Arithmetic Triangles and Pascal-Type Recurrence Relations

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O, Urania –

Bestow upon me the ability to discern those patterns which are hidden.

Allow me to set adrift the turbulent seas of Mathematics and revel in the unknown.

Permit me to scale the heights of abstraction and find the general in the specific.

Let the work commence.

– O, Urania

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ABSTRACT

The Arithmetic Triangle, commonly known as Pascal's Triangle, has been an object of interest for mathematicians since antiquity. The entries in the Arithmetic Triangle display interesting patterns while also having much combinatorial significance. We recount some of these patterns and exhibit a new construct on the Arithmetic Triangle: alternating products. Then, we generalize the construction of the Arithmetic Triangle by applying the same Pascal-type recurrence relation to different sets of seed values. We show that these Generalized Arithmetic Triangles still display many of the same interesting patterns as The Arithmetic Triangle, but with slight modifications determined by the seed values. By creating an order on the elements of Pascal's Triangle which captures the Pascal-type recurrence relation, we consider Pascal's Triangle through the lens of Order Theory. We conclude by considering the algebraic super-structure of the collection of all Generalized Arithmetic Triangles and see that there is a natural way to form the Arithmetic Triangle Group.

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CHAPTER 0

Introduction

In his essay "The Two Cultures of Mathematics", Dr. Gowers claims that those who practice mathematics fall, roughly and not mutually exclusively, into two camps: the Theory-Builders and the Problem-Solvers. [9] On one hand, the Theory-Builders attempt to create grand unified explanations for seemingly disparate areas of research and try to make sense of what is mathematics. On the other hand, Problem-Solvers focus on specific problems, hoping to explain seemingly curious coincidences. Dr. Gowers stresses that both paradigms are crucial to each other and may bolster one another. Results from Problem-Solvers generate those interesting connections the Theory-Builders attempt to explain, while new theories generate novel problems for the Problem-Solvers to attack.

In the present manuscript we move freely between these two modes of thinking. Motivated by a curious pattern of rational numbers, we delve deep into a mathematical object known since the time of antiquity. From this result we then turn to building a theory to explain why this pattern comes about, and in the process create a whole class of similar objects with recognizable yet novel properties. This leads to a swath of new problems which might shed more light on the underlying structures from which they spring.

Our main object of study will be Pascal's Triangle: a structure familiar to most students of mathematics. Pascal's Triangle is not really Pascal's, in that historical records indicate that what we now know as Pascal's Triangle was in fact known by the mathematical world far earlier than Blaise Pascal's (1623-1662) original publication. The Arithemetic Triangle (what is called now Pascal's Triangle) appears in the work of Chinese mathematician Chu Shih-chieh's work *Precious Mirror* (1303), in the Arabic-speaking world by way of Al-Kashi (1436) through his use of the Binomial Theorem, and earlier in Germany in Michael Stifel's work *Aritmetica integra*. [3]

However, the Arithmetic Triangle has not revealed all of its secrets yet. A puzzle for recreational mathematics [1], a device for asking and answering difficult modern questions [6], and an object of study in its own right [2]; The Arithmetic Triangle is still a well-spring of discoveries. When considering an object so rich in structure, there are a number of ways to tease out patterns.[19] Some works take the Arithmetic Triangle as a collection of binomial coefficients, seeing how it may generalize as an algebraic structure. [15] [16]. Others investigate the combinatorial aspects of the Arithmetic Triangle, giving explanations for why which numbers appear where based on counting problems. [18] [24]

One of the most interesting of the outstanding problems concerning the Arithmetic Triangle comes from the observation that each element in the Arithmetic Triangle is a binomial coefficient. [10] David Singmaster, in his work on binomial coefficients, conjectured that each integer greater than one appears a finite number of times in the Arithmetic Triangle. [20] [21]. In fact, he conjectured that not only is this number finite, but there is an upper bound for all integers. Presently only one integer has been found to appear eight times in the Arithmetic Triangle: 3003. No integer, so far, has been found to appear more than eight times. [17]

We begin our investigation into the Arithmetic Triangle by first recounting a selection of its properties. A unifying theme in these properties is Pascal's Rule, a recursive relationship between adjacent (in a particular way) entries of the triangle. Then, we delve into a new construction on elements of the Arithmetic Triangle: the alternating product.

From this construction, we create a generalization of The Arithmetic Triangle to consider a class of similar structures dubbed Right Arithmetic Triangles. Further, we introduce the Generalized Arithmetic Triangles. These structures turn out to be a fertile playground for many ideas in combinatorics, order theory, and algebra at large. We shall see that at the heart of these observations is the principle of recursion, in the form of a Pascal-type rule.

After exploring the numeric structure of these Generalized Arithmetic Triangles, we step back and consider the algebraic structure through the lens of Order Theory. Pascal's Rule is now what generates the order imposed upon the elements of the Arithmetic Triangle. From this inherited order, we show that the Arithmetic Triangle lattice has a surprising number of purely order-theoretic properties. By looking at our Generalized Arithmetic Triangles this way, we lose many of the combinatorial properties directly, but we also see that many can be made more apparent by considering the imposed order.

Finally, we consider the algebraic super-structure of the collection of all Generalized Arithmetic Triangles. By playing with these generalized triangles we see that there is a natural way to define the 'sum' of two Generalized Arithmetic Triangles, and this operation lets us talk about the group formed by all Generalized Arithmetic Triangles.

CHAPTER 1

Background

1.1 Properties of Pascal's Triangle

Pascal's Triangle is a recursively generated infinite array of integers with many interesting and well-known properties. Pascal's Triangle is named after the French mathematician Blaise Pascal; however, the triangle has been studied by mathematicians since antiquity. [3] In this first chapter we intend to collect together facts and properties of Pascal's Triangle. Many proofs given are standard in their logical argument but non-standard in the notation used.

