ELEMENTARY PROSPECTIVE TEACHERS AND THE NATURE OF MATHEMATICS: AN EXPLANATORY PHENOMENOLOGICAL STUDY

by

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This work is dedicated to my parents, Gordo and Rhonda.

Thank you for believing in me, supporting me, and loving me.

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ABSTRACT

Recently, there has been a resurgence of an effort to understand the nature of mathematics and what the construct means for mathematics education as a field and educators. Underlying teachers' understandings of the mathematics they teach are their conceptions of the nature of mathematics, and these conceptions provide a basis for the teacher's espoused and enacted models for teaching and learning mathematics. This explanatory phenomenological study sought to answer the following questions: What are prospective elementary teachers' conceptions of the nature of mathematics?; How do the lived experiences of the elementary prospective teachers inform their conceptions of the nature of mathematics?; and What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View? The results of this study revealed that (1) elementary prospective teachers viewed the nature of mathematics as a static-unified body of knowledge, but did not acknowledge mathematics as a discipline, (2) experiences with elementary prospective teachers' former teachers were most influential in forming their conceptions about the nature of mathematics, and (3) when presented with the Proposed Unified View of the nature of mathematics, prospective teachers experienced a dissonance in how they were expected to learn and how they wanted to teach mathematics.

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CHAPTER I: INTRODUCTION

Introduction

Throughout the history of mathematics education, the focus on school mathematics by mathematicians, mathematics educators, and psychologist has shifted between mathematics content and pedagogy and which of the two was more important for preparing teachers (National Council of Teachers of Mathematics [NCTM], 1989, 1991). However, content and pedagogy should not be viewed as separate constructs, but instead should work together. Content can answer the question of what to teach in mathematics and pedagogy can answer the question of how to teach mathematics (Grossman, Hammerness, & McDonald, 2009). That is, a balance between content and pedagogy is important to consider as mathematics educators and researchers come to terms with the history of mathematics education in the United States.

Historically, what was valued regarding appropriate mathematics content in schools changed drastically over the years with the foci shifting from drill-and-practice to meaningful arithmetic in the 1920s and 1930s, new math to drill-and-practice again in the 1960s and 1970s, and then problem solving which led to the early 1990s focus on standards and assessment (Lambdin & Walcott, 2007). Within each era of mathematics education, the theoretical underpinnings shifted. For example, the early eras (i.e., 1920s) focused on computation and memorization (Lambdin & Walcott, 2007). However, the memorization of procedures and quick computation of skill was not enough to say that a student had learned mathematics (Hiebert & Wearne, 2003; NCTM, 2000; National Research Council [NRC], 2001). As a result, later eras in mathematics education focused on discovery and problem solving (i.e., 1980s to now). These shifts in reform implied that one focus was more important than the other by emphasizing only one aspect of mathematics. For example, in the 1970s mathematics education reform focused on learning facts and procedures by repeated practice—a more procedural fluency with mathematics facts (Lambdin & Walcott, 2007). Although in previous years (i.e., 1960s), the focus was on making connections and understanding the structure of mathematics—a more conceptual understanding of mathematics (Lambdin & Walcott, 2007). The changing foci through the years created a disjointed representation of school mathematics, even though the focus of each era was important to mathematics. For example, problem solving cannot exist without the learning of mathematical skills and concepts. During the problem-solving phase of mathematics education in the 1980s, the focus was problem solving and mathematical thinking processes (Lambdin & Walcott, 2007) with a push to return to discovery and learning *through* problem solving with the implementation of meaningful whole class discussions (Smith & Stein, 2011).

The different foci throughout the decades of mathematics education reform focused on only one tenant of mathematics (i.e., skills or problem solving) instead of drawing out the importance of all parts of mathematics education. It is the combination of these foci presented in the different eras of mathematics education reform where a true understanding of mathematics occurs—to be mathematically proficient one must have skill in carrying out procedures, comprehension of mathematical concepts, an ability to problem solve, a capacity for logical thought, and an inclination to see mathematics as worthwhile (NRC, 2001). Unfortunately, balancing problem solving with the need for learning skills proves to be a difficult task in mathematics education.

As a way of beginning to think about the balance between the need for procedural skills and conceptual understanding, current reform documents continue to call for change in mathematics education through curriculum, instruction, and teacher preparation (Association of Mathematics Teacher Educators [AMTE], 2017; College Board of the Mathematical Sciences [CBMS], 2012; Common Core State Standards Initiative [CCSSI], 2010; NCTM, 2014). Teacher preparation is at the forefront of these newest standards documents because teaching is complex and requires teachers to be knowledgeable in many areas including, but not limited to, the discipline they teach (Ball, Thames, & Phelps, 2008). The mathematical knowledge required for teachers necessitates knowledge of not only mathematical facts and formulas but also specialized knowledge to include varying methods, different approaches, and "intramathematical connections that are the basic condition of meaningful learning" (Sfard, 2003, p. 386). Intramathematical connections refer to how a teacher chooses the mathematical tasks to implement in a classroom, how and why the teacher groups students in a certain way, and when to ask probing questions.

Teachers' specialized mathematical knowledge is vital to the effective teaching of mathematics. Underlying teachers' understandings of the mathematics they teach are their conceptions of the nature of mathematics (NOM)—what the teacher believes about mathematics as a discipline. A teacher's conception of NOM provides the basis for the teacher's models for the teaching and learning of mathematics (Ernest, 1989) as depicted in Figure 1. Ernest's (1989) placement of NOM at the top of Figure 1 signifies how influential one's personal philosophy of mathematics is to the teaching and learning of mathematics. It is teachers' conceptions of NOM which form their mental structures or espoused models of teaching and learning and ultimately what they believe in regards to the learning of mathematics and the teaching of mathematics, and how they might eventually enact those beliefs in the classroom.

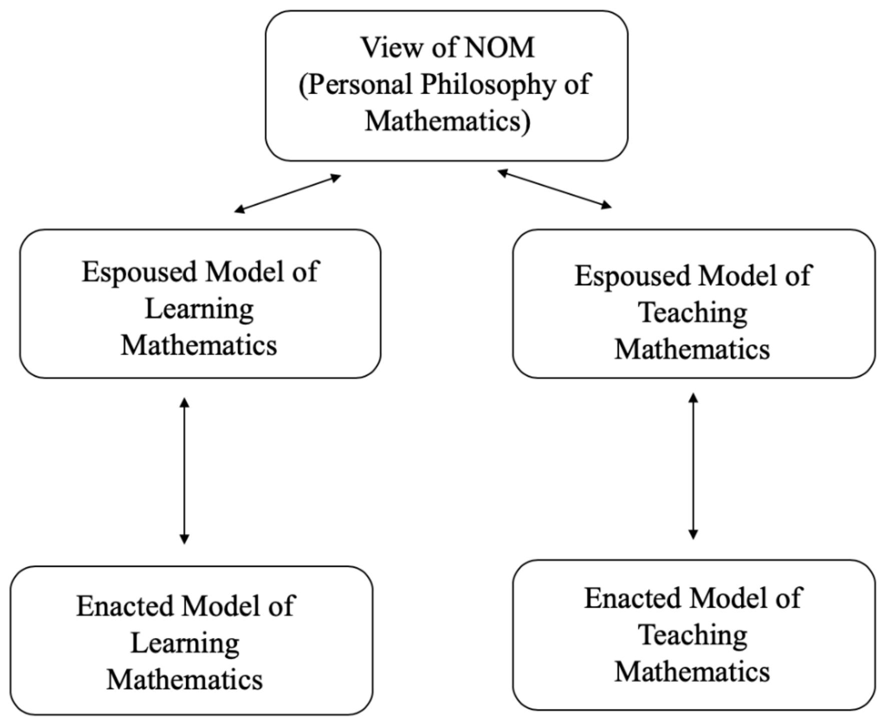

Figure 1. The influence of NOM on the teaching and learning of mathematics. Adapted from "The impact of beliefs on the teaching of mathematics" by P. Ernest, 1989, In C. Keitel, P. Damerow, A. Bishop, & P. Gerdes (Eds.), *Mathematics, Education, and Society* (pp. 99-101).

The significance of the bidirectional arrows is also vital in Figure 1, as it demonstrates how one's espoused or enacted models of teaching and learning mathematics can also influence her view of NOM. That is, a teacher might have a particular philosophy of mathematics based on how she engaged as a mathematics teacher in classrooms. Ernest's (1989) figure also illuminates the social constraints and opportunities which can influence one's model of teaching and learning. For example, fellow teachers, parents, national standards, and the use of certain curriculum (e.g., mathematics texts) are all social contexts that could influence a teacher's model for teaching and learning. With NOM at the top of the figure providing the basis for teacher's espoused and enacted models of teaching and learning mathematics, an examination of teachers' conceptions of NOM is an important consideration when discussing mathematics education because it influences the teaching and learning of mathematics at a time when mathematics education stakeholders are pushing for mathematics reform.

Background for the Study

A teacher's conception of NOM plays a critical role in the teaching and learning of mathematics (Beswick, 2012; Ernest, 1989; Phillip, 2007). Thus, there becomes a need to describe NOM for the field of mathematics education because the field promotes ideas important for teachers with regards to the discipline of mathematics. Also, as teachers were once doers of mathematics, then the individual teacher must understand NOM as their conceptions will almost certainly influence their classrooms. Like the conflicting ideas on content and pedagogy throughout the history of mathematics education, there are also conflicting conceptions about NOM. Mathematics is special because "mathematics is a discipline that enjoys a peculiar property: it may be loved or hated, understood or misunderstood, but everybody has some mental image of it" (Furinghetti, 1993, p. 34). These mental images of mathematics as a discipline (i.e., NOM) influence the teaching and learning of mathematics as depicted in Figure 1 (Ernest, 1989). For example, recall the different mental imagines in the history of mathematics education: computation and memorization of mathematical facts versus problem solving through discovery. Now, imagine two teachers, one who believes mathematics is about computation and memorization versus another who believes mathematics is about problem solving and discovery. These two teachers' different belief structures will affect their ideas about the teaching and learning of mathematics (Ernest, 1991; Philipp, 2007). Therefore, a look at teachers' beliefs about mathematics and different conceptions of NOM are important to consider due to the relationship between the two.

Teachers' Beliefs

Teachers' beliefs play a significant role in the teaching and learning of mathematics (Pajares, 1992; Phillip, 2007; Szydlik, 2013; Thompson 1992). Thus, a consideration of teachers' beliefs is important because teachers' conceptions of NOM influence their instructional practices (Beswick, 2012; Dossey, 1992; Lloyd, 2005; Mewborn & Cross, 2007). However, teachers are often not given ample opportunities to reflect on their beliefs about NOM. In fact, teachers hold different beliefs about school mathematics and mathematics as a discipline and they do not have an opportunity to develop their understandings about NOM in their teacher preparation programs (Beswick, 2012). These differing beliefs about school mathematics (i.e., what teachers are required to teach in school) and NOM (i.e., the discipline of mathematics) need to be made explicit to teachers and teacher educators. A look at teachers' understandings of NOM revealed that teachers had not previously considered how their tacit understandings of

NOM might eventually influence their instructional practices (Mewborn & Cross, 2007). Furthermore, teachers' beliefs can directly affect students' conceptions of NOM (Lloyd, 2005). Therefore, bringing these beliefs to the forefront and allowing teachers to explicitly reflect on and attend to their own beliefs is vital to understanding the influence those beliefs have on instructional practices and students.

There are many ways to conceptualize teacher beliefs and the ways in which teacher beliefs influence the teaching and learning of mathematics. For example, conceptions, philosophies, perceptions, and views are terms often used interchangeably to refer to the general understandings, values, and meanings used to define one's beliefs about mathematics. For the purposes of this study, the term conceptions will be used when referring to teacher beliefs regarding the nature of mathematics as a discipline. A teacher's conception of NOM refers to "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (Thompson, 1992, p. 132). When considering teachers' conceptions of NOM, it is nearly impossible to discuss the discipline of mathematics without discussing the teaching and learning of mathematics because of the context in which teachers place themselves, that is as teachers of mathematics and not just learners or doers of mathematics. Furthermore, the term belief has been used in such a variety of ways that it is often conflated with the term knowledge (Pajares, 1992). However, in this study, I use the term conception instead of beliefs to offer such a distinction. That is, because of the wide use of the term belief and its conflation with mathematics teaching, NOM, mathematical knowledge, and mathematical activities, I will use the term conception in this study as I try to describe and understand elementary prospective teachers (PTs) beliefs about the nature of mathematics separate from their views of teaching mathematics or mathematical knowledge. I will discuss further teachers', prospective teachers', and students' conceptions in Chapter Two. Additionally, Thompson (1992) offered the term conception as a way to connect to the historical, philosophical views of mathematics as a discipline, and I will further discuss the historical, philosophical views of NOM in the next section as well as in Chapter Two.

Nature of Mathematics

Helping teachers reflect on their conceptions of NOM is a significant and necessary step in improving mathematics education and breaking the back-and-forth cycle of reform that has previously occurred throughout mathematics education history (Beswick, 2012; Conner, Edenfield, Gleason, & Ersoz, 2011; Gold, 2011; White-Fredette, 2010). Consider that teachers' conceptions of NOM are not necessarily conscious constructs to the teachers themselves because NOM is not clearly defined. As with the various terms used when referring to teacher beliefs, there are different definitions and descriptions which categorize NOM. Mathematicians, mathematics teacher educators (MTEs), teachers, PTs, and students might all describe different definitions and conceptions about mathematics. For example, a pure mathematician may do mathematics for its own sake without direct application to another field (Browder, 1976; Pair, 2017) and might define NOM with respect to the theoretical practice of proving within mathematics. An MTE considers the teaching and learning of mathematics as well as the mathematics content (Tzur, 2001) and might define NOM based on standards documents (e.g., NCTM's (2000) Process Standards). Some mathematics teachers define NOM as facts and computations (Beswick, 2012). A PT considers NOM from a student perspective and also from the perspective of a future teacher and might define NOM as how to teach mathematics, instead of based on the discipline of mathematics itself (Bolden, Harries, Newton, 2010). Students define NOM as meaningless facts and rules to memorize (Hersh, 1997), while others define mathematics as a way to comprehend and change the world (Gutstein, 2016). These different groups—mathematicians, inservice teachers, prospective teachers, and students—all might define NOM based on what they value as important in mathematics and how they themselves learned mathematics (Pais, 2013). Since school mathematics values “knowledge and competence” (Pais, 2013, p. 16) it may not align with mathematicians or MTEs’ conceptions of NOM.

Categorizing the different conceptions of NOM becomes more challenging, because even within groups of people, conceptions may vary. For example, mathematicians’ conceptions of NOM may change depending on their perceived purpose of mathematics. A pure mathematician attempts to solve unknown problems whereas an applied mathematician uses mathematics to describe a relationship between mathematics and another discipline. These differences in peoples’ conceptions of NOM are also evident in teachers, students, and MTEs (Phillip, 2007; Szydlik, 2013; Thompson, 1992). People’s conceptions of NOM might depend on the context in which they are using mathematics. With the varying conceptions of NOM over the different groups, a consideration of one’s conceptions of NOM is imperative to begin to understand the different conceptions and what mathematical experiences influenced those conceptions.

There are a multitude of distinctions which can be made based on how one views NOM. Although a consensus view of characteristics of NOM does not exist in mathematics education, the remainder of this section will focus on the three most prevalent philosophical views of NOM: instrumentalism, Platonism, and fallibilism. An understanding and categorization of views is the first step to forming a foundation for the importance of understanding conceptions of NOM, because these conceptions are linked to teachers' espoused and enacted models of teaching and learning of mathematics (Ernest, 1989). The following views of NOM focus on mathematics as a discipline. Inherent within each of these views of NOM is the idea that proof is the heart of mathematics. However, in each of these views, the purpose of proof is different. The potential influence of the three views on the teaching and learning of mathematics will be discussed further in Chapter Two.

Instrumentalism. An instrumentalist view of NOM, sometimes called an absolutist view, refers to the idea that mathematics consists of isolated rules and truths which are unrelated to each other. That is, in this view of NOM, mathematics is used as a tool to solve problems (Chamberlin, 2013). More specifically, an instrumentalist view of NOM produces instrumental understanding and not relational understanding (Mellin-Olsen, 1981). Skemp (1976) associated relational understanding with mathematical understanding and instrumental understanding as recognizing a task or problem that one already knows how to solve by some rule. So, an instrumentalist would claim that once a proof is established, it is absolute (Ernest, 1991). Thus, mathematics is seen as a set of rules to memorize and as infallible, because one can establish certainty and disregard paradoxes.

Platonism. Platonism is perhaps the most widespread conception of NOM (Dossey, 1992; Hersh, 1997) and refers to the idea that mathematics is a discoverable body of knowledge. Platonism denotes the idea that mathematics exists outside the mind, outside of space or time, and independent of conscious thought. That is, mathematics exists in the world to be discovered. So, a Platonist might see mathematical knowledge as the discovery of truths already existing and therefore unchangeable (Hersh, 1997). Sometimes referred to as realism, this conception of NOM allows for the logical thought, understandable ideas, and connectedness between mathematical concepts, unlike instrumentalism which often does not value the logical connectedness of mathematical ideas. Proof is used in Platonism as a way to validate the truths that already exist. That is, proofs tell us the right answer in mathematics.

Fallibilism. Fallibilism is the conception that NOM is dynamic and created through exploration and problem solving (Hersh, 1976; Lakatos, 1978). Furthermore, the fallibilist view does not claim that mathematics is true, but instead that mathematics can be discovered through mistakes and is constantly open for revision (Ernest, 1991). That is, mathematics is a cycle of proof where, through communication and reflection, the doers of mathematics use mistakes as dialogue to consider ideas and revisions of ideas or proofs when necessary.

These three philosophical views of NOM are the most widespread and form a foundation for understanding conceptions of NOM. Building on these philosophical views of NOM, Thompson (1992) defined three conceptions of NOM because of their philosophical significance: a bag of tools (i.e., instrumentalism), a static-unified body of knowledge (i.e., Platonism), and a dynamic problem-driven view (i.e., fallibilism). A conception of NOM as a bag of tools aligns with instrumentalism by promoting “a set of unrelated but utilitarian rules and facts” (Thompson, 1992, p. 132). A conception of

NOM as a static-unified body of knowledge aligns with Platonism by defining NOM as "a monolithic, static, immutable product" (p. 132). Lastly, a conception of NOM as a problem-driven dynamic discipline aligns with fallibilism by defining NOM as "a continually expanding field of human creation and invention" (p. 132). Thompson used the more familiar language because it often appeared in how teachers spoke about NOM (Benacerraf & Putnam, 1964; Davis & Hersh, 1980; Ernest, 1989; Lakatos, 1976) as opposed to teachers discussing instrumentalism, Platonism, or fallibilism. I will also use Thompson's language when describing PTs' conceptions of NOM throughout this study.

In addition to discussing the philosophical tenants of NOM in a more familiar language, Thompson posited that an individual teacher's conceptions of NOM could include more than one aspect, even if conflicting, of the views of NOM. Therefore, I propose a continuum (see Figure 2) for thinking about the differences and connections between these three conceptions proposed by Thompson (1992), the three most prominent philosophical views of NOM. The continuum can also be used as an aid to help signify that a teacher may move forwards and backwards on the continuum.

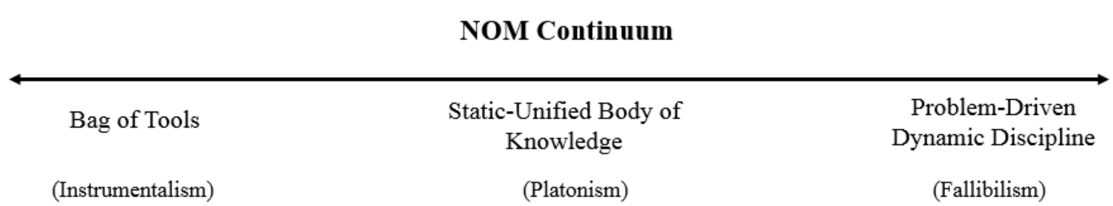

Figure 2. NOM continuum

In this continuum, the intent is to describe the overlapping nature of the three views of NOM and the conceptions of NOM proposed by Thompson. The use of the continuum was helpful throughout this study as a way to think about PTs ideas regarding NOM and how one might classify those ideas according to their conceptions of NOM. The continuum is not meant to suggest that one's conceptions of NOM progress through a linear sequence. For example, one PT may describe characteristics of NOM that most align with a bag of tools at one point and characteristics that align with a problem-driven dynamic discipline at a different point. However, another PT may in fact progress along the continuum linearly depending on specific experiences. PTs' conceptions of NOM will be different because individual PTs are different and have had different experiences with mathematics. Even with the differences between PTs, the NOM Continuum will be used in this study to help think about PTs' conceptions of NOM and why categorizing them might be important. Furthermore, while the goal is not to move everyone to a fallibilist perspective, the NOM Continuum shows how a fallibilist perspective must be appreciated and understood for teachers to consider teaching mathematics in a reformed way in the way that a fallibilist or problem-driven dynamic discipline is at one end of the continuum and thus in stark contrast to NOM as a bag of tools. That is, the field of mathematics education values ideas put forth by current reform documents which all focus on the characteristics inherent in fallibilist views of NOM (e.g., AMTE, 2017; CBMS, 2012; CCSSI, 2010; NRC, 2001; NCTM, 2000, 2014). For example, the five Process Standards (NCTM, 2000) include problem solving and proof, and two of the eight Standards for Mathematical Practice (CCSSM, 2010) require students to persevere in problem solving and construct viable arguments. These two standards align with the discoverability characteristic of NOM present in the fallibilist view. Thus, these practices put forth in the reform documents will be difficult to achieve if the teacher only holds non-fallibilist conceptions of NOM and has not considered the benefits of the characteristics which create the fallibilist view of NOM as an important part of mathematics as a discipline (Dossey, 1992). Moreover, bringing awareness to teachers' conceptions of NOM is vital in encouraging teachers to examine different ways of conceptualizing NOM and the implications those conceptions have on the teaching of mathematics, considering ideas inherent in the reform documents, and fostering meaningful mathematical knowledge in students.

The Problem Statement

The current vision for mathematics instruction calls for reform in curriculum, instruction, and teacher preparation (AMTE 2017; CBMS, 2012; CCSSI, 2010; NRC, 2001; NCTM, 2000, 2014). Effectively implementing reform-based practices is difficult. Classroom observations reveal a heavy reliance on instrumentalist views of NOM (Ball, Lubienski, & Mewborn, 2001; Dossey, 1992; Stigler & Hiebert, 1999). Providing teachers opportunities to consider different views of NOM is vital for bringing about reform in mathematics education, because "mathematics success for all cannot come about without radical change in instructional practices and an equally radical change in teachers' views of mathematics teaching and learning, as well as the discipline of mathematics itself" (White-Fredette, 2010, p. 21). However, because teachers are often unaware of their conceptions of NOM (Beswick, 2012; White-Fredette, 2010), and content and pedagogy are taught in separate courses, a disconnect is fostered among teachers, the discipline of mathematics, and the actual work of teaching (Grossman, Hammerness, & McDonald, 2009). Because a teacher's conception of NOM influences their instructional practices, then it is:

> Through reflection, [that] teachers learn new ways to make sense of what they observe, enabling them to see differently those things they had been seeing while developing the ability to see things previously unnoticed. While teachers are learning to see differently, they challenge their existing beliefs. (Phillip, 2007, p. 281)

Teachers must be provided opportunities to challenge their current conceptions of NOM and reflect on the influence of those conceptions. That is, to effectively begin the implementation of reform-based instruction, teachers need to first understand their own conceptions of NOM, consider alternative conceptions, experience dissonance as their conceptions are challenged, and then have the opportunity to restructure their understandings of NOM and the impact those conceptions have on their teaching.

Underlying teachers' conceptions of NOM is the understanding of mathematical content. Elementary PTs comprise a special group of PTs because they come to their teacher preparation programs assuming they already know the simple, fundamental mathematics involving basic arithmetic that is the foundation of elementary school mathematics (Ambrose, 2004; Ball, 1990; Richardson, 1996; Weinstein, 1989). This assumption can often lead PTs to underestimate the complexities required to teach (Ambrose, 2004). Additionally, the way in which PTs remember their own experiences from school also shape how they will teach in their future classrooms (Lortie, 1975; Shulman, 1986; Stigler & Hiebert, 1999). PTs' experiences as mathematics learners form their conceptions of mathematics as a discipline which then inform their models of the teaching and learning of mathematics (Ernest, 1989). Conceptions of NOM underlie PTs assumptions, and thus potentially interfere with the implementation of reform-based mathematics instruction (CBMS, 2012). Moreover, focusing on PTs is important due to the cognitive foundation that is developed in elementary students through their learning of mathematics. Students' experiences in elementary school provide a foundation for their future mathematical proficiency, and those foundations and dispositions developed later are often informed by their elementary school teachers (NRC, 2001, 2007, 2015). Consequently, examining and understanding PTs' conceptions of NOM is one possible direction to attempt to understand how those conceptions may eventually influence their instructional practices.

Previous studies suggested that teachers', students', and PTs' conceptions of NOM are not typically aligned with the fallibilist view (Beswick, 2012; Bolden et al., 2010; Jankvist, 2015; Rupnow, 2018; Sweeny, Ruef, & Willingham, 2018; Szydlik, 2013; Zazkis, 2015). Teachers' conceptions of mathematics vary depending on their use of mathematics. For example, as a learner of mathematics, teachers sometimes described mathematics as discovering truths (i.e., Platonism) or solving problems using equations (i.e., Instrumentalism), but expressed that in their classrooms they promoted an idea of mathematics as freedom to create (i.e., Fallibilism) (Beswick, 2012). Regarding students, authors expressed that by focusing on problem-solving approaches in the classroom, students showed a change in their conceptions from instrumentalism to fallibilism (Jankvist, 2015; Rupnow, 2018). PTs are unique in that they are students of mathematics and future teachers of mathematics, so similar ideas were found in studies with PTs. Additionally, regarding PTs, authors often reported PTs describing mathematics as mostly rote practices without any creativity and emphasized a need to expose PTs to different conceptions of mathematics (Bolden et al., 2010; Sweeny et al., 2018). In addition to discussing teachers', students', and PTs' conceptions of NOM, the authors also discussed the importance of the social context—including pedagogical approaches of the teacher and role of the students in the classroom. More specifically, Jankvist (2015) proposed a model that positioned NOM as influenced by the social context and described the social context as it relates to the students' experiences with mathematics.

With a focus on teachers', students', and PTs' conceptions of NOM and implications for classroom practices, and the proposed models of both Jankvist (2015) and Ernest (1989) which positioned NOM at the apex and thus important, it is unsettling that a consensus view of NOM does not exist. In Chapter Two, I present a Proposed Unified View of NOM as a way to begin conversation about the implicit ideas inherent in two standards-based reform documents in mathematics education. For the purposes of this current chapter, I present the nine characteristics associated with the synthesis of two standards-based documents and creation of the Proposed Unified View of NOM to provide relevance and context for the purpose of this study and research questions.

1. Mathematics involves exploration.
2. Mathematics involves multiple strategies.
3. Mathematical ideas are communicated and verified through proof/justification.
4. Mathematics requires justification of ideas to others.
5. Critique of mathematical ideas leads to refinement.
6. Structure and patterns are inherent in mathematics.
7. Mathematics uses multiple representations.

8. Mathematics is useful and worthwhile.
9. Anyone can be a learner of mathematics.

Statement of Purpose

To fully understand elementary PTs' views of the teaching and learning of mathematics, a consideration of PTs' views of NOM and how those views were formed is an important first step. Investigating conceptions and how conceptions were formed is vital for understanding how PTs' conceptions might influence their instructional practices and their future students' conceptions of NOM. Therefore, the purpose of this study was to investigate PTs' conceptions of NOM and the mathematics experiences that informed those conceptions. The following research questions guided the study:

1. What are elementary prospective teachers' conceptions of the nature of mathematics?
2. How do the lived experiences of the elementary prospective teachers inform their conceptions of the nature of mathematics?
3. What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View, and what are the implications of those connections, if any?

Significance of the Study

This study was significant in at least four ways. First, the study contributes to a larger body of knowledge on elementary PTs' beliefs (Pajares, 1992; Phillip, 2007; Thompson, 1992). In mathematics education, various terms are used interchangeably in association with PTs' beliefs about the teaching and learning of mathematics. This study parsed out the subtle differences in beliefs about the teaching and learning of mathematics compared to beliefs (i.e., conceptions) of mathematics as a discipline (i.e., NOM). Second, this study contributes to a larger body of knowledge of PTs' conceptions concerning NOM (Beswick, 2012; Mewborn & Cross, 2007; Szydlik, 2013). A characterization of PTs' conceptions regarding NOM is scarce in the literature. However, the consideration is important as the conceptions of NOM provide a foundation for the PTs' models of teaching and learning of mathematics. Third, for MTEs, understanding PTs' conceptions of NOM informs teacher preparation programs by helping support the alignment with reform-based mathematics practices for teaching and learning. MTEs can utilize the NOM continuum (See Figure 2) as a reflective tool for PTs in their teacher preparation program to help PTs understand their own conceptions of NOM as well as helping the MTE gain a deeper understanding of how the PTs might conceive of mathematics teaching and learning in their future classrooms. Last, through the literature review, I provide the beginning key characteristics of the Proposed Unified View for NOM based on current standards for teaching and learning mathematics.

Definitions

Throughout this dissertation I will refer to key terms. The following section is intended to support clarity of those meanings.

Conceptions

Conceptions will refer to a person's "conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (Thompson, 1992, p. 132).

High-Quality Mathematics Instruction

High-quality mathematics instruction engages students in the Standards for Mathematical Practice (CCSM, 2010) through the Mathematics Teaching Practices (NCTM, 2014). For example, high-quality instruction uses student thinking to advance the lesson and engages students in meaningful mathematical discourse.

Mathematics Learner

For the purposes of this study, a mathematics learner refers to students, teachers, mathematicians, or mathematics educators engaged in *doing* mathematics—including but not limited to conjecturing, justifying, and problem solving (CCSSM, 2010; Henningsen & Stein, 1997; NCTM, 2000).

Nature of Mathematics

The nature of mathematics refers to the key characteristics which comprise the discipline of mathematics, and asks the questions: what is mathematics, and how can we account for its nature (Ernest, 1991).

Reform-based Instruction

Throughout this study, the term reform-based instruction will refer to instruction that is described in documents that call for a change in mathematics education by providing evidence of the importance of the use of high-quality mathematics teaching and learning practices. These documents include but are not limited to the following: AMTE (2017); CBMS (2012); CCSSI (2010); NRC (2001); NCTM (2000, 2014).

Chapter Summary

In conclusion, NOM is an undefined construct where different people have different conceptions of NOM. Yet it is vital in understanding the teaching and learning of mathematics. Teachers' conceptions of NOM move beyond the well-researched study of teachers' beliefs by seeking to illuminate the detailed understanding teachers have of the discipline of mathematics (i.e., NOM). Elaborating on and understanding PTs' conceptions of NOM will forge the path of aligning PTs' conceptions of NOM with reform-based instruction.

CHAPTER II: LITERATURE REVIEW

Introduction

The purpose of this study is to describe elementary PTs' conceptions of NOM, understand their experiences that informed those conceptions, and connect PTs' conceptions of NOM to the Proposed Unified View of NOM. I propose the following three research questions:

1. What are elementary prospective teachers' conceptions of the nature of mathematics?
2. How do the lived experiences of the elementary prospective teachers inform their conceptions of the nature of mathematics?
3. What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View, and what are the implications of those connections, if any?

In this chapter, I provide a theoretical framing for investigating these questions by reviewing the literature in the following way. First, I discuss the literature related to NOM. Specifically, I provide descriptions of the ways the field of mathematics education has described NOM—highlighting the lack of a clear consensus on how the field talks about understandings of NOM. Second, I discuss relevant literature concerning teachers' conceptions of NOM. Lastly, I review the literature surrounding PTs' conceptions of NOM and how those conceptions can hinder their preparation to enter the teaching profession.

Nature of Mathematics

PTs are often unaware of their conceptions of NOM or even that NOM is a construct worthy of reflection (Thompson, 1992). This lack of attention to understanding NOM contributes to the disconnect between how students are expected to learn mathematics and how teachers are expected to teach mathematics (Grossman et al., 2009; Lampert, 2010). The lack of a clear description of NOM coupled with differing expectations of teaching and learning can arguably be traced to the varying and competing foci of the different eras of mathematics education reform in the United States. As evidenced by mathematics education history in the United States, at different times, different mathematical concepts and how to teach them were valued and emphasized in schools. Thus, standards documents were developed with the intent to provide important mathematical concepts and practices for students (i.e., CCSSI, 2010; NCTM, 2000) as well as practices for teachers (AMTE, 2017; NCTM, 2014). Although the standards documents of the last 30 years have been based on similar philosophies of mathematics education and NOM, neither the standards documents nor the literature surrounding them have made a concerted effort to come to a consensus on the question: What is mathematics? Asking this question refers to the nature of the discipline of mathematics that is NOM.

Understanding the nature of the discipline is paramount to moving forward with mathematics education reform, because an understanding of individuals' conceptions of NOM is the foundation upon which *all* of their mathematical activity will rest. In a seminal study that investigated a sixth-grader's conceptions of mathematics, Erlwanger (1973) revealed how the student (Benny) regarded mathematics as a set of rules invented by someone smart. Additionally, Erlwanger expressed that Benny's conceptions of mathematics could explain how he learned. He stated,

> Mathematics consists of different rules for different types of problems. These rules have all been invented. But they work like magic because the answers one gets from applying these rules can be expressed in different ways. Therefore, mathematics is not a rational, logical subject in which one has to reason, analyze, seek relationships, make generalizations, and verify answers. (Erlwanger, 1973, p. 54)

In an attempt to define NOM, or at least have a clear understanding of how different people (i.e., students, teachers, mathematicians, or mathematics teacher educators) view mathematics, the following section describes the different views of NOM throughout mathematics education and their implications for the teaching and learning of mathematics.

Before proceeding, it is important to note that not all authors refer to the same thing when they say NOM. The term NOM can refer to different aspects of mathematical knowledge, teaching mathematics, learning mathematics, or a combination of the three (Kean, 2012; Pair, 2017). While it is difficult to parse these inter-connected ideas, this study focused on NOM as it refers to mathematical knowledge. That is, I used NOM in reference to the question, "What is mathematics?" and focused on the content and nature of structures in mathematics and not the teaching and learning of mathematics. Additionally, I leveraged elementary PTs' personal relationships with mathematics as a way to understand their conceptions of mathematics content and structure (i.e., NOM).

More specifically, I tried to understand what mathematics looked like through the eyes of elementary PTs.

History of Nature of Mathematics

To begin to understand NOM as it relates to the question, "What is mathematics," I describe the varying views of NOM from a philosophical perspective. There are three main views of NOM: instrumentalism, Platonism, and fallibilism. Though the main purpose in discussing these views of NOM was to focus on the mathematical knowledge, these sections also described a likely example of mathematics instruction if the instructor holds each type of view of NOM. This was intended to provide more insight into the different views of NOM since these views inform teachers' models for the teaching and learning of mathematics (Ernest, 1989).

Instrumentalism. The instrumentalism view of NOM refers to the idea that mathematics consists of isolated rules and truths which are unrelated to each other. One whose conceptions align with this view of NOM sees mathematics as a utilitarian tool to solve problems (Chamberlin, 2013). The instrumentalism view of NOM values logic and proof. The logic idea inherent in the instrumentalism view is based on the usefulness of the mathematics as it relates to the problem being solved at a given time. For example, the previously mentioned student Benny believed mathematics was a set of rules, and as long as he knew those rules he could figure out the mathematics problem (Erlwanger, 1973). The idea of proof inherent in the instrumentalism view of NOM asserts that once a proof is established, it is absolute (Ernest, 1991).

Now consider how a teacher whose conception of NOM aligns mostly with the instrumentalist view of NOM might teach mathematics. An instrumental view of mathematics is likely to be associated with "the instructor model of teaching, and with strict following of a text or scheme. It is also likely to be associated with the child's compliant behavior and mastery of skills model of learning" (Ernest, 1989, p. 100). Additionally, a teacher holding an instrumentalist view will likely use a textbook and follow the prescribed trajectory for learning mathematics regardless of what is happening in the classroom (Dossey, 1992). An instrumentalist view of NOM produces instrumental understanding and not relational understanding (Mellin-Olsen, 1981). Skemp (1976) associated relational understanding with mathematical understanding and instrumental understanding as recognizing a task or problem that one already knows how to solve by some rule. Thus, a teacher whose conception of NOM aligns mostly with the instrumentalism view of NOM will produce students with an instrumental understanding of mathematics.

Platonism. Platonism refers to the idea that mathematics exists outside the mind, outside of space or time, and independent of conscious thought. That is, mathematics exists in the world to be discovered, and through discovery one can describe mathematical objects and the relationships and structure connecting them (Hersh, 1997). Platonism is more than just memorizing a set of rules. Platonism provides "a solution to a problem of the objectivity of mathematics. It accounts for its truths and the existence of its objects, as well as the apparent autonomy of mathematics, which obeys its own inner laws and logic" (Ernest, 1991, p. 30). So, a Platonist might see mathematical knowledge as the discovery of truths already existing and therefore unchangeable (Hersh, 1997). Like the instrumentalism view of NOM, the Platonism view of NOM also values logic. Platonism combines logical thought with understandable ideas and connectedness between mathematical concepts, unlike instrumentalism which often does not value the logical connectedness of mathematical ideas. Proof is used in Platonism as a way to validate the truths that already exist in the world. Consider a teacher who has a conception of NOM most aligned with the Platonism view of NOM. That teacher will likely conduct class as though there is only one answer and it is up to the students to discover the answer.

Fallibilism. Fallibilism is the conception that NOM is dynamic and discoverable through exploration and problem solving (Hersh, 1976; Lakatos, 1978). Furthermore, the fallibilist view does not expound that mathematics is true, but instead that mathematics can be discovered through mistakes and is always open to revision (Ernest, 1991). Lakatos (1976) explained, “Mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism” (Lakatos, 1976, p. 6). Consider a teacher who aligns with the fallibilist conception of NOM. This teacher’s classroom will allow students the opportunity to explore the mathematics, provide an argument, and then engage in discussion with other students about the agreement of the work. The teacher is not looking for one answer, but instead is focused on the students’ solutions, strategies, and discussions around validating the solutions and strategies.

Summary. Instrumentalism, Platonism, and fallibilism make up the three most prevalent philosophical views of NOM. Each of these three views of NOM emphasizes different aspects of what constitutes mathematics. However, each conception also involves the idea of proof and logic mathematics. In Table 1, I provided a summary of the important aspects of each view of NOM. A general agreement on which, if any, of these views is more valuable does not exist in mathematics education, and I argue the characteristics from each of the views of NOM are of equal importance because they promote different ideas of what mathematics is.

Table 1

Summary of the Philosophical Views of NOM

	Instrumentalist	Platonist	Fallibilist
Aspects of Each Philosophical View	Rules and facts	Outside of the mind	Problem solving
	Utilitarian	Discoverable	Dynamic and creative
	Truths exist	Truths exist	Constantly open for revision

The NRC (2001) explained that there are five strands which constitute mathematical proficiency: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning,* and *productive disposition.* These five strands are considered to be interwoven and interdependent. The same is true for the three conceptions of NOM. The conceptions do not stand alone, for inherit in each conception are important aspects of mathematics. For example, the NRC included procedural fluency in their definition of mathematical proficiency, and Instrumentalism includes the importance of rules and procedures. Moreover, the NRC focused on the importance of adaptive reasoning, which is inherent in Platonism. Thus, simply knowing or understanding only one conception of NOM is not sufficient to truly understand the interwoven and interdependent nature of the conceptions. It is this underdevelopment, looking at the conceptions as separate, unconnected pieces, which promotes the need for a holistic view of NOM. This holistic view of NOM does not currently exist, but is valuable to consider to answer the question, "What is mathematics?" The underdevelopment of a consensus for the construct of NOM requires a look into science, because science educators created a list of general characteristics of science which make up the nature of science (NOS).

Nature of Science

In science education, a consensus view exists regarding most elements of NOS, and this consensus view of NOS helps to guide and inform the question: What is science? (Lederman, 1998; McComas et al., 1998; Osborn et al., 2003). In addition to the consensus view aiming to answer that question also informs the teaching of science by helping teachers know what to teach and how to teach. The consensus view for NOS consists of a list of seven general characteristics of scientific knowledge (Abd-El-Khalick, 2012; Bartos & Lederman, 2014; Lederman, Abd-El-Khalick, Bell, & Schwartz, 2002; McComas, Clough, & Almazroa, 2002; NGSS Lead States, 2013) and are as follows:

1. Scientific knowledge is empirically based.
2. Observations differ from inferences.
3. There is a distinction between scientific theories and scientific laws.
4. Scientific knowledge is a product of a human imagination and creativity.
5. Scientific knowledge is theory-laden.
6. Scientific knowledge is affected by society and culture.
7. Scientific knowledge is tentative yet durable.

