \pi_G(e)$. Observe that $\overline{\{x, e\}} = \{x, e\} \cup \pi_G(e)$, and as we have seen a closed set with multiple vertices *cannot* be join-prime. Consequently, contraposition forces us to conclude that any join-prime closed subset must be the closure of a singleton.

It is sufficient to recall that the family

$$\mathtt{JS}(\mathtt{B}(\mathcal{G})) = \{\bigcup F : F \in \mathtt{Fin}(\mathtt{B}(\mathcal{G}))\}$$

is indeed a basis that satisfies (3), and to prove (4) suppose that $B(x) \cap B(y)$ is finite for every distinct $x, y \in G$. If we suppose $\mathcal{F} \in \mathtt{Fin}(\mathtt{B}(\mathcal{G}))$, we observe that if $\bigcup \mathcal{F} \subseteq E$, then $\mathcal{F}$ is a finite collection of finite edge-sets. Furthermore, if $\bigcup \mathcal{F} \cap G \neq \emptyset$, we know this intersection is finite and if $\bigcup \mathcal{F} \cap G = \{x_1, \ldots, x_n\}$ we must have $\bigcup_{i=1}^{n} B(x_i) \in \bigcup \mathcal{F}$. This in turn implies $\bigcap \mathcal{F} \subseteq \bigcap_{i=1}^{n} B(x_i)$, and by hypothesis this must be a finite (possibly empty) collection of edges. This means $\bigcap \mathcal{F} \in \mathtt{JS}(\mathtt{B}(\mathcal{G}))$ in either case. We may therefore conclude $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$ is closed under finite intersections, and hence is a sublattice of $\Omega(\mathcal{G})$ with the emptyset serving as a lower bound. $\square$

Readers familiar with topological representations of distributive lattices will recognize that Theorem 4.78 implies every graph topology is the *Stone space* associated with *some* distributive join semilattice. Indeed, under the mild requirement that distinct vertices be adjacent via at most finitely many edges, graph topologies are *spectral spaces*. For information on the subject of topological representations, we recommend starting with Gratzer [13].

In the sections to follow, we will establish that the graph topology on a hypergraph $\mathcal{G}$ is (homeomorphic to) the Stone space associated with the join semilattice generated by the basis $\mathtt{B}(\mathcal{G})$. The road leading to this conclusion is rather scenic; and along the way we will establish a number of results concerning hypergraphs. We begin by examining a well-known construct closely associated with hypergraphs.

4.2 Hypergraphs and Hypergraph Posets

Graph theorists have long exploited the fact that there is a simple way to associate a poset with any hypergraph — for a hypergraph $\mathcal{G} = (G, E)$, let $P_G = G \cup E$ and set $u \sqsubseteq v$ if and only if one of the following conditions hold:

- We have $u = v$.
- We have $u \in E$, $v \in G$, and $v \in \pi_G(u)$.

In this approach, the vertex set G and the edge set E are antichains in the poset, and every member of E is covered by at least one member of G. The pair $\mathcal{P}_G = (P_G, \leq)$ is often called the *incidence poset* or the *vertex-edge poset* for the hypergraph $\mathcal{G}$. (Incidence posets are also called *Levi graphs* in honor of Frederich Levi, an early investigator — see Levi [18].) An excellent bibliography on the early development of posets for simple graphs can be found in Trotter [31].

Definition 4.79. A poset $\mathcal{P} = (P, \leq)$ is called a *hypergraph poset* provided the following conditions are met.

1. There exist disjoint nonempty antichains $\mathtt{V}(\mathcal{P})$ and $\mathtt{E}(\mathcal{P})$ such that $P = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P})$.

2. The members of $\mathtt{V}(\mathcal{P})$ are maximal in $\mathcal{P}$, and the members of $\mathtt{E}(\mathcal{P})$ are minimal in $\mathcal{P}$.

3. Every member of $\mathtt{E}(\mathcal{P})$ is covered by at least one member of $\mathtt{V}(\mathcal{P})$. We will let $\mathtt{Cov}(y)$ denote the set of covers for y.

Note that the set $\mathtt{V}(\mathcal{P})$ represents the set of *all* maximal elements in a hypergraph poset $\mathcal{P}$. However, it does *not* have to be the case that $\mathtt{E}(\mathcal{P})$ represents the set of all minimal elements of $\mathcal{P}$. It is possible that some member of $\mathtt{V}(\mathcal{P})$ is also minimal in $\mathcal{P}$; this is the case for precisely those $x \in \mathtt{V}(\mathcal{P})$ such that $\downarrow x = \{x\}$. We will say that a hypergraph poset $\mathcal{P}$ is *social* provided $\downarrow x$ always contains at least two elements for any $x \in \mathtt{V}(\mathcal{P})$.

We will say that a hypergraph poset $\mathcal{P}$ is a *graph* poset provided $\mathtt{Cov}(j)$ contains at most two members for any $j \in \mathtt{E}(\mathcal{P})$.

Let $\mathcal{G} = (G, E)$ be a hypergraph. It is clear that the incidence poset $\mathcal{P}_G = (P_G, \leq)$ introduced above is a hypergraph poset that is a graph poset whenever $\mathcal{G}$ is a graph.

On the other hand, Suppose that $\mathcal{P} = (\mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}), \leq)$ is a hypergraph poset. Let

$$G_P = \mathtt{V}(\mathcal{P}) \quad \text{and} \quad E_P = \{(e, \mathtt{Cov}(e)) : e \in \mathtt{E}(\mathcal{P})\}$$

It is clear that $\mathcal{G}_P = (G_P, E_P)$ is a hypergraph.

Theorem 4.80. *If $\mathcal{P} = (\mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}), \leq)$ is a hypergraph poset, then $\mathcal{P}$ is order-isomorphic to $\mathcal{P}_{G_P}$.*

Proof. If we define a map $\varphi : \mathcal{P} \longrightarrow \mathcal{G}_P$ by the rule

$$\varphi(x) = \begin{cases} x, & x \in \mathtt{V}(\mathcal{P}); \\ (x, \mathtt{Cov}(x)), & x \in \mathtt{E}(\mathcal{P}). \end{cases}$$

from $\mathcal{P}$ and the induced graph G_P. This map is clearly bijective since $\varphi|_{\mathtt{V}(\mathcal{P})}$ is the identity map for $\mathtt{V}(\mathcal{P})$ provided its codomain is restricted to its range, and $e_1, e_2 \in \mathtt{E}(\mathcal{P})$ are distinct if and only if $(e_1, \mathtt{Cov}(e_1)), (e_2, \mathtt{Cov}(e_2)) \in E_P$ are as well.

Since $\mathcal{P}_{G_P} = G_P \cup E_P$, it remains to observe that if $x \sqsubseteq y$ where $x, y \in \mathcal{P}$, we either have $x = y$ (in which case $\varphi(x) = \varphi(y)$ in $\mathcal{P}_{\mathcal{G}}$) or

$$\begin{aligned} x \sqsubset_{\mathcal{P}} y \text{ where } x, y \in \mathcal{P} &\iff y \in G \cap \mathtt{Cov}(x) \text{ and } (x, \mathtt{Cov}(x)) \in E \\ &\iff (x, \mathtt{Cov}(x)) \sqsubset_{\mathcal{P}_{\mathcal{G}}} y \text{ where } (x, \mathtt{Cov}(x)), y \in \mathcal{P}_{\mathcal{G}} \end{aligned}$$

□

We will say that hypergraphs $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are *HG-isomorphic* provided there exists a bijection $f : G \cup E_G \longrightarrow H \cup E_H$ with the property that f and f^{-1} are both HG-morphisms. Note that in this case, we have $f(B(x)) = B(f(x))$ for all $x \in G$.

Theorem 4.81. *If $\mathcal{G} = (G, E)$ is a hypergraph, then $\mathcal{G}$ is HG-isomorphic to the hypergraph $\mathcal{G}_{\mathcal{P}_G} = (G_{P_G}, E_{P_G})$.*

Proof. First, suppose $\mathcal{G} = (G, E)$ is a hypergraph and define a map $\psi : \mathcal{G} \longrightarrow \mathcal{G}_{\mathcal{P}_G}$ defined under the rule as

$$\psi(x) = \begin{cases} x, & x \in G; \\ (x, \mathtt{Cov}(x)), & x \in E. \end{cases}$$

The argument that this map is bijective mirrors that in the previous proof; we will proceed to show this map and its inverse are HG-morphisms. To that end, observe that the first condition for HP-morphisms is satisfied for both ψ and its inverse since

$$G = \psi^{-1}(G_{P_G}) \text{ and } \psi(G) = G_{P_G}$$

Furthermore, Theorem 4.80 guarantees that both ψ and its inverse satisfy the second condition of HP-morphisms, hence the claim follows. □

This correspondence can be taken further.

Definition 4.82. Suppose that $\mathcal{P} = (\mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}), \leq)$ and $\mathcal{Q} = (\mathtt{V}(\mathcal{Q}) \cup \mathtt{E}(\mathcal{Q}), \sqsubseteq)$ are hypergraph posets. A mapping $f : \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}) \longrightarrow \mathtt{V}(\mathcal{Q}) \cup \mathtt{E}(\mathcal{Q})$ will be called an *HP-morphism* provided the following conditions are met.

1. We have $f(\mathtt{V}(\mathcal{P})) \subseteq \mathtt{V}(\mathcal{Q})$.

2. The function f is a strict order-homomorphism; that is, $x < y$ implies $f(x) \sqsubset f(y)$.

Lemma 4.83. *If f is an HG-morphism from the hypergraph $\mathcal{G} = (G, E_G)$ to the hypergraph $\mathcal{H} = (H, E_H)$, then f is also an HP-morphism from $\mathcal{P}_G$ to $\mathcal{P}_H$.*

Proof. Suppose f is an HG-morphism. By definition we have $f(\mathtt{V}(\mathcal{P}_G)) = f(G) \subseteq H = \mathtt{V}(\mathcal{P}_H)$, hence the first criterion for HP-morphisms is met.

Now suppose $x, y \in \mathcal{P}_G$ such that $x < y$. It follows that $y \in G$ and $x \in B(y)$. Since f is an HG-morphism, it follows that $f(B(y)) \subseteq B(f(y))$, which implies

$f(x) \in B(f(y))$; that is $f(y) \in \texttt{Cov}(f(x))$ This of course means $f(x) \sqsubset f(y)$.

Finally, we consider the case where $x, y \in G$ are distinct where $f(x) = f(y)$ and $e \in B(x) \cap B(y)$. Since f is an HG-morphism, we know that $f(B(x) \cap B(y)) \subseteq \texttt{Loop}(f(x))$, hence $f(e) = (i, \{f(x)\})$. From here it follows that $\uparrow f(e) = \{f(e), f(x)\}$ in the hypergraph poset $\mathcal{P}_H$, and therefore f satisfies that final criterion for HP-morphisms. $\square$

Lemma 4.84. *If we suppose f is an HP-morphism from the hypergraph poset $\mathcal{P} = (\texttt{V}(P) \cup \texttt{E}(P), \leq)$ to the hypergraph poset $\mathcal{Q} = (\texttt{V}(Q), \texttt{E}(Q), \sqsubseteq)$, then f' is an HG-morphism from the hypergraph $\mathcal{G}_P = (V_P, E_P)$ to the hypergraph $\mathcal{G}_Q = (V_Q, E_Q)$, where*

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V_P \\ (f(e_x), \texttt{Cov}(f(e_x))) & \text{if } x = (e_x, \texttt{Cov}(e_x)) \in E_P \end{cases}$$

Proof. Suppose $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is an HP-morphism and define f' as above. By definition $f'(V_P) = f'(\texttt{V}(P)) \subseteq \texttt{V}(Q) = V_Q$. We also observe that $f'(E_P) = \{(f(x), \texttt{Cov}(f(x))) : (x, \texttt{Cov}(x)) \in E_P\} \subseteq E_Q$, so the first criterion of HG-morphisms is satisfied.