$n=0$									1								
$n=1$								1		1							
$n=2$							1		2		1						
$n=3$						1		3		3		1					
$n=4$					1		4		6		4		1				
$n=5$				1		5		10		10		5		1			
$n=6$			1		6		15		20		15		6		1		
$n=7$		1		7		21		35		35		21		7		1	
$n=8$	1		8		28		56		70		56		28		8		1

Figure 1: Pascal's Triangle

In what follows, we denote the entirety of Pascal's Triangle as $\mathbf{\Delta}$ and the nth row of $\mathbf{\Delta}$ will be denoted as $\mathbf{\Delta}(n)$. We can think of $\mathbf{\Delta}(n)$ as a collection of, not necessarily distinct, $n+1$ elements or as a multiset where each element appears either exactly once or exactly twice. For instance, the 4th row of Pascal's Triangle will be written as $\mathbf{\Delta}(4) = \{1, 4, 6, 4, 1\}$. Thus, if we wish to speak of a particular entry in a particular row of Pascal's Triangle, we denote the kth element of the nth row as $\mathbf{\Delta}(n; k)$.
Many readers will have seen Pascal's Triangle in the context of Binomial Coefficients.

In this investigation, we develop a new notation for the entries in Pascal's Triangle to capture and extend the recurrence relation.
Now, with some terminology out of the way, we describe the standard recursive procedure for generating Pascal's Triangle.

Definition 1.1.1. ***Pascal's Rule***
Let $\mathbf{\Delta}(0) = \{1\}$ *and* $\mathbf{\Delta}(1) = \{1, 1\}$.
Then, for all $n \geq 2$ *define:*

$$\mathbf{\Delta}(n; k) = \begin{cases} \mathbf{\Delta}(n-1; k) + \mathbf{\Delta}(n-1; k-1) & 0 \leq k \leq n \\ \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

Implicit in this definition [2] of Pascal's triangle are three things, the nth row of Pascal's triangle is composed of $n+1$ elements, each row has infinitely many zero elements trailing the $n+1$st element and infinitely many zero elements preceding the 1st element, and all rows after the first are determined by the values of $\mathbf{\Delta}(1)$. We will exploit this fact later to consider a larger class of Pascal-Type Triangles.

Looking at the subset of $\mathbf{\Delta}$ given in Figure 1, we see that it is symmetric with respect to the middle column. This is our first interesting property of Pascal's Triangle.

Observation 1.1.2. *For each* $n \geq 1$ *and* $k \leq n$, $\mathbf{\Delta}(n; k) = \mathbf{\Delta}(n; n-k)$.

Proof. We show by induction on n using Pascal's Rule.
For the case of $n = 1$, we see the property holds due to our definition of $\mathbf{\Delta}$.
Suppose the result holds some $n \in \mathbb{Z}^+$, consider the element $\mathbf{\Delta}(n+1; k)$. Using Pascal's rule we have:

$$\mathbf{\Delta}(n+1; k) = \mathbf{\Delta}(n; k) + \mathbf{\Delta}(n; k-1)$$

Applying the induction hypothesis we can now write:

$$\mathbf{\Delta}(n+1;k) = \mathbf{\Delta}(n;n-k) + \mathbf{\Delta}(n;n-(k-1)) = \mathbf{\Delta}(n;n-k) + \mathbf{\Delta}(n;n+1-k)$$

Applying Pascal's Rule again, we achieve the desired result.

$$\mathbf{\Delta}(n+1;k) = \mathbf{\Delta}(n+1;n+1-k)$$

Thus, the claim holds for all $n \in \mathbb{Z}^+$ $\square$

Considering a single row of $\mathbf{\Delta}$, we can ask if there is some pattern lurking in the sum of the elements of a given row. Indeed, there is.

Observation 1.1.3. *For each* $n \in \mathbb{Z}^+$, $\displaystyle\sum_{k=0}^{n} \mathbf{\Delta}(n;k) = 2^n$.

Proof. We prove by induction. If $n = 0$, we have $\displaystyle\sum_{k=0}^{0} \mathbf{\Delta}(n;k) = 1 = 2^0$.

Suppose the result holds for n and consider the sum $\displaystyle\sum_{k=0}^{n+1} \mathbf{\Delta}(n+1;k)$. Using the recursive definition for $\mathbf{\Delta}$ we can write

$$\sum_{k=0}^{n+1} \mathbf{\Delta}(n+1;k) = \sum_{k=0}^{n+1} (\mathbf{\Delta}(n;k-1) + \mathbf{\Delta}(n;k)) = \sum_{k=0}^{n+1} \mathbf{\Delta}(n;k-1) + \sum_{k=0}^{n+1} \mathbf{\Delta}(n;k).$$

Since we know that $\mathbf{\Delta}(n;k) = 0$ for all $k \notin \{0, 1, .., n\}$ we can simplify and re-index the first sum to see that

$$\sum_{k=0}^{n+1} \mathbf{\Delta}(n+1;k) = \sum_{k=0}^{n} \mathbf{\Delta}(n;k) + \sum_{k=0}^{n} \mathbf{\Delta}(n;k) = 2\left(\sum_{k=0}^{n} \mathbf{\Delta}(n;k)\right)$$

Which, by our induction hypothesis lets us conclude that $\displaystyle\sum_{k=0}^{n+1} \mathbf{\Delta}(n+1;k) = 2(2^n) = 2^{n+1}$. Hence, by the Principle of Mathematical Induction we have that the result holds for all $n \in \mathbb{Z}^+$. $\square$

So, we see that even though the powers of two were never explicit in our construction of $\mathbf{\Delta}$, they appear when we impose a secondary construction on $\mathbf{\Delta}$. This leads us to ask what other 'nice' number patterns can be extracted from $\mathbf{\Delta}$ with elementary arithmetic operations?

One of the lesser-known but more curious observations about $\mathbf{\Delta}$ is the so-called 'Hockey Stick' property. [14] This says that if we start at any element of $\mathbf{\Delta}$ of the form $\mathbf{\Delta}(n;0)$ or $\mathbf{\Delta}(n;n)$ and traverse down the corresponding diagonal, the sum of these elements is given by the element adjacent to the final element of this diagonal. Figure 2 illustrates this observation. In Figure 2, the sum of the blue entries equals the red entry.