Though not all science scholars wholly accept the consensus view for NOS (see Abd-El-Khalick, 2012; Hodson, 2017), it remains a widely held belief that an understanding of

NOS (i.e., the seven characteristics) is imperative for the knowing, teaching, and understanding of science (McComas & Almazroa, 1998). Views of NOS have informed curriculum, instruction, and teacher preparation in science. The Next Generation Science Standards (NGSS Lead States, 2013) incorporated NOS in the practices and Crosscutting Concepts. Each NOS characteristic has a specific grade level representation in K-2, 3-5, middle, and high school. For example, the characteristic that scientific knowledge is empirically based is present in K-2, 3-5, middle, and high school by providing a matrix of practices associated with that characteristic. Thus, a third-grade student who understands that scientific knowledge is empirically based will be able to "use tools and technologies to make accurate measurements and observations" (NGSS Lead States, Appendix H, p. 5). The association of NOS characteristics with practices and Crosscutting Concepts represented in the NGSS emphasized the importance of the developing well-informed learners of science. That is, there is no specific characteristic of NOS which is more valuable than the other, but the characteristics together as a list represent the importance of NOS for learners of science and for teachers of science. However, teachers cannot possibly teach what they do not understand (Ball & McDiarmid, 1990; Shulman, 1987), and thus if a teacher is required to develop an understanding of NOS in her students as set forth by the NGSS, she must also understand NOS. Science educators argue that understanding of NOS is not an adequate condition, but a necessary condition for teachers and students alike (Abd-El-Khalick & Lederman, 2000).

Science education researchers explored PTs' and teachers' views on NOS. Teachers and PTs who generated well-developed arguments across various social issues related to science exhibited more informed understandings of NOS characteristics because they were more apt to look past their preconceived notions of science and beliefs about science to evaluate evidence from a clear perspective (Khishfe, Alshaya, BouJaoude, Mansour, & Alrudiyan, 2017; Kim & Helm, 2011; Mesci & Schwartz, 2017). Consequently, learning NOS not only influenced how the participants in the studies viewed science, but also how they viewed issues in the world. A deep understanding of NOS helped the participants become informed about science content. For example, one PT's growth in understanding of NOS revealed a shift in his belief that science could only be conducted using the scientific method. His deeper understanding of NOS allowed him to see and value different methods of research in a science setting. Understanding of NOS promotes the need to look past one's own beliefs to gather evidence and make informed decisions. NOS is neither universal nor stable, and thus individuals, specifically teachers and PTs, must be aware of the ideas which make up the consensus view of NOS as a way to promote student achievement and understanding in science (Lederman, 1992). An understanding of NOS, as defined by the consensus view, helps teachers teach science. Though a consensus view of NOM does not exist, a consideration of characteristics which make up mathematics is necessary to help teachers and PTs understand and teach mathematics. Ernest (1991) said,

> How mathematics is viewed is significant on many levels, but nowhere more so than in education and society. For if mathematics is a body of infallible, objective knowledge, then it can bear no social responsibility . . . On the other hand, if it is acknowledged that mathematics is a fallible social construct, then it is a process of inquiry and coming to know, a continually expanding field of human creation and invention, not a finished product. (p. xii)

Just as the consensus view of NOS has been helpful for studies in science education research, I drew on the reform documents themselves and mathematics education research on teachers' and PTs' beliefs about NOM to create a framework for this study.

Teachers' Conceptions of Nature of Mathematics

The consensus view of NOS aided science educators in their consideration of important aspects teachers should know about science. Without a consideration of the important aspects teachers should know about mathematics, I drew on the research regarding teachers' conceptions about NOM. Recall that a teacher's conceptions are defined as "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (Thompson, 1992, p. 132). These conceptions are dynamic structures susceptible to change based on experiences (Thompson, 1992). So, simply because a teacher espouses one conception at one moment in time does not necessarily mean this conception is never changing. Even though these conceptions may change, understanding these conceptions is the first step to helping teachers implement reform-based mathematics instruction.

Historically, teacher beliefs have been a widely researched area. However, teacher beliefs constitute a construct much broader than teachers' conceptions of NOM. One seminal mathematics education research review focusing on teacher beliefs highlighted different conceptions teachers held with regards to NOM (Thompson, 1992). Thompson expanded on the philosophical views of NOM by offering an explanation of the views as they relate to mathematics teachers. Thompson emphasized three conceptions of mathematics: (1) a dynamic, problem-driven discipline; (2) a static-unified body of knowledge; and (3) a bag of tools. Thompson's first view aligned with the fallibilist view of NOM as it emphasized the dynamic, problem-solving nature of NOM. The view of mathematics as a static body of knowledge aligned with the Platonist view of NOM because each promoted mathematics as unchanging truths to be discovered. The view of mathematics as a bag of tools aligned with the instrumentalist view of mathematics as tools to be used, often times without reason. In Chapter One, I defined the term conception and elaborated on the use of that term in this study instead of beliefs. Here, the term conception is used to relate, in Thompson's case, a teacher's conception of NOM to the philosophical views of NOM. The term conception denotes that the conceptions are most often associated with teachers' understandings of mathematics but are related to the broader philosophical views of NOM.

Teachers' beliefs about mathematics can create an obstacle when implementing reform-based instruction (Handal, 2003; Phillip, 2007). These beliefs held by teachers make implementing reform-based instruction difficult as the teachers often do not deviate from the more traditional approaches to mathematics. In fact, the perpetuation of the traditional view of mathematics—review homework, lecture on new material, practice multiple problems—is still pervasive in classrooms today (Ball, Lubienski, & Mewborn, 2001; Banilower et al., 2006; Dossey, 1992; Ertekin, 2010; Stigler & Hiebert, 1999; Throndsen & Turmo, 2013).

Beswick (2012) conducted a study which incorporated three views of NOM—the instrumentalist, Platonist, and fallibilist views. She sought to determine the disparities that might arise if teachers held differing views of NOM and school mathematics and the implications these disparities might have for teaching. One participant, Sally, had been teaching for 18 years, held beliefs consistent with the fallibilist view, and was consistent with problem solving in her classroom. However, when asked about mathematics as a broader discipline, she aligned more closely with the Platonist view. Beswick attributed Sally's differing views to her many years of teaching. That is, Sally likely held a more fallibilist view of school mathematics because she had been teaching and had significant exposure to reform documents. However, Sally admitted to rarely reflecting the broader discipline of mathematics, and this attributed to the differing views she held about school mathematics and NOM. The second participant, Jennifer, had been teaching for two years and held an instrumentalist/Platonist view of mathematics. Jennifer struggled to reconcile her instrumentalist/Platonist views of NOM with a self-expressed desire to teach through problem solving. However, there was no evidence that suggested Jennifer's beliefs about school mathematics were distinct from her beliefs about NOM.

Similarly, Garegae (2016) conducted a study seeking to answer the question, "What beliefs do teachers have about the nature of mathematics, its teaching and learning?" (p. 1). Like Beswick (2012), Garegae explained that teachers' views of NOM often are a combination of more than one view of NOM. More specifically, he stated that they are a combination of instrumentalist, Platonist, and fallibilist views. One teacher, Kgosing, held an instrumentalist/Platonist view of NOM. His view was evident by his teaching because he often provided students with a lecture followed by completion of problems emphasizing procedures and rules. A second teacher in the study, Thamo, enacted instrumentalist views in the classroom as he focused on mathematics as practicing skills. However, this was in conflict with his espoused beliefs about NOM. Thamo described mathematics as an exploration of ideas which promoted the discoverability of mathematics. So, his espoused beliefs more closely aligned with the fallibilist perspective. The third participant, Letsomane, believed that mathematics was pre-existing, and in school students had to discover the mathematics through investigations. Additionally, Letsomane believed that there are still concepts in the world to be discovered. However, his espoused Platonist view of mathematics differed from his view of mathematics as a discipline. As a discipline, Letsomane viewed mathematics as dynamic and expandable through exploration. In his practice, he allowed students to experiment and discover. He was wavering between Platonist view and fallibilist view.

Though sweeping generalizations cannot be made based on two studies, what is important from each is that the teachers' views of school mathematics and mathematics as a discipline were often in contrast to each other (Beswick, 2012; Garegae, 2016). This conflict led to discord between what was important in school and mathematics as a broader discipline. The teachers in both studies struggled between reconciling their conceptions of NOM with what was expected of them in school mathematics. Garegae suggested that a teacher might prioritize their conceptions based on the context (i.e., school mathematics or the broader NOM). With the discord between these teachers' conceptions, both Beswick and Garegae suggested a deeper reflection on one's own understandings and conceptions of NOM and experiences which influenced those conceptions as a way to reconcile the differences between conceptions about school mathematics and NOM.

Prospective Teachers' Conceptions of Nature of Mathematics

Beswick (2012) and Garegae (2016) both presented the contrasting views teachers held regarding school mathematics and NOM and argued for the teachers' need to reflect on NOM and the influence their conceptions may have on their classrooms. Additionally, in these two studies, the researchers each reported the influence of the teachers' mathematical experiences in school had on their conceptions of NOM. Subsequently, a look at PTs' conceptions of NOM was necessary because they are still participating in their mathematical experiences in school.

Students' Conceptions

In the following studies, the participants were general education students and not declared prospective teachers. However, the parallels between the two groups are comparable because they are both undergraduate students learning mathematics in a content course. Furthermore, because one of my aims in this study was to separate the often-intertwined ideas of mathematics as a discipline and the teaching and learning of mathematics, I argue that a consideration of general education students' conceptions of NOM as reported in empirical literature (Jankvist, 2015; Rupnow, 2018; Szydlik, 2013) will help in my study with the distinction between the nature of mathematics teaching and learning and the nature of mathematics as a discipline. Szydlik (2013), Jankvist (2015), and Rupnow (2018) conducted studies aimed at changing students' conceptions of NOM.

Szydlik (2013) conducted a study on undergraduate mathematics students' beliefs about NOM and if those beliefs might change during one classroom experience. As the instructor of the course, the author structured the classroom around small groups working on challenging mathematics tasks followed by whole class discussion of findings, strategies, solutions, and arguments. Szydlik (2013) used the Mathematics Belief Instrument (MBI) at the start and end of the course to assess their conceptions regarding NOM. Szydlik found that on the initial survey the students' conceptions of NOM aligned with the instrumentalist view. That is, the students viewed mathematics as disconnected facts and procedures. By the end of the course, however, the students' conceptions included aspects inherent in the fallibilist view because the students expressed value in the problem-solving aspects of mathematics and less emphasis on mathematics as a body of facts and procedures.

Like Szydlik (2013), Jankvist (2015) found that students' beliefs could change by the end of a mathematics course focused on a problem-solving approach to mathematics. Jankvist stated that reflection was a core piece in the process of changing students' beliefs about NOM. He noted that the more reflection from a student the more likely the change in beliefs would last. One student, Andrew, held a belief that mathematics was discovered—a characteristic of the Platonist and fallibilist view of NOM. However, throughout the course, Andrew questioned his belief about the discoverability of mathematics. He explained by saying, "This guy Mr. Pythagoras, for instance, he didn't invent the relations in a triangle, it was something he found. Of course, the basic numbers and series of numbers, they are of course invented" (p. 51). He elaborated and explained that today he did not think mathematics can be invented. Like Andrew, another student Gloria struggled between mathematics as discovery or invention, stating, "You can't come up with some brilliant mathematics thing now-because so much have already been created" (p. 52). She stated that the process of deep reflection was most valuable when considering her beliefs about NOM. She explained that without the process of reflection, she would not have developed an understanding of the different aspects of NOM and that it was more than simply school mathematics.

In a third study, Rupnow (2018) examined abstract algebra students' conceptions of NOM in two different courses. One course was taught using *inquiry-oriented* materials while the second class was taught as two days of lecture and one day where students were encouraged to discuss problems in groups. Rupnow reported that across the two courses, students' conceptions of NOM varied. She explained that some students described mathematics as numbers or a "practical problem-solving tool" (p. 1013) and more aligned with NOM as a bag of tools. Other students defined mathematics as "the study of logic" (p. 1013) and aligned closer with a conception of NOM as a static-unified body of knowledge. In her study, Rupnow also had students provide animal metaphors with their descriptions of mathematics and added an affective component to their descriptions of mathematics. One student in Rupnow's study likened mathematics to a cat because everyone either loves cats or hates them, and the same is true for mathematics. Through the thematic analysis, Rupnow reported that students held varying conceptions of NOM. Furthermore, Rupnow posited that the difference in the two courses influenced students' conceptions of NOM. She explained that the differences between the inquiry-based course and the lecture-based course caused students in the different courses to consider the mathematics differently and thus influenced how they described their conceptions of NOM. Additionally, Rupnow suggested that in addition to the classroom experience, the students also made comments about their backgrounds and previous mathematics courses. Rupnow explained that this was an area she planned to study to extend her explanation regarding students' conceptions of NOM.

In another study students also exhibited seemingly contradictory beliefs regarding the way students perceived their experiences regarding mathematics. Op't Eynde and colleagues (2002) presented a categorization of students' beliefs related to mathematics. The students in this study held contradictory beliefs similar to students in Jankvist (2015), Szydlik (2013), and Rupnow (2018). However, the authors focused on the structure of those beliefs and differentiated between students' beliefs about themselves in mathematics, beliefs about the social aspects of mathematics, and beliefs about mathematics education. Op't Eynde and colleagues (2002) posited that students held beliefs about themselves in mathematics based on self-efficacy, goal-orientation, and the usefulness of mathematics. With regards to the beliefs about the social context, students expressed the idea that the role of the teacher and of the students in the classroom made a difference. That is, one student expressed that in his class they had to establish who got to determine what counted as a different solution or acceptable explanation (the teacher or the students). Lastly, with regards to belief about the context of mathematics, students said it depended on if one meant the teaching of mathematics, the learning of mathematics, or the discipline of mathematics itself. The authors represented the dimensions of students' mathematics related beliefs as depicted in Figure 3. The authors explained that this figure represented how the students' beliefs were situated in and determined by their beliefs about themselves in mathematics, beliefs about the social aspects of mathematics, and beliefs about mathematics education.

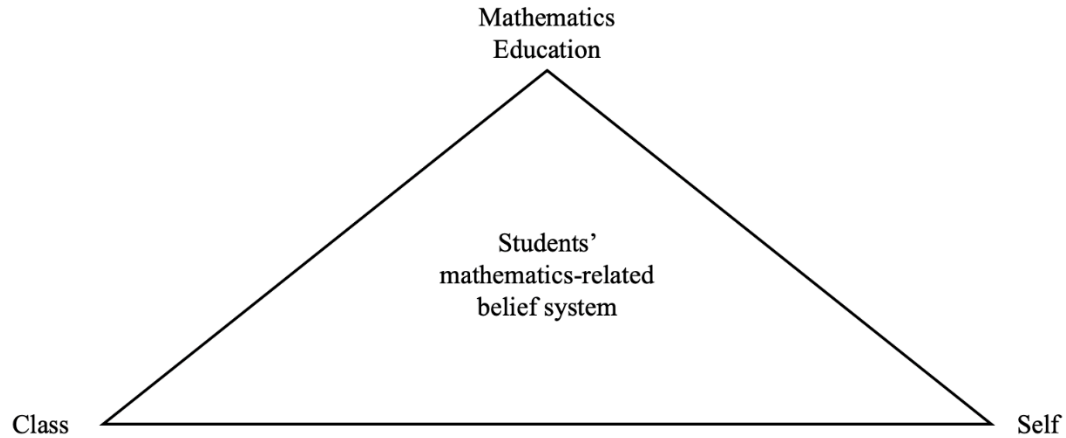

Figure 3. Students' mathematics-related beliefs. Adapted from “Framing students' mathematics-related beliefs” by P. Op ‘T Eynde, E. De Corte, & L. Verschaffel, 2002. In G. C. Leder, E. Pehkonen, & G. Tormer (Eds.) *Beliefs: A Hidden Variable in Mathematics Education* (pp. 13-37).

Jankvist (2015) expanded this figure to include mathematics as a discipline as depicted in Figure 4, Jankvist (2015) included the students' beliefs about NOM because it was “rather different than mathematics as a subject included in beliefs about mathematics education” (p. 45). He further explained that the development of a student's image about mathematics as a discipline can only develop through the interconnections they are making among mathematics education, themselves, and the social context of mathematics.

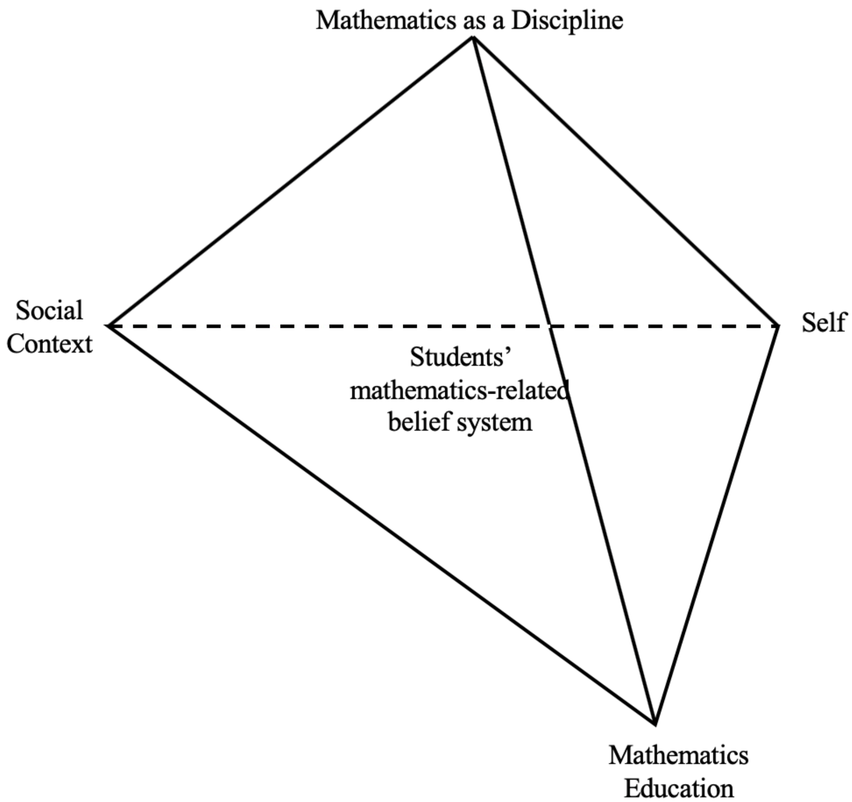

Figure 4. Inclusion of NOM in students' mathematics-related beliefs. Adapted from "Changing students' images of 'mathematics as a discipline' by U. T. Jankvist, 2015, *Journal of Mathematical Behavior, 38,* 41-56.

In each of these studies, students were given the opportunity to become aware of the conceptions of NOM, and the authors explained that while there was evidence of a shift in beliefs for students, that shift was not necessarily a lasting change. Jankvist (2015) argued that students needed access to ideas about NOM so they could reflect on and criticize them in regards to how they understand their beliefs about mathematics education, themselves, and the context of mathematics. Szydlik (2013) also suggested that students' conceptions of NOM can change when exposed to the experiences which promote fallibilist characteristics.

Prospective Teachers' Conceptions

Szydlik (2013) claimed that his initial findings about students' conceptions of NOM were consistent with those of PTs. In one study with elementary and middle PTs, Zazkis (2015) explored the PTs' relationship with mathematics through personification. The author provided PTs with the prompt,

> Your assignment is to personify Math. Write a paragraph about who Math is. This paragraph should address things such as: How long have you know each other? What does he/she/it look like? What does he/she/it act like? How has your relationship with Math changed over time? These questions are intended to help you get started. They should not constrain what you choose to write about. (p. 34)

The responses from this prompt elicited varying themes from the PTs, one of which related mathematics to a monster and the other to a former friend. Zazkis elaborated on this PT who personified mathematics as a former friend by equating mathematics as a sensible and understandable subject (i.e., a friend) and an increasingly complex subject (i.e., a monster). Within the themes that emerged, Zazkis related the personification themes to the *space of mathematics* which involved characteristics of NOM such as mathematics as creativity, level of enjoyment, complexity, and understandability. Though these themes discussed the nature of PTs' relationships with mathematics and not necessarily their conception of NOM as a discipline, this prompt can be used by researchers as a stepping stone to gain a deeper insight into PTs' conceptions of NOM.

For example, Sigley, Alqahtani, Zied, Widdall, and Hewer (2019) incorporated Zazkis' (2015) personification prompt by having students create a character and describe their relationship with mathematics as a way to study PTs' conceptions regarding mathematics. The authors reported that at the beginning of the class focused on teaching mathematics through a more conceptually based approach, the majority of PTs reported negative conceptions regarding mathematics. PTs stated that mathematics only occurred in schools and was “out to hurt or punish them” (p. 1027). As they described their negative relationships, PTs also portrayed mathematics as having one approach to a problem, a trait most often associated with a conception of NOM as a bag of tools. PTs in their study who described more positive relationships with mathematics reported that mathematics was logical, a trait that could be associated with any of the three conceptions of NOM depending on how the student described logic. At the end of the course, the author team had PTs complete the prompt again, and the team reported a stark contrast in how the PTs had described mathematics at the beginning of the course. The authors reported that PTs described a more positive relationship with mathematics and focused on multiple ways to approach a problem instead of rote memorization. Sigley et al. (2019) provided a description of the complex relationship that PTs have with mathematics and how using a personification prompt can allow researchers to better understand that complex relationship PTs described with mathematics that is not generally captured by the typical belief assessments.

Adding to the complexity of PTs’ descriptions of and relationships with mathematics, Sweeny, Ruef, and Willingham (2018) also incorporated the use of Zazkis’ (2015) personification prompt with the use of a Semantic Differential survey to answer the question, “What does it mean to be good at math?” Underlying this question were the PTs’ views about NOM. In this study, PTs were given a survey with 20 sets of paired words on a continuum and asked to place an x on each continuum when asked what they thought it meant to be good at math. One continuum had principles on one end and rules on the other. Sweeny et al. then conducted a matrix analysis to understand the commonalities in the components with the five strands of mathematical proficiency (NRC, 2001). After conducting the statistical analysis of the survey, authors explained the components of the survey related to the five strands of mathematical proficiency. For example, there was a high correlation between the continuum with principles and rules to the conceptual understanding strand. Another continuum with processes at one end and solutions at the other was highly correlated with the strategic competence strand. The writing and drawing prompts in this study documented the PTs' relationships with mathematics, like Zazkis' (2015) study. Similar to the Zazkis study, Sweeny and colleagues documented similar relationships with mathematics. Overall, the PTs' personal experiences with early mathematics created barriers that prevented them from engaging with and seeing the full scope and beauty of mathematics. That is, when the PTs expressed their conceptions of NOM, they were skewed towards rote practices indicative of an instrumentalist view of NOM (Sweeny et al., 2018, personal communication).

PTs' conceptions that mathematics is rote practices and not creative or beautiful is evident in one other study. In a study aimed at documenting PTs' conceptions of creativity in mathematics, the majority of PTs indicated they did not think mathematics was a creative subject (Bolden, Harries, & Newton, 2010). More specifically, the PTs valued other subjects over mathematics because other subjects—science and English—allowed for more opportunities to imagine, discuss, and explore. Some PTs even exclaimed that these subjects offered creativity because it did not depend on getting the correct answer like mathematics. These PTs' conceptions of NOM suggested they had instrumentalist views. After completing a class in which the PTs were expected to explore and create mathematics, the authors posited the PTs' conceptions had widened. One PT stated, "Now I know that maths [*sic*] is much more for understanding than for knowing, and whilst you can teach children the old, conventional ways it's nice to be able to offer them a couple of solutions, you know, to solve a mathematical problem" (Bolden et al., 2010, p. 153).

The previous studies suggested that PTs' conceptions of NOM are typically not aligned with the fallibilist view. Furthermore, in each of the four aforementioned studies (cf. Bolden et al., 2010; Sweeny et al., 2018; Szydlik, 2013; Zazkis, 2015) participants were required to reflect on their own conceptions by answering questions about mathematics as a discipline and engaging in thoughtful activities such as personifying mathematics or engaging in discussions about solving creative mathematical tasks. Without exposing PTs to different conceptions of NOM, often times through standards-based instruction (Bolden et al., 2010; Szydlik, 2013), they likely would not have explicitly been aware of their conceptions and how those conceptions were influencing their learning of mathematics. The PTs in the previous studies were exposed to mathematics in a way that was different than they were accustomed because fallibilist views of mathematics were at the forefront of teaching and PTs were often required to reflect on their conceptions of NOM. Through PTs' exposure to conceptions of NOM different from their own, by the end of the studies many of the PTs' conceptions of NOM had shifted. However, these changes may or may not be lasting changes. Thus, if a goal in mathematics education is to help teachers implement standards-based practices

(AMTE, 2017; CBMS, 2012) then bringing awareness to PTs regarding the fallibilist view and their own conceptions of NOM is a critical step in meeting this goal.

The characteristics encouraged through these studies align with the characteristics integral to the fallibilist perspective—that mathematics is dynamic, open to revision, and creative. The characteristics integral to the fallibilist view of NOM, as well as aspects of instrumentalist and Platonist views of NOM include aspects which are highlighted in mathematics standards documents (see NCTM, 2000; NRC, 2001). Additionally, these studies presented the ideas that when considering conceptions about NOM, PTs included examples about their social interactions, how they viewed themselves as mathematics learners, and how the classroom structure caused them to consider different ideas about NOM. Although the standards documents do not explicitly state the general characteristics of NOM, they do implicitly imply characteristics of NOM, as I will detail in the next section. Furthermore, the standards promote the alignment of school mathematics and mathematics as a discipline. NCTM stated that mathematics students should learn to appreciate mathematics as a discipline. Students should deepen their understanding of mathematics by realizing that mathematics is a human creation, is highly connected among mathematical topics but also non-mathematical contexts, and involves processes and skills which promote quantitative literacy (NCTM, 2000).

General Characteristics of the Nature of Mathematics

Fostering PTs' understandings of NOM will help them better understand the way that creatively solving problems, making connections between aspects of mathematics, and opening ideas to revision all relate to the nature of the discipline itself—a discipline that is not just made up of a body of facts to be memorized. With the goal of understanding PTs' conceptions of NOM and how PTs' lived experiences may have influenced those conceptions in mind and in the absence of a consensus view of NOM in the literature, I systematically analyzed reform documents and the literature to create a framework that represents a Proposed Unified View for NOM. This framework may be helpful to the broader field for two reasons: (1) to help MTEs understand how PTs' conceptions of NOM may be aligned with the Proposed Unified View of NOM and (2) to have a common list when referring to the construct of NOM. Recently, there has been a focus on NOM and the need for the field to have a unified view. For example, Norton (2018) expressed, "We need a definition of mathematics as a unified field of study rather than a collection of abstract sciences. What unifies mathematics? What are its objects of study? What is the basis for its reliability, utility, and ubiquity?" (p. 64). Norton (2019) continued to explain that in order to fully understand mathematics as a human creation and not just a Platonic idea requires the field to explicitly identify characteristics of mathematics. However, the field has not written explicitly and with agreement on what NOM is, and so I drew on what reform documents say, or imply, about NOM to have a working Proposed Unified View of NOM for this study.

Throughout the history of mathematics education, researchers placed importance on the ideas of what mathematics content to teach and how it should be taught. Sfard (2003) argued, "Mathematics is difficult. It is certainly among the most complex of human intellectual endeavors. As a school subject, it is often unmanageable. Much thought has been given over the years to how it can be successfully taught in spite of the difficulty" (p. 352). The thought given by the mathematics education community addressing the challenge to which Sfard refers is evident in current mathematics reform documents (see AMTE, 2017; CCSSM, 2010). AMTE (2017) expressed that, "For quite some time, professional organizations have called for opportunities for candidates to develop deep understandings and mathematical perspectives on the nature of mathematics as a discipline" (p. 89) in a practical and crucial way for teaching. However, unlike the science standards (NGSS Lead States, 2013) document, AMTE provides no explicit guidance for what teachers and students should know about NOM. That is, in the standards for preparing teachers, AMTE does not make explicit how to develop candidates' mathematical perspectives. Additionally, the Standards for Mathematical Practice (SMPs) outlined in the CCSSM (2010) hold implicit ideas about NOM. The SMPs rest on the foundation of the Process Standards (NCTM, 2000) and the Goal of Mathematical Proficiency (NRC, 2001). Therefore, the Process Standards and the Strands for Mathematical Proficiency, which are built on mathematics education research, are foundational to the ideas presented in AMTE (2017) standards as well as the CCSSM (2010) and provide a foundational list of characteristics inherent in the current vision for mathematics. These two documents will provide the basis for the Proposed Unified View of NOM.

Process Standards

NCTM's (2000) Process Standards provide a set of expectations that mathematics students should engage in when doing mathematics. These five processes include *problem solving, reasoning and proof, connections, communication,* and *representation* and represent processes students must engage in to learn mathematics. The process standards provide a means for instruction that should "enable students to know and do" (NCTM, 2000, p. 7). Problem solving, as outlined by NCTM, is not the traditional homework assignment of a list of problems to solve using a pre-taught strategy, but instead requires students to approach a problem or task with a strategy that is not previously known. That is, students must use prior mathematical knowledge to construct new ways of thinking about and eventually solving the problem. Problem solving allows students to develop more sophisticated ways of thinking about problems as well as persistence in solving problems. Problem solvers can then approach non-mathematical situations with the same questioning and discussion of strategies as they did in mathematical contexts. Lastly, problem solving allows students to change their strategy if needed. For example, it is not uncommon to consider their progress halfway through a problem and then adjust if needed.

Reasoning and proof engage students in recognizing patterns, making conjectures, and developing arguments. Reasoning is valuable for mathematical understanding because it helps students understand the importance of using evidence to support or refute an assertion. Reasoning and proof challenges the idea that there are magic tricks (i.e., keep-change-flip in regards to the algorithm for division of fractions) in mathematics. Engaging in mathematical reasoning and proof also allows students to discover mathematics because they can investigate conjectures while questioning the reasonableness of their arguments throughout problem solving.

Embedded in the previous two process standards is the idea of communication. Communication is essential for students to share ideas, understand reasoning of others, and develop their own mathematical understanding. Through communication students can begin to refine the mathematics they have written, describe their ideas coherently and clearly to others using appropriate mathematical language, listen to others' ideas, and critique the strategies. Communication is not limited to oral communication but also includes written communication also through proofs and written work during problem-solving tasks.

Problem solving, reasoning and proof, and communication foster students' ability to make connections across mathematical ideas. The connection of mathematical ideas builds a deeper mathematical understanding for students. Building connections can help students see that mathematics does not have to be a set of disconnected rules to be memorized, but instead a connected system of concrete or abstract ideas.

Lastly, representation provides different ways in which mathematical ideas can be displayed and therefore understood by students. In the process standards, representation refers to the product as well as the process. That is, representation refers to a mathematical model itself as well as the process of creating the model. It is vital to mathematics that students can represent their ideas in a way that makes sense to them and not by a prescribed model or form which may have little meaning to them. The use of representations and the discussions which can result from the connection of multiple representations allow students to build connections, understand others' ideas, and deepen their mathematical understanding. In Table *2*, I provided a summary of the key components of each of the five Process Standards.

Table 2

Summary of Process Standards

	Problem Solving	Reasoning and Proof	Communication	Connections	Representations
Characteristics of each Process Standard	Builds new knowledge	It is fundamental	Organize and consolidate thinking	Recognize and use connections	Create and use representations to organize, record, and communicate ideas
	Valuable for non-mathematical contexts	Make and investigate mathematical conjectures	Communicate thinking coherently and clearly	Understand how ideas interconnect and build on each other	Select, apply, and translate among representations
	Choice of strategy	Develop and evaluate mathematical arguments and proofs	Analyze and evaluate thinking of others	Recognize and apply in different contexts	Use representations to model and interpret
	Have opportunity to monitor and reflect on progress	Select and use various types of reasoning and methods of proof	Use mathematical language		

*Adapted from NCTM (2000)

Strands of Mathematical Proficiency

Similar to the process standards, NRC's (2001) Goal of Mathematical Proficiency focused on five strands that are essential for students to meaningfully learn mathematics. After reflecting on historical changes in mathematics education, reading mathematics education research, and experiencing mathematics as teachers and learners, the researchers agreed on the five strands which they considered to be a comprehensive view of mathematics learning. The five strands include: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning,* and *productive disposition.* Though these five strands are deeply interwoven and support one another, they also each have their own unique characteristics.

Conceptual understanding focuses on the idea that students learn more than memorized facts and procedures. Conceptual understanding requires students to understand the mathematical idea, how it came to be, and why it makes sense. This type of understanding promotes retention of mathematical ideas in ways that allow students to make connections among mathematical ideas and represent the mathematics in different ways.

The second strand of mathematical proficiency is procedural fluency. The base of this strand is that students must know the procedures of mathematics and how and when to appropriately use those procedures. When students are efficient and accurate in solving problems, their procedural fluency supports their conceptual understanding. Additionally, procedural fluency helps students see the structure of mathematics because a student with procedural fluency can see how procedures can be applied to not only individual problems but whole classes of problems, thus illuminating the structure of mathematics as a whole.

The third strand of mathematical proficiency is strategic competence. This refers to students' problem-solving facility. Strategic competence is important for mathematical proficiency because it refers to students' use of problem-solving skills in the mathematics classroom as well as in non-mathematical situations. Students should understand a variety of strategies and formulations of problems to solve a problem.

The fourth strand of mathematical proficiency is adaptive reasoning and refers to students' logical thinking about mathematical concepts. Adaptive reasoning is important for conceptual understanding, procedural fluency, and strategic competence because in each of these three strands, students must assess the reasonableness of their solution paths and results as well as the relationships forming between the different solution strategies and mathematics present in the problem. Adaptive reasoning does not only apply to formal mathematical proof, but also encompasses a broader idea of including informal logical thought along with formal mathematical proof.

The fifth, and last, strand of mathematical proficiency is productive disposition, which refers to the inclination to see mathematics as worthwhile and useful. Additionally, productive disposition encompasses students' views of themselves as a capable learner and doer of mathematics. Students must not only view themselves as capable but also recognize that through productive struggle and perseverance, one can figure out the mathematics. In Table 3, I provide a summary of the important aspects of the five strands of mathematical proficiency.

Table 3

Summary of Strands of Mathematical Proficiency

	Conceptual Understanding	Procedural Fluency	Strategic Competence	Adaptive Reasoning	Productive Dispositions
Characteristics of each Strand of Mathematical Proficiency	Comprehend mathematical concepts, operations, and relations	Skill in carryout procedures flexibly, accurately, efficiently, and appropriately	Ability to formulate, represent, and solve mathematical problems	Capacity for logical thought, reflection, explanation, and justification	Habitual inclination to see mathematics as sensible, useful, and worthwhile
	Provides basis for generation of new knowledge	Helps students assess reasonableness of results	Applies to non-routine and non-mathematical contexts	Not limited to formal mathematical proof	Belief in diligence and one's own efficacy
	Uses and values different representations		Important in every step of developing procedural fluency		Persevere through problem solving

*Adapted from NRC (2001)

The Process Standards describe a means of instruction, and the Strands of Mathematical Proficiency focus on outcomes of student learning. In addition to the clear focus in these two standards documents on the teaching and learning of mathematics, implied are the characteristics of the discipline of mathematics. For example, in both the Process Standards and Strands of Mathematical Proficiency, problem solving is evident. Problem solving in each of these documents focuses on the idea that mathematics involves working through non-routine problems and strategizing through different solution paths.

Overlapping Ideas of the Foundational Standards Documents

In addition to a focus on problem solving, there is an overlap of ideas between the Process Standards and Strands for Mathematical Proficiency. Consider the implied characteristics of NOM inherent within the Process Standards and the Strands for Mathematical Proficiency. The Process Standards and the Strands of Mathematical Proficiency have many commonalities, but there are four distinct ideas that overlap: problem solving, reasoning, communication, and connections. In addition to the ideas that overlap, the strands of mathematical proficiency offer the idea that mathematics also includes some sense of personal relationship. That is, one who understands mathematics sees the usefulness of mathematics, the sense of mathematics, and oneself as an effective learner and doer of mathematics. Thus, these five characteristics (i.e., problem solving, reasoning, communication, connections, and personal relationship) are the five characteristics which create a Proposed Unified View of NOM as defined by mathematics standards documents. I argue that these five characteristics, regardless of the context in which one does mathematics—as a teacher, as a mathematician, or as a student—do not change. I will present each characteristic and what they reveal about NOM.

Problem Solving

Problem solving allows for unique exploration of a task because there is no one strategy given to the student. Miller, Heeren, and Hornsby (2012) listed various problem-solving strategies such as making a chart/table, drawing a sketch, finding a pattern, and trial and error. Students have multiple entry points when attempting problem solving (Fi & Denger, 2012). Often times, problem solving is referred to based on the pedagogy which it promotes. That is, teaching *through* problem solving "engages students in problem solving as a tool to facilitate students' learning of important mathematics subject matter and mathematical practices" (Fi & Denger, 2012, p. 455). Problem solving allows one to make sense of a problem, persevere in solving the problem, and model with mathematics. Problem solving is inherent in NOM.

Reasoning

Mathematics is a reasonable discipline. That means in mathematics one can reason abstractly and quantitatively and find patterns within that reasoning that can lead one to create an argument. The idea of argument refers to the idea that at the heart of mathematics is proof. Through proof one can construct and justify an argument.

Communication

In mathematics, communicating precisely to others is essential, because after construction and justification of an argument, one must engage in a discussion with others to critique and refine the argument. This critique and refinement of ideas can lead to establishing results and truth in conjectures.

Connections

When one engages in mathematical communication, one will understand the connections across the ideas. Mathematics has a discernible structure or pattern, and one can attend to patterns and structure to understand different mathematical representations and how the representations are interrelated.

Personal relationships

One's personal relationship with mathematics is important to how one understands mathematics. For example, according to the Process Standards and Strands of Mathematical Proficiency, part of one's productive disposition is to see sense in mathematics, value mathematics as useful and worthwhile, and recognize oneself as a learner of mathematics.

More Characteristics of the Nature of Mathematics

Pair (2017) proposed a list of characteristics of pure mathematics that he claimed were inherent to the nature of pure mathematics. He expounded on the importance of the list as a way to make the goals in mathematics education explicit for students and teachers alike and that without explicitly stating the goals there would be no way to effectively move forward in research to best meet the goals. Through a heuristic inquiry and narratives from mathematicians and students to support this list, the researcher proposed the IDEA framework that consists of four characteristics which make up NOM (see Figure 5). Pair (2017) further explained the necessity of a "modest list" (p. 114) encompassing NOM because no one list could possibly outline everything one should understand about NOM.

The IDEA Framework for the Nature of Pure Mathematics

Our Mathematical Ideas and Practices are Part of Our **Identity**.

Mathematical Knowledge is **Dynamic** and Forever Changing.

Pure Mathematical Inquiry is an **Exploration** of Ideas.

Mathematical Ideas and Knowledge are Socially Vetted through **Argumentation**.

Figure 5. IDEA Framework. Adapted from *The Nature of Mathematics: A Heuristic Inquiry* (Unpublished doctoral dissertation) by J. Pair, 2017, Middle Tennessee State University, Murfreesboro, Tennessee.

Including ideas from the Process Standards, Strands of Mathematical Proficiency, and Pair's (2017) IDEA framework, I provide a summary of the list of statements associated with the Proposed Unified View of NOM as depicted in Figure 6. These nine key statements of a Proposed Unified View of NOM align closely with a framework suggested by Pair (2017). Problem solving aligns with the dynamic and exploration characteristics of the IDEA framework. I argue that mathematics is not just dynamic and changing, but it is the process of exploring problems which makes it so. The Process Standards (NCTM, 2000) and Strands for Mathematical Proficiency (NRC, 2001) support this by explaining that problem solving is a process in which learners explore the mathematics and begin to make conjectures in the path to a solution. In the IDEA framework, identity refers to the fact that an individual's mathematical practices are part of them. These identities learners have with the mathematics influence their relationship with mathematics. This means that learners have a personal relationship with mathematics. Argumentation in the IDEA framework aligns with the communication aspect of the Proposed Unified View, since inherent in argumentation is the idea that mathematics is inspected through social interactions.

Proposed Unified View
Mathematics involves exploration.
Mathematics involves multiple strategies.
Mathematical ideas are communicated and verified through proof/justification.
Mathematics requires justification of ideas to others.
Critique of mathematical ideas leads to refinement.
Structure and patterns are inherent in mathematics.
Mathematics uses multiple representations.
Mathematics is useful and worthwhile.
Anyone can be a learner of mathematics.

Figure 6. List of characteristics of a Proposed Unified View of NOM.

A Proposed Unified View of NOM includes characteristics that are applicable to mathematics regardless of the context of mathematics. That is, regardless of one's potential use of mathematics, the characteristics in the Proposed Unified View do not change. Thus, I used this Proposed Unified View as a conceptual framework for this study. Note that this Proposed Unified View is not intended to denote a consensus view regarding NOM in the field. Rather, this view is a starting point—a list of characteristics that are already implied in standards documents with regards to mathematics as a discipline. I am making this view explicit in this study as a way to investigate the alignment of PTs' views of NOM with this Proposed Unified View from influential literature in the field.

Conceptual Framework

The focus of this study was to understand PTs' conceptions about NOM and the experiences which influenced those conceptions. To guide the framing of this study, I used the dimensions of student beliefs presented by Jankvist (2015) and represented in Figure 4 as a way to think about NOM as an essential concept that relates to and influences the broader idea of mathematics-related beliefs. Second, I used the Proposed Unified View of NOM (see Figure 6) as a conceptual framework that draws on ideas based on the standards put forth by NCTM (2000) and NRC (2001) and the IDEA framework proposed by Pair (2017). The Proposed Unified View is a way to consider the importance of the construct of NOM as it is presented by Jankvist (2015).