Next, suppose suppose $v \in V_P$ and suppose $f'(x) \in f'(B(v))$. It follows that either $f'(x) = f'(v)$ (in which case $x = v$) or $x \in E(v)$. In the latter case, $x \in E(v)$ implies $x < v$, and since f is an HP-morphism, we must have $f(x) \sqsubset f(v)$. We may therefore conclude $f' \in B(f'(v))$. In either case $f'(B(v)) \subseteq B(f'(v))$ as desired.

Finally, suppose $x, y \in V_P$ are distinct where $f(x) = f(y)$ and $e \in \downarrow x \cap \downarrow y$. Since f is an HP-morphism, we know that $\uparrow f(e) = \{f(e), f(x)\}$, and hence $f'(e) = (f(e), \texttt{Cov}(f(x)) = (f(e), \{f(x)\})$ is a loop in $\mathcal{G}_Q = (V_Q, E_Q)$, hence Criterion 3 of HG-morphisms is satisfied, which completes the proof. $\square$

Definition 4.85. Let $\mathbf{HGraph}$ denote the category whose objects are hypergraphs and whose morphisms are HG-morphisms, and let $\mathbf{HPoset}$ denote the category whose objects are hypergraph posets and whose morphisms are HP-morphisms.

Theorem 4.86. *The category* $\mathbf{HGraph}$ *is equivalent to the category* $\mathbf{HPoset}$*. The equivalence is accomplished via the functors* $\mathtt{HP} : \mathbf{HGraph} \longrightarrow \mathbf{HPoset}$ *and* $\mathtt{HG} : \mathbf{HPoset} \longrightarrow \mathbf{HGraph}$ *defined by*

- $\mathtt{HP}[\mathcal{G}] = \mathcal{P}_G$ *and* $\mathtt{HP}[f] = f$.
- $\mathtt{HG}[\mathcal{P}] = \mathcal{G}_P$ *and* $\mathtt{HG}[f] = f'$.

Proof. Observe that Theorems 4.80 and 4.81 as well as Lemmas 4.83 and 4.84 tell us we can define natural transformations $e : \mathtt{Id}_{\mathbf{HGraph}} \longrightarrow \mathtt{HGHP}$ and $\epsilon : \mathtt{Id}_{\mathbf{HPoset}} \longrightarrow \mathtt{HPHG}$ (where $\mathtt{Id}_{\mathtt{C}}$ is the identity functor for a category $\mathtt{C}$) where each component is an isomorphism. We may therefore conclude $e : \mathtt{Id}_{\mathbf{HGraph}} \cong \mathtt{HGHP}$ and $\epsilon : \mathtt{Id}_{\mathbf{HPoset}} \cong \mathtt{HPHG}$ and that the following diagrams commute:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f"] \arrow[d, "e_G"'] & \mathcal{H} \arrow[d, "e_H"'] \\ \mathtt{HGHP}(\mathcal{G}) \arrow[r, "\mathtt{HGHP}(f)"] & \mathtt{HGHP}(\mathcal{H}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_G \arrow[r, "g"] \arrow[d, "\epsilon_G"'] & \mathcal{P}_H \arrow[d, "\epsilon_H"'] \\ \mathtt{HPHG}(\mathcal{P}_G) \arrow[r, "\mathtt{HPHG}(g)"] & \mathtt{HPHG}(\mathcal{P}_H) \end{tikzcd}$$

□

In the work to follow, we find it more convenient to work with hypergraph posets rather than hypergraphs; however, in light of the previous results, the two concepts are essentially interchangeable.

4.3 Lowersets of a Hypergraph Poset

If $\mathcal{G}$ is any hypergraph, then the following result tells us we may consider the lattices $\Omega(\mathcal{G})$ and $\mathtt{Low}(\mathcal{P}_G)$ to be interchangeable. This should come as no surprise given the results in Section 4.2.

Lemma 4.87. *Let $\mathcal{G} = (G, E)$ be a hypergraph endowed with the graph topology, and let $\mathcal{P}_G$ represent its hypergraph poset. For $X \subseteq G \cup E$, the following statements are equivalent.*

1. *We have $X \in \Omega(\mathcal{G})$.*
2. *We have $X \in \mathtt{Low}(\mathcal{P}_G)$.*

   Proof. It will be instructive to characterize that principal lowersets of $\mathcal{P}_G$. To that end, let $x \in \mathcal{G}$ and observe $\downarrow x = \{x\}$ whenever $x \in E$ and $\downarrow x = \{x\} \cup E(x) = B(x)$ whenever $x \in G$. From here we observe

$$\begin{aligned} U \in \mathtt{Low}(\mathcal{P}) &\iff U = \bigcup \{\downarrow x : x \in U\} \\ &\iff U = \bigcup \{\{e\} \cup B(x) : e \in U \cap E, x \in U \cap G\} \end{aligned}$$

   Recall that $\mathtt{B}(\mathcal{G}) = \{\{e\} : e \in E\} \cup \{B(x) : x \in G\}$ constitutes a basis for $\Omega(\mathcal{G})$, so we have shown that $U \in \mathtt{Low}(\mathcal{P})$ if and only if U can be expressed as the union of a family from the basis $\mathtt{B}((G))$ which proves the claim. $\square$

Recall that an element a of a lattice $\mathcal{L}$ with least element is called an *atom* of $\mathcal{L}$ provided $\perp \prec a$. Similarly, an element b of a lattice $\mathcal{L}$ with greatest element is a *coatom* of $\mathcal{L}$ provided it is an atom of the order-dual for $\mathcal{L}$.

Theorem 4.88. *Suppose that $\mathcal{P} = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P})$ is a hypergraph poset. If we let*

$$B_\perp = \{X : X \subseteq \mathtt{E}(\mathcal{P})\} \quad \textit{and} \quad B_\top = \{\mathtt{E}(\mathcal{P}) \cup Y : Y \subseteq \mathtt{V}(\mathcal{P})\}$$

then the following statements are true.

1. *Under subset inclusion, $B_\top$ and $B_\perp$ are complete, atomic Boolean sublattices of $\mathtt{Low}(\mathcal{P})$. Moreover, $B_\perp \cap B_\top = \mathtt{E}(\mathcal{P})$.*

2. *The set* $B_\perp$ *is a lowerset of* $\texttt{Low}(\mathcal{P})$*, and* $B_\top$ *is an upperset of* $\texttt{Low}(\mathcal{P})$*.*

3. *The atoms of* $B_\perp$ *are precisely the singleton subsets of* $\texttt{E}(\mathcal{P})$*.*

4. *The atoms of* $B_\top$ *are precisely the sets* $\texttt{E}(\mathcal{P}) \cup \{y\}$*, where* $y \in \texttt{V}(\mathcal{P})$*.*

5. *If* $y \in \texttt{V}(\mathcal{P})$ *is such that* $\downarrow y \notin B_\top$*, then* $\downarrow y$ *is minimal in* $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$*.*

6. *If* $x \in \texttt{E}(\mathcal{P})$ *and* $\texttt{Cov}(x) \neq \texttt{V}(\mathcal{P})$*, then* $\neg(\texttt{Cov}(x)) = P - (x \cup \texttt{Cov}(x))$ *is maximal in* $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$*.*

Proof. To prove (1), it is instructive to recall that every complete, atomic Boolean lattice is order-isomorphic to the powerset of some set, and conversely. Note that $B_\perp = \texttt{Su}(\texttt{E}(\mathcal{P}))$, and $B_\top$ is order isomorphic to $\texttt{Su}(\texttt{V}(\mathcal{P})$ via the map $\varphi : \texttt{Su}(\texttt{V}(\mathcal{P}) \longrightarrow B_\top$ defined $\varphi(S) = S \cup \texttt{E}(\mathcal{P})$ (we leave it to the reader to verify that φ is an order isomorphism). Observe that $\texttt{E}(\mathcal{P})$ is clearly the top element of $B_\perp$, and since $\varphi(\emptyset) = \texttt{E}(\mathcal{P})$, it also serves as the least element of $B_\top$. From here it is clear that $\texttt{E}(\mathcal{P}) = B_\perp \cap B_\top$ as desired.

In order prove (2), suppose $U_1 \in B_\perp$ and suppose $U_2 \in \texttt{Low}(\mathcal{P})$ such that $U_2 \subseteq U_1$. Since U_2 is a subset of U_1, it is a subset of $\texttt{E}(\mathcal{P})$ and consequently $U_2 \in B_\perp$. Similarly, if we suppose $V_1 \in B_\top$ and $V_2 \in \texttt{Low}(\mathcal{P})$ such that $V_1 \subseteq V_2$, the fact that $\texttt{E}(\mathcal{P}) \subseteq V_1$ guarantees $V_2 = \texttt{E}(\mathcal{P}) \cup T$ for some $T \in \texttt{Su}(\texttt{V}(\mathcal{P})$, which means $V_2 \in B_\top$, which verifies the claim.

Both (3) and (4) are easily verified by recalling that the atoms of any powerset are precisely the singletons, which proves (3) directly and (4) due to the fact that the sets $\texttt{E}(\mathcal{P}) \cup \{y\}$ are precisely the images of singletons under φ. We prove (5) by observing that for any $\downarrow y \in \texttt{Low}(\mathcal{P})$ such that $\downarrow y \notin B_\top$, we must have $\downarrow y = \{y\} \cup \texttt{E}(y)$ where $\texttt{E}(y) \subset \texttt{E}(\mathcal{P})$, hence $\downarrow y \notin B_\perp$. Furthermore, it is the *minimal* such lowerset to contain y, and any subset not including y is in $B_\perp$, therefore $\downarrow y$ is minimal in $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$.

Finally, to prove claim (6), if we suppose $x \in \texttt{E}(\mathcal{P})$ is such that $\texttt{Cov}(x) \neq \texttt{V}(\mathcal{P})$, we note that $\neg(\texttt{Cov}(x)) = P - (x \cup \texttt{Cov}(x))$ is necessarily nonempty. In particular, there exists $v \in \texttt{V}(\mathcal{P}) - \texttt{Cov}(x)$, hence $\neg(\texttt{Cov}(x))$ contains at least one vertex. If we suppose $z \in \mathcal{P}$ is such that $z \leq y$, then either $z = y$ or $z \in \texttt{E} - \{x\}$; in either case $z \in \neg\texttt{Cov}(x)$ and $\neg\texttt{Cov}(x) \in \texttt{Low}(\mathcal{P})$. Furthermore, since $x \notin \neg\texttt{Cov}(x)$, we must have $\neg\texttt{Cov}(x) \in \texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$. To see that $\neg\texttt{Cov}(x)$ is maximal in $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$, observe that $\texttt{E} - \{x\} \subseteq \neg\texttt{Cov}(x)$, which implies that $\neg\texttt{Cov}(x) \cup \{e\} \in B_\top$. Furthermore, if we have $U \in \texttt{Low}(\mathcal{P})$ such that $\neg\texttt{Cov}(x) \subseteq U$ and $U \cap \texttt{Cov}(x) \neq \emptyset$, we must have $x \in U$, hence $U \in B_\top$. From here we conclude (6) holds. $\square$

Corollary 4.89. *Suppose that $\mathcal{P} = \texttt{V}(\mathcal{P}) \cup \texttt{E}(\mathcal{P})$ is a hypergraph poset, and suppose that $U \in \texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$.*

1. *There is a maximal member of $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$ that contains U.*

2. *There is a minimal member of $\texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$ that is contained in U.*

3. *There is a smallest member of $B_\top$ that contains U.*

4. *There is a largest member of $B_\perp$ that is contained in U.*

Proof. If $U \in \texttt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$, we know that

- The set U can be represented as the union of principal lowersets.

- We have $\downarrow x \subseteq U$ for at least one $x \in \texttt{V}(\mathcal{P})$.

- We have $\texttt{E}(\mathcal{P}) - U \neq \emptyset$.