$n=0$									1								
$n=1$								1		1							
$n=2$							**1**		2		1						
$n=3$						1		**3**		3		1					
$n=4$					1		4		**6**		4		1				
$n=5$				1		5		10		**10**		5		1			
$n=6$			1		6		15		**20**		15		6		1		
$n=7$		1		7		21		35		35		21		7		1	
$n=8$	1		8		28		56		70		56		28		8		1

Figure 2: The Hockey Stick Identity

Observation 1.1.4. *(Hockey Stick) For* $n, j \in \mathbb{Z}^+$, $\sum_{k=0}^{j} \mathbf{\Delta}(n+k;k) = \mathbf{\Delta}(n+j+1;j)$.

Proof. Fix $n \in \mathbb{Z}^+$. We prove by induction on j. For the case of $j = 0$, the result is obvious since $\mathbf{\Delta}(n,0) = \mathbf{\Delta}(n+1,0)$.

Suppose the result holds up until j. We consider the sum $\sum_{k=0}^{j+1} \mathbf{\Delta}(n+k;k)$. From this we can pull out the final term of the sum and then apply our induction hypothesis to see

$$\sum_{k=0}^{j+1} \mathbf{\Delta}(n+k;k) = \mathbf{\Delta}(n+j+1;j+1) + \sum_{k=0}^{j} \mathbf{\Delta}(n+k;k) =$$
$$\mathbf{\Delta}(n+j+1;j+1) + \mathbf{\Delta}(n+j+1;j).$$

However, using Pascal's Rule, we have:

$$\mathbf{\Delta}(n+j+1;j+1)+\mathbf{\Delta}(n+j+1;j)=\mathbf{\Delta}(n+j+2;j+1).$$

Hence, the result holds for all $j \in \mathbb{Z}^+$. $\square$

Remark. It is important to observe that there are actually two Hockey Stick Identities, one beginning on the left-most diagonal and one beginning on the right-most diagonal. Due to the symmetry of $\mathbf{\Delta}$, as shown in Observation 1.1, we need only prove one of these identities to see both hold.

The Hockey Stick property suggests to us yet another avenue of inquiry regarding $\mathbf{\Delta}$. Instead of simply looking at rows, we might find interesting number sequences in the diagonals of $\mathbf{\Delta}$. We notice that the first diagonal $\{\mathbf{\Delta}(n;k)|n-k=1\}$ is simply a copy of the positive integers in their natural order. This is no coincidence; by examining our construction of $\mathbf{\Delta}$ we notice that the first diagonal off the edge (on either side) will contain the positive integers because we always add one to the previous integer.

If we now look at the second diagonal off the edge, we see another well-known sequence. That is, we consider the sequence $\{1,3,6,10,15,21,28,36,45,55,...\}$. Any student of combinatorics will recognize these numbers as the Triangular numbers. [22] The $n-$th Triangular number is given by the sum of the first n positive integers. Equivalently, we can say that the Triangular numbers are given by the reccurence

$$T_n = n + T_{n-1} \tag{2}$$

for $n>0$ with $T_0=0$.

We claim that the second diagonal of $\mathbf{\Delta}$ is precisely those Triangular numbers, in their natural order. To show the veracity of this claim we show that the elements in this diagonal follow the same reccurence relation as that given by Equation (2).

Observation 1.1.5. *For $n \geq 2$, the element $\mathbf{\Delta}(n, n-2)$ is the $n-1st$ Triangular number T_n. That is, $\mathbf{\Delta}(n; n-2) = T_{n-1}$.*

Proof. We prove by induction on n.
For the case of $n = 2$ we have $\mathbf{\Delta}(2; 0) = 1 = T_1$.
Next, suppose the result is true for $n = k$. So we have that $\mathbf{\Delta}(k; k-2) = T_{k-1}$. Now, by considering $\mathbf{\Delta}(k+1; k-1)$ we have by our recurrence relation for $\mathbf{\Delta}$ that

$$\mathbf{\Delta}(k+1; k-1) = \mathbf{\Delta}(k; k-2) + \mathbf{\Delta}(k; k-1)$$

But, from our earlier observation we have that $\mathbf{\Delta}(k; k-1) = k$. So, using our induction hypothesis we have then that

$$\mathbf{\Delta}(k+1; k-1) = T_{k-1} + k = T_k.$$

Hence, by the Principle of Mathematical Induction we have the desired result. $\square$

From the two preceding observations, we could make a conjecture that each diagonal of $\mathbf{\Delta}$ is some sort of interesting sequence. While 'interesting' is a purely subjective term, most in the math community would agree that the appearance of these seemingly unrelated sequences in $\mathbf{\Delta}$ must be more than a coincidence.

Stretching our deductive skills a little further, we consider the next set of diagonal elements $\{1, 4, 10, 20, 35, 56, ...\}$. We might notice that this sequence is intimately tied up with the Triangular numbers above. A search on the OEIS (Sequence A000292) reveals that these are precisely the Tetrahedral (Triangular Pyramidal) numbers [22].

The Tetrahedral numbers, Te_n, are given by the sum of the first $n-1$ Triangular numbers. We can think of the nth Tetrahedral number in terms of packing spheres in $\mathbb{R}^3$. The nth Tetrahedral number is given by the recurrence relation in 3.

$$Te_n = T_n + Te_{n-1} \tag{3}$$

for all $n > 1$ with $Te_1 = 1$.

With a formula in hand that describes the nth Tetrahedral number, we can now show that this diagonal of $\mathbf{\Delta}$ is precisely the Tetrahedral numbers.

