THESIS TITLE MTSU MATHEMATICAL

SCIENCES DEPARTMENT THESIS FORMAT

A Stone Duality Between Finite Hypergraphs and FCTH-Lattices

Presented to the Faculty of the Department of Mathematical Sciences

Middle Tennessee State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematical Sciences

---

by

Jake Elrod

July 10, 2020

Thesis Committee:

Dr. James Hart, Chair

Dr. Chris Stephens

Dr. Medha Sarkar

DEDICATION

I would like to dedicate this thesis to my two boys, Hunter and Anthony. They have given me so much more than I could ever give in return, and I would not be the man I am today without them.

ACKNOWLEDGMENTS

This thesis would have not been possible without the help of many individuals, all of whom have my sincerest appreciation for their time and support.

First, I would like to thank Dr. Medha Sarkar for the time she has invested in her review of my thesis and for her expert feedback. I'd like to thank Dr. Chris Stephens for the same as well as putting up with all my eccentricities as a graduate student at MTSU.

I'd like to thank my family, whose support has given me opportunities many don't receive and that I hardly deserve. I'd like to especially thank my father, Steve, for all the hours he put in taking care of my son, Anthony, while I was pursuing this degree.

Finally, I would like to give a special thanks to Dr. James Hart for his typeset handwriting, his seemingly inexhaustible patience, and his passion for both teaching and mathematics. The time and effort he has put into my education and into this thesis has been an invaluable resource, which I will forever be grateful for.

ABSTRACT

In this thesis we examine the relationship between hypergraphs and specialized frames called CTH-lattices. We find that these objects (under a finite context) and their associated homomorphisms form a duality, and therefore, are strongly related to each other.

CONTENTS

List of Tables

List of Figures

CHAPTER 1

INTRODUCTION

In the paper *A Duality between hypergraphs and cone lattices*, [Z. French [6]] shows that there is a duality between the category of hypergraphs under hypergraph homomorphisms and a specialized category whose objects are specialized frames and whose morphisms are specialized frame homomophisms. In this thesis we construct the same and show the duality of their finite subcategories.

In Chapter 2 we go over foundational topics required to show this duality. The topics cover a range of fields including Set Theory, Graph Theory, Order Theory, Topology, and Category Theory. While many of the results in Chapter 2 are standard, some of the definitions have been modified to serve the simplicity of this paper. The reader should find that such definitions are equivalent to their standard counterparts.

In Chapter 3 we introduce what we call hypergraph posets and hypergraph poset homomorphisms. You will notice that these posets and homomorphisms have very similar properties to hypergraphs and hypergraph homomorphisms. In fact, the category of such objects and morphisms end up being categorically equivalent to the category of hypergraphs under hypergraph homomorphisms, which we show in Chapter 4.

In Chapter 5 we introduce the classical topology on a hypergraph and a special frame generated from that topology. We then show various results around the same that will be used in the following Chapters.

Chapter 6 introduces what we call CTH-lattices and CTH-lattice homomorphisms, where we discover that the specialized frame from Chapter 5 ends up being a member of. In Chapter 7 we then show that we have a cateorgy under the objects CTH-lattices with CTH-lattice homomorphisms as our morphisms and show that we have a duality between the categories of finite hypergraphs and finite CTH-lattices.

CHAPTER 2

PRELIMINARIES

2.1 Introduction

In this chapter we will review various relevant topics from multiple disciplines including Set Theory, Graph Theory, Order Theory, Topology, and Category Theory. Many of the following definitions and theorems are foundational in each of their respective fields and for this thesis.

2.2 Set Theory

We will assume that the reader has a good understanding in the foundational concepts of Set Theory (specifically ZFC). However, there are a couple topics that we would like to highlight that will be relevant later on.

There was an important finding by Ernst Zermelo in the late 19th century and Bertrand Russell in the early 20th century that is now called Russell's Paradox (or, alternatively, the Russell-Zermelo Paradox). Simply stated, the paradox asks the following question:

$$\text{Given the set } R = \{x : x \notin x\}, \text{ is } R \in R?$$

If $R \in R$, then by the definition of R, $R \notin R$. Similary, if $R \notin R$, then $R \in R$ by the definition of R. So we have $R \in R \iff R \notin R$, which is impossible; and we thus have a paradox.

So where did we go wrong? Surely we can define a collection containing all sets that don't contain themselves? In order to circumvent this paradox (and others), we could no longer define a set as simply any collection of objects.

Definition 2.2.1. A *set* is a well-defined collection of objects.

If a collection of objects creates a paradox (such as Russell's Paradox) when considered a set, we instead call it a *class*. We may consider any set a class; however, not every class is a set. It is common practice to say a class is *small* if it can be considered a set. Alternatively, when a class may not be considered a set, we say it is *large* or that the class is a *proper class*. While proper class is not technically defined in ZFC (rather, in NBG), we will use the term here for convenience. Table 1 gives examples of sets and proper classes.

Set	Proper Class
Finite Sets	Sets
Real Numbers	Groups
Complex Numbers	Vector Spaces

Table 1: Example Sets and Proper Classes

2.3 Graph Theory

Definition 2.3.1. A *hypergraph* is a triple (V, E, ϕ) where V denotes a *vertex set*, E denotes an *edge set* that is disjoint from V, and ϕ is a mapping from E to $\mathscr{P}(V) \sim \{\emptyset\}$ where $\mathscr{P}(V)$ denotes the power set of V.

Before proceeding, it is useful to first discuss some terminology when describing graphs. A graph $\mathcal{G} = (V, E, \phi)$ is said to be *finite* if V and E are finite sets.

Given two vertices u and v, we say that u and v are *adjacent* if $u \neq v$ and there exists an edge, e, such that $u, v \in \phi(e)$. If two vertices are adjacent, we say they are *neighbors*. We say a vertex v is *incident* with an edge e if $v \in \phi(e)$. If a vertex is not incident with any edge, we say that it is *isolated*.

Let $\mathcal{G} = (V, E, \phi)$ be a graph. An *edge neighborhood* of a vertex $v \in V$ is the set $\{e \in E : v \in \phi(e)\}$, which is the set of all edges incident with v. We will denote E_N as the function mapping each vertex to its edge neighborhood. Similarly, an *edge ball* of v is defined as the set $\{v\} \cup E_N(v)$. We will denote E_B as the function mapping each vertex to its edge ball. A *vertex neighborhood* of an edge $e \in E$ is the set of all vertices that are incident with e, or equivalently, all vertices in $\phi(e)$. We define a vertex ball of e as $\{e\} \cup \phi(e)$ and denote V_B as the mapping from an edge to its vertex ball.

Definition 2.3.2. [Z. French [6]] Let $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G}, \phi_\mathcal{G})$ and $\mathcal{H} = (V_\mathcal{H}, E_\mathcal{H}, \phi_\mathcal{H})$ be hypergraphs. A function $\alpha : V_\mathcal{G} \cup E_\mathcal{G} \to V_\mathcal{H} \cup E_\mathcal{H}$ is a *hypergraph homomorphism* provided

1. $\alpha(V_\mathcal{G}) \subseteq V_\mathcal{H}$
2. $\alpha(E_\mathcal{G}) \subseteq E_\mathcal{H}$
3. For any $e \in E_\mathcal{G}$, $\alpha(\phi_\mathcal{G}(e)) = \phi_\mathcal{H}(\alpha(e))$

If α is a bijection and it's inverse is a hypergraph homomorphism we say that it α is a *hypergraph isomorphism.* If a hypergraph homomorphism exists between $\mathcal{G}$ and $\mathcal{H}$ we say they are *homomorphic.*

Theorem 2.3.1. [Z. French [6]] Let $\mathcal{A} = (V_\mathcal{A}, E_\mathcal{A}, \phi_\mathcal{A}), \mathcal{B} = (V_\mathcal{B}, E_\mathcal{B}, \phi_\mathcal{B}), \mathcal{C} = (V_\mathcal{C}, E_\mathcal{C}, \phi_\mathcal{C})$ be hypergraphs and let $\alpha : V_\mathcal{B} \cup E_\mathcal{B} \to V_\mathcal{C} \cup E_\mathcal{C}$ and $\beta : V_\mathcal{A} \cup E_\mathcal{A} \to V_\mathcal{B} \cup E_\mathcal{B}$ be hypergraph homomorphisms. Then $\alpha \circ \beta$ is a hypergraph homomorphism.

Proof. Since α and β both map vertices to vertices and edges to edges, we know that $\beta \circ \alpha$ must also map vertices to vertices and edges to edges. Thus we have the first two requirements for a hypergraph homomorphism. For the third requirement, let $e \in E$. Then we have

$$\begin{aligned}(\alpha \circ \beta)(\phi_{\mathcal{A}}(e)) &= \alpha(\beta(\phi_{\mathcal{A}}(e))) \\ &= \alpha(\phi_{\mathcal{B}}(\beta(e))) \\ &= \phi_{\mathcal{C}}(\alpha(\beta(e))) \\ &= \phi_{\mathcal{C}}((\alpha \circ \beta)(e)).\end{aligned}$$

□

It is easy to see that the composition of hypergraph isomorphisms is itself a hypergraph isomorphism since the composition of two bijections is also a bijection.

2.4 Order Theory

Order Theory is the foundation of the results in this thesis. We present a number of standard Order Theory results from [J. Hart and Z. French [4]] and [J. Snodgrass and C. Tsinakis [8]] that are relevant in later Chapters.

2.4.1 Posets

Definition 2.4.1. A *partially ordered set* (or poset for short) is an ordered pair $\mathcal{P} = (P, \leq)$ consisting of a set P and a binary relation $\leq$ on the set P satisfying the following conditions.

1. For all $x \in P$, we have $x \leq x$ (reflexivity).
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

The binary relation $\leq$ defined above is called a *partial ordering* on the set P.

As with the previous section, it is useful to discuss certain terminology when describing posets. For any poset $\mathcal{P} = (P, \leq)$, we say two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$. If x and y are not comparable, we say they are *incomparable*.

A poset $\mathcal{P}$ is called a *chain* if every element of P is comparable to every other element of P. If $\mathcal{P}$ is a chain it is said to be *totally ordered.* A poset $\mathcal{P}$ is said to be an *antichain* if every element is incomparable to every other element. In other words, in any antichain, $x \leq y \implies x = y$.

It is common to write $x < y$ if $x \leq y$ and $x \neq y$. If this is the case we say the inequality is *strict* or that x is *strictly* less than y. While most of the time it is convenient to write $x \leq y$, at other times it is more intuitive to write $y \geq x$. In cases where this occurs, we consider $y \geq x$ to simply mean $x \leq y$.

[J. Hart and Z. French [4]] We say $\mathcal{P}$ is *lower bounded* provided there exists some $\perp \in \mathcal{P}$ such that $\perp \leq x$ for all $x \in P$ and we call $\perp$ the *lower bound* or *least element* of $\mathcal{P}$. Similarly, $\mathcal{P}$ is *upper bounded* if there exists some $\top \in P$ such that $x \leq \top$ for all $x \in P$ and we call $\top$ the *upper bound* or *greatest element* of $\mathcal{P}$. We say $\mathcal{P}$ is *bounded* if it is both lower and upper bounded. When $\perp$ and $\top$ are used it will be assumed that they denote the lower and upper bounds (respectively) of the poset under consideration.

Definition 2.4.2. Let $\mathcal{P} = (P, \leq)$ be a poset. Given a set $Q \subseteq P$, the poset $\mathcal{Q} = (Q, \preceq)$ is said to be a *subposet* of $\mathcal{P}$ provided $q_1 \preceq q_2 \iff q_1 \leq q_2$ for any $q_1, q_2 \in Q$.

Given a subposet $\mathcal{Q} = (Q, \preceq)$ of a poset $\mathcal{P} = (P, \leq)$, it is common to use $\leq$ in place of $\preceq$. For example, given the poset $\mathcal{R} = (\mathbb{R}, \leq)$ where $\leq$ denotes the standard ordering on real numbers, one might define a subposet $(\mathbb{R}^+, \leq)$ where it is understood that $\leq$ is the restriction of the same ordering on $\mathcal{R}$ to $\mathbb{R}^+$.

Definition 2.4.3. Let $\mathcal{P}$ be a poset. A subposet $\mathcal{Q} = (Q, \leq)$ of $\mathcal{P}$ is a *directed poset* provided every finite subset of Q has an upper bound in Q.

Given a directed subposet $\mathcal{Q} = (Q, \leq)$ of a poset $\mathcal{P}$, it is common and often convenient to say Q is a *directed subset* of P or simply that Q is *directed* when the context is already established.

Definition 2.4.4. Let $\mathcal{P} = (P, \leq)$ be any poset. The order dual of $\mathcal{P}$ is defined to be the poset $\mathcal{P}^{op} = (P, \leq^{op})$ where $x \leq^{op} y \iff y \leq x$.

[J. Hart and Z. French [4]] Let $\mathcal{P} = (P, \leq)$ be any poset. A subset L of P is called a *lowerset* (or *order ideal*) of $\mathcal{P}$ provided $x \in L$ and $y \leq x$ together imply that $y \in L$. Similarly, a subset U of P is an *upper set* (or *order filter*) of $\mathcal{P}$ provided $x \in U$ and $x \leq y$ together imply that $y \in U$. We will let $Low(\mathcal{P})$ denote the poset of all lowersets of $\mathcal{P}$ and $Up(\mathcal{P})$ denote the poset of all uppersets of $\mathcal{P}$, both partially ordered by subset inclusion.

Definition 2.4.5. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. The set

$$\downarrow X = \{p \in P : p \leq x \text{ for some } x \in X\}$$

is called the *lowerset generated by* X in $\mathcal{P}$. Likewise, the set

$$\uparrow X = \{p \in P : x \leq p \text{ for some } x \in X\}$$

is called the *upperset generated by* X in $\mathcal{P}$.

[J. Hart and Z. French [4]] A lowerset generated by a singleton is called a *principal* lowerset; it is often denoted by $\downarrow x$ instead of $\downarrow \{x\}$.

Theorem 2.4.1. Let $\mathcal{P}$ be a poset. Then a set is a lowerset of $\mathcal{P}$ if and only if it is an upperset of $\mathcal{P}^{op}$. Similarly, a set is an upperset of $\mathcal{P}$ if and only if it is a lowerset of $\mathcal{P}^{op}$.

Proof. Suppose X is an upperset of $\mathcal{P} = (P, \leq)$. Then for any $x \in X, x \leq y \implies y \in X$. However, $x \leq y$ if and only if $y \leq_{op} x$ by the definition of $\mathcal{P}^{op}$. Therefore, for any $x \in X, y \leq_{op} x \implies y \in X$ and X must be a lowerset of $\mathcal{P}^{op}$.

Now suppose X is a lowerset of $\mathcal{P}^{op}$. Then for any $x \in X, y \leq_{op} x \implies y \in X$. However, $y \leq_{op} x$ if and only if $x \leq y$. Therefore, for any $x \in X, x \leq y \implies y \in X$ and X must be an upperset of $\mathcal{P}$. $\square$

It should be clear from Theorem 2.4.1 that for any poset $\mathcal{P}$, $Up(\mathcal{P}) = Low(\mathcal{P}^{op})$ and $Low(\mathcal{P}) = Up(\mathcal{P}^{op})$.

Theorem 2.4.2. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. Then $\downarrow X = \uparrow_{op} X$ and $\uparrow X = \downarrow_{op} X$ where $\uparrow_{op}$ and $\downarrow_{op}$ denote the upperset and lowerset of X with respect to $\mathcal{P}^{op}$.

Proof. Suppose $x \in \downarrow X$. Then $x \leq x'$ for some $x' \in X$. But $x \leq x'$ if and only if $x' \leq_{op} x$. Therefore, $x \in \uparrow_{op} X$ and $\downarrow X \subseteq \uparrow_{op} X$. Now suppose $x \in \uparrow_{op} X$. Then $x' \leq_{op} x$ for some $x' \in X$. But since $x' \leq_{op} x$ if and only if $x \leq x'$, x must also be in $\downarrow X$. Therefore we also have $\uparrow_{op} X \subseteq \downarrow X$ and we can conclude that $\downarrow X = \uparrow_{op} X$.

The proof for $\uparrow X = \downarrow_{op} X$ follows in the same manner, relying on the fact that for any $x, y \in P$, $x \leq y \iff y \leq_{op} x$. $\square$

Theorems 2.4.1 and 2.4.2 show that lowersets and uppsersets are dual notions. Because of this, they will have very similar properties, as we will see in the following theorems.

Theorem 2.4.3. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. Then X is a lowerset of $\mathcal{P}$ if and only if $X = \bigcup_{x \in X} \downarrow x$.

Proof. First, suppose X is a lowerset of $\mathcal{P}$. Since $x \in \downarrow x$, $X \subseteq \bigcup_{x \in X} \downarrow x$. So we need only show $\bigcup_{x \in X} \downarrow x \subseteq X$. If $y \in \bigcup_{x \in X} \downarrow x$, then $y \in \downarrow x$ for some $x \in X$. Therefore, $y \leq x$ for some $x \in X$, but that implies $y \in X$ since X is a lowerset. This gives us $\bigcup_{x \in X} \downarrow x \subseteq X$ and we can conclude $X = \bigcup_{x \in X} \downarrow x$.

Now suppose $X = \bigcup_{x \in X} \downarrow x$. If $x' \in X$, then $x' \in \downarrow x$ for some $x \in X$. Further, if $y \leq x'$, then $y \in \downarrow x$. Therefore, $y \in X$, and we can conclude that X is a lowerset of $\mathcal{P}$. $\square$

Theorem 2.4.4. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. Then X is an upperset of $\mathcal{P}$ if and only if $X = \bigcup_{x \in X} \uparrow x$.

Proof. Because of the duality of lowersets and uppersets, we know

$$\bigcup_{x \in X} \uparrow x = \bigcup_{x \in X} \downarrow_{op} x.$$

But by Theorem 2.4.3, X is an lowerset of $\mathcal{P}^{op}$ if and only if $X = \bigcup_{x \in X} \downarrow_{op} x$. Since X is a lowerset of $\mathcal{P}^{op}$ if and only if X is an upperset of $\mathcal{P}$, we can conclude that X is an upperset of $\mathcal{P}$ if and only if $X = \bigcup_{x \in X} \uparrow x$. $\square$

Theorem 2.4.5. The union of lowersets of a poset, $\mathcal{P} = (P, \leq)$, is a lowerset of $\mathcal{P}$.

Proof. Let $\mathcal{F}$ be a family of lowersets of $\mathcal{P}$. Suppose $x \in \bigcup_{X \in \mathcal{F}} X$. Then $x \in X$ for some $X \in \mathcal{F}$. If $y \leq x$, then $y \in X$ since X is a lowerset. This implies $y \in \bigcup_{X \in \mathcal{F}} X$ and we can now conclude $\bigcup_{X \in \mathcal{F}} X$ is a lowerset of $\mathcal{P}$. $\square$

Theorem 2.4.6. The union of an upperset of a poset, $\mathcal{P} = (P, \leq)$, is an upperset of $\mathcal{P}$.

Proof. This of course follows from the duality of lowersets and uppsersets. Since the union of uppersets of a poset is just the union of lowersets of the poset's order dual, it must then be a lowerset of the order dual and thus an upperset of the poset. $\square$

Definition 2.4.6. Let $\mathcal{P} = (P, \leq)$ be a poset. We say that p' is *minimal* in $\mathcal{P}$ if for any $p \in P$, $p' \leq p$ when p' is comparable to p. Similarly, we say p' is *maximal* in $\mathcal{P}$ if for any $p \in P$, $p \leq p'$ when p' is comparable to p.

Theorem 2.4.7. Let $\mathcal{P} = (P, \leq)$ be a poset. An element p of P is minimal in $\mathcal{P}$ if and only if $\downarrow p = \{p\}$.

Proof. Suppose p' is minimal in $\mathcal{P}$. Then for any $p \in P$, $p' \leq p$ when p' is comparable to p, which implies $\downarrow p' = \{p \in P : p \leq p'\} = \{p'\}$. Suppose $\downarrow p' = \{p \in P : p \leq p'\} = \{p'\}$. Then for any $p \in P$, $p' \leq p$ when p' is comparable to p. Therefore, p' is minimal in $\mathcal{P}$ if and only if $\downarrow p' = \{p'\}$. $\square$

Theorem 2.4.8. Let $\mathcal{P} = (P, \leq)$ be a poset. A element p of P is maximal in $\mathcal{P}$ if and only if $\uparrow p = \{p\}$.

Proof. Suppose p' is maximal in $\mathcal{P}$. Then for any $p \in P$, $p \leq p'$ when p' is comparable to p, which implies $\uparrow p' = \{p \in P : p' \leq p\} = \{p'\}$. Suppose $\uparrow p' = \{p \in P : p' \leq p\} = \{p'\}$. Then for any $p \in P$, $p \leq p'$ when p' is comparable to p. Therefore, p' is maximal in $\mathcal{P}$ if and only if $\uparrow p' = \{p'\}$. □

Definition 2.4.7. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. The *join* of X (if it exists in $\mathcal{P}$) is denoted $\bigvee X$ and is defined as the minimum element of the set $\{p \in P : x \leq p \text{ for all } x \in X\}$. Similarly, the *meet* of X (if it exists) is denoted $\bigwedge X$ and is defined as the maximum element of the set $\{p \in P : p \leq x \text{ for all } x \in X\}$

In many cases it is convenient to use $x \wedge y$ in place of $\bigwedge\{x, y\}$ and $x \vee y$ in place of $\bigvee\{x, y\}$. Both meet and join have many of the properties you would expect from their notation.

Theorem 2.4.9. Meet and join have the following properties:

1. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ **(Commutativity)**
2. $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ **(Associativity)**
3. $x \vee x = x$ and $x \wedge x = x$ **(Idempotency)**
4. $x \vee \perp = x$ and $x \wedge \top = x$ **(Identities)**
5. $x \vee \top = \top$ and $x \wedge \perp = \perp$ **(Annulment)**
6. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ **(Absortion)**

Proof.

1. Commutativity here is trivial since the definition of both meet and join is based on set membership.

2. Let $\mathcal{P} = (P, \leq)$ be a poset. Then

$$\begin{aligned}
x \vee (y \vee z) &= \min\{p \in P : x \leq p \text{ and } (y \vee z) \leq p\} \\
&= \min\{p \in P : x \leq p \text{ and } \min\{p \in P : y \leq p \text{ and } z \leq p\} \leq p\} \\
&= \min\{p \in P : x \leq p \text{ and } y \leq p \text{ and } z \leq p\} \\
&= \min\{p \in P : \min\{p \in P : x \leq p \text{ and } y \leq p\} \leq p \text{ and } z \leq p\} \\
&= (x \vee y) \vee z
\end{aligned}$$

and

$$\begin{aligned}
x \wedge (y \wedge z) &= \max\{p \in P : p \leq x \text{ and } p \leq (y \wedge z)\} \\
&= \max\{p \in P : p \leq x \text{ and } p \leq \max\{p \in P : p \leq y \text{ and } p \leq z\}\} \\
&= \max\{p \in P : x \leq p \text{ and } p \leq y \text{ and } p \leq z\} \\
&= \max\{p \in P : p \leq \max\{p \in P : p \leq x \text{ and } p \leq y\} \text{ and } p \leq z\} \\
&= (x \wedge y) \wedge z.
\end{aligned}$$

3. Let $\mathcal{P} = (P, \leq)$ be a poset. Then $x \vee x = \min\{p \in P : x \leq p\} = x$ and $x \wedge x = \max\{p \in P : p \leq x\} = x$.

4. Let $\mathcal{P} = (P, \leq)$ be a poset. Then

$$\begin{aligned}
x \vee \perp &= \min\{p \in P : x \leq p \text{ and } \perp \leq p\} \\
&= \min\{p \in P : x \leq p\} \\
&= x
\end{aligned}$$

and

$$\begin{aligned} x \wedge \top &= \max\{p \in P : p \leq x \text{ and } p \leq \top\} \\ &= \max\{p \in P : p \leq x\} \\ &= x. \end{aligned}$$

5. Since $\top$ is defined as the greatest element in a poset, it is the only element greater than or equal to both x and $\top$. Therefore, $x \vee \top = \top$. Similarly, since $\perp$ is defined as the least element in a poset, it is the only element less than or equal to both x and $\perp$. Therefore, $x \wedge \perp = \perp$.

6. Let $\mathcal{P} = (P, \leq)$ be a poset. Then

$$\begin{aligned} x \vee (x \wedge y) &= \min\{p \in P : x \leq p \text{ and } (x \wedge y) \leq p\} \\ &= \min\{p \in P : x \leq p \text{ and } \max\{p \in P : p \leq x \text{ and } p \leq y\} \leq p\} \\ &= \min\{p \in P : x \leq p\} \\ &= x \end{aligned}$$

and

$$\begin{aligned} x \wedge (x \vee y) &= \max\{p \in P : p \leq x \text{ and } p \leq (x \vee y)\} \\ &= \max\{p \in P : p \leq x \text{ and } p \leq \min\{p \in P : x \leq p \text{ and } y \leq p\}\} \\ &= \max\{p \in P : p \leq x\} \\ &= x. \end{aligned}$$

$\square$

Definition 2.4.8. Let $\mathcal{Q} = (Q, \leq)$ be a directed subposet of $\mathcal{P}$. Then the join of Q (if it exists) is called a *directed join*.

It is important to note that a directed join of a directed subposet need not be in the subposet. For example, given the poset $(\mathbb{R}, \leq)$ and the directed subposet $((0,1), \leq)$, we have $\bigvee(0,1) = 1 \notin (0,1)$. If a poset is closed under directed joins, we say the poset is *directed complete*.

Definition 2.4.9. Let $\mathcal{P} = (P, \leq)$ be a directed complete poset. We say an element $p \in P$ is *compact* if, given a directed subset $D \subseteq P$, $p \leq \bigvee D \implies p \leq d$ for some $d \in D$ whenever $\bigvee D$ exists.

Definition 2.4.10. Let $\mathcal{P} = (P, \leq)$ be a poset and $X \subseteq P$. We say $y \in P$ is a *cover* of (or covers) X if both of the following hold.

1. $y \geq x$ for all $x \in X$

2. $y > z \geq x$ for all $x \in X \implies z \in X$

The set of all covers of X we denote as $Cov(X)$. If X is a singleton set, $\{x\}$, we use $Cov(x)$ in place of $Cov(\{x\})$.

Theorem 2.4.10. Let $\mathcal{P} = (P, \leq)$ be a poset and let $X \subseteq P$. Then $y \in P$ is a cover of X provided y is minimal in $\mathcal{X} = (\uparrow X \sim X, \leq)$ where $\uparrow X \sim X$ denotes the set of all members of $\uparrow X$ that are not contained in X.

Proof. Suppose y is minimal in $\mathcal{X}$. $y > x$ for all $x \in X$ by definition of $\uparrow X \sim X$, so we have the first requirement of a cover. Now suppose z is an element in P such that $y > z \geq x$ for all $x \in X$. Then $\downarrow y = \{y\}$ in $\mathcal{X}$ by Theorem 2.4.7 and thus z must be in X, which satisfies the second requirement of a cover. Therefore, any element of P that is minimal in $\mathcal{X}$ is a cover of X. $\square$

For the rest of this thesis, given two sets U and V we will assume that $U \sim V$ has the same meaning as above which is to denote the set of all members in U that are not contained in V.

It is often useful to represent a poset visually using what is called a Hasse Diagram. In a Hasse Diagram of a poset $\mathcal{P} = (P, \leq)$, each element $x \in P$ is connected (by a line) to each of the elements of $Cov(x)$ with the elements of $Cov(x)$ located above x. See Figure 1 for an example Hasse Diagram.