Since there exists $e \in \texttt{E}(\mathcal{P}) - U$, we know that for every $\downarrow v \subseteq U \cap \texttt{Low}(\texttt{V}(\mathcal{P})$, we know $v \notin \texttt{Cov}(e)$, which in turn implies $\downarrow v \in \neg\texttt{Cov}(e)$, hence $U \subseteq \neg\texttt{Cov}(x)$ and (1) is proved. We observe that (2) is satisfied since every such $\downarrow v \in U$ is minimal in $\mathtt{Low}(\mathcal{P}) - (B_\perp \cup B_\top)$. Observe that $b_\top := \bigcup\{\downarrow v \cup \mathtt{E}(\mathcal{P}) :\downarrow v \subseteq U\}$ is such that $b_\top \in B_\top$, and is clearly the smallest such element of $B_\top$ by the way b is defined, so (3) is satisfied. Similarly, $b_\perp := \{e \in \mathtt{E}(\mathcal{P}) : e \in U\}$ is such that $b_\perp \in B_\perp$, and is the largest such member contained in U, hence proving (4). □

4.4 Cone Lattices

In the previous section, we described the structure of $\mathtt{Low}(\mathcal{P})$ for any hypergraph poset $\mathcal{P}$. In this section, we will prove that any lattice satisfying the same structural conditions is order-isomorphic to the lowerset lattice of some hypergraph. We begin with a definition that summarizes these structural conditions.

Definition 4.90. Let $\mathcal{L} = (L, \leq)$ be a bialgebraic, distributive lattice. We say that $\mathcal{L}$ is a *cone* lattice provided there exists an element $\perp < \eta < \top$ such that

1. The posets $B_\top =\uparrow \eta$ and $B_\perp =\downarrow \eta$ are complete atomic Boolean lattices.
2. The set $B_\perp$ is a lowerset of $\mathcal{L}$, and the set $B_\top$ is an upperset of $\mathcal{L}$.
3. Every member of $L-(B_\perp \cup B_\top)$ has a maximal upper bound in $L-(B_\perp \cup B_\top)$.
4. Every member of $L-(B_\perp \cup B_\top)$ has a minimal lower bound in $L-(B_\perp \cup B_\top)$.
5. If $x \in L-(B_\perp \cup B_\top)$, then $B_\perp \cap \downarrow x$ has a maximal member, and $B_\top \cap \uparrow x$ has a minimal member.

We will refer to the sublattice $B_\top \cup B_\perp$ as the *Boolean cone* of $\mathcal{L}$ and refer to the subposet $L - (B_\top \cup B_\perp)$ as the set of *suspended* elements in $\mathcal{L}$. We will say that $\mathcal{L}$ is a *proper* cone lattice whenever its set of suspended elements in nonempty.

In light of Theorem 4.88 and Corollary 4.89, we know that cone lattices are abundant structures. In particular, if $\mathcal{P} = (\mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}), \leq)$ is any hypergraph poset, then $\mathtt{Low}(\mathcal{P})$ is a cone lattice. Our next goal will be to prove that *every* cone lattice arises as the lowerset lattice of a hypergraph poset. We accomplish this task in several steps.

For any poset $\mathcal{P}$, the join-prime elements of $\mathtt{Low}(\mathcal{P})$ are precisely the principal lowersets of $\mathcal{P}$. Consequently, every element of $\mathtt{Low}(\mathcal{P})$ is the supremum of a set of compact, join-prime elements.

Recall that an element p in a complete lattice $\mathcal{L} = (L, \leq)$ is *completely meet-prime* provided it is completely join-prime in the order-dual of $\mathcal{L}$. In other words, p is completely meet-prime provided, whenever $X \subseteq L$ is such that $\bigwedge X \leq p$, then there exist $x \in X$ such that $x \leq p$. We will let $\mathtt{CMP}(\mathcal{L})$ denote the subposet of completely meet-prime members of $\mathcal{L}$. If $\mathcal{L}$ is complete, then it is clear that $\mathtt{CMP}(\mathcal{L}) \subseteq \mathtt{MP}(\mathcal{L})$.

For any complete lattice $\mathcal{L}$, we will let $\mathtt{CJP}(\mathcal{L})$ denote the subposet of compact, join-prime elements. It is well-known that an algebraic,distributive lattice $\mathcal{L}$ is bialgebraic if and only if $\mathcal{L}$ is order-isomorphic to $\mathtt{Low}[\mathtt{CJP}(\mathcal{L})]$. Another well-known quite useful fact about bialgebraic lattices is that every element can be represented as the join of a family from $\mathtt{CJP}(\mathcal{P})$; in particular if $x \in L$, we have $x = \bigvee \mathcal{J}_{\mathrm{x}}$ where $\mathcal{J}_{\mathrm{x}} := \{\ p \in \mathtt{CJP}(\mathcal{P}) : p \leq x\ \}$ (see Theorems 2.61 and 2.65). We will make use of these facts in this section.

Let $\mathcal{L}$ be a complete lattice. It is worth noting that $j \in \mathtt{CJP}(\mathcal{L})$ if and only if, for every $X \subseteq L$ such that $j \leq \bigvee X$, there exist $x \in X$ such that $j \leq x$. Compact, join-prime elements are often called *completely join-prime* for this reason.

Lemma 4.91. *Suppose that $\mathcal{L} = (L, \leq)$ is a cone lattice, and suppose that $j \in \mathtt{CJP}(\mathcal{L})$. If $j \vee \eta > \eta$, then $j \vee \eta$ is an atom of $B_{\top}$.*

Proof. It will be useful to characterize the necessary and sufficient condition(s) for one element to cover another in a bialgebraic lattice. First, observe that if $x, y \in L$ such that $x \prec y$, we must have $\mathcal{J}_{\mathrm{x}} \subset \mathcal{J}_{\mathrm{y}}$; that is, we necessarily have $j_y \in \mathcal{J}_{\mathrm{y}} - \mathcal{J}_{\mathrm{x}}$. Now, since $\mathcal{L}$ is complete, we know that $\mathcal{J}_{\mathrm{l}} := \mathcal{J}_{\mathrm{x}} \cup \{j_y\}$ has the property $\bigvee \mathcal{J}_{\mathrm{l}} = l$ for some $l \in L$. Since $\mathcal{J}_{\mathrm{x}} \subset \mathcal{J}_{\mathrm{l}}$, we must have $x < l$, but since $\mathcal{J}_{\mathrm{l}} \subseteq \mathcal{J}_{\mathrm{y}}$ we must have $l \leq y$. From here we notice that we must have $l = y$ since we y covers x by hypothesis, so we see it is also sufficient for $\mathcal{J}_{\mathrm{y}}$ to contain *precisely* one join-prime element not contained in $\mathcal{J}_{\mathrm{x}}$ to cover x. From here it is easy to see that our claim is a corollary of this result. To see why, observe that if $j \vee \eta > \eta$, then it must be true that $j \notin J_\eta$ and $a = j \vee \eta$ has the property $a \in \uparrow \eta = B_\top$ and $\mathcal{J}_{\mathrm{a}} = J_\eta \cup \{j\}$. $\square$

Lemma 4.92. *Suppose that $\mathcal{L} = (L, \leq)$ is a cone lattice. Every minimal member of $L - (B_\perp \cup B_\top)$ is completely join-prime in $\mathcal{L}$.*

Proof. Suppose $x \in L - (B_\perp \cup B_\top)$ is minimal. Since $x \in (B_\perp \cup B_\top)$, it must be true that $\mathcal{J}_{\mathrm{x}} - B_\perp \neq \emptyset$ since $B_\perp$ is a complete sublattice of $\mathcal{L}$. This in turn implies there exists $x_0 \in \mathcal{J}_{\mathrm{x}} - B_\perp$. Note that we have $x_0 \leq x$. Observe that if $x_0 \in B_\top$, we have $x \in \uparrow x_0 \subseteq B_\top$; contrary to hypothesis. This means $x_0 \in L - (B_\perp \cup B_\top)$. However, since x is minimal in $L - (B_\perp \cup B_\top)$, we must have $x_0 = x$ and x is completely join-prime as a consequence. $\square$

For any cone lattice $\mathcal{L} = (L, \leq)$, let $\texttt{MinSus}(\mathcal{L})$ denote the set of minimal suspended elements of $\mathcal{L}$. Note that every member of $\texttt{MinSus}(\mathcal{L})$ is completely join-prime in $\mathcal{L}$.

Lemma 4.93. *Suppose that $\mathcal{L} = (L, \leq)$ is a cone lattice and let $j \in L - (B_\perp \cup B_\top)$. The element j is completely join-prime in $\mathcal{L}$ if and only if $j \in \texttt{MinSus}(\mathcal{L})$.*

Proof. In light of Lemma 4.92 it is sufficient to show that every completely join-prime element of $\mathcal{L}$ in $L - (B_\perp \cup B_\top)$ is necessarily minimal. To that end, suppose

$j \in \mathtt{CJP}(\mathcal{L} \cap [L - (B_\perp \cup B_\top)]$. We know there exists $k \in \mathtt{MinSus}(\mathcal{L})$ such that $k \leq j$, and $B_\perp$ is a lowerset of $\mathcal{L}$. It follows $\eta < \eta \vee j$ and $\eta < \eta \vee k$, and consequently $\eta \vee j$ and $\eta \vee k$ are both atoms of $B_\top$ by Lemma 4.91. But we also observe that $k \leq j$ implies $\eta \vee k \leq \eta \vee j$, and since j is completely join-prime we conclude $j = k$ since $j \leq k \vee \eta$.

□

Corollary 4.94. *Suppose that $\mathcal{L} = (L, \leq)$ is a cone lattice.*

1. *If $j \in B_\top$ is completely join-prime in $\mathcal{L}$, then j must be an atom of $B_\top$.*
2. *An atom a in $B_\top$ is completely join-prime in $\mathcal{L}$ if and only if $\downarrow a \cap \mathtt{MinSus}(\mathcal{L}) = \emptyset$.*

Proof. Observe that if $j \in B_\top$ is completely join-prime, Lemma 4.91 guarantees j is an atom of $B_\top$ since $j \in \uparrow \eta - \{\eta\}$ by assumption. On the other hand, $a \in B_\top$ is completely join-prime if and only if $\downarrow a \cap \mathtt{MinSus}(\mathcal{L}) = \emptyset$; other wise $a = \eta \vee j$ for some $j \in \mathtt{MinSus}(\mathcal{L})$. □

Theorem 4.95. *If $\mathcal{L} = (L, \leq)$ is a cone lattice, then $\mathcal{L}$ is order-isomorphic to $\mathtt{Low}(\mathcal{P})$ for some hypergraph poset $\mathcal{P}$.*

Proof. Recall that in the introduction to this section, we highlighted the well-known fact that $\mathcal{L} \cong \mathtt{Low}(\mathtt{CJP}(\mathcal{L}))$; we will show that $(\mathtt{CJP}(\mathcal{L}), \leq)$ constitutes a hypergraph poset. First, it is well known that the atoms of a Boolean lattice are precisely its completely join-prime elements , and since the atoms of $B_\perp$ cover the least element of $\mathcal{L}$ it follows that these are completely join-prime in $\mathcal{L}$ as well. For convenience, let $\mathcal{A}_\perp$ denote the set of atoms for $B_\perp$. This of course implies $\mathtt{CJP}(\mathcal{L}) - \mathcal{A}_\perp \subseteq \mathcal{L} - B_\perp$. That is, if $j \in \mathtt{CJP}(\mathcal{L}) - \mathcal{A}_\perp$, then in light of Lemma 4.93 and Corollary 4.94 we know either $j \in \mathtt{MinSus}(\mathcal{L})$ or j is an atom of $B_\top$ and $\downarrow j \cap \mathtt{MinSus}(\mathcal{L}) = \emptyset$.

If we let $\mathtt{V}_{\mathcal{L}} = \mathtt{MinSus}(\mathcal{L}) \cup \mathcal{A}'_{\top}$ where $\mathcal{A}'_{\top}$ is the set of all the atoms of $B_{\top}$ that are completely join-prime in $\mathcal{L}$, we observe that $\mathtt{V}_{\mathcal{L}}$ is an antichain. Furthermore, we find that $\mathtt{V}_{\mathcal{L}} \cup \mathcal{A}_{\perp} = \mathtt{CJP}(\mathcal{L})$, and $\mathtt{V}_{\mathcal{L}} \cup \mathcal{A}_{\perp} = \emptyset$. Note that every member of $\mathcal{A}_{\perp}$ is minimal since they cover the least element of $\mathcal{L}$, and as a consequence the elements of $\mathtt{V}_{\mathcal{L}}$ are maximal.