Jankvist (2015) drew on an earlier work describing three tenets of students' mathematics-related beliefs—the social context, oneself, and mathematics education. Jankvist's (2015) expansion of the earlier work was necessary because the original figure diminished the importance of a student's beliefs about mathematics as a discipline and instead focused on them as part of the teaching and learning of mathematics. In Chapter One, I referred to the difficulty in distinguishing mathematics as a discipline from the teaching and learning of mathematics (Thompson, 1992) and that separating ideas of the teaching and learning of mathematics from mathematics as a discipline (i.e., How I am defining NOM in this study) was a sub goal of the study. Jankvist (2015) further posited that a student may come to understand or develop an image of mathematics as a discipline by considering the interplay of the student's beliefs about teaching and learning of mathematics, themselves in mathematics, and the context (i.e., social aspects) of mathematics. This idea aligns with the image put forth by Ernest (1989) suggesting that NOM influences the mental structures teachers have of teaching and learning mathematics. Jankvist's (2015) intentionally created Figure 4 as a tetrahedron, and not a square, to show that NOM (i.e., the discipline of mathematics) is at the apex of the tetrahedron with the original triangle as the base. Including NOM at the apex of the tetrahedron brought attention to NOM as an integral aspect of students' mathematics-related beliefs.

The emphasis of NOM in both the frameworks that Jankvist (2015) and Ernest (1991) presented was their intentional way to posit the importance of the construct of NOM. That is, NOM is at the top of the figure to emphasize the importance of NOM when considering mathematics-related beliefs. However, without a Proposed Unified View of NOM, the emphasis on NOM will continue to be implicit and potentially invisible as a construct. Therefore, I used the Proposed Unified View of NOM (see Figure 6) as a conceptual framework as it describes the different aspects that should be considered in the construct of NOM. Since the goal of the study was to describe elementary PTs' conceptions of NOM, understand the experiences that influenced those conceptions, and connect PTs' conceptions of NOM to the Proposed Unified View of NOM, I used the conceptual framework as a definition of clear characteristics of NOM. The Proposed Unified View is intended to illuminate and clarify the different ideas surrounding NOM to make the construct of NOM less ambiguous. That is, in order to understand the phenomenon of NOM and elementary PTs' conceptions of NOM, and how their experiences inform it, NOM must have a clear meaning. The Proposed Unified

View is not intended to be an exhaustive list of characteristics of NOM, but instead a list of general characteristics that any *doer* of mathematics would agree upon.

Chapter Summary

In conclusion, in this chapter I provided an overview of the relevant literature concerning NOM, which focused on three main views of NOM (instrumentalist, Platonist, and fallibilist), their importance for the teaching of mathematics, and how NOS can inform mathematics education. I also described the contrasting nature of teachers' conceptions about NOM and school mathematics and how they struggle to reconcile those differences through reflection and based on their mathematical experiences. Because PTs' are still engaged in their mathematical experiences, a look at PTs' conceptions of NOM was necessitated. Research studies have shown that PTs' conceptions of NOM are generally not aligned with the fallibilist view. However, the PTs' reflection on non-fallibilist aspects of NOM was critical to creating a dissonance which fostered reflection on NOM and provided opportunities for PTs to consider non-fallibilist perspectives. It is this exposure to differing conceptions of NOM which can help PTs implement standards-based instruction in their future classrooms. In considering the importance of teachers' and PTs' consideration of NOM, I also provided a list of characteristics which make up the Proposed Unified View of NOM based on the ideas put forth by two standards documents. Finally, I concluded this chapter with a description of the framework which guided this study. The Proposed Unified View of NOM was also used as an analytical framework, which will be described further in Chapter Three.

CHAPTER III: METHODOLOGY

Introduction

Mathematics reform-documents (e.g., AMTE, 2017; CBMS, 2012; CCSM, 2010; NCTM, 2000, 2014) cast an ambitious vision for school mathematics—a vision that promotes the alignment of school mathematics and the discipline of mathematics. This is an enormous challenge and "meeting it is essential" (NCTM, 2000, p. 4) for the successful mathematics education of each and every student. As evidenced in Chapter Two, although the vision for school mathematics is set through standards documents, PTs' conceptions of mathematics are often not aligned with ideas in the documents (Sweeny et al., 2018; Szydlik, 2013; Zazkis, 2015). In fact, PTs' alternative conceptions about NOM, lack of overall consensus regarding NOM, and the need to improve teacher preparation programs create further impediments to achieving this ambitious vision. Understanding elementary PTs' conceptions about NOM and how these conceptions came to be is vital to bring awareness to elementary PTs themselves about the conceptions they hold regarding NOM and also to MTEs seeking to improve mathematics education.

Elementary PTs often assume they already know the content they are expected to teach and, therefore, underestimate the complexities required for teaching elementary school (Ambrose, 2004; Ball, 1990; Richardson, 1996; Weinstein, 1989). It is reasonable to assume that a similar dynamic might be at play with respect to NOM. If PTs have never reflected upon their own conceptions of NOM, they will likely fail to understand the dynamic nature of mathematics and allow those understandings to inform their teaching. Thus, this study aimed to understand elementary PTs' conceptions of NOM and how those conceptions were influenced by previous mathematics experiences. This study was a first step to understanding how PTs' conceptions of NOM could inform their instructional practices. The following sections provide an overview of the research methodologies I employed in this study.

Research Overview

I used an explanatory phenomenological design for this study. First, an explanatory sequential design allowed me to collect and interpret quantitative data before collecting and interpreting qualitative data (Creswell & Plano Clark, 2011). Second, I employed the use of phenomenological aspects to understand how PTs' experiences informed their conceptions of NOM and how those conceptions were connected, if at all, to the Proposed Unified View of NOM (Moustakas, 1994). This explanatory phenomenological design helped me investigate elementary PTs' conceptions of NOM and the experiences that informed those conceptions throughout various points in the PTs' mathematics career. Research questions included the following:

1. What are elementary prospective teachers' conceptions of the nature of mathematics?
2. How do the lived experiences of the elementary prospective teachers inform their conceptions of the nature of mathematics?
3. What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View, and what are the implications of those connections, if any?

With a goal to understand elementary PTs' conceptions of NOM and bring elementary PTs' awareness of their own conceptions of NOM as the first step in potentially improving elementary PTs' implementations of standards-based instruction, I chose an explanatory, sequential phenomenological mixed-methods approach as a way to incorporate diverse viewpoints (Creswell & Plano Clark, 2011; Patton, 2015; Smith, Flowers, & Larkin, 2009). This mixed-methods design was helpful in order to gain a clearer, fuller understanding of PTs' conceptions of NOM. More specifically, the explanatory sequential design was used as a way to use the quantitative data collected as a building block for the qualitative data (Creswell & Plano Clark, 2011). In this design, the quantitative portion is followed by a qualitative portion as a way to elaborate on findings from the quantitative portion in more depth and detail. I chose to collect both qualitative and quantitative data and sequentially have the data build on one another as a way to better explain and understand PTs' conceptions of NOM in more depth by giving priority to both forms of data in four phases (See Figure 7).

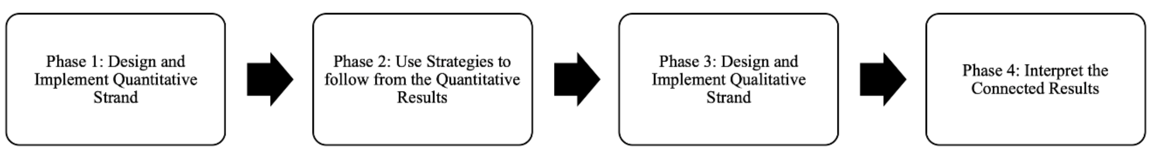

Figure 7. Four phases of the study adapted from "Designing and Conducting Mixed Method Research, 2nd ed." By J. W. Creswell, & V. L. Plano Clark, 2011, Thousand Oaks, California, SAGE Publications.

First, I collected quantitative data via surveys and writing prompts from a large group of PTs enrolled in mathematics content and methods courses. More specifically, phase one began the quantitative portion of the mixed-methods design and included the dissemination of the quantitative surveys and writing prompts. Second, based on the survey results and writing prompts, I selected participants from the larger group of elementary PTs to explain and offer insights into their survey answers and writing prompt via interviews. In phase two, I focused on analysis of the surveys and the ideas and themes revealed through the analysis about PTs' overall conceptions of NOM. The analysis of the surveys provided insight into the subgroup of PTs to select for interviews. In the qualitative phase, phase three, I continued analyzing writing prompts and also conducted interviews with a subgroup of PTs to gain a deeper insight into their conceptions and how those conceptions were formed based on their experiences with mathematics. In this qualitative phase, I relied on a phenomenological approach to focus on how PTs "make sense of experience and transform experience into consciousness" (Patton, 2015, p. 115). That is, the phenomenological approach provided me with an opportunity to gain insight into the direct or indirect experiences which shaped PTs' conceptions of NOM by asking questions and considering the first-hand voice of the participants. More specifically, a phenomenological approach allowed me to explore a phenomenon in which a group of individuals have all experienced (Moustakas, 1994)—being a student of mathematics. The goal in the phenomenological methods was to describe *essence* of the experience for PTs and interpret what they experienced as mathematics students and future teachers (Creswell, 2013). In this study, the PTs shared the experience of being enrolled in either a content or methods course at the same university. However, PTs also shared similar experiences as lifelong mathematics students and through the interviews offered additional insights into their previous mathematics experiences. Additionally, a phenomenological approach to this study allowed the opportunity for *intentionality,* by asking PTs to deliberately reflect on their ideas about mathematics and experiences with mathematics in order to gain a deeper insight of those conceptions and experiences (Husserl, 1927; Smith, Flowers, & Larkin, 2009). Intentionally asking PTs to reflect on and consider their mathematical experiences permitted me, the researcher, to describe PTs' conceptions of NOM and experiences that influenced those conceptions.

With an overall goal of describing PTs' conceptions of NOM, understanding PTs' experiences that influenced their conceptions of NOM, and connecting their conceptions of NOM to the Proposed Unified View, dividing this study into four phases was important for three reasons. First, by collecting the survey data initially, I was able to use it to help choose participants for the interviews. Second, the four phases allowed me to build my analysis and focus on one particular idea at a time. Last, by focusing on one idea at a time, I was able to create a more cohesive story and see connections among emergent themes.

For the following sections in this chapter, I will present the overall context of the study, including a description of the university where the study took place, a description of the courses in which the elementary PTs were enrolled, and an overall time frame for data collection and analysis. Next, I will provide the details for each of the four phases of the study outlined by Creswell and Plano Clark (2011) and detailed in Figure 7. After the context of the study and description of each phase of the study, I will then provide my view as a researcher and attend to issues of trustworthiness and credibility.

Research Context

This study involved elementary PTs enrolled in mathematics content and methods courses at a public southeastern university during the spring semester 2019. The following two sections provide more detail about the university and the courses.

University

The university where the study took place was a public university located in the southeastern United States. At the time of this study, the university had a total population of 22,000 undergraduate and graduate students with a diverse student body of 34% non-white or underrepresented minority groups and 55% female.

Courses

The courses in which the participants were enrolled during the spring 2019 semester for the study were at the undergraduate level and were required for all students pursuing a degree in early childhood education, elementary education, or special education. These courses were part of the degree plan to prepare PTs to teach elementary grades (i.e., PreK-5). The three courses included two content courses and one methods course and are described further in Table 4.

Table 4

Description of Required Courses for PTs

Course Number	Course Title	Course Description
Course 1	Concepts and Structure of Elementary School Mathematics	Problem solving, set theory, functions, number theory
Course 2	Informal Geometry	Plane, solid, coordinate, and motion geometry as well as constructions, congruence, similarity, and conceptions of measurements
Course 3	Mathematics Methodology	Preparation for elementary and middle school PTs with a focus on pedagogy and field-based experiences

Participant Selection

I visited each section of the mathematics content courses and each section of the mathematics methods courses to introduce myself and the study to the elementary PTs so that all PTs enrolled in either a mathematics content or mathematics methods course in the spring 2019 semester were given the opportunity to participate in this study. Of the 221 students enrolled in the three courses (See Table 5) 178 consented to participate in the study.

Table 5

PTs during Spring 2019

	Number of Sections	Number of PTs Enrolled	Number of PTs who Consented
Course 1	4	93	60
Course 2	3	72	69
Course 3	3	55	49
Totals	10	220	178

Of the 178 PTs that consented to participate in the study, the number of PTs who completed each data source varied. In Figure 8, I show the breakdown of PTs and which mathematics courses they had previously completed to give a broad overview of their mathematical background.

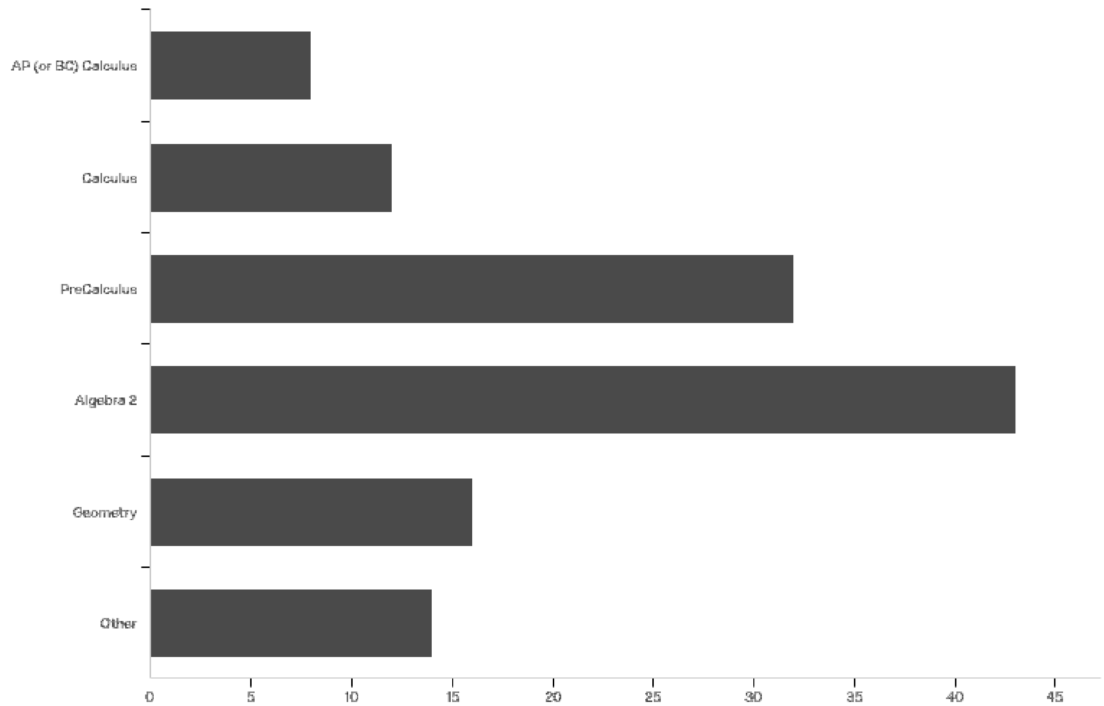

Figure 8. PTs' highest level of mathematics taken in high school.

Data Sources

After obtaining consent from the PTs, I distributed two surveys via Qualtrics, the Mathematics Belief Instrument (MBI) and the Semantic Differential to gain insight into PTs' conceptions of NOM. In addition to the two surveys, participants also had access to a writing prompt where they were asked to personify mathematics. The details of these data sources follow below.

Mathematics Belief Instrument

The MBI (Szydlik, 2013; see Appendix B) consisted of one open-ended question asking, "What is mathematics," and 10 Likert-scale questions regarding beliefs about NOM. The 10 statements included in the MBI survey were both positively and negatively worded phrases. For example, one statement one the MBI was, *Mathematics* *is mostly a body of facts and procedures*, and the participants could rate between 1 (strongly disagree) to 5 (strongly agree). The MBI was designed to measure the degree of alignment of views of the discipline of mathematics based on mathematicians' views. The MBI has a possible range of -20 to 20 points because each response garners a score between -2 to 2, with negative scores indicating disagreement with characteristics of the mathematics community and positive scores indicating agreement. Thus, if a PT strongly agreed that mathematics is mostly a body of facts and procedures, that PT would be given a score of -2 for that statement indicating disagreement with the mathematics community. If a PT strongly disagreed that mathematics is mostly a body of facts and procedures, that PT would be given a score of 2. If the PT rated neutral for the statement mathematics is mostly a body of facts and procedures, they would be given a score of 0.

The Proposed Unified View presented in Chapter Two described general characteristics of NOM based on two foundational standards documents and other mathematics education research literature, thus the MBI survey also served as an indication of PTs' alignment or misalignment with the Proposed Unified View. Of the 178 PTs who consented to participate in the study, 108 completed the MBI.

Semantic Differential Survey

The Semantic Differential (Sweeny et al., 2018; see Appendix C) consisted of 20 paired words (e.g., fast/accurate or relationships/recall) and asked PTs to place an X along a continuum closer to which word best described someone who is good at mathematics. Researchers designed the Semantic Differential to align with the strands of mathematical proficiency (Sweeny et al., 2018). Together with the MBI, the Semantic

Differential provided more description of PTs' conceptions regarding NOM. In Figure 9, I show an example of a PT's possible response on two items.

fast ______:__X____:_______:______:______ accurate

relationships ______:______:_______:___X___:______ recall

Figure 9. Example of PT's response on the Semantic Differential

The response in Figure 9 would indicate that the PT thinks that to be good at mathematics one needs to be somewhat fast and should be able to recall facts. The scores for the Semantic Differential range from 20 to 100, because each of the paired words garners a score of 1 to 5. Like the MBI that includes positively and negatively worded statements, the order of words presented in the Semantic Differential switches so that all the words that align with the ideas from the Strands of Mathematical Proficiency are not on one side of the continuum. Thus, for scoring purposes, based on the response in Figure 9, the PTs would receive 2 for fast/accurate pairing and a 2 for the relationship/recall pairing. Overall for the Semantic Differential, a score closer to 20 indicates misalignment with the strands of mathematical proficiency where a score closer to 100 indicates alignment. Additionally, similar to the MBI, since the Semantic Differential was created based on the strands for mathematical proficiency and the Proposed Unified View was also generated using the strands of mathematical proficiency as the foundation, the Semantic Differential provided evidence of PTs' conceptions of NOM aligned or misaligned with the Proposed Unified View. Of the 178 PTs who consented to participate in the study, 133 completed the Semantic Differential. The results from these two surveys contributed to answering research question one, what are elementary PTs' conceptions of NOM?

Writing Prompt

In addition to the two surveys, in phase one, the instructors of each course disseminated the following personification writing prompt to all students:

> Your assignment is to personify Math. Write a paragraph about who Math is. This paragraph should address things such as: How long have you known each other? What does he/she/it look like? What does he/she/it act like? How has your relationship with Math changed over time? These questions are intended to help you get started. They should not constrain what you choose to write about. (Zazkis, 2015, p. 34)

Zazkis explained that through prompt and *eliciting personification* he obtained a richer image of the PTs' affect and dispositions toward mathematics. In this study, I used this prompt to help elaborate on, understand, and provide a richer description of PTs' conceptions of NOM that can sometimes be lacking through the Likert-scale survey items. The prompt helped answer all three research questions.

Semi-structured Interviews

Patton (2015) explained, "We interview people to find out from them those things we cannot directly observe and to understand what we've observed. The fact of the matter is that we cannot observe everything. We cannot observe feelings, thoughts, and intentions" (p. 426). I conducted the interviews with the intention of elaborating on PTs' answers provided on the MBI, SD, and the writing prompt. The main purpose of the semi-structured interview was to provide PTs with an opportunity to clarify answers on the MBI and Semantic Differential, provide more insight about their conceptions of and experiences NOM, and reflect on a Proposed Unified View of NOM.

The interview consisted of general questions (i.e., name, major, classification), broad questions regarding mathematics and their experiences as mathematics students, and questions regarding the Proposed Unified View of NOM (See Figure 6). I asked PTs to describe past and current mathematical experiences. Additionally, I used the two surveys and the writing prompt to ask follow-up questions to clarify answers from PTs and also to elaborate on answers if needed (see Appendix D). For example, "On the MBI you answered that mathematics is [insert student quote]. Why do you think that?" By asking PTs to elaborate on their survey responses I was able to get a deeper insight into their answers and Likert-scale ratings. I was also able to ask them to elaborate on their writing prompts, which helped to gain a deeper insight into their relationship with mathematics. I audio recorded each interview and transcribed each interview to support data analysis.

Time Frame

Upon approval from the internal review board (see Appendix A), I began data collection the first week of the Spring 2019 semester. I attended all the courses to introduce myself and my study to the potential participants. After obtaining consent from the PTs, I then granted them access to the online surveys and the personification prompt. In Table 6, I provided an overview of the time frame of this study, including data collection and analysis.

Table 6

Time Frame

Phase of Study	Date	Data Collection and Analysis
One	January 14-January 28	Dissemination of quantitative surveys and writing prompt
Two	January 28-February 25	Quantitative analysis of surveys
Three	February 25-March 22	Analysis of writing prompts Interview participants selected Emails distributed to potential interview participants Interviews Conducted
Four	March 22-May 1	Analysis of interviews Connected overall results

I distributed the two surveys and writing prompts to PTs during the first and second weeks of the spring semester. PTs' were encouraged to participate by the instructors of the courses in three ways. First, some instructors allowed students to complete the surveys during class. Second, some instructors assigned the personification prompt as a class assignment and gave a completion grade. Last, one professor offered PTs extra credit on a test score for completing both the survey and personification prompt. I gave PTs approximately three weeks to complete the surveys and submit the writing prompts before beginning analysis. After quantitative analysis, I selected participants for the interviews. I present the details of each phase of the study in the remaining sections of this chapter.

Phase One and Phase Two

Phase one and phase two of this study consisted of the dissemination of the surveys and writing prompts and the quantitative analysis of the surveys. With a goal of finding a logical, common structure amongst PTs' conceptions of NOM, I focused quantitative analysis of the MBI and the Semantic Differential and qualitative analysis of the writing prompts. The quantitative analysis of the surveys consisted of reporting descriptive statistics for the PTs as well as more in-depth statistical analysis of the surveys both separately and together. The initial qualitative analysis of the writing prompts began with open coding. The next three sections describe this analysis in more detail.

Mathematics Belief Instrument Analysis

First, I assigned each PT a score between -20 and 20 based on her answers on the MBI. I used each PTs' score to report the descriptive statistics for the MBI, including the mean, standard deviation, median, mode, and interquartile ranges (IQR). Once each PT was assigned a score, I conducted a one-way ANOVA to see if there were any differences in MBI scores of PTs enrolled in the different courses. A one-way ANOVA was appropriate because the samples were independent, the sample sizes were not equal, the sample was fairly normally distributed, and there were more than two groups. If a statistically significant difference was found, I conducted a Tuckey post-hoc analysis to determine which means differed and by how much (i.e. effect size).

In addition to providing descriptive statistics of PTs' overall MBI scores, I also used the conceptions of NOM (Thompson, 1992) to categorize individual MBI statements. Seven of the ten statements on the MBI included different aspects of

Thompson's (1992) conceptions of NOM, and I show this categorization in Table 7. With the help of the knowledgeable other, we individually categorized the seven MBI statements related to NOM and then met to discuss our categorizations. The knowledgeable other and I agreed on all but one statement, *mathematics reveals hidden structures that help us understand the world around us*. After discussion of the ideas, we decided to code this as both static-unified body of knowledge and problem-driven dynamic discipline because depending on context of PTs' non-survey related data, we felt this statement could lend itself to either category. In order for this to be considered a problem-driven dynamic discipline statement, a PT needed to attend both parts of the statement (i.e. structures and understanding the world).

Table 7

MBI Statements Aligned with Conceptions of NOM

	MBI Statement	Conception of NOM
Item 1	To know mathematics means remembering and applying the correct rule or technique to solve a given problem.	Bag of Tools
Item 3	Learning mathematics is mostly memorizing and practicing procedures.	Bag of Tools
Item 7	There is usually only one correct way to solve a mathematics problem.	Bag of Tools
Item 8	Mathematics is mostly a body of facts and procedures.	Bag of Tools
Item 2	In mathematics everything goes together in a logical and consistent way.	Static-Body of Knowledge
Item 4	Mathematics reveals hidden structures that help us understand the world around us.	Static-Body of Knowledge/Problem-Driven Dynamic Discipline
Item 5	Mathematics is as much about patterns as it is numbers.	Problem-Driven Dynamic Discipline

Semantic Differential Analysis

First, I assigned each PT a score between 20 and 100 based on her answers on the Semantic Differential. I used each PTs' score to report the descriptive statistics for the Semantic Differential, including the mean, standard deviation, median, and mode. Once each PT was assigned a score, I conducted a one-way ANOVA to see if there were any differences in Semantic Differential scores of PTs enrolled in the different courses. A one-way ANOVA was appropriate because the samples were independent, the sample sizes were not equal, the sample was fairly normally distributed, and there were more than two groups. If a statistically significant difference was found, I conducted a Tuckey post-hoc analysis to determine which means differed and by how much (i.e. effect size).

In addition to providing descriptive statistics of PTs' overall Semantic Differential scores, I deconstructed the pairs and aligned the individual words with the appropriate Thompson (1992) conceptions of NOM as shown in Table 8. For example, instead of keeping relationships and recall together as a pair, relationships appear in the column for static-body of knowledge, and recall appears in the column for bag of tools. Fitting each of the words on the SD in distinct categories proved difficult as there can be different interpretation of the words. For example, there were two specific cluster of words (i.e. (1) connection, relationships, explanations and (2) sense making, understanding, and reasoning) that initially aligned with the problem-driven dynamic discipline category. However, upon further inspection, I noticed that those words in cluster 1 were nouns where the words in cluster 2 were verbs. The nouns (i.e., connection, relationships, explanations) are more about the network of knowledge where the verbs (i.e. sense making, understanding, and reasoning) are about applications of the network of knowledge. Furthermore, the idea that cluster 1 is nouns (i.e. static) and cluster 2 are verbs (i.e. moving parts) additionally supports the placement in static body of knowledge and problem-driven dynamic discipline. I did not want to misrepresent the intention of the semantic differential, and I reached out to one of the creators of to ask his opinion on the sorting. He responded, "I think this is a great idea, and I've spent a good bit of time this morning working on my own sorting. Overall, I think we have a high degree of alignment." (Personal Communication).

Table 8

Semantic Differential words Aligned with Conceptions of NOM

	Thompson Categorization		
	Bag of Tools	Static Body of Knowledge	Problem-driven Dynamic Discipline
Individual Words	Recall Facts Step-by-step Rules Ability Applying Operations Procedures Repetition Calculating Best-Approach	Connections Relationships Principles Reproduction Knowing Explanations Concepts Algorithms Memorization Solutions Answers	Ideas Flexible Processes Strategies Invention Justifying Creating Learning Multiple Methods Sense Making Understanding Reasoning

Connections Between the Mathematics Belief Instrument and the Semantic Differential

I ran a correlation to examine the relationship between PTs' scores on the MBI and Semantic Differential. For the correlation I provided a scatterplot as a visual representation of the correlation between the PTs' MBI scores and Semantic Differential scores. I also ran a simple linear regression to see if the PTs' scores on the MBI could predict the PTs' scores on the Semantic Differential.

The Open-ended Question Analysis

In addition to the statistical analyses of the two surveys, I also systematically analyzed PTs' open-ended responses for the question, "What is mathematics?" from the MBI. First using R, I created a word cloud to show common words used by participants to define mathematics. I removed all common English words (e.g., and, the, of) from the word count, and I wrote the code to recognize the same root word. For example, solved and solves were not counted as separate words. I also used R to run a Bray-Curtis hierarchical analysis to see if any words, or parings of words, were distinct and reported the distinct clusters with a cluster dendogram. Second, I analyzed coded individual PTs' definitions of mathematics based on Thompson's (1992) three views of NOM to help further explain PTs' conceptions of NOM.

Phase Three

The surveys included questions regarding characteristics of NOM and allowed me to gain an initial, though broad, insight into PTs' conceptions of NOM. For example, PTs read the following statement, "Mathematics is mostly a body of facts and procedures" and then chose how to rate that on a scale from strongly disagree to strongly agree. The PTs' survey answers revealed PTs' overall conceptions of NOM. In phrase three, I analyzed the writing prompts and interviewed the elementary PTs.

Qualitative Analysis of the Writing Prompts

After analysis of the surveys, I began reading and open coding PTs personification writing prompts. As a phenomenological study, I focused on bringing out the voices of the participants. When I presented the results in Chapter Four, I will use vignettes as a tool to help the reader understand the voice of the PTs as they describe NOM in their own words. In the writing prompts, I open-coded many statements regarding the appearance of mathematics, PTs' relationships with mathematics, and how PTs described mathematics as a character, and experiences PTs reported about mathematics each of which helped me understand and describe PTs' conceptions of mathematics.

Through my analysis, my goal was to fully capture, in the most authentic and un-biased way possible, PTs' descriptions of NOM. With this goal in mind, I employed the assistance of a knowledgeable other. The knowledgeable other was qualified to help analyze the data in this study for three reasons. First, because he has a Master of Science in mathematics and was pursuing his Doctorate of Philosophy in mathematics education at the time of this study, he is an expert in both the mathematics content and educational aspects present in this study. Second, he previously taught elementary PTs in content courses making him familiar with the sample of participants in this study as well as the content courses. Third, he was well-informed regarding the literature surrounding PTs' beliefs about mathematics and NOM. We independently open-coded individual PTs' writing prompts. After this, we met and resolved any disagreements by adding to or refining the coding.

Iteration one of coding consisted of individually naming each PTs "Math-character" (Zazkis, 2015, p. 34). That is each PT described mathematics using different characters (e.g. friend, cousin, witch, mosquito, shape-shifter) in their writing prompts. In this initial analysis of coding PTs' math-characters, agreement was not necessary as the knowledgeable other and I used PTs' own words. However, we then independently separated the math-characters into groups, and we did come to agreement about these groups and what characters should be included in the specific groups. For example, mom, step-dad, cousin, brother, and sister became a group because the characters all described members of one's family. Overall, categorizing PTs math-characters helped us further describe PTs' overall relationships with mathematics.

After the first iteration of coding, the knowledgeable other and I noticed the PTs also attended to their relationship with mathematics and their experiences with mathematics in the writing prompts. Thus, the next iterations of coding the writing prompts included coding the PTs' relationships with mathematics. That is, the PTs reported either positive, negative, negative to positive, positive to negative, or roller coaster relationships with mathematics. The knowledgeable other and I independently coded the PTs' writing prompts again but this time looking for and coding the PTs' relationships with mathematics. After we coded independently, we met to discuss our codes and come to agreement. In this particular iteration of coding, we discussed positive, negative, negative to positive, positive to negative, and roller-coaster to come to agreement about which statements we coded as such. We discussed that the relationship codes felt obvious at times and thus it was appropriate to label them with one of the five relationship codes. However, what stood out to us in this iteration of coding, is there were some statements that did not seem to fit one of the five relationship codes. Thus, we added a code here that was improves life/appreciation. This code focused on the idea that PTs did not describe their relationship with mathematics as positive, negative, or anything in between. Instead it was as if they had no relationship with mathematics except to value it for its importance in everyday life. Ellsworth and Bus (2000) incorporated an indifferent relationship with mathematics into their coding for a negative relationship with mathematics. Upon discussion however, the knowledgeable other and I agreed, that we did not want to describe this indifference, or appreciation as negative, because it did not have a negative connotation in their writing. Instead it was an overall appreciation for the use of mathematics in everyday life. Furthermore, we also did not feel comfortable grouping this idea as an overall positive relationship with mathematics, and we added a new code—improves life/appreciation.

The knowledgeable other and I focused a third iteration of coding on how PTs' were describing NOM in their writing prompts. We first captured the statements in the writing prompts that described NOM, and then we used Thompson's (1992) conceptions of NOM—bag of tools, static-unified body of knowledge, and problem-driven dynamic Discipline—to classify the PTs' statements in their writing prompts. Once again, the knowledgeable other and I coded each writing prompt independently and then met to discuss our codes and come to agreement. Overall many of the PTs' statements in their writing prompts fit nicely into one of Thompson's (1992) three conceptions of NOM. So, the majority of the discussion between myself and the knowledgeable other focused on the idea that at times in individual writing prompts the PT made statements. that were in more than one of Thompson's conceptions. This is aligned with other literature that states one can hold two different, even if contradictory, beliefs at the same time (Thompson, 2007).

In the last iteration of coding, the knowledgeable other and I coded the participants' statements regarding their experiences with mathematics. Participants described experiences with teachers, specific mathematics courses, and in some cases the specific context (e.g., in school or out of school). More specifically, the PTs described experiences that illuminated different experiences with mathematics. For example, PTs discussed how their experiences with mathematics changed as they progressed through school. The focus on PTs' experiences was to gain more insight into how their experiences potentially influenced their different conceptions of NOM.

As the knowledgeable other and I reflected on the four iterations—*math-character,* relationships, conceptions of NOM, and experiences—of coding, I went back to the literature and created an analytical framework, as depicted in Table 9, to help describe the analysis process. The analytical framework in this study helped me assess the data after the iterations of coding, specified directions that helped me to focus analysis, and provided me with insight on how the PTs were describing NOM compared to previous literature.

Table 9

Analytical Framework

Coding Categories	Specific Ideas in each Category
Math Character (Zazkis, 2015)	Open coding summarized each PTs' math-character Compiled characters by group
Type of Relationship (Ellsworth & Bus, 2000; Sweeny et al., 2018;)	Positive Described a moderate to strong reaction to mathematics Negative Described indifferent to strong unfavorable reaction to mathematics Positive to Negative or Negative to Positive Reflected a transition from beginning to end Roller-Coaster Reflected overall back and forth relationship in regards to mathematics Neutral Described an overall appreciation of mathematics as useful, but did not mention a like or dislike of the discipline
Conceptions of NOM (Thompson, 1992)	Bag of Tools NOM is an accumulation of unrelated facts, rules, and skills to get to an end. Static-Unified Body of Knowledge NOM is interconnecting structures and truths bound by logic and reasoning. It is discovered. Problem-driven Dynamic Discipline NOM is human creation driven by patterns, a process of inquiry, and open to revisions.
Experiences (Drake, 2006; Ellsworth & Bus, 2000)	What happened, when did it happen, what mathematics was Involved? Impact of teachers or family on one's mathematical experiences

First, each writing prompt was read and read, coded and recoded, and grouped by similarities making each transcript the unit of analysis. In the beginning iterations of analysis, I did not use the analytic framework. Instead, I read the writing prompts and used open-coding as a way to suspend my judgement from the data and examine the transcripts as they were originally intended. This idea, epoché or bracketing, is inherent to phenomenological analysis (Moustakas, 1994). The purpose of this type of phenomenological analysis was to describe PTs' conceptions of NOM and understand PTs' experiences with mathematics. After grouping similar aspects, or codes, I examined why these particular phenomena related (i.e., if a PT viewed NOM one way what did that mean for how they described their relationship with mathematics). My goal in analysis was to find a logical, common structure (Smith et al., 2009) that could help relate PTs conceptions of NOM, their experiences with mathematics, and the Proposed Unified View of NOM. In Figure 10, I provided a screenshot as an example of one PTs' writing prompt and the corresponding codes. This is meant to illustrate the complexity of the codes associated with the writing prompts. I will refer to the analytical framework throughout the remaining sections and chapters of this study as a way to keep the ideas related to the literature.

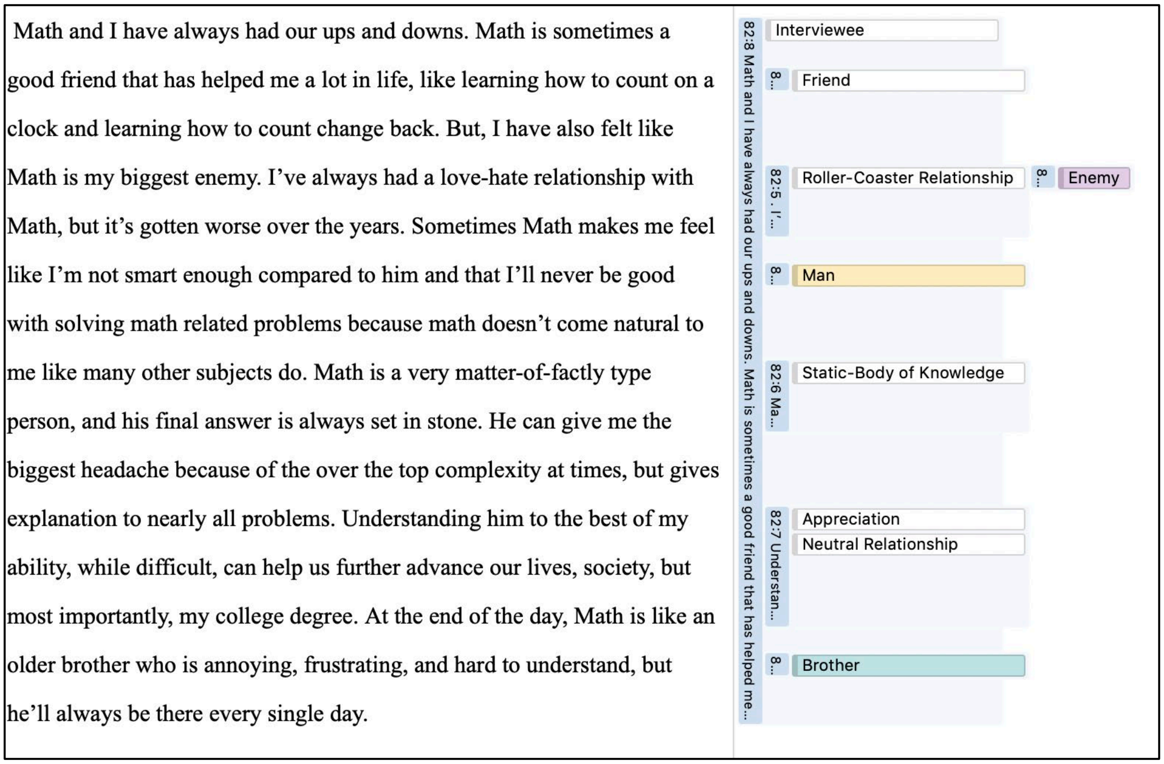

Figure 10. Screenshot of one PTs' Writing Prompt

Participant Selection for Interviews

After quantitative analysis of the MBI and Semantic Differential and qualitative analysis of the writing prompts, I used the results from the two surveys to choose interview participants. I created a scatter plot (See Figure 11) with PTs' Semantic Differential and MBI scores. From the scatter plot, I chose a representative sample of the PTs who participated in this study based on their scores on the MBI and Semantic Differential. In the scatter plot, the blue points represent all PTs who answered both the MBI and Semantic Differential and their subsequent scores on each. The pink points on the scatter plot represent the 33 purposefully selected PTs that I emailed and requested an interview.

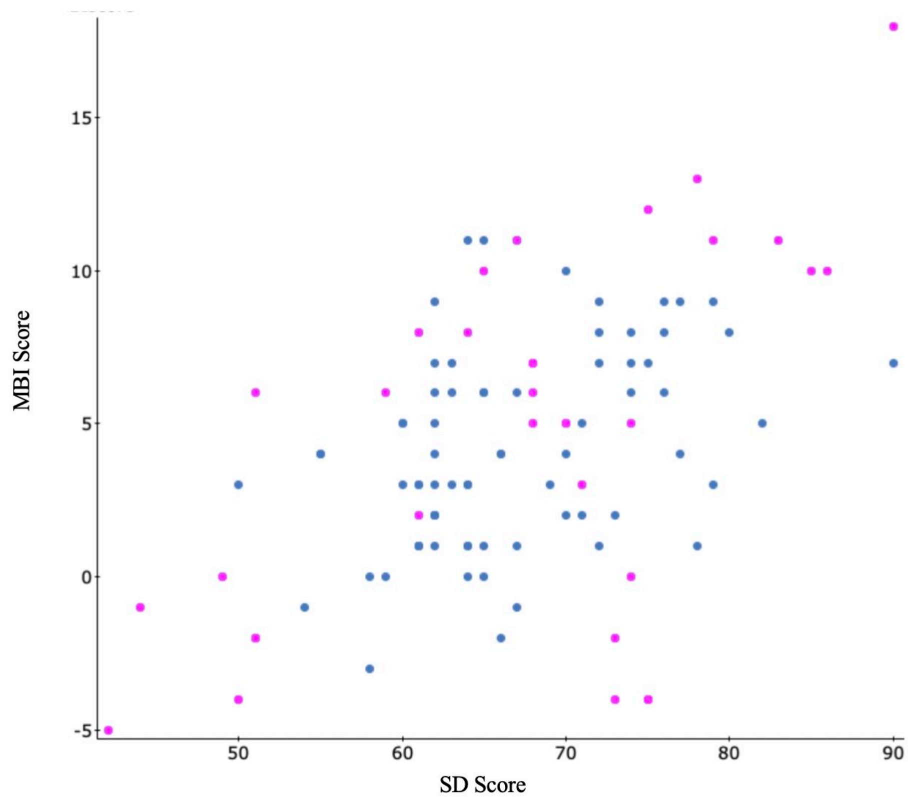

Figure 11. Scatter plot of PTs' survey scores

I purposefully selected participants to have a representative sample of the PTs in my study. First, I selected potential interviewees based on scores (i.e., high scores on both, low scores on both, middle scores on both, and high on one, low on the other). In addition to considering PTs' MBI and Semantic Differential scores, I also made sure to have an appropriate number of PTs in the content and methods courses. Overall, there was a 2:1 ratio of PTs enrolled in a content course (i.e., course 1 or course 2) to PTs enrolled in a methods course (i.e., course 3). Thus, I kept the ratio of interviewees to 2:1. After soliciting all 33 potential interviewees, 13 responded and agreed to participate in the interview. In Figure 12, I provide a visual representation for my process of choosing interview participants, why some were excluded, and how many PTs were actually interviewed.