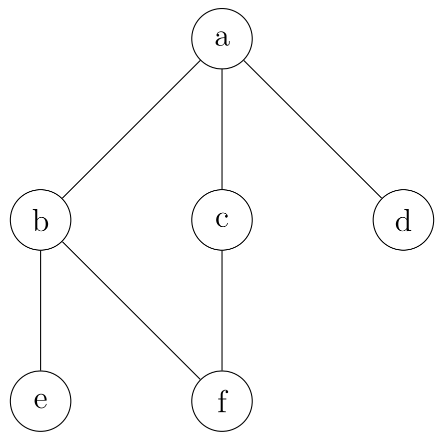

Figure 1: *Example Hasse Diagram of poset* $\mathcal{P} = (\{a, b, c, d, e, f\}, \leq)$ *where* $b < a, e < b, f < b, f < c < a,$ *and* $d < a$.

Definition 2.4.11. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets. A function $f : P \to Q$ is said to be an *order homomorphism* (or that f is *isotone*) if $p_1 \leq p_2$ implies $f(p_1) \preceq f(p_2)$. A bijective order-homomorphism is called an *order isomorphism* provided is inverse is also an order homomorphism.

Theorem 2.4.11. Let $\mathcal{A} = (A, \leq_{\mathcal{A}})$, $\mathcal{B} = (B, \leq_{\mathcal{B}})$, and $\mathcal{C} = (C, \leq_{\mathcal{C}})$ be posets and let $\alpha : B \to C$ and $\beta : A \to B$ be order homomorphisms from $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{A}$ to $\mathcal{B}$, respectively. Then $\alpha \circ \beta : A \to C$ is an order homomorphism from $\mathcal{A}$ to $\mathcal{C}$.

Proof. Since β is an order homomorphism we know $a_1 \leq_{\mathcal{A}} a_2 \implies \beta(a_1) \leq_{\mathcal{B}} \beta(a_2)$. However, since α is an order homomorphism and $\beta(a_1), \beta(a_2) \in B$, we know $\beta(a_1) \leq_{\mathcal{B}} \beta(a_2) \implies \alpha(\beta(a_1)) \leq_{\mathcal{C}} \alpha(\beta(a_2))$. Therefore, we have $a_1 \leq_{\mathcal{A}} a_2 \implies (\alpha \circ \beta)(a_1) \leq_{\mathcal{C}} (\alpha \circ \beta)(a_2)$ and we can now conclude that $\alpha \circ \beta$ is an order homomorphism. $\square$

It is easy to see that the composition of two order isomorphisms is itself an order isomorphism since the composition of two bijections is itself a bijection.

Theorem 2.4.12. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets and let $f : P \to Q$ be a surjective function from P to Q. Then the following statements are equivalent.

1. f is an order isomorphism

2. $p_1 \leq p_2 \iff f(p_1) \preceq f(p_2)$ for any $p_1, p_2 \in P$

Proof. Suppose f is an order isomorphism. Then $p_1 \leq p_2 \implies f(p_1) \preceq f(p_2)$ for any $p_1, p_2 \in P$. Since f is an order isomorphism, we know $f(p_1) \preceq f(p_2) \implies f^{-1}(f(p_1)) \leq f^{-1}(f(p_2)) \implies p_1 \leq p_2$. Therefore, $p_1 \leq p_2 \iff f(p_1) \preceq f(p_2)$.

Now suppose $p_1 \leq p_2 \iff f(p_1) \preceq f(p_2)$ for any $p_1, p_2 \in P$. We already know that f is surjective and have the necessary inequality implication of an order homomorphism, so we need only show that f is injective. If $f(p_1) = f(p_2)$, then $f(p_1) \preceq f(p_2)$ and $f(p_2) \preceq f(p_1)$. But this implies $p_1 \leq p_2$ and $p_2 \leq p_1$. Thus $p_1 = p_2$ and f must be injective. No we can conclude that f is an order isomorphism. □

Theorem 2.4.13. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets where $P \subseteq Q$. Then $\mathcal{P} = \mathcal{Q}$ if and only if the function $f : P \to Q$ defined by $f(x) = x$ is an order isomorphism.

Proof. Suppose $\mathcal{P} = \mathcal{Q}$. Then P and Q must be the same set and $\leq$ and $\preceq$ must be the same partial ordering. Therefore, f must be a bijection such that for any $x, y \in P = Q$, $x \leq y \iff f(x) = x \preceq y = f(y)$ and we can conclude that f must be an order isomorphism.

Now suppose f is an order isomorphism. Since $P \subseteq Q$, P and Q must be the same set in order for $f(x) = x$ to define a bijection. Further, we know $x \leq y \iff x = f(x) \preceq f(y) = y$ by Theorem 2.4.12. Therefore, $\leq$ and $\preceq$ define the same partial ordering and we have $\mathcal{P} = \mathcal{Q}$. □

Theorem 2.4.14. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets and let $f : P \to Q$ be a function such that $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$. Then f is an order homomorphism.

Proof. Suppose $x \leq y$. Then $x \vee y = y$ and we have $f(y) = f(x \vee_{\mathcal{P}} y) = f(x) \vee_{\mathcal{Q}} f(y)$. But this implies $f(x) \preceq f(y)$ and thus f is an order homomorphism. □

Theorem 2.4.15. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets and let $f : P \to Q$ be a function such that $f(\bigwedge_{\mathcal{P}} X) = \bigwedge_{\mathcal{Q}} f(X)$. Then f is an order homomorphism.

Proof. Suppose $x \leq y$. Then $x \wedge y = x$ and we have $f(x) = f(x \wedge_{\mathcal{P}} y) = f(x) \wedge_{\mathcal{Q}} f(y)$. But this implies $f(x) \preceq f(y)$ and thus f is an order homomorphism. $\square$

Definition 2.4.12. A poset $\mathcal{P} = (P, \leq)$ is called *bipartite* if there exists nonempty disjoint antichains A and B such that $P = A \cup B$.

Definition 2.4.13. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets. We say the functions $f : P \to Q$ and $g : Q \to P$ form an *adjunction* provided $f(p) \preceq q \iff p \leq g(q)$ for any $p \in P$ and $q \in Q$. When f and g form an adjunction between $\mathcal{P}$ and $\mathcal{Q}$ we write $(f, g) : P \rightleftharpoons Q$ and say that f is the *left adjoint* of g and g is the *right adjoint* of f.

Theorem 2.4.16. Let $(f, g) : P \rightleftharpoons Q$ be an adjunction between posets $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$. Then

1. $f(g(q)) \preceq q$ and $p \leq g(f(p))$ for any $p \in P$ and $q \in Q$.

2. f and g are order homomorphisms.

3. $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$ and $g(\bigwedge_{\mathcal{Q}} Y) = \bigwedge_{\mathcal{P}} g(Y)$ for all $X \subseteq P$ and $Y \subseteq Q$

4. $f \circ g \circ f = f$ and $g \circ f \circ g = g$

Proof.

1. Suppose $q \in Q$. Then $g(q) \leq g(q)$ and thus $f(g(q)) \preceq q$ by the definition of adjunction. Similarly, if $p \in P$, then $f(p) \preceq f(p)$ and $p \leq g(f(p))$ by the definition of adjunction.

2. Suppose (by way of contradiction) that f is not isotone. Then there exist $p_1, p_2 \in P$ such that $p_1 \leq p_2$ and $f(p_1) \npreceq f(p_2)$. By the definition of adjunction, this gives us $p_1 \nleq g(f(p_2))$. But this is a contradiction since by (1) we know

$p_1 \leq p_2 \leq g(f(p_2))$. Similarly, if we assume that g is not isotone, then there must exist $q_1, q_2 \in Q$ such that $q_1 \preceq q_2$ and $g(q_1) \not\leq g(q_2)$. This implies $f(g(q_1)) \not\preceq q_2$. But $f(g(q_1)) \leq q_1 \leq q_2$ by (1) and we have another contradiction. Therefore, both f and g must be isotone.

3. Since f is isotone and $x \leq \bigvee_{\mathcal{P}} X$ for all $x \in X \subseteq P$, we know $\bigvee_{\mathcal{Q}} f(X) \preceq f(\bigvee_{\mathcal{P}} X)$. So we need only show $f(\bigvee_{\mathcal{P}} X) \preceq \bigvee_{\mathcal{Q}} f(X)$. Suppose (by way of contradiction) that $f(\bigvee_{\mathcal{P}} X) \not\preceq \bigvee_{\mathcal{Q}} f(X)$. Then $\bigvee_{\mathcal{P}} X \not\leq g(\bigvee_{\mathcal{Q}} f(X))$ by the definition of adjunction. Further, $g(\bigvee_{\mathcal{Q}} f(X)) \leq g(f(\bigvee_{\mathcal{P}} X)$ since g is isotone. So $\bigvee_{\mathcal{P}} X \not\leq g(f(\bigvee_{\mathcal{P}} X))$. But this is a contradicion since by (1) we know $\bigvee_{\mathcal{P}} X \leq g(f(\bigvee_{\mathcal{P}} X))$. Therefore, $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$.

   Similarly, since g is isotone and $\bigwedge_{\mathcal{Q}} Y \preceq y$ for all $y \in Y \subseteq Q$, we know $g(\bigwedge_{\mathcal{Q}} Y) \leq \bigwedge_{\mathcal{P}} g(Y)$. So we need only show $\bigwedge_{\mathcal{P}} g(Y) \leq g(\bigwedge_{\mathcal{Q}} Y)$. Suppose (by way of contradiction) that $\bigwedge_{\mathcal{P}} g(Y) \not\leq g(\bigwedge_{\mathcal{Q}} Y)$. Then $f(\bigwedge_{\mathcal{P}} g(Y)) \not\preceq \bigwedge_{\mathcal{Q}} Y$ by the definition of adjunction. Further, $f(g(\bigwedge_{\mathcal{Q}} Y) \preceq f(\bigwedge_{\mathcal{P}} g(Y))$ since f is isotone. So $f(g(\bigwedge_{\mathcal{Q}} Y)) \not\preceq \bigwedge_{\mathcal{Q}} Y$. But this is a contradiction since by (1) we know $f(g(\bigwedge_{\mathcal{Q}} Y)) \preceq \bigwedge_{\mathcal{Q}} Y$. Therefore, $g(\bigwedge_{\mathcal{Q}} Y) = \bigwedge_{\mathcal{P}} g(Y)$.

4. Let $p \in P$. We already know from (1) that $f(g(f(p))) \leq f(p)$ so to prove $f \circ g \circ f = f$ we need only show that $f(p) \leq f(g(f(p)))$. Suppose (by way of contradiction) that $f(p) \not\leq f(g(f(p)))$. Then since f is isotone (2), $p \not\leq g(f(p))$, which is a contradiction (of (1)).

   Now let $q \in Q$. Again, we know from (1) that $g(q) \leq g(f(g(q)))$ so to prove $g \circ f \circ g = g$ we need only show that $g(f(g(q))) \leq g(q)$. Suppose (by way of contradiction) that $g(f(g(q))) \not\leq g(q)$. Then since g is isotone, $f(g(q)) \not\leq q$, which is a contradiction.

□

Theorem 2.4.17. [J. Hart and Z. French [4]] Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q}(Q, \preceq)$ be posets and let $f : P \to Q$ and $g : Q \to P$ be functions. Then the following are equivalent:

1. f and g form an adjunction between $\mathcal{P}$ and $\mathcal{Q}$.

2. g is isotone and $g^{-1}(\uparrow_{\mathcal{P}} p) = \uparrow_{\mathcal{Q}} f(p)$ for any $p \in P$.

3. f is isotone and $f^{-1}(\downarrow_{\mathcal{Q}} q) = \downarrow_{\mathcal{P}} g(q)$ for any $q \in Q$.

Proof.

(1 $\implies$ 2)
Let f and g form an adjunction between $\mathcal{P}$ and $\mathcal{Q}$. We know that g is isotone from Theorem 2.4.16, so we need only show that $g^{-1}(\uparrow_{\mathcal{P}} p) = \uparrow_{\mathcal{Q}} f(p)$ for any $p \in P$.

Suppose $q \in g^{-1}(\uparrow_{\mathcal{P}} p)$ for some $p \in P$. Then $g(q) \in \uparrow_{\mathcal{P}} p$. This implies that $p \leq g(q)$, which gives us $f(p) \preceq q$ by the definition of adjunction. Therefore, $q \in \uparrow_{\mathcal{Q}} f(p)$ and we have $g^{-1}(\uparrow_{\mathcal{P}} p) \subseteq \uparrow_{\mathcal{Q}} f(p)$.

Now suppose $q \in \uparrow_{\mathcal{Q}} f(p)$. Then $f(p) \preceq q$. By the definition of adjunction we then have $p \leq g(q)$. But this implies $g(q) \in \uparrow_{\mathcal{P}} p$ and thus $q \in g^{-1}(\uparrow_{\mathcal{P}} p)$. Therefore, $\uparrow_{\mathcal{Q}} f(p) \subseteq g^{-1}(\uparrow_{\mathcal{P}} p)$ and we can now conclude that $g^{-1}(\uparrow_{\mathcal{P}} p) = \uparrow_{\mathcal{Q}} f(p)$ for any $p \in P$.

(2 $\implies$ 3)
Let g be isotone and $g^{-1}(\uparrow_{\mathcal{P}} p) = \uparrow_{\mathcal{Q}} f(p)$ for any $p \in P$.

Suppose $x \leq y$ for some $x, y \in P$. Then $\uparrow_{\mathcal{P}} y \subseteq \uparrow_{\mathcal{P}} x$, which implies $g^{-1}(\uparrow_{\mathcal{P}} y) \subseteq g^{-1}(\uparrow_{\mathcal{P}} x)$. But this gives us $\uparrow_{\mathcal{Q}} f(y) \subseteq \uparrow_{\mathcal{Q}} f(x)$, and this implies $f(x) \preceq f(y)$. Therefore, f is isotone.

Suppose $p \in f^{-1}(\downarrow_{\mathcal{Q}} q)$ for some $q \in Q$. Then $f(p) \in \downarrow_{\mathcal{Q}} q$, which implies $f(p) \preceq q$. Therefore, $\uparrow_{\mathcal{Q}} q \subseteq \uparrow_{\mathcal{Q}} f(p)$. But $\uparrow_{\mathcal{Q}} f(p) = g^{-1}(\uparrow_{\mathcal{P}} p)$, so this gives us $\uparrow_{\mathcal{Q}} q \subseteq g^{-1}(\uparrow_{\mathcal{P}} p)$. This implies $g(\uparrow_{\mathcal{Q}} q) \subseteq \uparrow_{\mathcal{P}} p$. But $q \in \uparrow_{\mathcal{Q}} q$, so $g(q) \in \uparrow_{\mathcal{P}} p$, which gives us $p \leq g(q)$. Therefore, $p \in \downarrow_{\mathcal{P}} g(q)$ and we have $f^{-1}(\downarrow_{\mathcal{Q}} q) \subseteq \downarrow_{\mathcal{P}} g(q)$.

Now suppose $p \in \downarrow_{\mathcal{P}} g(q)$. Then $p \leq g(q)$, which implies $\uparrow_{\mathcal{P}} g(q) \subseteq \uparrow_{\mathcal{P}} p$. Therefore, we have $g^{-1}(\uparrow_{\mathcal{P}} g(q)) \subseteq g^{-1}(\uparrow_{\mathcal{P}} p)$. But $g^{-1}(\uparrow_{\mathcal{P}} p) = \uparrow_{\mathcal{Q}} f(p)$, so we have $g^{-1}(\uparrow_{\mathcal{P}} g(q)) \subseteq \uparrow_{\mathcal{Q}} f(p)$. Further, $g(q) \in \uparrow_{\mathcal{P}} g(q)$, so $g^{-1}(g(q)) \subseteq \uparrow_{\mathcal{Q}} f(p)$, which gives us $q \in \uparrow_{\mathcal{Q}} f(p)$. But this implies $f(p) \preceq q$. Therefore, $\downarrow_{\mathcal{Q}} f(p) \subseteq \downarrow_{\mathcal{Q}} q$. Taking the inverse image of f on both sides gives us $f^{-1}(\downarrow_{\mathcal{Q}} f(p)) \subseteq f^{-1}(\downarrow_{\mathcal{Q}} q)$. But like before, $f(p) \in \downarrow_{\mathcal{Q}} f(p)$, so $f^{-1}(f(p)) \subseteq f^{-1}(\downarrow_{\mathcal{Q}} q)$. Therefore, $p \in f^{-1}(\downarrow_{\mathcal{Q}} q$ and we have $\downarrow_{\mathcal{P}} g(q) \subseteq f^{-1}(\downarrow_{\mathcal{Q}} q)$. We can now conclude that $f^{-1}(\downarrow_{\mathcal{Q}} q) = \downarrow_{\mathcal{P}} g(q)$ for any $q \in Q$.

$(3 \implies 1)$
Let f be isotone and $f^{-1}(\downarrow_{\mathcal{Q}} q) = \downarrow_{\mathcal{P}} g(q)$ for any $q \in Q$.

Suppose $f(p) \preceq q$ for some $p \in P$ and $q \in Q$. Then $\downarrow_{\mathcal{Q}} f(p) \subseteq \downarrow_{\mathcal{Q}} q$. Taking the inverse image of f on both sides gives us $f^{-1}(\downarrow_{\mathcal{Q}} f(p)) \subseteq f^{-1}(\downarrow_{\mathcal{Q}} q)$. But $f^{-1}(\downarrow_{\mathcal{Q}} q) = \downarrow_{\mathcal{P}} g(q)$, so we have $f^{-1}(\downarrow_{\mathcal{Q}} f(p)) \subseteq \downarrow_{\mathcal{P}} g(q)$. Further, $f(p) \in \downarrow_{\mathcal{Q}} f(p)$, so $p \in f^{-1}(\downarrow_{\mathcal{Q}} f(p))$. Therefore, $p \in \downarrow_{\mathcal{P}} g(q)$, which implies $p \leq g(q)$.

Now suppose $p \leq g(q)$. Then $\downarrow_{\mathcal{P}} p \subseteq \downarrow_{\mathcal{P}} g(q)$. But $\downarrow_{\mathcal{P}} g(q) = f^{-1}(\downarrow_{\mathcal{Q}} q)$, so $\downarrow_{\mathcal{P}} p \subseteq f^{-1}(\downarrow_{\mathcal{Q}} q)$. Taking the image of f on both sides gives us $f(\downarrow_{\mathcal{P}} p) \subseteq \downarrow_{\mathcal{Q}} q$. Further, $p \in \downarrow_{\mathcal{P}} p$, so $f(p) \in f(\downarrow_{\mathcal{P}} p)$, which gives us $f(p) \in \downarrow_{\mathcal{Q}} q$. Therefore, $f(p) \preceq q$. Now we can conclude $f(p) \preceq q$ if and only if $p \leq g(q)$ for any $p \in P$ and $q \in Q$, and thus f and g form an adjunction between $\mathcal{P}$ and $\mathcal{Q}$.

□

Theorem 2.4.18. Let $(f, g) : P \rightleftharpoons Q$ be an adjunction between posets $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$. Then f and g uniquely determine each other and we have

$$f(p) = \bigwedge\nolimits_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} p)$$
$$g(q) = \bigvee\nolimits_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} q)$$

for all $p \in P$ and $q \in Q$.

Proof. Theorem 2.4.18 follows directly from Theorem 2.4.17:

$$f(p) = \bigwedge_{\mathcal{Q}} \uparrow_{\mathcal{Q}} f(p) = \bigwedge_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} p)$$

$$g(q) = \bigvee_{\mathcal{P}} \downarrow_{\mathcal{P}} g(q) = \bigvee_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} q).$$

$\square$

For any left adjoint, α, we will denote the unique right adjoint of α as τ_α.

Theorem 2.4.19. Let $(f, \tau_f) : Q \rightleftharpoons R$ and $(g, \tau_g) : P \rightleftharpoons Q$ be adjunctions between posets $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$. Then $\tau_{f \circ g} = \tau_g \circ \tau_f$.

Proof. First we will show that $[\tau_g \circ \tau_f](x) \leq_{\mathcal{P}} \tau_{f \circ g}(x)$ for any $x \in R$. Suppose (by way of contradiction) that there is some x such that $\tau_f(\tau_g(x)) \not\leq_{\mathcal{P}} \tau_{f \circ g}(x)$. Then by the definition of adjunction we have $[f \circ g](\tau_g(\tau_f(x))) = f(g(\tau_g(\tau_f(x)))) \not\leq_{\mathcal{R}} x$. We know $g(\tau_g(\tau_f(x))) \leq_{\mathcal{Q}} \tau_f(x)$, and since f is isotone, we have $f(g(\tau_g(\tau_f(x)))) \leq_{\mathcal{R}} f(\tau_f(x))$. But $f(\tau_f(x)) \leq_{\mathcal{R}} x$; therefore, we have $f(g(\tau_g(\tau_f(x)))) \leq_{\mathcal{R}} x$, which is a contradiction. Therefore, $[\tau_g \circ \tau_f](x) \leq_{\mathcal{P}} \tau_{f \circ g}(x)$.

We will now show that $\tau_{f \circ g}(x) \leq_{\mathcal{P}} [\tau_g \circ \tau_f](x)$. Let $x \in R$ We know from Theorem 2.4.18 that

$$\tau_{f \circ g}(x) = \bigvee\nolimits_{\mathcal{P}} [f \circ g]^{-1}(\downarrow_{\mathcal{R}} x) = \bigvee\nolimits_{\mathcal{P}} g^{-1}(f^{-1}(\downarrow_{\mathcal{R}} x)).$$

It is easy to see that

$$f^{-1}(\downarrow_{\mathcal{R}} x) \subseteq \downarrow_{\mathcal{Q}} \bigvee\nolimits_{\mathcal{Q}} f^{-1}(\downarrow_{\mathcal{R}} x),$$

which implies

$$g^{-1}(f^{-1}(\downarrow_{\mathcal{R}} x)) \subseteq g^{-1}\left(\downarrow_{\mathcal{Q}} \bigvee\nolimits_{\mathcal{Q}} f^{-1}(\downarrow_{\mathcal{R}} x)\right).$$

Therefore, we have

$$\begin{aligned}
\tau_{f\circ g}(x) &= \bigvee_{\mathcal{P}} g^{-1}(f^{-1}(\downarrow_{\mathcal{R}} x) \\
&\leq_{\mathcal{P}} \bigvee_{\mathcal{P}} g^{-1}\left(\downarrow_{\mathcal{Q}} \bigvee_{\mathcal{Q}} f^{-1}(\downarrow_{\mathcal{R}} x)\right) \\
&= \bigvee_{\mathcal{P}} g^{-1}(\downarrow_{\mathcal{Q}} \tau_f(x)) \\
&= \tau_g(\tau_f(x)) \\
&= [\tau_g \circ \tau_f](x).
\end{aligned}$$

We can now conclude that $\tau_{f\circ g} = \tau_g \circ \tau_f$.

□

2.4.2 Lattices

Definition 2.4.14. A *lattice* is a poset $\mathcal{L} = (L, \leq)$ such that for any $x, y \in L$, $\bigvee\{x, y\}$ and $\bigwedge\{x, y\}$ are in L.

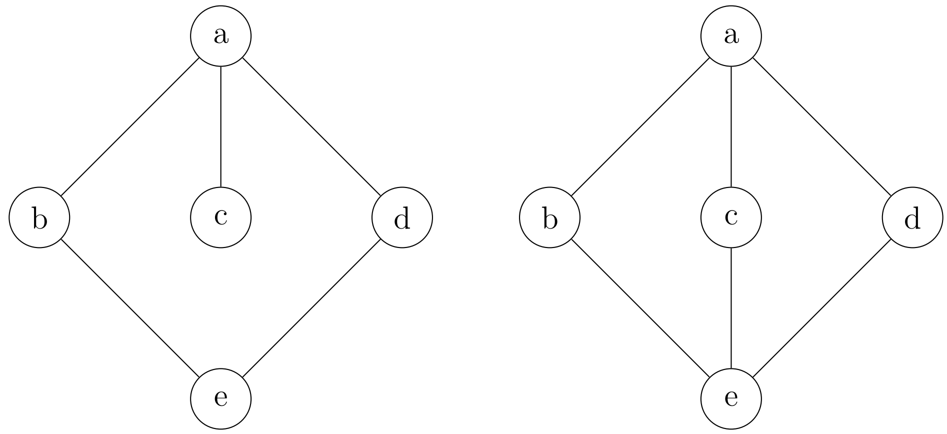

Figure 2: *Hasse Diagrams of a poset $\mathcal{P}$ that is not a lattice (left) and a poset $\mathcal{L}$ that is a lattice (right). As you can see, $b \wedge c, c \wedge d$, and $c \wedge e$ exist in $\mathcal{L}$ but not in $\mathcal{P}$.*

Theorem 2.4.20. Let $\mathcal{L} = (L, \leq)$ be a lower bounded lattice. Then $\bigvee \emptyset = \perp$.

Proof. By definition of join we have $\bigvee \emptyset = \min\{l \in L : x \leq l \text{ for all } l \in \emptyset\}$. But all $l \in L$ are vacuously greater than or equal to every member of $\emptyset$. Therefore, $\bigvee \emptyset = \min(L) = \perp$. $\square$

Theorem 2.4.21. Let $\mathcal{L} = (L, \leq)$ be an upper bounded lattice. Then $\bigwedge \emptyset = \top$.

Proof. By definition of meet we have $\bigwedge \emptyset = \max\{l \in L : l \leq x \text{ for all } l \in \emptyset\}$. Like in the pervious proof, all $l \in L$ are vacuosly less than or equal to every member of $\emptyset$. Therefore, $\bigwedge \emptyset = \max(L) = \top$. $\square$

Theorem 2.4.22. Every member of a finite lattice is compact.

Proof. Let $\mathcal{L} = (L, \leq)$ be a finite lattice. Then every directed subset $D \subset L$ must necessarily contain $\bigvee D$ since D is finite and D contains an upper bound of D. Now suppose $l \leq \bigvee D$. Since $\bigvee D \in D$, we have $l \leq d$ for some $d \in D$ trivially. Therefore, all members of $\mathcal{L}$ are compact. $\square$

Definition 2.4.15. Let $\mathcal{L} = (L, \leq)$ be a lattice. We say $\mathcal{L}$ is a *complete lattice* (or that it is *complete*) provided the following holds.

1. $\forall X \subseteq L, \bigvee X \in L$

2. $\forall X \subseteq L, \bigwedge X \in L$

It follows from induction that any finite lattice is complete.

Theorem 2.4.23. Let $\mathcal{P}$ be a complete lattice and let $\mathcal{Q}$ be a poset. Then a function $f : P \to Q$ has a right adjoint if and only if $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$ for any $X \subseteq P$.

Proof. Suppose f has a right adjoint, g. Let $X \subseteq P$. We know that for any $x \in X$, $x \leq \bigvee_{\mathcal{P}} X$. Since f and g form an adjunction, they must both be order homomorphisms. Therefore, $f(x) \preceq f(\bigvee_{\mathcal{P}} X)$. But this implies that $\bigvee_{\mathcal{Q}} f(X) \preceq f(\bigvee_{\mathcal{P}} X)$. Further, we know $f(x) \preceq \bigvee_{\mathcal{Q}} f(X)$. By the definition of adjunction, this gives us $x \leq g(\bigvee_{\mathcal{Q}} f(X))$, which implies $\bigvee_{\mathcal{P}} X \leq g(\bigvee_{\mathcal{Q}} f(X))$. Using the definition of adjunction once again gives us $f(\bigvee_{\mathcal{P}} X) \preceq \bigvee_{\mathcal{Q}} f(X)$ and we can conclude that $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$.

