Three-Linkage on the Projected Plane

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ABSTRACT

This thesis will provide some brief background information in the fields of Graph Theory and Algebraic Topology, a short history of linkage problems, and then a new result for 3-linked graphs embedded in the projected plane. First we will prove some Algebraic Topology results, importantly that a punctured projective plane is homoemorphic to an open möbius band. Then in the Graph Theory section, we will provide a proof of Menger's Theorem. Next, we will discuss the results near to that of this thesis, including the 2-linkage theorem and an extremal function of k-linkage. Finally, we will describe one structure which ensures three-linkage on the projected plane. Three-linkage is defined as follows: a graph G is three-linked if for any three pairs (s_1, t_1), (s_2, t_2), (s_3, t_3) of vertices in G, there exist vertex disjoint paths P_1, P_2, P_3 such that for $1 \leq i \leq 3$, P_i links s_i to t_i. In this paper we will provide a classification of 5 conected graphs embedded in the projected plane with face-width at least 5. Namely, we will prove that if G is a 5-connected graph embedded in the projective plane with face width at least 5, then G is three-linked if and only if $G - s_3$ has has a specific structure.

CONTENTS

List of Figures

CHAPTER 1

Topology Background

The following definitions and theorems are standard in topology; see e.g. [10].

Definition 1.1. A *topology* T on a set X is a collection of subsets of X such that $\emptyset, X \in T$, T is closed under infinite unions and T is closed under finite intersections. The pair (X, T) is called a Topological space.

Definition 1.2. A *homeomorphism* between two topological spaces **X** and **Y** is a continous bijection $f : X \to Y$ with a continous inverse.

Definition 1.3. A *manifold* is a topological space **X** such that for every $x \in \mathbf{X}$ there is a neighborhood around x that is homeomorphic to the open unit ball in $\mathbb{R}^n$.

Definition 1.4. A *homotopy* between two functions $f$and g from a topological space **X** to a space **Y** is a surjective continuous map G from $X \times [0,1] \to Y$ such that G(x,0)=f(x) and G(x,1)=g(x).

Definition 1.5. A subspace **A** of **X** is called a *deformation retract* of **X** if there is a homotopy $F : X \times I \to X$ (called a retract) such that for all x in X and a in A, $F(x, 0) = x, F(x, 1) \in A, F(a, 1) = a$.

Definition 1.6. The *Möbius band* is defined to be the space obtained from the unit square $I \times I$ by identifying two opposite edges $I \times \{1\}$ and $I \times \{0\}$ in reverse order. That is, let $(x, 0) \sim (1 - x, 1)$ and define the Möbius band to be the quotient space $(I \times I)/\sim$. See Figure 1.

Definition 1.7. The *open Möbius band* is defined to be the space obtained from $(0, 1) \times I$ by identifying two opposite edges $(0, 1) \times \{1\}$ and $(0, 1) \times \{0\}$ in reverse order. That is, let $(x, 0) \sim (1 - x, 1)$ and define the Möbius band to be the quotient space $((0, 1) \times I)/\sim$.

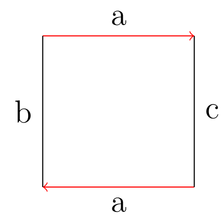


Figure 1: A Möbius band from identifications on the edges of a square

Definition 1.8. The *real projective plane* $\mathbb{R}P^2$ is defined as the space obtained from the unit square $I \times I$ by identifying antipodal points. That is, let $(x, 0) \sim (1 - x, 1)$ and let $(0, y) \sim (1, 1 - y)$, and define $\mathbb{R}P^2$ to be the quotient space $(I \times I)/\sim$. See Figure 2.

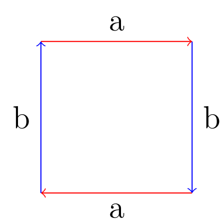


Figure 2: The real projective plane from identifications along the edges of a square

Theorem 1.9. *A disc is homotopic to a singular point.*

Theorem 1.10. $\mathbb{R}P^2 - x$ *is homeomorphic to the open Möbius band.*

Proof. $\mathbb{R}P^2 - x$ deformation retracts onto the boundary circle of the disc with antipodal points identified, $\mathbb{R}P^2 = S^1/(x \sim (-x))$. Then take the map $S^1 \to \mathbb{R}P^2$ given by $z \to z^2$. This is a continuous bijection, thus a homeomorphism. So the punctured projective plane is homotopy equivalent to S^1.