Observation 1.1.6. *For $n \geq 1$, the element $\mathbf{\Delta}(n+2; n-1)$ is the nth Tetrahedral number Te_n. That is, $\mathbf{\Delta}(n+2; n-1) = Te_n$.*

Proof. We prove by induction on n.
For the case $n = 1$ we have $\mathbf{\Delta}(3; 0) = 1 = Te_1$.
Next, suppose the result is true for $n = k$ and consider $\mathbf{\Delta}((k+1)+2; (k+1)-1) = \mathbf{\Delta}(k+3; k)$. Using the recursive definition of $\mathbf{\Delta}$ we have

$$\mathbf{\Delta}(k+3; k) = \mathbf{\Delta}(k+2; k-1) + \mathbf{\Delta}(k+2; k).$$

By noticing that $\mathbf{\Delta}(k+2; k)$ is precisely the kth Triangular number and applying our induction hypothesis we have then that

$$\mathbf{\Delta}(k+3; k) = Te_k + T_k = Te_{k+1}.$$

Ergo, by the principle of Mathematical Induction we have that the nth Tetrahedral number is $\mathbf{\Delta}(n+2; n-1)$. □

Before moving onto the main properties of $\mathbf{\Delta}$ which we wish to investigate, we first show how $\mathbf{\Delta}$ contains one of the most well-known recurrence relations in mathematics: The Fibonacci numbers.

The history of the Fibonacci sequence is intertwined with that of the Arithmeitic Triangle [3], showing up in mathematics in a wide variety of locales while also being a pattern observed in nature. Typically, the Fibonacci sequence is given by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \tag{4}$$

for all $n > 2$ and $F_1 = F_2 = 1$.

Theorem 1.1.7. *The nth Fibonacci number F_n is the sum of the shallow diagonals of $\mathbf{\Delta}$. That is for $n \geq 1$, $F_n = \displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathbf{\Delta}(n-k-1;k)$*

Proof. This proof is adapted from a similar one given in [4].

First, we recall that the Fibonacci numbers are given by the recurrence relation $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. We prove by induction on the value of n. For the cases of $n = 1, 2$ we have the result easily. More specifically :

$$\sum_{k=0}^{\lfloor \frac{1-1}{2} \rfloor} \mathbf{\Delta}(1-k-1;k) = \mathbf{\Delta}(0;0) = 1 = F_1$$

and

$$\sum_{k=0}^{\lfloor \frac{2-1}{2} \rfloor} \mathbf{\Delta}(2-k-1;k) = \mathbf{\Delta}(1;0) = 1 = F_2.$$

Next, suppose the result holds for $n-1$ and n where n is an even integer greater than 2. From this, we prove that the result holds for $n+1$ and $n+2$ and then based on the Principle of Mathematical Induction we have the result for all $n \in \mathbb{Z}^+$.

In our induction hypothesis we will assume that

$$F_{n-1} = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \mathbf{\Delta}(n-k-2;k)$$

and that

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathbf{\Delta}(n-k-1;k)$$

Now, we endeavor to show the result holds for the $n+1$ case. Observe that

$$\sum_{k=0}^{\lfloor \frac{n+1-1}{2} \rfloor} \mathbf{\Delta}(n+1-k-1;k) = \sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k)$$

Assuming that n is even, we can split up this sum thusly.

$$\sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k) = \mathbf{\Delta}(n;0) + \mathbf{\Delta}\left(\frac{n}{2};\frac{n}{2}\right) + \sum_{k=1}^{\frac{n}{2}-1} \mathbf{\Delta}(n-k;k).$$

Applying Pascal's Rule to the inner sum we have

$$\sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k) = \mathbf{\Delta}(n;0)+\mathbf{\Delta}\left(\frac{n}{2};\frac{n}{2}\right)+\sum_{k=1}^{\frac{n}{2}-1}(\mathbf{\Delta}(n-k-1;k)+\mathbf{\Delta}(n-k-1;k-1)).$$

Next, we decompose this sum into two smaller sums in order to use our induction hypotheses. After reindexing the second sum we have that,

$$\sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k) = \mathbf{\Delta}(n;0)+\mathbf{\Delta}(\frac{n}{2};\frac{n}{2})+\sum_{k=1}^{\frac{n}{2}-1}\mathbf{\Delta}(n-k-1;k)+\sum_{k=0}^{\frac{n}{2}-2}\mathbf{\Delta}(n-k-2;k).$$

Last, we can bring the two front terms into each sum to achieve

$$\sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k) = \sum_{k=0}^{\frac{n}{2}-1}\mathbf{\Delta}(n-k-1;k)+\sum_{k=0}^{\frac{n}{2}-1}\mathbf{\Delta}(n-k-2;k) = F_n + F_{n-1}.$$

Thus, by the recurrence relation for the Fibonacci numbers we have that

$$\sum_{k=0}^{\frac{n}{2}} \mathbf{\Delta}(n-k;k) = F_{n+1}$$

The case for $n+2$ follows similarly, so that we can say

$$\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \mathbf{\Delta}(n-k+1;k) = F_{n+2}$$

Ergo, we have that the sum of the shallow diagonals of $\mathbf{\Delta}$ are the Fibonacci numbers.

□

Summing up the previous discussion, we have shown that $\mathbf{\Delta}$ contains many well-known and interesting integer sequences. Finding these sequences and rephrasing them in the notation developed shows us that in some way $\mathbf{\Delta}$ acts as an generator for these combinatorial constructions.

Next we look at a few of the properties of $\mathbf{\Delta}$ which are most important to us. Recall that we have already seen that the sum of any row of $\mathbf{\Delta}$ is always a power of two, that is for every non-negative integer n, $\sum \mathbf{\Delta}(n) = 2^n$. One way to think of this observation is that we infixed a $(+)$ between each two adjacent elements of $\mathbf{\Delta}$ and then evaluated the result on a row-by-row basis. Clearly, if we were to do the same type of construction but with $(-)$ we would arrive at the negative of a power of two.