Now suppose $f(\bigvee_{\mathcal{P}} X) = \bigvee_{\mathcal{Q}} f(X)$. Let $g : Q \to P$ be a function defined by $g(q) = \bigvee_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} q)$. If $f(x) \preceq y$ for some $x \in P$ and $y \in Q$, then we have

$$\begin{aligned} f(x) \preceq y &\iff f(x) \in \downarrow_{\mathcal{Q}} y \\ &\iff x \in f^{-1}(\downarrow_{\mathcal{Q}} y) \\ &\implies x \leq \bigvee_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} y) = g(y). \end{aligned}$$

Unfortunately, going the other way is not as direct. If $x \leq g(y)$, then we have $x \leq \bigvee_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} y)$. f must be isotone since it preserves arbitrary joins; therefore, $f(x) \preceq f(\bigvee_{\mathcal{P}} f^{-1}(\downarrow_{\mathcal{Q}} y))$. But since f preserves joins, this gives us

$$f(x) \preceq \bigvee\nolimits_{\mathcal{Q}} f(f^{-1}(\downarrow_{\mathcal{Q}} y)) = \bigvee\nolimits_{\mathcal{Q}} \downarrow_{\mathcal{Q}} y = y.$$

Therefore, we have $f(x) \preceq y \iff x \leq g(y)$ for any $x \in P$ and $y \in Q$ and g is the right adjoint of f. $\square$

Theorem 2.4.24. Let $\mathcal{Q}$ be a complete lattice and let $\mathcal{P}$ be a poset. Then a function $g : Q \to P$ has a left adjoint if and only if $g(\bigwedge_{\mathcal{Q}} Y) = \bigwedge_{\mathcal{P}} g(Y)$ for any $Y \subseteq Q$.

Proof. Suppose g has a right adjoint, f. Let $Y \subseteq Q$. We know that for any $y \in Y$, $\bigwedge_{\mathcal{Q}} Y \preceq y$. Since f and g form an adjunction, they must both be order homomorphisms. Therefore, $g(\bigwedge_{\mathcal{Q}} Y) \leq g(y)$. But this implies that $g(\bigwedge_{\mathcal{Q}} Y) \leq \bigwedge_{\mathcal{P}} g(Y)$. Further, we know $\bigwedge_{\mathcal{P}} g(Y) \leq g(y)$. By the definition of adjunction, this gives us $f(\bigwedge_{\mathcal{P}} g(Y)) \preceq y$, which implies $f(\bigwedge_{\mathcal{P}} g(Y)) \preceq \bigwedge_{\mathcal{Q}} Y$. Using the definition of adjunction once again gives us $\bigwedge_{\mathcal{P}} g(Y) \leq g(\bigwedge_{\mathcal{Q}} Y)$ and we can conclude that $g(\bigwedge_{\mathcal{Q}} Y) = \bigwedge_{\mathcal{P}} g(Y)$.

Now suppose $g(\bigwedge_{\mathcal{Q}} Y) = \bigwedge_{\mathcal{P}} g(Y)$. Let $f : P \to Q$ be a function defined by $f(p) = \bigwedge_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} p)$. If $x \leq g(y)$ for some $x \in P$ and $y \in Q$, then we have

$$\begin{aligned} x \leq g(y) &\iff g(y) \in \uparrow_{\mathcal{P}} x \\ &\iff y \in g^{-1}(\uparrow_{\mathcal{P}} x) \\ &\implies f(x) = \bigwedge_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} x) \preceq y. \end{aligned}$$

Unfortunately, going the other way is not as direct. If $f(x) \preceq y$, then we have $\bigwedge_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} x) \preceq y$. g must be isotone since it preserves arbitrary meets; therefore, $g(\bigwedge_{\mathcal{Q}} g^{-1}(\uparrow_{\mathcal{P}} x)) \preceq g(y)$. But since g preserve meets, this gives us

$$x = \bigwedge_{\mathcal{P}} \uparrow_{\mathcal{P}} x = \bigwedge_{\mathcal{P}} g(g^{-1}(\uparrow_{\mathcal{P}} x)) \leq g(y).$$

Therefore, we have $f(x) \preceq y \iff x \leq g(y)$ for any $x \in P$ and $y \in Q$ and f is the left adjoint of g. $\square$

Theorem 2.4.25. Let $\mathcal{P} = (P, \leq)$ be a poset. Then $Low(\mathcal{P})$ and $Up(\mathcal{P})$ are both complete lattices.

Proof. We know that $\emptyset$ and P must be in both $Low(\mathcal{P})$ and $Up(\mathcal{P})$. Therefore, any pair of elements in $Low(\mathcal{P})$ or $Up(\mathcal{P})$ must have both an infimum and supremum. So we can conclude that both $Low(\mathcal{P})$ and $Up(\mathcal{P})$ are lattices.

Let $X \subseteq Low(\mathcal{P})$. Suppose $x \in \bigcup X$. Then x must be in some lowerset contained in X. So if $y \leq x$, then y must be in the lowerset containing x, which implies $y \in \bigcup X$. Therefore, $\bigcup X$ is a lowerset of $\mathcal{P}$. Further, it is easy to see that $\bigcup X$ is the least upper bound of X in $Low(\mathcal{P})$.

Now suppose $x \in \bigcap X$. Then x is in every lowerset contained in X. So if $y \leq x$, then y must be in every lowerset contained in X. Therefore, $y \in \bigcap X$ and $\bigcap X$ must be a lowerset of $Low(\mathcal{P})$. Since $\bigcap X$ is the greatest upper bound of X in $Low(\mathcal{P})$, we can conclude that $Low(\mathcal{P})$ is complete.

The proof for $Up(\mathcal{P})$ follows in the same manner since $Low(\mathcal{P})$ and $Up(\mathcal{P})$ are dual notions. $\square$

Definition 2.4.16. A lattice $\mathcal{L} = (L, \leq)$ is *distributive* provided $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$.

Theorem 2.4.26. A lattice $\mathcal{L} = (L, \leq)$ is *distributive* if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

Proof. First, suppose $\mathcal{L}$ is distributive. Then we know $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

for all $x, y, z \in L$. Using this and the other properties of meet and join we get

$$\begin{aligned}
x \vee (y \wedge z) &= [x \vee (x \wedge z)] \vee (y \wedge z) \\
&= x \vee [(x \wedge z) \vee (y \wedge z)] \\
&= x \vee [(z \wedge x) \vee (z \wedge y)] \\
&= x \vee [z \wedge (x \vee y)] \\
&= [x \wedge (x \vee y)] \vee [z \wedge (x \vee y)] \\
&= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\
&= (x \vee y) \wedge (x \vee z).
\end{aligned}$$

Similarly, if we assume $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$, we get

$$\begin{aligned}
x \wedge (y \vee z) &= [x \wedge (x \vee z)] \wedge (y \vee z) \\
&= x \wedge [(x \vee z) \wedge (y \vee z)] \\
&= x \wedge [(z \vee x) \wedge (z \vee y)] \\
&= x \wedge [z \vee (x \wedge y)] \\
&= [x \vee (x \wedge y)] \wedge [z \vee (x \wedge y)] \\
&= [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \\
&= (x \wedge y) \vee (x \wedge z).
\end{aligned}$$

Therefore, $\mathcal{L}$ must be distributive. $\square$

Theorem 2.4.37 has a secondary (somewhat obvious) consequence: A lattice is distributive if and only if its order dual is distributive. This will be useful later.

Theorem 2.4.27. Let $\mathcal{L} = (L, \leq)$ be a lattice. Then $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ for any $x, y, z \in L$.

Proof. We know $x \wedge y \leq x$ and $x \wedge z \leq x$. Therefore, $(x \wedge y) \vee (x \wedge z) \leq x \vee x = x$. But $x \leq x \wedge (y \vee z)$, so we have $(x \wedge y) \vee (x \wedge z) \leq x \leq x \wedge (y \vee z)$. □

This simple result shows us that in order to show a lattice $\mathcal{L} = (L, \leq)$ is distributive, we only need to show $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$.

Definition 2.4.17. Given a lattice, $\mathcal{L}$, we say j is *join-irreducible* in $\mathcal{L}$ if $j \neq \perp$ (if it exists) and $j = a \vee b \implies j = a$ or $j = b$. Similarly, m is *meet-irreducible* in $\mathcal{L}$ provided $m \neq \top$ (if it exists) and $m = a \wedge b \implies m = a$ or $m = b$.

Given a lattice, $\mathcal{L}$, we will denote $Ji(\mathcal{L})$ as the poset of all join-irreducible elements of $\mathcal{L}$ and $Mi(\mathcal{L})$ as the poset of all meet-irreducible elements of $\mathcal{L}$.

Theorem 2.4.28. Let $\mathcal{L} = (L, \leq)$ be a lattice. Then $j \in L$ is join-irreducible in $\mathcal{L}$ if and only if j is meet-irreducible in $\mathcal{L}^{op}$.

Proof. Let $a, b \in L$. Then for any $x \in L$, we have

$$\begin{aligned} x \text{ is join-irreducible in } \mathcal{L} &\iff (x = a \vee b \implies x = a \text{ or } x = b) \\ &\iff (x = a \wedge_{op} b \implies x = a \text{ or } x = b) \\ &\iff x \text{ is meet-irreducible in } \mathcal{L}^{op}. \end{aligned}$$

□

Theorem 2.4.29. Given a lattice, $\mathcal{L} = (L, \leq)$, an element $j \in L$ is join-irreducible if and only if $|Cov^{-1}(j)| = 1$.

Proof. Let j is join-irreducible in $\mathcal{L}$. Suppose (by way of contradiction) $|Cov^{-1}(j)| \neq 1$. If $|Cov^{-1}(j)| = 0$, then j must be minimal in $\mathcal{L}$, and therefore, $j = \perp$, which is a contradiction. If $|Cov^{-1}(j)| \geq 2$, then there must be some $a, b \in Cov^{-1}(j)$ such that $a \neq b$. However, this means that $j = a \vee b$ where $j \neq a$ and $j \neq b$, which is a contradiction since j is join-irreducible. Therefore, $|Cov^{-1}(j)| = 1$.

Now let $|Cov^{-1}(j)| = 1$. Suppose $j = a \vee b$ for some $a, b \in L$. We have a few cases to consider here:

1. Consider $a, b \in Cov^{-1}(j)$. Then $a = b$ since $|Cov^{-1}(j)| = 1$ and thus $j = a \vee b = a \vee a = a$. This implies $j \in Cov^{-1}(j)$, which is a contradiction.

2. Consider $a, b \notin Cov^{-1}(j)$. Then $Cov^{-1}(j) = \{c\}$ for some $c \in L$ where $c \neq a$ and $c \neq b$. But $j = a \vee b$ implies $j \geq a$ and $j \geq b$, which means $c \geq a$ and $c \geq b$. This gives us $j > c \geq a \vee b$, which is a contradiction.

3. WLOG, consider $a \in Cov^{-1}(j)$ and $b \notin Cov^{-1}(j)$. Since $j = a \vee b$, we know $j \geq a$ and $j \geq b$. This implies $j = b$ or $j > a > b$ since $Cov^{-1}(j) = \{a\}$. But if $j > a > b$, we have $j = a \vee b = a$, which is a contradiction since $Cov^{-1}(j) = \{a\}$. Therefore, $j = b$.

Therefore, $j = a \vee b \implies j = a$ or $j = b$ and j is join-irreducible. □

Theorem 2.4.30. Given a lattice, $\mathcal{L} = (L, \leq)$, an element $m \in L$ is meet-irreducible if and only if $|Cov(m)| = 1$.

Proof. This follows from Theorems 2.4.28 and 2.4.29:

$$\begin{aligned} m \text{ is meet-irreducible in } \mathcal{L} &\iff m \text{ is join-irreducible in } \mathcal{L}^{op} \\ &\iff |Cov_{op}^{-1}(m)| = 1 \\ &\iff |Cov(m)| = 1 \end{aligned}$$

□

Theorems 2.4.29 and 2.4.30 can be used to easily identify join-irreducible and meet-irreducible elements in a Hasse diagram. Figure 3 shows a Hasse diagram of a lattice with its join-irreducible and meet-irreducible elements highlighted.

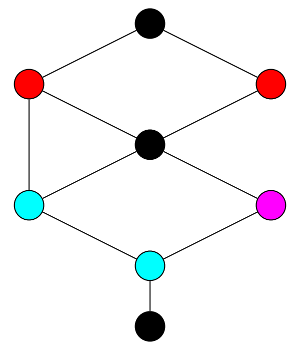

Figure 3: *Join-irreducible elements are blue and meet-irreducible elements are red. The purple element is both join-irreducible and meet-irreducible. Black elements are neither join-irreducible nor meet-irreducible.*

Definition 2.4.18. Let $\mathcal{L}$ be a lattice. We say j is *completely join-irreducible* in $\mathcal{L}$ provided for any subset $S \subseteq L$, $j = \bigvee S \implies j = s$ for some $s \in S$. Similarly, we say m is *completely meet-irreducible* in $\mathcal{L}$ provided for any subset $S \subseteq L$, $\bigwedge S = m \implies s = m$ for some $s \in S$.

Given a lattice, $\mathcal{L}$, we will denote $Cji(\mathcal{L})$ as the poset of all completely join-irreducible elements of $\mathcal{L}$ and $Cmi(\mathcal{L})$ as the poset of all completely meet-irreducible elements of $\mathcal{L}$.

Definition 2.4.19. Given a lattice, $\mathcal{L}$, we say j is *join-prime* in $\mathcal{L}$ if $j \neq \perp$ (if it exists) and $j \leq a \vee b \implies j \leq a$ or $j \leq b$. Similarly, m is *meet-prime* in $\mathcal{L}$ provided $m \neq \top$ (if it exists) and $a \wedge b \leq m \implies a \leq m$ or $b \leq m$.

Given a lattice, $\mathcal{L}$, we will denote $Jp(\mathcal{L})$ as the poset of all join-prime elements of $\mathcal{L}$ and $Mp(\mathcal{L})$ as the poset of all meet-prime elements of $\mathcal{L}$.

Theorem 2.4.31. Every join-prime member of a lattice is also join-irreducible.

Proof. Let $\mathcal{L} = (L, \leq)$ be a lattice and let j be join-prime in $\mathcal{L}$. Suppose $j = a \vee b$. Then $j \leq a \vee b$ and $j \geq a \vee b$. Since j is join-prime, we know $j \leq a$ or $j \leq b$. But $j \geq a \vee b$ tells use that $j \geq a$ and $j \geq b$. Which gives us $j = a$ or $j = b$ and we can conclude that j is join-irreducible. $\square$

Theorem 2.4.32. Every meet-prime member of a lattice is also meet-irreducible.

Proof. Let $\mathcal{L} = (L, \leq)$ be a lattice and let m be meet-prime in $\mathcal{L}$. Suppose $m = a \wedge b$. Then $a \wedge b \leq m$ and $a \wedge b \geq m$. Since m is meet-prime, we know $a \leq m$ or $b \leq m$. But $a \wedge b \geq m$ tells use that $a \geq m$ and $b \geq m$. Which gives us $m = a$ or $m = b$ and we can conclude that m is meet-irreducible. $\square$

Theorem 2.4.33. Let $L = (L, \leq)$ be a distributive lattice. Then every join-irreducible member of $\mathcal{L}$ is also join-prime in $\mathcal{L}$. Similarly, every meet-irreducible member of $\mathcal{L}$ is also meet-prime in $\mathcal{L}$.

Proof. Let j be join-irreducible in $\mathcal{L}$. Suppose $j \leq a \vee b$ for some $a, b \in L$. Then $j = j \wedge (a \vee b)$. Since $\mathcal{L}$ is distributive, we have $j = j \wedge (a \vee b) = (j \wedge a) \vee (j \wedge b)$. This implies $j = j \wedge a$ or $j = j \wedge b$ since j is join-irreducible. But if $j = j \wedge a$, then $j \leq a$, and if $j = j \wedge b$, then $j \leq b$. Therefore, we have $j \leq a$ or $j \leq b$ and we can conclude that j is also join-prime.

The proof for meet-irreducible members follows similarly. Let m be meet-irreducible in $\mathcal{L}$. Suppose $a \wedge b \leq m$ for some $a, b \in L$. Then $m \vee (a \wedge b) = m$. Since $\mathcal{L}$ is distributive, we have $(m \vee a) \wedge (m \vee b) = m$, which implies $m = m \vee a$ or $m = m \vee b$ since m is meet-irreducible. If $m = m \vee a$, then $m \geq a$, and if $m = m \vee b$, then $m \geq b$. Therefore, $m \geq a$ or $m \geq b$ and m must also be meet-prime. $\square$

Theorem 2.4.34. Let $\mathcal{L} = (L, \leq)$ be a lattice. An element, $j \in L$, is join-prime if and only if for any finite subset $F \subseteq L$, $j \leq \bigvee F \implies j \leq x$ for some $x \in F$. Similarly, an element, $m \in L$, is meet-prime if and only if for any finite subset $F \subseteq L$, $\bigwedge F \leq m \implies x \leq m$ for some $x \in F$.

Proof. Let j be join-prime in $\mathcal{L}$. By the definition of join-prime, we know that for any $F \subseteq L$ where $|F| = 2$, $j \leq \bigvee F \implies j \leq x$ for some $x \in F$. Now suppose for any $F \subseteq L$ where $|F| = n$, $j \leq \bigvee F \implies j \leq x$ for some $x \in F$. Let $F' \subseteq L$ where $|F'| = n + 1$ and let $x' \in F$. Then $j \leq \bigvee F \setminus \{x'\} \implies j \leq x$ for some $x \in F \setminus \{x'\}$. But $\bigvee F \setminus \{x'\} \leq \bigvee F$, so we have

$$
\begin{aligned}
j \leq \bigvee F &\implies j \leq \bigvee F \setminus \{x'\} \\
&\implies j \leq x \text{ for some } x \in F \setminus \{x'\} \\
&\implies j \leq x \text{ for some } x \in F.
\end{aligned}
$$

Therefore, for any finite subset $F \subseteq L$, $j \leq \bigvee F \implies j \leq x$ for some $x \in F$. We can now conclude that if j is join-prime, then for any finite subset $F \subseteq L$, $j \leq \bigvee F \implies j \leq x$ for some $x \in F$. The converse holds trivially.

Now let m be meet-prime in $\mathcal{L}$. Then m must be join-prime in $\mathcal{L}^{op}$. Therefore, for any finite subset $F \subseteq L$, $m \leq_{op} \bigvee_{op} F \implies m \leq_{op} x$ for some $x \in F$. But $m \leq_{op} \bigvee_{op} F \iff \bigwedge F \leq m$ and $m \leq_{op} x \iff x \leq m$. Therefore, we have $\bigwedge F \leq m \implies x \leq m$ for some $x \in F$. So if m is meet-prime in $\mathcal{L}$, then for any subset $F \subseteq L$, $\bigwedge F \leq m \implies x \leq m$ for some $x \in F$. The converse again holds trivially. □

Theorem 2.4.35. Let $\mathcal{L} = (L, \leq)$ be a lattice and let $j, m \in L$. Then if $\uparrow j \cap \downarrow m = \emptyset$ and $\uparrow j \cup \downarrow m = L$ then j is join-prime in $\mathcal{L}$ and m is meet-prime in $\mathcal{L}$.

Proof. Suppose $x \wedge y \leq m$ for some $x, y \in L$. Then $x \wedge y \in \downarrow m$. If x or y is in $\downarrow m$, then we have $x \leq m$ or $y \leq m$ and m must be meet-prime. Suppose x and y are not in $\downarrow m$. Then x and y must be in $\uparrow j$ since $\uparrow j \cup \downarrow m = L$. Therefore, $j \leq x$ and $j \leq y$, which implies $j \leq x \wedge y$. This gives us $x \wedge y \in \uparrow j \cap \downarrow m$, which is a contradiction. Therefore, either x or y (possibly both) are in $\downarrow m$, which gives us $x \leq m$ or $y \leq m$.

Now suppose $j \leq x \vee y$ for some $x, y \in L$. Then $x \vee y \in \uparrow j$. If x or y is in $\uparrow j$, then we have $j \leq x$ or $j \leq y$ and j must be join-prime. Suppose x and y are not in

$\uparrow j$. Then x and y must be in $\downarrow m$, which implies $x \vee y \in \downarrow m$. But then we have $x \vee y \in \uparrow j \cap \downarrow m$, which is a contradiction. Therefore, either x or y must be in $\uparrow j$ and we have $j \leq x$ or $j \leq y$. $\square$

Theorem 2.4.36. [J. Snodgrass and C. Tsinakis [8]] Let $\mathcal{L} = (L, \leq)$ be a finite lattice and let $j \in Jp(\mathcal{L})$. Then $m = \bigvee\{l \in L : j \notin \downarrow l\}$ is meet-prime in $\mathcal{L}$.

Proof. Suppose $j \leq m \bigvee\{l : j \notin \downarrow l\}$. Then since j is join-prime, $j \leq l$ for some $l \in \{l \in L : j \notin \downarrow j\}$. But $j \not\leq l$ by definition of m so we have a contradiction. Therefore, $j \not\leq m$.

Let $x \in L$. We will show $x \in \downarrow m$ if and only if $x \notin \uparrow j$. Suppose $x \in \uparrow j$, then $j \leq x$. If $x \leq m$, then we would have $j \leq x \leq m$, which is impossible. Therefore, $x \not\leq m$, so $x \notin \downarrow m$. So we have $x \in \uparrow j \implies x \notin \downarrow m$. The contrapositive of this statement gives us $x \in \downarrow m \implies x \notin \uparrow j$. Now suppose $x \notin \uparrow j$. Then $j \not\leq x$, which implies $j \notin \downarrow x$. Therefore, $x \in \{l : L : j \notin \downarrow l\}$, which gives us $x \leq \bigvee\{l \in L : j \notin \downarrow l\} = m$. So we then have $x \in \downarrow m$.

Since $x \in \downarrow m \iff x \notin \uparrow j$, we can conclude that $\uparrow j \cap \downarrow m = \emptyset$ and $\uparrow j \cup \downarrow m = L$, so m must be meet-prime. $\square$

Definition 2.4.20. Let $\mathcal{L}$ be a lattice. We say j is *completely join-prime* in $\mathcal{L}$ provided for any subset $S \subseteq L$, $j \leq \bigvee S \implies j \leq s$ for some $s \in S$. Similarly, we say m is *completely meet-prime* in $\mathcal{L}$ provided for any subset $S \subseteq L$, $\bigwedge S \leq m \implies s \leq m$ for some $s \in S$.

Given a lattice, $\mathcal{L}$, we will denote $Cjp(\mathcal{L})$ as the poset of all completely join-prime elements of $\mathcal{L}$ and $Cmp(\mathcal{L})$ as the poset of all completely meet-prime elements of $\mathcal{L}$.

It should be evident that in any lattice, completely join-prime elements are also join-prime and completely meet-prime elements are also meet-prime. Similarly, by Theorem 2.4.34, in a finite lattice every join-prime element is also completely join-prime and every meet-prime element is also completely-meet-prime.

Definition 2.4.21. [J. Hart and Z. French [4]] Let $\mathcal{L} = (L, \leq)$ be a finite lattice and $X \subseteq L$. We say X is *join-dense* in $\mathcal{L}$ if every element of L is the join of a (possibly empty) subset of X. Similarly, we say X is *meet-dense* in $\mathcal{L}$ if every element of L is the meet of a (possibly empty) subset of X.

Theorem 2.4.37. Let $\mathcal{L}$ be a finite lattice. Then the following are equivalent.

1. $\mathcal{L}$ is distributive.
2. $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$.
3. $Mp(\mathcal{L})$ is meet-dense in $\mathcal{L}$.

Proof. Let $\mathcal{L}$ be a finite lattice.

$(1 \implies 2)$

We will assume that $\mathcal{L}$ is distributive and suppose (by way of contradiction) that $Jp(\mathcal{L})$ is not join-dense in $\mathcal{L}$. Then there exists some $l \in L$ such that $\bigvee J \neq l$ where $J = \{j \in Jp(\mathcal{L}) : j \leq_{\mathcal{L}} l\}$. Therefore, $\bigvee J <_{\mathcal{L}} l$ and $l \notin Jp(\mathcal{L})$. But since $\mathcal{L}$ is distributive, we know if $l \notin Jp(\mathcal{L})$, then $l \notin Ji(\mathcal{L})$ and thus $|Cov^{-1}(l)| > 1$. But $|Cov^{-1}(l) \cap J| \leq 1$; otherwise, $\bigvee J = l$. So there must be some $X \subseteq Cov^{-1}(l)$ such that $X \cap J = \emptyset$. For each $x \in X$, we can follow the same line of reasoning to conclude that $|Cov^{-1}(x)| > 1$ but $|Cov^{-1}(x) \cap J| \leq 1$. In fact, this line of reasoning continues downward until you reach an element whose inverse cover has all join-prime elements, which we are guaranteed to reach since $Cov(\perp) \subseteq Jp(\mathcal{L})$. Thus we have a contradiction and we can conclude that $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$.

$(2 \implies 1)$

Suppose $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$. Let $J = \{j \in Jp(\mathcal{L}) : j \leq x \wedge (y \vee z)\}$. Then $\bigvee J = x \wedge (y \vee z)$ since $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$. Further, we know that for any $j \in J$, $j \leq x$ and $j \leq y \vee z$. But since $j \in Jp(\mathcal{L})$ we know $j \leq y \vee z \implies j \leq y$ or $j \leq z$. Therefore, either $j \leq x$ and $j \leq y$ or $j \leq x$ and $j \leq z$. But this implies $j \leq x \wedge y$ or $j \leq x \wedge z$, which gives us $j \leq (x \wedge y) \vee (x \wedge z)$. Therefore, $x \wedge (y \vee z) = \bigvee J \leq (x \wedge y) \vee (x \wedge z)$. By Theorem 2.4.27 we have distributivity.

$(1 \iff 3)$

This follows from the dual natures of distribution and of meet-prime and join-prime elements. $\mathcal{L}$ is distributive if and only if $\mathcal{L}^{op}$ is distributive. But we know $1 \iff 2$, so $\mathcal{L}^{op}$ is distributive if and only if $Jp(\mathcal{L}^{op})$ is join-dense in $\mathcal{L}^{op}$. Since $Jp(\mathcal{L}^{op})$ is join-dense in $\mathcal{L}$ if and only if $Mp(\mathcal{L})$ is meet-dense in $\mathcal{L}$, we can now conclude that $\mathcal{L}$ is distributive if and only if $Mp(\mathcal{L})$ is meet-dense in $\mathcal{L}$.

□

Theorem 2.4.38. Let $\mathcal{L} = (L, \leq)$ be a finite distributive lattice. Then $\mathcal{L}$ is order isomorphic to $Low(Jp(\mathcal{L}))$ and $Up(Mp(\mathcal{L}))$ under subset inclusion.

Proof. Let $f : L \to Low(Jp(\mathcal{L}))$ be a function defined by $f(x) = \{j \in Jp(\mathcal{L}) : j \leq x\}$. It is easy to see that for any x, $f(x)$ is a lowerset of $Jp(\mathcal{L})$ since if $k \leq j$ for some $j \in f(x)$, then $k \leq x$ and therefore $k \in f(x)$ by the definition of f. Therefore, we know f is well-defined.