Now the open Möbius band deformation retracts directly to S^1. Then we can use the identity map as our continuous bijection. Thus the open Möbius band is homeomorphic to S^1.

Therefore we have that $\mathbb{R}P^2$ is homeomorphic to the open Möbius band. $\square$

CHAPTER 2

Graph Theory Background

The following definitions are standard and come from Diestel's "Graph Theory" [4]. A graph $G = (V, E)$ is defined to be a pair of sets V and E, with $E \subseteq [V]^2$. In general, $E \cap V$ is empty. The number of vertices on a graph is called its *order*, notated $|G|$. We define the *edge set* as $E(G)$ and the *vertex set* as $V(G)$. A *subgraph* of a graph G is a graph H with the property that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $A \in V(G)$, then $G - A$ is the subgraph obtained from G by deleting A, and $G(A)$ is the subgraph of G *induced* by A, i.e. $G(A) = G - [V(G)/A]$. An edge e of G is a subgraph of G with the property that $e = a \cup l \cup b$ where $a, b \in V(G)$ and $l \in E(G)$. A *path* P from v_1 to v_2 of G is a subgraph of G consisting of a union of edges $e_1, e_2, \dots, e_n$ that starts at v_1 and end at v_2 where the second vertex of e_i is the same as the first vertex of e_{i+1}, with the property that any $v \in V(P)$ appears exactly once in P as a second vertex and once as a first vertex.

Definition 2.11. We will notate a path on G from v_1 to v_2 by $[v_1, v_2]$.

A *walk* on G from v_1 to v_2 is a union of edges from G, $e_1, e_2, \dots, e_n$, that start at v_1 and end at v_2, in which the second vertex of e_i is the same as the first vertex of e_{i+1}. Note that the definition of walk allows for the repetition of edges and vertices while the definition of a path does not. A *cycle* of G is a path of G that has the same starting and ending point. A *closed walk* of G is a walk of G with the same starting and ending points.

Definition 2.12. We say a vertex has degree k if that vertex appears as a starting and ending vertex in $E(G)$ exactly k times.

If a vertex has degree zero, it is said to be *disconnected* from the graph. If G is a connected graph and R, S, T are vertex sets, we say that R *separates* S from T if each of $S \setminus R$ and $T \setminus R$ is non-empty, and every $S - T$ path of G (i.e. a path with ends in S and T, respectively) contains a vertex of R.

Definition 2.13. We say a graph is *k-connected* if it has more than k vertices and removal of fewer than k vertices does not result in a disconnected graph.

A *plane* graph (finite or infinite) is a graph drawn in the plane such that any two edges have at most an end in common. A *planar* graph is an abstract graph isomorphic to a plane graph. A *facial cycle* of a plane graph is a cycle whose interior or exterior does not intersect the graph.

Definition 2.14. We say that a graph G embedded in some manifold M has *face-width* k if every non-contractable surface curve of M passes through at leask k faces of the embedding of G.

Definition 2.15. A graph G is said to be *k-linked* if for every k distinct pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ we can find paths $P_1, ..P_k$ that are vertex disjoint such that P_i has s_i and t_i as its endpoints.

Definition 2.16. Let $A : a_1 \to b_1$ be a path. Then $A[a_1, b_1]$ will mean the subgraph of A induced by the vertices of A from a_1 to b_1. We will use this regularly throughout the proof for clarity purposes.

Definition 2.17. Consider a plane graph G_0 such that the unbounded face is bounded by a 4 -cycle $S_0 : x_1x_2y_1y_2x_1$ and such that every other face is bounded by a 3-cycle. Suppose in addition that G_0 has no separating 3-cycle (i.e. a 3-cycle which is not a facial cycle). For each 3-cycle S of G_0 we add K^S, a possible empty complete graph disjoint from G_0, and we join all vertices of K^S to all vertices of S. The resulting graph G is called an $(x_1, x_2, y_1, y_2)-$*web* with frame S_0 and rib G_0. [3]