The next property of $\mathbf{\Delta}$ is a startling result. Instead of summing the values in a row, we take the alternating sum of the elements in a given row. At first glance this might not look to produce anything of interest, but if we consider a few small cases we notice something.

Obviously, for the 0th row, the alternating sum is simply 1, since there is only one element in this row. In the 1st row the alternating sum is 0, as to be expected since the two elements in this row are identical. For the 2nd row we also see that the alternating sum is 0. Does this pattern continue? The answer is in fact, yes.

The following proof is typical coursework for any number theory or combinatorics course. For example, see [4].

Observation 1.1.8. $\sum_{k=0}^{n}(-1)^k\mathbf{\Delta}(n;k) = 0$ *for all* $n \in \mathbb{Z}^+$.

Proof. We prove by induction on n. For the case of $n = 1$, we have already seen that the result is true. Now, suppose the result holds for some $n \in \mathbb{Z}^+$ such that $n > 1$ and consider the sum

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k)$$

First, we pull out the initial and final terms so that we have

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) = \mathbf{\Delta}(n+1;0) + \sum_{k=1}^{n}(-1)^k\mathbf{\Delta}(n+1;k) + (-1)^{n+1}\mathbf{\Delta}(n+1;n+1)$$

Now, we apply Pascal's rule which gives us

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) =$$

$$\mathbf{\Delta}(n+1;0) + \sum_{k=1}^{n}(-1)^k\left[\mathbf{\Delta}(n;k-1) + \mathbf{\Delta}(n;k)\right] + (-1)^{n+1}\mathbf{\Delta}(n+1;n+1).$$

The next step is the non-obvious one. We wish to somehow simplify the interior sum. If one were to play with examples or write out the terms of the sum, it would become apparent that we actually have a telescoping series. So, if we factor a (-1) from the whole sum we can write

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) =$$
$$\mathbf{\Delta}(n+1;0) - \sum_{k=1}^{n}\left[(-1)^{k-1}\mathbf{\Delta}(n;k-1) - (-1)^k\mathbf{\Delta}(n;k)\right] + (-1)^{n+1}\mathbf{\Delta}(n+1;n+1)$$

Now we take advantage of the telescoping sum, keeping only the first and last terms of the sum so that now we have

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) =$$
$$\mathbf{\Delta}(n+1;0) - (-1)^0\mathbf{\Delta}(n;0) + (-1)^n\mathbf{\Delta}(n;n) + (-1)^{n+1}\mathbf{\Delta}(n+1;n+1).$$

However, we know that any entry in $\mathbf{\Delta}$ on either the left most diagonal or the right-most diagonal is identically 1, so we can simplify to say

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) = 1 - 1 + (-1)^n + (-1)^{n+1}.$$

Last, since precisely one of n and $n+1$ is even, and the other is odd we have that $(-1)^n + (-1)^{n+1} = 0$. Thus, we have the result that

$$\sum_{k=0}^{n+1}(-1)^k\mathbf{\Delta}(n+1;k) = 0$$

□

Remark. Observation 1.1.8 is a standard property in combinatorics. However, our proof is non-standard in that it uses a telescoping series argument instead of the traditional Binomial Theorem argument. A proof using the Binomial Theorem may be found in [4].

Now, instead of simply adding a single operator between adjacent elements of $\mathbf{\Delta}$ we imagine what happens if we use the elements of each row as a very primitive sort of generating function. In a particular row, $\mathbf{\Delta}(n)$, infix before each element both an addition sign as well as a power of 10. That is, use the numbers in $\mathbf{\Delta}(n)$ as the coefficients in the base-10 expansion of some number. We ask then: what numbers result?

Observation 1.1.9. *When read as a number in base 10, $\mathbf{\Delta}(n)$ is equal to (with some carrying) 11^n.*

Proof. This is yet another standard properties of the entries of Pascal's Triangle. We give a novel proof using the notation developed henceforth, but a more standard proof will be given after formally introducing binomial coefficients.

Stated more formally, we claim that

$$\sum_{j=0}^{n} \mathbf{\Delta}(n; n-j)10^j = 11^n.$$

We show this by induction on n.

For $n = 0$, we have the result by inspection. So, suppose the result holds for $n = k$ and consider the sum

$$\sum_{j=0}^{k+1} \mathbf{\Delta}(k+1; k+1-j)10^j$$

Using Pascal's Rule, we can break up this interior sum to read

$$\sum_{j=0}^{k+1} \left(\mathbf{\Delta}(k; k+1-j) + \mathbf{\Delta}(k; k-j)\right) 10^j$$

Which gives us two separate sums we can deal with

$$\sum_{j=0}^{k+1} \mathbf{\Delta}(k; k+1-j)10^j + \sum_{j=0}^{k+1} \mathbf{\Delta}(k; k-j)10^j$$

Since our Arithmetic Triangle coefficients are 0 when the second coordinate is less than zero, we can throw away the last term in our second sum. Likewise, the Arithmetic Triangle coefficients are 0s when the second coordinate is strictly greater than the first, thus, we can throw out the first summand in the first term. This leaves us with

$$\sum_{j=1}^{k+1} \mathbf{\Delta}(k; k+1-j)10^j + \sum_{j=0}^{k} \mathbf{\Delta}(k; k-j)10^j$$

Next, we can re-index the sum on the left. Letting $u = j - 1$, we now have

$$\sum_{u=0}^{k} \mathbf{\Delta}(k; k-u)10^{u+1} + \sum_{j=0}^{k} \mathbf{\Delta}(k; k-j)10^{j}$$

Now, pull out a factor of 10 from the first sum and apply our induction hypothesis twice to obtain

$$10\sum_{u=0}^{k} \mathbf{\Delta}(k; k-u)10^{u} + \sum_{j=0}^{k} \mathbf{\Delta}(k; k-j)10^{j} = 10 * 11^k + 11^k = 11^{k+1}$$

Hence, we have the result for all $n \in \mathbb{N}$ $\square$

There is another, simpler way of showing the above result. However, to do so we first introduce the connection between the elements of $\mathbf{\Delta}$ and the binomial coefficients. A larger discussion of binomial coefficients may be found in Section 1.2 and [23].