Let $J \in Low(Jp(\mathcal{L}))$. There must be some $l \in L$ such that $l = \bigvee J$. But this implies that for any $j \in J$, $j \leq l$. Therefore, $j \in f(l)$ and we know $J \subseteq f(l)$. Suppose $x \in f(l)$. Then $x \leq l = \bigvee J$. This gives us $x = x \wedge l = x \wedge \bigvee J = \bigvee_{j \in J} x \wedge j$. Therefore, $x \wedge j \leq x$ for all $j \in J$, which implies $j \leq x$ for all $j \in J$. We have two cases to now consider:

1. Suppose $j = x$ for some j. Then $x \in J$ trivially.

2. Suppose $j < x$ for all $j \in J$. Then $\bigvee J \leq x$, which gives us $x = \bigvee J = l$. So only elements of J are strictly less than l. Further, since $x = l$, l must be join-prime and $l \in f(l)$. Since every join-prime element is also join irreducible, $|Cov^{-1}(l)| = 1$. But this implies that $\bigvee J = j$ for some $j \in J$ where $j < x = l$, which is a contradiction since $l = \bigvee J$. Therefore, there must exist $j \in J$ such that $j \geq x$. Since J is a lowerset, $x \leq j \implies x \in J$.

In either case, we find that $x \in J$. Therefore, $f(l) \subseteq J$ and we have $J = f(l)$. We can conclude that f is surjective.

Now suppose $x, y \in L$ and $x \leq y$. If $j \in f(x)$, then $j \leq x \leq y$, and $y \in f(y)$ as well. Therefore, $f(x) \subseteq f(y)$. Similarly, if $f(x) \subseteq f(y)$, then for any $j \in f(x)$, j must also be in $f(y)$. Therefore, $j \leq x \implies j \leq y$. Since we know that the join-prime members of $\mathcal{L}$ are join-dense in $\mathcal{L}$ (by Theorem 2.4.37) and $f(x)$ includes all the join-prime elements less than or equal to x, we know $x = \bigvee f(x)$. But $\bigvee f(x) \leq y$ since $j \leq x \implies j \leq y$. Therefore, $x \leq y$ and we can conclude that $x \leq y \iff f(x) \subseteq f(y)$ and f must be an order isomorphism by Theorem 2.4.12.

The proof for $Up(Mp(\mathcal{L}))$ follows similarly. $\square$

2.4.3 Frames

Definition 2.4.22. A lattice $\mathcal{L} = (L, \leq)$ is called a *frame* provided

1. $\mathcal{L}$ is closed under arbitrary joins.

2. $\mathcal{L}$ is closed under finite meets.

3. $\forall l \in L, S \subseteq L, l \wedge \bigvee S = \bigvee\{l \wedge s : s \in S\}$

Definition 2.4.23. Let $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \preceq)$ be frames. Then the function $\alpha : L \to M$ is called a *frame homomorphism* provided

1. $\forall S \subseteq L, \alpha(\bigvee_{\mathcal{L}} S) = \bigvee_{\mathcal{M}}(S)$

2. $\forall S \subseteq L \ni S$ is finite and $\alpha(\bigwedge_{\mathcal{L}} S) = \bigwedge_{\mathcal{M}} \alpha(S)$

Theorem 2.4.39. The composition of two frame homomorphisms is a frame homomorphism.

Proof. Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be frames, let $f : M \to N$ and $g : L \to M$ be frame homomorphism, and let $X \subseteq L$. Then we have

$$[f \circ g](\bigvee_{\mathcal{L}} X) = f(g(\bigvee_{\mathcal{L}} X)) = f(\bigvee_{\mathcal{M}} g(X)) = \bigvee_{\mathcal{N}} f(g(X))$$

Similarly, if X is finite we have

$$[f \circ g](\bigwedge_{\mathcal{L}} X) = f(g(\bigvee_{\mathcal{L}} X)) = f(\bigvee_{\mathcal{M}} g(X)) = \bigvee_{\mathcal{N}} f(g(X)).$$

Therefore, $f \circ g$ is a frame homomorphism. $\square$

Theorem 2.4.40. Let $\mathcal{L}$ and $\mathcal{M}$ be lower bounded frames and $f : L \to M$ be a frame homomorphism from $\mathcal{L}$ to $\mathcal{M}$. Then $f(\bot_{\mathcal{L}}) = \bot_{\mathcal{M}}$.

Proof. This follows directly from the fact that frame homomorphisms preserve arbitrary joins:

$$f(\bot_{\mathcal{L}}) = f(\bigvee_{\mathcal{L}} \emptyset) = \bigvee_{\mathcal{M}} f(\emptyset) = \bigvee_{\mathcal{M}} \emptyset = \bot_{\mathcal{M}}.$$

$\square$

Theorem 2.4.41. Let $\mathcal{L}$ and $\mathcal{M}$ be upper bounded frames and $f : L \to M$ be a frame homomorphism from $\mathcal{L}$ to $\mathcal{M}$. Then $f(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$.

Proof. Similarly to the previous theorem, this follows directly from the fact that frame homomorphisms preserve finite meets:

$$f(\top_{\mathcal{L}}) = f(\bigwedge_{\mathcal{L}} \emptyset) = \bigwedge_{\mathcal{M}} f(\emptyset) = \bigwedge_{\mathcal{M}} \emptyset = \top_{\mathcal{M}}.$$

$\square$

Theorem 2.4.42. Let $(f, g) : L \rightleftharpoons M$ be an adjunction between bounded frames $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \preceq)$ such that f is a frame homomorphism. Then $m \in Mp(\mathcal{M}) \implies g(m) \in Mp(\mathcal{L})$.

Proof. Let $m \in Mp(\mathcal{M})$. If $g(m) = \top_{\mathcal{L}}$, then by the definition of adjunction we have $f(\top_{\mathcal{L}}) \preceq m$. But $f(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$, which gives us $\top_{\mathcal{M}} \preceq m$, which contradicts the fact that m is meet-prime. Therefore, $g(m) \neq \top_{\mathcal{L}}$.

Now suppose $x \wedge_{\mathcal{L}} y \leq g(m)$. Then $f(x \wedge_{\mathcal{L}} y) \preceq m$. But f preserves finite meets, so we have $f(x \wedge_{\mathcal{L}} y) = f(x) \wedge_{\mathcal{M}} f(y)$. This gives us $f(x) \wedge_{\mathcal{M}} f(y) \preceq m$. But m is meet-prime, so $f(x) \preceq m$ or $f(y) \preceq m$, which implies $x \leq g(m)$ or $y \leq g(m)$. $\square$

2.5 Topology

While most of this thesis relies on Order Theory, Topology plays a small, but pivotal role in the results. We present some standard definitions and results (most from [J. Munkres [2]]) that will be required in Chapters 5 through 7.

Definition 2.5.1. [J. Munkres [2]] A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties.

1. $\emptyset$ and X are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*. Any subset of X that is an element of $\mathcal{T}$ is said to be *open* in $\mathcal{T}$.

Definition 2.5.2. [J. Munkres [2]] If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Note that each basis element is itself an element of $\mathcal{T}$.

Theorem 2.5.1. [J. Munkres [2]] Let $\mathcal{T}$ be a topology on a set X and let $\mathcal{B}$ be a basis of $\mathcal{T}$. Then the collection of all unions of elements of $\mathcal{B}$ is precisely the topology $\mathcal{T}$.

Proof. We will follow closely with Munkres [2] proof. Let $U_{\mathcal{B}}$ denote the collection of all unions of elements of $\mathcal{B}$. Since each element of $\mathcal{B}$ is open in $\mathcal{T}$, we know that any arbitrary union of elements of $\mathcal{B}$ must also be open in $\mathcal{T}$. Therefore, $U_{\mathcal{B}} \subseteq \mathcal{T}$. Now suppose U is open in $\mathcal{T}$. Then for each $x \in U$, there exists a basis element B_x such that $x \in B_x$ and $B_x \subseteq U$. Therefore, $U = \bigcup_{x \in U} B_x$. Since each open set in $\mathcal{T}$ can be represented as a union of elements of $\mathcal{B}$, we know $\mathcal{T} \subseteq U_{\mathcal{B}}$. Therefore, $U_{\mathcal{B}} = \mathcal{T}$ and we can conclude that the collection of all unions of elements of $\mathcal{B}$ is $\mathcal{T}$. $\square$

An important point of Theorem 2.5.1 is that any open set in a topology can be represented as a union of basis elements of the topology. This fact will be useful later on.

Theorem 2.5.2. Let $\mathcal{T}$ be a topology on a set X. Then $(\mathcal{T}, \subseteq)$ is a frame.

Proof. It is easy to see that the first two properties of a frame are satisfied by the definition of a topology since joins and meets in $(\mathcal{T}, \subseteq)$ are precisely the unions and intersections (respectively) of the sets under consideration. Further, for $t \in \mathcal{T}$ and $S \subseteq \mathcal{T}$, $t \wedge \bigvee S = t \cap \bigcup S = \bigcup \{t \cap s : s \in S\} = \bigvee \{t \wedge s : s \in S\}$. Therefore, $(\mathcal{T}, \subseteq)$ is a frame. $\square$

2.6 Category Theory

Category Theory forms the basis of our results in this thesis. While many of the definitions and results presented are a composite of a number of resources, [T. Leinster [3]] was a key influencer of the definitions and results presented in this section.

Definition 2.6.1. A *category* $\mathcal{C}$ is an ordered triple $(O, M, \circ)$, where O denotes a class of objects, M denotes a class of morphisms between objects in O, and $\circ$ is a binary composition operation taken on elements of M such that the following properties hold.

1. $\forall \alpha, \beta \in M, \alpha \circ \beta$ is defined if the domain of α is the same object as the codomain of β.

2. $\forall \alpha, \beta \in M$, if $\alpha \circ \beta$ is defined, then $\alpha \circ \beta \in M$

3. $\forall X \in O, \exists 1_X \in M \ni$ if $\alpha : X \to A$ and $\beta : B \to X$ are morphisms in M, then $\alpha \circ 1_X = \alpha$ and $1_X \circ \beta = \beta$ for any $A, B \in O$.

4. $\forall \alpha, \beta, \delta \in M$, $\alpha \circ (\beta \circ \delta) = (\alpha \circ \beta) \circ \delta$ provided each composition is defined.

Given a category $\mathcal{C} = (O, M, \circ)$, it is common to write $Ob(\mathcal{C})$ to represent O and $Hom(\mathcal{C})$ to represent M. Further, $Hom(A, B)$ is used to represent the set of all morphisms from A to B where A and B are elements of O. $\mathcal{C}$ is said to be *small* if M is small; otherwise, $\mathcal{C}$ is said to be *large*. We say that $\mathcal{C}$ is *locally small* if for any $A, B \in O$, $Hom(A, B)$ is small. Every small category is necessarily locally small since $Hom(A, B) \subseteq M$ for any category $\mathcal{C} = (O, M, \circ)$ and $A, B \in O$.

It is often useful to use diagrams when discussing morphisms. Figure 4 shows example diagrams of morphisms.

$$\begin{tikzcd} o_1 \arrow[r, "m_1"] \arrow[dr, "m_2"'] & o_2 \arrow[d, "m_3"] \\ & o_3 \end{tikzcd} \qquad \begin{tikzcd} o_1 \arrow[r, "m_1"] \arrow[d, "m_2"'] & o_2 \arrow[d, "m_3"] \\ o_3 \arrow[r, "m_4"'] & o_4 \end{tikzcd}$$

Figure 4: *Example Morphism Diagrams. When* $m_3 \circ m_1 = m_2$ *(left) or* $m_4 \circ m_2 = m_3 \circ m_1$ *(right), we say that the diagram is commutative (or commutes).*

Definition 2.6.2. Let $\mathcal{C} = (O, M, \circ)$ be a category. A *subcategory* of $\mathcal{C}$ is a catogory $\mathcal{S} = (O_S, M_S, \circ)$ such that

1. $O_S \subseteq O$.

2. $M_S \subseteq M$

Definition 2.6.3. A subcategory $\mathcal{S}$ of $\mathcal{C}$ is said to be *full* provided $Hom_{\mathcal{S}}(o_1, o_2) = Hom_{\mathcal{C}}(o_1, o_2)$ for any $o_1, o_2 \in Ob(\mathcal{S})$.

Definition 2.6.4. Let $\mathcal{C} = (O, M, \circ)$ be a category and let $\alpha : A \to B$ be a morphism in M. α is said to be an *isomorphism* provided there exists a morphism $\beta : B \to A$ in M such that $\alpha \circ \beta = 1_B$ and $\beta \circ \alpha = 1_A$. If there exists an isomorphism between objects A and B in O, then A and B are said to be *isomorphic* and we write $A \cong B$.

Theorem 2.6.1. Each object in a category is isomorphic to itself.

Proof. Let $\mathcal{C} = (O, M, \circ)$ be a category. Suppose $X \in O$. Then there exists $1_X \in M$ such that for any $\alpha : X \to A$ and $\beta : B \to X$ in M, $\alpha \circ 1_A = \alpha$ and $1_X \circ \beta = \beta$ for any $A, B \in O$. Since 1_X is a morphism from X to X, $1_X \circ 1_X = 1_X$. Therefore, 1_X is an isomorphism from X to itself and we can conclude that X is isomorphic to itself. $\square$

Definition 2.6.5. A *covariant functor*, $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a mapping from a category $\mathcal{C}_1 = (O_1, M_1, \circ_1)$ to another category $\mathcal{C}_2 = (O_2, M_2, \circ_2)$ such that the following properties hold.

1. For every $A \in O_1$, $F(A) \in O_2$.

2. If $\alpha : A \to B$ is a morphism in M_1, then $F(\alpha) : F(A) \to F(B)$ is a morphism in M_2 with the following properties.

   (a) $F(1_X) = 1_{F(X)}$ for any $X \in O_1$.

   (b) $F(\alpha \circ_1 \beta) = F(\alpha) \circ_2 F(\beta)$ for all $\alpha, \beta \in M_1$ if $\alpha \circ_1 \beta$ is defined.

A *contravariant functor* is defined in nearly the same way, with the only difference being that $F(\alpha \circ_1 \beta) = F(\beta) \circ_2 F(\alpha)$ for any $\alpha, \beta \in M_1$ where $\alpha \circ_1 \beta$ is defined. When the term *functor* is used in isolation it is assumed that we are referring to a covariant functor. Given a functor (covariant or contraviariant) $F : \mathcal{A} \to \mathcal{B}$, we say

$\mathcal{A}$ is F's domain and $\mathcal{B}$ is F's codomain. Like functions, existence of composition of two functors is dependent upon their domains and codomains. For example, given categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$, the composition $F \circ G$ exists but $G \circ F$ does not since F maps to $\mathcal{C}$ and G maps from $\mathcal{A}$.d

Definition 2.6.6. Let $\mathcal{C}_1 = (O_1, M_1, \circ_1)$ and $\mathcal{C}_2 = (O_2, M_2, \circ_2)$ be locally small categories and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor. For any $A, B \in O_1$ define

$$F_{A,B} : Hom_{\mathcal{C}_1}(A, B) \to Hom_{\mathcal{C}_2}(F(A), F(B))$$

to be the function induced by F. We say F is *faithful* provided $F_{A,B}$ is injective for any $A, B \in O_1$ and *full* if $F_{A,B}$ is surjective for any $A, B \in O_1$. If $F_{A,B}$ is bijective for all $A, B \in O_1$, we say F is *fully faithful.*

Definition 2.6.7. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is *essentially surjective* if for each object B in $Ob(\mathcal{C}_2)$ there exists an object A in $Ob(\mathcal{C}_1)$ such that $F(A) \cong B$.

Definition 2.6.8. Let $\mathcal{C}_1 = (O_1, M_1, \circ_1)$ and $\mathcal{C}_2 = (O_2, M_2, \circ_2)$ be categories and let F_1 and F_2 be functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. A *natural transformation* $N : F_1 \to F_2$ is the collection of morphisms of the form $m_o : F_1(o) \to F_2(o)$ (called *components* of N) such that for each morphism $m : o \to o'$ in M_1, $m_{o'} \circ_2 F_1(m) = F_2(m) \circ_2 m_o$.

This definition of a natural transformation is not very intuitive. Another common (and equivalent) approach to defining a natural transformation is to say that a natural transformation $N : F_1 \to F_2$ is a collection of morphisms of the form $m_o : F_1(o) \to F_2(o)$ such that for each morphism $m : o \to o'$ in M_1 the diagram in Figure 5 commutes.

$$\begin{tikzcd} F_1(o) \arrow[r, "F_1(m)"] \arrow[d, "m_o"'] & F_1(o') \arrow[d, "m_{o'}"] \\ F_2(o) \arrow[r, "F_2(m)"'] & F_2(o') \end{tikzcd}$$

Figure 5: *Natural Transformation Diagram*

Let $\mathcal{C}_1 = (O_1, M_1, \circ_1)$ and $\mathcal{C}_2 = (O_2, M_2, \circ_2)$ be categories, let F be the class of all functors from $\mathcal{C}_1$ to $\mathcal{C}_2$, and let N be the class of all natural transformations between functors in F. It is well know that $(F, N, \circ_2)$ forms a category. These categories of functors are intuitively called *functor categories*. A functor category from $\mathcal{C}_1$ to $\mathcal{C}_2$ is commonly denoted $Fun(\mathcal{C}_1, \mathcal{C}_2)$ or $[\mathcal{C}_1, \mathcal{C}_2]$.

Definition 2.6.9. [3] Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. A *natural isomorphism* between functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ is an isomorphism in $Fun(\mathcal{C}_1, \mathcal{C}_2)$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories, let F_1 and F_2 be functors from $\mathcal{C}_1$ to $\mathcal{C}_2$, and let $N : F_1 \to F_2$ be a natural transformation. Leinster [3] provides a simple proof that N is a natural isomorphism if and only if $m_o : F_1(o) \to F_2(o)$ is an isomorphism for all $o \in Ob(\mathcal{C}_1)$.

Definition 2.6.10. [3] An *equivalence* between categories $\mathcal{A}$ and $\mathcal{B}$ consists of a pair of functors, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, together with natural isomorphisms $\eta : 1_{\mathcal{A}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathcal{B}}$. If there exists an equivalence between $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*, and write $\mathcal{A} \simeq \mathcal{B}$.

[T. Leinster [3]] claims and it is well known that a functor forms an equivalence if and only if the functor is full, faithful, and essentially surjective.

[T. Leinster [3]] also defines the concept of a *dual* or *opposite* of a category, which is used to construct what is called a *dual equivalence* or a *duality* between two categories. However, for the sake of brevity, we will define a *duality* using a well known result.

If a category is equivalent to the *dual* of another category we say that there is a *dual equivalence* or a *duality* between the two categories. This is nearly equivalent to the above definition with the only exception being that F and G be contravariant functors.

Definition 2.6.11. A *duality* or *dual equivalence* between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of a pair of contravariant functors, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, together with natural isomorphisms $\eta : 1_{\mathcal{A}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathcal{B}}$.

One can also show equivalence (or duality) of two categories by showing that there exist commutative covariant (alternatively, contravariant) functors between the two categories. Also, it is common to be more concerned with the equivalence and duality of structure in categories rather pointwise equivalence and duality. This thesis will use both of these practices in its final chapter.

CHAPTER 3

HYPERGRAPH POSETS

3.1 Introduction

In this chapter we will define new mathematical objects called hypergraph posets as well as hypergraph poset homomorphisms. Once defined, we will show how these constructs are related to both hypergraphs and hypergraph homomorphisms. The theorems in this chapter will build the foundation to eventually show that hypergraphs and hypergraph posets are effectively the same mathematically.

3.2 Hypergraph Posets

[M. Wiese [7]] originally presented the order dual of hypergraph posets. In this thesis we consider hypergraph posets as the order dual of the posets in [M. Wiese [7]] in order to align more with the standard definition of an incidence poset.

Definition 3.2.1. A *hypergraph poset* is a bipartite poset $\mathcal{P} = (V \cup E, \leq)$ such that for all $e \in E, 1 \leq |Cov^{-1}(e)|$.

Theorem 3.2.1. Let $\mathcal{P} = (V \cup E, \leq)$ be a hypergraph poset. Then for any $x, y \in V \cup E, x < y$ if and only if $x \in V, y \in E$ and $x \in Cov^{-1}(y)$.

Proof. Let $x < y$. If $x \in E$, then $y \in V$ since $\mathcal{P}$ is bipartite. But $1 \leq |Cov^{-1}(x)|$, which implies that there exists $z \in V$ such that $z \leq x$. But then $z < y$, which is a contradiction since $\mathcal{P}$ is bipartite. So $x \in V$. This implies that $y \in E$ since $\mathcal{P}$ is bipartite. Now suppose $x \notin Cov^{-1}(y)$. Then there must be some $z \in P$ such that $x < z < y$. If $z \in V$, then we have two comparable members of V and if $z \in E$, we have two comparable members of E. In either case we have a contradiction. Therefore, $x \in Cov^{-1}(y)$. We can now conclude the $x \in V, y \in E$ and $x \in Cov^{-1}(y)$. The converse holds trivially since $x \in Cov^{-1}(y) \implies x < y$. $\square$

Theorem 3.2.2. Let $\mathcal{P} = (V \cup E, \preceq)$ be a hypergraph poset. Then $\{\uparrow x : x \in V \cup E\}$ is the order dual of a hypergraph poset under the set inclusion partial ordering.

Proof. We know that $\{\uparrow x : x \in V \cup E\} = \{\uparrow v : v \in V\} \cup \{\uparrow e : e \in E\}$. But V is an antichain, so $\{\uparrow v : v \in V\}$ must also be an antichain under set inclusion. This is because if $v_1 \neq v_2$ for some $v_1, v_2 \in V$, then $v_1 \notin \uparrow v_2$ and $v_2 \notin \uparrow v_1$. The same holds for $\{\uparrow e : e \in E\}$. Therefore, we have a bipartite poset under set inclusion. For any $e \in E$ we know that $1 \leq |Cov^{-1}(e)|$. Therefore, there must exist some $v \in V$ such that $v \preceq e$ and thus $\uparrow e \subseteq \uparrow v$. This implies that $\uparrow v \in Cov(\uparrow e)$ which gives us $1 \leq |Cov(\uparrow e)|$. Finally, taking the order dual of this poset we find that for any $e \in E$, we have $1 \leq |Cov_{op}^{-1}(\uparrow e)|$. Therefore, the order dual of $\{\uparrow x : x \in V \cup E\}$ under set inclusion is a hypergrpah poset. $\square$

Theorem 3.2.3. [M. Wiese [7]] Let $\mathcal{P} = (V \cup E, \preceq)$ be a hypergraph poset. Then $\{\downarrow x : x \in V \cup E\}$ is a hypergraph poset under the set inclusion partial ordering.

Proof. Both $\{\downarrow v : v \in V\}$ and $\{\downarrow e : e \in E\}$ are antichains under the same reasoning as Theorem 3.2.2, so again we have a bipartite poset under set inclusion. However, unlike Theorem 3.2.2, for any $e \in E$, there is some $v \in V$ such that $\downarrow v \subseteq \downarrow e$. Therefore, we have $1 \leq |Cov^{-1}(\downarrow e)|$ without the need to take the order dual. $\square$

Definition 3.2.2. An *incidence poset* of a hypergraph $\mathcal{G} = (V, E, \phi)$, denoted $IP_{\mathcal{G}}$, is the bipartite poset $\mathcal{P} = (V \cup E, \leq)$ where $x \leq y$ provided either of the following are true.

1. $x = y$
2. $x \in V$, $y \in E$, and $x \in \phi(y)$.

Theorem 3.2.4. The incidence poset of a hypergraph is unique. In other words, given two graphs $\mathcal{G}$ and $\mathcal{H}$, $IP_{\mathcal{G}} = IP_{\mathcal{H}} \implies \mathcal{G} = \mathcal{H}$.

Proof. Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be hypergraphs. Suppose $IP_{\mathcal{G}} = IP_{\mathcal{H}}$ where $IP_{\mathcal{G}} = (V_G \cup E_G, \leq)$ and $IP_{\mathcal{H}} = (V_H \cup E_H, \preceq)$. Then $V_G \cup E_G = V_H \cup E_H$ and $x \leq y \iff x \preceq y$ for any $x, y \in V_G \cup E_G = V_H \cup E_H$.

Let $y \in V_G$. Suppose $y \in E_H$. Then there exists some $x \in V_H$ such that $x \in \phi_H(y)$. This implies that $x \prec y$, which in turn implies $x < y$. Therefore, $x \in V_G, y \in E_G$, and $x \in \phi_G(y)$. But this is a contradiction since $y \in V_G$ and V_G and E_G are disjoint. Therefore, $y \in V_H$ and we know $V_G \subseteq V_H$. Through the same process we can show that $V_H \subseteq V_G$, so we have $V_G = V_H$. Further, since $V_G \cup E_G = V_h \cup E_H$, we have $E_G = E_H$ as well.

Now we need only show that $\phi_G = \phi_H$. Let $e \in E_G$. Suppose $v \in \phi_G(e)$. Then $v < e$, which implies $v \prec e$. But this means that $v \in V_H, e \in E_H$, and $v \in \phi_H(e)$. Therefore, $\phi_G(e) \subseteq \phi_H(e)$. Through the same process we can also see that $\phi_H(e) \subseteq \phi_G(e)$, thus $\phi_G(e) = \phi_H(e)$. Since $E_G = E_H$, we can now conclude that $\phi_G = \phi_H$, and therefore, $\mathcal{G} = \mathcal{H}$. $\square$

Theorem 3.2.5. [M Wiese [7]] The incidence poset of a hypergraph is a hypergraph poset.

Proof. Let $\mathcal{G} = (V, E, \phi)$ be a hypergraph and $\mathcal{P} = (V \cup E, \leq)$ be its incidence poset. By definition of an incidence poset we know $\forall x, y \in V, x \leq y \implies x = y$, so V must be an antichain. Since the same holds for all elements of E, we know that E is an antichain as well. Further, we know $1 \leq |\phi(e)|$ for any edge in a hypergraph since ϕ maps edges to $\mathscr{P}(V) \sim \emptyset$. This implies that $1 \leq |Cov^{-1}(e)|$ for all $e \in E$. Therefore, $\mathcal{P}$ is a hypergraph poset. $\square$

Theorem 3.2.6. [M. Wiese [7]] Let $\mathcal{P} = (V \cup E, \leq)$ be a hypergraph poset. Then there exists a hypergraph $\mathcal{G}$ such that $IP_{\mathcal{G}}$ is order isomorphic to P.

Proof. Let $\mathcal{G} = (V, E, \phi)$ be a hypergraph such $\phi(e) = Cov_{\mathcal{P}}^{-1}(e)$ for all $e \in E$, let $IP_{\mathcal{G}} = (V \cup E, \preceq)$ be the incidence poset on $\mathcal{G}$, and let $f : V \cup E \to V \cup E$ be a function such that $f(x) = x$. It is easy to see that f is a bijection, so we need only show that f and its inverse are order homomorphisms.

Suppose $x \preceq y$. Then $x = y$ or $x \in V, y \in E$ and $x \in \phi(y)$ by the definition of an incidence poset. If $x = y$, then $f(x) = x = y = f(y)$. If $x \in V, y \in E$ and $x \in \phi(y)$, then $x \in Cov_{\mathcal{P}}^{-1}(y)$ by the definition of $\mathcal{G}$. Therefore, $f(x) = x \leq y = f(y)$. Therefore, $x \preceq y \implies f(x) \leq f(y)$ and f is an order homomorphism.