Theorem 2.18 (2-linkage theorem (Thomassen/Seymour 1980 [3] [14]))**.** *Let x_1, y_1, x_2, y_2 be vertices of a graph G. If G has no x_1, x_2, y_1, y_2 linkage and the addition of any edge to G results in a graph containing an x_1, x_2, y_1, y_2 linkage, then G is an x_1, x_2, y_1, y_2-web. Conversely, any x_1, x_2, y_1, y_2-web is maximal with respect to the property of not containing an x_1, x_2, y_1, y_2-linkage.*

Theorem 2.19. *Menger's Theorem: Let* $G = (V, E)$ *be a graph and* $A, B \subseteq V$ *. Then the minimum number of vertices separating* A *from* B *in* G *is equal to the maximum number of disjoint* $A - B$ *paths in* G*.*

CHAPTER 3

Introduction and History

One of the oldest linkage type of problems is the Königsberg bridge problem. In the eighteenth century, in the old Prussian city of Königsberg (now Kaliningrad, Russia), the waters of the Pregel (Pregolya) River surrounded two central landmasses connected by a bridge. Additionally, the first landmass was an island that was connected by two bridges to the lower bank of the Pregel and also by two bridges to the upper bank. The other landmass was connected to the lower bank by one bridge and to the upper bank by one bridge, for a total of seven bridges. This gave rise to the question of whether a person could navigate the city crossing each bridge exactly once.[12]

In 1735 Leonhard Euler concluded that such a walk was impossible. He provided an early example of a proof by contradiction: "To confirm this, suppose that such a walk is possible. In a single encounter with a specific landmass, other than the initial or terminal one, two different bridges must be accounted for: one for entering the landmass and one for leaving it. Thus, each such landmass must serve as an endpoint of a number of bridges equaling twice the number of times it is encountered during the walk. Therefore, each landmass, with the possible exception of the initial and terminal ones if they are not identical, must serve as an endpoint of an even number of bridges." [12]

Moving to the early 20th century, we arrive at Menger's theorem. Karl Menger introduced his theorem in 1927 as part of his work in topology and combinatorial geometry. This was motivated by his evolving interests in the structure of networks. The theorem has been vital in the development of modern graph theory, influencing later work in combinatorial optimization, network flows, and connectivity. Paul Erdős and László Lovász used this theorem in their subsequent work on infinite graphs [13].

More recently, Carsten Thomassen and P.D Seymour proved equivalent vesriosnthe 2-linkage theorem independently in 1980 [3][14]. This is the beginning of our current research. One avenue for further investigation is to generalize Thomassen's theorem to an arbitrary k-linkage. In 2006, K. Kawarabayashi, A. Kostochka and G. Yu [7] found a bound such that high connectivity forces a k-linkage. This was in no way an optimal bound, however. The improved bound was found in 2005 by Thomas and Wollan. They found it to be an edge bound of $5kn$[15] Then, in 2007, R. Thomas and P. Wollan proved that every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked [5], and that this is best possible in terms of number of edges (i.e., there exist 6-connected graphs with $5n - 15$ edges which are not 3-linked).

In this paper we will investigate three-linked graphs. In particular, we characterize the three-linked graphs on the projective plane with connecitivity 5 and face width 5.

3.1

In 2020, Stephens and Ye [personal communication] conjectured that a 5-connected graph with face width 5 embedded on the projective contains a $\{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$ linkage unless there is a contractible cycle with $s_1, s_2, s_3, t_2, t_1, t_3$ in that order, with none of the $s_i \to t_i$ paths present in the interior. See Figure 3. This conjecture remains open. It is hoped that this thesis is a step toward the proof of that conjecture of Stephens and Ye.

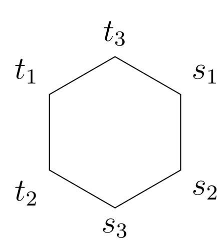


Figure 3: Bad Hexagon

CHAPTER 4

Results

4.1 Supporting Theorems and Lemmas

For reference we state the following condition.

$\boldsymbol{G}$ is a 5-connected graph on the projected plane with face width at least 5. $(*)$

Without loss of generality, we will let $(s_1, t_1), (s_2, t_2), (s_3, t_3)$ be the pairs of our nominated vertices.