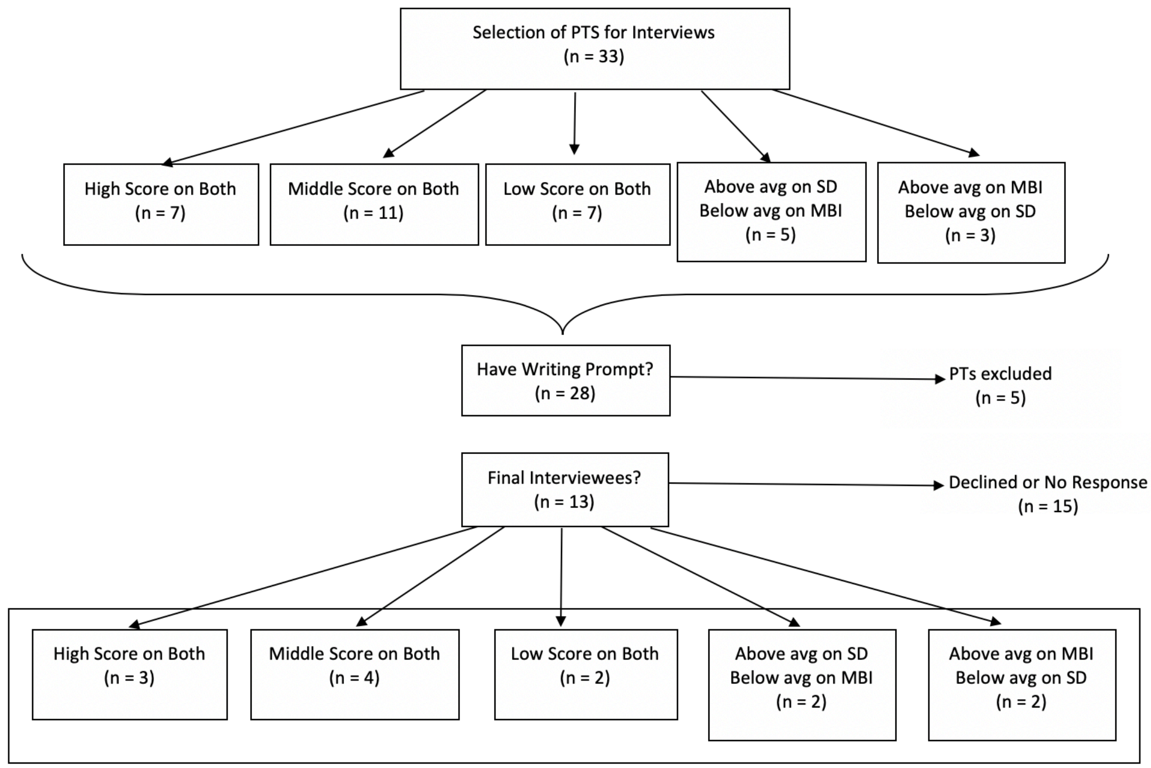

Figure 12. Process of PT selection for interviews

Phase one of the study consisted of dissemination of surveys and the writing prompt to all elementary PTs enrolled in either a mathematics content or mathematics methods course in the Spring 2019 semester. All PTs enrolled in the content or methods courses were given the opportunity to participate in the study as a way to gain a broad overview of PTs' conceptions of NOM at a specific university. The instructors of the courses allowed me to introduce myself and describe the study to the PTs in the first week of classes. In this first week, students were given the link to the Qualtrics survey as well as the personification prompt.

Due to the large numbers of elementary PTs completing the surveys and writing prompt, I did not assign each individual PT a pseudonym. However, I will refer to their data throughout by explaining the course in which they were enrolled at the time of this study. After I collected data, I blinded each PT's survey responses and writing prompt with a format of PTX.Y. The X refers to the course that the PT was enrolled: 1 for course 1, 2 for course 2, or 3 for course 3. The Y refers to the number assigned to that PT once their data was submitted. When I refer to a PT that I did not interview and therefore only had a survey and writing prompt you will see PT1.25 or PT3.52. For the 13 PTs I interviewed, I assigned each of the 13 interviewees a pseudonym either starting with "O" for course 1, "T" for course 2, or "M" for course 3—the methods course. I chose to assign pseudonyms to only the 13 PTs who participated in the interview for two reasons. First, creating pseudonyms for all PTs in this study was not a valuable use of my time as there were 130 total PTs. Second, I thought it was important to provide pseudonyms for some of the PTs as a way to help the reader relate to their stories throughout this study. In Table 10, I provide more background information for the thirteen interviewees, including their pseudonym, major, highest level of mathematics they had taken, the course in which they were enrolled at the time of the study, and their score on the MBI and Semantic Differential surveys. I included the survey scores in this table because this is how I chose the interviewee participants. Notice there are low scores (n = 2), middle scores (n = 4), high scores (n = 3), and then over average and below average on one or the other (n = 4). Each interviewee completed both the surveys and the writing prompt. Throughout

Chapter Four, because the 13 interviewees completed all data sources in this study, I will refer to them by their pseudonym.

Table 10

Background Information of Interviewees

Name	Major	Highest Level of Mathematics	Current Course	MBI/SD Score
Olive	Elementary Education	Precalculus	Course 1	Middle on both
Odette	Early Childhood	Geometry	Course 1	Low on both
Olga	Early Childhood	Geometry	Course 1	Middle on both
Ophelia	Elementary Education	AP/BC Calculus	Course 1	Low on both
Octavia	Special Education	AP/BC Calculus	Course 1	Over avg on MBI, below avg on SD
Tabby	Elementary Education	Geometry	Course 2	Over avg on SD, below avg on MBI
Tess	Early Childhood	Precalculus	Course 2	Middle on both
Tatum	Special Education	Geometry	Course 2	High on both
Tia	Elementary Education	Algebra 2	Course 2	Over avg on SD, below avg on MBI
Mia	Interdisciplinary Studies	Precalculus	Course 3	Over avg on MBI, below avg on SD
Maggie	Elementary Education	Precalculus	Course 3	Middle on both
Millie	Special Education	Geometry	Course 3	High on both
Margot	Elementary Education	Precalculus	Course 3	High on both

Phase Four

Phase four consisted of analysis of the interviews. In phase four, I also organized emergent themes from each of the data sources to find overall themes from the study as a whole instead individual themes for each data source. The following three sections provide more detail about the analysis that occurred in this phase of the study as well as an overall summary of analysis.

Qualitative Analysis of the Interviews

I transcribed the interviews and uploaded them to the qualitative analysis software. I went through four iterations of deductive coding. First, I focused on the ideas in the analytical framework, I focused each iteration of coding on one of the four specific categories (i.e., math-character, relationship, conceptions of NOM, and experiences). One goal of the interviews was to have PTs elaborate on their answers to the survey questions as well as their writing prompts. Thus, it was important to use the same codes from the writing prompts to gain a deeper understanding of PTs' conceptions of NOM.

A second goal of the interview was to probe PTs regarding their experiences with mathematics. Part of the analytical framework provided a basis for some of the codes used to describe PTs' experiences. For example, Drake (2006) and Ellsworth and Bus (2000) provided the codes what happened, when did it happen, and what math was involved. Additionally, the Drake (2006) and Ellsworth and Bus (2000) provided the code teacher experience or family experience when at PT described an experience with a teacher or family member that influenced their conception of NOM. In addition to the codes formed based on results from the studies of Drake (2006) and Ellsworth and Bus (2000), PT discussed different types of experiences. For example, PTs were asked to describe a good day and a bad day in mathematics class. In addition to describing a good day and a bad day in a mathematics class, PTs also discussed their ideas regarding NOM in terms of being future teachers. So, in the interviews, the analytical framework was used to build on to ideas PTs discussed in the writing prompt. However, the analytical framework did not limit the analysis process to a set of codes, but allowed for inductive coding as well.

A third goal of the interview was to provide the PTs with an explicit opportunity to reflect on the Proposed Unified View of NOM. Until this opportunity for reflection, PTs were implicitly attending to conceptions of NOM. I presented the PTs with the statements that make up the Proposed Unified View of NOM (See Figure 6), and asked the PTs to reflect on the statements by providing the PTs with a copy of the characteristics in the Proposed Unified View. After reading the characteristics, I asked participants if they would add to or take away anything from the list. I also asked them to elaborate on specific statements they made. For example, a PT asked about critique of mathematical ideas leads to refinement because she was confused about the meaning. So, I first asked her to explain what she thought it meant, then I explained my perspective, and we then had a conversation about the idea. For these transcriptions, I used iterations of inductive coding. In Table 11, I provide a list of codes and emergent themes from analysis of the interviews. I will extend this table in Appendix E with the codes and emergent themes from all phases of the study.

Table 11

Open Codes and Emergent Themes by Data Collection Stage

Phase Four Codes and Themes	
Open Codes	Emergent Themes
Mathematician The Discipline School vs Non-School	Evidence PTs did not view school mathematics the same as mathematicians
Math Class Teacher Experience Student Experience	Evidence PTs' experiences with teachers, as students, and in mathematics classes influenced their conceptions of NOM.

The knowledgeable other did not help in the coding process with the interviews. Therefore, after an initial coding, I created a summary paragraph of the ideas and sent them to each of the interviewees for a member check. Not all of the interviewees emailed me back to confirm or dispute my interpretation of their writing prompts and interviews, and I took this as an agreement by abstention. Additionally, I asked questions during the interview such as, "What I hear you saying is," or "I am interpreting what you are saying as," to give the interviewees a chance to correct me in the moment instead of as an afterthought.

Researcher as Instrument

As the primary researcher in this study, it was my interpretation of the literature and data collected in this study which propelled the analysis and discussion of the results forward. Therefore, I will elaborate on my own theoretical perspective in order to provide insight into my view as a researcher, because "what we choose to research and the way in which we carry out that research are constructions determined, among other factors, by who we are and how we choose to engage in academic inquiry" (Valero, 2004,

p. 2). As a mathematics student and mathematics educator, I have witnessed firsthand the differences in how I was expected to learn mathematics and how I was expected to teach mathematics. As a student, I sat in lecture-based classes where the teacher filled the board with notes and then assigned homework problems that I was expected to master for the test (Stigler & Hiebert, 1999). As a mathematics educator, I am expected to teach in a way that aligns with reform-based standards through the implementation of instruction that promotes discovery and problem solving and is student-centered (NCTM, 2014). The struggle between how I learned mathematics and how I have come to teach mathematics caused me to question NOM. Once I began questioning NOM, my motivation became not necessarily to define NOM, but instead to make sure each person (student, teacher, prospective teacher, mathematics educator, or mathematician) knew and could explain and understand their own conceptions of NOM. That is, answering the question, "What is mathematics?" is important for individuals regardless of their current role, because reflection on these ideas can cause refinement or change in those ideas (Bolden, 2010; Szydlik, 2013), and then perhaps the disconnect I experienced learning mathematics and teaching mathematics will cease to exist.

For me, how I teach mathematics changes through reflection on past teaching experiences and ideas research documents put forth as best practices. In my experience, teaching in a way that aligns with reform documents and characteristics of the fallibilist perspective of NOM provides students with an opportunity to develop a deeper understanding of mathematics. By teaching in a way that aligns with best practices, students can construct their own meaning about the mathematics. Thus, with a goal to understand different conceptions of NOM, I align with a radical constructivist approach to teaching and learning. Radical constructivism allows for the implication that knowledge is constructed in the minds of the learner and that the construction of knowledge happens based on the learners' experiences (von Glasersfeld, 1995). My view of radical constructivism does not discount the role that social interactions have on learning, but instead gives more importance to the individuals' reflection on learning within that social context and subsequently the idea that reflection is essential to the way a person comes to know (von Glasersfeld, 1995). Hence, why I believe it is important for elementary PTs to reflect on and understand their own conceptions of NOM—without reflecting, they might not come to know.

Thus, my lens of radical constructivism allows for the idea that reflection leads to the way a person comes to know. This point is important as this study not only helps mathematics educators understand elementary PTs' conceptions of NOM, but also provides an opportunity for elementary PTs to understand their own conceptions as these are sometimes unconscious thoughts. My personal perspective was also important as I was an instrument in the data collection of the qualitative portion of this study (Patton, 2015). My willingness to share my theoretical perspective is one way to maintain credibility of this study. In order to add to the credibility of this study, I also maintained a reflective journal in which I documented my thoughts throughout each phase of data collection. This journal was more useful for the qualitative portion as a way for me document my presence during data collection and the potential effect it could have on participants (Patton, 2015).

Trustworthiness, Limitations, and Delimitations

The analysis relied heavily on my interpretation of the data, and thus it was important to establish trustworthiness of the study. In order to establish trustworthiness of this study, I was forthcoming with my theoretical perspective and its potential influence on conducting research. Throughout this study, I consistently added to an analytical memo any notes or thoughts I deemed could be important or useful in describing the methodology used in this study. By keeping notes of this process, I was able to note where my views influenced analysis and why I chose to add codes or make changes. I took the four steps during data collection and analysis regarding credibility, transferability, dependability, and authenticity (Lincoln & Guba, 1985; Patton, 2015). First, I maintained a reflective journal through the memos and notes documenting activities or conversations related to the study, personal reflections, and methodological decisions made by the researcher. The reflective journal served as an accurate representation of the study during the data collection and analysis phases. I used the reflective journal to document difficulties in the study, changes and the justification of those changes, if they were made, as well as potential interferences with the data. Additionally, the reflective journal provided an account of the analysis process, documenting choices made during the analysis process. Second, I used multiple data sources to support the triangulation of data and give a holistic interpretation of the data being studied and PTs' conceptions of and experiences with NOM. Third, I employed the use of an expert other throughout all iterations of coding the writing prompts. The use of an expert other provided strength in agreement on codes as well as pushed me, the primary researcher, to consider alternative ideas regarding PTs' conceptions and experiences. Lastly, I conducted member checks with each participant to review his/her story. That is, once I established a narrative for each participant, I allowed each PT the opportunity to read his or her corresponding narrative to make sure it is an accurate reflection of the PT's conception of NOM and experiences that influenced those conceptions. This helped ensure the participant stories were accurately portrayed by myself, the researcher.

In continuing to establish trustworthiness as a researcher and for the purposes of this study, I will elaborate on the delimitations and limitations of the study. There were two significant delimitations of this study: the use of PTs in content and methods courses and PTs' conceptions of NOM at one university. This study investigated PTs' conceptions of NOM when the PTs were enrolled in either a mathematics content courses or mathematics methods course. The choice of investigating PTs at these two times in their teacher preparation program was to gain an understanding of PTs' conceptions of NOM before they are teaching in their own classrooms and while they are in various points of their teacher preparation program. Second, the choice to use PTs from one southeastern public university might not be representative of the population of PTs throughout the United States, especially considering the wide variation of teacher education programs across universities. However, the choice to examine PTs' conceptions at one university was intentional because the qualitative aspect of this study necessitated familiarity with the environment and context in order to try to understand the essence of PTs' conceptions about mathematics.

One significant limitation of the study was the use of survey instruments. The MBI (Szydlik, 2013) and Semantic Differential Survey (Sweeny et al., 2018) do not allow the researcher to know how the subject interpreted the statements or how important each statement was to the subject. However, the advantages, in this case, outweighed the limitations since both surveys offered were relatively short thus reducing subject fatigue when taking the surveys. The MBI asked one open-ended question, "What is mathematics," which was used in part of the qualitative data analysis. As a way to combat the limitation of using Likert-type surveys, I analyzed each statement on the surveys based on Thompson's (1992) categorizations of NOM. This grouping allowed me to analyze individual statements on the surveys as they related to elementary PTs' conceptions of NOM. The Semantic Differential Survey was piloted and validated by Sweeny and colleagues (2018) with 242 PTs. The use of this survey was important to my study as it was designed for use with PTs and it explicitly aims at having the PTs consider aspects of NOM inherent in the three main conceptions of NOM—instrumentalist, Platonist, and fallibilist. Additionally, the survey has strong correlations between the paired words and the strands of mathematical proficiency. However, there were three paired words that had lower communalities based on a factor analysis. That is, three sets of paired words were not well explained by the strands of mathematical proficiency. Thus, the team is modifying the survey in an attempt to strengthen the low communalities. I am in communication with this team, and will use the latest version of their semantic differential and will share the data collected in this study using their instrument so that they can make further improvements to the instrument.

Chapter Summary

In this chapter, I outlined the methodology to be utilized in this study. The details I included regarding the explanatory sequential, phenomenological study were intended to explain the research methodology in detail and support the use of this specific methodology. An explanatory sequential, phenomenological design was appropriate for this study because I analyzed different data sources at different phases in the study to gain an understanding of the conceptions of NOM held by elementary PTs and of how their lived experiences influenced those conceptions.

CHAPTER IV: RESULTS

Introduction

Mathematics reform-documents (e.g., AMTE, 2017; CBMS, 2012; CCSSI, 2010; NCTM, 2000; 2014) cast an ambitious vision for school mathematics—a vision that promotes the alignment of school mathematics and the discipline of mathematics. This is an enormous challenge and "meeting it is essential" (NCTM, 2000, p. 4) for the successful mathematics education of each and every student. As evidenced in Chapter Two, foundational standards documents propose an ambitious vision for mathematics in schools, but PTs' conceptions of mathematics are often misaligned with the mathematics promoted by the standards documents. In fact, PTs' conceptions about NOM and lack of overall consensus about NOM across groups. Understanding PTs' conceptions about NOM and experiences which informed the conceptions is vital to bring awareness to PTs themselves about the conceptions they hold regarding NOM and also to MTEs seeking to improve mathematics education.

Elementary PTs often perceive they already know the mathematical content in elementary school, and therefore PTs can underestimate the complexities required for teaching elementary school (Ambrose, 2004; Ball, 1990; Richardson, 1996; Weinstein, 1989). It is reasonable to assume that a similar dynamic might be at play with respect to NOM. If PTs do not have opportunities to reflect upon their own conceptions of NOM, they will likely fail to understand the dynamic nature of mathematics expounded upon in the standards documents and allow those understandings to influence their teaching. The purpose of this study was to describe elementary PTs' conceptions of NOM, understand experiences that influenced those conceptions, and connect PTs' conceptions of NOM with the Proposed Unified View of NOM. This study was a first step to understanding how PTs' conceptions of NOM could inform their instructional practices. The purpose of this study was to answer the following three research questions:

1. What are elementary prospective teachers' conceptions of the nature of mathematics?
2. How do the lived experiences of elementary prospective teachers inform their conceptions of the nature of mathematics?
3. What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View, and what are the implications of those connections, if any?

This chapter contains the details of my investigation into PTs' conceptions of NOM, their experiences that influenced those conceptions, and how their conceptions of NOM are related to the Proposed Unified View presented in Chapter Two. I followed a design consistent with an explanatory phenomenological methodology. In this chapter, I will present the results from the data analysis outlined in Chapter Three. I will begin with a brief introduction of the participants and particular units of analysis.

In Chapter Three, I presented the methodology and analysis associated with each of the four phases in the study. However, it is my intention to present the results in this chapter based on the research design that guided this study which focused on the use of qualitative results to help explain and elaborate on the quantitative results. The three research questions that guided this study and my analysis had overlapping themes. Building the qualitative results on to the quantitative results allowed me to provide a full, descriptive, detailed story of the PTs' conceptions of NOM, experiences with NOM, and connections to the Proposed Unified View of NOM. There were overlapping ideas put forth by the PTs in both the quantitative surveys and qualitative writing prompts and interviews. For example, in her writing prompt, Mia wrote her own story about mathematics. She said,

> When I was a little girl, I first met Math. My preschool teacher never properly introduced us, but I could always see his shadow creeping through the window. Kindergarten was when we first officially met. His colorful attire and zany attitude was [*sic*] most appealing to my classmates, but I knew there was something different about him. Unlike my peers, I knew he was up to no good. The teacher would have us impress Math with our number recognition and basic function problems to which he would respond with elaborate leaps of joy and hysterical laughs. Everybody loved him. . .except for me. As we grew older, my classmates began to see that Math wasn't all he was cracked up to be. His energetic personality became too off the wall and seemed to complicate the work we were trying to complete for him. Furthermore, his exaggerated praise only seemed to make us feel less adequate in his presence: when he would leap for joy, new problems and approaches would be revealed, and his hysterical laughs only mocked the processes we were using in mastering content. We couldn't stand him. However, some of our peers were so entranced by his unique behavior and yearned so much to understand his comedic praise, that they were drawn to the subject. With much practice, they learned to predict the problems in which would be revealed and thrived from the mockery Math would throw at them. The rest of us dreaded his presence. We realized that Math had been a set up all along, but

> everything we saw in the outside world reminded us of him. The sun reminded us of his golden hair and the blue sky looked like his vibrant shoes. Grass textured his vest and clouds filled his pockets. He was everywhere. As much as we hated it, we could and will never escape Math. (Mia, Writing Prompt)

In her story, Mia described a complicated and challenging relationship with mathematics. She called Math zany and colorful and said she dreaded his presence. Mia's description of mathematics and her relationship with mathematics provided me with a unique view into Mia's conceptions of NOM and experiences with NOM which illuminated the interconnectedness of the ideas on which my research questions were based. Mia was not the only PT to describe mathematics in such a rich, detailed way. But, with a story such as Mia's, with so many twists and turns, delights and fears, what can one understand about her relationship with mathematics and conceptions of NOM? Reading Mia's story and 129 others like it, combined with the two surveys and interviews, I began an analysis that would begin to answer these questions.

As a way to organize the results and emergent themes from data analysis as they related to the research questions in this study, I present Figure 13. Additionally, Figure 14 represents the structure of this chapter. That is, for each research question I will discuss the emergent themes, and then I will provide evidence from the data sources to support that theme as well as how and why they helped answer that specific research question.

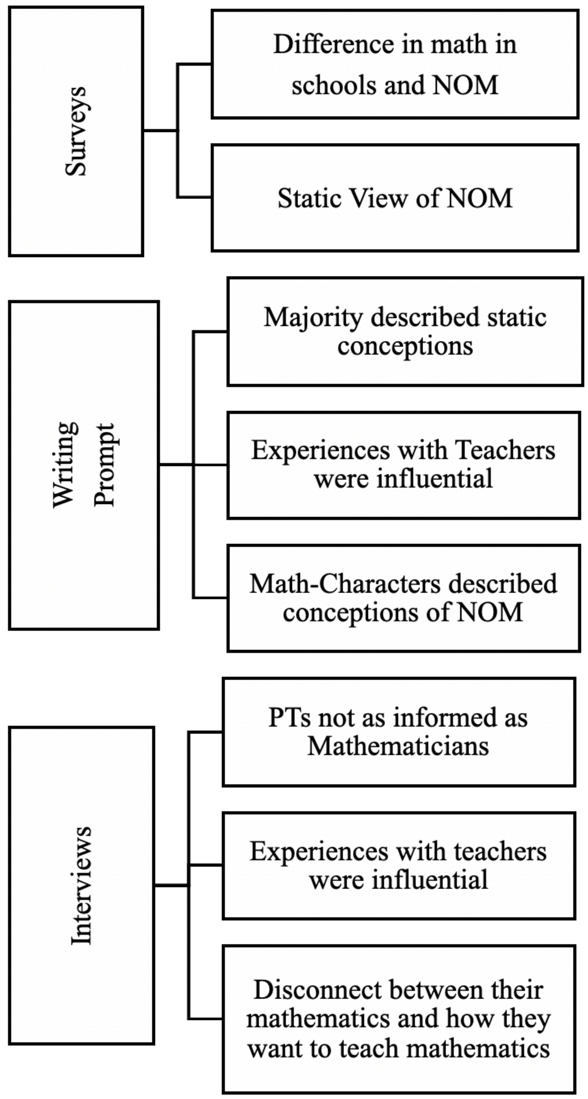

Figure 13. Emergent Themes from each Data Source

Prospective Teachers' Conceptions of the Nature of Mathematics

Overall in this study, PTs described conceptions of NOM most closely aligned with NOM as a static-unified body of knowledge. PTs' answers from the MBI and Semantic Differential illuminated their overall conceptions. However, PTs descriptions of mathematics in their definitions and writing prompts provided evidence that individual PTs' conceptions of NOM were not always aligned with NOM as a static-unified body of knowledge. In the following sections, I provide results from the surveys and writing prompts that provided evidence of PTs' conceptions of NOM.

Mathematics Belief Instrument

Analysis of the MBI (n = 108) revealed that PTs' scores had a mean overall score of 4.23 out of a scale of -20 to 20 and a standard deviation of 4.32. Since this instrument was designed to measure how one's beliefs align with the characteristics of mathematics laid out by professional mathematicians, a low score of -20 would indicate misalignment with mathematicians and therefore be closer aligned with a conception of NOM as a bag of tools and a high score of 20 would indicate alignment with mathematicians' views and therefore be closer aligned with a conception of NOM as a problem-driven dynamic discipline. Thus, a mean score of 4.23 indicated that the on average PTs in this study aligned more with the mathematicians than not. The relatively low standard deviation of 4.32 indicates that PTs' scores on the MBI were also relatively close to the mean and still falling on the NOM Continuum closest to a static-unified body of knowledge. When considering the NOM Continuum, the results from the MBI indicated that, on average, PTs' conceptions of NOM were more closely aligned with NOM as a static-unified body of knowledge (see Figure 14), and therefore a majority of PTs were not considering the dynamic and creative nature of discipline.

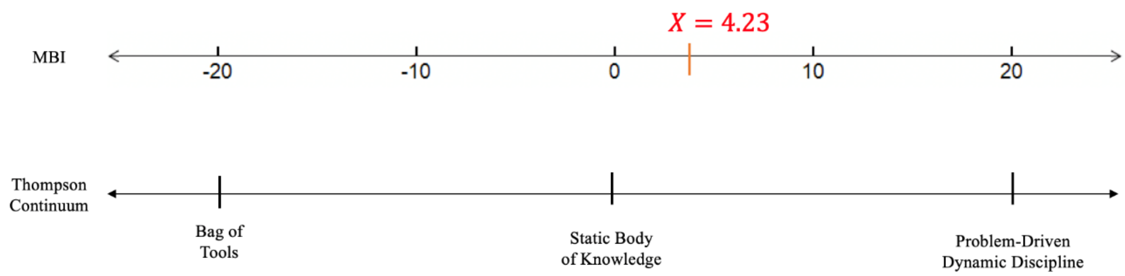

Figure 14: MBI score and NOM Continuum

The distribution of PTs' MBI scores is shown in Figure 15. As seen in the boxplot for the overall scores, with the exception of one PT (i.e., the outlier), all PTs scored below 14 on the MBI. Only one PT scoring above a 14 indicated that the PTs in this study did not tend to view mathematics as substantially involving creativity and problem solving and instead were likely to define mathematics consistent with NOM as a bag of tools or a static-unified body of knowledge. Approximately 25% of the PTs scored relatively high (between 7 and 13) indicating that these PTs were approaching an agreement with the fallibilist characteristic of NOM. At least 75% of PTs scored above 1 on the MBI, and 50% of the PTs scored between 1 and 7. Even the bottom 25% of PTs, who had a range of scores between -5 and 1, still fell in the middle of the NOM Continuum and thus aligned closer to NOM as a static-unified body of knowledge. Therefore, no PT demonstrated a conception of NOM as a bag of tools on the MBI.

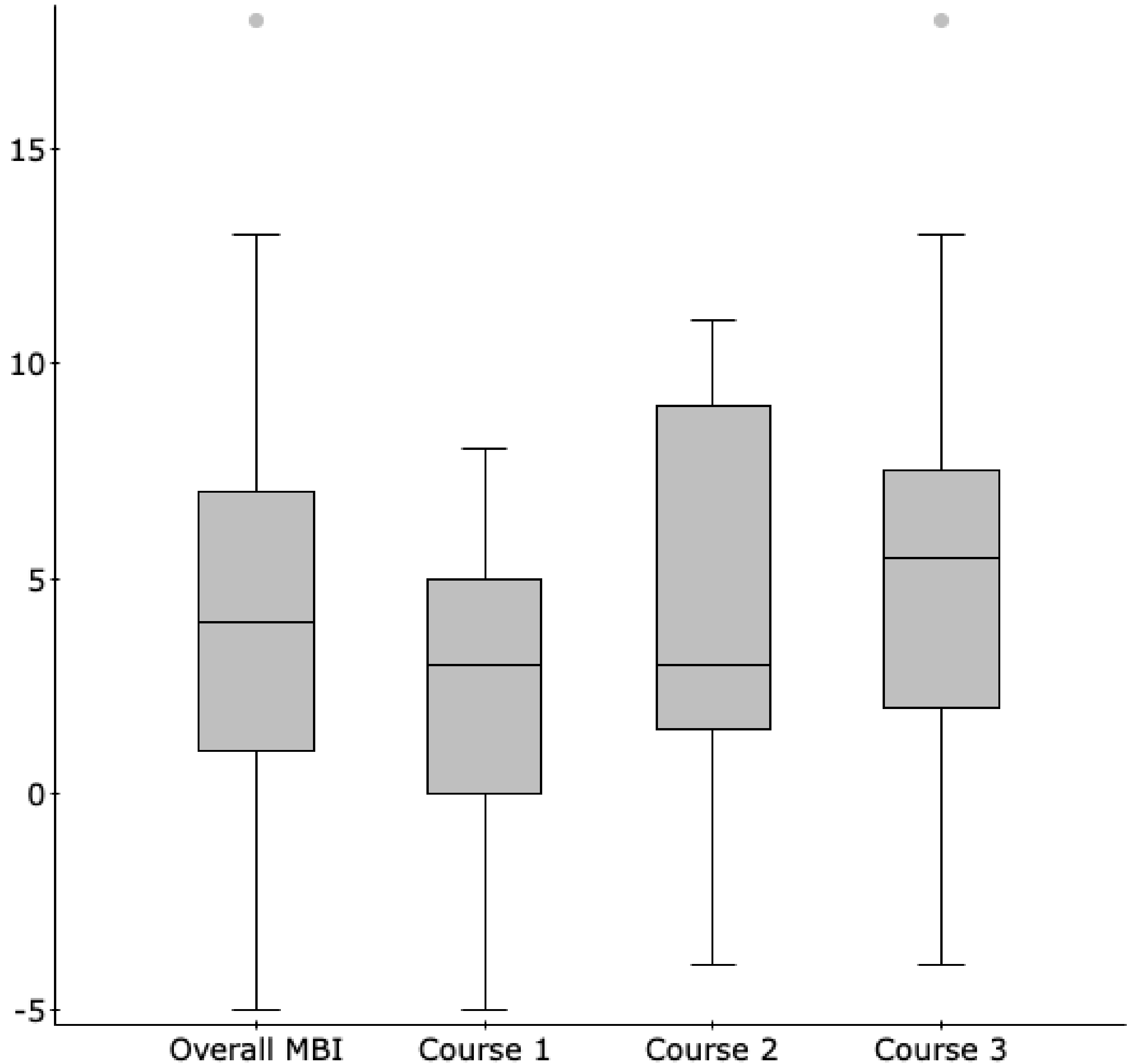

Figure 15. Distribution of MBI scores.

In Table 12, I provided a summary of the results from a one-way ANOVA using PTs' MBI scores. Because PTs take course 1, course 2, and course 3 in succession during their teacher preparation program, I kept course 1 and course 2 separate even though they were both content courses. An examination of PTs' MBI scores by course showed a slight increase in average score as PTs progressed through the content and methods courses. The course that the PT was enrolled in had a significant impact on PTs' overall MBI scores, $F(2, 105)=4.66$, $p=.011$. With a moderate effect size of .082 between groups, the course that the PT was enrolled explained 8.2% of the variation in the MBI scores. Post-hoc analysis revealed differences between MBI scores of PTs enrolled in course 1 and course 3 ($p = .009$) with a mean difference of 3.02. Other differences between groups were not statistically significant.

Table 12

ANOVA Results for MBI

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	163.03	2	81.52	4.66	0.012	3.08
Within Groups	1837.07	105	17.50			
Total	2000.10	107				

The alignment of PTs' conceptions of NOM with a static-unified body of knowledge was further evident when considering the individual statements on the MBI. The MBI consisted of 10 statements designed to assess conceptions of NOM and, thus, had positively and negatively worded statements about mathematics. As a reminder, 7 of the 10 statements on the MBI lent themselves to the different aspects of Thompson's (1992) three levels as depicted in Table 7.

In Figure 16, I represented the seven individual statements on the MBI that were categorized based on Thompson's conceptions of NOM and the PTs' responses to each statement with a stacked bar graph. Items 1, 3, 5,7, and 8 represent the statements from the MBI which were negatively worded and therefore blue and orange represent alignment with the Proposed Unified View of NOM. Items 2, 4, and 6 represent the positively worded statements on the MBI and therefore green and yellow represent alignment with the Proposed Unified View of NOM. The two statements that most closely represented NOM as a static-unified body of knowledge were: *everything in mathematics goes together in a logical and consistent way* (Item 2) and *mathematics reveals hidden structures that help us understand the world around us* (Item 4). An examination of Figure 16 revealed that on Items 2 and 4 the majority of PTs either agreed or strongly agreed. Item 2 lent itself to the conception of NOM as a static-unified body of knowledge, because at the core of NOM as a static body of knowledge is the idea of NOM as a monolith with interconnected structures. Item 4 was also categorized as promoting mathematics as a static-unified body of knowledge. The idea that mathematics can help one understand the world around them supports the interconnectedness of mathematical ideas present when defining NOM as a static-unified body of knowledge. The majority agreement of PTs regarding Item 2 and Item 4 supported the overall analysis from the MBI that PTs' conceptions of NOM , in that it became evident PTs' conceptions regarding NOM were slightly to the right of the middle or slightly right of static-unified body of knowledge. Overall, PTs falling further to the right of the NOM Continuum expressed conceptions of NOM that aligned more with professional mathematicians and therefore the standards documents. PTs were beginning to see mathematics as not simply a bag of tools to be memorized and used in order to obtain one answer, but as a body of logically connected structures that can help one understand the word.

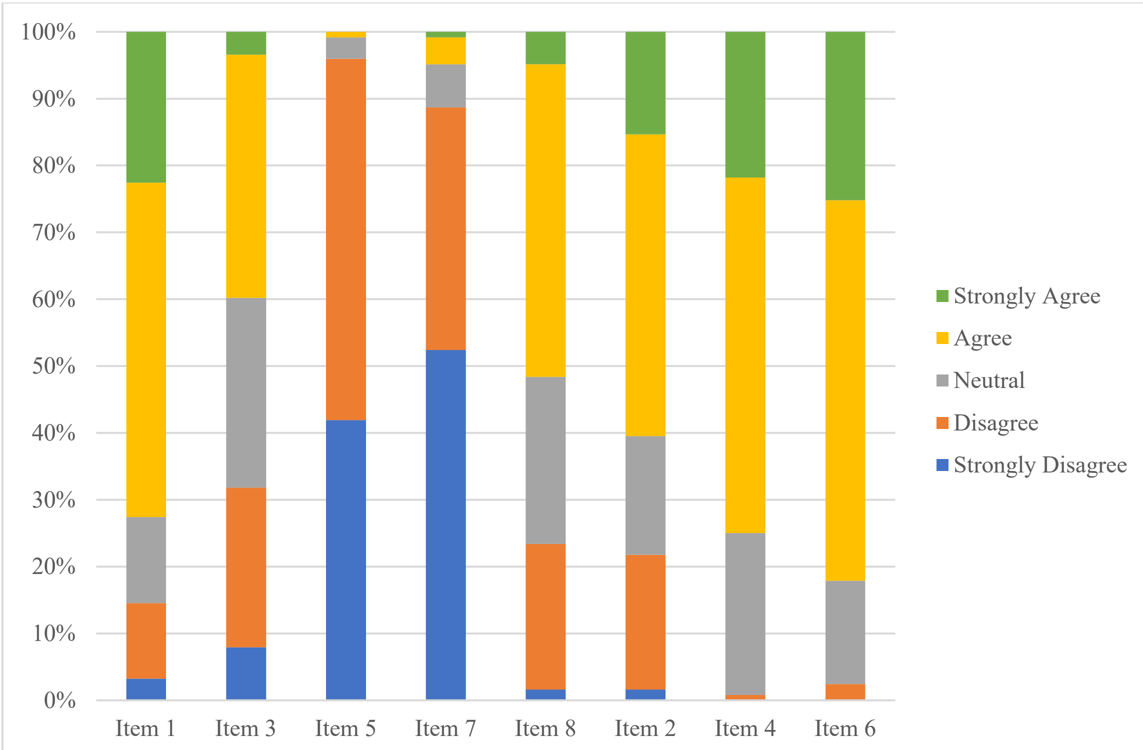

Figure 16. PTs' MBI answers by individual statement on MBI.

However, every statement on the MBI did not necessarily represent NOM as a static-body of knowledge. For example, the statement *mathematics is as much about patterns as it is numbers* (Item 6) more closely aligned with mathematics as a problem-driven dynamic discipline. Contrastingly, the statement *to know mathematics means remembering and applying the correct rule or technique to solve a given problem* (Item 1) more closely aligned with mathematics as a bag of tools. The majority of PTs either strongly agreed or agreed with Items 1 and 6. Since for both Item 1 and Item 6 a majority of the PTs rated the statements agree or strongly agree suggested that the PTs struggled with the ideas of mathematics as a discipline (i.e., NOM) because these statements are aligned with different conceptions of NOM (see Figure 16). Simply because a PT expressed a conception of NOM as a bag of tools at one point in time did not mean that they were restricted to holding a conception of NOM as a bag of tools. At any time, a PT could espouse conceptions that would fall on different parts of the NOM Continuum. To be clear, I do not intend to use the NOM Continuum to place PTs, but instead to place their ideas along the continuum as a way help explain the dynamic nature of the PTs' conceptions regarding NOM. Additionally, the placement of PTs' ideas along the continuum can aid in discussions regarding the contrasting ideas of the different conceptions of NOM and help PTs reflect on their own (sometimes conflicting) conceptions of NOM and the potential influence in the PTs' future classrooms. While overall descriptive results from the MBI revealed PTs' conceptions of NOM were more closely aligned with a static-unified body of knowledge, the individual analyses of the individual MBI statements indicated that students also agreed with aspects of NOM that did not support a static-unified body of knowledge. Thus, relying only on the MBI survey did not provide enough detail to say that PTs always held a conception of NOM as a static-unified body of knowledge. Therefore, I will also present the results from the semantic differential.

Semantic Differential

Analysis of the results from the Semantic Differential revealed that PTs' scores had a mean overall score of 66.76 out of 100 and a standard deviation of 8.75. Since this instrument was designed to measure alignment with the characteristics of mathematics laid out by the mathematics education community-specifically the strands of mathematical proficiency—a low score of 20 indicated misalignment with the strands and more alignment with NOM as a bag of tools and a high score of 100 indicated alignment with the strands and more alignment with NOM as a problem-drive dynamic discipline.

Therefore, the mean score of 66.76 indicated that the PTs aligned more with the mathematics education community than not. The relatively low standard deviation of 8.75 did not change this categorization as the PTs' Semantic Differential scores were relatively close to the mean. Additionally, the results also indicated that on average PTs' conceptions of NOM were more closely aligned with NOM as a static-unified body of knowledge (See Figure 17), and they likely did not consider characteristics of the dynamic, problem-driven nature of mathematics as a discipline.

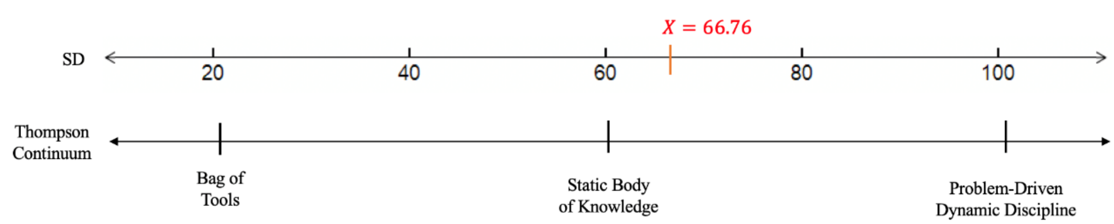

Figure 17. Semantic Differential alignment with NOM continuum.

The distribution of PTs' Semantic Differential scores are shown in Figure 18. As seen in the boxplots for the overall scores, with the exception of the two outliers, all PTs scored below 87 on the Semantic Differential. Only two PTs scored above 86 indicating that overall the PTs did not tend to view mathematics as substantially involving creativity and problem solving and, instead, were likely to define mathematics consistent with NOM as a bag of tools or static-unified body of knowledge. There were some PTs who scored relatively high (between 73 and 86) when compared to the other 75% of PTs, indicating that these PTs had a stronger agreement with the fallibilist characteristics of NOM. At least 75% of PTs scored above a 62 on the Semantic Differential, and 50% of the PTs scored between 62 and 73. Even the bottom 25% of PTs, who had a range of scores between 49 and 62, still fell in the middle of the NOM Continuum and, thus, aligned closer to NOM as a static-unified body of knowledge. The two outliers who scored below 49 on the Semantic Differential, still had scores that suggested they were moving away from a conception of NOM as a bag of tools with a score of 42 and 44. Thus, based on PTs' scores on from the semantic differential, no PT had a score that would be associated with the conception of NOM as a bag of tools. This is evident when considering the alignment of NOM Continuum and the PTs scores on the Semantic Differential, because it is the PTs ideas being placed on the continuum and not the PTs themselves.