From here we have established that $\mathtt{CJP}(\mathcal{L})$ can be expressed as the union of two disjoint antichains of minimal and maximal elements; it remains to show that if $a \in \mathcal{A}_{\perp}$ then $a < v$ for some $v \in \mathtt{V}_{\mathcal{L}}$. Observe that if $\mathcal{A}'_{\top}$ is nonempty, the result follows immediately since $B_{\perp} \subseteq \downarrow a$ whenever a is an atom of $B_{\top}$. If $\mathcal{A}'_{\top} = \emptyset$ we must have $\mathtt{V}_{\mathcal{L}} = \mathtt{MinSus}(\mathcal{L})$. Since $\mathtt{MinSus}(\mathcal{L})$ and $\mathcal{A}_{\perp}$ are disjoint sets of completely join-prime elements, it will suffice to show $\bigvee \mathtt{MinSus}(\mathcal{L}) \in B_{\top}$ since $\bigvee \mathcal{A}_{\perp} = \eta$. To that end, suppose $a \in \mathcal{A}_{\top}$. By hypothesis, $\mathcal{J}_{\mathrm{a}} = \mathcal{A}_{\perp} \cup \mathcal{S}$ for some nonempty $\mathcal{S} \subseteq \mathtt{MinSus}(\mathcal{L})$. But then

$$a = \bigvee \mathcal{J}_{\mathrm{a}} = \bigvee \mathcal{A}_{\perp} \vee \bigvee \mathcal{S} = \eta \vee \bigvee \mathcal{S}$$

From here it is easy to see that we must have $a = \bigvee \mathcal{S}$, and from here it follows $\bigvee \mathtt{MinSus}(\mathcal{L}) = \bigvee \mathcal{A}_{\top} = \top$. This of course means $b < \bigvee \mathtt{MinSus}(\mathcal{L})$ for every $b \in \mathcal{A}_{\perp}$, and since these elements are completely join-prime, we must have $b \leq j$ for at least one $j \in \mathtt{MinSus}(\mathcal{L})$. That is, every element in $\mathcal{A}_{\perp}$ is covered by at least one member of $\mathtt{V}_{\mathcal{L}}$ and $(\mathtt{CJP}(\mathcal{L}), \leq)$ constitutes a hypergraph poset.

$\square$

In light of Theorem 4.87, we have established the following result.

Theorem 4.96. *For a bialgebraic, distributive lattice $\mathcal{L}$, the following statements are equivalent.*

1. *$\mathcal{L}$ is a cone lattice.*

2. *The compact, join-prime elements of $\mathcal{L}$ form a hypergraph poset.*

3. *There is a hypergraph* $\mathcal{G}$ *such that* $\mathcal{L}$ *is order-isomorphic to* $\Omega(\mathcal{G})$.

CHAPTER 5

THE DUAL EQUIVALENCE OF THE CATEGORIES $\mathbf{HGraph}^*$ AND $\mathbf{Cone}$

5.1 The Prime Spectrum of a Cone Lattice

We will let $\mathtt{MP}(\mathcal{L})$ denote the subposet of meet-prime members of $\mathcal{L}$. If $\mathcal{L}$ is algebraic, then $\mathtt{MP}(\mathcal{L})$ is often called the *spectrum* of $\mathtt{Com}(\mathcal{L})$. It should be noted that in order theory contexts, the term "spectrum" has traditionally referred to the family of prime ideals (or prime filters) of a distributive join semilattice containing at least two elements (see Gratzer [13] for example). However, since every algebraic lattice is isomorphic to the ideal completion of its join sub semilattice of compact elements, the traditional understanding does not conflict with our definition. (Recall that the ideal completion of a poset $\mathcal{P}$ is the family of directed lowersets of $\mathcal{P}$, ordered by set-inclusion.)

With a nod to the term's origin in ring theory, some lattice theorists use "spectrum" to refer to the meet-prime elements of an algebraic lattice only when the lattice is distributive and its family of compact elements form a sublattice (and hence a commutative ring).

Recall that a join semilattice $\mathcal{J} = (J, \leq)$ is *distributive* provided for all $a, b, c \in J$ such that $a \leq b \vee c$, there exist $u \in\downarrow b$ and $v \in\downarrow c$ such that $a = u \vee v$. It is well-known that a (lower-bounded) join semilattice is distributive if and only if its ideal completion is a distributive lattice. (See Gratzer [13].)

The *Stone space* for a distributive join semilattice containing at least two elements is a topology defined on its spectrum. In particular, suppose that $\mathcal{J} = (J, \leq)$ is a lower-bounded, distributive join semilattice containing at least two elements and let $\mathtt{Idl}(\mathcal{J})$ represent its ideal completion. For each $x \in J$, let

$$\sigma(x) = \{P \in \texttt{MP}(\texttt{Idl}(\mathcal{J})) : x \notin P\}$$

The collection $\mathbb{B} = \{\sigma(x) : x \in J\}$ forms a basis for a topological space on the set $\texttt{MP}(\texttt{Idl}(\mathcal{J}))$ known as the *Stone space* for $\mathcal{J}$. The Stone space for a lower-bounded, distributive join semilattice $\mathcal{J}$ always has the following properties:

1. The space has the T_0 separation property.

2. The space has a lower-bounded basis of compact-open sets.

3. Every join-prime closed subset is the closure of a singleton.

In the abstract sense, we will say a topological space is a *Stone* topology provided it satisfies these properties. (The definition of Stone space varies somewhat in the literature; we are following Gratzer [13].) If a Stone topology has a compact-open basis that is a lower-bounded sublattice of the open set lattice, we will call this space a *spectral* topology. It is well-known that every Stone topology is homeomorphic to the Stone space of some lower-bounded, distributive join semilattice — namely the Stone space for its lower-bounded compact-open basis.

It is known that $\texttt{CJP}(\mathcal{L})$ and $\texttt{CMP}(\mathcal{L})$ are order-isomorphic for any complete lattice $\mathcal{L}$. (See for example Snodgrass and Tsinakis [26].) The isomorphism is accomplished via the mappings $\phi : \texttt{CMP}(\mathcal{L}) \longrightarrow \texttt{CJP}(\mathcal{L})$ and $\zeta : \texttt{CJP}(\mathcal{L}) \longrightarrow \texttt{CMP}(\mathcal{L})$ defined by

$$\phi(m) = \bigwedge\{x \in L : x \not\leq m\} \qquad \zeta(j) = \bigvee\{y \in L : j \not\leq y\}$$

Let us consider what this tells us about completely meet-prime elements in $\texttt{Low}(\mathcal{P})$ for any hypergraph poset $\mathcal{P}$. Suppose that U is completely meet-prime in $\texttt{Low}(\mathcal{P})$. This means that $U = \zeta(\downarrow p)$ where $\downarrow p \in \texttt{CJP}(\texttt{Low}(\mathcal{P}))$, hence

$$U = \bigcup\{I \in \texttt{Low}(\mathcal{P}) : \downarrow p \not\subseteq I\} = \bigcup\{I \in \texttt{Low}(\mathcal{P}) : p \notin I\}$$

Now, either $p \in \mathtt{E}(\mathcal{P})$ or $p \in \mathtt{V}(\mathcal{P})$. If $p \in \mathtt{E}(\mathcal{P})$, we know $\downarrow p = \{p\}$, so it follows that $U = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}) - (\downarrow p \cup \mathtt{Cov}(p))$. For simplicity, let

$$\neg(\mathtt{Cov}(p)) = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}) - (\downarrow p \cup \mathtt{Cov}(p))$$

On the other hand, if $p \in \mathtt{V}(\mathcal{P})$ we have $U = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}) - \{p\}$ since $\downarrow p$ is maximal in $\mathtt{Low}(\mathcal{P})$. In all that follows we can let $\gamma(v) = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P}) - \{v\}$ whenever $v \in \mathtt{V}(\mathcal{P})$. Observe that we have sufficiently proven the following theorem.

Theorem 5.97. *Let $\mathcal{P}$ be a hypergraph poset. A member U of $\mathtt{Low}(\mathcal{P})$ is completely meet-prime if and only if $U = \gamma(y)$ for some $y \in \mathtt{V}(\mathcal{P})$ or $U = \neg(\mathtt{Cov}(x))$ for some $x \in \mathtt{E}(\mathcal{P})$.*

If $\mathcal{L}$ is a complete Boolean lattice, then $\mathtt{CMP}(\mathcal{L}) = \mathtt{MP}(\mathcal{L})$. (This result may be deduced from the well-known fact that the prime ideals of any Boolean lattice are precisely the maximal ideals of that lattice.) This observation gives us the following result.

Lemma 5.98. *Let $\mathcal{P}$ be a hypergraph poset, and suppose $U \in B_\top$. If $U \neq \gamma(y)$ for some $y \in \mathtt{V}(\mathcal{P})$, then U is not meet-prime in $\mathtt{Low}(\mathcal{P})$.*

Proof. First, observe that the hypothesis that $U \in B_\top$ guarantees $\mathtt{E}(\mathcal{P}) \subset U$. As a consequence $U \neq \neg(\mathtt{Cov}(x))$ for *any* $x \in \mathtt{E}(\mathcal{P})$. A direct result of Theorem 5.97 tells us that if $U \neq \gamma(y)$ for any $y \in \mathtt{V}(\mathcal{P})$, then U can't be completely meet-prime in $\mathtt{Low}(\mathcal{P})$. Of course this means that U is not meet-prime in $B_\top$, so it can't be meet-prime in $\mathtt{Low}(\mathcal{P})$ either. $\square$

Suppose that $\mathcal{L} = (L, \leq)$ is any lattice, and suppose that $x, y \in L$. If x is meet-prime in $\mathcal{L}$, then either $x \wedge y = y$, or $x \wedge y$ is meet-prime in the sublattice $\downarrow y$. To see why this is so, let $z = x \wedge y$ and suppose $z < y$. If $u, v \in \downarrow y$ are such that $u \wedge v \leq z$, then clearly $u \wedge v \leq x$ as well; hence, we know $u \leq x$ or $v \leq x$. Suppose $u \leq x$. Since

$u \wedge y = u$, it follows that $u \leq z$ as well; and we may conclude that z is meet-prime in the sublattice $\downarrow y$. This observation gives us the following result.

Lemma 5.99. *Let $\mathcal{P} = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P})$ be a hypergraph poset. If $U \in \mathtt{Low}(\mathcal{P})$ is meet-prime, then $U \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P})$, or $U \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P}) - \{x\}$ for some $x \in \mathtt{E}(\mathcal{P})$.*

Proof. Clearly $U \cap \mathtt{E}(\mathcal{P}) \in B_\perp$ in either case. Since U is meet-prime, the above argument guarantees that either $U \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P})$, or $U \cap \mathtt{E}(\mathcal{P})$ is meet-prime in $\mathtt{Low}(\mathcal{P})$. In the latter case, it must be meet-prime in $B_\perp$, which is true if and only if $U \cap \mathtt{E}(\mathcal{P})$ is a co-atom of $B_\perp$; that is $U \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P}) - \{x\}$ for some $x \in \mathtt{E}(\mathcal{P})$. □

Theorem 5.100. *Let $\mathcal{P} = \mathtt{V}(\mathcal{P}) \cup \mathtt{E}(\mathcal{P})$ be a hypergraph poset. If $U \in \mathtt{Low}(\mathcal{P})$ is meet-prime but not maximal, then $U = \neg(\mathtt{Cov}(x))$ for some $x \in \mathtt{E}(\mathcal{P})$. In particular, U is completely meet-prime.*

Proof. Observe that every (completely) meet-prime element of $B_\top$ is maximal. This means that if U is meet-prime but not maximal, it cannot be a member of $B_\top$. It therefore follows that $M = U \cap \mathtt{E}(\mathcal{P}) \subset \mathtt{E}(\mathcal{P})$, and since M is meet-prime in $B_\perp$, Lemma 5.99 guarantees that $U \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P}) - \{\text{x}\}$ for some $x \in \mathtt{E}(\mathcal{P})$.