Theorem 4.20. *Let G be as in (*). Then $M(G) = \boldsymbol{G} - s_3$ is embedded in a Möbius band.*

Proof. Let $\boldsymbol{G}$ be as in (*). There exists a smallest open region around s_3 , call it R. Then R is homeomorphic to a singular point x on the projective plane by 1.9. Then $\mathbb{RP}^2 - \{\text{x}\}$ is homeomorphic to a Möbius band by 1.10. This implies that $G - s_3$ is embedded on a Möbius band. □

Definition 4.21. Define C to be the cycle obtained by taking the symmetric difference of the faces of the embedding of G incident with s_3 (this is a cycle due to the connectivity and face-width conditions). Note that by Theorem 4.20, C bounds a Möbius band. We will refer to C as the *boundary* of the graph $G - s_3 = M(G)$.

Definition 4.22. A *Möbius system of paths* on $M(G)$ is a set of pairwise disjoint paths $M_1, M_2, \ldots, M_\ell$ in $M(G)$ such that each M_i has endpoints a_i and b_i on C, and such that these endpoints appear on C in the following order:

$$a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n.$$

Lemma 4.23. *With G as in (*), $M(G)$ has a Möbius system containing at least 5 paths.*

4.2 Three Linkage on the Projected Plane

Theorem 4.24. *Suppose G is a 5 connected graph with face-width 5 on the projective plane. Then G is has a 3-linkage connecting s_i to t_i for $1 \leq i \leq 3$ if and only if $G - s_3$ has a Möbius system of two paths M_1 and M_2 such that for some paths $P_1 : s_1 \to t_1$ and $P_2 : s_2 \to t_2$, $M_1 \cap (P_1 - s_1 - t_1)$ and $M_2 \cap (P_2 - s_2 - t_2)$ are nonempty.*

Proof. Define condition (**) to be:

> $G - s_3$ has a Möbius system of two paths M_1 and M_2 such that for some paths $P_1 : s_1 \to t_1$ and $P_2 : s_2 \to t_2$, $M_1 \cap (P_1 - s_1 - t_1)$ and $M_2 \cap (P_2 - s_2 - t_2)$ are nonempty.

Let $\boldsymbol{G}$ meet $*$, with prescribed vertices $s_1, t_1, s_2, t_2, s_3, t_3$.

($\Longleftarrow$) We consider two cases: $G - s_3$ contains a 2-linkage connecting s_1 to t_1 and s_2 to t_2, or not.

Case 1. $G - s_3$ does not contain such 2-linkage.

Assume $G - s_3$ does not contain such a 2-linkage. Then by the 2-linkage theorem, s_1, s_2, t_1, t_2 either appear on C in that order or are in a facial cycle within $M(G)$ in that order [3].

Case 1.1. s_1, s_2, t_1, t_2 appear on C in that order.

First let our nominated vertices be on C. See Figure 4.

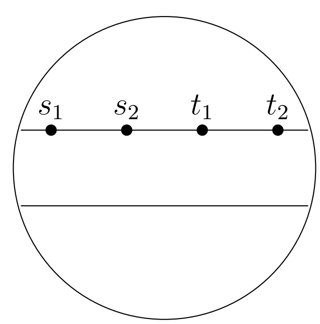


Figure 4: Nominated Vertices Appear on C

Now assume by way of contradiction that G does not have the desired 3-linkage, but condition (**) is met. Then we have $M_1 : a_1 \to b_1, M_2 : a_2 \to b_2$, meeting P_1 and P_2, respectively.

We split into cases by the locations of the endpoints a_1 and b_1, up to symmetry.

Case 1.1.1. a_1 **in** $C(s_1, s_2]$ **and** a_2 **in** $C[t_1, t_2)$**.**

If a_1 is in $C(s_1, s_2]$ and a_2 in $C[t_1, t_2)$, then take $P_1 : C[s_1, b_2] \cup M_2 \cup C[a_2, t_1]$ and $P_2 : C[s_2, a_1] \cup M_1 \cup C[b_1, t_2]$. We have found a two linkage in $G - s_3$, thus contradicting our assumption.

Case 1.1.2. a_1 **in** $C(s_1, s_2]$ **and** a_2 **in** $C(s_2, t_1)$**.**

If a_1 is in $C(s_1, s_2]$ and a_2 in $C(s_2, t_1)$, then take $P_1 : C[s_1, b_2] \cup M_2 \cup C[a_2, t_1]$ and $P_2 : C[s_2, a_1] \cup M_1 \cup C[b_1, t_2]$. We have found a two linkage in $G - s_3$, thus contradicting our assumption.