Now suppose $x \leq y$. Then $x = y$ or $x \in V, y \in E$ and $x \in Cov_{\mathcal{P}}^{-1}(y)$. If $x = y$, then $f^{-1}(x) = x = y = f^{-1}(y)$. If $x \in V, y \in E$ and $x \in Cov_{\mathcal{P}}^{-1}(y)$, then $f^{-1}(x) = x \in V, f^{-1}(y) = y \in E$, and $x \in \phi(e)$ by definition of $\mathcal{G}$. Therefore, $f^{-1}(x) = x \preceq y = f^{-1}(y)$ and f^{-1} must be an order homomorphism. □

Theorem 3.2.7. [M. Wiese [7]] Let $\mathcal{P}$ be a hypergraph poset. Then there exists a unique hypergraph whose incidence poset is $\mathcal{P}$.

Proof. Let $\mathcal{P} = (V \cup E, \leq)$ be a hypergraph poset. As in the proof for Theorem 3.2.6, let $\mathcal{G} = (V, E, \phi)$ be a hypergraph such that $\phi(e) = Cov_{\mathcal{P}}^{-1}(e)$ for all $e \in E$. The proof for Theorem 3.2.6 showed us that the function $f : V \cup E$ defined by $f(x) = x$ is an order isomorphism between the posets $IP_{\mathcal{G}}$ and $\mathcal{P}$. Therefore, we know $IP_{\mathcal{G}} = \mathcal{P}$.

Now suppose there exists another hypergraph $\mathcal{G}'$ such that $IP_{\mathcal{G}'} = \mathcal{P}$. Then $IP_{\mathcal{G}} = IP_{\mathcal{G}'}$, which implies that $\mathcal{G} = \mathcal{G}'$. Therefore, there must exist a unique hypergraph whose incidence poset is $\mathcal{P}$. □

For any poset, $\mathcal{P}$, we will denote the unique hypergraph whose incidence poset is $\mathcal{P}$ as $\mathcal{G}_{\mathcal{P}}$. It follows from Theorems 3.2.4 and 3.2.7 that, given a hypergraph $\mathcal{H}$, $\mathcal{G}_{IP_{\mathcal{H}}} = \mathcal{H}$.

Theorem 3.2.8. [M. Wiese [7]] Let $\mathcal{P} = (V \cup E, \leq)$ be a hypergraph poset. If $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ are hypergraphs such that $IP_{\mathcal{G}}$ and $IP_{\mathcal{H}}$ are both order isomorphic to $\mathcal{P}$, then $\mathcal{G}$ and $\mathcal{H}$ are hypergraph isomorphic.

Proof. If $IP_{\mathcal{G}}$ and $IP_{\mathcal{H}}$ are order isomorphic to the same poset, then they must be order isomorphic to each other. Therefore, there exists an order isomorphism $\alpha : V_G \cup E_G \to V_H \cup E_H$ from $IP_{\mathcal{G}}$ to $IP_{\mathcal{H}}$. Since α is an order isomorphism it must also map maximal elements to maximal elements and minimal elements to minimal elements. Therefore, we have $\alpha(V_G) = V_H$ and $\alpha(E_G) = E_H$, giving us the first two requirements for α and it's inverse being hypergraph homomorphisms. Further, we can see that $\alpha(\phi_G(e)) = \phi_H(\alpha(e))$ for any $e \in E_G$ because of the following:

$$\begin{aligned} v \in \phi_G(e) &\iff v <_G e \\ &\iff \alpha(v) <_H \alpha(e) \\ &\iff \alpha(v) \in \phi_H(\alpha(e)). \end{aligned}$$

Similarly, we can see $\alpha^{-1}(\phi_H(e)) = \phi_G(\alpha^{-1}(e))$ for any $e \in E_H$ since

$$\begin{aligned} v \in \phi_H(e) &\iff v <_H e \\ &\iff \alpha^{-1}(v) <_G \alpha^{-1}(e) \\ &\iff \alpha^{-1}(v) \in \phi_H(\alpha^{-1}(e)). \end{aligned}$$

Therefore, α must also be a hypergraph isomorphism. □

Theorem 3.2.9. [Z. French [6]] Let $\mathcal{G}_1 = (V_1, E_1, \phi_1)$ and $\mathcal{G}_2 = (V_2, E_2, \phi_2)$ be hypergraphs and let $\alpha : V_1 \cup E_1 \to V_2 \cup E_2$ be a function. If α is a hypergraph homomorphism, then α is a order homomorphism with respect to $IP_{\mathcal{G}_1}$ and $IP_{\mathcal{G}_2}$.

Proof. Suppose α is a hypergraph homomorphism. Let $x \leq_1 y$. We need to show that $\alpha(x) \leq_2 \alpha(y)$. We know $x \leq_1 y$ if and only if $x = y$ or $x \in V_1, y \in E_1$, and $x \in \phi_1(y)$. If $x = y$, then $\alpha(x) \leq_1 \alpha(y)$ holds trivially. If $x \in \phi_1(y)$, then by the definition of a hypergraph homomorphism, we know $\alpha(x) \in \phi_2(\alpha(y))$. Therefore, $\alpha(x) <_2 \alpha(y)$ and we have an order homomorphism. □

It is easy to see that Theorem 3.2.9 holds for hypergraph isomorphisms and order isomorphisms since the only other requirement is that α be a bijection. The converse of Theorem 3.2.9 is not true and a counter example can be easily made by mapping any isolated vertex to an edge. In this case, we can still have an order homomorphism but not a hypergraph homomorphism.

3.3 Hypergraph Poset Homomorphisms

Hypergraph Poset Homomorphisms are originally presented in [Z. French [6]]. You will notice that hypergraph poset homomorphisms and hypergraph homomorphisms have very similar properties. In fact, we will show that hypergraph poset homomorphisms are hypergraph homomorphisms and vice versa.

Definition 3.3.1. Let $\mathcal{H}_1 = (V_1 \cup E_1, \leq)$ and $\mathcal{H}_2 = (V_2 \cup E_2, \preceq)$ be hypergraph posets. A function $\alpha : V_1 \cup E_1 \to V_2 \cup E_2$ is a *hypergraph poset homomorphism* provided

1. $\alpha(V_1) \subseteq V_2$.
2. $\alpha(E_1) \subseteq E_2$.
3. For any $e \in E$, $\alpha(Cov^{-1}(e)) = Cov^{-1}(\alpha(e))$

Criterion (3) ensures that any hypergraph poset homormophism is also an order homomorphism.

For the sake of brevity we will use the term *HP-homomorphism* instead of hypergraph poset homomorphism. A bijective HP-homomorphism whose inverse is also an HP-homomorphism is a *hypergraph poset isomorphism* (or *HP-isomorphism*). If a HP-homomorphism exists between two hypergraph posets, we say they are *hypergraph poset homomorphic* (or *HP-homomorphic*). Similarly, if an HP-isomorphism exists between two hypergraph posets, we say they are *hypergraph poset isomorphic* (or *HP-isomorphic*).

Theorem 3.3.1. [Z. French [6]] Let $\mathcal{P}_1 = (V_1 \cup E_1, \leq)$ and $\mathcal{P}_2 = (V_2 \cup E_2, \preceq)$ be hypergraph posets and let $\alpha : V_1 \cup E_1 \to V_2 \cup E_2$ be a function. Then α is an HP-homomorphism if and only if it is a hypergraph homomorphism with respect to $\mathcal{G}_{\mathcal{P}_1}$ and $\mathcal{G}_{\mathcal{P}_2}$.

Proof. HP-homomorphisms and hypergraph homomorphisms have the same first two requirements, so we need only show that the third requirement for each implies the other.

Let $e \in E_1$ and suppose α is an HP-homomorphism. Then we have

$$\begin{aligned} v \in \alpha(\phi_1(e)) &\iff v \in \alpha(Cov^{-1}(e)) = Cov^{-1}(\alpha(e)) \\ &\iff v <_2 \alpha(e) \\ &\iff v \in \phi_2(\alpha(e)). \end{aligned}$$

Therefore, $\alpha(\phi_1(e)) = \phi_2(\alpha(e)$.

Now suppose α is a hypergraph homomorphism. Then we have

$$\begin{aligned} v \in \alpha(Cov^{-1}(e)) &\iff v \in \alpha(\phi_1(e)) = \phi_2(\alpha(e)) \\ &\iff v <_2 \alpha(e) \\ &\iff v \in Cov^{-1}(\alpha(e)). \end{aligned}$$

Therefore, $\alpha(Cov^{-1}(e)) = Cov^{-1}(\alpha(e))$. $\square$

Theorem 3.3.2. [Z. French [6]] Let $\mathcal{A} = (V_A \cup E_A, \leq_A), \mathcal{B} = (V_B \cup E_B, \leq_B), \mathcal{C} = (V_C \cup E_C, \leq_C)$ be hypergraph posets and let $\alpha : V_B \cup E_B \to V_C \cup E_C$ and $\beta : V_A \cup E_A \to V_B \cup E_B$ be HP-homomorphisms. Then $\alpha \circ \beta$ is an HP-homomorphism.

Proof. Since $\beta(V_A) \subseteq V_B$ and $\alpha(V_B) \subseteq V_C$, we know that $\alpha(\beta(V_A)) \subseteq V_C$. Similarly, since $\beta(E_A) \subseteq E_B$ and $\alpha(E_B) \subseteq E_C$, we know that $\alpha(\beta(E_A)) \subseteq E_C$. So we have the first two requirements for an HP-homomorphism. For the third requirement we have

$$\begin{aligned} (\alpha \circ \beta)(Cov^{-1}(e)) &= \alpha(\beta(Cov^{-1}(e))) \\ &= \alpha(Cov^{-1}(\beta(e))) \\ &= Cov^{-1}(\alpha(\beta(e))) \\ &= Cov^{-1}((\alpha \circ \beta)(e)). \end{aligned}$$

Therefore, $\alpha \circ \beta$ is an HP-homomorphism. $\square$

As with order isomorphisms, the composition of HP-isomorphisms are HP-isomorphisms since the composition of two bijections is also a bijeciton.

Theorem 3.3.3. [Z. French [6]] Let $\mathcal{G} = (V_G \cup E_G, \leq_G)$ and $\mathcal{H} = (V_H \cup E_H, \leq_H)$ be hypergraph posets and let $\alpha : V_G \cup E_G \to V_H \cup E_H$ be an order isomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then α is an HP-isomorphism.

Proof. For the first requirement of an HP-homomorphism, suppose that there is some $v \in V_G$ such that $\alpha(v) \notin V_H$. Then $\alpha(v)$ is necessarily a member of E_H. This implies that there must be some $v' \in V_H$ such that $v' < \alpha(v)$. This, in turn, gives us $\alpha^{-1}(v) < v$, which is a contradiction since there are no elements less than members of V_G. Therefore $\alpha(v) \in V_H$ and we have $\alpha(V_G) \subseteq V_H$.

For the second requirement, suppose there is some $e \in E_G$ such that $\alpha(E_G) \notin E_H$. Then $\alpha(e)$ is necessarily a member of V_H. We know there must be some $v \in V_G$ such that $v < e$, which implies that $\alpha(v) < \alpha(e)$. But this is a contradiction since $\alpha(v) \in V_H$ and V_H is an antichain. Therefore, $\alpha(e) \in E_H$ and we have $\alpha(E_G) \subseteq E_H$.

For the third requirement, let $e \in E_G$. Then we have

$$\begin{aligned} v \in \alpha(Cov^{-1}(e)) &\iff \alpha^{-1}(v) \in Cov^{-1}(e) \\ &\iff \alpha^{-1}(v) <_G e \\ &\iff v <_H \alpha(e) \\ &\iff v \in Cov^{-1}(\alpha(e)). \end{aligned}$$

Therefore, $\alpha(Cov^{-1}(e)) = Cov^{-1}(\alpha(e))$ for any $e \in E_G$ and α must be an HP-homomorphism.

□

CHAPTER 4

CATEGORIES HG AND HGP

4.1 Introduction

In this chapter we will introduce the categories of hypergraphs and hypergraph posets and will use concepts from previous chapters to show that they are equivalent. The material in this chapter is fairly straightforward and was originally presented in [Z. French [6]].

4.2 HG: The Category of Hypergraphs

Definition 4.2.1. Let O be the class of all hypergraphs, let M be the class of all hypergraph homomorphisms, and let $\circ$ be traditional functional composition. We define HG as the ordered triple $(O, M, \circ)$.

Theorem 4.2.1. *HG is a category.*

Proof. Suppose $\alpha, \beta \in M$. Since α and β are hypergraph homomorphisms, $\alpha \circ \beta$ is defined provided the domain of α is the same set as the codomain of β. Thus we have the first requirement of a category.

By Theorem 2.3.1 we know that the composition of two graph homomorphisms is itself a hypergraph homomorphism. Thus we have the second requirement of a category.

Let $\mathcal{G} = (V, E, \phi) \in O$ and let $1_{\mathcal{G}} : V \cup E \to V \cup E$ be a function such that $1_{\mathcal{G}}(x) = x$ for all $x \in V \cup E$. $1_{\mathcal{G}}(V) = V$ and $1_{\mathcal{G}}(E) = E$, which gives us the first two requirements of a hypergraph homomorphism. Further, for any $v \in V$, if $v \in \phi(e)$ for some $e \in E$, then $1_{\mathcal{G}}(v) = v \in \phi(e) = \phi(1_{\mathcal{G}}(e))$, so we have the third requirement of a hypergraph homomorphism. Thus, $1_{\mathcal{G}}$ is a hypergraph homomorphism. Also, given hypergraph homomorphisms $\alpha : A \to V \cup E$ and $\beta : V \cup E \to B$ in M, we have $1_{\mathcal{G}} \circ \alpha = \alpha$ and $\beta \circ 1_{\mathcal{P}} = \beta$. Therefore, we have the third requirement of a category.

The fourth requirement of a category holds trivially since all functions (and thus graph homomorphisms) are associative under the composition operation. Now we can conclude that HG is a category. □

Definition 4.2.2. Let O be the set of all finite hypergraphs, let M be the set of all hypergraph homomorphisms between finite hypergraphs, and let $\circ$ be traditional functional composition. We define FHG as the ordered triple $(O, M, \circ)$.

It should be no surprise that FHG is a subcategory of HG since $Ob(FHG) \subseteq Ob(HG)$, $Hom(FHG) \subseteq Hom(HG)$, and FHG intuitively inherits all of the properties of a category from HG.

4.3 HGP: The Category of Hypergraph Posets

Definition 4.3.1. Let O be the class of all hypergraph posets, let M be the class= of all HP-homomorphisms of elements of O, and let $\circ$ be traditional function composition. We define HGP as the ordered triple $(O, M, \circ)$.

Theorem 4.3.1. [Z. French [6]] HGP is a category

Proof. Suppose $\alpha, \beta \in M$. Since α and β are HP-homomorphisms, $\alpha \circ \beta$ is defined provided the domain of α is the same set as the codomain of β. Thus we have the first requirement of a category.

By Theorem 3.3.2 we know that the composition of two HP-homomorphisms is itself an HP-homomorphism. Thus we have the second requirement of a category.

Let $\mathcal{P} = (V_P \cup E_P, \leq) \in O$ and let $1_{\mathcal{P}} : V_P \cup E_P \to V_P \cup E_P$ be a function such that $1_{\mathcal{P}}(x) = x$ for all $x \in V_P \cup E_P$. It is easy to see that $1_{\mathcal{P}}$ is an HP-homomorphism from $\mathcal{P}$ to itself. Further, given HP-homomorphisms $\alpha : A \to V_P \cup E_P$ and $\beta : V_P \cup E_P \to B$ in M, we have $1_{\mathcal{P}} \circ \alpha = \alpha$ and $\beta \circ 1_{\mathcal{P}} = \beta$. Therefore, we have the third requirement of a category.

The fourth requirement of a category holds trivially since all functions (and thus HP-homomorphisms) are associative under the composition operation. Now we can conclude that HGP is a category. $\square$

Definition 4.3.2. Let O be the set of all finite hypergraph posets, let M be the set of all HP-homomorphisms between elements of O, and let $\circ$ be traditional functional composition. We define FHG as the ordered triple $(O, M, \circ)$.

Like with FHG, $Ob(FHGP) \subseteq Ob(HGP)$, $Hom(FHGP) \subseteq Hom(HG)$, and $FHGP$ inherits all the properties of a category from HGP. Therefore, $FHGP$ is a subcategory of HGP.

4.4 HG vs. HGP

Theorem 4.4.1. [Z. French [6]] The mapping $F : HGP \to HG$ defined below is a functor.

1. $\forall \mathcal{P} \in Ob(HGP), F(\mathcal{P}) = \mathcal{G}_{\mathcal{P}}$

2. $\forall \alpha \in Hom(HGP), F(\alpha) = \alpha.$

Proof. We have the first requirement for F to be a functor by the definition of F since $\mathcal{G}_{\mathcal{P}}$ is a hypergraph for any $\mathcal{P} \in Ob(HGP)$. Suppose $\alpha : A \to B$ be a morphism in $Hom(\mathcal{A}, \mathcal{B})$ (i.e. α is a HP-homomorphism from $\mathcal{A}$ to $\mathcal{B}$). Then $F(\alpha) = \alpha$ must also be a hypergraph homomorphism from $F(\mathcal{A}) = \mathcal{G}_{\mathcal{A}}$ to $F(\mathcal{B}) = \mathcal{G}_{\mathcal{B}}$. Therefore $F(\alpha) : F(\mathcal{A}) \to F(\mathcal{B})$ is a hypergraph homomorphism in $Hom(HG)$.

Now let $\mathcal{P} = (V \cup E, \leq) \in Ob(HGP)$. Then $1_{\mathcal{P}} : V \cup E \to V \cup E$ is the HP-homomorphism such that $\forall x \in V \cup E, 1_{\mathcal{P}}(x) = x$. However, we also know that $1_{\mathcal{P}}$ must be a hypergraph homomorphism from $\mathcal{G}_{\mathcal{P}}$ to itself. Further, it is easy to see that $1_{\mathcal{P}} = 1_{\mathcal{G}_{\mathcal{P}}}$. Therefore, $F(1_{\mathcal{P}}) = 1_{\mathcal{P}} = 1_{\mathcal{G}_{\mathcal{P}}} = 1_{F(\mathcal{P})}$ and we have the first part of the second requirement of a functor.

Finally, let $\mathcal{A} = (V_A \cup E_A, \leq_A), \mathcal{B} = (V_B \cup E_B, \leq_B), \mathcal{C} = (V_C \cup E_C, \leq_C) \in Hom(HGP)$ and let $\alpha : V_B \cup E_B \rightarrow V_C \cup E_C$ and $\beta : V_A \cup E_A \rightarrow V_B \cup E_B$ be HP-homomorphisms in $Hom(HGP)$. Then $\alpha \circ_{HGP} \beta$ is defined and therefore in $Hom(HGP)$. Further, $F(\alpha \circ_{HGP} \beta) = \alpha \circ_{HGP} \beta = \alpha \circ_{HG} \beta = F(\alpha) \circ_{HG} F(\beta)$. Thus we have the second part of the second requirement for a functor and we can now conclude that F is a functor. $\square$

Theorem 4.4.2. [Z. French [6]] HG and HGP are categorically equivalent.

Proof. The domains and codomains of hypergraph homomorphisms and HP homomorphisms are defined as sets. Therefore, any function (which by definition is a subset of the cross product of the domain and codomain) that is a hypergraph homomorphism or HP homomorphism must also be a set. This tells us that both HG and HGP are locally small. Therefore, we can find a fully faithful, essentially surjective functor to show that they are equivalent. Let F be the functor defined in Theorem 4.4.1. Let $\mathcal{A} = (V_A \cup E_A, \leq)$ and $\mathcal{B} = (V_B \cup E_B, \preceq)$ be members of $Ob(HGP)$ and let $F_{\mathcal{A},\mathcal{B}} : Hom_{HGP}(\mathcal{A}, \mathcal{B}) \rightarrow Hom_{HG}(\mathcal{G}_{\mathcal{A}}, \mathcal{G}_{\mathcal{B}})$ be the function induced by F. We must show that $F_{\mathcal{A},\mathcal{B}}$ is bijective and F is essentially surjective.

First, let α and β be distinct morphisms in $Hom_{HGP}(\mathcal{A}, \mathcal{B})$ and suppose $F_{\mathcal{A},\mathcal{B}}(\alpha) = F_{\mathcal{A},\mathcal{B}}(\beta)$. By the definition of F, we know $F_{\mathcal{A},\mathcal{B}}(\alpha) = \alpha$ and $F_{\mathcal{A},\mathcal{B}}(\beta) = \beta$. Therefore, we have $\alpha = F_{\mathcal{A},\mathcal{B}}(\alpha) = F_{\mathcal{A},\mathcal{B}}(\beta) = \beta$ and we can conclude that $F_{\mathcal{A},\mathcal{B}}$ is injective.

Now suppose $\eta \in Hom_{HG}(\mathcal{G}_{\mathcal{A}}, \mathcal{G}_{\mathcal{B}})$. Then $\eta : V_A \cup E_A \rightarrow V_B \cup E_B$ is a hypergraph homomorphism, and thus also a HP-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Thus, $\eta \in Hom(\mathcal{A}, \mathcal{B})$ and we can conclude that $F_{\mathcal{A},\mathcal{B}}$ is surjective.

Finally, let $\mathcal{H} = (V, E, \phi) \in Ob(HG)$. We know $F(IP_{\mathcal{H}}) = \mathcal{G}_{IP_{\mathcal{H}}} = \mathcal{H}$. Since $\mathcal{H}$ is isomorphic to itself, we have $F(IP_{\mathcal{H}}) \cong \mathcal{H}$ and we can conclude that F is essentially surjective. $\square$

We can restrict the functor from Theorem 4.4.1 to $FHGP$ and FHG to see that FHG and $FHGP$ are categorically equivalent as well.

CHAPTER 5

CLASSICAL TOPOLOGY ON A HYPERGRAPH

5.1 Classical Topology of a Hypergraph

For any hypergraph, $\mathcal{G} = (V, E, \phi)$, there is a naturally associated topology, called the *classical topology* on $\mathcal{G}$ (first introduced by [Antoine Vella [9]]). This topology is generated by the basis $\{\{e\} : e \in E\} \cup \{E_B(v) : v \in V\} = \{\uparrow x : x \in IP_{\mathcal{G}}\}$, which is the union of the collection of all singleton sets of edges in $\mathcal{G}$ and all edge balls in $\mathcal{G}$. We will denote $CTH(\mathcal{G})$ as the classical topology on $\mathcal{G}$ and $\Omega(\mathcal{G})$ as the poset of all open sets in this topology, ordered by set inclusion.

Theorem 5.1.1. [M. Wiese [7]] Let $\mathcal{G} = (V, E, \phi)$ be a hypergraph. Then $\Omega(\mathcal{G}) = Up(IP_{\mathcal{G}})$

Proof. Suppose $X \in \Omega(\mathcal{G})$. Then X must be the union of edge balls and singleton sets of edges of $\mathcal{G}$. But each edge ball is an upperset of $IP_{\mathcal{G}}$ and each singleton set containing an edge of $\mathcal{G}$ is an upperset of $IP_{\mathcal{G}}$. Since the union of uppersets of $IP_{\mathcal{G}}$ must also be an upperset of $IP_{\mathcal{G}}$, we have $X \in Up(IP_{\mathcal{G}})$ and $\Omega(\mathcal{G}) \subseteq Up(IP_{\mathcal{G}})$. Now suppose $X \in Up(IP_{\mathcal{G}})$. Then $X = \bigcup_{x \in X} \uparrow x$. If $x \in V$, then $\uparrow x = B(x)$ (the edge ball around x). If $x \in E$, then $\uparrow x = \{e\}$ (a singleton set containing an edge). Therefore, X is the union of edge balls and singleton sets of edges of $\mathcal{G}$, which implies $X \in \Omega(\mathcal{G})$ and $Up(IP_{\mathcal{G}}) \subseteq \Omega(\mathcal{G})$. We can now conclude that $\Omega(\mathcal{G}) = Up(IP_{\mathcal{G}})$. $\square$

Theorem 5.1.2. [M. Wiese [7]] Given a finite hypergraph, $\mathcal{G} = (V, E, \phi)$, $Cjp(\Omega(\mathcal{G})) = \{\uparrow x : x \in IP_{\mathcal{G}}\}$

Proof. Suppose $x \in Cjp(\Omega(\mathcal{G})) = Jp(\Omega(\mathcal{G}))$. Then $|Cov^{-1}(x)| = 1$ since $\Omega(\mathcal{G})$ is a finite, distributive lattice. Because of this, x cannot be the join members of the basis $\mathcal{B} = \{\uparrow x : x \in IP_{\mathcal{G}}\}$ of $\Omega(\mathcal{G})$. But x must be the join of members of $\mathcal{B}$; therefore, x itself must be a member of $\mathcal{B}$.

Now suppose $x \in \mathcal{B}$. We know that $\mathcal{B}$ consists of edge balls and singleton sets of edges of $\mathcal{G}$. The edge singleton sets are necessarily in $Cov(\emptyset)$. Therefore, the inverse cover of each of these sets must have a magnitude of one. Thus the edge singleton sets are completely join-prime. We have two cases to consider for the edge balls:

1. Let v be an isolated vertex. Then v is a singleton set and therefore, v must be in the cover of $\emptyset$. Therefore, $\uparrow v = E_B(v) = v$ is completely join-prime.

2. Suppose v is not isolated. Then v's n edge ball $E_B(v)$ contains precisely v and the edge neighborhood of v, $E_N(v)$. But since the singleton sets of the edges of $E_N(v)$ are also in $\mathcal{B}$, we know that $\bigcup E_N(v) \in \Omega(\mathcal{G})$. Since $\{v\} \notin \mathcal{B}$, we can conclude that $Cov^{-1}(v) = \bigcup E_N(v)$. Therefore, $E_B(v) =\uparrow v$ is completely join-prime in $\Omega(\mathcal{G})$.

□

Theorem 5.1.3. [Z. French [6]] Let $\mathcal{G} = (V, E, \phi)$ be a finite hypergraph and let the function $\alpha : IP_{\mathcal{G}} \to Cjp(\Omega(\mathcal{G}))^{op}$ be defined by $\alpha(x) =\uparrow x$. Then α is an HP-isomorphism.

Proof. Theorem 5.1.3 follows directly from Theorem 5.1.2. □

Theorem 5.1.4. Let $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism from $\mathcal{G} = (V_G, E_G, \phi_G)$ to $\mathcal{H} = (V_H, E_H, \phi_H)$. Then f is continuous under the classical topologies on $\mathcal{G}$ and $\mathcal{H}$.

Proof. Let O be open in $CTH(\mathcal{H})$. Suppose (by way of contradication) that $f^{-1}(O)$ is not open in $CTH(\mathcal{G})$. Since singleton edge sets are basis elements, there must be a vertex $v \in f^{-1}(O)$ such that $E_B \nsubseteq f^{-1}(O)$. This implies that there is an edge $e \in E_{\mathcal{G}}$ such that $v \in \phi_G(e)$ and $e \notin f^{-1}(O)$. Since $v \in f^{-1}(O), f(v) \in O$ and $E_B(f(v)) \subseteq O$ since O is open. By the definition of a hypergraph homomorphism, $v \in \phi_G(e) \implies f(v) \in \phi_H(f(e))$. Therefore, $f(e) \in E_B(f(v)) \subseteq O$. But this is a contradiction since $e \notin f^{-1}(O)$. So $f^{-1}(O)$ must be open in $CTH(\mathcal{G})$ and we can conclude that f is continuous. □

Theorem 5.1.5. Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be hypergraphs and $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism. The function $\omega_f : \Omega(\mathcal{H}) \to \Omega(\mathcal{G})$ defined by $\omega_f(U) = f^{-1}(U)$ is well-defined and the following properties hold.