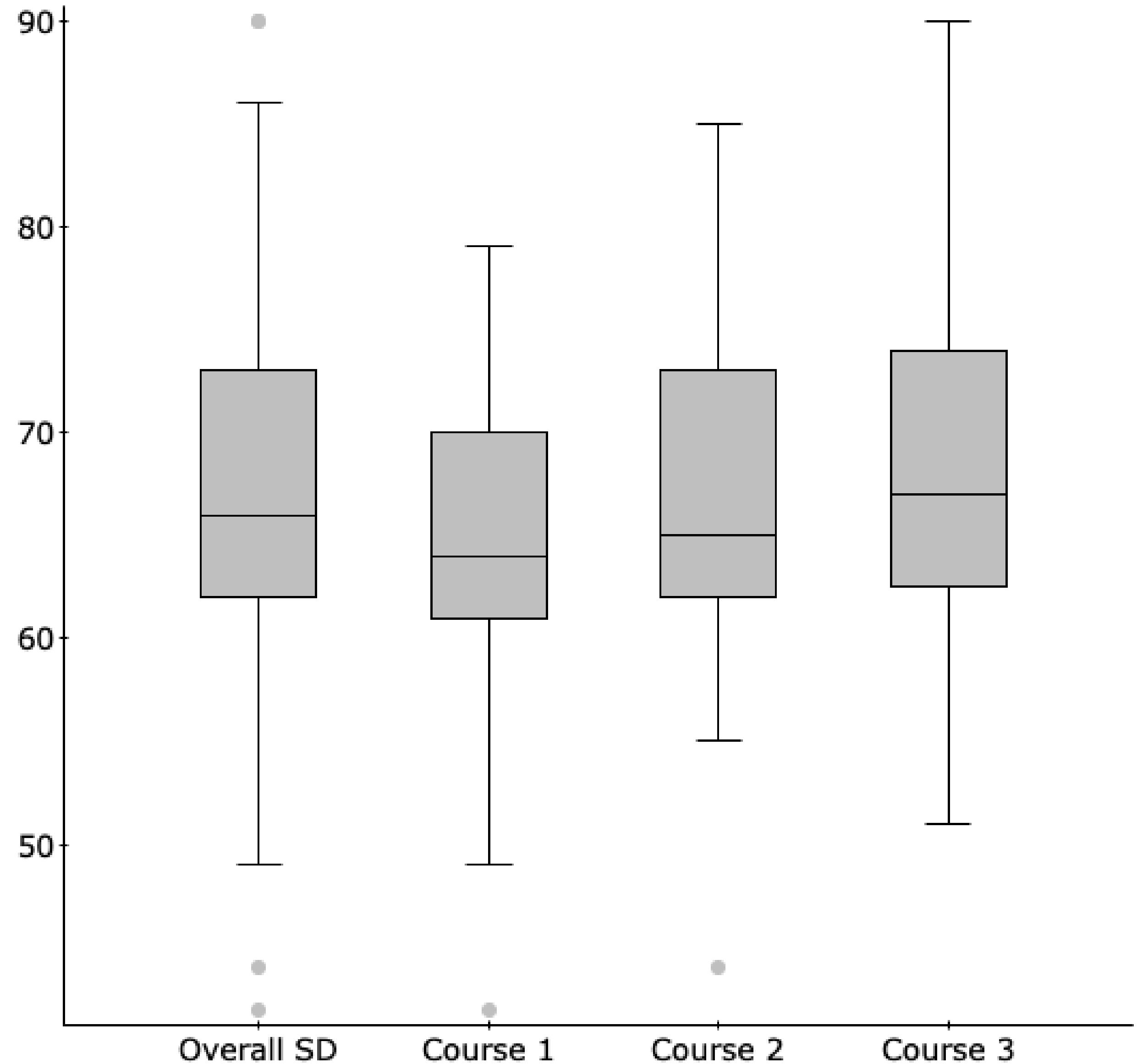

Figure 18. Distribution of Semantic Differential scores.

An examination of PTs' Semantic Differential scores by course showed a slight increase in average score as PTs progressed through the content and methods courses. The course a PT was enrolled in had a moderate impact on PTs' overall Semantic Differential scores and were approaching significance, $F(2, 105)$=3.04, p=.052 (See Table 13). With a moderate effect size of .058 between groups, the course that the

PT was enrolled in explained 5.8% of the variation in scores on the Semantic Differential. Post-hoc analysis revealed differences between Semantic Differential scores of PTs enrolled in course 1 and course 3 (p = .047) with a mean difference of 5.057. Other differences between groups were not statistically significant.

Table 13

ANOVA Results for Semantic Differential

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	477.50	2	238.75	3.04	0.052	3.08
Within Groups	8237.41	105	78.45			
Total	8714.92	107				

As a reminder, the SD consisted of 20 paired words designed to assess aspects that makes one good at mathematics. I categorized the different words from the Semantic Differential based on the conceptions of NOM (See Table 8). PTs' conceptions of NOM aligning with a static-unified body of knowledge was further evident when considering the individual word pairs on the Semantic Differential. Using this categorization, it was helpful to see PTs' individual responses for each of these paired words and how those responses also supported their overall conception that NOM is a static-unified body of knowledge. I Figure 20, I represented the individual word pairings from the Semantic Differential and PTs' responses to each pairing with a stacked bar graph. When answering the question, what does it mean to be good at mathematics with the paired words connections and memorization, almost half of PTs' rated either a 4 or 5 closer to connections. More interesting is that, when considering the words connections and memorization, only 19 out of the 123 PTs, or 15%, stated that memorization is important to mathematics. In contrast, 104 out of the 123, or 85%, agreed that connections are very important or somewhat important, or they were neutral. This aligned with the idea that mathematics is a static-unified body of knowledge because, in general, PTs thought connections were important. A similar trend was shown with the paired words procedures and concepts. Again, it is important to note that only 15 out of the 123 PTs, or 12%, stated that procedures are important or somewhat important to being good at mathematics. In addition, 52 of the 122 PTs, or 43%, reported that concepts are important or very important.

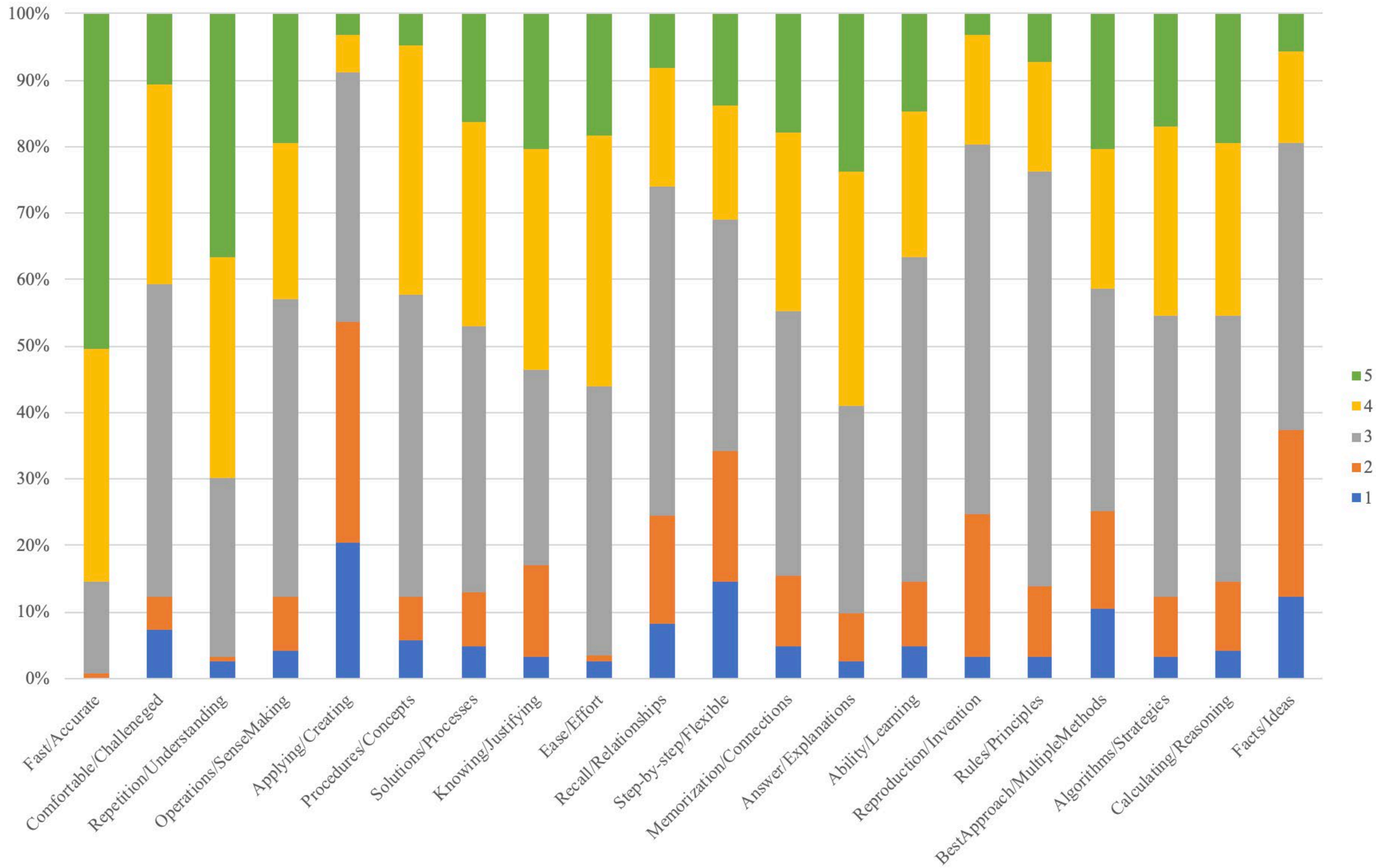

Figure 19. PTs' answers to individual paired words on the Semantic Differential.

Unlike the MBI, more PTs rated items on the SD as neutral. For example, with the paired words step-by-step (i.e., bag of tools) and flexible (i.e., problem-driven dynamic discipline) almost 35% of PTs rated directly in the middle, meaning that problem-solving in mathematics required sometimes following steps and sometimes utilizing flexibility. Similarly, the same was true for the words learning (i.e., problem-driven dynamic discipline) and ability (i.e., bag of tools). PTs rated directly in the middle, meaning that some learning and some ability are important in mathematics. The PTs' ratings of 3 on so many of the individual paired words further supported the idea that PTs' had conceptions of NOM that have aspects of both a bag of tools and a problem-driven dynamic discipline. When considering this idea on the NOM Continuum, PTs' choice of a rating of 3 on the individual paired words indicate that they are moving away from a conception of NOM as a bag of tools, but perhaps not fully convinced of NOM as a problem-driven dynamic discipline. I will discuss this idea more when I compare results from the SD and MBI. The overall descriptive results from the Semantic Differential revealed that PTs' conceptions of NOM closely aligned with NOM as a static-unified body of knowledge. The consideration of the PTs' responses to the individual word pairings also showed that, overall, PTs considered mathematics as a static-unified body of knowledge.

Connections between the Mathematics Belief Instrument and the Semantic Differential

Results from the MBI and the Semantic Differential both revealed that PTs' conceptions of NOM were most closely aligned with NOM as a static-unified body of knowledge. This was evident when looking at the overall scores on the MBI (n = 108, m

= 4.23, *sd* = 4.32) and the SD (*n* = 123, *m* = 66.76, *sd* = 8.75). It was also evident in the item analysis of individual statements on the MBI and individual word pairings from the Semantic Differential.

The relationship between PTs' scores on the MBI and Semantic Differential is shown in Figure 20. From the scatter plot, it is evident there is a small positive correlation between PTs' scores on the MBI and Semantic Differential. Further analysis revealed that the model was statistically significant with $F_{(1, 1106)} = 31.925$, $p < .001$, $R^2 = .231$. The unstandardized beta was equal to 1.004 and standardized beta was .481, indicating that for every point increase in PTs' score on the MBI, a PTs' score on the SD increased by 1.004 points, or for every standard deviation increase in a PTs MBI score, there was a .481 standard deviation increase in PTs' SD scores. The Durbin-Watson test was 2.215 suggesting that there was no threat of autocorrelation in the data.

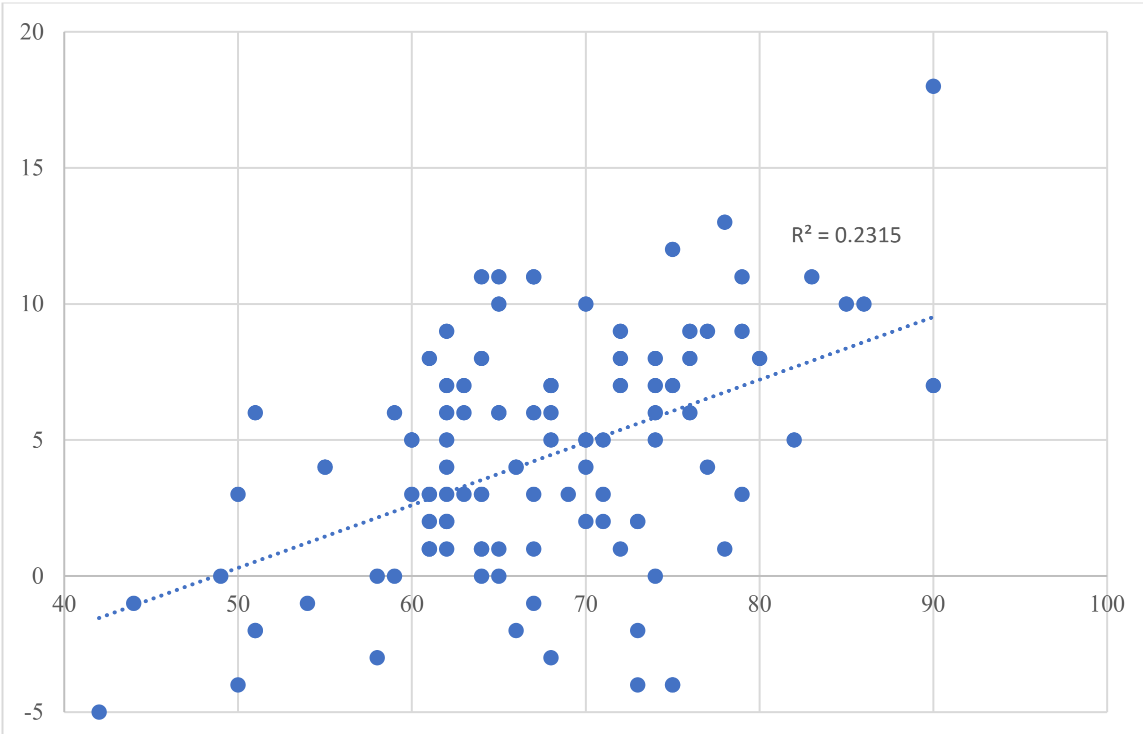

Figure 20. Scatterplot of MBI and SD scores with trend line.

Additionally, the correlation results also indicated an overall moderate positive significant relationship (*r*=.481, *p*=.01) between the PTs' scores on the MBI and the SD. This suggests that a PT with a relatively high score on the MBI will also have a high score on the SD. This alignment of PTs scores on the MBI and Semantic Differential suggest that regardless of the survey, PTs were consistent in describing their conceptions of NOM.

While, it was encouraging to see that PTs' scores on the MBI and Semantic Differential have an overall positive relationship, further examination of individual PTs' scores revealed that there were some PTs who scored relatively low on one of the surveys and relatively high on the other. For example, one PT scored a 75 on the Semantic Differential (above the average) and a -4 on the MBI (below the average). A score of 75 on the Semantic Differential indicated that the PT rated more closely to words associated with NOM as a problem-driven dynamic discipline, but a score of -4 on the MBI, 2 standard deviations below the mean, indicated that the PT was rating statements consistent with a conception of NOM between a bag of tools and static-unified body of knowledge.

I showed the distribution of PTs' responses to individual statements on the MBI and word pairings on the Semantic Differential as a way to see how PTs in this study answered individual questions and how responses on those individual statements helped describe PTs' conceptions of NOM. The individual item analysis revealed that PTs rated closely to static-unified body of knowledge for some statements, but also confirmed that there were PTs who rated statements closely to NOM as a bag of tools or a problem-driven dynamic discipline. In Figure 16, there was very little neutral ratings (gray coloring) compared to strongly agree and agree (green and yellow colorings). Contrastingly, in *Figure 19*, there is a lot of gray, or neutral. This distinction between the two figures is important, because a neutral rating for the MBI meant that the PT was impartial, meaning that the PT did not support the positively or negatively aligned statement. However, a PT who rated neutral on the Semantic Differential meant that the PT agreed that to be good at mathematics a person needs to have some of both. For example, for the paired words applying and creating, 46 of the 123, or 37%, rated neutral meaning that to be good at mathematics one sometimes applied facts or skills to solve a problem (i.e., bag of tools) and sometimes was able to be creative in problem solving (i.e., problem-driven dynamic discipline). Although the use of both the MBI and Semantic Differential seemed to confirm the finding that PTs' conceptions of NOM are most closely related to a static-unified body of knowledge, individual statements revealed that often PTs were neutral or agreed with aspects of both bag of tools and problem-driven dynamic discipline. Therefore, a more in-depth examination of PTs' conceptions of NOM was required in order to describe them more accurately. I will discuss the PTs' answers to the open-ended question on the MBI which provided more insight to PTs' conceptions of NOM.

Definitions of Mathematics

In addition to the quantitative items present in the MBI and Semantic Differential, PTs elaborated on their conceptions of NOM by answering the following open-ended question on the MBI: "Different people describe mathematics in different ways. How would you answer the question, what is mathematics?" Initial analysis of PTs' responses to this question provided further description of and elaboration on their conceptions of NOM.

Using the PTs' responses to the open-ended question, I created a word cloud in R. When creating this word cloud, all common English words (e.g., and, the, of) were removed completely from analysis. Additionally, I wrote code to recognize the same root words. For example, in the word cloud, solve, solving, and solved became "solv." By recognizing the same root words, words were not double counted when creating the frequency count. Additionally, I wrote the code to remove the words math, mathematics, or any variation. Because the prompt asked PTs to define what is mathematics, generating a count of the students who used mathematics or math was not beneficial in the analysis of the PTs' open-ended responses. In Figure 21, I present the word cloud generated in R using all 123 PTs' definitions of mathematics. Most prominent in this word cloud are the following words: number, problem, solv, use, and equat.

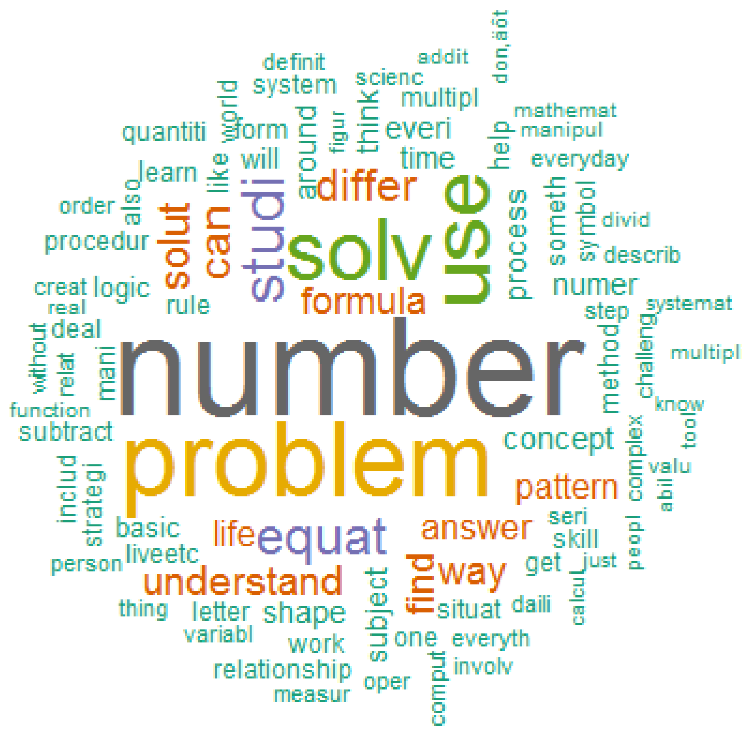

Figure 21. World cloud based on PTs' definitions of mathematics.

A frequency count of the words PTs used to define mathematics revealed that more PTs used words often associated with NOM as a bag of tools, such as number, problem, solv, use, and equat to describe mathematics, and did not use words often associated with NOM as a static-unified body of knowledge or problem-driven dynamic discipline, such as create, logic, patterns, and changing (See Table 14).

Table 14

Frequency Count of Words in PTs' definitions of Mathematics

Word in PTs' Definitions	Frequency
Number	82
Problem	60
Solv	47
Use	31
Equat	23
Logic	7
Creat	4
Change	1
Patterns	12

However, further examination of individual PT's definitions revealed that the PTs focused more on NOM as a static-body of knowledge. For example, one PT said, "I believe mathematics is number and letter based. Mathematics uses patterns, imaginary numbers, etc. in order to find a solution. Mathematics more times than none has a solution to every problem." Initially, this PT described mathematics as numbers and letters which could lend itself to the idea that mathematics is a bag of tools. However, the fact that she then described mathematics as patterns means she recognized that in mathematics there can also be recurring ideas which could lend itself to the idea that mathematics is a problem-driven discipline. Furthermore, the PT surmised in her definition that mathematics almost always has a solution. The idea that mathematics almost always has a solution provided evidence that this PT was thinking of mathematics as something to be discovered. Additionally, on the two surveys the PT scored in the middle of both, representing scores most aligned with the static-unified body of knowledge. This was evidence that the PT viewed mathematics as having a conception of NOM that most closely aligned with a static-unified body of knowledge.

Another PT defined mathematics as, "multiple processes and methods of logical thinking. Logic is math, and algorithms are results of others' logical thinking processes to simplify things. Most math problems can be solved using logic to manipulate numbers." Here, this PT decided that most mathematics problems can be solved, and therefore mathematics is a finished product. This PT also focused her definition of mathematics on the idea that it is a highly logical process. The two main ideas in this PT's definition of mathematics, that (1) mathematics is logical and (2) mathematics as a finished product, aligned with the conception of NOM as a static-body of knowledge. These two main ideas surfaced in other PTs' definitions. I provide 10 different PTs' definitions in Table *15*, as a representation of definitions that aligned with NOM as a static-unified body of knowledge. I bolded words or phrases that support the conception of NOM as a static-unified body of knowledge.

Table 15

PTs' Definitions that Support NOM as a Static-Unified Body of Knowledge

Logic Feature	Discovery Feature
Mathematics deals with the **logic** of shapes, arrangements and quantities.	Using numbers and problem solving to **find solutions.**
Mathematics is multiple processes and **logical** thinking.	All math truly is just **finding a pattern** or formula
Mathematics is critically thinking by **logical** means.	It is about different strategies you can use to **find the solution** to the problem.
Mathematics focuses on **logical reasoning**.	My personal definition of mathematics is **finding definite solutions** to problems or equations while also creating individual/unique approaches.
Mathematics is a **systematic** way of problem solving.	
I think mathematics is **the relationship of** the numbers to the real world.	

The PTs' attention to mathematics as logical clearly fell under the conception of NOM as a static-unified body of knowledge. The phrase finding a solution might also be categorized as a bag of tools. However, PTs' definitions that were coded as a bag of tools used different terms than finding a solution. For example, one PT defined mathematics as "procedures and functions we do to numbers to complete calculations." In this definition the PT did not attend to the idea of finding a solution or discovering a solution to the mathematics problem, instead she referred to the solution as a calculation. Another PT defined mathematics as, "numbers used to solve problem." Again, in this definition, the PT did not discuss finding a solution, instead she used a tool (i.e., numbers) to solve a problem. This is a distinction that was important when coding the PTs' definitions. PTs who used the language of "finding a solution" did not use the language of “solving.” The words find and solution appeared so often together and without appearing with the word solv that a hierarchical cluster analysis in R revealed distinct clusters as depicted in Figure 22.

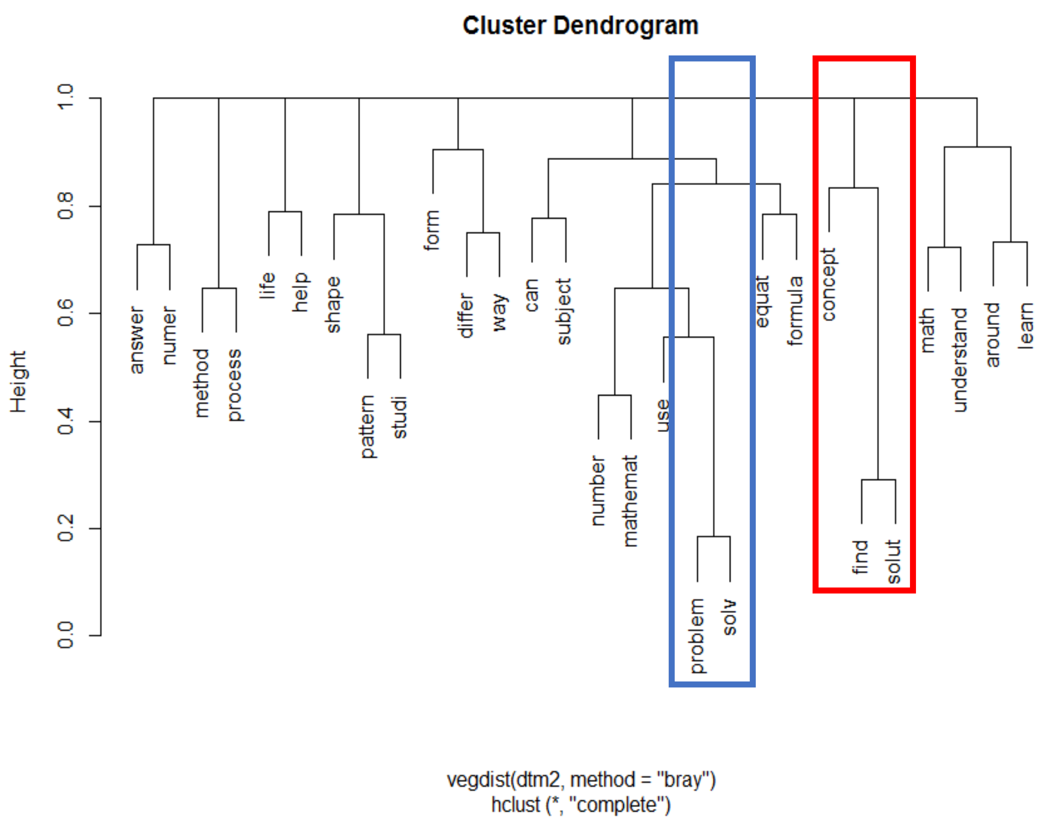

Figure 22. Cluster dendogram revealed that find and solv were two distinct clusters.

In Figure 22, no cluster appears at a height of 0, meaning that no words in the PTs’ definitions of mathematics were said the same number of times together. Find and solut—the words in the distinct cluster in the red box—appear at the same height, meaning that these words appeared together more often than concept and solut or concept and find. As represented in the PTs’ definitions of mathematics in Table 15, PTs used a combination of the words find and solut to describe the discoverability aspect of mathematics. PTs also used the words problem and solv together in their definitions, but most often in relation to making a calculation or solving a problem. The fact that these two word pairs (i.e., find/solut and problem/solv) appeared in two distinct clusters confirmed that the PTs were using the terms to describe mathematics in different ways. That is, find and solut were typically associated with the discoverability aspect of NOM as a static-unified body of knowledge, whereas problem and solv were associated with the calculability aspect of NOM as a bag of tools. Therefore, because these word pairings appeared in two distinct clusters and were associated with different conceptions of NOM, it follows that they were using the words in different ways. The clustering of the two words find and solut in PTs' definitions of mathematics, as well as PTs' definitions of mathematics, aligned with NOM as a static body of knowledge. In the writing prompt, PTs continued to make statements that aligned their conceptions of NOM with a static-unified body of knowledge.

The Writing Prompt

In the writing prompt, I asked PTs to personify mathematics by asking them to write about who Math is. I gave suggestions of possible ideas to include, such as how does Math look and what is your relationship like. However, these suggestions were not meant to constrain the PTs' writing. Through inductive and deductive qualitative analysis of the writing prompts, I coded many statements regarding the appearance of Math, PTs' relationships with Math, and how PTs described the character of Math, each of which helped to understand PTs' conceptions of mathematics.

The ideas central to NOM as a static-unified body of knowledge were widespread among the PTs' writing prompts. Of the 130 writing prompts, the knowledgeable other

82, 63%, PTs described aspects of mathematics that most closely aligned with a static-unified body of knowledge, 43, 33%, PTs described aspects of mathematics that most closely aligned with a bag of tools, and only 5, 4%, PTs described aspects of mathematics as a problem-driven dynamic discipline. To illustrate the majority of PTs who described mathematics as a static-unified body of knowledge, I selected representative quotes from a number of participants and present them as if the PTs were in a conversation with one another. As a phenomenological study, I focused on bringing to light the lived experiences of PTs through their voices. Creating vignettes is a tool that will allow the reader to understand the voice of the PTs as they further describe their conceptions of NOM. By giving voice to the PTs, it is my intention that the reader might gain both an understanding as well as a sense of how the participants' described their conceptions of NOM.

Vignette 1. PTs described mathematics as discoverable.

PT1.40: She looks like a gorgeous blonde model that you can't help but be jealous of due to all the knowledge she holds.

PT2.04: And, she is always learning too, but her steps and processes never change.

PT2.05: Yeah. You can always come back to a conversation with Math and find parts that you did not discover before.

PT1.30: He [Math] doesn't change with time but learns more and more and shares it with everyone he can.

PT 3.11: Yes, I am looking forward to learning new ways of figuring out the answer.

PT2.14: But, Math can also be mysterious. There is still so much I do not know about her. She has formulas and equations to whom I have never been introduced.

Octavia: I agree. He's also very unpredictable . . . sort of like a mystery. Math intrigues me and always teaches me new things. We get each other, him and I. Whenever he speaks to me, I understand.

PT3.31: He is a tricky one to figure out. To love math . . . is to weave past his fluff and find his hidden messages.

Present in Vignette 1 is the idea of mathematics as a person holding knowledge and eventually, even if circuitously, sharing it with others. This idea of a person holding mathematical knowledge and eventually sharing it lent itself to the belief that mathematics has a product that can be discovered. In the previous conversation, the PTs focused on the idea that mathematics is a body of knowledge that exists and someone must share or that they must discover a solution by interpreting a hidden message. Finding hidden messages, figuring out solutions, and sharing information are all clues in the PTs' conversation about their personification essays that described an idea of mathematics as having a finished product to be discovered. That is, within these phrases, I viewed the PTs' responses as aligning more with NOM as a static-body of knowledge.

The discoverability idea present within PTs' writing prompts was only one of the two main descriptions of NOM as a static-unified body of knowledge. The second main description of NOM as a static-body of knowledge according to Thompson (1992) is that mathematics is logical. Many PTs also attended to this piece of the definition in their writing prompts and interviews.

Vignette 2. PTs described mathematics as logical.

PT3.20: Yeah. But, the more I am around her I feel like she is becoming more organized and I understand her more.

PT2.02: Yeah. Math looked like numbers I understood, rulers, puzzles, and things that I was all comfortable with and that made logical sense.

PT2.05: For sure, Math is logical and reasonable.

PT2.56: In reality, he is strategic and makes sense.

Here, PTs described mathematics as being logical, reasonable, and making sense. These words aligned with the part of NOM as a static-unified body of knowledge being logical and making sense. Generally, in this study, a PT who described mathematics as a static-body of knowledge in one data source had similar descriptions of NOM as a static body of knowledge in all the data sources. In the following vignette, I chose representative samples of three PTs who described mathematics as a static-unified body of knowledge in their writing prompts.

Vignette 3. PTs described mathematics as a static-unified body of knowledge.

Olga: Math is a very matter-of-factly type person, and his final answer is always set in stone. He can give me the biggest headache because of the over the top complexity at times, but he gives explanation to nearly all problems.

Tabby: Hmm. I'm still not sure whether Math was responsible for our lapsed friendship or I was, but it felt like he started creating all sorts of rules for our relationship and I couldn't keep track of them all. Then one day, just a few months ago, Math and I came together again through a mutual acquaintance who helped us both to see that all those rules and algorithms we had believed were necessary for us to be friends actually weren't so necessary.

Margot: Yeah, I was taught that math can be more about discussing and problem solving than just repeating facts and procedure.

When Olga defined mathematics on the MBI, she said, "I believe mathematics is number and letter based. Mathematics uses patterns, imaginary numbers, etc. in order to find a solution. Mathematics more times than none has a solution to every problem." In this vignette, Olga's idea of mathematics as matter of fact substantiates her definition of mathematics. Here, Olga's description of matter of fact (i.e., logical) meant that Olga was always getting an explanation because there was always a solution in mathematics (i.e., a final product). Though Tabby did not explicitly talk about mathematics as logical or discoverable, she was moving away from the idea that mathematics was only rules and algorithms (i.e., a bag of tools). Tabby's survey score of over average for the Semantic Differential and below average of the MBI confirmed that she was moving away from the idea of NOM as a bag of tools and closer to NOM as a static-unified body of knowledge. In Vignette 3, Margot, who scored in the middle of both surveys, said that mathematics can be about discussion and problem solving, but also valued the facts and procedures at times. First, Olga clearly stated two ideas consistent with Thompson's (1992) definition in that mathematics is logical and has a final product. Second, Tabby talked about her conception of mathematics moving away from rules and algorithms. Third, Margot described mathematics as involving aspects of repeating facts and procedures and problem solving, thus aligning her conception of NOM with a static-unified body of knowledge.

The three vignettes provided more evidence that further supported the PTs' overall conceptions that mathematics was discoverable and logical. Mathematics as discoverable and logical align with the conception of NOM as a static-unified body of knowledge. Of the 130 writing prompts, 82, 63%, PTs described mathematics as a static-unified body of knowledge, 43, 33%, PTs described mathematics as a bag of tools, and only 5, 4%, PTs described mathematics as a problem-driven dynamic discipline. One-third of PTs' did not describe mathematics as a static-unified body of knowledge, indicating that some of the PTs had contrasting views to the 82 who described mathematics as a static-unified body of knowledge. Thus, I examined PTs' choice of *math-character* which helped me elaborate on and further understand PTs' conceptions of NOM on an individual level.

Math-Characters. In addition to making general statements about the logical and discoverable features of mathematics, PTs used adjectives to describe their *math-characters* that also provided more detail about their conceptions of NOM. Overall, PTs described 24 different characters (see Table 16).

Table 16

Initial Character Codes

Math-Character	Frequency	Math- Character	Frequency
Acquaintance	5	Jerk	1
Baby Mosquito	1	Mean Girl	2
Boyfriend	1	Mentor	2
Brother	2	Model	1
Bully	12	Monster	13
Cousin	1	Mother	1
Devil	3	Rain Cloud	1
Enemy	6	Sister	3
Friend	43	Step-Dad	1
Family	15	Teacher	2
Grandfather	1	Tricky Person	5
Grumpy Old Man	1	Witch	7

With 130 writing prompts and only 24 characters described, there was a lot of overlap among PTs' *math-characters*. For example, 43 PTs' described mathematics as a friend and 15 PTs' personified mathematics as a family member. In an analytic memo I wrote,

> Use characters to parse apart what that means about PTs' relationship with mathematics. Zazkis (2015) reported a PT who described math as a sensible best friend. Sensible means readily perceived and so can be related to the understandability of math. (Analytic memo, January 22, 2019)

Therefore, in the following sections, I will describe the three distinct groupings—friend, family, and antagonist—which resulted from the character coding as well as how the characters depicted helped me further understand and describe PTs' conceptions of NOM. In the following sections describing PTs' *math-character*, I will provide excerpts from their writing prompts where the PTs describe the character, and then I will discuss the adjectives used by the PTs and how those adjectives further describe PTs' conceptions of NOM.

Friend Group. Forty-three, or 33%, of PTs described mathematics as their friend when asked to personify mathematics. Some PTs described a lifelong friendship, others wrote about the struggle to become friends, and others said their friendships ended. To illustrate this, I selected representative quotes from a number of other participants collected during writing prompts and interviews and present them as if the participants were in a conversation with one another.

Vignette 4. PTs described mathematics as their friend.

PT2.04: To me, math is an old, wise friend. Someone whom has been around forever and everyone knows- almost like a town legend. She is wise beyond her years but she always keeps up with new trends.

Ophelia: I agree. Math is a good friend to me because I can always rely on her. I have known Math most of my life and our relationship has gotten more complex as we have gotten older. Math can be very difficult to be around sometimes, but I always remember that I enjoy Math most of the time.

PT1.56: I thought I was Math's only friend. He was nice to me and I was nice to him. My relationship with Math hasn't really changed that much. There were sometimes when he messed with me, but we were ok afterwards. I still enjoy my friendship with Math. I'm still his only friend because no one else wants to be friends with him. Probably because he still wears neckties and has his shirt tucked in and wears glasses.

PT3.27: Yeah! Math has always been my friend, but we don't always get a long, because he tricks me all the time.

PT3.35: Math is an old friend of mine, too. I felt like I knew everything about him, but I was smart enough to know there were always more tricks up his sleeve.

PT3.19: Well, we were best friends in elementary and middle school, but we grew apart in high school. Now, we have rekindled what I hope will be a lifelong friendship. Math is a friend that I talk to every day.

PT2.50: Yeah, Math has been a friend for thirty years, we have had a few disagreements at times, but always team back up. His friendship is continually rewarding.

Maggie: Math loves to present a challenge to some of us, but he loves when the problem is finally solved. Math is an encourager to never give up. Best of all, Math is my friend who I enjoy seeing every day.

In this passage, all the PTs referred to mathematics as their friend, but the adjectives they each used to describe mathematics were different. For example, two PTs stated Math was reliable and they enjoyed seeing him (or her). Furthermore, these PTs, explained that even though sometimes there might be misunderstandings in their friendships that those misunderstandings were always resolved and PTs remained friends with mathematics. The PTs' use of the word reliable when describing their friend Math pointed to the idea that mathematics is consistent and therefore unchanging. The idea that mathematics is unchanging aligns with the discoverability aspect of mathematics. Contrasting this idea were PT1.56, PT 3.27, and PT3.35 who stated that Math messed with her or played tricks. PT3.35 explained further that she could always figure out Math's tricks because he would tell her "please excuse my dear aunt sally" (Writing Prompt). The PT's use of the phrase "messed with" implies a conception of NOM as a bag of tools. Phrases such as messed with, play tricks, and sneaky appeared in 27 other times in the writing prompts. In each reference, the PT was referring to mathematics as a bag of tools by describing the memorization of a rule or formula. Lastly in this passage, two PTs described Math as having disagreements but teaming back up, speaking daily, and presenting challenges, here the PTs descriptions point to mathematics as problem driven. Both of these PTs referred to the challenges of having Math as a friend, but persevering through them to figure out the problem. Maggie further elaborated that Math was an integral part of the world, and PT2.50 explained that the disagreements were unlocking clues to continue to solve the puzzles. Both of these descriptors align with mathematics as a problem-driven dynamic discipline. So, an examination of the adjectives PTs used to describe their *math-character* revealed that their conceptions are much more complex than what the quantitative measure revealed. In Vignette 4, even though all the PTs wrote about their friend Math, the PTs descriptors of Math revealed different aspects of their conceptions of NOM. The same was true for PTs who personified mathematics as a family member.

Family Group. Of the 24 original *math-character* codes, six were familial relationships and 22, or 17%, of PTs described a familial *math-character.* In Figure 23, I show the different familial relationships (i.e. sister, brother, mother, step dad, grandfather, and cousin) PTs described in their writing prompts and the network I created to show these familial relationships.

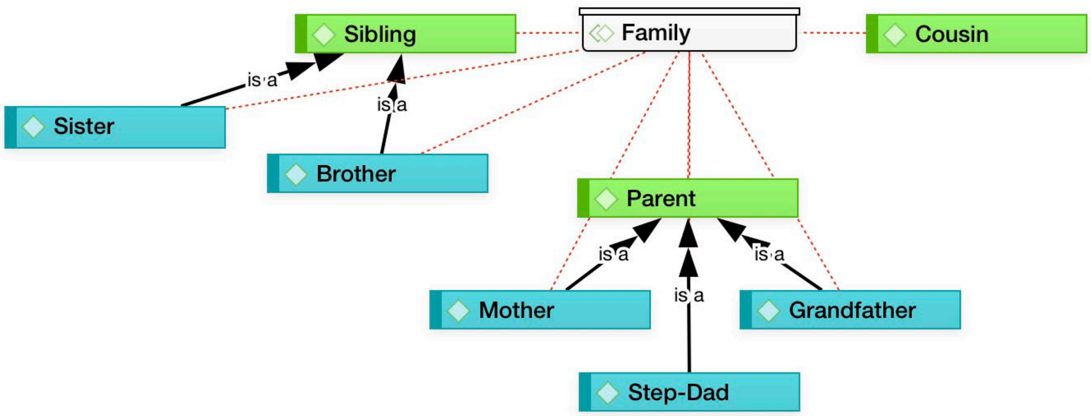

Figure 23. Familial emergent theme from Writing Prompt

The vignette below draws on a representative sample of PTs who described a familial relationship with mathematics. Though 22 PTs described a familial *math-character*, in the following vignette, I focus on a representative sample of PTs who described mathematics using different family members.

Vignette 5. PTs described mathematics as a family member.

Olive: Math is like your mom. She's always there. Your mom asks you what you want for dinner you tell her homemade lasagna. You wake up late and she rushes out the door, speeds down the highway, and gets you to your first class on time. After school you have a football game and your mom is cheering you on.

PT1.30: I think that Math is a distant cousin that I seldom talk to. Me and my cousin Math would see each other on thanksgiving and give a nod of awareness of each other's presence and then stay in our circles of comfort until it's time to leave.