Observation 1.1.10. *For each $n, k \in \mathbb{W}$, we have that $\binom{n}{k} = \mathbf{\Delta}(n; k)$.*

Proof. Pascal's rule for Binomial coefficients says that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. We notice that this is exactly the recurrence relation that generates the elements of $\mathbf{\Delta}$. Since the elements $\mathbf{\Delta}(n; k)$ and $\binom{n}{k}$ are generated from the same recurrence relation, we have that $\mathbf{\Delta}(n; k) = \binom{n}{k}$.

$\square$

Thus, we can see that those elements in $\mathbf{\Delta}$ are precisely the Binomial Coefficients. Theorem 1.1.11 is the Binomial Theorem and provides a concrete way of expanding powers of sums in any commutative ring.

Theorem 1.1.11. *For any $a, b \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$, we have that*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{5}$$

Proof. We prove by induction on the exponent n.

Base Case: $n = 1$

We have $\sum_{k=0}^{1}\binom{1}{k}a^{1-k}b^k = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1$. By the definition of binomial coefficients, we have then $\sum_{k=0}^{1}\binom{1}{k}a^{1-k}b^k = a+b$. Thus, the result holds for $n=1$.

Induction Step: We assume the result holds for $n=m$ for some $m \in \mathbb{Z}^+$ and consider $(a+b)^{m+1}$. Expanding this product and then employing our induction hypothesis we have:

$$(a+b)^{m+1} = (a+b)(a+b)^m = (a+b)\left(\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k\right).$$

By employing the distributive property we have:

$$(a+b)\left(\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k\right) = a\left(\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k\right) + b\left(\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k\right).$$

Pulling each a and b into the sum and simplifying exponents we see:

$$(a+b)^{m+1} = \sum_{k=0}^{m}\binom{m}{k}a^{m-k+1}b^k + \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^{k+1}.$$

Expanding these sums out, we can combine like terms using the associative property of addition and commutative property of multiplication. First, expanding we have:

$$(a+b)^{m+1} = \left[\binom{m}{0}a^{m+1} + \binom{m}{1}a^mb + \binom{m}{2}a^{m-1}b^2 ... + \binom{m}{m}ab^m\right] + \\ \left[\binom{m}{0}a^mb + \binom{m}{1}a^{m-1}b^2 + \binom{m}{2}a^{m-2}b^3 ... + \binom{m}{m}b^{m+1}\right].$$

Now, collecting like terms, we see that the coefficient of $a^{m+1-k}b^k$ is $\left[\binom{m}{k} + \binom{m}{k-1}\right]$. If we write these binomial coefficients in factorial form we have:

$$\binom{m}{k} + \binom{m}{k-1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!}$$

Now, we multiply each term by a well-chosen 1 so they have a common denominator, allowing us to simplify the fractions.

$$\binom{m}{k} + \binom{m}{k-1} = \frac{m!(m-k+1)}{k!(m-k+1)!} + \frac{k(m!)}{k!(m-k+1)!}$$

Employing the distributive property, we can factor out to achieve:

$$\binom{m}{k} + \binom{m}{k-1} = \left[\frac{m!}{k!(m-k+1)!}\right](m-k+1+k) = \left[\frac{m!}{k!(m-k+1)!}\right](m+1) = \frac{(m+1)m!}{k!(m-k+1)!} = \frac{(m+1)!}{k!(m-k+1)!}$$

Last, we notice this is exactly the binomial coefficient $\binom{m+1}{k}$. Hence, we have that the coefficient of $a^{m-k+1}b^k$ is exactly $\binom{m+1}{k}$. Therefore, the result holds for $n = m+1$. So, by the Principle of Mathematical Induction we can say for all positive integers n we have:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

□

The Binomial Theorem gives us a new set of tools to work with elements of $\mathbf{\Delta}$. We can see this by reconsidering Observation 1.1.9.

Corollary 1.1.12. *When read as a number in base 10, $\mathbf{\Delta}(n)$ is equal to (with some carrying) 11^n [4]*

Proof. We begin by noting that $11^n = (10+1)^n$. Applying the binomial theorem we have:

$$11^n = (1+10)^n = \sum_{k=0}^{n} \mathbf{\Delta}(n;k)10^k = \sum_{k=0}^{n} \binom{n}{k} 10^k$$

This is exactly a base-10 representation, since each coefficient in the expansion is multiplied by a unique power of 10. □

This concludes our introductory survey of properties of the Arithmetic Triangle $\mathbf{\Delta}$. We shall see direct analogues of many of the observations above in our future study of Generalized Arithmetic Triangles. For further reading on elementary properties of the Arithmetic Triangle we direct the interested reader to [2].

1.2 Combinatorial Preliminaries

We'd like to develop a few tools from the subfield of combinatorics. These are definitions and identities that will be useful in the future. Many are concerned with reducing formulas for binomial coefficients, but we also introduce the notion of a double factorial. Then, we extend this notion of double factorial to the little studied notion of the modular factorial. Last, we introduce product notation and use it to prove a surprisingly simple and useful lemma.

First, we introduce an alternative (yet more standard) way of working with binomial coefficients; then we use this to give a useful reducing formula.

Definition 1.2.1. *For any non-negative integer n, we define the factorial of n, denoted $n!$ as*

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n(n-1)!, & \text{if } n > 0 \end{cases} \tag{6}$$

Remark. The factorial function is well known and widely used [10].
The notation for the factorial was introduced by French mathematician Christian Kramp [13]. Combinatorially, we can think of $n!$ as the number of distinct ways of arranging n objects. We make the convention that $0! = 1$ since there is exactly 1 way to arrange 0 objects - the empty arrangement!

One might be interested in arranging n objects, but k of them must be in certain positions. This restriction of the factorial function is captured by the dual notions of rising and falling factorials. Both the notation and widespread usage of these variants of the factorial are due to Knuth [10].