1. $\omega_f(U_1 \bigwedge_{\Omega(\mathcal{H})} U_2) = \omega_f(U_1) \bigwedge_{\Omega(\mathcal{G})} \omega_f(U_2)$ for any $U_1, U_2 \in \Omega(\mathcal{H})$

2. $\omega_f(\bigvee_{\Omega(\mathcal{H})} D) = \bigvee_{\Omega(\mathcal{G})} \omega_f(D)$, for any $D \subseteq \Omega(\mathcal{H})$.

3. Given a compact element $h \in \Omega(\mathcal{H})$, $\omega_f(h)$ is compact in $\Omega(\mathcal{G})$.

[Gratxer G. [11]] introduces a more generalized result with frame homomorphisms and [W.H. Cornish [10]] presents a more general result of (3) characterizing "spectral" mappings.

Proof. By Theorem 5.1.4, f is continuous and ω_f must be well-defined. For the first property of ω_f, let $U_1, U_2 \in \Omega(\mathcal{H})$. We know that for any X, Y, where X and Y are both in $\Omega(\mathcal{G})$ or both in $\Omega(\mathcal{H})$, we have $X \bigwedge Y = X \cap Y$. Therefore,

$$\begin{aligned}
x \in \omega_f(U_1 \bigwedge_{\Omega(\mathcal{H})} U_2) &\iff x \in \omega_f(U_1 \cap U_2) \\
&\iff x \in f^{-1}(U_1 \cap U_2) \\
&\iff f(x) \in U_1 \cap U_2 \\
&\iff f(x) \in U_1 \text{ and } f(x) \in U_2 \\
&\iff x \in f^{-1}(U_1) \text{ and } x \in f^{-1}(U_2) \\
&\iff x \in \omega_f(U_1) \text{ and } x \in \omega_f(U_2) \\
&\iff x \in \omega_f(U_1) \cap \omega_f(U_2) \\
&\iff x \in \omega_f(U_1) \bigwedge_{\Omega(\mathcal{G})} \omega_f(U_2)
\end{aligned}$$

Therefore, $\omega_f(U_1 \bigwedge_{\Omega(\mathcal{H})} U_2) = \omega_f(U_1) \bigwedge_{\Omega(\mathcal{G})} \omega_f(U_2)$ for any $U_1, U_2 \in \Omega(\mathcal{H})$.

For the second property, let D be a subset of $\Omega(\mathcal{H})$. Since topologies are closed under arbitrary unions, we know $\bigvee X = \bigcup X$ for any X that is a subset of $\Omega(\mathcal{G})$ or $\Omega(\mathcal{H})$. Therefore, we have

$$\begin{aligned}
x \in \omega_f\left(\bigvee_{\Omega(\mathcal{H})} D\right) &\iff x \in \omega_f(\bigcup D) \\
&\iff x \in f^{-1}\left(\bigcup D\right) \\
&\iff f(x) \in \bigcup D \\
&\iff f(x) \in d \text{ for some } d \in D \\
&\iff x \in f^{-1}(d) \text{ for some } d \in D \\
&\iff x \in \bigcup_{d \in D} f^{-1}(d) \\
&\iff x \in \bigcup_{d \in D} \omega_f(d) \\
&\iff x \in \bigcup \omega_f(D) \\
&\iff x \in \bigvee_{\Omega(\mathcal{G})} \omega_f(D).
\end{aligned}$$

For the third property, let $h \in \Omega(\mathcal{H})$ be compact. Let D be a directed subset of $\Omega(\mathcal{G})$. We already know $\bigvee_{\Omega(\mathcal{G})} D = \bigcup D$ exists since $\Omega(\mathcal{G})$ is closed under arbitrary unions. Therefore, we have

$$\begin{aligned}
\omega_f(h) \subseteq \bigvee_{\Omega(\mathcal{G})} D &\iff \omega_f(h) \subseteq \bigcup D \\
&\iff f^{-1}(h) \subseteq \bigcup D \\
&\iff h \subseteq f\left(\bigcup D\right) \\
&\iff h \subseteq \bigcup_{d \in D} f(d) \\
&\iff h \subseteq \bigcup \{f(d) : d \in D\} \\
&\iff h \subseteq \bigvee_{\Omega(\mathcal{H})} \{f(d) : d \in D\}.
\end{aligned}$$

Before preceding further, we must show that $F_D = \{f(d) : d \in D\}$ is a directed subset of $\Omega(\mathcal{H})$. Let S be a finite subset of F_D. Then for any $s \in S$, $s = f(d_s)$ for some $d_s \in D$. Let D_s be the set of all d_s's. Then D_s must be a finite subset of the directed subset D. Therefore, D_s has some upper bound d' in D. This implies that $d_s \subseteq d'$ for all $s \in S$, which in turn gives us $s \subseteq f(d')$ for all $s \in S$. Therefore $f(d')$ is an upper bound of S which is in F_D and F_D must be a directed subset of $\Omega(\mathcal{H})$. Now that we know F_D is directed we have

$$\begin{aligned}
\omega_f(h) &\iff h \subseteq \bigvee_{\Omega(\mathcal{H})} F_D \\
&\implies h \subseteq f(d) \text{ for some } d \in D \\
&\iff f^{-1}(h) \subseteq d \text{ for some } d \in D \\
&\iff \omega_f(h) \subseteq d \text{ for some } d \in D.
\end{aligned}$$

Therefore, $\omega_f(h)$ is also compact in $\Omega(\mathcal{G})$. □

For the remainder of this thesis, given a hypergraph homomorphism, f, we will denote ω_f as the function defined in Theorem 5.1.5.

CHAPTER 6

CTH-LATTICES

6.1 Introduction

In this chapter we will define new mathematical objects called CTH-lattices as well as CTH-lattice homomorphisms. Once defined, we will show how these constructs are related to both classical topologies on hypergraphs and CTH-homomorphisms. CTH-lattices are implicit in [M. Wiese [7]] but first appear in [J. Hart and B. Frazier [1]] where they are referred to as ”graph lattices”. CTH-lattice homomorphisms are originally introduced in [Z. French [6]].

6.2 CTH-Lattices

Definition 6.2.1. A frame $\mathcal{L} = (L, \leq)$ is a *CTH-lattice* provided its completely join-prime elements are join-dense in $\mathcal{L}$ and form the order dual of a hypergraph poset.

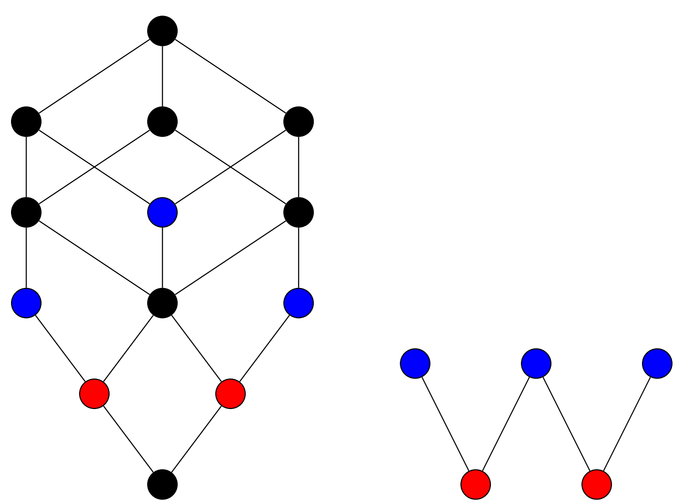

Figure 6: *Example cth-lattice and its hypergraph poset dual of completely join-prime elements.*

Theorem 6.2.1. Let $\mathcal{G}$ be a finite hypergraph. Then $\Omega(\mathcal{G})$ is a CTH-lattice.

Proof. Theorem 5.1.2 tells us that the basis $\mathcal{B} = \{\uparrow_{IP_{\mathcal{G}}} x : x \in IP_{\mathcal{G}}\}$ of $\Omega(\mathcal{G})$ are precisely the completely join-prime members of $\Omega(\mathcal{G})$. Since every member of $\Omega(\mathcal{G})$ is the union of members of $\mathcal{B}$, we can conclude that the completely join-prime members of $\Omega(\mathcal{G})$ are join dense in $\Omega(\mathcal{G})$. By Theorem 3.2.2 we know that $\{\uparrow_{IP_{\mathcal{G}}} x : x \in IP_{\mathcal{G}}\}$ is the order dual of a hypergraph poset. Therefore, $\Omega(\mathcal{G})$ is a CTH-lattice. □

Theorem 6.2.2. Let $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \preceq)$ be CTH-lattices such that $Cjp(\mathcal{L})$ is order isomorphic to $Cjp(\mathcal{M})$. Then $\mathcal{L}$ is order isomorphic to $\mathcal{M}$.

Proof. For any $l \in L$, we will let $J_{\mathcal{L}}(l) = \{j \in Cjp(\mathcal{L}) : j \leq l\}$. Similarly, for any $m \in M$, $J_{\mathcal{M}}(m) = \{j \in Cjp(\mathcal{M}) : j \preceq m\}$. Finally, let $\alpha : Cjp(\mathcal{L}) \to Cjp(\mathcal{M})$ be an order isomorphism and let $\beta : L \to M$ such that $\beta(l) = \bigvee_{\mathcal{M}} \alpha(J_{\mathcal{L}}(l))$.

Suppose $\beta(x) = \beta(y)$. Then $J_{\mathcal{M}}(\beta(x)) = J_{\mathcal{M}}(\beta(y))$. But $J_{\mathcal{M}}(\beta(x)) = \alpha(J_{\mathcal{L}}(x))$ and $J_{\mathcal{M}}(\beta(y)) = \alpha(J_{\mathcal{L}}(y))$, so we have $\alpha(J_{\mathcal{L}}(x)) = \alpha(J_{\mathcal{L}}(y))$. Since α is a bijection, $J_{\mathcal{L}}(x) = J_{\mathcal{L}}(y)$, which gives us $x = \bigvee_{\mathcal{L}} J_{\mathcal{L}}(x) = \bigvee_{\mathcal{L}} J_{\mathcal{L}}(y) = y$. Therefore, β must be injective.

Let $m \in M$. Then $m = \bigvee_{\mathcal{M}} J_{\mathcal{M}}(m)$. But since α is an order isomorphism, we know that there is some $J \subseteq Cjp(\mathcal{L})$ such that $\alpha(J) = J_{\mathcal{M}}(m)$. Therefore, $m = \bigvee_{\mathcal{M}} \alpha(J)$. However, since every CTH-lattice is a frame, we know $\bigvee_{\mathcal{L}} J \in L$. Thus $\beta(\bigvee_{\mathcal{L}} J) = \bigvee_{\mathcal{M}} \alpha(J_{\mathcal{M}}(m)) = m$ and β must be surjective.

Suppose $x \leq y$ for some $x, y \in L$. We know that $x = \bigvee_{\mathcal{L}} J_{\mathcal{L}}(x)$ and $y = \bigvee_{\mathcal{L}} J_{\mathcal{L}}(y)$ since $Cjp(\mathcal{L})$ is join-dense in $\mathcal{L}$. Further, $J_{\mathcal{L}}(x) \subseteq J_{\mathcal{L}}(y)$ since $x \leq y$. This implies

$\alpha(J_{\mathcal{L}}(x)) \subseteq \alpha(J_{\mathcal{L}}(y))$, which gives us

$$\begin{aligned}\beta(x) &= \beta\left(\bigvee_{\mathcal{L}} J_{\mathcal{L}}(x)\right)\\ &= \bigvee_{\mathcal{M}} \alpha\left(J_{\mathcal{L}}(x)\right)\\ &\preceq \bigvee_{\mathcal{M}} \alpha(J_{\mathcal{L}}(y))\\ &= \beta(y).\end{aligned}$$

Finally, suppose $w \preceq z$ for some $w, z \in M$. Then $J_{\mathcal{M}}(w) \subseteq J_{\mathcal{M}}(z)$, which implies $\alpha^{-1}(J_{\mathcal{M}}(w)) \subseteq \alpha^{-1}(J_{\mathcal{M}}(z))$ and thus $\beta^{-1}(w) = \bigvee_{\mathcal{L}} \alpha^{-1}(J_{\mathcal{M}}(w)) \preceq \bigvee_{\mathcal{L}} \alpha^{-1}(J_{\mathcal{M}}(z)) = \beta^{-1}(z)$. Therefore, β^{-1} is an order homomorphism. $\square$

Theorem 6.2.3. [M. Wiese [7]] Let $\mathcal{L}$ be a CTH-lattice. Then there exists a graph $\mathcal{G}$ such that $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$.

Proof. Let $\mathcal{H}$ be the hypergraph poset whose order dual is subposet of completely join-prime elements of $\mathcal{L}$. Then by Theorem 3.2.7 we know that there is a unique hypergraph $\mathcal{G} = (E, V, \phi)$ such that $\mathcal{H} = IP_{\mathcal{G}}$. We already know by Theorem 6.2.1 that $\Omega(\mathcal{G})$ is a CTH-lattice. Thus, by Theorem 6.2.2, $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$. $\square$

Theorem 6.2.4. [J. Snodgrass and C. Tsinakis [8]] Let $\mathcal{L}$ be a finite CTH-lattice and let $m \in Mp(\mathcal{L})$. Then $m \in \max(Mp(\mathcal{L}))$ if and only if there exists a unique $j \in Jp(\mathcal{L})$ such that $j \not\preceq m$. This unique join-prime element, j, is necessarily maximal in $Jp(\mathcal{L})$.

Proof. If $\mathcal{L}$ is finite, then $Cjp(\mathcal{L}) = Jp(\mathcal{L})$, which implies that $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$ by the definition of a CTH-lattice. Therefore, $\mathcal{L}$ is a finite distributive lattice and thus $Mp(\mathcal{L})$ is meet-dense in $\mathcal{L}$. Further, we know that an element $l \in L$ is meet-prime if and only if $|Cov(l)| = 1$ since $\mathcal{L}$ is a finite distributive lattice. Therefore, $Cov^{-1}(\top) \subseteq Mp(\mathcal{L})$ since $\top$ is the only member of the covers of elements in $Cov^{-1}(\top)$.

It's easy to see that $\max(Mp(\mathcal{L})) = Cov^{-1}(\top)$ since $\downarrow Cov^{-1}(\top) = L \sim \{\top\}$. Under the same reasoning we can conclude that $\min(Jp(\mathcal{L})) = Cov(\perp)$.

Now let $m \in \max(Mp(\mathcal{L}))$. We know $m \neq \bigvee Jp(\mathcal{L})$ since $\bigvee Jp(\mathcal{L}) = \top$ and $m \neq \top$. Suppose there exists $j_1, j_2 \in Jp(\mathcal{L})$ such that $j_1 \not\leq m$ and $j_2 \not\leq m$. Since m is a maximal meet-prime element we know $m \vee j_1 = m \vee j_2 = \top$. Therefore, $j_1 \leq m \vee j_2$ and $j_2 \leq m \vee j_1$. Since j_1 and j_2 are join-prime, this implies that $j_1 \leq m$ or $j_1 \leq j_2$ and $j_2 \leq m$ or $j_2 \leq j_1$. But $j_1 \not\leq m$ and $j_2 \not\leq m$, so we must conclude that $j_1 \leq j_2$ and $j_2 \leq j_1$. Therefore, $j_1 = j_2$ and we know that there is a unique join-prime element j such that $j \not\leq m$.

Now suppose this unique join-prime element is not maximal in $Jp(\mathcal{L})$. Then there must be some $j' \in Jp(\mathcal{L})$ such that $j < j'$. But $j' \leq m$ since j is unique, which gives us $j < j' \leq m$ and we once again have a contradiction. Therefore, j must be maximal.

Finally, let $m \in L$ and j is the unique join-prime element in L such that $j \not\leq m$. Suppose $m \notin \max(Mp(\mathcal{L}))$. Then there exists some $m' \in \max(Mp(\mathcal{L}))$ such that $m < m'$. We know that $m = \bigvee Jp(\mathcal{L}) \sim \{j\}$. But since $m < m'$, m' must be the join of a proper superset of $Jp(\mathcal{L}) \sim \{j\}$. However, this is impossible since the only proper superset of $Jp(\mathcal{L}) \sim \{j\}$ is $Jp(\mathcal{L})$ and $\bigvee Jp(\mathcal{L}) = \top$. Therefore, $m \in \max(Mp(\mathcal{L}))$ □

It should be evident that if we have distinct maximal meet-prime elements m and n with unique join-prime elements j_m and j_n where $j_m \not\leq m$ and $j_n \not\leq n$, then $j_m \neq j_n$. Otherwise, we would have $m = \bigvee Jp(\mathcal{L}) \sim \{j_m\} = \bigvee Jp(\mathcal{L}) \sim \{j_n\} = n$ and m and n would not be distinct.

Theorem 6.2.5. [B. Frazier [5]] Let $\mathcal{G} = (V, E, \phi)$ be a finite hypergraph. Then for any $m \in \max(Mp(\Omega(\mathcal{G})))$, $m = [V \cup E] \sim \{v\}$ for some $v \in V$.

Proof. Let $m \in \max(Mp(\Omega(\mathcal{G})))$. Then there must be a unique $j \in Jp(\Omega(\mathcal{G}))$ such that $j \nsubseteq m$ and this j must necessarily be maximal in $Jp(\Omega(\mathcal{G}))$. But the maximal join-prime elements of $\Omega(\mathcal{G})$ are precisely the edge balls of $\mathcal{G}$ and the minimal, non-maximal join-prime elements are the singleton edge sets of $\mathcal{G}$. Since m must necessarily contain all of the minimal, non-maximal join-prime elements of $\Omega(\mathcal{G})$, we know that it contains all the edges of $\mathcal{G}$ and thus all the edges contained in j. Further, it must contain all the vertices of $\mathcal{G}$ but the single vertex in j since it contains all the maximal join-prime elements of $\Omega(\mathcal{G})$ excluding j. Therefore, the vertex in j is the only element of $V \cup E$ not in m and we can conclude that $m = [V \cup E] \sim \{v\}$ for some $v \in V$. $\square$

Theorem 6.2.6. [B. Frazier [5]] Let $\mathcal{L}$ be a finite CTH-lattice and let $m \in Mp(\mathcal{L})$. Then $m \in \min(Mp(\mathcal{L}))$ if and only if there exists a unique $j \in \min(Cjp(\mathcal{L}))$ such that $j \nleq m$. Further, if $m \notin \max(Mp(\mathcal{L}))$, then $j \notin \max(Cjp(\mathcal{L}))$.

Proof. We know $\mathcal{L}$ must be distributive since it is finite and its join-prime elements are join-dense in $\mathcal{L}$. Suppose $m \in \min(Mp(\mathcal{L})) \cap \max(Mp(\mathcal{L}))$. Then by Theorem 6.2.4 there must exist a unique $j \in Jp(\mathcal{L})$ such that $j \nleq m$. This j must necessarily be maximal in $Jp(\mathcal{L})$. Now suppose $j \notin \min(Jp(\mathcal{L}))$. Then there must exist some $j' \in Jp(\mathcal{L})$ such that $j' < j$. Further, we know that $j' \leq m$ since m is maximal and j' is strictly minimal. Theorem 2.4.36 tells us that $m' = \bigvee\{l \in L : j' \notin\downarrow l\}$ is meet-prime.

Suppose $m' \nleq m$. Then since m' is the join of join-prime elements, there must be some join-prime element j^* such that $j^* \leq m'$ but $j^* \nleq m$. But j is the only join-prime element such that $j \nleq m$. So $j^* = j$, which gives us $j \leq m'$. But this is impossible since $j' \nleq m'$ and $j' < j$. So $m' \leq m$.

If $m' < m$, then m cannot be a minimal meet-prime element, so then $m' = m$. But then $j' \leq m'$, which is another contradiction. This tells us that our other supposition ($j \notin \min(Jp(\mathcal{L}))$) is false as well. Therefore, j is necessarily a minimal and maximal join-prime element.

Now let $m \in \min(Mp(\mathcal{L})) \sim \max(Mp(\mathcal{L}))$. Then there must be some $m' \in \max(Mp(\mathcal{L}))$ such that $m < m'$. Further, there must be some unique $j' \in Jp(\mathcal{L})$ such that $j' \nleq m'$ and it is necessary that this j' be maximal in $Jp(\mathcal{L})$. Suppose

$j' \in \min(Jp(\mathcal{L}))$. Then $m' \wedge j' = \perp$, which gives us $m' \wedge j' \leq m$. Since m is meet-prime, this implies $m' \leq m$ or $j' \leq m$. However, $m < m'$ and if $j' \leq m$, then $j' \leq m'$. In either case we have a contradiction, so $j' \notin \min(Jp(\mathcal{L}))$. Since j' is not a minimal join-prime element, there must be minimal, non-maximal join-prime elements $J \subseteq \min(Jp(\mathcal{L})) \sim \max(Jp(\mathcal{L}))$ such that $j < j'$ for all $j \in J$. We also know that $j \leq m'$ for all $j \in J$. Therefore, $j \leq m' \wedge j'$ for all $j \in J$ and thus $m' \wedge j' > \perp$. Further, we know that $m' \wedge j'$ must be the join of elements in J since $m' \wedge j' < j'$. Specifically, if there is some join-prime element j^* outside of J such that $j^* \leq m \wedge j'$, then j^* is necessarily maximal since J is all the minimal, non-maximal join-prime elements less than j'. But this implies $j^* \leq m' \wedge j' < j'$, which is impossible since j^* must be a maximal join-prime element.

Now suppose $j \leq m$ for all $j \in J$. Then $m' \wedge j' \leq \bigvee J \leq m$, which implies $m' \leq m$ or $j' \leq m$. Since neither case is possible, we can conclude that there must be some $j \in J$ such that $j \not\leq m$.

Now we must show that j is unique. Let $m \in Mp(\mathcal{L})$ and let j be a minimal join-prime element such that $j \not\leq m$. Suppose there exists some other $j^* \in \min(Jp(\mathcal{L})$ such that $j \not\leq m$. Since j and j^* are minimal join-prime prime elements, we know $j \wedge j^* = \perp$. But $\perp \leq m$, which gives us $j \wedge j^* \leq m$. Therefore, since m is meet-prime, $j \leq m$ or j, which is a contradiction. Therefore, j must be unique.

Finally, for the converse, suppose there exists some unique $j \in \min(Jp(\mathcal{L}))$ such that $j \not\leq m$. Suppose (by way of contradiction) $m \notin \min(Mp(\mathcal{L}))$. Then there must exist some $m' \in \min(Mp(\mathcal{L}))$ such that $m' < m$. This gives us $m' \vee (j \wedge m) = m' \vee \perp = m'$. Further, we know $m < m' \vee j$ since $j \not\leq m$, which gives us $(m' \vee j) \wedge (m' \vee m) = (m' \vee j) \wedge m = m$. This, however, contradicts the distributivity of $\mathcal{L}$. Therefore, $m \in \min(Mp(\mathcal{L}))$. $\square$

Theorem 6.2.7. [B. Frazier [5]] Let $\mathcal{G} = (V, E, \phi)$ be a finite hypergraph. Then for any $m \in \min(Mp(\Omega(\mathcal{G}))) \sim \max(Mp(\Omega(\mathcal{G})))$, $m = [V \cup E] \sim V_B(e)$ for a unique $e \in E$.

Proof. Since $m \in \min(Mp(\Omega(\mathcal{G}))) \sim \max(Mp(\Omega(\mathcal{G})))$, we know there must be a unique $j \in \min(Jp(\Omega(\mathcal{G}))) \sim \max(Jp(\Omega(\mathcal{G})))$ such that $j \not\subseteq m$. Since j is a minimal, non-maximal join-prime element of $\Omega(\mathcal{G})$, it must necessarily be a unique singleton edge set $\{e\}$ of $\mathcal{G}$. Therefore, m must not contain all maximal join-prime elements that contain e, which are precisely the edge balls containing e, each containing a vertex that is incident with e. Therefore, m does not contain $V_B(e)$. Since $\{e\}$ is the only minimal join prime element that m does contain, m must contain all other edges of $\mathcal{G}$.

Now let m' be the maximal meet-prime such that $m < m'$. Then there is some edge ball $E_B(v')$ such that $E_B(v') \not\subseteq m'$ and $e \in E_B(v')$. The only members of we have not considered yet are the vertices not in $\phi(e)$. Let $v \notin \phi(e)$ and suppose (by way of contradiction) that $v \notin m$. Then $E_B(v) \not\subseteq m$ since $v \in E_B(v)$ and $E_B(v) \wedge \{e\} = \emptyset$ since $v \notin \phi(e)$. But this gives us

$$\begin{aligned}
E_B(v') &= E_B(v') \cup \emptyset \\
&= E_B(v') \cup [E_B(v) \cap \{e\}] \\
&= [E_B(v') \cup E_B(v)] \cap [E_B(v') \cup \{e\}] \\
&= [E_B(v') \cup E_B(v)] \cap E_B(v') \\
&= [E_B(v') \cap E_B(v')] \cup [E_B(v) \cap E_B(v')] \\
&= E_B(v') \cap \emptyset \\
&= \emptyset,
\end{aligned}$$

which gives us a contradiction. Therefore, $v \in m$. We can now conclude that $m = [V \cup E] \sim V_B(e)$. $\square$

Theorem 6.2.8. [M. Wiese [7]] Let $\mathcal{L}$ be a finite CTH-lattice and let $m_1 \in \min(Mp(\mathcal{L}))$ and $m_2 \in \max(Mp(\mathcal{L}))$ such that $m_1 < m_2$. Then if $j_1 \in \min(Jp(\mathcal{L}))$ such that

$j_1 \not\leq m_1$ and $j_2 \in \max(Jp(\mathcal{L}))$ such that $j_2 \not\leq m_2$, then $j_1 < j_2$

Proof. Theorem 6.2.6 tells us that $j_1 \notin \max(Jp(\mathcal{L}))$, so we know $j \neq j_2$. Suppse $j_1 \not< j_2$. Then since j_1 is a minimal join-prime element, we know $j_1 \wedge j_2 = \perp$. Therefore, $j_1 \wedge j_2 \leq m_1$. Since m_1 is meet-prime, this tells us that $j_1 \leq m_1$ or $j_2 \leq m_1$, both of which are impossible. Therefore, $j_1 < j_2$. ☐

Theorem 6.2.9. [M. Wiese [7]] Let $\mathcal{L}$ be a finite CTH-lattice. Then $Mp(\mathcal{L}) = \min(Mp(\mathcal{L})) \cup \max(Mp(\mathcal{L}))$

Proof. Suppose (by way of contradiction) that $Mp(\mathcal{L}) \neq \min(Mp(\mathcal{L})) \cup \max(Mp(\mathcal{L}))$. We know $\min(Mp(\mathcal{L})) \cup \max(Mp(\mathcal{L})) \subseteq Mp(\mathcal{L})$, so then there must be some $m^* \in Mp(\mathcal{L})$ such that $m^* \notin \min(Mp(\mathcal{L})) \cup \max(Mp(\mathcal{L}))$. Therefore there must exist some $m \in \min(Mp(\mathcal{L}))$ and $M \in \max(Mp(\mathcal{L}))$ such that $m < m^* < M$. Further, we know that there is a unique $j_m \in \min(Jp(\mathcal{L}))$ such that $j_m \not\leq m$ and a unique $j_M \in Jp(\mathcal{L})$ such that $j_m < j_M < M$.