We know that $\neg(\mathtt{Cov}(x)) \cap \mathtt{E}(\mathcal{P}) = \mathtt{E}(\mathcal{P}) - \{x\}$ by definition, which gives us $\neg(\mathtt{Cov}(x)) \cap \mathtt{E}(\mathcal{P}) = U \cap \mathtt{E}(\mathcal{P})$, which means $\neg(\mathtt{Cov}(x)) \cap \mathtt{E}(\mathcal{P}) \subseteq U$. This in turn implies $\neg(\mathtt{Cov}(x)) \subseteq U$ since U is meet-prime. On the other hand, a symmetric argument guarantees $U \subseteq \neg(\mathtt{Cov}(x))$ since $\neg(\mathtt{Cov}(x))$ is meet-prime as well, hence the claim is proved. □

For any hypergraph poset $\mathcal{P}$, we have now established that $\mathtt{MP}(\mathtt{Low}(\mathcal{P})) = \mathtt{CMP}(\mathtt{Low}(\mathcal{P}))$. Consequently, we know that $\mathtt{MP}(\mathtt{Low}(\mathcal{P}))$ is a hypergraph poset that is order-isomorphic to $\mathcal{P}$. In light of this observation, we have the following result.

Corollary 5.101. *If $\mathcal{L} = (L, \leq)$ is a bialgebraic cone lattice, then* $\mathtt{Low}(\mathtt{MP}(\mathcal{L}))$ *is order isomorphic to $\mathcal{L}$; in particular,* $\mathtt{CJP}(\mathcal{L})$ *is order-isomorphic to* $\mathtt{MP}(\mathcal{L})$*.*

This result has important implications for hypergraphs and their associated graph topologies, since it tells us the incidence poset for any hypergraph may be viewed as either the family of compact join-prime members or the family of meet-prime members of its open set lattice under the graph topology.

Suppose that $\mathcal{L} = (L, \leq)$ is a bialgebraic cone lattice. We may consider the hypergraph associated with $\mathcal{L}$ to be the pair $\mathcal{G}_L = (\mathtt{V}(\mathtt{MP}(\mathcal{L})), \mathtt{E}(\mathtt{MP}(\mathcal{L})))$. A basis for the graph topology Ω_{G_L} would be the family

$$\mathcal{B} = \{\{p\} : p \in \mathtt{E}(\mathtt{MP}(\mathcal{L}))\} \cup \{B(q) : q \in \mathtt{V}(\mathtt{MP}(\mathcal{L}))\}$$

where $m \in B(q) - \{q\}$ if and only if m is covered by q in $\mathtt{MP}(\mathcal{L})$.

Let us see what implications this has for the Stone topology associated with the join semilattice $\mathtt{Com}(\mathcal{L})$ of a bialgebraic cone lattice $\mathcal{L}$. We will let the Stone topology on $\mathtt{Com}(\mathcal{L})$ be represented by the pair $\mathtt{Stone}_L = (\mathtt{MP}(\mathcal{L}), \Omega^*)$. Since every compact member of $\mathcal{L}$ is the union of a (finite) family of compact join-prime members, it follows that the collection

$$\mathbb{B} = \{\sigma(u) : u \in \mathcal{P}_L\}$$

constitutes a basis for the Stone space, where $\sigma(u) = \{p \in \mathtt{MP}(\mathcal{L}) : u \not\leq p\}$.

Suppose that$\mathcal{L} = (L, \leq)$ is a bialgebraic cone lattice. Suppose that $u \in \mathtt{CJP}(\mathcal{L})$ and consider the (completely) meet-prime element

$$\zeta(u) = \bigvee \{y \in L : u \not\leq y\}$$

It is clear that $\zeta(u) \in \sigma(u)$. On the other hand, if $p \in \sigma(u)$, then we must have $p \leq \zeta(u)$.

Now, suppose that $u \in \mathtt{E}(\mathtt{CJP}(\mathcal{L}))$. It follows that $\zeta(u)$ is minimal in $\mathtt{MP}(\mathcal{L})$; therefore, we must conclude that $\sigma(u) = \{\zeta(u)\}$.

On the other hand, suppose that $u \in \mathtt{V}(\mathtt{CJP}(\mathcal{L}))$. It follows that $\zeta(u)$ is maximal in $\mathtt{MP}(\mathcal{L})$; therefore, those members p of $\mathtt{MP}(\mathcal{L})$ such that $p \leq \zeta(u)$ are precisely those minimal members of $\mathtt{MP}(\mathcal{L})$ that are covered by $\zeta(u)$. Consequently, we may conclude that $\sigma(u) = B(\zeta(u))$.

We have proven the following result.

Corollary 5.102. *If $\mathcal{L} = (L, \leq)$ is a bialgebraic cone lattice, then $\mathtt{Stone}_L$ is homeomorphic to the graph topology on $\mathcal{G}_L$. Furthermore, if we identify the vertices and edges of $\mathcal{G}_L$ with the meet-prime elements of $\mathcal{L}$, then the topologies are equal.*

At this point, it is worth noting that a well-known forbidden substructure condition characterizes those bialgebraic cone lattices associated with graphs, and we conclude this section by introducing this condition.

A lower-bounded, distributive lattice $\mathcal{L}$ is *relatively n-normal* provided every prime ideal of $\mathcal{L}$ is contained in at most n maximal ideals for some fixed positive integer n. Lattices satisfying the order-dual of this definition were first introduced in Cornish [7], and relatively 1-normal lattices (usually referred to as "relatively normal" lattices) have been extensively studied (see for example Mandelker [21] or Hart and Tsinakis [14]).

Definition 5.103. An algebraic, distributive lattice $\mathcal{L} = (L, \leq)$ is *weakly relatively n-normal* provided every meet-prime element is exceeded by at most n maximal members of $\mathcal{L}$ for some fixed positive integer n. We say $\mathcal{L}$ is *strongly* relatively n-normal provided it is weakly relatively n-normal and $\mathtt{Com}(\mathcal{L})$ is a sublattice of $\mathcal{L}$.

Note that an algebraic, distributive lattice $\mathcal{L}$ is strongly relatively n-normal if and only if $\mathtt{Com}(\mathcal{L})$ is relatively n-normal. The following result is a direct consequence of Corollary 5.101 and Theorem 4.78.

Corollary 5.104. *The following statements are true for any hypergraph $\mathcal{G} = (G, E)$.*

1. *$\mathcal{G}$ is a graph if and only if $\Omega(\mathcal{G})$ is weakly 2-normal.*

2. *$\mathcal{G}$ is a finitely based graph if and only if $\Omega(\mathcal{G})$ is strongly 2-normal.*

Proof. To prove (1), let $\mathcal{G} = (G, E)$ be a hypergraph and let $\Omega(\mathcal{G})$ be its corresponding open-set lattice under the graph topology. If we suppose $U \in \mathcal{MP}(\Omega(\mathcal{G}))$ is not maximal. As we have seen, Theorem 5.100 guarantees $U = \neg\texttt{Cov}(x) = \mathcal{G} - (\{\mathrm{x}\} \cup \pi(\mathrm{x}))$ for some $x \in E$. From here we observe that $\mathcal{G}$ is a graph if and only if we can assume $\pi(x) = \{y_1, y_2\}$ for every $x \in E$ (where $y_1, y_2 \in G$ are not necessarily distinct). Observe that this is true if and only if U is exceeded by at most two distinct maximal elements; namely $\mathcal{G} - \{\mathrm{y}_1\}$ or $\mathcal{G} - \{\mathrm{y}_2\}$.

To prove (2) it is sufficient to observe that if $\mathcal{G}$ is a graph, it is finitely based if and only if it satisfies part (4) from Theorem 4.78, which are precisely the conditions under which $\texttt{Com}(\mathcal{L})$ is a meet-semilattice (and therefore a sublattice of $\Omega(\mathcal{G})$). $\square$

5.2 The Categories HGraph*, and Cone

Let $\mathcal{G}$ be a hypergraph. The fact that $\Omega(\mathcal{G})$ is the Stone space associated with the join-semilattice generated by hypergraph poset $\mathcal{P}_G$ along with the categorical equivalence between **HGraph** and **HPoset** strongly suggest the existence of a Stone-type duality between graphs and cone lattices. In this section, we exploit the results presented in previous sections to verify that this is indeed the case for a *subcategory* of **HGraph** and a category of cone lattices. This material draws upon well-known results from the realm of representation theory (see Johnstone [17]). We begin by summarizing the well-known "big picture" germane to our purposes.

Suppose that $\mathcal{L} = (L, \leq)$ is an algebraic frame. For each $x \in \texttt{Com}(\mathcal{L})$, let $\sigma_L(x) = \{p \in \texttt{MP}(\mathcal{L}) : x \nleq p\}$. It is well-known that the set

$$\mathbb{B}_L = \{\sigma_L(x) : x \in \texttt{Com}(\mathcal{L})\}$$

forms the basis for a Stone topology $\texttt{Stone}_L = (\texttt{MP}(\mathcal{L}), \Omega_L)$. Moreover, it is a routine exercise to show that $\mathcal{L}$ is order-isomorphic to the lattice $(\Omega_L, \subseteq)$.

On the other hand, suppose that $\mathcal{S} = (S, \Omega)$ is a Stone topology. Since $\mathcal{S}$ has a compact-open basis that is a join-semilattice, it is clear that $\mathcal{L}_S = (\Omega, \subseteq)$ is an algebraic frame. Furthermore, we know that $\mathcal{S}$ is homeomorphic to the space $\texttt{Stone}_{\mathcal{L}_S}$.

Suppose that $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are algebraic frames, and suppose that $f : L \longrightarrow M$ is a *frame homomorphism*. (That is, f preserves finite meets and arbitrary joins.) Since $\mathcal{L}$ and $\mathcal{M}$ are complete lattices, the assumption that f preserves arbitrary joins tells us f has an *upper adjoint* $\tau_f : M \longrightarrow L$. These functions satisfy the relationship

$$f(x) \sqsubseteq y \iff x \leq \tau_f(y)$$

It is well-known that the function τ_f is defined by

$$\tau_f(y) = \bigvee\{x \in L : f(x) \sqsubseteq y\}$$

Furthermore, it is well-known that the function τ_f preserves meet-prime elements. We provide a proof of this critical fact for completeness.

Lemma 5.105. *The function τ_f defined above preserves meet-prime elements.*

Proof. Suppose $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are algebraic frames and $f : L \longrightarrow M$ is a frame homomorphism. Let $m \in M$ be meet-prime and consider $l = \tau_f(m) \in L$. If we suppose $\mathcal{F} = \{x_1, \ldots, x_n\} \subseteq L$ is such that $\bigwedge \mathcal{F} \leq l$, we observe that since f

is a frame homomorphism we have

$$f(\bigwedge \mathcal{F}) = f(\bigwedge_{i=1}^{n} x_i) = \bigwedge_{i=1}^{n} f(x_i) \sqsubseteq m$$

Of course, this means there exists $x_i \in \mathcal{F}$ such that $f(x_i) \sqsubseteq m$, and as a consequence of the isotone nature of adjoints we must have $x_i \leq l$. This proves l is meet-prime in $\mathcal{L}$. $\square$

Let $\tau_f^* : \mathtt{MP}(\mathcal{M}) \longrightarrow \mathtt{MP}(\mathcal{L})$ denote the restriction of τ_f to the meet-prime members of $\mathcal{M}$. If we make the additional assumption that the function f preserves compact elements, then τ_f^* is continuous relative to $\mathtt{Stone}_M$ and $\mathtt{Stone}_L$. To see why this is so, observe

$$\begin{aligned} [\tau_f^*]^{-1}(\sigma_L(j)) &= \{p \in M^* : \tau_f^*(p) \in \sigma_L(j)\} \\ &= \{p \in M^* : j \not\leq \tau_f^*(p)\} \\ &= \{p \in M^* : f(j) \not\leq p\} \\ &= \sigma_M(f(j)) \end{aligned}$$

Since the inverse image under τ_f^* of a basic-open set is itself basic-open, we may conclude that τ_f^* is continuous relative to the Stone topologies. Of course, we have actually proven more than this — we have shown that the inverse image under τ_f^* of a *compact-open* set is *compact-open*. In topological parlance, continuous functions having this property are called *spectral* maps.