Olga: Well, Math and I have always had our ups and downs. At the end of the day, Math is like an older brother who is annoying, frustrating, and hard to understand, but he'll always be there every single day.

PT1.50: Yeah, and growing up has made me realize that he's [Math] more like a grandfather with all the wisdom and strength. I still don't like to spend much time with Math, but I understand his importance in my life and I appreciate his complexity. Math is everywhere in some way, helping to build foundations or finding missing items. That guy might come off as boring and irritating, but he is an important guy with talents that are unbelievable.

In this passage, PTs described mathematics as a mother who is always there no matter what happens in life, a distant cousin, an annoying and frustrating brother, and a wise and strong grandfather. Aside from the PTs describing their *math-character* as a family member, the adjectives they used to describe the different family members also provided insight into their individual conceptions of NOM. For example, Olive described mathematics as her mother, gave examples of the everyday activities that she associated with her mother (i.e. Math). Olive's account of the utilitarian helpfulness of mathematics—baking, speeding, or telling time—indicated that Olive views mathematics as useful for some external end which is a characteristic that most closely aligns with

NOM as a bag of tools. Olive's account of the utilitarian aspect of mathematics is different than PT1.30.

PT1.30 personified mathematics as a distant cousin. When she elaborated on her cousin Math, she explained a cousin that was constantly changing. She first said her cousin Math was "rigid and by the book" and phrase most likely associated with NOM as a bag of tools. However, later in her writing prompt, the PT said that her cousin Math was "reliable and will always have your back" and "he doesn't change with time, but always shares what he learns." Olga, who described mathematics as her brother who is always around, explained that mathematics was always set in stone and her brother could always find an answer. In these two excerpts, mathematics is described as reliable, unchanging, and has knowledge to share suggesting that mathematics is static and discoverable and thus aligning with the conception of NOM as a static-unified body of knowledge.

Different than the other three PTs in Vignette 5, PT1.50 described mathematics as her wise grandfather, and the adjectives she used such as foundational, talented, complex, and important suggest that the PT viewed mathematics as more than a utilitarian or static discipline. She elaborated and said, "He always felt like an important presence in my life, but I didn't understand why too much, until my freshman year of high school." This PTs overall description of her grandfather Math, suggested an overall appreciation for the complexity of mathematics, the wisdom of mathematics, and the foundational aspects of mathematics and thus more closely aligned conception of NOM as a problem-driven dynamic discipline.

Antagonist Group. Lastly, 44, or 34%, of PTs related mathematics to an antagonist. In Figure 24, I show the different antagonistic relationships (i.e., mosquito, monster, jerk, bully, enemy, frenemy, and mean girl) PTs described in their writing prompts and the network I created to show that these were antagonistic relationships.

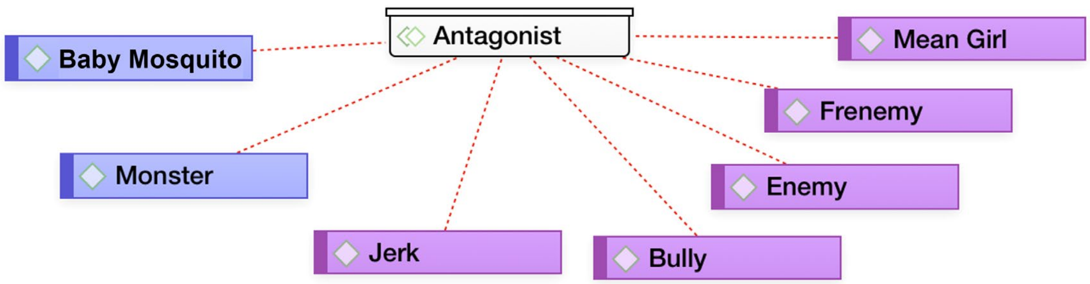

Figure 24. Antagonist emergent theme from writing prompts.

The vignette below draws on a representative sample of PTs who described an antagonistic relationship with mathematics.

Vignette 6. PTs described mathematics as an antagonist.

PT1.11: Math is the monster that lives under my bed. Always hiding in the shadows, waiting for the perfect time to pop out and smack me in the face.

Mia: Same for me! I could always see his shadow creeping through the window . . . Unlike my peers, I knew he was up to no good. They would have us impress Math with our number recognition and basic function problems to which he would respond with elaborate leaps of joy and hysterical laughs. Everybody loved him . . . except for me.

PT1.01: I think it [math] has always been like a baby mosquito from high school until now. I feel it sucking my blood and at some points can catch it in my sight, but as soon as I look away or focus too hard it disappears.

PT2.10: For me, Math is a grumpy old man who bullied me throughout elementary and middle school. I met him twenty-one years ago in kindergarten, and I have not been able to get away from him. He used to bully me and make me feel like I was not smart enough. He often compared me to other students who understood him better than I did.

PT1.47: I have unfortunately known Math since Kindergarten. I was a mere six years old when she was introduced to me and quickly took over my life. I like to compare Math to the stereotypical mean girl in schools. No one actually likes her, but everyone has to put up with her without really knowing why.

In this vignette, the PTs described mathematics as an unfriendly creature or person up to no good. Unlike the PTs who used very specific adjectives to describe their *math-character* as friend or family, the PTs in Vignette 6 in most cases did not use specific adjectives, but instead painted a broader picture of the scariness of mathematics. Mia's account of needing to impress Math with skills is a key feature of mathematics as a bag of tools. At another point in her writing prompt, Mia explained that even when she was older she and her classmates had to "predict problems" to try to impress Math. PT1.11's description of Math supported Mia's when she elaborated in her writing prompt about mathematics as a monster, she explained that she could "not think in the way mathematicians require you to." It is unclear exactly how PT1.11 believes mathematicians think, but it is clear that she believes mathematicians do something different than what she is doing, and so her choice of mathematics as the monster under her bed does not support a fallibilist view of NOM.

PT2.10 described mathematics as a bully, and described not feeling smart enough. She explained that mathematics was a bully because he forced her to memorize facts about him. The idea of memorizing facts is part of the conception of NOM as a bag of tools. The PT did explain that sometimes Mr. Math was there to guide her about the logical aspects of mathematics, and thus in this case her use of the math-character and adjectives to describe him aligned most closely with the conception of NOM as a static-unified body of knowledge. Lastly, this PT described that she did not want to know many things about the bully, but that in time she came to appreciate his "numbers, patterns, and problem-solving techniques." The PTs appreciation for the patterns and problem-solving techniques align with NOM as a problem-driven dynamic discipline. It seemed that the use of the *math-character* as a bully for PT2.10 explained a frustrating and changing relationship with mathematics that embodied aspects of NOM as a bag of tools, static-unified body of knowledge, and a problem-driven dynamic discipline. PT1.47 echoed PT2.10's *math-character* as a bully. Like PT2.10, PT1.47 wrote that there was always a reason behind bullies, suggesting that there is some logical argument to mathematics. She further explained that once she learned about Math that "Math became less difficult once I really tried to see the good in her." The good the PT referred to was the connectedness of mathematics to so many aspects of her life. The idea of connectedness aligns with the conception of NOM as a static-unified body of knowledge.

Section Summary

With 82 of the 130 PTs describing general characteristics of NOM as logical and discoverable, the PTs' descriptions of mathematics in their writing prompt confirmed the results from the MBI and Semantic Differential. That is, overall PTs held conceptions of NOM that most closely aligned with NOM as a static-unified body of knowledge. However, an examination of the individual *math-characters* and adjectives used to personify mathematics by PTs in their writing prompts revealed no clear connection between PTs and their *math-characters.* For example, the PTs who described mathematics as a friend used adjectives associated with all three conceptions of NOM. The same was true for PTs who described mathematics as a family member or antagonist. Therefore, even though overall the majority of PTs could be classified as holding a conception of NOM most closely aligned with a static-unified body of knowledge, the details PTs provided in their writing prompts point to the complexity of the construct of NOM and PTs' conceptions concerning NOM. Thus, a deeper examination of PTs conceptions and experiences that influenced those conceptions was warranted.

Prospective Teachers' Experiences with the Nature of Mathematics

The PTs results on the MBI and Semantic Differential, their definitions of mathematics, and parts of their writing prompt described mathematics as a static-unified body of knowledge. However, a more detailed examination of the PTs' *math-characters* revealed conceptions of NOM that aligned with bag of tools, static-unified body of knowledge, and problem-driven dynamic discipline. The writing prompts highlighted the complexity of the PTs' conceptions of NOM. Therefore, in the following section, I will describe the complexities of PTs' conceptions of NOM by discussing the relationships with mathematics PTs described in their writing prompts as a way to help understand their experiences with mathematics that influenced their conceptions of NOM.

Prospective Teachers' Relationship with mathematics.

The PTs' descriptions of their *math-character* helped describe their conceptions of NOM as well as their overall relationships with mathematics. I used the PTs' *math-characters* to help describe their overall relationship with mathematics. That is, based on how the PTs personified mathematics I analyzed to see if there was a pattern with other codes. For example, I analyzed for patterns with the PTs' *math-character* and the relationship codes used to analyze their writing prompts. Thirty-six PTs described an overall positive relationship with mathematics in their writing prompt. In Vignette 4, six PTs from the passage as described an overall positive relationship with mathematics. PT3.19 expressed a rekindling of the friendship, which suggested at one point they were not friends, and the expert other and I coded this as overall positive. She said,

> I have known Math my whole life. She is beautiful and mystical. Many have called her complicated, but she holds the mysteries of the world. She has made me wake up with excitement ready for school, but she has also made me wake up with dread. We were best friends in elementary and middle school, but we grew apart in high school. Now, we have rekindled what I hope will be a lifelong friendship. Math is a friend that I talk to every day. I'm hoping she comes with me to my future classroom and that my future students will learn to love her as much as I do. (PT3.19, writing prompt)

That is, in PT3.19's writing prompt there is only one indication of the PT not being friends with mathematics, but she expressed not only a rekindling, but also a hope for her future students to also be friends with mathematics. In our coding, we believed this warranted an overall positive relationship. Furthermore, when I asked Maggie to elaborate on her friend, Math, she explained,

> I've always enjoyed math. So, the image of the number 10, it has like big bug eyes and he's always smiling because he's always like hey let's learn something new today kind of thing… but yeah, some people see math in a negative view, and I just have never seen math and a negative view. So, when I think of something that's going to make me excited and want to learn, I think of something big bug eyes and smile. (Maggie, Interview)

And so, generally, PTs in this study who described mathematics as a friend had an overall positive relationship with mathematics.

Nine PTs described neutral relationships with mathematics. Among the four PTs represented in Vignette 5, the knowledgeable other and I coded Olive, PT1.30, and PT1.50 as an overall neutral relationship with mathematics. First, when Olive described mathematics as a mother she began by stating that a mother is always present in different situations (i.e. cooking for you, helping you, supporting you). Second, at no point in her writing prompt did Olive say she liked or disliked mathematics, and often this was found with other PTs that we coded as positive or negative. Furthermore, when I interviewed Olive, I asked her to elaborate more on why she described mathematics as a mother and what that meant to her. She explained,

> I just thought that like your mom's always there for you, at least in my case. I know some people don't have that. But in my case my mom’s always there. My mom is always the one that's there to pick me up from cheerleading practice, or to

> help me with my math homework or whatever homework I had. So, it's kind of like I said, math is all around us and your mom is always around whether you know it or not, because moms know everything! I just said that…your mom's always there, math is kind of there too. (Olive, Interview)

We also coded PT1.50 and PT1.30 as a neutral relationship with mathematics. Like Olive, these two PTs neither described mathematics as overall negative or positive. Instead one said mathematics is not someone he really wants to be around, but he understands it is everywhere and is important (PT1.50), and the other expressed a toleration for mathematics (PT1.30).

Contrast these ideas of a neutral relationship with Olga in the prior conversation. Olga described a roller-coaster relationship with mathematics. First, Olga said that she and mathematics had their ups and downs. Again, in the prompt she described always having a "love-hate" relationship with mathematics (Olga, writing prompt). I asked Olga if she still felt like she still had a love-hate relationship with mathematics. She explained,

> Yes. I do. I do. I do. It's so weird because last semester when I took course 1, I hated it. I don't know why I hated it so much, but I just dreaded going. I don't know? I feel like I just kind of gave up. In this class now [course 2], I have three friends and I feel like that makes it much better. It's weird because I like certain types of math too. When you get into Precalculus and stuff, no thanks. But I feel like when it's more of geometry that type of stuff in like algebra and that stuff is just like really simple and I could grasp that pretty well and I could feel like my confident in that. (Olga, Interview)

Forty-three other PTs also described roller-coaster relationships with mathematics. In this passage, not only did Olga directly say that she does have a roller-coaster relationship with mathematics, but she specifically describes what she likes and what she does not. This is in contrast to Olive and the other two PTs who never directly say they like or dislike mathematics, but only that it is around and perhaps useful. So overall, PTs who described a *math-character* as a family member either had a neutral relationship with mathematics or a roller-coaster relationship.

Aside from being together in group of antagonistic *math-characters*, I reflected on what these PTs' personifications had in common and if their personification might describe their overall relationship with mathematics. Sixteen total PTs described an overall negative relationship with mathematics. Specifically, the knowledgeable other and I coded PT1.11, Mia, and PT1.01 as an overall negative relationship with mathematics. PT1.11 only at the end of her writing prompt described mathematics as necessary, at no other point did she discuss anything else positive about mathematics and so this is why she was overall coded as a negative relationship. The same is true for PT1.01 she still feels the same about mathematics today. Similarly, Mia at the end of her writing prompt wrote about mathematics as necessary and therefore having some positive potential. In an interview, I asked Mia to elaborate on her personification of mathematics. Below is an excerpt of our conversation:

Interviewer: Can you elaborate [on your personification]?

Mia: Yeah. Like a clown. Okay, so, there's the kids that love him like he looks like so much fun. Where I would always be skeptical like why is his face painted? What's he hiding?—that type of thing. So, I'm just like super skeptical about his whole presence. Why does he want us to like him so much?

Interviewer: Okay, then you said that everyone loved him except for you, and then you talked about getting older and it got more complicated. Can you elaborate on that at all?

Mia: The exceptions. I don't have like an actual example. I just know that when there are exceptions in math…so you're doing something, you feel so great about it, and all of a sudden, they're like well it doesn't work when you do this. And you're like oh my gosh and now your world has just turned upside down. So, he [Math] just laughs at you, 'Haha, I tricked you'" (Mia, Interview).

In this passage from Mia's interview she eludes to Math as a person who tries to trick you. Furthermore, she said that even though a lot of friends liked mathematics, she never did. Thus, Mia was coded as having an overall negative relationship with mathematics.

Where Mia and PT1.11 were described as an overall negative relationship with mathematics, PT2.10 was described as having a change from a negative relationship to a positive one. She clearly states in Vignette 6 that mathematics is a grumpy old man who bullied her. She elaborated, "I thought I would never get along with him until I reached high school. He was suddenly my best friend" (PT1.01, Writing Prompt). Not only did this PT eventually refer to mathematics as friend, but she also said that she is great at understanding him, Math guides her, and she appreciates everything that Math has taught her. Contrast this to PT1.47 who was coded as having a roller-coaster relationship with mathematics. After describing mathematics as a mean girl and a bully, she stated,

> Yet, somehow, there is always reasoning behind bullies. Bullies were sometimes mean to people because they have trouble at home or they don't get enough attention. Around the time I was in high school, I decided to test my theory about Math, the bully. I began to dig deep and really learn a lot about Math even when other didn't see why. I began to realize why I saw Math everywhere and how important Math really was. Math became less difficult to deal with once I really tried to see the good in her. I now have a mostly positive relationship with Math, even though sometimes she can be a real witch. (PT1.47, Writing Prompt)

In this passage, this PT elaborates on her relationship with mathematics. While first she described mathematics as a mean girl and bully, she then says she has a positive relationship with mathematics. At first, the expert other and I coded this as a change from negative to positive. However, upon further discussion, we agreed on roller-coaster because she still, in the end, calls mathematics a "real witch." We agreed this back and forth was enough to warrant a roller-coaster relationship code. So overall, PTs who described an antagonistic *math-character* either had a negative relationship, roller-coaster relationship, or change from negative to positive.

Where there seemed to be no connection to PTs' conception of NOM and the *math-character* they described, there did seem to be a connection to PTs' *math-character* and their relationship with mathematics. First, PTs described mathematics as a friend. In this group, overall, the PTs were coded as having a positive relationship with mathematics. Second, PTs described mathematics as a family member. In this group, the PTs were coded as having either a neutral relationship with mathematics or a roller-coaster relationship. Last, PTs described mathematics as an antagonistic character. In the third group, PTs were coded as having either a negative, roller-coaster, or negative to positive relationship with mathematics. This connection of *math-character* and relationship with mathematics pointed to the different experiences of the PTs and how those experiences influenced their views of mathematics. In my analytical notes I wrote,

> Knowing what I know about PTs' overall conceptions of NOM and their relationships with mathematics, I am now wondering what happened to the PTs to make them believe this. Specifically, if they viewed NOM as static, why? What were their experiences in school. Or if they viewed NOM as a monster, why? (Analytic memo, March 25, 2018)

In the following sections, I aim to describe PTs' experiences with mathematics and also understand how those experiences with mathematics influenced their conceptions of NOM. In the previous section of this chapter, I focused on an overall description of all of the PTs who participated in the study (n=130). However, for the following section, I limit the discussion of experiences to the 13 PTs that I interviewed (See **Error! Reference source not found.**). Though some PTs referenced experiences in their writing prompts, I was able to ask probing questions about these experiences in the interviews thereby furthering the PT's recollection of the experiences and aiding in my understanding of their experiences. Furthermore, by discussing the 13 PTs who completed the surveys, the writing prompt and an interview, I was able to triangulate the data and have a richer, more descriptive representation of the PTs' own voices in regards to their experiences with mathematics (Moustakas, 1994). In the following sections, I leveraged PTs' results on the two surveys as well as how they described their conceptions of NOM and relationships with mathematics in their writing prompts to help understand their experiences with NOM.

Analysis of the writing prompts and interviews of these 13 PTs provided me with detailed descriptions and understandings related to their experiences. These 13 PTs discussed—in their writing prompts and interviews—how current and previous teachers played a role in their mathematical journey in regards to PTs' conceptions of and relationships with mathematics. Thus, in the following sections I will provide evidence to support these two main ideas as reported by the PTs: (1) teachers played a major role in shaping PTs' conceptions of NOM and (2) teachers played a major role in shaping PTs' relationships with mathematics.

Teachers Influenced Prospective Teachers' Conceptions of the Nature of Mathematics

Of the 13 PTs selected for an interview, their conceptions of NOM ranged from Bag of Tools to Problem-driven Dynamic Discipline. In the scatter plot below (see Figure 25), I labeled the MBI and SD scores with the corresponding 13 PTs. In this scatterplot, it is important to note that some of the cases (i.e. Odette, Mia, and Tina) are labeled as bag of tools and in line with the classification in **Error! Reference source not found.**. Similarly, the same is true for Millie, Tatum, and Margot—the scatterplot shows them as closer to problem-driven dynamic discipline as well as their classification in **Error! Reference source not found.**. However, there are discrepancies in the other 7 PTs. This was expected, and is why the following sections in this chapter focus only on those 13 PTs. That is, relying on only one data source, such as the MBI or Semantic

Differential, to describe and understand PTs' experiences with mathematics did not provide a rich, detailed view into the essence of the PTs stories.

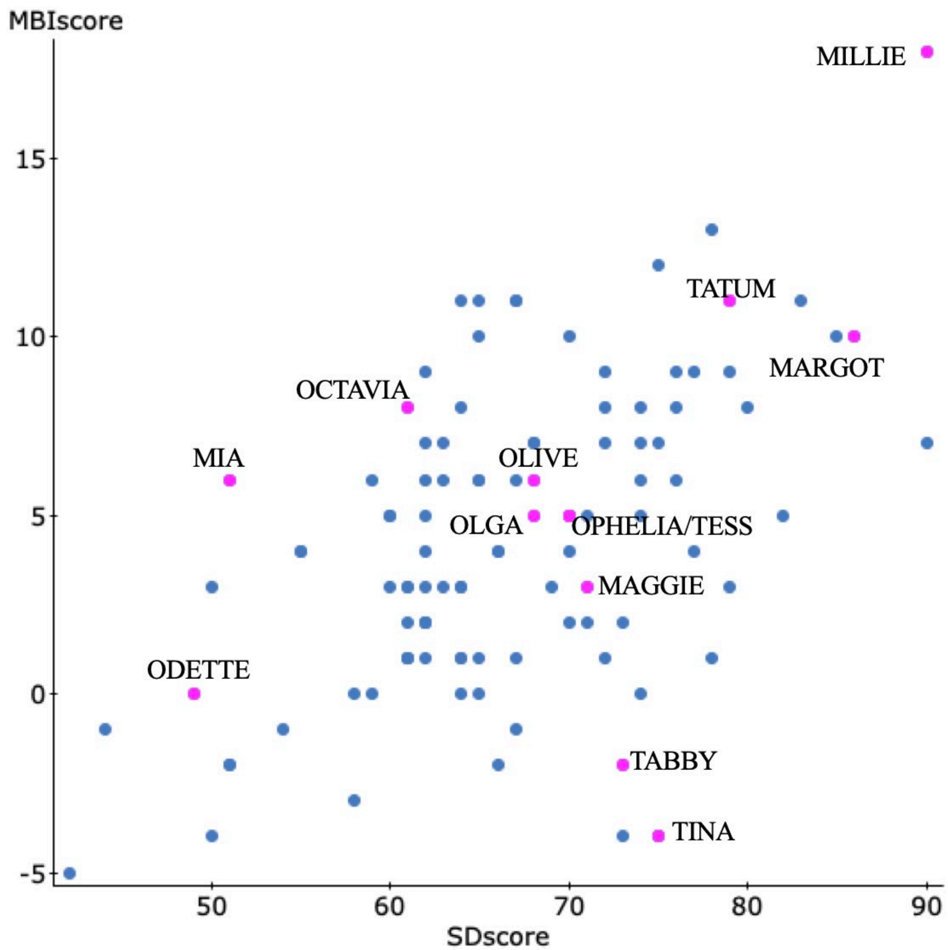

Figure 25. Scatterplot with corresponding interviewees.

I provide a more in-depth view of the subgroup of 13 PTs through vignettes. Similar to previous sections, to illuminate these ideas, I have selected representative quotes from the 13 PTs collected from the writing prompts and interviews and present them as if the PTs were in a conversation with one another.

Prospective teachers described experiences without understanding. Through their writing prompts and interviews, PTs described experiences with teachers that influenced the conception of NOM towards NOM as a bag of tools. PTs explained experiences that focused on memorization, retelling, and no understanding. I created the following vignette to illustrate the ideas described by PTs in their writing prompts and interviews regarding experiences with teachers and NOM as a bag of tools.

Vignette 7. PTs with a Bag of Tools conception of NOM described interactions with teachers.

Odette: One time in sixth grade, my teacher decided not to teach anymore. He said, “We're going to go through the workbook and here are all the assignments you need to do. If you can't do them, come up to me one by one.” Well, there were 28 of us going up to him one by one saying, “I don't know how to do this one.” We were all on different pages. That is when I stopped learning math—because he stopped teaching it. I went from a grade where I learned all the foundations, all the basics to a class now all they [the class] know is our teacher said we flipped the fraction to change it because that's what we're supposed to do.

Ophelia: Whenever I didn't understand it [Calculus] at first, she [the teacher] would always just give different problems. Sometimes she would just try to go through the same problem again and again, and I was still like, “Oh, I don't understand it.” So, she would give me a different problem.

Tina: I just remember it being more hands-on in elementary school. I guess things were laid out more plainly. And then it was their [the teachers'] way or the highway.

Mia: My overall experiences aren't super great, because it's not consistent from year to year or teacher to teacher. A good day in a math class would be where the teacher shows you the equation and explains to you why it works, but doesn't ask you why it works. Because I don't know, and I can make something up and prove it, but that doesn't make it right. If they [the teacher] would tell me why it works and then let me do it a bunch of times to prove to myself that it really is effective. Then the test would be very black-and-white, like doing the same type of practice.

In this vignette, Odette, Ophelia, Tina, and Mia alluded to a similar action from their mathematics teachers. That is, each of these four PTs discussed the idea of the teacher explaining a problem and giving examples to practice. Odette talked about how her teacher stopped teaching, and it was no longer learning mathematics about foundational knowledge, but instead it became a game of memorization. When I pressed Odette on this idea of memorization, she further explained that because of this she expected all her homework problems to be the same. Specifically, she said that if she worked some out in class, then she should be able to work the homework problems outside of class in the exact same way. She grew frustrated when this did not always work, and she said,

> When I get home, I look at it [the homework] and it doesn't even look like the same things that we went over in class. It looks totally different. In class one plus one is two, and I get homework and it's how do you calculate the amount of muffins on Mars? (Odette, Interview)

Odette elaborated more and said that most of the time in her classes she was working independently. Occasionally, she said the teacher(s) would allow group work. In this group work, she grew more frustrated, because she said, "Everybody can't be a teacher" (Odette, Interview). She explained that in group work she considered her peers as teachers, but they all did the problem differently and that was confusing to her. She wanted one way, a characteristic associated with NOM as a bag of tools.

Like Odette, Mia declared that her overall experiences with mathematics were not great because of the inconsistencies. She explained that the teacher would show her an example and she would practice and then she expected tests to look the same. When I asked Mia to elaborate she said she grew frustrated when her teacher would ask her how to solve a problem without showing Mia first. She said she felt like she was "pulling numbers and letters out of thin air trying to make stuff work" (Mia, Interview). In this example, Mia focused on solving a problem with the intent of getting a correct answer and no understanding of the reasons behind the process, a characteristic associated with NOM as a bag of tools.

Ophelia, Mia, Tina, and Odette experienced teachers who focused on the memorization of facts, rules, and skills by transmitting information and having students practicing problems. This shared experience, often of which were accounts of past occurrences with previous teachers, influenced these four PTs' current conceptions of mathematics.

Prospective teachers described experiences with connections and logic. Olive, Olga, Octavia, and Tess, however, reported different experiences with teachers and more closely aligned with a conception of NOM as a static-unified body of knowledge. Each of these PTs described teachers who promoted more than memorization and rule following, and instead helped the PTs see the logical connectedness of mathematics.

Vignette 8. PTs with a Static conception of NOM described interactions with teachers.

Olive: In class lately, we've been learning about the science behind adding and stuff. When I was younger—in elementary school—and I was learning to add I just did what the teacher taught us. I never knew the science behind it. In high school, I know that whenever my teacher would teach something she'd have it on the projector and she'd be working it out with us and explaining it to us. Now, she [my teacher] always says math is patterns—that's what she says every day in class. And that's true because it's different patterns, whether it's a set way of doing things or it's an actual, literal pattern.

Tess: A lot of my teachers just use the board and I really didn't figure out I have to look at something to see the connections. Until I got to high school and I found geometry it made sense. And my precalculus teacher she would only let us do our math on light colored pieces of paper because she said that math was an art form and that it needed to flow. She was right.

Octavia: When I didn't understand the stuff that came after that [some Calculus topic] and he [the teacher] had to sit down with me and start from that first building block of calculus up until where we were. I was very excited to know that I was on the same page as everybody else and that I'm good for the next step—the next building block. And it just keeps going.

Olga: I feel like it really has to do with what type of teacher you have. I'm already not a fan of math just because it's just so much. The instructor really plays a huge role in how a person views math, because if you have a sucky professor or a teacher in math, then it'll make you not like math or want to go to math class because you don't understand what they're saying. Therefore, you don't understand the work you know? It's very broad and you can get to any answer and it could be wrong.

In this vignette, Olive explained at different times in school and the actions or words of her teachers at those times. First, in elementary school, she said she learned mathematics by doing whatever the teacher told her to do. Here, Olive's elaboration about doing what the teacher taught provided evidence that she did not have a conceptual understanding of adding, but instead a procedural understanding. When Olive elaborated on this idea, she mentioned high school, where her teacher would explain problems while working them. This was a shift from simply doing what the teacher taught to beginning to understand further by the explanations her teacher would offer. Olive ended by explaining that she currently had a teacher who everyday focuses on the patterns in mathematics. When I interviewed Olive, I asked about her overall experiences with mathematics, and she elaborated on when she finally gained a conceptual understanding of adding. She said,

> Now that I'm in this class [course 1] where I'm learning. Instead of carrying just a one you're carrying a set of 10 or 100. When I'm a teacher one day and I have like students, I'm going to teach them the way that I'm learning in college. Because it's then you actually understand it and why you do it rather than just doing. (Olive, Interview)

In Olive's elaboration of learning to add, she expressed the desire to teach the way that made her finally understand that mathematics—or adding in her case—was not just something to memorize from the teacher (i.e., a bag of tools) but was logical and had meaning (i.e., a static-unified body of knowledge). In this brief description from Olive, she illustrated the actions of the teacher and how they informed her conception of NOM.

Similarly, Tess discussed a transition of teachers' actions at different points in her schooling. First, Tess explained that in her earlier years of school the norm in her mathematics classes was a teacher putting problems on the board. When I interviewed Tess, she explained that problems on the board meant a teacher would first show an example, then the teacher and class would work through an example together with the teacher at the board and students dictating the next step, ending with the teacher assigning homework problems from a textbook. In these moments of Tess' teachers putting problems on the board, Tess expressed that she did not see any connections, and that it was not until she took a geometry class that she began to see mathematics as a connected subject. As Tess and I talked about her experiences, she said she started to see the connections in geometry because her teacher would use "visual aids to show how ideas in geometry were connected" (Tess, Interview). Tess also said she began to see the connections in other mathematics classes such as precalculus because her precalculus teacher told her mathematics was an art form that needed to flow, and in the interview, Tess said she agreed. Like Olive, through Tess' teachers' actions and words, she began to see the connectedness of mathematics and the logical flow (i.e., static-unified body of knowledge).

Echoing Olive and Tess' sentiments of finally understanding how mathematics was connected, Octavia explained coming to this same realization when her Calculus teacher helped her see that Calculus was made of "building blocks" (Octavia, Interview). Octavia's descriptions of Calculus as building blocks described the logic in mathematics. That is, Octavia said that in order to progress in Calculus, she needed to understand one part before moving to the next. Octavia recognized the logic in Calculus describing her progression of understanding from one building block to the next. Octavia's descriptions of the building blocks of Calculus described the logical static nature of mathematics (i.e., static-unified body of knowledge). Although Octavia, Olive, and Tess discussed specific teacher actions and words that inevitably influenced their conceptions of mathematics, Olga's discussion of teachers was vaguer.

When I asked Olga to talk about her experiences with mathematics, she said it was all about the type of teacher, and in this explanation, she did not give any specific examples. However, Olga's words were still important in helping me understand how PTs' experiences with teachers—past and present—influenced their views about NOM. Olga expressed how important the teachers' role was to either liking or hating mathematics. She elaborated that the teacher could determine not just if a person likes or hates mathematics but even if the person will want to attend the mathematics class. Olga explained that a student might not attend class because that student did not understand what the teacher was saying. Specifically, she said mathematics is broad and has many possible answers, some of which could be incorrect. Olga believed that these aspects of mathematics did not make it easy for teachers to explain, and that was why students could begin to like or hate mathematics. At first, this statement of mathematics as broad and potentially incorrect answers did not lend itself to the idea that Olga viewed mathematics as a static-body of knowledge. However, when I asked her to elaborate she explained,

> I feel like in math there's different ways you can go about math. For example, a different person could answer this question a different way and but still get the same answer. Because I don't think there is one correct way to solve a math problem. Because when you do math you have to be flexible because not everything is step-by-step. You'll have to reread it over and over again and you may have to do the same step over again or a different one. You have to be flexible in how you approach math. (Olga, Interview)

As Olga elaborated in our interview, she discussed that, to her, broad meant the possibility of solving a problem in different ways and that one must be flexible because not everything in mathematics is step-by-step. I interpreted this passage and her elaboration on mathematics as broad and potentially incorrect to be unaligned with mathematics as a bag of tools. Furthermore, Olga’s scores on the MBI and Semantic Differential placed her closer to aligning with NOM as a static-unified body of knowledge.

Prospective teachers described experiences about creativity. Olga described experiences with former teachers that helped her see mathematics as a subject that is not just step-by-step where the doer must be flexible. Olive, Tess, and Octavia experienced teachers who helped them see the logical connectedness and meaning behind mathematics. These four PTs' accounts of occurrences with teachers, influenced these four PTs' current conceptions of mathematics. Millie, Tatum, Margot, Maggie, and Tabby, however, reported different experiences with teachers and more closely aligned with a conception of NOM as a dynamic problem-solving discipline. I selected representative quotes from the PTs who described NOM as a problem-driven discipline and present them in the following vignette.

Vignette 9. PTs with a problem-solving view of NOM described interactions with teachers.

Millie: They [the teachers] would present a problem and we might work the entire class on the problem and not even figure it out, but I liked being challenged and I liked being out of my comfort zone. There's not a right or wrong answer and that's what I think a good day is. I always hated when they [the teacher] just gave me the answer. Or in middle school they would give me a similar problem that had nothing to do with it, but would give me the right answer. Those were bad days.

Margot: I think my teachers would just like work a few problems I don't ever really remember using manipulatives ever. Eventually, I was taught that math can be more about discussing and problem solving that just repeating facts and procedures.

Tatum: So, I would bring it [homework] home to my mom, but my mom would teach me a whole completely different way and my teacher would get super upset with me because I wasn't doing it the way she taught. Since then I've always felt like I have to do it exactly the way the teacher says. Now, I can say that I've had teachers that actually . . . instead of just shutting me down and saying no you're wrong, they've seen where I was actually coming from in the problem. I don't want to make kids just learn one technique.

Maggie: I had one class where the teacher would give us a problem and say, "Okay show me how you would get your students to the answer this. Is your answer? How did you get from this to this?" That's when I first realized that what learning really is you have to understand it and be able to explain it.

Tabby: I had a teacher, he said "You didn't do it the way that I told you to, so this is wrong." So, I would ask questions because my brain didn't think about math in the same way that a teacher that was teaching it thought it. Instead of trying to understand how I was thinking about it teachers would basically just repeat what they had already said. Math that I remember, was teachers at the front. They used the overhead, they wrote up stuff, and everyone sat in rows. Nobody ever talked about anything. But then I learned that I can use the way that I think about

> the world and the way that I'm thinking about math, and it's not the wrong—that was really a pivotal moment for me. My professor last semester in [course 1] taught it [mathematics] in such a way that made a lot of the younger students very uncomfortable, because they had all these rules and ideas memorized and I didn't. I used what I know about the real world and thought about it.

Although there are similarities in how these five PTs discussed teachers' actions and words as in the previous two vignettes, I will focus on the differences and the reasons these were classified as problem-driven dynamic discipline. Thompson (1992) defined this view as a,

> Continually expanding field of human creation and invention in which patterns are generated and then distilled into knowledge. Thus, math is a process of inquiry and coming to know, adding to the sum of knowledge. Math is not a finished product for its results remain open to revisions." (p. 132)

Thompson's definition was evident in Millie's description of a good day in a mathematics class. Millie explained that sometimes in her mathematics course an entire class might be spent on one problem. She emphasized, rather enthusiastically, that even when an entire class period was spent on a problem that did not necessarily mean the class found a solution. Not finding a solution did not bother her, in fact, she claimed that she liked the challenge and the fact that there might not necessarily be a right or wrong answer. In her explanation Millie was unperturbed that she did not find an answer, and she viewed the experience of working through one problem as challenging. Furthermore, when Millie described the opposite of a good day in mathematics course, she claimed it was when the teacher just gave her an answer, suggesting that she appreciated the process of coming to know herself, thus aligned with Thompson's (1992) definition. Like Millie, Margot recognized through her experiences with teachers that mathematics was not limited to memorizing facts and procedures, but instead included problem solving.

When I pressed Margot on what she meant by problem solving and discussing, she explained that she had experiences where teachers expected her to do mathematics only the way they taught. However, she elaborated and said that once she had a teacher who gave her more freedom and that act helped her understanding of mathematics. Echoing Margot, Tatum explained,

> I find my own way to do it. When I'm creating it [the mathematics concept] sticks with me better. So, that's where I feel like if you were to come up with a concept by yourself, then you're going to be able to remember it better than anyone else. (Tatum, Interview)

Despite Tatum having teachers in the past who expected her to do mathematics one way, the teachers who gave her freedom to create ultimately changed the way she viewed mathematics. Tatum echoed Millie's and Margot's views of mathematics and the realization of the importance of creating something mathematical and not simply following the teacher. Furthering the idea of freedom to create in mathematics, Maggie explained the power of a teacher asking her to think about how future students might answer a question. Maggie's engagement with anticipating student responses opened her eyes to the many different ways that students may approach a problem and the importance of valuing those ideas and using those ideas to progress student thinking. Additionally, Maggie likened her future students to her own experiences with teachers in mathematics. She explained that when her own teachers presented problems and answers that she could do the mathematics and understand it, but that she felt comfortable. Maggie said she would rather tell someone, “This problem challenged me, and I tried to complete it” (Maggie, Interview) rather than being able to reproduce an answer the teacher had already given.

Further echoing the idea of being creative when solving problems in mathematics, Tabby explained that it was her interaction with a teacher that led her to the realization that she could use her own thinking instead of relying on the memorization of facts. In Tabby’s account of this experience, she not only mentioned what the teacher was doing, but she discussed in detail the mathematics task (See Appendix E) that she was assigned in her group. She said,

> There was a point where we had this chocolate milk problem. Well, I sat at home, and I must have sat there for hours just trying to decide how to work it. All of a sudden, I just had a lightbulb and I got it. Then I went to class, and I thought I had this really interesting way, and everybody’s answers were totally different than mine. I was like, “Crap, I don’t have it.” Then I was like, “You know, I do have this, this is the right answer.” I convinced my team. We put my solution up for the class. No one else had our answer or solved it in the same way.

Even in her retelling of this experience in course 1, about a year before our interview together, she explained that she would never have done that in the past, because the way she was taught in the past was to do it the way the teacher said and only that way. Tabby explained that her teacher in course 1 was very patient and allowed her to struggle through problems without judgement. Instead, she said she remembered her teacher the most because her teacher would just ask her questions about a step she did or an idea, and that it was those questions that helped her and gave her confidence in the mathematics class. Ultimately, it was that teacher who helped her realized the type of teacher Tabby wanted to be in the future.

Tabby's experience with one mathematics task and a teacher who valued her ideas in the classroom allowed Tabby to experience mathematics as a problem-driven dynamic discipline. Maggie, Tatum, Margot, and Millie also recounted experiences with teachers who allowed them to view mathematics as more than a bag of tools and instead as a discipline where creation and freedom to create could be valued. Octavia, Olive, Olga, and Tess also elaborated on their experiences with teachers who ultimately shaped their view of mathematics as a static-unified body of knowledge. Whereas, Mia, Odette, Ophelia, and Tina who experienced teachers who all promoted mathematics as a bag of tools. These 13 PTs all discussed experiences with teachers—past and present—and the impact those teachers had on their conception of mathematics. In the previous vignettes, I provided examples that illustrated each of the PTs' similar experiences with their mathematics teachers that shaped their subsequent conception of NOM of mathematics and drew on the similarities in those teachers' instructional practices. It is also important to note here that not one PT enrolled in course 1 aligned with the conception of NOM as a problem-driving dynamic discipline. In X, I provide a classification of each of the interviewees' conceptions of NOM based on their surveys, writing prompts, and interviews.

Table 17

Interviewees' Conceptions of NOM

Conception of NOM	Interviewee
Bag of Tools	Odette, Ophelia, and Tina
Bag of Tools/Static-Unified	Tabby
Static-Unified	Olive, Olga, Octavia, and Tess
Static-Unified/Problem-Driven Dynamic	Tatum and Mia
Problem-Driven Dynamic	Maggie, Millie, and Margot

Recall, from a previous section that the course in which a PT was enrolled explained some of the variation in scores on the MBI and Semantic Differential. I noted that since the MBI and Semantic Differential ultimately measure PTs' alignment with standards documents it was encouraging to see a significant increase as PTs progressed through the content and methods courses.

As discussed above, the 13 PTs all discussed experiences with teachers and how those experiences informed their conceptions about mathematics. First, in Vignette 7, PTs aligning with NOM as a bag of tools described experiences with teachers and explained that the teachers focused on one way to solve problems, lectured from the front of the classroom, and assigned homework problems for practice. In Vignette 8, PTs aligning with NOM as a static-unified body of knowledge described experiences with teachers and explained that teachers focused on the logical progression of mathematical ideas and making sense in mathematics. Last, in Vignette 9, PTs aligning with NOM as a problem-driven dynamic discipline described experiences with teachers and explained that teachers allowed them the freedom to use their own mathematical ideas to solve problems and indicated that mathematics could be creative. These PTs' experiences with teachers informed their conceptions of NOM and relationships with mathematics. In addition to the PTs recounting experiences with teachers and how those teachers informed their conceptions of mathematics, the PTs also talked about how the teachers influenced their relationship with mathematics.

Teachers Influenced Prospective Teachers' Relationships with the Nature of Mathematics

In the previous section, I provided statements from the 13 PTs to illuminate how they perceived previous experiences with their mathematics teachers and how those experiences influenced their conception of NOM. In this section, I will provide statements from the 13 PTs to illuminate how their experiences with teachers influenced their overall relationship with mathematics. Ellsworth and Bus (2000) reported that in their study PTs reported teachers as most influential in the effect—both positive and negative—they had on PTs' attitudes towards mathematics. The authors further explained that the PTs viewed their teachers as either an authority or facilitator. Similarly, the 13 PTs in this study also credited past and present teachers as either an authority in the classroom telling them mathematical facts or as a facilitator in the classroom promoting problem solving. Thus, the PTs all had similar experiences that they discussed as influential in forming their overall relationships with NOM.