Definition 1.2.2. *For any non-negative integers n, k we denote the rising and falling factorials, denoted respectively as $n^{\overline{k}}$ and $n^{\underline{k}}$. The rising factorial is given by*

$$n^{\overline{k}} = \prod_{j=0}^{k-1}(n+j)$$

While the falling factorial is given by

$$n^{\underline{k}} = \prod_{k=0}^{k-1}(n-j)$$

Remark. We can give a combinatorial interpretation of the falling factorial. If we have a set consisting of n objects, and we wish to make all the permutations of length k, the number of distinct permutations is given by $n^{\underline{k}}$.
It should also be noted, that the constructions of the factorial, and rising/falling factorial carries over to any commutative ring A. Thus, we could extend the discussion from strictly integers to elements coming from A.

Now we give a more thorough explanation of the binomial coefficients introduced in Theorem 1.1.11. The binomial coefficient is typically denoted $\binom{n}{k}$ and is read as "n choose k".

Definition 1.2.3. *For non-negative integers n, k we define $\binom{n}{k}$ as*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \tag{7}$$

The standard combinatorial interpretation of the binomial coefficient $\binom{n}{k}$ is the number of ways of choosing k objects from a group of n objects if order is ignored. By using rising and falling factorial notation, we can write the binomial coefficient in two more ways. Namely

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{(n-k+1)^{\overline{k}}}{k} \tag{8}$$

When using these two alternative variations on the formula for binomial coefficients, the first formulation will be much more useful. Also, it is easier to see here how $\binom{n}{k}$ represents the ways of picking k things from n objects since the falling factorial in the numerator gives us the number of permutations of length k of the n objects and the denominator counts the number of permutations on k objects.

We now give a small techninal lemma that will be useful in Chapter 2. We also give this lemma here to show the flavor of algebraic proofs using binomial coefficients.

Lemma 1.2.4. *For any $n, k \in \mathbb{Z}$ such that $n \geq k$ we have:*

$$\binom{n+2}{k} = \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}\binom{n}{k} \tag{9}$$

Proof. We show by direct computation, taking advantage of the property of factorials given in Equation 6.

We have

$$\binom{n+2}{k} = \frac{(n+2)!}{k!(n+2-k)!} = \frac{(n+2)(n+1)n!}{k!(n+2-k)(n+1-k)(n-k)!}$$

$$\binom{n+2}{k} = \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}\binom{n}{k}$$

□

Thinking about the factorial function from an algebraic point of view, we can represent the factorial function as

$$n! = \prod_{j=0}^{n-1}(n-j). \tag{10}$$

This gives us a new way to work with the factorial. Namely, we can say that the factorial is the product over all non-negative integers less than or equal to n. A rather natural, yet perhaps unexpected, generalization of this is to look at the product over all the integers less than n with the same parity. This is known as the double factorial, and represented by $n!!$.

Definition 1.2.5. *For a non-negative integer n define the double factorial*

$$n!! = \begin{cases} 1, & \text{if } n = 0, 1 \\ n(n-2)!!, & \text{otherwise} \end{cases} \tag{11}$$

As before, we can give an equivalent characterization in terms of a product. For a non-negative integer n we can write

$$n!! = \prod_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2j) \tag{12}$$

Typically we will work with Definition 1.2.5, as then it will not be necessary to work with the floor function. Notice here that the double factorial function has slightly different behavior depending on the parity of n. This disparity will be crucial in our future work and will lead to some interesting generative effects.

We can observe that the product of the double factorial of two consecutive integers is simply the factorial of the larger of those two integers.

Lemma 1.2.6. *For every positive integer n, we have that $n! = (n)!!(n-1)!!$*

Proof. The quantity on the left, $n!$, is the product of all integers less than or equal to n. On the right, we have that $n!!$ is the product of all the integers less than or equal to n with the same parity. Likewise, $(n-1)!!$ is the product of all integers less than n with the opposite parity. Since each integer either has the same or opposite parity as n, we have that the two sides are equal. □

The double factorial is only one variant on the factorial function. We can think about how the two functions relate to one another. For instance, if we are given an even number, the double factorial of that number can be written as the factorial of a single number along with an appropriate power of two. We make this more precise in lemma 1.2.7.

Lemma 1.2.7. *For every integer of the form $2k$ where $k \in \mathbb{Z}$, we have $(2k)!! = 2^k k!$*

Proof. We prove by induction.

Base Case: $k = 1$.

Clearly $2!! = 2 = 2^1 \cdot 1!$

Induction Step: Assume true for $k = j$.

Consider $(2(j+1))!!$. We have:

$$(2(j+1))!! = (2j+2)!! = (2j+2)(2j)!! = 2(j+1)2^j j! = 2^{j+1}(j+1)!$$

Hence, the result. $\square$

Unfortunately, there is not a statement as simplistic for the odd integers. However, we can give a characterization of the double factorial of an odd integer in terms of the typical factorial function.

Corollary 1.2.8. *For every integer of the form $2k+1$ where $k \in \mathbb{Z}$, we have $(2k+1)!! = \dfrac{(2k+1)!}{(2k)!!}$*

Proof. This follows quickly from the lemma 1.2.6. We have that $(2k+1)! = (2k+1)!!(2k+1-1)!! = (2k+1)!!(2k)!!$. Dividing both sides by $(2k)!!$ we achieve the result. $\square$

Thus, by combining these two lemmas we have an immediate corollary.

Corollary 1.2.9. *For every integer of the form $2k+1$ where $k \in \mathbb{Z}$, we have $(2k+1)!! = \dfrac{(2k+1)!}{2^k k!}$*

Remark. Note that the lemmas above tell us that given a double factorial, we can write it in terms of the factorial function and vice-versa. Essentially we are creating a toolbox for working with factorial functions.