On the other hand, suppose that $\mathcal{S} = (S, \Omega_S)$ and $\mathcal{T} = (T, \Omega_T)$ are Stone spaces. If $f : S \longrightarrow T$ is a spectral map with respect to these topologies, then it is easy to see that the function $\varphi_f : \Omega_T \longrightarrow \Omega_T$ defined by $\varphi_f(U) = f^{-1}(U)$ is a frame homomorphism that preserves compact elements.

Let **AFrame** denote the category consisting of algebraic frames coupled with frame homomorphisms that preserve compact elements, and let **Stone** denote the category consisting of Stone spaces coupled with spectral maps. The stage has now been set to present a classic categorical duality.

Theorem 5.106. *The category* $\mathbf{AFrame}$ *is dually equivalent to the category* $\mathbf{Stone}$. *The duality is accomplished via the contravariant functors*

- $\mathtt{FRM} : \mathbf{Stone} \longrightarrow \mathbf{AFrame}$ *defined by* $\mathtt{FRM}[\mathcal{S}] = (\Omega_S, \subseteq)$ *and* $\mathtt{FRM}[f] = \varphi_f$
- $\mathtt{STN} : \mathbf{AFrame} \longrightarrow \mathbf{Stone}$ *defined by* $\mathtt{STN}[\mathcal{L}] = \mathtt{Stone}_L$ *and* $\mathtt{STN}[\varphi] = \tau^*_\varphi$

The proof is a straightforward category theory exercise; we refer the reader to Johnstone [17] for details. Let us consider how we may exploit this result in the context of our study on hypergraphs.

WHAT WE ALREADY KNOW

- The graph topology on any hypergraph $\mathcal{G} = (G, E)$ is homeomorphic to the Stone topology associated with the join-semilattice $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$.
- For any hypergraph $\mathcal{G} = (G, E)$, the algebraic frame that is the ideal completion of $\mathtt{JS}(\mathtt{B}(\mathcal{G}))$ is a cone lattice.
- If $\mathcal{L} = (L, \leq)$ is any cone lattice, then the Stone space associated with $\mathtt{Com}(\mathcal{L})$ is homeomorphic to the graph topology on some hypergraph, namely the hypergraph whose incidence poset is isomorphic to $\mathtt{CJP}(\mathcal{L})$.

Suppose $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are hypergraphs, and suppose that $f : G \cup E_G \longrightarrow H \cup E_H$ is an HG-morphism. What additional conditions must we place on f in order to guarantee it is spectral with respect to the graph topologies?

Let $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ be hypergraphs, and suppose $f : G \cup E_G \longrightarrow H \cup E_H$ is an HG-morphism. For any $y \in H$, we will say that $\epsilon \in E_G$ is an *incidence-orphan* relative to y provided $y \notin f(\pi_G(\epsilon))$. (In other words, $f(\epsilon)$ is incident to y, but no vertex incident to ϵ is mapped to y by f.) For any $y \in H$, we will let $\mathtt{IO}(y)$ represent the set of all incidence-orphans relative to y. It is clear that

$$f^{-1}(B(y)) = \bigcup\{B(x) : x \in f^{-1}(\{y\})\} \cup \mathtt{IO}(y)$$

Definition 5.107. Let $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ be hypergraphs, and suppose $f : G \cup E_G \longrightarrow H \cup E_H$ is an HG-morphism.

- We will say that f is *anchored* provided $\mathtt{IO}(y)$ is finite for all $y \in H$.
- We will say that f is *finite-based* provided $f^{-1}(\{y\})$ is finite for all $y \in H \cup E_H$.

Theorem 5.108. *Suppose that $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H,)$ are hypergraphs. An HG-morphism from $\mathcal{G}$ to $\mathcal{H}$ is finite-based and anchored if and only if it is spectral relative to the graph topologies.*

Proof. First, suppose $f : G \longrightarrow H$ is a spectral map with respect to the graph topologies and let $y \in H \cup E_H$. We observe that if $y \in E_H$, then we know that since singleton edges are compact open under the graph topology, by hypothesis we must have $f^{-1}(\{y\}) = \bigcup_{i=1}^{n}\{e_i\}$ where $\{e_i\}_{i=1}^{n} \subseteq E_G$. On the other hand, if $y \in H$ and observe that if $f^{-1}(y) = \emptyset$, it is vacuously finite-based. If $f^{-1}(y) \neq \emptyset$, recall $B(y) \in \mathtt{B}(\mathcal{H})$ is compact, so $f^{-1}(B(y))$ is compact as well since f is a spectral map. This means there is a finite family $\mathcal{F} := \{B(x_i)\}_{i=1}^{n} \subseteq \mathtt{B}(\mathcal{G})$ such that $f^{-1}(B(y)) \subseteq \bigcup_{i=1}^{n} B(x_i)$. From here we clearly have $f(x_i) = y$ for each i since f is an HG-morphism, hence $f^{-1}(y) = \{x_i : B(x_i) \in \mathcal{F}\}$ and f is finite-based.

Furthermore, observe that y has no incidence-orphans in this case; indeed if $e \in E_G$ is such that $y \in \pi_H(f(e))$, we must have $f(e) \in f(B(x_e))$ for some $x_e \in f^{-1}(y)$. This means that $\mathtt{IO}(y)$ is empty. Furthermore, if $f^{-1}(y) = \emptyset$, it is vacuously finite-based, we need only verify $\mathtt{IO}(y)$ is finite. Indeed, if $\mathtt{IO}(y)$ is nonempty, the fact that $\mathtt{IO}(y) \subseteq f^{-1}(B(y))$ and $f^{-1}(B(y)) \subseteq E_G$ tells us there is a finite $\{e_i\}_{i=1} \subseteq E_G$

such that $\bigcup_{i=1}^{n}\{e_i\}$. We may therefore conclude that f is anchored.

Now suppose that f in *not* spectral. This means there exists $U \in \texttt{Com}(\Omega_H)$ such that $f^{-1}(U)$ is not compact under Ω_H. Without loss of generality we may assume $U = \bigcup_{i=1}^{n}\{e_i\} \cup \bigcup_{j=1}^{m} B(y_j)$ for some finite collections $\{e_1, \ldots, e_n\} \subseteq E_H$ and $\{y_1, \ldots, y_m\} \subseteq H$. It follows from basic set theory that

$$f^{-1}(U) = \bigcup_{i=1}^{n} f^{-1}(\{e_i\}) \cup \bigcup_{j=1}^{m} f^{-1}(B(y_j))$$

Since by hypothesis this collection is not compact, the preimage of at least one $\{e_i\}$ or $B(y_j)$ must possess a cover which does not yield a finite subcover. If this is the case for some $\{e_i\}$, then $f^{-1}(\{e_i\}) \subseteq E_G$ is infinite and e_i is not finitely based. Similarly, if this is true for some $B(y_j)$, then $f^{-1}(B(y_j))$ can be represented as the infinite union of edge-balls from $\mathcal{G}$; consequently this implies $f^{-1}(y_j) = f^{-1}(B(y_j)) \cap G$ is infinite, hence f *cannot* be finite-based.

Since U is compact, we may assume that $U = \bigcup \mathcal{F}_E \cup \bigcup \mathcal{F}_H$ where $\mathcal{F}_E \subseteq E_H$ and $\mathcal{F}_H \subseteq \{B(x) : x \in H\}$. In the first case, the fact that $f^{-1}(U) \subseteq E_G$ is not compact implies that any cover of $f^{-1}(U)$ must be infinite (and therefore not finite-based). In the second case we may assume $U = \bigcup_{i=1}^{n} B(y_i)$ where $\{y_i\}_{i=1}^{n} \subseteq H$. In this case we have

$$f^{-1}(U) = \bigcup_{i=1}^{n} f^{-1}(B(y_i)) = \bigcup_{i=1}^{n}\{B(x_i) : x_i \in f^{-1}(y_i)\} \cup \bigcup_{i=1}^{n} \texttt{IO}(y_i)$$

The hypothesis that $f^{-1}(U)$ forces us to conclude that either $B_U = \{B(x_i) : x_i \in f^{-1}(y_i)\}$ or $\texttt{IO}(y_i)$ is not finite. The first case is true if and only if $\{x_i \in G : x_i \in f^{-1}(y_i)\}$ is infinite (i.e. f is not finite-based) and the second case implies $\texttt{IO}(y_i)$ is infinite for at least one $i \in \{1, 2, \ldots, n\}$. Observe that f fails to be anchored in the latter case, and the proof is complete. $\square$

Definition 5.109. Suppose $\mathcal{L}$ and $\mathcal{M}$ are cone lattices and $\varphi : L \longrightarrow M$ is a frame homomorphism. We say the map φ is *grounded* provided for every $p \in \max(\texttt{MP}(\mathcal{M}))$ there exists a unique $j \in \texttt{CJP}(\mathcal{L})$ such that $\varphi(j) \not\leq p$.

In the work to follow, we will let **HGraph*** represent the category whose objects are hypergraphs and whose morphisms are finite-based anchored HG-morphisms. Note that **HGraph*** is a subcategory of **HGraph** but is not a *full* subcategory of this category.

Now, if $\mathcal{G} = (G, E_G)$ and $\mathcal{H} = (H, E_H)$ are hypergraphs and $f : G \cup E_G \longrightarrow H \cup E_H$ is a finite-based anchored HG-morphism, then it follows from Theorem 5.106 that the function $\varphi_f : \Omega(\mathcal{H}) \longrightarrow \Omega(\mathcal{G})$ is a frame homomorphism that preserves compact elements.

On the other hand, suppose that $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are cone lattices; and suppose that $\varphi : L \longrightarrow M$ is a frame homomorphism that preserves compact elements. We know that the function $\tau_\varphi^* : \texttt{MP}(\mathcal{M}) \longrightarrow \texttt{MP}(\mathcal{L})$ is spectral with respect to the graph topologies on the hypergraphs $\mathcal{G}_{\texttt{MP}(\mathcal{M})}$ and $\mathcal{G}_{\texttt{MP}(\mathcal{L})}$. The question remains, however — is the function an HG-morphism? We answer this question through the following results.

The next results rely heavily on the fact that, if $\mathcal{L}$ and $\mathcal{M}$ are cone lattices, then $\texttt{MP}(\mathcal{L})$ and $\texttt{MP}(\mathcal{M})$ are graph posets that are order-isomorphic to $\texttt{CJP}(\mathcal{L})$ and $\texttt{CJP}(\mathcal{M})$, respectively.

Lemma 5.110. *Suppose that $\mathcal{L} = (L, \leq)$ is a cone lattice, and suppose that $p \in \texttt{MP}(\mathcal{L})$.*

1. *We have p maximal in $\texttt{MP}(\mathcal{L})$ if and only if there is exactly one CJP element m_p such that $m_p \not\leq p$. This element is necessarily maximal in $\texttt{CJP}(\mathcal{L})$.*

2. *We have p minimal in $\texttt{MP}(\mathcal{L})$ if and only if there is exactly one CJP element m_p such that m_p is minimal in $\texttt{CJP}(\mathcal{L})$ and $m_p \not\leq p$. Furthermore, if p is not maximal in $\texttt{MP}(\mathcal{L})$, then m_p is not maximal in $\texttt{CJP}(\mathcal{L})$.*

Proof. Recall the order-isomorphisms defined by the maps ϕ and ζ defined in the previous section where $\phi(m) = \bigwedge\{x \in L : x \not\leq m\}$. Observe that p_1 is maximal in $\mathtt{MP}(\mathcal{L})$ if and only if $\phi(p) = j_{p_1}$ is maximal in $\mathtt{CJP}(\mathcal{L})$. Similarly p_2 is minimal in $\mathtt{MP}(\mathcal{L})$ if and only if $\phi(p) = j_{p_2}$ is minimal in $\mathtt{CJP}(\mathcal{L})$. □

Theorem 5.111. *Suppose that $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are cone lattices and suppose that $\varphi : L \longrightarrow M$ is a grounded frame homomorphism.*

1. *If $j \in \mathtt{CJP}(\mathcal{M})$ is minimal in this poset, then $\varphi(\tau_\varphi(j)) = j$.*
2. *If $p \in \mathtt{MP}(\mathcal{M})$ is maximal in this poset, then $\tau_\varphi(p)$ is maximal in $\mathtt{MP}(\mathcal{L})$.*
3. *If $p, q \in \mathtt{MP}(\mathcal{M})$ and $p \sqsubset q$, then $\tau_\varphi(p) < \tau_\varphi(q)$.*
4. *If $p \in \mathtt{MP}(\mathcal{M})$ is minimal but not maximal in this poset, then $\tau_\varphi(p)$ is minimal but not maximal in $\mathtt{MP}(\mathcal{L})$.*

Proof. To prove (1), recall that we have established that $\varphi, \tau_\varphi : L \rightleftharpoons M$ defines an adjunction between L and M. We know this is true if and only if $\tau_\varphi \circ \varphi$ and $\varphi \circ \tau_\varphi$ form closure and kernel operators, respectively. Since $\varphi \circ \tau_\varphi$ is a kernel operator, it follows $\varphi \circ \tau_\varphi(m) \leq m$ for every $m \in M$. If we suppose $j \in \mathtt{CJP}(\mathcal{M})$ is minimal, we observe that $\varphi \circ \tau_\varphi(j) \leq j$. We also recall that since cone lattices are bialgebraic, every element can be expressed as the join of completely join-prime elements. This means there exists $j^* \in \mathtt{CJP}(\mathcal{M})$ such that $j^* \sqsubseteq \varphi \circ \tau(j)$. Since we chose j to be minimal, we must have $j^* = j$, hence (1) is established.