Recall the details of the 13 interviewees described in Table 10, I provided details about the 13 PTs who I interviewed. In this table, I provided PTs' survey scores, the *math-character* described in their writing prompt, their overall relationship with mathematics, and their view of NOM. In this section, I present vignettes that group the 13 PTs by their overall relationship with mathematics—positive, negative, or roller-coaster. Note that I will include PTs who described a change in their relationship from negative to positive in the positive group and the PTs who described a change in their relationship from positive to negative in the negative group. The vignettes provide an in-depth view of how the PTs' experiences influenced their relationships with mathematics. I utilized the same strategy for constructing the vignettes in this section that I described in previous sections.

Vignette 10. PTs with negative relationships with mathematics described interactions with teachers.

Mia: My preschool teacher never properly introduced us [math and I], but I could always see his shadow creeping through the window. My overall experiences aren't super great because it's not consistent from year to year or teacher to teacher. The teaching method, the assessments, the practices . . . they're not consistent.

Odette: When I went to school, suddenly, Math became unintelligible. Math and I could no longer communicate. No one had taken the time to teach me where I needed to be taught. Teachers tended to wave me off, because I am smart in other subjects and I do have a grasp on what's going on. But because I struggle with math it's hard for them to believe, so they wave me off because they figure I'll get it eventually. And I don't.

In this short vignette, Mia and Odette both described inconsistent teachers. In her writing prompt, Mia said her preschool teacher never properly introduced her to mathematics. When I asked her to elaborate on what she meant in our interview, she talked about the inconsistency of that elementary school teacher, but also the inconsistency of future teachers. Notably, she was quite emphatic about how she felt dismissed by her mathematics teachers, and she even felt as though her teachers did not care whether or not she learned the mathematics itself. Odette explained, "Teachers were pressed for time with end of the year exams. She taught us how to do everything on calculator so we could pass the test" (Odette, Writing prompt and Interview). Mia and Odette both described instances with teachers which influenced their overall negative relationships with mathematics. In the next vignette, I will present the PTs with an overall positive relationship with mathematics.

Vignette 11. PTs with positive relationships with mathematics described interactions with teachers.

> Tabby: I never thought we [math and I] would find a way back to the friendship we once had. Honestly it wasn't the math that helped me find my way back, it was the teacher. She gave me very good feedback and improved my confidence to do math.
>
> Tess: We [math and I] started out rocky. A lot of the reason I didn't like math was because of my teacher. But, the only good teacher I ever had went out of her way to help me early before school or after school. She devoted a lot of her time to helping me. Then, I understand what Math was trying to tell me all those years, and I had a different perspective.

Tatum: My understanding of and relationship with math has changed drastically. When I was a child, I did not understand math like some of my classmates. Recently, in college, that has changed. I have been taught there can be multiple ways of thinking, and this has reopened my mind towards mathematics. My teachers, and family, have taken more time to actually explain to me what I'm doing. I've had teachers that actually listen to me, instead of just shutting me down.

Margot: Our relationship [mine and math's] changed when I was taught it can be more about discussing and problem solving. The good teachers interacted with me and tried to form relationships with me.

Ophelia: Math is a great friend that I can always rely on. In [course 1], she [the teacher] asked us who liked math, and most of the class is like, "No, we don't." I never understood that. But then she asks us questions and you can tell she's passionate about it. That helps.

Maggie: I have just never seen math in a negative view. I've always had teachers who ask questions and provide good examples. So when I walked out of the classroom I felt like I really understood whatever topic we went over that day.

Millie: My teacher would always be walking around and observing when we worked in groups—giving hints or asking questions. I really hated it when a teacher would just tell me something. Being able to figure it out helped me like math and understand it.

Tabby, Tess, Tatum, and Margot all expressed a change in their relationship with mathematics from negative to positive. First, Tabby, Tess, Tatum, and Margot discussed ideas similar to Mia and Odette in Vignette 10. That is, like Mia and Odette, they portrayed an initial negative relationship with mathematics as a result of feeling as though teachers did not value their ideas as learners. In each of Tabby, Tess, Tatum, and Margot's explanations they described a change from negative to positive, and they all attributed the change to an interaction with a teacher. These four PTs described different teacher actions such as providing productive feedback, allowing opportunities for multiple strategies, creating relationships with students as experiences with teachers that impacted a change in their relationship with mathematics from a negative to positive relationship with mathematics. Unlike Tabby, Tess, Tatum, and Margot, the other three PTs in Vignette 11 never discussed negative relationships with mathematics. Ophelia, Maggie, and Millie discussed always having positive relationships with mathematics. When I asked Maggie in the interview why she described mathematics as her friend, she explained that she always enjoyed mathematics. She said, "It's my teachers. I've always enjoyed my math classes, but especially since junior year in high school. I really started enjoying math more. I always enjoy it. I would do math before any subject any day" (Maggie, Interview).

This vignette illuminated how specific pieces of the PTs' experiences with their mathematics teachers—both past and present—helped them have a positive relationship with mathematics. Even the four PTs who spoke of an initial negative relationship with mathematics (i.e., Tabby, Tess, Tatum, and Margot) claimed that specific teachers or specific teacher actions helped them understand mathematics more and therefore had a more positive relationship with mathematics. Teacher actions such as providing feedback and valuing student ideas helped these PTs form a positive relationship with mathematics. These teacher actions were different than what Odette and Mia reported in Vignette 11. Odette and Mia talked about an overall negative relationship with mathematics because they had teachers who did not value their ideas as students. However, relationships with mathematics are not an either/or situation, and Olga, Octavia, and Tina described a continually changing relationship with mathematics. In the following vignette, I will present PTs who had an overall roller-coaster relationship with mathematics.

Vignette 12. PTs with roller-coaster relationships with mathematics described interactions with teachers.

Olga: I've always had a love-hate relationship with mathematics. It really has to do with the type of teacher you have. I had this really good math teacher—it was my geometry teacher—and he made it fun. Then, my precalculus teacher, he just pushed me into it and it made me very fearful. I didn't like that at all. When math got harder, my teachers had a hard time explaining it. If my precalculus teacher doesn't get it, how the heck am I supposed to get it? But then I had a very knowledgeable professor in college which made it ten times more enjoyable.

Octavia: We [math and I] developed a love-hate relationship. First, I have to be comfortable with my teacher. My calculus teacher went really fast and didn't allow me to get it. He would just explain the same thing over and over. But then, when we were one-on-one it would be better, and I would be comfortable again, and proud that I figured something out.

Tina: We [math and I] have a rocky on/off relationship. I loved [course 1] because my professor was amazing and he explained everything super well and math started to make sense again. He went back to the basics. But now in [course 2], it's just not explained and I think he gets frustrated we don't know what's going on. Then, I get frustrated because he's frustrated.

First, in Olga, Odette, and Tina's descriptions they each stated directly that they had a back and forth, or roller-coaster, relationship with mathematics. In my initial round of coding, I coded some relationships as roller-coaster; then, in my notes, I wrote,

> I found a study that talks about a change in relationship from negative to positive or positive from negative. I need to go back through the passages I coded as roller-coaster to decide if they could be grouped in either a change from positive to negative or negative to positive or if in fact they are really a roller-coaster. (Analytical memo, February 20, 2019)

Consequently, in the next round of coding, I focused on the roller coaster relationships and decided that, in fact, that should be a separate code because the way the PTs described their relationship never seemed to be settled.

Tina, who had no *math-character* and a bag of tools conception of NOM, described her roller coaster relationship with mathematics,

> I have known math for all of my life, but we have a rocky and on/off relationship. When we were younger, we were in love. Our relationship was easy. But as time went on, our relationship got a lot more complicated. I felt like I didn't have time for math and all math wanted to do was stress me out and make me mad. I guess I could've tried harder, but I didn't care to. As I have gotten older, I am beginning to better understand the importance of our relationship. I need math to succeed in life and now, after years of knowing each other, I am willing to work on our relationship. I want to fall back in love with math. (Tina, Writing prompt)

Notice three specific phrases in this prompt—"I want to," "I am beginning to," and "I am willing to work"—that provide insight into her relationship with mathematics as non-static. That is, I interpreted these three phrases as having a dynamic quality meaning that her relationship with mathematics was changing, or she wanted it to change, and this was a roller-coaster. When Tina to elaborated on her prompt during the interview, she said mathematics was good in elementary school and rocky in middle and high school. Then, she started to enjoy it again in course 1. She explained that after taking course 1 she really hoped she would still continue to enjoy it, but that currently in course 2, she was going back to not enjoying mathematics. When I asked her why, she told me about her two teachers and the differences she felt between them. Like Tina, Olga and Octavia also talked about different experiences with teachers and how those experiences influenced their relationship with mathematics.

Overall, PTs described teachers as influential in forming their relationship with mathematics. First, in Vignette 10, I provided evidence from two PTs who had an overall negative relationship with mathematics, and the PTs described teachers who did not value them as students and learners of mathematics by dismissing their mathematical ideas. Second, in Vignette 11, I provided evidence from seven PTs who had an overall positive relationship with mathematics, and these seven PTs described teachers who did value them as students and learners of mathematics by providing meaningful feedback and devoting time to help the PTs understand the mathematics. Lastly, in Vignette 12, I provided evidence from three PTs who had roller-coaster relationships with mathematics, and these three PTs all described different teachers who made mathematics enjoyable or understandable or teachers who were frustrated and did not explain ideas.

Section Summary

For the subgroup of 13 PTs each of them discussed experiences with teachers—former and current—and how those experiences shaped their conceptions of NOM and relationships with mathematics. PTs whose conception of NOM aligned more with a bag of tools reported experiences with teachers who lectured versus PTs whose conception of NOM aligned more with a problem-driven dynamic discipline reported experiences where teachers provided them with opportunities to create and problem solve. Additionally, PTs reported that their experiences with teachers—former and current—influenced their overall relationship with mathematics. PTs with a positive relationship with mathematics reported experiences with teachers who valued their ideas and formed relationships with the PTs. Contrastingly, PTs with a negative relationship with mathematics reported experiences with teachers who did not value their ideas or themselves as individuals. PTs who described a roller-coaster relationship with mathematics reported experiences with teachers that were never consistent.

Note, that in the previous section, I did not include Olive in any of the vignettes.

In the subgroup of 13 PTs, Olive was the only one who was coded as having a neutral relationship with mathematics. Unlike Ellsworth and Bus (2000) who included neutral relationship with mathematics in with an overall positive relationship, the expert other and I kept these codes separate. We agreed the separation of neutral and positive is important in this study because it reflected neither a positive or negative relationship with mathematics but an overall all appreciation for mathematics. In my reflective notes, I wrote,

> We used a relationship code (i.e., positive, negative, roller-coaster, or change between) if it felt obvious and appropriate. But what if the PTs' writing and descriptions only felt appreciative? The expert other used the code *improves life,* where I used the code *appreciates math*. Together we decided these ideas represent a neutral relationship. There seems to be a distinction when the PTs talk about mathematics in school versus out of school. (Analytic memo, March, 25, 2019).

Once the expert other and I agreed to keep the code *neutral* as separate, we returned to the data to better understand the neutral code. For example, one of the characteristics of the Proposed Unified View states that *mathematics is useful and worthwhile.* The *neutral* code was used when a PT referred to the usefulness and applicability of mathematics as well as a general appreciation for the discipline. Overall, we found that the *neutral* code related more to the PTs' conceptions of NOM as a discipline as it was connected to the Proposed Unified View of NOM. This next section describes how the neutral code is connected to PTs' conceptions of NOM as they related to the Proposed Unified View of NOM.

Prospective Teachers Conceptions of the Nature of Mathematics and the Proposed Unified View of the Nature of Mathematics

I categorized PTs' conceptions of NOM according to Thompson's (1992) variations of conceptions. Specifically, based on the PTs' survey scores, definitions of mathematics, writing prompt, and interviews, I interpreted the various data points into one of the three conceptions—bag of tools, static-unified body of knowledge, or problem-driven dynamic discipline. I made claims about the PTs' overall conceptions of NOM, their relationships with NOM, and experiences that influenced their conceptions of NOM based on how they answered they survey questions, defined mathematics, and personified mathematics. Through the surveys and writing prompt the PTs implicitly reflected on their ideas regarding NOM. Through the interviews, PTs explicitly reflected on their ideas regarding NOM when confronted with their own definitions of mathematics and the Proposed Unified View of NOM. In the following sections, I will elaborate on PTs' explicit reflections of NOM. Additionally, I will further describe the *neutral* code and how the code pertained to the PTs' reflection on NOM.

As a reminder, I described the conceptual framework of this study—The Proposed Unified View of NOM (See Figure 6). The Proposed Unified View of NOM is intended to provide a list of the ideas and processes central to mathematics as put forth by the mathematics education community. The Proposed Unified View includes the following statements:

Mathematics involves exploration.

Mathematics involves multiple strategies.

Mathematical ideas are communicated and verified through proof/justification.

Mathematics requires justification of ideas to others.

Critique of mathematical ideas leads to refinement.

Structure and patterns are inherent in mathematics.

Mathematics uses multiple representations.

Mathematics is useful and worthwhile.

Anyone can be a learner of mathematics.

In the section that follows, I will present the major emergent theme from this study's analysis. Specifically, I will discuss how I encouraged PTs in this study to reflect on ideas central to the discipline of mathematics (i.e. NOM) through their definitions of mathematics from the MBI survey, the categorization from the *neutral* code, and their reflection on a Proposed Unified View of NOM. This reflection revealed that, overall, the PTs did not view their conceptions of NOM in the same way that they conceived of mathematics as a discipline.

Prospective Teachers Explicitly Reflect on their Definition of Mathematics

As part of the MBI survey, PTs were asked to answer the question, what is mathematics. Specifically, the prompt read, "Different people describe mathematics in different ways. How would you answer the question, what is mathematics?" In Table 18, I provide a list of the 13 named PTs and their responses to the question.

Table 18

PTs' Responses to What is Mathematics?

PT	What is mathematics?
Olive	Mathematics is all around us. Every step we take creates distance. Every time we get in a car to go to the store we are driving a speed limit. Every time we wake up late and rush to class we are fighting against time. Every time we budget our money, we are doing equations in our head. How much can I spend this week? Math is in everything we do.
Odette	Different ways to problem solve through equations and critical thinking.
Olga	I believe mathematics is number and letter based. Mathematics uses patterns, imaginary numbers, etc. in order to find a solution. Mathematics more times than none has a solution to every problem.
Ophelia	Mathematics is a way of solving different problems/situations through different equations.
Octavia	Mathematics is the process of using numbers, in a variety of forms, and when necessary to solve given problems.
Tabby	Mathematics is a system by which things are quantified using a variety of tools and methods to assess height, width, depth, angles, percentages, averages, etc.
Tess	It's the study of numbers and patterns. If there's one thing I've learned from math is that most times everything revolves around a pattern of some sort.
Tatum	Math is using formulas to solve for a definitive answer that cannot be challenged due to the laws of mathematics. The answers that you receive if they are calculated correctly will not be able to be challenged. Mathematics is used in our daily lives from algebra, geometry, or simple math.
Tina	The study of numbers and the relationship of those numbers with other numbers.
Mia	Mathematics is the study of numbers and the functions use to compute equations and solve problems.
Maggie	Mathematics involves numbers, equations, formulas, and the processes one takes to determine the answer to a problem.
Millie	Problem solving using numerical values. Not necessarily a procedure or equation, but any kind of problem solving.

(continued)

Table 18 continued

PT	What is mathematics?
Margot	Mathematics is the process of thinking abstractly and quantitatively about various mathematical ideas (like numbers, shapes, etc.) Math can vary from basic operations like addition and subtraction to thinking abstractly about shapes and planes.

I discussed various PTs' definitions of mathematics as they related to Thompson's (1992) three conceptions of mathematics. Specifically, I elaborated on the definitions that focused on mathematics as a static-unified body of knowledge to support my claim that overall, the majority of PTs in this study aligned with that view. In this section, I will focus on how the PTs elaborated on their definitions of mathematics in the interview. In the interviews, I first asked PTs, "What is mathematics?" After they answered, I then showed them a copy of their previous answer from the MBI survey. In most cases, the PTs' definitions were identical. Some changed slightly, but upon seeing their initial definition they added that they still agreed and that what they said in the interview was in addition to and not instead of their previous definition. Next, I asked PTs if they thought others would agree with their definitions, specifically mathematicians. I will focus this section on the PTs' responses to this question.

Vignette 13. PTs thought mathematicians would not agree with their definitions.

Olive: I think they would add a lot because they're more professional than I am. But, I think that they would agree that math is everywhere it is everything. Whether it's an amount of something or zero it's still number and it's still math.

Ophelia: But they'd probably add more and have a more advanced definition because they know more about math than I do.

Octavia: A mathematician might add fancy words. I think mine's just kind of dumbed down.

Tabby: Probably not because they would feel more comfortable with math. When you're comfortable with something you tend to view it in totally different ways. So, I would think that if you understood something on a more fundamental level you wouldn't necessarily need those systems you would almost sort of just intuitively know how you do it.

Tess: Probably not. In high school my precalculus teacher only let us do our math on colored pieces of paper because she said that math was an art form and that it needed to flow.

Tatum: I would say yes and no, because I think that you're always solving for like an exact answer, but I think a mathematician—they would say that, let's say I have two people and one of them is trying to challenge the other one and maybe they're using a different formula but I mean from what I've learned in math, there's formulas but there's different ways of solving it. So, I wouldn't think the mathematicians are necessarily challenging that answer, but the way that it's solved, the process, because the way the comprehend the math in it is different.

Tina: No. Theirs is probably just more specific and detailed.

Mia: Oh, I think they have a lot more detail about the ways that you can use math because I'm more vague. Also, because I don't really have as many real-world applications and stuff as they would.

Maggie: Probably not. I feel like people who are just solely for that study have a deeper definition of it and have a separate meaning than just a generalization. I don't think your average people who don't enjoy math would have that answer, but I'm not somebody who is really intellectually smart in math. So, I just feel like those who are, will have a deeper answer for it.

Millie: I think some people, but not all of them. But what I hear more of is people agreeing with what I've just said.

Margot: When I think of a mathematician I think of someone who's very smart and can solve any equation. So maybe not.

In each of these PT's responses to the question, "Do you think a mathematician would agree with your definition of mathematics?" there was a hesitation in their response. In Vignette 13, not one PT thought a mathematician would agree with their definition of mathematics. Each of the PTs agreed that a mathematician would likely add on to their definition in various ways. Olive said that mathematician would have a more advanced definition because a mathematician is more professional. Maggie implied that a mathematician is *intellectually smart* in mathematics, where she is not, and therefore would have a deeper definition of mathematics. Two PTs provided more detail when they described why a mathematician might not agree with their definition. Mia agreed a mathematician would add more detail because a mathematician knows more real-world applications. Tatum said that a mathematician would care more about the process of mathematics and not just finding an answer. Tess talked about the mathematician's mathematics as an art form that should flow.

I did not ask the PTs to define a mathematician or talk about what a mathematician does. However, this would have been an illuminating question, because PTs' use of words and phrases such as more formal, deeper definition, professional, and fancy suggested that the mathematics of mathematicians is in some way different than the mathematics of the PTs. However, standards documents suggest that the mathematics taught in school should be consistent with the mathematics done by mathematicians (see CBMS, 2012; NCTM, 2000). For example, NCTM (2000) stated that high school students should be able to present mathematical arguments acceptable to mathematicians and that even children in elementary school should learn the norms of logical deduction used by mathematicians. However, in this study, the PTs did not view their mathematics as the same as that of a mathematician. Often in the data, when the PTs discussed mathematics they referred to the everyday use of mathematics and not to the mathematics learned in school or mathematicians' mathematics. This idea related to the neutral relationship code the knowledgeable other and I added in our analysis.

Prospective Teachers Described Neutral Feelings Towards Mathematics

As previously described, the knowledgeable other and I did not include what we categorized as a neutral relationship with positive relationship with mathematics. We agreed that this added to Ellsworth and Bus' (2000) codes of positive, negative, roller-coaster, positive to negative, and negative to positive in a new and different way. In this study, we assigned a code of neutral to PTs' data sources when we agreed that the PT described neither a positive or negative relationship with mathematics, but instead an overall appreciation for mathematics. In the following vignette, I will present representative quotes from PTs' writing prompts as if they were in conversation with one another to illustrate the commonalities present within writing prompts coded with a neutral assignment.

Vignette 14. PTs described an overall neutral relationship with mathematics.

PT1.03: Although Math is not always nice, and we have not always gotten along, I can say that I am somewhat thankful for him. Math is someone that I need in order to do what I want to do in life. Without Math I couldn't make it through school or learn how to teach children for a living.

Olive: Math is like your mom. She's always there. Your mom asks you what you want for dinner and you tell her homemade lasagna. She has to collect her ingredients and follow your great grandmother's secret recipe, doubling the amounts to feed your large family. Throughout the whole process math is hiding in the kitchen waiting to be baked into a masterpiece. You wake up late and have twenty minutes to get ready and get to school. Your mom rushes out the door to start the car with you following and speeds down the highway to beat Math as she rings at the start of your first class. After school you have a football game and your mom is cheering you on from the stands. You are the quarterback and you throw into the air yards away, as your teammate rushes into the end zone. The announcers and crowd cheer as they repeat the final score. '27-13, Math wins.'

PT2.24: Math is used every day of your life whether you enjoy it or not. Math is an everyday concept.

PT3.14: I know that math is important to my understanding of the world around me. I appreciate Math for who it is.

PT3.02: I know now that I will always need math in my life and that they are not going anywhere.

In Vignette 14, the commonalities in theses PTs personifications of mathematics are in the way they described an importance of mathematics in life and the world. Each of the PTs represented in this vignette described neither a like or dislike for mathematics but instead a realization that mathematics is all around them and it would be pointless to deny that fact. I included Olive's entire writing prompt to illustrate that, overall, she described mathematics as helpful to ordinary, daily activities. Unlike PTs in previous vignettes (see Vignette 10, Vignette 11, or Vignette 12), Olive did not make any type of statement about mathematics as a discipline. When I interviewed Olive, I asked her to elaborate on what she was thinking about when she wrote her personification prompt. Olive said, "Math is all around us and your mom is always around whether you like know it or not. Yeah, your mom's always there and math is there." I began the interview with Olive asking her to elaborate on the question from the MBI survey, what is mathematics. In her elaboration she continued to focus on the idea that mathematics was everywhere. She said, "I think math is everywhere. Like this table—the volume of it or the width of it or the length of it. Or when you're driving down the road—the speed limit. It's everywhere around you. Everything deals with math. Everything is a quantity." Thus, in this study the neutral code represented PTs who described an overall appreciation of mathematics without expressing a like or dislike for mathematics. PTs who expressed a like or dislike for mathematics often times also commented on the usefulness or described an appreciation of mathematics.

Often in the analysis, the expert other and I noticed that the codes positive, negative, or roller-coaster relationship and neutral relationship often times came together. For example, Olga who described her overall relationship with mathematics as a roller-coaster, also described an appreciation for mathematics. In Figure 26, I provided a screenshot of Olgas' writing prompt which depicts the coding of both a roller-coaster relationship with mathematics, but also a neutral relationship with mathematics.

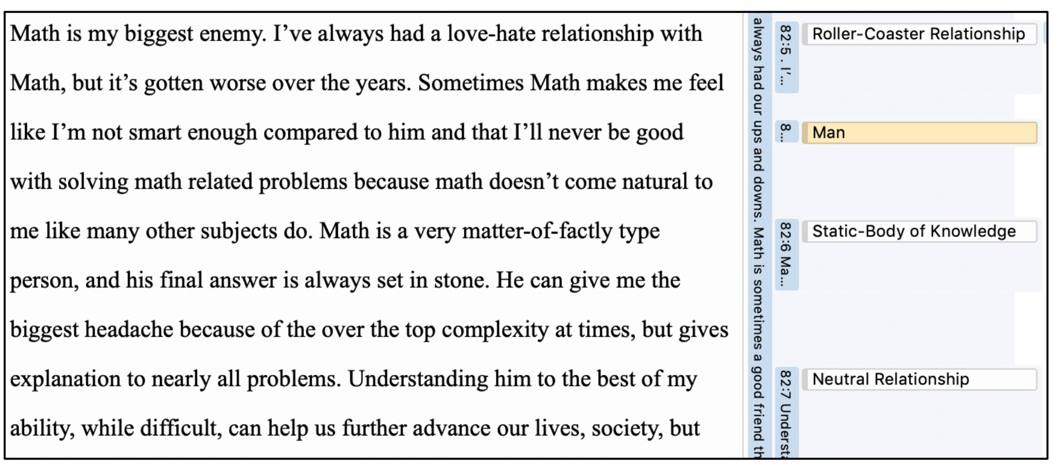
Math is my biggest enemy. I've always had a love-hate relationship with Math, but it's gotten worse over the years. Sometimes Math makes me feel like I'm not smart enough compared to him and that I'll never be good with solving math related problems because math doesn't come natural to me like many other subjects do. Math is a very matter-of-factly type person, and his final answer is always set in stone. He can give me the biggest headache because of the over the top complexity at times, but gives explanation to nearly all problems. Understanding him to the best of my ability, while difficult, can help us further advance our lives, society, but

Figure 26. Screenshot of Olga's writing prompt and subsequent codes.

Olga described her roller-coaster relationship with mathematics in her writing, and continued this sort of description in her interview. In contrast, at no point in the interview or writing prompts did Olive express a like or dislike for mathematics. This was different than Olive. Thus, throughout the coding process, there were two possibilities. One, PTs could have been assigned both a relationship code of positive, negative, roller-coaster a change and neutral code. If this was the case, like Olga, then the overall relationship was coded as positive, negative, roller-coaster, or a change. Two, PTs could have been assigned only a code of neutral because they never mentioned a like or dislike for mathematics, like Olive. If this was the case, then, like Olive the neutral code was used to describe the PTs' relationship. The other PTs presented in Vignette 14 with Olive did not describe a like or dislike for mathematics, and thus were coded as having a neutral relationship with mathematics.

PTs' descriptions of their appreciation towards mathematics, or a neutral relationship, provided insight into how the PTs perceived the usefulness of mathematics. Additionally, PTs' descriptions of the usefulness of mathematics in daily life align with one characteristics of the Proposed Unified View of NOM (See Figure 6) which states that, "Mathematics is useful and worthwhile." It was evident with the neutral relationship code that PTs attended to at least one aspect of mathematics as a discipline. However, their elaborations of their definitions of mathematics along with stating that a mathematician would likely not agree with their definition suggested that PTs' conceptions of mathematics was not aligned with a mathematician's mathematics or mathematics as a discipline. To more fully understand the implications of this finding, I turned to PTs' reflections on their conceptions of NOM when presented with the Proposed Unified view of mathematics. The findings of this analysis are presented in the next section.

Prospective Teachers Reflected on the Proposed Unified View of NOM

AMTE (2017) expressed that “for quite some time, professional organizations have called for opportunities for candidates to develop deep understandings and mathematical perspectives on the nature of mathematics as a discipline” (p. 89) in a practical and crucial way for teaching. In Chapter Two, I suggested a framework for NOM, the Proposed Unified View (see Figure 6). In this framework, I presented nine statements, which represent mathematics as a discipline based on two foundational standards documents for mathematics education (NCTM, 2000; NRC, 2001). I argued that these nine statements, regardless of the context in which one is doing mathematics—as a teacher, as a mathematician, or as a student—do not change and are representative of how the field of mathematics education conceives of NOM. After analyzing the data sources in this study, I noticed that, at times, PTs seemed to be considering ideas of NOM as defined by the Proposed Unified View. For example, PTs discussed the usefulness and worthwhileness of mathematics without prompting in an interview. That is, PTs often described the usefulness of mathematics, and I coded this as a neutral relationship (see Vignette 14). However, I also noticed that PTs seemed to view the mathematics they learned as different from a mathematician’s mathematics (see Vignette 13).

Since the Proposed Unified View of NOM is intended to be a list of characteristics that are inherent to the discipline of mathematics that have the potential to be agreed upon by all people, including mathematicians, I asked PTs to reflect on the Proposed Unified View of NOM in the interview. Therefore, in the forthcoming vignette, I will provide representative quotes from the PTs’ interviews when they were asked to reflect on the statements which make up the Proposed Unified View of NOM.

Vignette 15. PTs reflected on the Proposed Unified View of NOM.

Olive: Oh yeah. I agree with that, because it kind of covers everything.

Odette: This would be a pretty decent set of what math is.

Tabby: I actually really like that it. It sort of hits on everything.

Maggie: So, I think I have to agree with all of those. I think that is a very good list of characteristics of math.

Octavia: I see all of these a lot in my teacher now, and I'm relating a lot of this to [course 1], which is taught by somebody who knows math and how to teach math.

Millie: I agree with problem-solving. I like the multiple strategies. I really agree with personal relationships, because I find that a lot of kids aren't motivated, but I think creating your personal relationship with math makes you more motivated and it's also an important part of discovery.

Tina: Just thinking about different teachers and professors that I've had, I think it would be nice for everyone to know these. For teachers, where it says mathematics involves multiple strategies—I know when I was younger it was "my way or the highway." The other thing I like is math involves exploration and anyone can be a learner of math. I think like these are good things for people who teach math to consider.

In this Vignette, the PTs echoed each other and said they agreed with the characteristics of the Proposed Unified View of NOM. It seemed that, regardless of the

PT's conception of NOM, they expressed agreement with the characteristics present in the Proposed Unified View of NOM. For example, Odette whom I categorized as holding a conception of NOM most aligned with a bag of tools, stated that she agreed with the Proposed Unified View of NOM. As Odette reflected on the Proposed Unified View, she seemed to be really considering each individual statement, what that meant for mathematics, and how it reconciled with her own conceptions of mathematics. She said,

> That last one got me, because really anyone can be a learner of mathematics if they focus. Mathematics definitely requires justification of ideas by others! I would sum up math to be these nine things. But the problem is, there are some maths that even if you have a proof there are actually two answers, and that bothers me. Facts are facts. (Odette, Interview)

As Odette reflected on the ideas, she stated that she agreed with them all, but she also struggled to believe that some of the statements could actually be true of mathematics. When she stated that she was bothered by the potential of two answers, she was thinking about the statement *Mathematical ideas are communicated and verified through proof/justification*. She explained that a proof to her is when you check your answer, and in that process of checking the answer should be correct and thus have only one answer. Odette asked clarifying questions and we discussed each individual statement one by one. After this, she agreed, "Yeah, this would be a pretty decent set of what math is."

Octavia, whom I classified as having a conception of NOM most aligned with NOM as a static-unified body of knowledge, also expressed an overall agreement with the statements in the Proposed Unified View of NOM. As Octavia reflected on the nine statements, she related them back to teaching. She explained,

> So, the first part—the exploration and solving and multiple strategies—we look at a lot of real-life student work in my [course 1] from real elementary schools, and there is indeed multiple strategies that they could get. There was only one that I saw that had the wrong answer, but the rest of them had the right answer even though none of the work or the numbers that they showed or the crossing out—none of it looked the same. (Octavia, Interview)

Until she was provided with authentic student work as a class assignment, she had never considered that elementary students could invent their own ways of solving problems and that those would be considered valuable and productive in the learning and understanding of mathematical ideas. Reviewing the nine statement in the Proposed Unified View reminded her of that task, and she realized that the Proposed Unified View was applicable to her as a future teacher. Octavia also reconciled the Proposed Unified View against her own conceptions as a mathematics student. She discussed how she also agreed with the statement that *Mathematics uses multiple representations,* by explaining an example of how she used base-ten blocks to represent the number 14, but that another representation of 14 would be 28 halves. Octavia expressed the value in the Proposed Unified View of NOM both for teachers of mathematics and students of mathematics.

Like Odette and Octavia, Maggie also expressed agreement with the nine statements that make up the Proposed Unified View of NOM. I classified Maggie as holding a conception that most aligned with NOM as a problem-driven dynamic discipline. When reflecting on the Proposed Unified View of NOM, Maggie elaborated on two of the nine statements. First, she pointed out that she wholeheartedly agreed with

Anyone can be a learner of mathematics and *Mathematics is useful and worthwhile.* She explained,

> I think . . . it's just some people put a negative view on stuff like math and science, because it doesn't come as easy. It involves a multiple steps and variables and numbers. It intimidates a lot of people. But anyone can be a learner of mathematics, they just have to flip that switch in their mind and say math is good. (Maggie, Interview)

When I pushed Maggie to continue to reflect on the nine statements, she referred back to her definition of mathematics. Maggie defined mathematics as any type of problem solving. When she reflected on the Proposed Unified View, she explained that she believed the nine statements reflected her definition to some extent. She said, in problem solving, reasoning and proof are inherent to problem solving. As the other PTs reflected on the Proposed Unified View of NOM, they all agreed with the nine statements.

Section Summary

PTs naturally included aspects of NOM in their descriptions of mathematics without being explicitly asked to do so. For example, in the writing prompts, many PTs mentioned the usefulness of mathematics and I coded this as a neutral relationship with mathematics. However, when I asked PTs to reflect on their definitions of mathematics, many stated that a mathematician would not agree with them, suggesting that they believed their mathematics was somehow different than the mathematician's mathematics. When the PTs were asked to explicitly reflect on the characteristics inherent to NOM through the Proposed Unified View, even those PTs with different conceptions of NOM agreed that the Proposed Unified View represented ideas that they all found valuable in mathematics.

Chapter Summary

In this chapter, I presented the results from this explanatory phenomenological study by describing how PTs' conceptions of NOM, understanding the experiences that influenced their conceptions of NOM, and making connections between the PTs conceptions of NOM and the Proposed Unified View of NOM. The results indicated that in this study, overall, PTs held a conception of NOM most consistent with NOM as a static-unified body of knowledge. Furthermore, the PTs *math-characters* were classified into three main categories—friendly, familial, or antagonistic—and PTs used specific adjectives to describe the characters and elaborate on the conceptions of NOM. The PTs use of the different *math-character* aligned with different conceptions of NOM. Also, the PTs' *math-characters* helped illuminate their relationships with mathematics. For example, the PTs who described mathematics as their friend, also described an overall positive relationship with mathematics. PTs who described mathematics as a family member, described their overall relationship with mathematics to be either neutral or a roller-coaster. Last, PTs who described mathematics as an antagonistic character described either a negative, roller-coaster, or negative to positive relationship with mathematics. In regards to PTs experiences with mathematics, PTs most often credited experiences with teachers as most influential in forming their conceptions of NOM and relationships with mathematics. Lastly, PTs discussed experiences as mathematics students and future mathematics teachers that aligned with the Proposed Unified View of NOM. When the PTs explicitly reflected on NOM, PTs agreed with the characteristics of the Proposed Unified View of NOM. This finding was surprising given the fact that PTs often expressed that their mathematics was different than a mathematician's mathematics. I will share a summary and discussion of results from this study in the next chapter.

CHAPTER V: SUMMARY AND DISCUSSION

Introduction

Despite a call in mathematics education for students, prospective teachers, and practicing teachers to develop a perspective on the nature of mathematics as a discipline (NOM) over the last two decades, there is much to be accomplished (AMTE, 2017; CBMS, 2012; NCTM, 2000, 2014). Extensive research bases suggest that underlying teachers' understandings of the mathematics they teach are their conceptions of NOM (Ernest, 1989; Jankvist, 2015; Lerman, 1990), that the teachers' views of school mathematics and mathematics as a discipline were often in contrast to each other (Beswick, 2012; Garegae, 2016), and that PTs must be provided opportunities to reflect on mathematics as a discipline so that the PTs understand their own conceptions and can consider alternative conceptions of NOM (Bolden et al., 2010; Sweeny et al., 2018; Szydlik, 2013; Zazkis, 2015). However, regarding PTs, little empirical investigation has been conducted in order to address PTs underlying conceptions of NOM and how those conceptions were formed through experiences with mathematics. Providing PTs opportunities to reflect on their conceptions of NOM is a significant and necessary step in improving mathematics education and breaking the back-and-forth cycle of reform that has previously occurred throughout mathematics education history (Beswick, 2012; Conner, Edenfield, Gleason, & Ersoz, 2011; Gold, 2011; White-Fredette, 2010).

The purpose of this study was to describe PTs' conceptions of NOM, understand the experiences that influenced those conceptions, and connect PTs' conceptions of NOM to the Proposed Unified View of NOM by answering the following three research questions:

1. What are elementary prospective teachers' conceptions of the nature of mathematics?
2. How do the lived experiences of the elementary prospective teachers inform their conceptions of the nature of mathematics?
3. What are the connections, if any, to prospective teachers' conceptions of the nature of mathematics and the Proposed Unified View, and what are the implications of those connections, if any?

As an aid to the reader, the final chapter contains a restatement of the research problem, a review of methodology utilized in the study, and a summary of the results of the study. This review will be followed by a discussion of the results of the study, which will include its connections to prior research, theoretical and practical implications, and recommendations for future research.

The Research Problem

Teachers struggle to effectively implement reformed-based practices, and classroom observations often reveal a heavy reliance on instrumentalist conceptions of NOM (Ball, Lubienski, & Mewborn, 2001; Dossey, 1992; Stigler & Hiebert, 1999). Thus, teachers do not have the opportunity to consider a conception of NOM aligned with Thompson's (1992) problem-driven dynamic discipline. Providing teachers with opportunities to consider the different views of NOM is vital for bringing about reform in mathematics education, because "mathematics success for all cannot come about without radical change in instructional practices and an equally radical change in teachers' views of mathematics teaching and learning, as well as the discipline of mathematics itself" (White-Fredette, 2010, p. 21). However, because teachers are often unaware of their conceptions of NOM (Beswick, 2012; White-Fredette, 2010), and content and pedagogy are taught in separate courses, a disconnect is fostered among teachers, the discipline of mathematics, and the actual work of teaching (Grossman et al., 2009). Because a teacher's conception of NOM influences their instructional practices, it is,

> Through reflection, [that] teachers learn new ways to make sense of what they observe, enabling them to see differently those things they had been seeing while developing the ability to see things previously unnoticed. While teachers are learning to see differently, they challenge their existing beliefs. (Philipp, 2007, p. 281)

Teachers must be provided opportunities to challenge their current conceptions of NOM and reflect on the influence of those conceptions. That is, in order to effectively begin the implementation of reform-based instruction, teachers need to first understand their own conceptions of NOM, consider alternative conceptions, experience dissonance as their conceptions are challenged, and then have the opportunity to restructure their understandings of NOM and the impact those conceptions have on their teaching.

Underlying teachers' conceptions of NOM is the understanding of the mathematical content. Elementary PTs make up a special group of PTs because they come to their teacher preparation programs assuming they already know the simple, fundamental mathematical content that is the foundation of elementary school mathematics (Ambrose, 2004; Ball, 1990; Richardson, 1996; Weinstein, 1989). This assumption can often lead elementary PTs to underestimate the complexities required to teach (Ambrose, 2004). Additionally, the way in which elementary PTs remember their own experiences from school also shape how they will teach in their future classrooms

(Lortie, 1975; Shulman, 1986; Stigler & Hiebert, 1999). Elementary PTs' experiences as mathematics learners form their conceptions of mathematics as a discipline which then inform their models of the teaching and learning of mathematics (Ernest, 1991). It is the underlying conceptions of NOM which help form these assumptions, and thus interfere with implementation of reform-based mathematics instruction (CBMS, 2012). Moreover, focusing in on elementary PTs is important due to the cognitive foundation that is developed in elementary students through their learning of mathematics. Students' experiences in elementary school provide a foundation for their future mathematical proficiency, and those foundations and dispositions developed later on are often informed by their elementary school teachers (NRC, 2001; 2007; 2015). Consequently, examining and understanding elementary PTs' conceptions of NOM is one avenue to attempt to understand how those conceptions may eventually influence their instructional practices. I developed an explanatory phenomenological design to address these concerns.

Review of Methodology

I utilized an explanatory, phenomenological design to describe and understand PTs' conceptions of NOM, consider the mathematical experiences of the PTs that influenced those conceptions, and examine the connections between the PTs' conceptions of NOM and the Proposed Unified View of NOM proposed in Chapter Two. One hundred and thirty elementary PTs enrolled in either a mathematics content or methods course in their teacher preparation program at a public southeastern university were selected for this study. Multiple sources of data including one Likert-scale survey, a semantic differential survey, a writing prompt, interviews, and a researcher's reflective journal were used to develop a rich, detailed description of elementary PTs conceptions of NOM, experiences with NOM, and connection to the Proposed Unified View of NOM. Through iterations of both inductive and deductive coding, the data generated emergent themes representing overall PTs' conceptions of and experiences with NOM, as well as descriptions of and experiences of a subgroup of PTs selected for the interview. Results of the full study are provided in the previous chapter and are summarized in the next section.

Summary of Results

I presented the results in Chapter Four of this study as emergent themes that resulted from analysis considering each research question. First, regarding research question one, what are elementary PTs' conceptions of NOM, the two emergent themes were that (1) overall, PTs described conceptions of NOM as most closely aligned with a static-unified body of knowledge and (2) overall, PTs described either a friendly, familial, or antagonistic relationship in their personifications of mathematics which provided additional insight to their conceptions of NOM. Second, regarding research question two, how do the lived experiences of elementary PTs inform their conceptions of mathematics, one theme emerged, that teachers were the most influential in forming PTs' conceptions of and relationships with mathematics. Last, regarding research question three, What are the connections, if any, of elementary PTs' conceptions of NOM to the Proposed Unified View of NOM, the two emergent themes were that (1) explicit reflection revealed PTs did not consider their mathematics to be the same as a mathematician's mathematics, but they agreed that the statements included in the Proposed Unified View represented a comprehensive view of mathematics and (2) without explicit reflection on NOM, PTs unknowingly discussed connections to the

Proposed Unified View. Brief summaries of the results from each of these themes constitute the remainder of this section.