Suppose (again, by way of contradiction) $j_m \not\leq m^*$. We know $m < m^*$ and m is greater than or equal to every other minimal join-prime element of $\mathcal{L}$, so there must be some $j^* \in \max(Jp(\mathcal{L}))$ such that $j^* \not\leq m$ and $j^* \leq m^*$ since $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$. Further, we know that $j^* \notin \min(Jp(\mathcal{L}))$ since all minimal join-prime elements besides j_m are less than or equal to m. Therefore, there must be some $a \in \min(Jp(\mathcal{L}))$ such that $a < j^* < m^*$. Since $a < m^*$ and $j_m \not\leq m^*$, $m^* < a \vee j_m$. This gives us $m^* \wedge (j_m \vee a) = m^*$ while $(m^* \wedge j_m) \vee (m^* \wedge a) = \perp \vee a = a$, which contradicts the distributivity of $\mathcal{L}$. Therefore, $j_m \leq m^*$.

Now that we know $j_m \leq m^*$, we have $m^* \vee (M \wedge j_M) = m^* \vee j_m = m^*$. But $(m^* \vee M) \wedge (m^* \vee j_M) = M \wedge \top = M$, which again contradicts distributivity. Therefore, $Mp(\mathcal{L}) = \min(Mp(\mathcal{L})) \cup \max(Mp(\mathcal{L}))$. ☐

Theorems 6.2.4, 6.2.6, 6.2.8, and 6.2.9 tell us that there is a natural mapping from the meet-prime elements of a finite CTH-lattice to its join-prime elements. Given a CTH-lattice, $\mathcal{L}$, we will denote this mapping as $\lambda_{\mathcal{L}}$ defined by $\lambda_{\mathcal{L}} : Mp(\mathcal{L}) \to Jp(\mathcal{L})$ such that

1. If $m \in \max(Mp(\mathcal{L}))$, then $\lambda_{\mathcal{L}}(m)$ is the unique $j \in \max(Jp(\mathcal{L}))$ such that $j \not\leq m$.

2. If $m \in \min(Mp(\mathcal{L})) \sim \max(Mp(\mathcal{L}))$, then $\lambda_{\mathcal{L}}(m)$ is the unique $j \in \min(Jp(\mathcal{L})) \sim \max(Jp(\mathcal{L}))$ such that $j \not\leq m$.

[J. Snodgrass and C. Tsinakis [8]] show a more generalized version of this result.

Theorem 6.2.10. Let $\mathcal{L}$ be a finite CTH-lattice. Then $\lambda_{\mathcal{L}}$ is a bijection.

Proof. First we will show that $\lambda_{\mathcal{L}}$ is an injection. Let $\lambda_{\mathcal{L}}(x) = \lambda_{\mathcal{L}}(y)$ for some $x, y \in Mp(\mathcal{L})$.

Suppose $\lambda_{\mathcal{L}}(x) = \lambda_{\mathcal{L}}(y) \in \max(Mp(\mathcal{L}))$. Then $x, y \in \max(Mp(\mathcal{L}))$. Since $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$, we know $x = \bigvee JP(\mathcal{L}) \sim \{\lambda_{\mathcal{L}}(x)\}$ and $y = \bigvee JP(\mathcal{L}) \sim \{\lambda_{\mathcal{L}}(y)\}$. But $\lambda_{\mathcal{L}}(x) = \lambda_{\mathcal{L}}(y)$, so we have $x = \bigvee Jp(\mathcal{L}) \sim \{\lambda_{\mathcal{L}}(x)\} = \bigvee Jp(\mathcal{L}) \sim \{\lambda_{\mathcal{L}}(y)\} = y$.

Suppose $\lambda_{\mathcal{L}}(x) = \lambda_{\mathcal{L}}(y) \in \min(Jp(\mathcal{L})) \sim \max(Jp(\mathcal{L}))$. Then $x, y \in \min(Mp(\mathcal{L})) \sim \max(Mp(\mathcal{L}))$. This implies that there must be some $m_x \in \max(Mp(\mathcal{L}))$ such that $x < m_x$. If $y \not\leq m_x$, then $m_x \vee y = \top$. But $\lambda_{\mathcal{L}}(m_x)$ is the unique join-prime element such that $m_x \vee \lambda(m_x) = \top$, so then $\lambda_{\mathcal{L}}(m_x) \leq y$. But this is a contradiction since Theorem 6.2.8 tells us that $\lambda_{\mathcal{L}}(y) = \lambda_{\mathcal{L}}(x) < \lambda_{\mathcal{L}}(m_x)$ and $\lambda_{\mathcal{L}}(y) \not\leq y$. Therefore, $y \leq m_x$. This implies that $y \vee j_{m_x} = m_x \vee j_{m_x} = \top$, which gives us

$$\begin{aligned} x &= x \wedge \top \\ &= x \wedge (y \vee j_{m_x}) \\ &= (x \wedge y) \vee (x \wedge j_{m_x}). \end{aligned}$$

If $x \neq y$, then $x \wedge j_{m_x}$ must necessarily be greater that $\perp$ for $x = (x \wedge y) \vee (x \wedge j_{m_x})$. But then there must be some $j \in \min(Jp(\mathcal{L}))$ such that $j \leq x$ and $j < j_{m_x}$, which gives us

$$\begin{aligned}
j_{m_x} &= j_{m_x} \vee j \\
&= j_{m_x} \vee (x \wedge j) \\
&= (j_{m_x} \vee x) \wedge (j_{m_x} \vee j) \\
&= \top \wedge j \\
&= j
\end{aligned}$$

which is a contradiction since $j < j_{m_x}$. Therefore, $x = y$ and we can conclude that $\lambda_{\mathcal{L}}$ is injective.

Now we will show that $\lambda_{\mathcal{L}}$ is surjective. Let $j \in Jp(\mathcal{L})$. Since $\mathcal{L}$ is distributive, we know that its meet-prime elements are meet-dense in $\mathcal{L}$. Therefore, $j = \bigwedge M_j$ where $M_j = \{m \in Mp(\mathcal{L}) : j \leq m\}$. We know there must be some $m \in Mp(\mathcal{L})$ such that $m \notin M_j$; otherwise, $\bigwedge M_j = \bot$. Since $m \notin M_j$, we know $j \nleq m$. If m is a maximal meet-prime element, then j is necessarily the unique join-prime element such that $j \nleq m$ and thus $\lambda_{\mathcal{L}}(m) = j$. Suppose $m \in \min(Mp(\mathcal{L})) \sim \max(Mp(\mathcal{L}))$. If $j \in \min(Jp(\mathcal{L}))$, then j must be the unique minimal join-prime element of $\mathcal{L}$ such that $j \nleq m$, which implies $\lambda_{\mathcal{L}}(m) = j$. If $j \in \max(Jp(\mathcal{L})) \sim \min(Jp(\mathcal{L}))$, then there must be some $n \in Mp(\mathcal{L})$ distinct from m such that $j \nleq n$. If $n \in \max(Mp(\mathcal{L}))$, then $\lambda_{\mathcal{L}}(n) = j$, so we can assume that $n \in \min(Mp(\mathcal{L})) \sim \max(Mp(\mathcal{L}))$. $\square$

Theorem 6.2.11. [J. Snodgrass and C. Tsinakis [8]] Let $\mathcal{L}$ be a finite CTH-lattice. Then $\lambda_{\mathcal{L}}$ is an order isomorphism.

Proof. We already know that $\lambda_{\mathcal{L}}$ is a bijection, so we need only show that $\lambda_{\mathcal{L}}$ and its inverse are order homomorphisms. We know from Theorem 6.2.8 that for any $x, y \in Mp(\mathcal{L})$ that $x < y \implies \lambda_{\mathcal{L}}(x) < \lambda_{\mathcal{L}}(y)$. Therefore, we have $x \leq y \implies \lambda_{\mathcal{L}}(x) \leq \lambda_{\mathcal{L}}(y)$ and $\lambda_{\mathcal{L}}$ is isotone.

Now let $j_1, j_2 \in Jp(\mathcal{L})$ and suppose $j_1 \leq j_2$. If $j_1 = j_2$, then $\lambda_{\mathcal{L}}^{-1}(j_1) = \lambda_{\mathcal{L}}^{-1}(j_2)$, so we will assume that $j_1 < j_2$. Then $j_1 \in \min(Jp(\mathcal{L})) \sim \max(Jp(\mathcal{L}))$ and $j_2 \in \max(Jp(\mathcal{L})) \sim \min(Jp(\mathcal{L}))$. Since $\lambda_{\mathcal{L}}$ is a bijection, we know there must be some unique $m_2 \in \max(Mp(\mathcal{L}))$ such that $j_2 \not\leq m_2$. Similarly, there must be some $m_1 \in \min(Mp(\mathcal{L}))$ such that $j_1 \not\leq m_1$. Suppose $m_1 \not\leq m_2$. Then since $m_2 \in \max(Mp(\mathcal{L}))$, $m_2 \vee m_1 = \top$ and $m_2 \vee j_2 = \top$. But this implies that $j_2 \leq m_1$, which is a contradiction since $j_1 < j_2$ and $j_1 \not\leq m_2$. Therefore, $m_1 \leq m_2$ and $\lambda_{\mathcal{L}}^{-1}$ must be isotone. □

Theorem 6.2.12. Let $\mathcal{L} = (L, \leq)$ be a finite CTH-lattice, let $m \in Mp(\mathcal{L})$, and let $l \in L$. Then $l \in \downarrow m$ if and only if $l \notin \uparrow \lambda_{\mathcal{L}}(m)$.

Proof. Suppose $l \in \downarrow m$. Then $l \leq m$. If $l \in \uparrow \lambda_{\mathcal{L}}(m)$, then $\lambda_{\mathcal{M}}(m) \leq l \leq m$, which is impossible. So $l \notin \uparrow \lambda_{\mathcal{L}}(m)$.

Now suppose $l \notin \uparrow \lambda_{\mathcal{L}}(m)$. Then $\lambda_{\mathcal{L}}(m) \not\leq l$. If $\lambda_{\mathcal{L}}$ is a minimal join-prime element, then $x \wedge \lambda_{\mathcal{L}} = \bot \leq m$. Since m is meet-prime, this implies $x \leq m$ or $\lambda_{\mathcal{L}}(m) \leq m$. We know the latter is not possible; therefore, $x \leq m$ and we have $x \in \downarrow m$. If $\lambda_{\mathcal{L}}$ in a maximal, non-minimal join-prime element, then $x \wedge \lambda_{\mathcal{L}}(m)$ is either $\bot$ or the join of minimal join-prime elements. Therefore, $x \wedge \lambda_{\mathcal{L}}(m) \leq m$ and we once again have $x \leq m$ and thus $x \in \downarrow m$. □

Theorem 6.2.12 tells us that the meet-prime elements with their associated join-prime elements effective split a CTH-lattice. This will be useful in our final proof.

Theorem 6.2.13. Let $\mathcal{L} = (L, \leq)$ be a finite CTH-lattice. Then the function $\varepsilon_{\mathcal{L}} : L \to \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$ defined by $\varepsilon_{\mathcal{L}}(l) = \downarrow l \cap Jp(\mathcal{L})$ is an order isomorphism.

Proof. First we will show $\varepsilon_{\mathcal{L}}$ is an order homomorphism. Let $x, y \in L$ and suppose $x \leq y$. Then since $Jp(\mathcal{L})$ is join-dense in $\mathcal{L}$, we know $x = \bigvee J_x$ and $y = \bigvee J_y$ where J_x and J_y are subsets of $Jp(\mathcal{L})$. Further, we know $J_x \subseteq J_y$ since $x \leq y$. Therefore, we have

$$\begin{aligned}
\varepsilon_{\mathcal{L}}(x) &= \downarrow x \cap Jp(\mathcal{L}) \\
&= J_x \\
&\subseteq J_y \\
&= \downarrow y \cap Jp(\mathcal{L}) \\
&= \varepsilon_{\mathcal{L}}(y).
\end{aligned}$$

Thus $\varepsilon_{\mathcal{L}}$ is an order homomorphism.

Next we will show that $\varepsilon_{\mathcal{L}}$ is an injection. Suppose $\varepsilon_{\mathcal{L}}(x) = \varepsilon_{\mathcal{L}}(y)$. Then $\downarrow x \cap Jp(\mathcal{L}) = \downarrow y \cap Jp(\mathcal{L})$, which tells us that the join-prime elements in $\downarrow x$ are precisely the join-prime elements in $\downarrow y$. Therefore, $x = y$ and $\varepsilon_{\mathcal{L}}$ must be injective.

To show that $\varepsilon_{\mathcal{L}}$ is surjective, suppose $U \in \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$. Since $\Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}}) = Up(Jp(\mathcal{L})^{op})$, we know $U \in Up(Jp(\mathcal{L})^{op})$, which implies $U \in Low(Jp(\mathcal{L}))$. Therefore, $U \subseteq Jp(\mathcal{L})$. Let $x = \bigvee U$. Then since $U \subseteq Jp(\mathcal{L})$, we know $U \subseteq \downarrow x \cap Jp(\mathcal{L}) = \varepsilon_{\mathcal{L}}(x)$. Suppose $\varepsilon_{\mathcal{L}}(x) \not\subseteq U$. Then there must exist some $j \in \varepsilon_{\mathcal{L}}(x) = \downarrow x \cap Jp(\mathcal{L})$ such that $j \notin U$. Let $m_j = \lambda_{\mathcal{L}}^{-1}(j)$. We know $x \in \uparrow j$, so by Theorem 6.2.12 we know $x \notin \downarrow m_j$ and thus $x \not\leq m_j$. Since U is a lowerset of join-prime elements of $\mathcal{L}$, we know $j \not\leq u$ for all $u \in U$ (otherwise, $j \in U$). Therefore, $u \notin \uparrow j$ for all $u \in U$. By Theorem 6.2.12, each $u \in U$ must then be in $\downarrow m_j$. But this implies $\bigvee U \in \downarrow m_j$, which is a contradiction since $x = \bigvee U$. Therefore, $\varepsilon_{\mathcal{L}}(x) \subseteq U$, which gives us $U = \varepsilon_{\mathcal{L}}(x)$ and thus $\varepsilon_{\mathcal{L}}$ is surjective.

Finally, we will show that $\varepsilon_{\mathcal{L}}^{-1}$ is an order homomorphism. Suppose $\varepsilon_{\mathcal{L}}(x) \subseteq \varepsilon_{\mathcal{L}}(y)$ for some $x, y \in L$. Then $\downarrow x \cap Jp(\mathcal{L}) \subseteq \downarrow y \cap Jp(\mathcal{L})$. Therefore $\downarrow x \subseteq \downarrow y$, which implies $x \in \downarrow y$ and thus $x \leq y$. We can now conclude that $\varepsilon_{\mathcal{L}}^{-1}$ is an order homomorphism. $\square$

6.3 CTH-Lattice Homomorphisms

Definition 6.3.1. Let $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \preceq)$ be CTH-Lattices. A frame homomorphism $\alpha : A \to B$ is called a *CTH-lattice homomorphism* provided the restriction of its right adjoint to the meet-prime elements of $\mathcal{B}$ is an HP-homomorphism under the order dual of the partial ordering of the meet-prime elements. This restriction we will denote as τ^*_α.

If α is a bijection and its inverse is a CTH-lattice homomorphism, we say that α is a *CTH-lattice isomorphism*. It is worth noting that in a finite CTH-lattice all elements of the lattice are compact and therefore, the first requirement for a CTH-lattice homomorphism is not necessary for finite CTH-lattices.

Since τ^*_α is an HP-homomorphism and meet-prime elements of a CTH-lattice are order isomorphic to the join-prime elements, we know that τ^*_α induces a hypergraph homomorphism from the $\mathcal{G}_{Jp(\mathcal{M})^{op}}$ to $\mathcal{G}_{Jp(\mathcal{L})^{op}}$. This hypergraph homomorphism we will denote as μ_α and is defined by $\mu_\alpha = \lambda_{\mathcal{L}} \circ \tau^*_\alpha \circ \lambda^{-1}_{\mathcal{M}}$.

Theorem 6.3.1. [Z. French [6]] Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be finite hypergraphs and $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism and let ω_f be defined as above. Then τ_{ω_f} maps maximal meet-prime elements of $\Omega(\mathcal{H})$ to maximal meet-prime elements of $\Omega(\mathcal{G})$ and minimal, non-maximal meet-prime elements of $\Omega(\mathcal{G})$ to minimal, non-maximal meet-prime elements of $\Omega(\mathcal{H})$.

Proof. It's easy to see that the meet-prime elements of $\Omega(\mathcal{G})$ and $\Omega(\mathcal{H})$ form order duals of hypergraph posets since they are both CTH-lattices and the join-prime and meet-prime elements of finite CTH-lattices are order isomorphic. We also know that right adjoints of frame homomorphisms between bounded frames preserve meet-prime elements. So $\tau_{\omega_f}(Mp(\Omega(\mathcal{G})) \subseteq Mp(\Omega(\mathcal{H}))$.

We will first show that maximal meet-prime elements in $\Omega(\mathcal{G})$ map to maximal meet-prime elements in $\Omega(\mathcal{H})$ under τ_{ω_f}. Let $m \in \max(Mp(\Omega(\mathcal{G}))$. We must show that there is some unique edge ball (maximal join-prime element) in $\Omega(\mathcal{H})$ that is not contained in $\tau_{\omega_f}(m)$. We know from the previous section that $m = [V_G \cup E_G] \sim \{v\}$ for some $v \in V_G$. Let us consider $E_B(f(v))$. Suppose $E_B(f(v)) \subseteq \tau_{\omega_f}(m)$. Then $f(v) \in \tau_{\omega_f}(m)$ and we have

$$\begin{aligned} f(v) \in \tau_{\omega_f}(m) &\implies v \in f^{-1}(\tau_{\omega_f}(m)) \\ &\implies v \in \omega_f(m). \end{aligned}$$

But $\omega_f(\tau_{\omega_f}(m)) \subseteq m$. Therefore, $v \in m$, which is a contradiction. So we now have a maximal join-prime element that is not contained in τ_{ω_f}. We must now show it is unique. Suppose there exists some other edge ball $E_B(v')$ in $\Omega(\mathcal{H})$ that is not contained in $\tau_{\omega_f}(m)$ such that $v' \neq f(v)$. Then $\omega_f(E_B(v')) \nsubseteq m$ and $E_B(v') \neq E_B(f(v))$. Further, we have $v \notin f^{-1}(v')$ since $v' \neq f(v)$. But $f^{-1}(v') \subseteq f^{-1}(E_B(v')) = \omega_f(E_B(v'))$. Therefore, $v \notin \omega_f(E_B(v'))$. Since $m = [V_G \cup E_G] \sim \{v\}$ and $v \notin \omega_f(E_B(v'))$, we have $\omega_f(E_B(v')) \subseteq m$. Therefore, there is no other edge ball (maximal join-prime element) in $\Omega(\mathcal{H})$ that is not contained in $\tau_{\omega_f}(m)$. We can now conclude that $\tau_{\omega_f}(m)$ is a maximal meet-prime element in $\Omega(\mathcal{H})$.

Now we will show that minimal, non-maximal meet-prime elements in $\Omega(\mathcal{G})$ map to minimal, non-maximal meet-prime elements in $\Omega(\mathcal{H})$. Let n be a minimal, non-maximal element of $Mp(\Omega(\mathcal{G}))$. Then there must be some $m \in \max(Mp(\Omega(\mathcal{G})))$ such that $n < m$. Since τ_{ω_f} is isotone, we know $\tau_{\omega(\mathcal{G})}(n) \subseteq \tau_{\omega(\mathcal{G})}(m)$. So we need only show $\tau_{\omega_f}(n) \subset \tau_{\omega_f}(m)$. Suppose $\tau_{\omega_f}(n) = \tau_{\omega_f}(m)$. Then $\tau_{\omega_f}(n)$ must be a maximal join-prime element since $\tau_{\omega_f}(m)$ is. We know from the previous section that $n = [V_G \cup E_G] \sim V_B(e)$ for some $e \in E_G$. Further, since $f(e)$ is an edge in $\mathcal{H}$, we know $\{f(e)\}$ is a minimal join-prime element of $\Omega(\mathcal{H})$. Since $\tau_{\omega_f}(n)$ is maximal, we have $\{f(e)\} \subseteq \tau_{\omega_f}(n)$. Therefore, $\omega_f(\{f(e)\}) \subseteq n$. This gives us $f^{-1}(\{f(e)\}) \subseteq n$ by the definition of ω_f. But $e \in f^{-1}(\{f(e)\})$, so then $e \in n$, which is a contraction since $n = [V_G \cup E_G] \sim V_B(e)$. Therefore, $\tau_{\omega_f}(n) < \tau_{\omega_f}(m)$ and thus $\tau_{\omega_f}(n)$ must be a minimal, non-maximal meet-prime element in $\Omega(\mathcal{H})$. $\square$

Theorem 6.3.2. [Z. French [6]] Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be finite hypergraphs and $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism. Then the restriction of τ_{ω_f} to the meet-prime elements of $\Omega(\mathcal{G})$ is an HP-homomorphism under the dual orderings of the meet-prime elements of $\Omega(\mathcal{G})$ and $\Omega(\mathcal{H})$.

Proof. Since τ_{ω_f} maps maximal meet-prime elements to maximal meet-prime elements and minimal, non-maximal meet-prime elements to minimal, non-maximal meet-prime elements, we now have the first two requirements of an HP-homomorphism. We must now show that $\tau_{\omega_f}(Cov_{\Omega(\mathcal{G})}(e)) = Cov_{\Omega(\mathcal{H})}(\tau_{\omega_f}(e))$ for any $e \in E_G$.

Let e be a minimal, non-maximal meet-prime member of $\Omega(\mathcal{G})$. If $v \in Cov(e)$, then we have $e \subset v$. Since τ_{ω_f} is isotone, we then know that $\tau_{\omega_f}(e) \subseteq \tau_{\omega_f}(v)$. But we know that τ_{ω_f} maps maximal meet-prime elements and minimal, non-maximal meet-prime elements to minimal, non-maximal meet-prime elements. Therefore, $\tau_{\omega_f}(v)$ must a be maximal element in $Mp(\Omega(\mathcal{H}))$ and $\tau_{\omega_f}(e)$ must be a minimal, non-maximal element in $Mp(\Omega(\mathcal{H}))$, which tells us that $\tau_{\omega_f}(e) \subset \tau_{\omega_f}(v)$. So every element in the cover of e maps to an element of the cover of $\tau_{\omega_f}(e)$, or equivalently, $\tau_{\omega_f}(Cov_{\Omega(\mathcal{G})}(e)) \subseteq Cov_{\Omega(\mathcal{H})}(e)$.

Now suppose $x \in Cov_{\Omega(\mathcal{H})}(\tau_{\omega_f}(e))$. Then x must be a maximal meet-prime element in $\Omega(\mathcal{H})$. Therefore, there must exist a unique join-prime element j_x of $\Omega(\mathcal{H})$ such that $j_x \not\subseteq x$. This join-prime element must necessarily be a maximal join-prime element, so $j_x = E_B(v_x)$ for some $v_x \in V_H$. Since $\tau_{\omega_f}(e) \subset x$, we know $j_x \not\subseteq \tau_{\omega_f}(e)$, which implies $\omega_f(j_x) \not\subseteq e$. Therefore, $\omega_f(j_x) > \perp_{\Omega(\mathcal{G})} = \emptyset$. We also know that there is a unique minimal, non-maximal join-prime element $j_{\tau_{\omega_f}(e)}$ such that $j_{\tau_{\omega_f}(e)} \not\subseteq \tau_{\omega_f}(e)$. By the definition of adjunction, we once again have $\omega_f(j_{\tau_{\omega_f}(e)}) \not\subseteq e$, so $\omega_f(j_{\tau_{\omega_f}(e)}) > \perp_{\Omega(\mathcal{G})} = \emptyset$. Since $\tau_{\omega_f}(e) \subset x$, we know $j_{\tau_{\omega_f}(e)} \subset j_x = E_B(v_x)$ We also know that $j_{\tau_{\omega_f}(e)}$ is a singleton edge set of containing an edge in E_H, so $\omega_f(j_{\tau_{\omega_f}(e)}) = f^{-1}(j_{\tau_{\omega_f}(e)}) = E'$ where $E' \subseteq E_G$. For each $e' \in E'$, we know $f(\phi_G(e')) = \phi_H(f(e'))$ since f is a hypergraph homomorphism. But $f(\{e'\}) = j_{\tau_{\omega_f}(e)}$, so we have $f(\phi_G(e')) = \phi_H(j_{\tau_{\omega_f}(e)})$. Since $j_{\tau_{\omega_f}(e)} \subset j_x = E_B(v_x)$, we know $v_x \in \phi_H(j_{\tau_{\omega_f}(e)})$. Therefore, $v_x \in f(\phi_G(e'))$. This implies that there must be some $v' \in \phi_G(e')$ such that $f(v') = v_x$. So then we have

$$\begin{aligned} v' &\in f^{-1}(v_x) \\ &\subseteq \omega_f(E_B(v_x)) \\ &= \omega_f(j_x). \end{aligned}$$

But if $v' \in \omega_f(j_x)$, then $E_B(v') \subseteq \omega_f(j_x)$ since $E_B(v')$ is the least element of $\Omega(\mathcal{G})$ containing v'. Let $V' = \{v' \in V_G : v' \in \phi_G(e') \text{ for some } e' \in E'\}$. This gives us

$$\begin{aligned} \bigcup\{\{e'\} : e' \in E'\} &= E' \\ &= \omega_f(j_{\tau_{\omega_f}(e)}) \\ &\subseteq \bigcup\{E_B(v') : v' \in V'\} \\ &\subseteq \omega_f(j_x) \\ &= \omega_f(E_B(v_x)). \end{aligned}$$

We know that there is some unique $j_e \in \min(Jp(\Omega(\mathcal{G})))$ such that $j_e \not\subseteq e$ since e is a minimal, non-maximal meet-prime element in $\Omega(\mathcal{G})$. Further, since $\omega_f(j_{\tau_{\omega_f}(e)}) \not\subseteq e$, we know that $j_e \in \omega_f(j_{\tau_{\omega_f}(e)}) = E'$. Let v_{j_e} be a vertex in V_G such that $j_e \subset E_B(v_{j_e})$ and let m_{j_e} be the unique maximal meet-prime element in $\Omega(\mathcal{G})$ such that $E_B(v_{j_e}) \not\subseteq m_{j_e}$. Note that $v_{j_e} \in V'$ (this will be used later). Since $j_e \subset E_B(v_{j_e})$, we know $e \subset m_{j_e}$, or equivalently, $m_{j_e} \in Cov_{\Omega(\mathcal{G})}(e)$. Therefore, if $\tau_{\omega_f}(m_{j_e}) = x$, then we know that $x \in \tau_{\omega_f}(Cov_{\Omega(\mathcal{H})}(e))$, which is what we will now show.

Since $e \subset m_{j_e}$, we know $\tau_{\omega_f}(e) \subseteq \tau_{\omega_f}(m_{j_e})$. Let j be a join-prime element of $\Omega(\mathcal{H})$ such that $j \subseteq \tau_{\omega_f}(m_{j_e})$. We already know that $j_x \not\subset j$ since j_x is a maximal join-prime element. Suppose $j_x = j$. Then we have

$$\begin{aligned} E_B(v_{j_e}) &\subseteq \bigcup\{E_B(v') : v' \in V'\} \\ &\subseteq \omega_f(j_x) \\ &= \omega_f(j) \\ &\subseteq e \\ &\subset m_{j_e}. \end{aligned}$$

However, this is a contradiction since $E_B(v_{j_e}) \nsubseteq m_{j_e}$. Therefore, $j_x \neq j$. Thus $j \subseteq x$ since x contains all other join-prime members of $\Omega(\mathcal{H})$. But this means that x contains all the join-prime elements contained in $\tau_{\omega_f}(m_{j_e})$. Thus $\tau_{\omega_f}(m_{j_e}) \subseteq x$. Since τ_{ω_f} maps maximal meet-prime elements to maximal meet-prime elements, we know that $\tau_{\omega_f}(m_{j_e}) \not\subset x$. Therefore, $\tau_{\omega_f}(m_{j_e}) = x$, which tells us that $x \in \tau_{\omega_f}(Cov_{\Omega(\mathcal{G})}(e))$.