Case 1.1.3. a_1 **is in** $C[s_2, t_1]$ **and** a_2 **in** $C[s_2, t_1]$**.**

If a_1 is in $C[s_2, t_1]$ and a_2 in $C[s_2, t_1]$, then take $P_1 : C[s_1, b_2] \cup M_2 \cup C[a_2, t_1]$ and $P_2 : C[s_2, a_1] \cup M_1 \cup C[b_1, t_2]$. We have found a two linkage in $G - s_3$, thus contradicting our assumption.

Case 1.1.4. a_1, a_2 **both in** $C(s_1, s_2]$**.** Suppose a_1 is in $C(s_1, s_2)$ and a_2 in $C(s_1, s_2)$. Now t_1 must have only edges that lead to paths with an intersection to $C[s_2, t_2]$, and the vertices of these paths must similarly only have edges that lead to paths an intersection in $C[s_2, t_2]$. Then s_3, s_2, t_2 seperates $C[s_2, t_2]$ from G, contradicting that G is 5 connected.

This concludes case 1.1. Now we consider the case where s_1, s_2, t_1, t_2 are on a facial cycle F^* in the interior of $M(G)$ in that order. See Figure 5.

Case 1.2. s_1, s_2, t_1, t_2 **are on a facial cycle** F^* **in the interior of** $M(G)$**.**

First suppose there exists a Möbius system of paths M_1, M_2, such that M_1 contains s_1 and s_2 and M_2 contains t_1 and t_2. In this case $M_1[a_1, s_2] \cup F^*[s_2, s_1] \cup M_1[s_1, b_1]$ and $M_2[a_2, t_1] \cup F^*[t_1, t_2] \cup M_2[t_2, b_2]$ forms the desired 2-linkage, contrary to our assumption.

Similarly, if M_1 contains t_1 and s_2 and M_2 contains t_2 and s_1, we can find our desired 2-linkage.

Thus we may assume no such Möbius system exists that contains the nominated vertices in either of these ways.

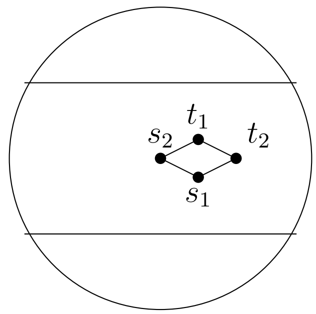


Figure 5: Nominated Vertices Appear on a Facial Cycle

Assume condition (**) holds for some Möbius system M_1, M_2 and paths $P_1 : s_1 \rightarrow t_1$, $P_2 : s_2 \rightarrow t_2$.

Now by four-connectivity, there exist four paths Q_1, Q_2, Q_3, Q_4 from s_1, s_2, t_1, t_2 to some vertices q_1, q_2, q_3, q_4, respectively, in C. Up to reversal, there are only two cyclic orderings the endpoints may have on C. Either the "planar" ordering q_1, q_2, q_3, q_4, or the "Möbius" ordering q_1, q_2, q_3, q_4. If the endpoints have the Möbius ordering, then the desired 2-linkage is present, contradicting our assumption.

So we may assume the q_i's are in the planar ordering on C (see Figure 6). By condition (**), there exists a Möbius system of paths M_1 and M_2 and paths $P_1 : s_1 \rightarrow t_1$, $P_2 : s_2 \rightarrow t_2$ such that M_i intersects P_i. In particular, there must be some path from s_1 (using part of M_1) to the "other half" of C, and likewise a path from t_2 to the other half of C. See Figure 7. But now taking b_1 as the new q_1 and b_2 as the new

q_4, we have the q's in the Möbius order, giving the desired 2-link and contradicting our assumption.