To establish (2), if we suppose $p \in \mathtt{MP}(\mathcal{M})$ is maximal, we know by Lemma 5.110 there exists a unique maximal $j_p \in \mathtt{CJP}(\mathcal{M})$ such that $j_p \not\leq p$. Note that it is necessarily true that $p < \top_M$, and since τ preserves meet-prime elements it follows that $\tau(p) < \top_L$, hence there exists $j_l \in \mathtt{CJP}(\mathcal{L})$ such that $j_l \not\leq \tau(p)$. Of course, this means $\varphi(j_l) \not\sqsubseteq p$. Since φ is grounded we may conclude that j_l is unique and

$\varphi(j_l) = j_m$. This means there is precisely one $j_l \in \mathtt{CJP}(\mathcal{L})$ such that $j \not\leq \tau(p)$, hence $\tau(p)$ is maximal in $\mathtt{MP}(\mathcal{L})$.

To establish (3), we observe that if $p, q \in \mathtt{MP}(\mathcal{M})$ and $p \sqsubset q$, Theorems 9 and 10 guarantee that p is minimal (but not maximal) whereas q is maximal in $\mathtt{MP}(\mathcal{M})$. Again, Lemma $p \sqsubset q$ guarantees there exists a unique minimal $j_m \in \mathtt{CJP}(\mathcal{M})$ such that $j_p \not\sqsubseteq p$. Since j_p is minimal but not maximal, we know $j_p \sqsubseteq q$. By (1), it follows $\varphi(\tau_\varphi(j_m)) = j_m$, and since τ_φ is isotone we must have $\tau_\varphi(j_m) \not\leq \tau(p)$. It follows that there must exist some $k \in \mathtt{CJP}(\mathcal{L}$ such that $k \leq \tau_\varphi(j_m)$ but $k \not\leq \tau_\varphi(p)$. Since $k \leq \tau_\varphi(q)$, we must conclude $\tau_\varphi(p) < \tau_\varphi(q)$. We observe that (4) also directly follows from the preceding argument. $\square$

We will let **Cone** represent the directed graph whose objects are cone lattices and whose morphisms are those grounded frame homomorphisms that preserve compact elements. We will verify that **Cone** is in fact a category; to do so we need only verify that $\mathtt{Ar}_{\mathbf{Cone}}$ is closed under composition.

Lemma 5.112. *Suppose $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are cone lattices where $f : \mathcal{L} \longrightarrow \mathcal{M}$ and $g : \mathcal{M} \longrightarrow \mathcal{N}$ are grounded frame homomorphisms. Then $g \circ f : \mathcal{L} \longrightarrow \mathcal{N}$ is a grounded frame homomorphism as well.*

Proof. Let $f : L \longrightarrow M$ and $g : M \longrightarrow N$ be grounded maps. If p is maximal in $\mathtt{MP}(\mathcal{N})$, then Theorem 5.111 guarantees there exists a unique $j \in \mathtt{CJP}(L)$ with the property that $f(j) \not\leq \tau_g(p)$. Consequently, this guarantees $g(f(j)) \not\leq p$ and we may conclude $g \circ f$ is grounded. $\square$

In light of the preceding lemma, we conclude **Cone** is indeed a category.

Suppose $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are cone lattices and suppose $\varphi : L \longrightarrow M$ is a grounded frame homomorphism. Let τ_φ^* represent the restriction of the upper adjoint τ_φ to the meet-prime elements of $\mathcal{M}$. Theorem 5.111 tells us that τ_φ^* is an

HP-morphism from the hypergraph poset $\mathcal{M}^* = (\mathtt{MP}(\mathcal{M}), \sqsubseteq)$ to the hypergraph poset $\mathcal{L}^* = (\mathtt{MP}(\mathcal{L}), \leq)$. Consequently, Theorem 4.86 tells us the function $\mathtt{HG}[\tau_\varphi^*]$ is an HG-morphism from the hypergraph $\mathcal{G}_{\mathcal{M}^*}$ to the hypergraph $\mathcal{G}_{\mathcal{L}^*}$. We have now proven the following result.

Corollary 5.113. *Suppose $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \sqsubseteq)$ are cone lattices and suppose $\varphi : L \longrightarrow M$ is a grounded frame homomorphism. If φ preserves compact elements, then $\mathtt{HG}[\tau_\varphi^*]$ is a finite-based anchored HG-morphism.*

5.3 Dual Adjunction and Equivalence

In order to extend what we have already seen, we need one more result. This result, which is a minor extension of the work done by Matthew Wiese in his 2016 thesis, establishes a necessary class of isomorphisms in the category $\mathbf{HGraph}^*$ which we will use to complete the duality.

Lemma 5.114. *Suppose $\mathcal{G}$ is a hypergraph and $\Omega(\mathcal{G})$ is its associated graph topology open lattice. Then there exists a hypergraph $\mathcal{G}_{\Omega(\mathcal{G})}$ associated with $\Omega(\mathcal{G})$ such that $\mathcal{G}$ is isomorphic to $\mathcal{G}_{\Omega(\mathcal{G})}$.*

Proof. If $\mathcal{G}$ is a hypergraph and $\Omega(\mathcal{G})$ is its associated graph topology open lattice, then $\mathtt{MP}(\Omega(\mathcal{G}))$ is a hypergraph poset. Observe that $\mathcal{P}_\mathcal{G}$ is the hypergraph poset associated with $\mathcal{G}$, and we know that $\mathcal{P}_\mathcal{G}$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$. Since $\mathtt{MP}(\Omega(\mathcal{G}))$ is a hypergraph poset, there exists a hypergraph $\mathcal{G}_{\Omega(\mathcal{G})}$ where $\mathcal{P}_{\mathcal{G}_{\Omega(\mathcal{G})}}$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$. This in turn implies $\mathcal{P}_\mathcal{G} \cong \mathcal{P}_{\mathcal{G}_{\Omega(\mathcal{G})}}$, and therefore $\mathcal{G} \cong \mathcal{G}_{\Omega(\mathcal{G})}$. ∎

In this section, we will establish the dual adjunction between the categories $\mathbf{Cone}$ and $\mathbf{HGraph}^*$. First, we define the maps $\mathtt{Co} : \mathbf{HGraph}^* \longrightarrow \mathbf{Cone}$ and $\mathtt{HG}^* : \mathbf{Cone} \longrightarrow \mathbf{HGraph}^*$. Suppose $\mathcal{G}, \mathcal{H} \in \mathbf{HGraph}^*$ are finite-based hypergraphs and

$f : G \cup E_G \longrightarrow H \cup E_H$ is a finite based anchored HG-morphism, we shall define $\mathtt{Co}(G) = \Omega(G)$ where $\Omega(G)$ is the cone lattice induced by G; $\mathtt{Co}(H) = \Omega(H)$ is of course identically defined. From here we define $\mathtt{Co}(f) = \varphi_f : \Omega(H) \longrightarrow \Omega(G)$ as the compact element preserving grounded frame homomorphism induced by f.

Next, suppose $\mathcal{L}, \mathcal{M} \in \mathbf{Cone}$ are cone lattices and $\varphi : L \longrightarrow M$ is a grounded frame homomorphism that preserves compact elements. From here we define $\mathtt{HG}^* : \mathbf{Cone} \longrightarrow \mathbf{HGraph}^*$ such that $\mathtt{HG}^*(L) = \mathcal{G}_{\mathcal{L}^*}$ where $\mathcal{G}_{\mathcal{L}^*}$ is the hypergraph induced by the cone lattice $\mathcal{L} = (L, \leq)$ via the hypergraph poset $\mathcal{L}^* = (\mathtt{MP}(\mathcal{L}), \leq)$. Of course, $\mathtt{HG}^*(M) = \mathcal{G}_{\mathcal{M}^*}$ is again identically defined as the hypergraph induced by the cone lattice $\mathcal{M} = (M, \sqsubseteq)$ via the hypergraph poset $\mathcal{M}^* = (\mathtt{MP}(\mathcal{M}), \sqsubseteq)$. If we suppose $\varphi : \mathcal{L} \longrightarrow \mathcal{M}$ is a compact element preserving grounded frame homomorphism, we will define $\mathtt{HG}^*(\varphi) = \mathtt{HG}[\tau_\varphi^*]$.

Lemma 5.115. *The maps* $\mathtt{Co}$ *and* $\mathtt{HG}^*$ *define contravariant functors between the categories* $\mathbf{HGraph}^*$ *and* $\mathbf{Cone}$*.*

Proof. First, suppose $\mathcal{G}$ and $\mathcal{H}$ are hypergraphs and $f : \mathcal{G} \longrightarrow \mathcal{H}$ is a finite-based HG-morphism. As we have seen, $\Omega(G)$ and $\Omega(H)$ are indeed cone lattices, and since f is finite-based (and therefore a spectral map) there exists a grounded frame homomorphism $\varphi_f : \Omega(H) \longrightarrow \Omega(G)$ that preserves compact elements. This verifies that $\mathtt{Co}$ is indeed a functor, and it satisfies the first condition of Definition 3.68.

To verify the remaining conditions, if we suppose $\mathcal{G}, \mathcal{H}, \mathcal{J} \in \mathbf{HGraph}^*$ and $u : \mathcal{G} \longrightarrow \mathcal{H}$ and $v : \mathcal{H} \longrightarrow \mathcal{J}$ are finite-based HG-morphisms, we have $v \circ u : \mathcal{G} \longrightarrow \mathcal{J}$

is also a finite-based HG-morphism. Observe that

$$\begin{aligned}\mathtt{Co}(v \circ u) &= \varphi_{v \circ u} : \Omega(J) \longrightarrow \Omega(G) \\ &= \varphi_u \circ \varphi_v \\ &= \mathtt{Co}(u) \circ \mathtt{Co}(v)\end{aligned}$$

Consequently, if $\mathtt{Id}_G : \mathcal{G} \longrightarrow \mathcal{G}$ is any identity map in $\mathbf{HGraph}^*$ and $f : \mathtt{dom}(f) \longrightarrow \mathcal{G}$ and $g : \mathcal{G} \longrightarrow \mathtt{cod}(g)$ are finite-based HG-morphisms, it follows

$$\begin{aligned}\mathtt{Co}(f) &= \mathtt{Co}(\mathtt{Id}_G \circ f) \\ &= \mathtt{Co}(f) \circ \mathtt{Co}(\mathtt{Id}_G) \\ &= \varphi_f \circ \varphi_{\mathtt{Id}_G} \\ &= \varphi_f \text{ since } \mathtt{Co}(f) = \varphi_f\end{aligned} \qquad \begin{aligned}\mathtt{Co}(g) &= \mathtt{Co}(g \circ \mathtt{Id}_G) \\ &= \mathtt{Co}(\mathtt{Id}_G) \circ \mathtt{Co}(g) \\ &= \varphi_{\mathtt{Id}_G} \circ \varphi_g \\ &= \varphi_g \text{ since } \mathtt{Co}(g) = \varphi_g\end{aligned}$$

From here we conclude $\mathtt{Co}(\mathtt{Id}_G) = \varphi_{\mathtt{Id}_G} = \mathtt{Id}_{\Omega(G)}$, which completes the proof that $\mathtt{Co}$ is a contravariant functor from $\mathbf{HGraph}^*$ to $\mathbf{Cone}$.