Prospective Teachers' Conceptions of Nature of Mathematics

I will answer research question one, what are elementary prospective teachers' conceptions of the nature of mathematics. Overall, PTs in this study were found to have conceptions of NOM that closely aligned with NOM as a static-unified body of knowledge. Analysis of the MBI (n = 108, m = 4.23, sd = 4.32) revealed that PTs' conceptions of NOM are most closely aligned with Thompson's (1992) static-body of knowledge conception. There was a statistically significant difference in the means of PTs' MBI scores when grouped by course, $F(2, 105) = 4.66$, $p = .011$. Post-hoc analysis revealed differences between MBI scores of PTs enrolled in course 1 and course 3 (p = .009) with a mean difference of 3.02. Other differences between groups were not statistically significant.

Though there was a slight increase in mean of PTs' MBI scores enrolled in different courses, the differences were not statistically significant. An item analysis of individual statements from the MBI also showed that PTs often agreed or strongly agreed with the statements associated with NOM as a static-unified body of knowledge. However, the item analysis also showed that PTs agreed with ideas associated with NOM as a bag of tools and a dynamic problem-drive discipline. Thus, the results from the Semantic Differential aided confirming the results from the MBI and helping to answer research question one.

Overall analysis of the Semantic Differential (n = 123, m = 66.76, SD = 8.75) also revealed that PTs' conceptions of NOM were most closely aligned with NOM as a static-unified body of knowledge. Like the MBI, there was a slight increase in PTs' average scores across the three courses, and these differences were approaching statistical significance, $F(2, 105) = 3.04$, $p = .052$. Post-hoc analysis revealed differences between Semantic Differential scores of PTs enrolled in course 1 and course 3 ($p = .047$) with a mean difference of 5.057. Other differences between groups were not statistically significant. Further analysis of individual words on Semantic Differential also revealed that by question the majority of students often agreed or strongly agreed with the statements most associated with NOM as a static-unified body of knowledge. However, PTs also rated words associated with NOM as a bag of tools or a dynamic-problem driven discipline as important to being good at mathematics as well. Thus, I discussed results from the MBI open-ended question, what is mathematics, as a way to provide detail of the results from the MBI and Semantic Differential.

In their definitions of mathematics PTs attended to two characteristics of NOM as a static-unified body of knowledge: logic and discovery. PTs defined mathematics as logical reasoning, thinking logically, and systematic. PTs attended to the discovery aspect of NOM in their definitions by using phrases such as find a solution. This idea of finding a solution was coded as static-unified body of knowledge and not bag of tools. PTs who defined mathematics as a bag of tools did not use the phrase find a solution, but instead used phrases such as "complete calculations" or "solve a problem." A hierarchical cluster revealed that these words—calculate/solve and find/solutions—were distinct clusters indicating that in this study PTs used them to mean something different. This analysis helped categorize the finding solution language as static-unified body of knowledge instead of bag of tools. PTs continued to attend to their conceptions of NOM through their statements in their writing prompts. Specifically, in their writing prompts, PTs attended to NOM as logical and discoverable. In the writing prompts PTs described their math-character as a person who shares information, never changes, or has hidden messages. These phrases alluded to the discoverability aspect of NOM as a static-unified body of knowledge. PTs also used phrases to describe their *math-character* such as she's more organized, he's strategic, and Math is logical, all of which described the logical aspect of NOM as a static-body of knowledge. So, based on overall quantitative results from the MBI and SD as well as a qualitative analysis of the open-ended MBI question and PTs' writing prompts, PTs' conceptions of NOM aligned most closely with NOM as a static-unified body of knowledge.

Prospective Teachers' Experiences with the Nature of Mathematics

The subgroup of 13 PTs each discussed experiences with teachers—former and current—and how those experiences shaped their conceptions of NOM and relationships with mathematics. PTs whose conception of NOM aligned more with a bag of tools reported experiences with teachers who lectured versus PTs whose conception of NOM aligned more with a problem-driven dynamic discipline reported experiences where teachers provided them with opportunities to create and problem solve. Additionally, PTs reported that their experiences with teachers—former and current—influenced their overall relationship with mathematics. PTs with a positive relationship with mathematics reported experiences teachers who valued their ideas and formed relationships with the PTs. Contrastingly, PTs with a negative relationship with mathematics reported experiences with teachers who did not value their ideas or themselves as individuals. PTs who described a roller-coaster relationship with mathematics reported experiences with teachers that were never consistent.

Prospective Teachers' Conceptions of the Nature of Mathematics and Connections with the Proposed Unified View of the Nature of Mathematics

PTs naturally included aspects of NOM in their descriptions of mathematics without being explicitly asked to do so. For example, in the writing prompts, many PTs mentioned the usefulness of mathematics and I coded this as a neutral relationship with mathematics. However, when I asked PTs to reflect on their definitions of mathematics, many stated that a mathematician would not agree with them, suggesting that they believed their mathematics was somehow different than the mathematician's mathematics. The PTs often stated that mathematicians' math was somehow different from their own, not just in the language they used to describe the mathematics, but in the way to PTs described mathematics in general. That is, the PTs often referred to the usefulness of mathematics for daily activities. However, when the PTs were asked to explicitly reflect on the characteristics inherent in NOM through the Proposed Unified View, all PTs agreed with the characteristics in the Proposed Unified View of NOM, regardless of their described conception of NOM.

Discussion of Results

The results of this study provided insights and contributions to the mathematics education community. First, I will share how the results are connected to the literature through the exiting theories regarding PTs' beliefs about mathematics and relationships with mathematics. Second, I will share the theoretical implication of the suggested model for the Proposed Unified View of NOM and implications this has for teacher education as well as the field of mathematics education. Last, I will share the practical implications by presenting the use of the Semantic Differential and MBI surveys as reflective models to use for mathematics students, PTs, and inservice teachers and the importance of using these surveys as reflective tools.

Connections to Prior Research

The results of this study connect to prior research in two important ways. First, they offer a rich description of PTs' conceptions of NOM, relationship with mathematics, and the experiences that influenced those conceptions as called for in recent literature examining PTs' beliefs regarding NOM (Beswick, 2012; Di Martino & Zan, 2010; Sigley et al., 2019; White-Fredette, 2010). Second, the results reinforce the associations presented in Jankvist's (2015) model (See Figure 4) connecting the student's image about mathematics education, themselves as learners of mathematics, and the social context of mathematics as the relate to NOM and the importance of reflecting on their own conceptions.

Prospective Teachers' Conceptions of NOM. In this study, overall, PTs' conceptions of mathematics (i.e. NOM) most closely aligned with Thompson's (1992) idea that mathematics is a static-unified body of knowledge that is logical, has meaning, and is discovered not created. As discussed in Chapter Two, previous empirical research reported that PTs most often described mathematics as rote practices (i.e. a bag of tools) and not creative or beautiful (Bolden et al., 2010; Sweeny et al., 2018). Only one study reported that PTs did hold a belief that mathematics was a logical domain (Chamberlin, 2013). Additionally, Sigley and colleagues (2019) reported PTs' relationships with mathematics changed after they were immersed in a mathematics class that focused on reform-based instruction. This was evident in the study when PTs described their experiences with mathematics. Often, the PTs who have overall negative relationship with mathematics experienced teachers who focused on non reform-based instruction as opposed to the PTs who had positive relationships with mathematics and described experiences with teachers that valued their own mathematical ideas as a way for progress the learning forward and think deeply about those mathematical ideas. However, a PTs' described relationship with mathematics did not necessarily have a connection with their conception of NOM. Overall in this study, PTs described conceptions of NOM that were most aligned with Thompson's (1992) conception of NOM as a static-unified body of knowledge. This was evident by the scores on the MBI and SD survey as well as in their written definitions and descriptions of mathematics. This was different than reported in most literature regarding PTs conceptions of mathematics. Furthermore, PTs who held a conception of NOM most closely aligned with a static-unified body of knowledge reported relationships as changing from negative to positive, roller-coaster, neutral, and negative. The same was true of the PTs whose conception of NOM most aligned with a bag of tools, they reported relationships ranging from positive to negative and in between. This suggest the complicated nature of a PTs conception of NOM and relationship with mathematics. Of the 13 PTs I interviewed, it was encouraging to see that only 2 of the 13 were classified as having a negative view of mathematics, because often students who described a negative relationship with mathematics also described a conception of NOM most closely aligned with a bag of tools.

Prospective Teachers' Reflection on NOM. Jankvist (2015) explained that the conception a PT holds or a relationship the PT describes was not necessarily the most important aspect to consider with regards to NOM. Instead, Jankvist explained that providing the PTs with an opportunity to reflect on their own conceptions and reconsider their own conceptions if they experienced a conflict or contradictions was vital to bringing awareness to PTs conceptions of NOM. This study provided PTs with an opportunity to explicitly reflect on the ideas inherent to the nature of mathematics by providing statements on surveys where PTs had to agree or disagree, eliciting personification of mathematics as a character, and asking questions in the interview that focused on why they rated certain survey items and experiences they had as mathematics students. In this study, I made it very clear to the PTs who participated that there was no right or wrong answer, and that the study was meant to describe and understand their own conceptions and understanding of mathematics to help the field of mathematics education strengthen ideas in the field (i.e. NOM) and better understand how PTs conceptions came to be since they would be future mathematics educators.

Theoretical Implications

The results in this study are significant for the theoretical contributions to the field of mathematics education in two ways. First the results add to the body of literature regarding NOM by proposing the model for a Proposed Unified View of NOM and implications this has for the field of mathematics education. Secondly, in this study, I utilized the Semantic Differential which is still being developed and refined by Sweeny and colleagues (2018). Therefore, the use of this survey in this study provided important validity aspects for the creators of the survey to consider and build upon.

Proposed Unified View. There has been a call in mathematics education to have all learners of mathematics consider and deeply understand NOM (AMTE, 2017; White-

Fredette, 2010). However, few documents have proposed an actual definition of NOM for consideration. One author (Pair, 2017) proposed the IDEA Framework for the nature of pure mathematics. In Chapter Two, I incorporated aspects of Pair's IDEA framework as well as two foundational standards documents to create a list of characteristics of NOM in general and not just the nature of pure mathematics. That is, the Proposed Unified View of NOM (See Figure 6) is intended to be a list of characteristics that no matter the purpose one has as a doer of mathematics, the Proposed Unified View will be applicable. Pair expounded on a potential use of the IDEA framework by explaining that,

> "I believe students may benefit if we structure pure mathematics classrooms so that the mathematical ideas are at the heart of students' work and class discussions. Students should understand mathematics is an exploration of ideas. Students should develop confidence in creating and sharing their own personal ideas (even though their ideas will be subject to criticism). Students should understand that their ideas will be forever refined as long as they continue to study mathematics" (p. 191).

I proposed an expansion of Pair's (2017) ideas of pure mathematics and propose that not only should students in pure mathematics classes consider these ideas, but so should also PTs throughout their teacher preparation programs, MTEs as they prepare to teach future teachers, and current teachers at the K-12 level. If school mathematics is intended to promote the ideas of mathematics as a discipline (NCTM, 2000), then anyone involved in the teaching of mathematics needs to consider the ideas inherent to mathematics as a discipline (i.e., NOM). For example, a Proposed Unified View of NOM incorporated the idea of exploration as a key characteristic of mathematics, and it is important for doers of mathematics to explore and not simply apply a memorized fact or formula. Exploration is a key aspect for pure mathematicians, students of mathematics, PTs, and MTEs. Also, the idea that mathematics is a dynamic discipline is present in the Proposed Unified View with the characteristic that critique of ideas leads to refinement. This statement suggests that mathematics can change as ideas are continually refined. This is important for doers of mathematics to understand about NOM so that they do not have a conception of NOM and unchanging. The Proposed Unified View of NOM presented in Chapter Two is a starting framework for all involved in mathematics education or mathematics to consider, critique, and expand on with the intent of refining a list that will encompass the true ideas of mathematics as a discipline. In this study, PTs found value in the Proposed Unified View, regardless of where their ideas fell on the NOM Continuum. This provides evidence that the Proposed Unified View actually has potential to be a consensus view for NOM. Continuing to refine a list of characteristics of NOM is useful to the field of mathematics education to help promote a new vision of mathematics. That is, instead of implying ideas about mathematics in standards documents, Proposed Unified View of NOM offers explicit ideas about NOM for each and every person to consider and reflect upon.

Prospective Teachers' Understandings of Terms. At the time of this study, it was clear that PTs had alternative understandings of the terms in standards documents. For example, PTs explanations of the paired words *applying* and *creating* during the interviews, illuminated some alternative interpretations of the words. The PTs seemed to define applying in the same way an MTE would define creating. PTs struggled with the term creating in mathematics, because they believed one needed some mathematical knowledge in order to problem solve and therefore one could not simply create ideas. Often times their use of the word create was synonymous with the idea of making up knowledge, and this frustrated them as they thought mathematics was based on at least some prior knowledge. One PT explained why she chose closer to applying rather than creating, and she said it was because a person cannot simply make up mathematical knowledge, but that one must slowly gain knowledge and then it is how that person applies that knowledge that is important to mathematics. When I asked her to elaborate on what she meant, she explained that she viewed applying as a way to creatively solve problems, and creating as making up nonsense (Olga, Interview). Here, Olga actually interpreted the term applying as having a conceptual idea behind it and being able to thoughtfully think about how to solve a problem rather than just applying a rule or procedure. Her definition of applying was closer to the intended definition of creating. PTs' lack of understanding of words essential to defining NOM is important to consider in order to reach the vision set out by standards documents for each and every student to learn mathematics as a discipline (AMTE, 2017; NCTM, 2000; NRC, 2001). PTs' alternative definitions of the terms associated with standards documents indicated that PTs did not interpret the important terms in mathematics education as they are intended. Thus, how can PTs sincerely and deeply reflect on the ideas inherent in mathematics if their interpretation of basic terms used to describe mathematics in the standards documents (i.e. NRC, 2001) are misaligned?

A secondary implication of PTs' alternate understandings of words commonly found in standards documents is the influence on the use of the Semantic Differential. I shared survey results of the PTs in this study with the creators of the Semantic

Differential to help the creators to continue to revise and improve their survey. Sharing the results in this study allowed the creators to analyze findings of a larger sample size of PTs as well as analyze trends in how the PTs answered the individual survey items. This will help with the validity of the instrument by providing the creators with a larger sample size when considering the results of the survey. Furthermore, since PTs often had alternative understandings of terms in standards documents, PTs may not be interpreting the terms in the Semantic Differential in the same way the creators intended. Since the Semantic Differential was created based on the ideas present in NRC's (2001) strands of mathematical proficiency, the ideas inherent in that document are what researchers and MTEs intended to convey with the use of the words. Thus, the use of the survey illuminated that, at times, PTs' interpretations of the words were often misaligned with the intent of the creators.

Practical Implications

With a growing body of literature surrounding ideas that create the nature of mathematics, the results of this study have at least one practical implication regarding opportunities to reflect on the nature of mathematics. Jankvist (2015) claimed that simply categorizing one's conception of NOM was not as important as providing opportunities for that person to reflect on their own understandings of NOM, specifically when a contradiction or conflict arose. In Chapter Four, I discussed a categorization of individual statements on both the MBI and Semantic Differential surveys into Thompson's (1992) three conceptions of NOM as a way to further understand PTs' conceptions of NOM. The expert other and I agreed on the sorting of the MBI statements, and after discussion, one creator of the Semantic Differential and I also agreed on the sorting of the words from the Semantic Differential. Beyond the use of these two instruments to understand and interpret someone else's conceptions of NOM (i.e. how I used the instruments in this study), I propose that the sorting exercise itself would be a nice reflective tool for PTs, math teacher educators, teachers, and students to deeply consider the ideas of mathematics as a discipline. Attention to explicit tasks and conversations need to occur in teacher education and this is one way. First, these two instruments provide statements that one must genuinely consider in order to sort them using Thompson's (1992) three conceptions. For example, the statement, "*mathematics is mostly a body of facts and procedures"* (MBI statement) would allow for a discussion to occur on how to place that statement if a person agree, disagrees, or says they are neutral-then what might that imply about NOM. Second, if done with a partner or group, the sorting activity would allow for discussion contemplation when the partners or groups do not agree on sorting. For example, if PTs were sorting the words *apply* and *creating* into Thompson's categories, they would have to define each of those words. There might be alternative definitions, as was evident in this study when interviewing the PTs, and this conflict or contradiction to one's own understanding of the word would provide an opportunity for the two PTs to discuss the differences to settle on a categorization of the words. Furthermore, if used as a sorting activity for PTs, then the instructor of the course can include a change for whole class discussion where they can elaborate on the MBI statements or words on the Semantic Differential in a way that is consistent with ideas of mathematics as a discipline, thus providing opportunities for PTs to explicitly reflect on NOM. Providing PTs with opportunities to explicitly reflect on NOM will allow PTs to consider conceptions of NOM that may not always align with their own conceptions and therefore cause some cognitive dissonance when considering characteristics inherent to NOM. Creating a cognitive dissonance in PTs regarding their conceptions of NOM is the first step in helping PTs understand the dynamic problem-driven nature of mathematics.

Recommendations for Future Research

This study included 130 personification prompts written by elementary PTs enrolled in either a mathematics content or mathematics methods course. My analysis of the writing prompts focused on answering my research questions in this study. Furthermore, my theoretical perspective as a researcher narrowed my lens when analyzing data. As the principal investigator in this study, I focused on data that helped answer the research questions proposed in this study about how PTs described their conceptions of NOM, how the lived experiences of the PTs influenced their conceptions of NOM, and the connections to a Proposed Unified View of NOM. Therefore, my own theoretical perspective, my selection of an analytical framework, and how I chose to categorize the data influenced the results of this study. Although this study provided tentative answers to research questions, it also leaves many questions to be addressed. First, I will describe interesting ideas that surfaced in the analysis of the data, but that were not common enough among all the PTs and thus did not help answer the research questions in this study. Then, I will discuss three specific areas of exploration that should continue based on findings from analysis in this study that were interesting, but did not answer the research questions specifically and that I believe to be of utmost importance for the field of mathematics education: exploring PTs' characteristics and pronouns used to describe mathematics, further examination and study of the characteristics which make up the nature of mathematics, and the equity issues associated with proposing a framework to describe mathematics.

General Questions. While reading, rereading, and analyzing the various data sources in this study, I personally reflected on some of the nuanced ideas that the PTs seemed to discuss in their survey, writing prompts, or interviews. These smaller ideas did not meaningfully provide insight into my research questions for this study, but I believe provide insight into future directions for exploring PTs' conceptions of NOM and experiences that influenced and related to those conceptions of NOM. In this section, I will discuss the questions I asked myself through analysis and synthesis of the data in this study and explain why I believe they would be important questions for future research.

As reported in Chapter Four, PTs did not attend to mathematics as a discipline. Overall, when PTs discussed mathematics they did so in a way that made clear they focused on school mathematics and the daily uses of mathematics—mathematics with a purpose. More specifically, as future teachers, the PTs often brought up ideas of how they would teach mathematics or expect students to learn mathematics in their future classrooms. From this wondering, I arrived at two potential research questions (1) how do PTs separate the constructs nature of mathematics, nature of teaching, and nature of learning if at all? And (2) Does the lens through which a person considers mathematics influence how that person views mathematics as a discipline (i.e., a teacher of mathematics, a student of mathematics, a professional mathematician)?

Secondly, PTs sometimes mentioned the mathematics itself. For example, in her interview, Mia said "I'm better at algebra than geometry. That's a content thing. In elementary school you learn the same content, you just add on each year" (Interview).

Some of the PTs mentioned specific mathematics content when they discussed their experiences with mathematics. When this idea emerged, I wondered if the mathematical content truly influenced the PTs' conceptions of NOM or if it was how that mathematical content was presented in the classrooms where the PTs were learning that content. From this wondering, I arrived at another potential research question, what features of a mathematics classroom influence students' conceptions of NOM? A sub question might examine different mathematics content classes (e.g., Algebra, Geometry, Calculus, Statistics) and if the content specific to those courses actually influence one's conceptions of NOM—as some PTs suggested in this study—or if the way the content is being presented in those different courses is actually influencing one's conceptions. This question is similar to asking how overall experiences influence one's conceptions of NOM, but narrows the experiences to the specific mathematics classes.

Last, present in the Proposed Unified View of NOM is the characteristic that anyone can be a learner of mathematics. I associate this statement with a person's identity as a mathematics learner or how one might perceive someone else's identity as a mathematics learner. Pair (2017) also included identity in the IDEA framework as "our mathematical ideas and practices" being part of one's identity. The idea of identity appeared in some PTs' data, but not often. When discussing solving mathematics problems, one PT said,

> I think that once you solve a problem you either build—maybe not physically but emotionally or psychologically—when you solve a problem you feel more confident in yourself and students feel more confident in themselves and they build as a person and then get more interested in math. But then also it works in

> engineering, if they solve a problem, they can then move on to physically building. (Millie, Interview)

She interpreted building as building in mathematics three different ways: first as a way to physically apply a mathematical concept to building a structure (i.e. in in engineering), building onto mathematical ideas and making connections, and building one's confidence as a doer of mathematics. Millie's elaboration on the emotions associated with problem solving (i.e. building confidence as a doer of mathematics) seems to enforce the ideas of included identity into a framework for NOM. From this wondering, I arrived at another potential research question, how does incorporating one's personal identity in a framework for the nature of mathematics help explain mathematics as a discipline?

Characteristics and Pronouns. In addition to the potential research question above, there were three very specific ideas I considered in relation to the data in this study that was present for a majority of PTs instead of singular ideas as mentioned above. In Chapter Four, I presented a breakdown of the characters PTs used to personify mathematics. With 130 PTs' descriptions, I then grouped the characters by commonalities that resulted in three main groups—friends, family, and antagonists. When PTs described their *math-character*, they also included specific characteristics. Figure 27 represents a sampling of the characteristics the PTs included in their writing prompts.

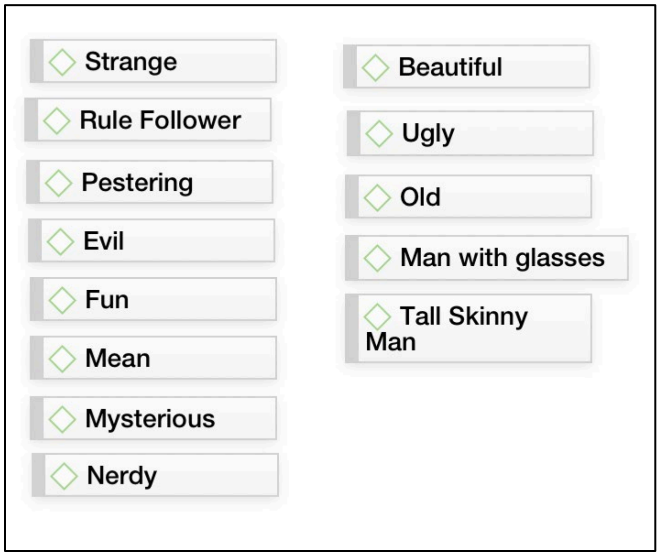

Figure 27. Characteristics PTs used to describe their math-characters

The characteristics described relate to the physical traits of mathematics as well as the personality traits of mathematics. Based on these types of characteristics, future questions can and should be asked about how the characteristics one chooses are connected to or elaborate on the PTs' relationship with mathematics and their conceptions of NOM. For example, the PT who described her *math-character* as a rule follower said that, "Math has always been tall and confident, but can be too serious on the rules sometimes" (PT1.33, Writing Prompt). It is likely that this PT might have a conception that closely aligns with NOM as a bag of tools.

PTs also described mathematics with varying pronoun choices. The majority of PTs described mathematics as either male or female. Some of the PTs chose to describe mathematics using the pronoun it or they. Only 2 of the 130 PTs did not include a male or female gendered pronoun in their personification of mathematics. One PT stated, "Math is definitely a girl" (PT2.21, Writing Prompt). Another exclaimed, "I would definitely say Math is a boy" (PT2.06, Writing Prompt). A different theoretical perspective (i.e. Critical Theory such as Race or Feminist) would allow for a different examination of this data and the relationships to PTs' descriptions of and understandings of NOM as well as how their experiences with mathematics influenced these descriptions. Further examination of the characteristics PTs used to describe their *math-characters* is important to gain a deeper and better understanding of how PTs categorize and describe mathematics. Additionally, examining the data from a different perspective can help future researchers consider the repercussions of the mental images PTs have of mathematics and how those images impact the field of mathematics education.

Equity Matters. Above, I proposed that one potential direction to extend this study could be to consider how one's lens might affect one's conception of NOM. Here, I consider three ways that the choice of a lens can affect a NOM study:

1. The choice of participants in the study (i.e., a PT, a MTE, student, teacher, or mathematician).
2. The theoretical perspective from which the researcher considers the data, and
3. The type of instruction that is valued in mathematics.

Completing the same study with different participants might provide vastly different insights on the MBI, Semantic Differential, writing prompt, and interview questions and would help to connect the conceptions of NOM across various groups as a way to look at common connections among the groups. Examining and interpreting the data in this study through a different theoretical lens might offer different results. For example, what questions would a critical theorist ask and how might that critical theorist interpret the data differently? Lastly, this study presents standards-based mathematics instruction as a foundational idea. Specifically, I used ideas presented in two foundational standards-documents (NCTM, 2000; NRC, 2001) to help create the Proposed Unified View of NOM. However, if the field of mathematics education, standards-based documents have been criticized as contributing to the inequities in mathematics (Gutierrez, 2008; Rubel, 2017). In her presentation at the NCTM Annual Conference, Picha (2019) proposed a representation of the relationship between four different pedagogies and the progression towards teaching mathematics for social justice as depicted in Figure 28.

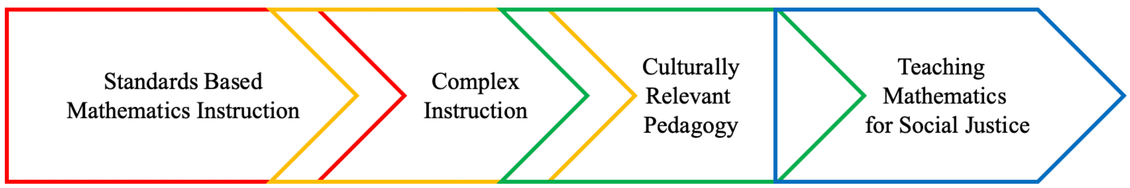

Figure 28: Nested relationship for different types of mathematics instruction. Adapted from "Strategies to teach math for social justice" by G. Picha, 2019, Presented at National Council of Teachers of Mathematics Annual Conference, San Diego, Ca.

Present in the proposed nested relationship in Figure 26 are four main equity-directed instructional practices in mathematics education. For standards-based mathematics instructions, the focus is on five practices—problem solving, connections, reasoning and proof, communication, and representations—that provide each and every student an opportunity to conceptually understand mathematics (NCTM, 2000). Rubel (2017) claimed that this type of instruction presents a challenge because teachers must view students as "possessing the prerequisite mathematical skills, literacy abilities, and problem-solving dispositions." (p. 71). Rubel (2017) explained this was a challenge at times given teachers constructions of some minorities. Therefore, a consideration of the nested relationship in Figure 28 is important when considering how the main ideas in the other mathematics education relevant pedagogies might change a PTs' experiences with school mathematics and thus influence their conceptions of NOM. For example, complex instruction (CI) views diversity among students as a way to improve instruction instead of an obstacle of instruction as sometimes perceived with SBMI (Cohen, 1994; Rubel, 2017). Culturally relevant pedagogy (CRP) takes CI a step further and attempts to connect mathematics instruction to students' experiences. Lastly, teaching mathematics for social justice (TMfSJ) includes ideas inherent in SBMI, CI, and CRP, but add to the ideas by including the development of students social and cultural identities so they can see themselves as agents of change (Gutstein, 2003; Rubel 2017). So, how would the incorporation of ideas from the other equity-based pedagogies (i.e., CI, CPR, and TMfSJ) change this study, if at all? Additionally, how would the ideas of experiences, identities, and social justice help define the nature of mathematics as a discipline versus the nature of mathematics teaching?

Characteristics of the Nature of Mathematics. In Chapter Two, I offered a Proposed Unified View of NOM (See Figure 6) and the characteristics associated with the discipline of mathematics. The Proposed Unified View is based on two foundational standards documents in mathematics education—NCTM's (2000) *Principals and Standards for School Mathematics* and NRC's (2001) *Adding it Up.* The posited model for the Proposed Unified View as a framework for NOM prompts two questions, in what ways, if any, is the discussion and proposal of a Proposed Unified View of NOM beneficial for the field of mathematics education? And whose math does the Proposed Unified View support? To continue to move the field of mathematics education forward more research should be conducted on revising and creating the list of characteristics in a

Proposed Unified View. Thus, I propose one future direction of this study is to incorporate the ideas from the science literature and how scientists and science educators came to develop a list of characteristics included in NOS. McComas and Almazroa (1998) acknowledged a lack of consensus regarding science and how science works, but explained that there should be significant consensus "regarding fundamental issues in the nature of science relevant to science education" (p. 512). Therefore, to extend the work of this study as it relates to the Proposed Unified View of NOM, I propose an inclusion of not only standards documents, but mathematicians, mathematics educators, and teachers of mathematics in the development of a list of characteristics of NOM. An implication is to use this Proposed Unified View in other NOM studies and see if other groups in the mathematics and mathematics education communities agree with the Proposed Unified View in similar ways as PTs in this study agreed with the Proposed Unified View. The inclusion of the Proposed Unified View in future studies could perhaps yield a future version that would serve as a consensus view and allow the field to speak more explicitly about NOM. Additionally, steps towards a potential consensus view by use of the Proposed Unified View can help describe what students and teachers should understand about NOM, and therefore could inform curriculum design and future standards documents. Creating and revising a list of characteristics of NOM is important for the field of mathematics education in two ways. First, a list of NOM characteristics can help doers of mathematics, specifically mathematics students, understand mathematics as a discipline and not just a topic required in schools. Second, a list of characteristics of NOM can provide curriculum writers, teachers, MTEs, and mathematicians with common ground to support the goal of each and every student learning mathematics.

Chapter Summary

In this chapter, I discussed the background of the study, reviewed the methodology, and summarized results. Additionally, I connected the results to prior literature and the theoretical framework, and I expounded on theoretical and practical implications for the field of mathematics education. Lastly, I provided many different directions for future research and why those questions might be important to consider regarding the development of the nature of mathematics.

Prospective teachers' conceptions of NOM, relationships with mathematics, and experiences with teachers are complicated and interconnected. This study examined the conceptions and experiences as reported by 130 elementary PTs enrolled in either a mathematical content or methods course. A rich description of PTs *math-characters*, relationships with mathematics, and experiences with mathematics were provided as well as connections between those constructs when applicable. In this study, prospective teachers most often described their conception of NOM as a static-unified body of knowledge and they most often credited experiences with teachers as the most influential in not only forming their conception of NOM but their overall relationship with NOM. This study was just one step forward in understanding the complicated connections between PTs' conceptions of and experiences with mathematics.

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APPENDICES

Appendix A: Internal Review Board Approval

IRB

INSTITUTIONAL REVIEW BOARD
Office of Research
Compliance, 010A
Sam Ingram Building,
2269 Middle
Tennessee Blvd
Murfreesboro, TN
37129

IRBN001 - EXPEDITED PROTOCOL APPROVAL NOTICE

Tuesday, March 12, 2019

Principal Investigator	**Lucy A. Watson** (Student)
Faculty Advisor	Jeremy Strayer
Co-Investigators	NONE
Investigator Email(s)	*law6z@mtmail.mtsu.edu; jeremy.strayer@mtsu.edu*
Department	Mathematical Sciences
Protocol Title	***Prospectives teachers' conceptions about the nature of mathematics***
Protocol ID	**19-2128**

Dear Investigator(s),

The above identified research proposal has been reviewed by the MTSU Institutional Review Board (IRB) through the **EXPEDITED** mechanism under 45 CFR 46.110 and 21 CFR 56.110 within the category *(7) Research on individual or group characteristics or behavior*. A summary of the IRB action and other particulars in regard to this protocol application is tabulated below:

IRB Action	**APPROVED for ONE YEAR**		
Date of Expiration	**1/31/2020**	Date of Approval	1/14/19
Sample Size	200 (TWO HUNDRED)		
Participant Pool	Primary Classification: **General Adults (18 or older)** Specific Classification: **MTSU students enrolled in math courses**		
Exceptions	1. Contact information for conducting the study is permitted. 2. Voice recording and handwriting samples for data collection are allowed.		

Restrictions	**1. Mandatory signed informed consent; the participants must have access to an official copy of the informed consent document signed by the PI.** **2. Data must be deidentified once processed.** **3. Identifiable data must be destroyed as described in the protocol. This includes audio/video data, photo images, handwriting samples, contact information and etc.**
Comments	NONE

This protocol can be continued for up to THREE years (**1/31/2022**) by obtaining a continuation approval prior to **1/31/2020**. Refer to the following schedule to plan your annual project reports and be aware that you may not receive a separate reminder to complete your continuing reviews. Failure in obtaining an approval for continuation will automatically result in cancellation of this protocol. Moreover, the completion of this study MUST be notified to the Office of Compliance by filing a final report in order to close-out the protocol.

Post-approval Actions

The investigator(s) indicated in this notification should read and abide by all of the post-approval conditions imposed with this approval. Refer to the post-approval guidelines posted in the MTSU IRB's website. Any unanticipated harms to participants or adverse events must be reported to the Office of Compliance at (615) 494-8918 within 48 hours of the incident. Amendments to this protocol must be approved by the IRB. Inclusion of new researchers must also be approved by the Office of Compliance before they begin to work on the project.

Continuing Review (**Follow the Schedule Below:**)
Submit an annual report to request continuing review by the deadline indicated below and please be aware that **REMINDERS WILL NOT BE SENT.**

Reporting Period	Requisition Deadline	IRB Comments
First year report	12/31/2019	The protocol will expire on 06/01/2019 as requested by PI unless a continuing review request is submitted
Second year report	12/31/2020	NOT COMPLETED
Final report	12/31/2021	NOT COMPLETED

Post-approval Protocol Amendments:
Only two procedural amendment requests will be entertained per year. *In addition, the researchers can request amendments during continuing review. This amendment restriction does not apply to minor changes such as language usage and addition/removal of research personnel. .*

Date	Amendment(s)	IRB Comments
03/08/2019	Samuel Reed (sdr4m - CITI5730541) has been approved to join the research team as a co-investigator..	Minor Amendment

Other Post-approval Actions:

Date	IRB Action(s)	IRB Comments
NONE	NONE.	NONE

Mandatory Data Storage Requirement: All of the research-related records, which include signed consent forms, investigator information and other documents related to the study, must be retained by the PI or the faculty advisor (if the PI is a student) at the secure location mentioned in the protocol application. The data storage must be maintained for at least three (3) years after study has been closed. Subsequent to closing the protocol, the researcher may destroy the data in a manner that maintains confidentiality and anonymity.

IRB reserves the right to modify, change or cancel the terms of this letter without prior notice. Be advised that IRB also reserves the right to inspect or audit your records if needed.

Sincerely,

Institutional Review Board
Middle Tennessee State University

Quick Links:

Click here for a detailed list of the post-approval responsibilities. More information on expedited procedures can be found here.

IRBN001 – Expedited Protocol Approval Notice

Appendix B: Mathematics Beliefs Instrument

Please give your best definition of mathematics by completing the sentence:

Mathematics is…

Now for each of the following items,

1. please indicate your level of agreement: Strongly Disagree (SD), Disagree (D), Neutral/Not Sure (NS), Agree (A), Strong Agree (SA)
2. Then give a brief explanation of why you answered as you did.

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1. To know mathematics means remembering and applying the correct rule or technique to solve a given problem.
2. In mathematics everything goes together in a logical and consistent way.
3. Learning mathematics is mostly memorizing or practicing procedures.
4. Mathematics reveals hidden structures that help us understand the world around us.
5. Mathematics is as much about patterns as it is about numbers.
6. There is usually only one correct way to solve a mathematics problem.
7. Mathematics is mostly a body of facts and procedures.
8. Ordinary students cannot expect to understand mathematics.
9. I understand what it means to make a sound mathematical argument.
10. I am capable of making sound mathematical arguments most of the time.

Appendix C: Semantic Differential Survey

For each pair of words, place an X in the blank that
best tells how you feel about the question.

Example:

What do you think about school?

like _______:___X___:_______:_______:_______ hate

important ___X___:_______:_______:_______:_______ insignificant

work _______:_______:___X___:_______:_______ play

These responses would indicate that the person likes school but is not crazy about it. The person thinks school is very important and that school means some work and some play.

What do you think it means to be good at math?

fast _______:_______:_______:_______:_______ accurate

relationships _______:_______:_______:_______:_______ recall

comfortable _______:_______:_______:_______:_______ challenged

flexible _______:_______:_______:_______:_______ step-by-step

connections _______:_______:_______:_______:_______ memorization

repetition _______:_______:_______:_______:_______ understanding

operations _______:_______:_______:_______:_______ sense making

explanations _______:_______:_______:_______:_______ answers

applying _______:_______:_______:_______:_______ creating

learning _______:_______:_______:_______:_______ ability

invention _______:_______:_______:_______:_______ reproduction

principles_______:_______:_______:_______:_______ rules

procedures_______:_______:_______:_______:_______ concepts

solutions _______:_______:_______:_______:_______ processes

multiple methods _______:_______:_______:_______:_______ best approach

step-by-step _______:_______:_______:_______:_______ flexible

strategies _______:_______:_______:_______:_______ algorithms

reasoning _______:_______:_______:_______:_______ calculating knowing _______:_______:_______:_______:_______ justifying

ease _______:_______:_______:_______:_______ effort

ideas _______:_______:_______:_______:_______ facts

Appendix D: Interview Protocol

1. General Questions
    a. Name and major?
    b. Where are you in program (freshman not yet admitted to teacher education program, admitted, doing student teaching?)
    c. What class are you enrolled in now?
2. How would you describe your past mathematics experiences?
3. How would you describe your current mathematics experiences?
4. How would you answer the question, “What is Mathematics?”
    a. This question has been asked on the initial MBI survey. Make sure to have interviewee’s response to prompt if needed or see if aligned with what they say again?
5. In mathematics, how do you know when you solved a problem?
6. Questions regarding personification of mathematics prompt.
    a. What did you mean when you say [whatever they said]?
    b. Can you tell me more about what you were thinking?
    c. How did you interpret that question?
7. Questions regarding MBI survey/Semantic Differential Survey.
    a. You marked [whatever they marked]. Why did you answer like that?
    b. Can you tell me more about what you were thinking?
8. How do you think your life would be if you did not have to take mathematics courses?
9. What are the differences between a good day and a bad day in mathematics class?

10. Show PTs the list of characteristics in the Proposed Unified View of NOM and ask them to reflect. Would you add anything? Why? What you take away anything? Why?
11. In thinking back to the original survey, writing the personification story, and this interview, how do you think any of the ideas you have thought about, or have been brought up, have influenced what you think about mathematics?

Appendix E: Codes and Themes

Open Codes	Emergent Themes
Friend	Majority of PTs described math as a friend.
Teacher	
Mentor	
Boyfriend	
Model	
Cool Person	
Baby Mosquito	PTs described antagonistic *math-characters.*
Bully	
Enemy/Frenemy	
Jerk	
Mean Girl	
Monster	
Brother	PTs described familial *math-characters.*
Cousin	
Grandfather	
Mother	
Parent	
Sibling	
Sister	
Step-Dad	
Family	
Bag of Tools (1)	PTs described different conceptions of NOM.
Static Body (2)	
Dynamic (3)	
(1) and (2)	PTs were not always aligned with only one conception, but instead at time described aspects of the different conceptions.
(2) and (3)	
(1) and (3)	
Positive	PTs described varying relationships with mathematics.
Negative	
Roller-Coaster	Neutral code was different than most ideas reported in the literature. Represented neither a like or dislike of mathematics, but an overall appreciation for mathematics.
P to N	
N to P	
Neutral	
	Neutral code marked PTs statements about different mathematics than mathematicians.
Man	PTs assigned their *math-character* with a specific gender.
Woman	
It	
They	
Mathematician	Evidence PTs did not view school mathematics the same as mathematicians
The Discipline	
School vs Non-School	

Math Class Good Day Bad Day	Evidence PTs' experiences with teachers, as students, and in mathematics classes influenced their conceptions of NOM.
Teacher Experience Student Experience	Experiences with teachers were most influential
Strange Pestering	PTs used adjectives to describe their *math-characters.*
Evil Fun Rule-Follower Mean	In some cases, the adjectives helped elaborate on their conceptions of NOM (i.e. Rule-follower closely aligned with. Bag of tools).
Mysterious Nerdy Beautiful Old Ugly Glasses-wearing Skinny Fat Changing Appearance	PTs adjective use, in general, would be interesting to look at from a different theoretical perspective.

Appendix F: Tabby's Mathematics Problem

Jenny and Leslie made chocolate milk. Jenny used 1/3 glass of chocolate syrup. Leslie, whose glass is twice as large as Jenny's, used ¼ glass of syrup. They decide to combine their drinks into a larger pitcher. What part of the combined mixture would be syrup?

Found in:

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