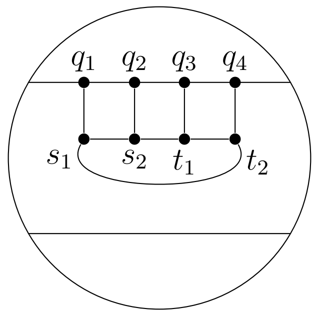


Figure 6: The endpoints q_i occur in the planar ordering

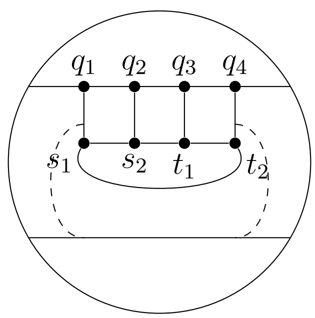
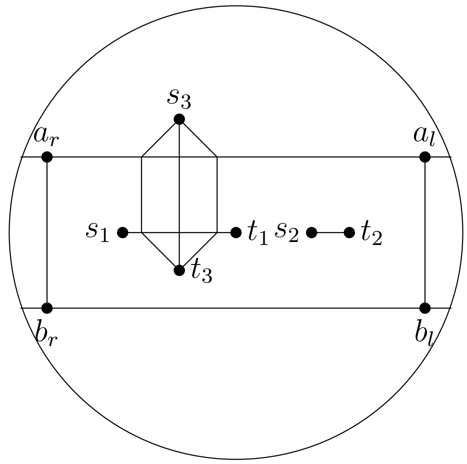


Figure 7: Using the Möbius system to change the ordering

Case 2. G **contains a 2-linkage** $s_1 \to t_1$**,** $s_2 \to t_2$**, but it does not extend to a 3-linkage including** $s_3 \to t_3$**.**

Let $P_1 : s_1 \to t_1$ and $P_2 : s_2 \to t_2$ be a 2-linkage in $G - s_3$. By 5-connectivity of G, there exist at least 5 pairwise internally disjoint paths from s_3 to t_3, and all of them must intersect either P_1 or P_2. Suppose without loss of generality that three $s_3 \to t_3$ paths, say P_3, P_3', P_3'', intersect P_1. Let r be the vertex of $P_1 \cap P_3$ closest to s_1, and

Figure 8: 2-linked, but not 3-linked



r'' be the vertex of $P_1 \cap P_3''$ closest to t_1. Let q, q', q'' be the intersections of C with P_3, P_3', P_3'', respectively, and let Q be the segment of C from q to q'' not containing q'.

Case 2.1. $P_2 \cap C = \emptyset$**.**

In this case $P_1[s_1, r] \cup P_3[r, q] \cup Q[q, q''] \cup P_3''[q'', r''] \cup P_1[r'', t_2]$, P_3', P_2 is the desired 3-linkage.

Case 2.2. $P_2 \cap C \neq \emptyset$.

Let $A =: C[a, a']$ be the smallest subpath of C containing all the vertices of $C \cap P_2$. Let Q' be the subpath of C between q and q'' containing q'.

Suppose there is a path W from t_1 to some w on $C[a', q]$. Then $W[t_1, w] \cup C[w, q] \cup P_3[q, r] \cup P_1[r, s_1]$, P_3', P_2 is the desired 3-linkage. This can be extended to the case that W links any vertex on $C[q'', a) \cup P_3''[q'', r''] \cup P_1[t_1, r''] \cup P_3''[r'', t_3)$ to any vertex on $P_3[q, r] \cup P_1[r, s_1] \cup P_3[r, t_3)$.

So we may assume there is no such path W. Let $B := C[b, b']$ be the set of all endpoints of paths from t_3 to C missing the union $P_3 \cup P_3' \cup P_3''$, where b' is nearer to Q' and b is nearer to A.

Then we may find a new path from s_3 to t_3: $P_3[s_3, q] \cup C[q, b'] \cup Z[b', t_3]$ misses P_1 and P_2, and provides the desired 3-linkage, unless P_1 includes q.

Thus we may assume P_1 is on C.

We may also assume s_3 has no adjacency in $C \setminus (A \cup Q)$, otherwise this would give a new P_3* missing P_1 and P_2.

If there is no path outside the union of the P_3's from t_3 to A, then there is a face containing a and b', a contradiction to face-width 5. Thus we may assume $A \subseteq B$. But in this case t_3 is a "pinch point" such that every C-chord passes through t_3, again a contradiction to face-width.

This concludes the proof that condition (**) implies the desired 3-linkage.