Now suppose $\mathcal{L}$ and $\mathcal{M}$ are cone lattices and $\varphi : \mathcal{L} \longrightarrow \mathcal{M}$ is a compact element preserving grounded frame homomorphism. We have verified that $\mathcal{G}_{\mathcal{L}^*}$ and $\mathcal{G}_{\mathcal{M}^*}$ are hypergraphs, and Corollary 5.113 verified that $\mathtt{HG}^*(\varphi) = \mathtt{HG}[\tau^*_\varphi]$ is indeed a finite-based HG-morphism.

Again, we have verified that $\mathtt{HG}^*$ is a functor that satisfies the first condition of Definition 3.68. To verify the remaining conditions, observe that if we suppose $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathbf{Cone}$ and suppose $s : \mathcal{L} \longrightarrow \mathcal{M}$ and $t : \mathcal{M} \longrightarrow \mathcal{N}$ are compact element preserving grounded frame homomorphisms. Lemma 5.115 guarantees that $t \circ s : \mathcal{L} \longrightarrow \mathcal{N}$ is also a compact element preserving grounded frame homomorphisms, and we observe

$$\begin{aligned}\mathtt{HG}^*(t \circ s) &= \mathtt{HG}[\tau^*_{t \circ s}] : \mathcal{G}_{\mathcal{N}^*} \longrightarrow \mathcal{G}_{\mathcal{L}^*} \\ &= \mathtt{HG}[\tau^*_s] \circ \mathtt{HG}[\tau^*_t] \\ &= \mathtt{HG}^*(s) \circ \mathtt{HG}^*(t)\end{aligned}$$

If $\mathtt{Id}_L : \mathcal{L} \longrightarrow \mathcal{L}$ is any identity map in **Cone** and $\psi : \mathtt{dom}(\psi) \longrightarrow \mathcal{L}$ and $\gamma : \mathcal{L} \longrightarrow \mathtt{cod}(\gamma)$ are compact element preserving grounded frame homomorphisms, it follows

$$\begin{aligned}\mathtt{HG}^*(\psi) &= \mathtt{HG}^*(\mathtt{Id}_L \circ \psi) & \mathtt{HG}^*(\gamma) &= \mathtt{HG}^*(\gamma \circ \mathtt{Id}_L) \\ &= \mathtt{HG}^*(\psi) \circ \mathtt{HG}^*(\mathtt{Id}_L) & &= \mathtt{HG}^*(\mathtt{Id}_L) \circ \mathtt{HG}^*(\gamma) \\ &= \mathtt{HG}[\tau^*_\psi] \circ \mathtt{HG}[\tau^*_{\mathtt{Id}_L}] & &= \mathtt{HG}[\tau^*_{\mathtt{Id}_L}] \circ \mathtt{HG}[\tau^*_\gamma] \\ &= \mathtt{HG}[\tau^*_\psi] \text{ since } \mathtt{HG}^*(\psi) = \mathtt{HG}[\tau^*_\psi] & &= \mathtt{HG}[\tau^*_\gamma] \text{ since } \mathtt{HG}^*(\gamma) = \mathtt{HG}[\tau^*_\gamma]\end{aligned}$$

From here we conclude $\mathtt{HG}^*(\mathtt{Id}_L) = \mathtt{HG}[\tau^*_{\mathtt{Id}_L}] = \mathtt{Id}_{\mathcal{G}_\mathcal{L}}$, which completes the proof that $\mathtt{HG}^*$ is a contravariant functor from **Cone** to **HGraph***.

□

If we suppose $\mathcal{G} \in \mathbf{HGraph}^*$, we observe that as defined above $\mathtt{Co}(\mathcal{G}) = \Omega(G)$, so it follows from Lemma 5.114 that $\mathtt{HG}^*\mathtt{Co}(\mathcal{G}) = \mathtt{HG}^*(\Omega(G))$ is order isomorphic to $\mathcal{G}$. It therefore follows that for every such $\mathcal{G}$ there exists an HG-isomorphism $e_G : \mathcal{G} \longrightarrow \mathtt{HG}^*\mathtt{Co}(\mathcal{G})$. Similarly we observe that for $\mathcal{L} \in$ **Cone**, since cone lattices are uniquely determined by their underlying hypergraphs that there exists a similar isomorphism $\epsilon_\mathcal{L}$ from $\mathcal{L}$ to $\mathtt{CoHG}^*(\mathcal{L})$. To see why, suppose $\mathcal{L}$ is induced by the hypergraph $\mathcal{H}$. Observe that, since we know $\mathtt{HG}^*(\mathcal{L}) = \mathcal{H}_{\mathcal{L}^*}$ and $\mathcal{H} \cong \mathcal{H}_{\mathcal{L}^*}$, then it follows that $\mathtt{CoHG}^*(\mathcal{L}) = \Omega(\mathcal{G}_{\mathcal{L}^*}) \cong \mathcal{L}$. From here, we define two natural transformations $e : \mathtt{Id}_{\mathbf{HGraph}^*} \longrightarrow \mathtt{HG}^*\mathtt{Co}$ and $\epsilon : \mathtt{Id}_{\mathbf{Cone}} \longrightarrow \mathtt{CoHG}^*$ as $e(\mathcal{G}) = \mathtt{HG}^*\mathtt{Co}(\mathcal{G}) = e_G$

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f"] \arrow[d, "e_G"'] & \mathcal{H} \arrow[d, "e_H"'] \\ \mathtt{HG}^*\mathtt{Co}(\mathcal{G}) \arrow[r, "\mathtt{HG}^*\mathtt{Co}(f)"] & \mathtt{HG}^*\mathtt{Co}(\mathcal{H}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\varphi"] \arrow[d, "\epsilon_{\mathcal{L}}"'] & \mathcal{M} \arrow[d, "\epsilon_{\mathcal{M}}"'] \\ \mathtt{CoHG}^*(\mathcal{L}) \arrow[r, "\mathtt{CoHG}^*(\varphi)"] & \mathtt{CoHG}^*(\mathcal{M}) \end{tikzcd}$$

Figure 6: Natural Transformations e and ϵ.

and $\epsilon(\mathcal{L}) = \mathtt{CoHG}^*(\mathcal{L}) = \epsilon_{\mathcal{L}}$. As we have seen, each component of both natural transformations are isomorphisms, hence Definition 3.69 tells us $e : \mathtt{Id}^*_{\mathbf{HGraph}} \cong \mathtt{HG}^*\mathtt{Co}$ and $\epsilon : \mathtt{Id}_{\mathbf{Cone}} \cong \mathtt{CoHG}^*$ define natural isomorphisms. The following commuting diagram illustrates these natural transformations:

From here, it is worth noting that the first condition of Definition 3.70 is fulfilled. We will verify that the second condition also holds, which will establish the following theorem:

Theorem 5.116. *The contravariant functors* $\mathtt{Co}$ *and* $\mathtt{HG}^*$ *give a dual-equivalence between the categories* $\mathbf{Cone}$ *and* $\mathbf{HGraph}^*$.

Proof. Suppose $\mathcal{G}$ is a finite-based hypergraph and $\mathcal{L}$ is a cone lattice and let $f \in \hom(\mathcal{G}, \mathtt{HG}^*(\mathcal{L}))$ and $\varphi_f \in \hom(\mathcal{L}, \mathtt{Co}(\mathcal{G}))$ correspond to f. That is, $f : \mathcal{G} \longrightarrow \mathcal{G}_{\mathcal{L}^*}$ is a finite-based HG-morphism and $\varphi_f : \mathcal{L} \longrightarrow \Omega(G)$ is the unique compact element preserving grounded frame homomorphism induced by f.

Let $e : \mathtt{Id}_{\mathbf{HGraph}^*} \longrightarrow \mathtt{HG}^*\mathtt{Co}$ be the natural isomorphism defined previously and observe $\mathtt{HG}^*(\varphi_f) = \mathtt{HG}[\tau^*_\varphi] : \mathcal{G}_{\Omega(G)^*} \longrightarrow \mathcal{G}_{\mathcal{L}^*}$, and since $\mathcal{G}_{\Omega(G)^*} = \mathtt{HG}^*\mathtt{Co}(\mathcal{G}) \cong_{e_G} \mathcal{G}$, it follows that $f \circ e_G^{-1} = \mathtt{HG}^*(\varphi_f)$ since φ_f is uniquely determined by f. Naturally, this implies $f = \mathtt{HG}^*(\varphi) \circ e_G$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "f"] \arrow[d, "e_G"'] & \texttt{HG}^*(\mathcal{L}) \\
\texttt{HG}^*\texttt{Co}(\mathcal{G}) \arrow[ur, "\texttt{HG}^*(\varphi)"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{L} \arrow[r, "\varphi"] \arrow[d, "\epsilon_L"'] & \texttt{Co}(\mathcal{G}) \\
\texttt{CoHG}^*(\mathcal{L}) \arrow[ur, "\texttt{Co}(f)"] &
\end{tikzcd}$$

Figure 7: Commuting diagrams for Theorem 5.116

Of course, as we have seen, not only does f uniquely determine φ, but φ uniquely determines (up to isomorphism) the map f. If we let $\epsilon : \texttt{Id}_{\mathbf{Cone}} \longrightarrow \texttt{CoHG}^*$ be as previously described, we observe that $\texttt{Co}(f)$ is equal to the map $\varphi_f^* : \Omega(\mathcal{G}_\mathcal{L}) \longrightarrow \Omega(\mathcal{G})$ where $\Omega(\mathcal{G}_\mathcal{L}) = \texttt{CoHG}^*(\mathcal{L}) \cong_{\epsilon_L} \mathcal{L}$ and $\varphi_f \circ \epsilon_\mathcal{L}^{-1} = \texttt{Co}(f)$, we conclude $\varphi_f = \texttt{Co}(f) \circ \epsilon_L$, which verifies that the diagrams in Figure 7 commute. This verifies that Condition 2 of Definition 3.70 is satisfied, completing the proof.

□

We have now established $\langle \texttt{Co}, \texttt{HG}^*, e, \epsilon \rangle$ is a dual equivalence between categories **HGraph*** and **Cone**. Of course, we can say more that this, since Theorem 3.71 guarantees the following:

Theorem 5.117. *Let $\mathcal{G}$ and $\mathcal{H}$ be finite-based hypergraphs and $\mathcal{L}$ and $\mathcal{M}$ be cone lattices. If we suppose $f : \mathcal{G} \longrightarrow \mathcal{H}$ is a finite-based HG-morphism and $\varphi : \mathcal{L} \longrightarrow \mathcal{M}$ is a grounded frame homomorphism that preserves compact elements. The following statements are true:*

1. *There is a cone lattice $\mathcal{L}$, namely $\mathcal{L} = \texttt{Co}(\mathcal{G})$, such that $\mathcal{G} \cong \texttt{HG}^*(\mathcal{L})$.*
2. *There is a finite-based hypergraph $\mathcal{G}$, namely $\mathcal{G} = \texttt{HG}^*(\mathcal{L})$, such that $\mathcal{L} \cong \texttt{Co}(\mathcal{G})$.*
3. *Both $\texttt{HG}^*$ and $\texttt{Co}$ are full and faithful.*
4. *The finite-based HG-morphism f is an isomorphism if and only if $\texttt{Co}(f)$ is an isomorphism, and $\mathcal{G} \cong \mathcal{H}$ if and only if $\texttt{Co}(\mathcal{G}) \cong \texttt{Co}(\mathcal{H})$.*

5. *The compact-element preserving grounded frame homomorphism φ is an isomorphism if and only if $\mathtt{HG}^*(\varphi)$ is an isomorphism, and $\mathcal{L} \cong \mathcal{M}$ if and only if $\mathtt{HG}^*(\mathcal{L}) \cong \mathtt{HG}^*(\mathcal{M})$.*

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