Now that we have shown that $\tau_{\omega_f}(Cov_{\Omega(\mathcal{G})}(e)) = Cov_{\Omega(\mathcal{H})}(\tau_{\omega_f}(e))$ for any minimal, non-maximal meet-prime element $e \in \Omega(\mathcal{G})$, we know that $\tau_{\omega_f}(Cov^{-1}_{\Omega(\mathcal{G})^{op}}(e)) = Cov^{-1}_{\Omega(\mathcal{H})^{op}}(\tau_{\omega_f}(e))$, which gives us the third requirement for an HP-homomorphism. $\square$

Theorem 6.3.3. [Z. French [6]] Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be finite hypergraphs and let $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then the function $\omega_f : \Omega(\mathcal{H}) \to \Omega(\mathcal{G})$ defined by $\omega_f(U) = f^{-1}(U)$ is a CTH-lattice homomorphism.

Proof. We know by Theorem 5.1.5 that ω_f is a frame homomorphism, and Theorem 6.3.2 tells us that $\tau^*_{\omega_f}$ is an HP-homomorphism. Therefore, ω_f must be a CTH-lattice homomorphism. $\square$

Theorem 6.3.4. Let $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$ be finite hypergraphs, let $f : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism from $\mathcal{G}$ to $\mathcal{H}$, and let

$\kappa_G : IP_{\mathcal{G}} \to Cjp(\Omega(\mathcal{G}))$ and $\kappa_H : IP_{\mathcal{H}} \to Cjp(\Omega(\mathcal{H}))$ be the order isomorphisms defined by $\kappa_G(x) = \uparrow_{IP_{\mathcal{G}}} x$ and $\kappa_H(x) = \uparrow_{IP_{\mathcal{H}}} x$. Then $f = \kappa_H^{-1} \circ \mu_{\omega_f} \circ \kappa_G$.

Proof. Let $v_G \in V_G$. Since f is a hypergraph homomorphism, we know that $f(V_G) \subseteq V_H$ and thus $f(v_G) = v_H$ for some $v_H \in V_H$. We also know that $f(\phi_G(e)) = \phi_H(f(e))$ for any $e \in E_G$. This tells us that $E_B(v_G) \subseteq f^{-1}(E_B(v_H)) = \omega_f(E_B(v_H))$. We have also shown there is a unique maximal meet-prime member $x \in \Omega(\mathcal{G})$ such that $x = V_G \cup E_G \sim \{v_G\}$ (or equivalently, $E_B(v_G) \not\subseteq x$).

Now suppose $E_B(v_H) \subseteq \tau_{\omega_f}(x)$. Then $\omega_f(E_B(v_H)) \subseteq x$. But $E_B(v_G) \subseteq \omega_f(E_B(v_H))$ and $E_B(v_G) \not\subseteq x$, so we have a contradiction. Therefore, $E_B(v_H) \not\subseteq \tau_{\omega_f}(x)$. This tells us that if $f(v_G) = v_H$, then

$$\mu_{\omega_f}(E_B(v_G)) = [\lambda_{\Omega(\mathcal{H})} \circ \tau_{\omega_f} \circ \lambda_{\Omega(\mathcal{G})}^{-1}](E_B(v_G)) = E_B(v_H).$$

Equivalently, we have

$$\mu_{\omega_f}(\kappa_G(v_G)) = \kappa_H(v_H),$$

which gives us

$$\begin{aligned}[\kappa_H^{-1} \circ \mu_{\omega_f} \circ \kappa_G](v_G) &= \kappa_H^{-1}(\mu_{\omega_f}(\kappa_G(v_G)) \\ &= v_H \\ &= f(v_G).\end{aligned}$$

Now let $e_G \in E_G$. Again, since f is a hypergraph homomorphism, we know that $f(E_G) \subseteq E_H$ and thus $f(e_G) = e_H$ for some $e_H \in E_H$. It's easy to see that $\{e_G\} \subseteq f^{-1}(\{e_H\}) = \omega_f(\{e_H\})$ and we also know there is a unique minimal, non-maximal meet-prime member of $\Omega(\mathcal{G})$, y such that $\{e_G\} \not\subseteq y$.

Suppose $\{e_H\} \subseteq \tau_{\omega_f}(y)$. Then $\omega_f(\{e_H\}) \subseteq y$. But $\{e_G\} \subseteq \omega_f(\{e_H\})$ and $\{e_H\} \not\subseteq y$, so we have a contradiction. Therefore, $\{e_H\} \not\subseteq \tau_{\omega_f}(y)$. Once again, this tells us that if $f(e_G) = e_H$, then

$$\mu_{\omega_f}(\{e_G\}) = [\lambda_{\Omega(\mathcal{H})} \circ \tau_{\omega_f} \circ \lambda_{\Omega(\mathcal{G})}^{-1}](\{e_G\}) = \{e_H\}.$$

Equivalently, we have

$$\mu_{\omega_f}(\kappa_G(e_G)) = \kappa_H(e_H),$$

which gives us

$$\begin{aligned}[\kappa_H^{-1} \circ \mu_{\omega_f} \circ \kappa_G](e_G) &= \kappa_H^{-1}(\mu_{\omega_f}(\kappa_G(e_G)) \\ &= e_H \\ &= f(e_G).\end{aligned}$$

Now that we have considered cases for both vertices and edges, we can conclude that $f = \kappa_H^{-1} \circ \mu_{\omega_f} \circ \kappa_G$. □

Theorem 6.3.5. [Z. French [6]] The composition of two CTH-lattice homormophisms is a CTH-lattice homomorphism.

Proof. Let $\mathcal{L} = (L, \leq_L)$, $\mathcal{M} = (M, \leq_M)$, and $\mathcal{N} = (N, \leq_N)$ be CTH-lattices and let $\alpha : M \to N$ and $\beta : L \to M$ be CTH-lattice homomorphisms. We know the composition of frame homomorphisms is a frame homomorphism, so we need only show that $\tau^*_{\alpha\circ\beta}$ is an HP-homomorphism. We know τ^*_α and τ^*_β must be HP-homomorphisms by the definition of CTH-lattice homomorphism. Further, we know $\tau_{\alpha\circ\beta} = \tau_\beta \circ \tau_\alpha$. But the composition of two HP-homomorphisms is also an HP-homomorphism. Therefore, $\tau_{\alpha\circ\beta}$ must be an HP-homomorphism. We can not conclude that $\alpha \circ \beta$ is a CTH-lattice homomorphism. □

It follows from Theorem 6.3.5 that the composition of two CTH-lattice isomorphisms is also a CTH-lattice isomorphism.

CHAPTER 7

DUALITY OF FHG & FCTH-LATTICES

7.1 The Category of FCTH-Lattice

Definition 7.1.1. Let O be the set of all finite CTH-lattices, let M be the set of CTH-lattice homomorphisms between finite CTH-lattices, and let $\circ$ be traditional functional composition. We define $FCTH$ as the ordered triple $(O, M, \circ)$.

Theorem 7.1.1. [Z. French [6]] $FCTH$ is a category.

Proof. $FCTH$ satisfies the first and fourth requirements of a category, trivially, through the properties of functions. Further, by Theorem 6.3.5, we know that morphism composition in $FCTH$ is closed. So we need on show that the third requirement of a category holds for $FCTH$.

Let $\mathcal{L} = (L, \leq) \in Ob(FCTH)$, let $\alpha : L \to M$ and $\beta : N \to L$ be finite CTH-lattice homomorphisms, and let $1_{\mathcal{L}} : L \to L$ be a function defined by $1_{\mathcal{L}}(x) = x$ for all $x \in L$. It is easy to see that that the restriction of the right adjoint of $1_{\mathcal{L}}$ is an HP-isomorphism since it maps the meet-prime elements of $\mathcal{L}$ to themselves. Further, for any $x \in L$, we have $(\alpha \circ 1_{\mathcal{L}})(x) = \alpha(1_{\mathcal{L}}(x)) = \alpha(x)$ and $(1_{\mathcal{L}} \circ \beta)(x) = 1_{\mathcal{L}}(\beta(x)) = \beta(x)$. We can now conclude that $FCTH$ is a category. □

7.2 DUALITY OF FHG AND FCTH

Theorem 7.2.1. [Z. French [6]] The mapping $F : FHG \to FCTH$ defined below is a contravariant functor.

1. $\forall \mathcal{G} \in Ob(FHG), F(\mathcal{G}) = \Omega(\mathcal{G})$
2. $\forall \alpha \in Hom(FHG), F(\alpha) = \omega_{\alpha}$.

Proof. Theorems 6.2.1 and 6.3.3 tell us that the mappings of objects in FHG are CTH-lattices and mappings of morphisms in FHG are CTH-lattice homomorphisms. Thus we have the first two properties of a functor.

For the third property, let $\mathcal{G} = (V, E, \phi) \in Ob(FHG)$. Then since FHG is a category, we know there exists an identity function $1_{\mathcal{G}} : V \cup E \to V \cup E$ defined by $1_{\mathcal{G}}(x) = x$. It is easy to see that the inverse image of $1_{\mathcal{G}}$ of any open set under any topology will be itself. Thus, for any open set $U \in \Omega(\mathcal{G})$, $1_{\mathcal{G}}^{-1}(U) = U$. Therefore, $F(1_{\mathcal{G}})(U) = \omega_{1_{\mathcal{G}}}(U) = U$ for any $U \in \Omega(\mathcal{G})$. But the identity mapping for a CTH-lattice, $\mathcal{L} = (L, \leq)$, is simply $1_{\mathcal{L}} : L \to L$ defined by $1_{\mathcal{L}}(l) = l$. So we have $1_{\Omega(\mathcal{G})}(U) = U = F(1_{\mathcal{G}})(U)$. Therefore $F(1_{\mathcal{G}}) = 1_{F(\mathcal{G})}$ and we have the third property of a functor.

Finally, for the fourth property, let $\mathcal{A} = (V_A, E_A, \phi_A)$, $\mathcal{B} = (V_B, E_B, \phi_B)$, and $\mathcal{C} = (V_C, E_C, \phi_C)$ be finite hypergraphs and let $\alpha : V_B \cup E_B \to V_C \cup E_C$ and $\beta : V_A \cup E_A \to V_B \cup E_B$ be hypergraph homomorphisms from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{A}$ to $\mathcal{B}$, respectively. Then $F(\alpha \circ \beta) = \omega_{\alpha\circ\beta}$ where $\omega_{\alpha\circ\beta} : \Omega(\mathcal{A}) \to \Omega(\mathcal{C})$ is defined by $\omega_{\alpha\circ\beta}(U) = (\alpha\circ\beta)^{-1}(U)$. But $(\alpha \circ \beta)^{-1}(U) = \beta^{-1}(\alpha^{-1}(U)) =$ for any $U \in \Omega(\mathcal{C})$, so we have $F(\alpha \circ \beta)(U) = \omega_{\alpha\circ\beta}(U) = (\alpha \circ \beta)^{-1}(U) = \beta^{-1}(\alpha^{-1}(U)) = \omega_\beta(\omega_\alpha(U)) = F(\beta)(F(\alpha)(U)) = (F(\beta) \circ F(\alpha))(U)$. Thus we have the fourth property of a contravariant functor. □

Theorem 7.2.2. The mapping $G : FCTH \to FHG$ defined below is a contravariant functor.

1. $\forall \mathcal{L} \in Ob(FCTH), G(\mathcal{L}) = \mathcal{G}_{Jp(\mathcal{L})^{op}}$

2. $\forall \alpha \in Hom(FCTH), G(\alpha) = \mu_\alpha$.

Proof. We know that $Jp(\mathcal{L})$ is the order dual of a finite hypergraph poset. Therefore, $JP(\mathcal{L})^{op}$ must be a hypergraph poset and $\mathcal{G}_{Jp(\mathcal{L})^{op}}$ is well defined and is the unique finite hypergraph determined by $Jp(\mathcal{L})^{op}$.

We know from Chapter 6 that for any CTH-lattice homomorphism $\alpha : L \to M$ from CTH-lattices $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \preceq)$, that the function μ_α is precisely a hypergraph homomorphism from $G(\mathcal{L}) = \mathcal{G}_{Jp(\mathcal{L})^{op}}$ to $G(\mathcal{M}) = \mathcal{G}_{Jp(\mathcal{M})^{op}}$.

Let $\mathcal{L} = (L, \leq) \in Ob(FCTH)$. It is easy to see that $1_{\mathcal{L}} : L \to L$ defined by $1_{\mathcal{L}}(l) = l$ is the identity function for $\mathcal{L}$ in $Hom(FCTH)$. Further, the right adjoint of $1_{\mathcal{L}}$ is simply it's inverse, $1_{\mathcal{L}}^{-1}$, so $G(1_{\mathcal{L}}) = \mu_{1_{\mathcal{L}}} = \lambda_{\mathcal{L}} \circ \tau_{1_{\mathcal{L}}} \circ \lambda_{\mathcal{L}}^{-1}$, which is simply a mapping of each of $\mathcal{L}$'s join-prime elements to themselves, which is a hypergraph isomorphism from $G(\mathcal{L}) = \mathcal{G}_{Jp(\mathcal{L})^{op}}$ to itself that is the identity function for $\mathcal{G}_{Jp(\mathcal{L})^{op}}$. In other words, we have $G(1_{\mathcal{L}}) = 1_{\mathcal{G}_{Jp(\mathcal{L})^{op}}} = 1_{G(\mathcal{L})}$.

For the last criterion for a contravariant functor, let $\alpha : M \to N$ and $\beta : L \to M$ be CTH-lattice homomorphisms between CTH-lattices $\mathcal{L} = (L, \leq_{\mathcal{L}})$, $\mathcal{M} = (M, \leq_{\mathcal{M}})$, and $\mathcal{N} = (N, \leq_{\mathcal{N}})$. $\alpha \circ \beta$ is clearly defined here, so we need only show that $G(\alpha \circ \beta) = G(\beta) \circ G(\alpha)$.

$$
\begin{aligned}
G(\alpha \circ \beta) &= \mu_{\alpha \circ \beta} \\
&= \lambda_{\mathcal{L}} \circ \tau^*_{\alpha \circ \beta} \circ \lambda_{\mathcal{N}}^{-1} \\
&= \lambda_{\mathcal{L}} \circ (\tau^*_{\beta} \circ \tau^*_{\alpha}) \circ \lambda_{\mathcal{N}}^{-1} \\
&= (\lambda_{\mathcal{L}} \circ \tau^*_{\beta}) \circ (\tau^*_{\alpha} \circ \lambda_{\mathcal{N}}^{-1}) \\
&= (\lambda_{\mathcal{L}} \circ \tau^*_{\beta}) \circ (\lambda_{\mathcal{M}}^{-1} \circ \lambda_{\mathcal{M}}) \circ (\tau^*_{\alpha} \circ \lambda_{\mathcal{N}}^{-1}) \\
&= (\lambda_{\mathcal{L}} \circ \tau^*_{\beta} \circ \lambda_{\mathcal{M}}^{-1}) \circ (\lambda_{\mathcal{M}} \circ \tau^*_{\alpha} \circ \lambda_{\mathcal{N}}^{-1}) \\
&= \mu_{\beta} \circ \mu_{\alpha} \\
&= G(\beta) \circ G(\alpha).
\end{aligned}
$$

We can now conclude that G is a contravariant functor. $\square$

Theorem 7.2.3. [Z. French [6]] For any $\mathcal{G} \in Ob(FHG)$ and any $\mathcal{L} \in Ob(FCTH)$, $G(F(\mathcal{G})) \cong \mathcal{G}$ and $F(G(\mathcal{L})) \cong \mathcal{L}$.

Proof. We know from Theorem 5.1.3 that $IP_{\mathcal{G}}$ is HP-isomorphic to $Jp(\Omega(\mathcal{G}))^{op}$; therefore, $\mathcal{G}$ must be hypergraph isomorphic to $G(F(\mathcal{G})) = \mathcal{G}_{Jp(\Omega(\mathcal{G}))}$.

Again, using Theorem 5.1.3, we know that $IP_{\mathcal{G}_{Jp(\mathcal{L})^{op}}}$ is HP-isomorphic to $Jp(\Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}}))^{op}$. But $IP_{\mathcal{G}_{Jp(\mathcal{L})^{op}}}$ is precisely $Jp(\mathcal{L})^{op}$, so by Theorem 6.2.2 we have $F(G(\mathcal{L})) = \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$ is order isomorphic to $\mathcal{L}$. ☐

Theorem 7.2.4. For any $\alpha \in Hom(FHG)$, $G(F(\alpha)) = \alpha$ (up to isomorphism).

Proof. Let $\alpha : V_G \cup E_G \to V_H \cup E_H$ be a hypergraph homomorphism between finite hypergraphs $\mathcal{G} = (V_G, E_G, \phi_G)$ and $\mathcal{H} = (V_H, E_H, \phi_H)$. Then $F(\alpha) = \omega_\alpha$. We also know that for any $U \in \Omega(\mathcal{G})$,

$$\begin{aligned} \tau_{\omega_\alpha}(U) &= \bigcup \omega_\alpha^{-1}(\downarrow_{\Omega(\mathcal{G})} U) \\ &= \bigcup \{\omega_\alpha^{-1}(V) : V \in \downarrow_{\Omega(\mathcal{G})} U\} \\ &= \bigcup \{\alpha(V) : V \in \downarrow_{\Omega(\mathcal{G})} U\} \end{aligned}$$

Since $V \subseteq U$ for any $V \in \downarrow_{\Omega(\mathcal{G})} U$, we have $\bigcup\{\alpha(V) : V \in \downarrow_{\Omega(\mathcal{G})} U\} = \alpha(U)$. Therefore, $\tau_{\omega_\alpha}(U) = \alpha(U)$ for any $U \in \Omega(\mathcal{G})$. This gives us

$$\begin{aligned} G(F(\alpha)) &= G(\omega_\alpha) \\ &= \mu_{\lambda_{\Omega(\mathcal{G})}} \\ &= \lambda_{\Omega(\mathcal{H})} \circ \tau^*_{\omega_\alpha} \circ \lambda^{-1}_{\Omega(\mathcal{G})} \\ &= \lambda_{\Omega(\mathcal{H})} \circ \alpha^* \circ \lambda^{-1}_{\Omega(\mathcal{G})} \end{aligned}$$

where α^* is the mapping of α over the open sets in $CTH(\mathcal{G})$ which are meet-prime in $\Omega(\mathcal{G})$. We know that $IP_\mathcal{G}$ is HP-isomorphic to $Jp(\Omega(\mathcal{G}))^{op}$ and $IP_\mathcal{G}$ is HP-isomorphic to $Jp(\Omega(\mathcal{H}))^{op}$, so there must be HP-isomorphisms $I_\mathcal{G} : IP_\mathcal{G} \to Jp(\Omega(\mathcal{G}))^{op}$ and $I_\mathcal{H} : Jp(\Omega(\mathcal{H})) \to IP_\mathcal{H}$. Using these HP-isomorphisms we have

$$\alpha = I_\mathcal{H} \circ \lambda_{\Omega(H)} \circ \alpha^* \circ \lambda^{-1}_{\Omega(\mathcal{G})} \circ I_\mathcal{G}.$$

We can now conclude that $G(F(\alpha))$ is α up to the isomorphisms between their domains and codomains. ☐

Theorem 7.2.5. For any $\beta \in Hom(FCTH)$, $F(G(\beta)) = \beta$ (up to isomorphism).

Proof. Let $\mathcal{L} = (L, \leq)$ and $\mathcal{M} = (M, \preceq)$ be CTH-lattices. We will define functions $\varepsilon_{\mathcal{L}} : L \to Up(Jp(\mathcal{L})^{op})$ and $\varepsilon_{\mathcal{M}} : M \to Up(Jp(\mathcal{M})^{op})$ such that $\varepsilon_{\mathcal{M}}(l) = \downarrow_{\mathcal{L}} l \cap Jp(\mathcal{L})$ and $\varepsilon_{\mathcal{M}}(m) = \downarrow_{\mathcal{M}} m \cap Jp(\mathcal{M})$. We know that $\Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}}) = Up(Jp(\mathcal{L})^{op})$ and $\Omega(\mathcal{G}_{Jp(\mathcal{M})^{op}}) = Up(Jp(\mathcal{M})^{op})$, so it is equivalent to define $\varepsilon_{\mathcal{L}}$ and $\varepsilon_{\mathcal{M}}$ as $\varepsilon_{\mathcal{L}} : L \to \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$ and $\varepsilon_{\mathcal{M}} : M \to \Omega(\mathcal{G}_{Jp(\mathcal{M})^{op}})$.

Let $\beta : L \to M$ be a CTH-lattice homomorphism. We claim

$$F(G(\beta)) = \omega_{\mu_\beta} = [\varepsilon_{\mathcal{M}} \circ \beta \circ \varepsilon_{\mathcal{L}}^{-1}].$$

Let $x \in \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$. Suppose $y \in \varepsilon_{\mathcal{M}}(\beta(\varepsilon_{\mathcal{L}}^{-1}(x)))$. Then $y \in \downarrow_{\mathcal{M}} \beta(\varepsilon_{\mathcal{L}}^{-1}(x)) \cap Jp(\mathcal{M})$, which tells us that y must be a join-prime member of $\mathcal{M}$ such that $y \preceq \beta(\varepsilon_{\mathcal{L}}^{-1}(x))$. Let $m_y = \lambda_{\mathcal{M}}^{-1}(y)$. Then by definition of $\lambda_{\mathcal{M}}$, we know $y \not\preceq m_y$. Since $\beta(\varepsilon_{\mathcal{M}}(x)) \in \uparrow_{\mathcal{L}} y$, by Theorem 6.2.12 we know $\beta(\varepsilon_{\mathcal{M}}(x)) \notin \downarrow_{\mathcal{M}} m_y$. Therefore, $\beta(\varepsilon_{\mathcal{M}}(x)) \not\preceq m_y$. This gives us $\varepsilon_{\mathcal{L}}^{-1}(x) \not\leq \tau_\beta(m_y) = \tau_\beta^*(m_y)$. Let $j_{\tau_\beta^*(m_y)} = \lambda_{\mathcal{L}}(\tau_\beta^*(m_y))$. We know

$$\lambda_{\mathcal{L}}(\tau_\beta^*(m_y)) = \lambda_{\mathcal{L}}(\tau_\beta^*(\lambda_{\mathcal{M}}^{-1}(y))) = \mu_\beta(y).$$

Therefore, $j_{\tau_\beta^*(m_y)} = \mu_\beta(y)$. Using Theorem 6.2.12, we know $j_{\tau_\beta^*(m_y)} \leq \varepsilon_{\mathcal{L}}^{-1}(x)$. But this tells us that $j_{\tau_\beta^*(m_y)} \in x$ since $j_{\tau_\beta^*(m_y)}$ is join-prime and in the principal lowerset $\downarrow_{\mathcal{L}} \varepsilon_{\mathcal{L}}^{-1}(x)$. Since $j_{\tau_\beta^*(m_y)} = \mu_\beta(y)$, we have $\mu_\beta(y) \in x$ and thus $y \in \mu_\beta^{-1}(x) = \omega_{\mu_\beta}(x)$. Therefore, $\varepsilon_{\mathcal{M}}(\beta(\varepsilon_{\mathcal{L}}^{-1}(x) \subseteq \omega_{\mu_\beta}(x)$.

Now suppose $y \in \omega_{\mu_\beta}(x)$. Then $y \in \mu_\beta^{-1}(x)$, which gives us $\mu_\beta(y) \in x$ and thus $\mu_\beta(y) \leq \varepsilon_{\mathcal{L}}^{-1}(x)$. Note that μ_β is a function over the join-prime members of $\mathcal{M}$, which implies that y must be join-prime in $\mathcal{M}$. Let $m_y = \lambda_{\mathcal{M}}^{-1}(y)$ and $m_{\mu_\beta(y)} = \lambda_{\mathcal{L}}^{-1}(\mu_\beta(y))$. Then we have

$$\lambda_{\mathcal{L}}^{-1}(\mu_\beta(y)) = \lambda_{\mathcal{L}}^{-1}(\lambda_{\mathcal{L}}(\tau_\beta^*(\lambda_{\mathcal{M}}^{-1}(y)))) = \tau_\beta^*(\lambda_{\mathcal{M}}^{-1}(y)) = \tau_\beta^*(m_y).$$

Therefore, $\mu_\beta(y) \not\leq m_{\mu_\beta(y)} = \tau_\beta^*(m_y)$. Theorem 6.2.12 tells us that $\varepsilon_{\mathcal{L}}^{-1}(x) \not\leq \tau_\beta^*(m_y)$. By the definition of adjunction, we have $\beta(\varepsilon_{\mathcal{L}}^{-1}(x)) \not\preceq m_y$. Again, using Theorem 6.2.12, we have $y \preceq \beta(\varepsilon_{\mathcal{L}}^{-1}(x))$. Since y is join-prime, this gives us $y \in \downarrow_{\mathcal{M}} \beta(\varepsilon_{\mathcal{L}}^{-1}(x)) \cap$

$Jp(\mathcal{M}) = \varepsilon_{\mathcal{M}}(\beta(\varepsilon_{\mathcal{L}}^{-1}(x)))$. We can now conclude that $F(G(\beta))(x) = \omega_{\mu_\beta}(x) = [\varepsilon_{\mathcal{M}} \circ \beta \circ \varepsilon_{\mathcal{L}}^{-1}](x)$ for all $x \in \Omega(\mathcal{G}_{Jp(\mathcal{L})^{op}})$. $\square$

We have now shown that F and G are contravariant functors that commute with respect to their object mappings and commute with respect to their morphism mappings up to isomorphic composition. This allows us to conclude that we have a duality between the category FHG and FCTH.

BIBLIOGRAPHY

[1] James B. Hart, Brian Frazier, *Finite Simple Graphs and Their Associated Graph Lattices*, Theory and Applications of Graphs,**5**, (2018)

[2] J. R. Munkres, *Topology, 2nd Ed.*, Pearson, (2000).

[3] Tom Leinster,*Basic Category Theory*, (2014)

[4] J. Hart, and Z. French, *An introduction to order theory*, Journal of Inquiry-Based Learning in Mathematics, No. 44 (March 2017).

[5] B. Frazier, *Lattice structures in finite graph topologies*, Middle Tennessee State University, ProQuest Publishing Services, 1588247, (2015).

[6] Z. French, *A Duality between hypergraphs and cone lattices*, Middle Tennessee State University, ProQuest Publishing Services, 10784141, (2018).

[7] M. Wiese, *The classical topology and cone lattices*, Middle Tennessee State University, ProQuest Publishing Services, 10240429, (2016).

[8] J. Snodgrass, and C. Tsinakis, *Finite-valued algebraic lattices*, Algebra Universalis **30**, pp. 311 - 318 (1993).

[9] A. Vella, *A Fundamentally Topological Perspective on Graph Theory*, Ph.D. Dissertation, University of Waterloo, (2005).

[10] W. H. Cornish, *On H. Priestley's dual of the category of bounded distributive lattices*, Matematički Vesnik, **12 (27)**, 329 - 332, (1975).

[11] Gratzer G. *Lattice Theory: Foundation*, Springer Basel (2010).