($\Longrightarrow$) Now we will show that if G has the desired 3-linkage, then we have condition (**). From Theorem 4.20, $G - s_3$ is Möbius band. Let the three paths be $P_1 : s_1 \rightarrow$

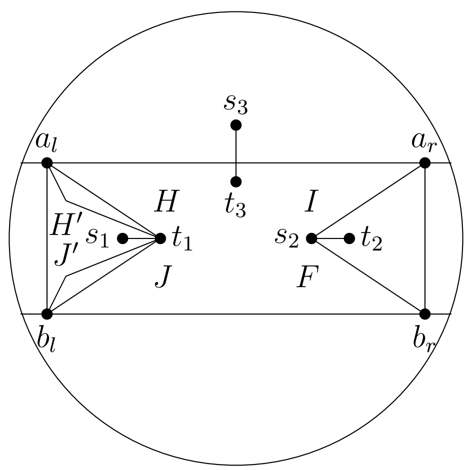


Figure 9: Assuming the 3-linkage exists

$t_1, P_2 : s_2 \rightarrow t_2, P_3 : s_3 \rightarrow t_3$. s_3 is exterior to $M(G)$ and t_3 is interior to $M(G)$, so P_3 intersects C at some point k. We know by Lemma 4.23 that a Möbius system of at least 5 paths is contained in $M(G)$. Now if two of these intersect P_1 and P_2, respectively, we have our result, so there must two of them, M_r and M_l that miss, say, P_2. Now we can find vertex disjoint paths $H : a_l \rightarrow t_1, H' : a_l \rightarrow t_1$ that do not include a_r, s_2, b_r. And paths $J : t_1 \rightarrow b_l, J' : t_1 \rightarrow b_l$ that do not include a_r, s_2, b_r. Now we have $I : a_r \rightarrow s_2, F : s_2 \rightarrow b_r$ that do not include a_l, t_1, b_l. If I, F are disjoint from H, H', J, J' then we have our two Möbius chords by definition.

Case 3.1 I intersects either H' or J' only, and that F intersects either H' or J' only. Then without loss of generality we can find a cycle $M_R \cup H \cup J$ that seperate P_2 from H' and J'. So any paths I, F that connect s_2 to H' or J' on vertices x, y respectively must intersect M_R at some vertices i, f respectively. Then we can use $M_1 : H \cup J$ and $M_2 : M_r[a_r, i] \cup I[i, s_2] \cup F[s_2, f] \cup [f, b_r]$ as our Möbius chords.

Lemma 4.25. *If we dont have I intersect $H, H'; H, J'; H'J; H'J'; H, J$, up to symmetry with F , we have our two Möbius chords.*

Proof. Assume we dont have an intersection between I and any of the followuing intervals. Then without loss of generality, I intersects H or H' and J, J' or niether. Then M_1 can take H' if I intersects H and then take J if I interscts J' and J if I intersects J', then $M_2 : I \cup F$. Similarly we can find such a M_1 if F only intersects H or H' and J, J' or niether.

□

Lemma 4.26. *If I intersects H, J only, up to symmetry with F, then we have our two Möbius chords. Similarly If I intersects H', J' only, up to symmetry with F, then we have our two Möbius chords.*

Proof. Assume that I only intersects H, J, Then M_1 can use H' and J' and M_2 can use I and F. Similarly if I only intersects H', J', Then M_1 can use H and J and M_2 can use I and F □

Case 3.2 Now let I intersect H and H',up to symmetry with F. Then I either has vertices in the order $i_{H1}, i_{H'}, i_{H2}$ or in the order $i_H, i_{H'}$. If $i_{H1}, i_{H'}, i_{H2}$ then We cause use $H[i_{H1}, i_{H2}]$ to form I^* that does not intersect H. Then from 4.25 we have our two Möbius chords. So The vertices must appear in the order $i_H, i_{H'}$. Then $M_2 : [a_l, i_{H'}] \cup [i_{H'}, s_2] \cup F$ and $M_1 : [a_r, i_H] \cup [i_H, t_1] \cup J$ as our disjoint Möbius Chords.
Case 3.3 I intersect H and J' , up to symmetry with F. Then I either has vertices in the order $i_H, i_{H'}, i_{J'}$ or $i_H, i_{M_r}, i_{J'}$. If in the order $i_H, i_{H'}, i_{J'}$, then by 4.25 we have our two Möbius chords. So they appear in the ordrer $i_H, i_{M_r}, i_{J'}$. We can

create $I^* : M_r[a_r, i_{m_r}] \cup I[i_{M_r}, s_2]$ that does not intersect H, so by 4.25 we have our two disjoint Möbius chords. Similarly, if I intersect H' and J , up to symmetry with F, we have our two disjoint Möbius chords.

Therefore if G is three linked we have condition **. Thus we have G is a 5 connected, graph with face width 5 on the projected plane. Then G is three linked if and only if $G - s_3$ has 2 Möbius chords M_1 and M_2 such that for some paths $P_1 : s_1 \to t_1$ and $P_2 : s_2 \to t_2$, $M_1 \cap (P_1 - s_1)$ and $M_2 \cap (P_2 - t_2)$ are nonempty.

□

CHAPTER 5

Conclusion

Origonally, we set out to prove that a Graph G meeting condition $*$ was 3-linked only if G did not contain the closed walk seen in Figure 10 , which we called a Bad Hexagon. One key note of the bad hexagon is it only prohibits a 3-linkage if there is not a path contained within the face created by the hexagon that connected $s_i \to t_i$.

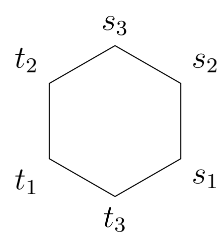


Figure 10: Bad Hexagon

Initally we attempted to remove one path from $s_3 \to t_3$, and then show that either the resulting graph was 2-linked or conatined a Bad Hexagon. Although we have not proved this yet, it is a project we intend to continue in the future.

BIBLIOGRAPHY

[1] Menger, Karl. "Zur allgemeinen Kurventheorie." *Fundamenta Mathematicae* **10** (1927) 96-115.

[2] Munkres, J. *Topology.* 2nd ed. Prentice Hall, Upper Saddle River, NJ, 2000.

[3] C. Thomassen, "2-Linked Graphs," *Europ. J. Combinatorics* **1** (1980), 371–378. Available online: https://core.ac.uk/download/pdf/81926143.pdf.

[4] R. Diestel, *Graph Theory*, 6th ed., Springer-Verlag, Heidelberg, 2024. Available online: https://www.emis.de/monographs/Diestel/en/GraphTheoryII.pdf.

[5] R. Thomas and P. Wollan, "The extremal function for 3-linked graphs," *J. Comb. Theory, Ser. B* **98** (2008), 939–971.

[6] G. Chen, R. Gould, K. Kawarabayashi, F. Pfender, and B. Wei, "Graph minors and linkages," *J. Graph Theory* **49** (2005), 75–91.

[7] K. Kawarabayashi, A. Kostochka, and G. Yu, "On sufficient degree conditions for a graph to be kkk-linked," *Combin. Probab. Comput.* **15** (2006), 685–694.

[8] J. H. C. Whitehead, "Combinatorial homotopy. I," *Bull. Amer. Math. Soc.* **55** (1949), 213–245.

[9] University of Warwick, "Lecture 22," available online: https://homepages.warwick.ac.uk/staff/Martin.Lotz/files/topology/lect22.pdf, accessed October 21, 2024.

[10] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. ISBN 0-521-79540-0.

[11] H. W. Broer et al., "Preliminaries of Dynamical Systems Theory," in *Handbook of Dynamical Systems*, Elsevier Science, September 24, 2010. Available online: https://www.sciencedirect.com/science/article/pii/S1874575X10003097.

[12] "Königsberg Bridge Problem," *Encyclopædia Britannica*, Encyclopædia Britannica, Inc., available online: https://www.britannica.com/science/Konigsberg-bridge-problem, accessed October 21, 2024.

[13] R. Bellman, *Applications of Menger's Graph Theorem*, The Department of Pure Mathematics, The University, Sheffield, England. Available online: https://core.ac.uk/download/pdf/82704343.pdf, accessed October 21, 2024.

[14] P.D. Seymour et al. Graph Minors. VII. Disjoint Paths on a Surface, Department of Mathematics, Ohio State University

[15] R. Thomas and P. Wollan, An Improved Extremal Function for Graph Linkages, European J. Combinatorics 26 (2005), 309–324.