When constructing the double factorial function we were taking the product over all integers less than or equal to n in the same residue class mod 2. Extending this notion to the residue classes mod p, for some integer p, gives the notion of a modular factorial. There is little use of the modular factorial in present literature, and as of yet no standardized notation. We propose to represent n factorial modular p as $n!_p$ and give a formal definition.

Definition 1.2.10. *Given non-negative integers n, p we define the modular factorial of n with modulus p, denoted $n!_p$ as*

$$n!_p = \begin{cases} 1, & \text{if } 0 \leq n < p \text{ or } p = 0 \\ \\ n(n-p)!_p, & \text{otherwise} \end{cases} \tag{13}$$

Similar to the double and classic factorial functions, we can write the modular factorial of n in product notation.

$$n!_p = \prod_{j=0}^{\lfloor \frac{n-1}{p} \rfloor} (n - pj) \tag{14}$$

This is a bit messier than before, but it captures the notion that we are looking at every integer in the same residue class mod p as n more clearly.

This extension to any "level" of factorial allows us to work in a much more general setting. The modular factorial induces the same parity considerations as the double factorial, and allows $n!$ to be decomposed in a similar way.

Lemma 1.2.11. *For any non-negative integers n, p we have*

$$n! = \prod_{k=0}^{p-1} (n-k)!_p$$

Proof. The proof for this follows in the same argument as Lemma 1.2.6, but now we note that each integer falls into exactly one of the $p-1$ residue classes mod p. Applying the pigeonhole principle we see that we have equality. $\square$

These combinatorial constructions will become vital once we begin to investigate the alternating product of binomial coefficients. It is perhaps surprising that the modular factorial appears frequently in said alternating products.

1.3 Order Theory Preliminaries

Next, we shift from thinking combinatorialy to thinking strictly algebraically. When dealing with many algebraic objects, there is an embedded notion of order between two objects. This order allows us to compare two objects, giving us more insight into their relationship. We begin from the ground-up in defining a partial order on a set, and gradually work our way up to the idea of a lattice. This terminology will be important later when considering $\mathbf{\Delta}$ no longer as simply a recursively generated array of numbers, but instead as a set of objects with an order imposed upon them that captures the recursive generation. Many of the proofs that follow are standard in the field and some may be found in [7] [11].

First, we give a precise definition of a partial order on a set.

Definition 1.3.1. *Let X be a non-empty set, then a partially ordered set (poset) is a pair $(X, \leq)$. Where $\leq$ on X is a binary relation on X that satisfies the following properties for all $a, b, c \in X$:*

1. *$a \leq a$ (Reflexivity)*
2. *If $a \leq b$ and $b \leq a$, then $a = b$ (Antisymmetry)*
3. *If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitivity)*

Remark. It is important to note, that in the definition above there is no guarantee that for arbitrary $a, b \in X$ that one of $a \leq b$ or $b \leq a$ holds. The antisymmetry of $\leq$ only guarantees that if both $a \leq b$ and $b \leq a$ hold, then a and b are the same element.

It could be the case that a and b are *incomparable* elements. When this is the case, we will write $a||b$ and say that a is incomparable to b.
If it is the case that for any $a, b \in X$ that either $a \leq b$ or (inclusive) $b \leq a$, then we say that X is a chain, or total ordering.

Definition 1.3.2. *If $(X, \leq)$ is a partial order such that for every $a, b \in X$, at least one $a \leq b$ or $b \leq a$ holds, then $(X, \leq)$ is a chain (total order).*

Just as with any algebraic structure, we can consider sub-structures. If we let $U \subset X$ and restrict $\leq$ to U, then we call U a subposet of X. In a sub-poset U, the order relation between two elements coincides with their order relation in the larger set X.
Instead of simply restricting the order on X to the order on an arbitrary subset, we can consider a few special kinds of sub-posets. .

Definition 1.3.3. *A lowerset D of X is a sub-poset of X such that if $x \in D$ and $y \in X$ such that $y \leq x$, then $y \in D$. An upperset U of X is defined similarly.*
A principal lowerset is denoted $\downarrow x$ and defined as $\downarrow x = \{y \in X : y \leq x\}$. A principal upperset is denoted $\uparrow x$ and defined as $\uparrow x = \{y \in X : x \leq y\}$.

The concept of lower-set and upper-set let us talk about specific sub-posets of X. We can draw an analogy here to rays on the real line, in that they have a beginning but no end. If we are working in a bounded poset, then every lower-set or upper-set is necessarily bounded, but this need not be true in a general poset. It will be helpful in the future to characterize a lower-set by its cardinality. We introduce the following concepts so that we have the terminology needed when it is most helpful.

Definition 1.3.4. *Let $\mathbf{P}$ be a poset. If $\downarrow x$ is finite for each $x \in P$, we say that the poset $\mathbf{P}$ is finitary. Likewise, we define upward-finitary.*

Continuing the analogy with connected subsets of the real line, instead of considering rays we can consider intervals. The next construction gives a direct analogy to intervals of elements of $\mathbb{R}$.

Definition 1.3.5. *Let $\mathbf{P}$ be a poset and $\alpha, \beta \in \mathbf{P}$, then we say that the interval from α to β is the set of elements greater than or equal to α and less than or equal to β. We denote this as $[\alpha, \beta]$. Further, we have*

$$[\alpha, \beta] = (\uparrow \alpha) \cap (\downarrow \beta)$$

If every interval $[\alpha, \beta] \subset P$ is finite, we say that $\mathbf{P}$ is locally finite.

As an example of these preceding concepts, consider the poset described below.

Let X be some non-empty set and consider the powerset of X: $\wp(X)$. We show that $\subseteq$ is a partial order on $\wp(X)$.

1. (Reflexivity) Clearly if $A \in \wp(X)$, then $A \subseteq A$.
2. (Antisymmetry) If $A, B \in \wp(X)$ such that $A \subseteq B$ and $B \subseteq A$, then from the definition of set equality we have that $A = B$
3. (Transitivity) If $A, B, C \in \wp(X)$ such that $A \subseteq B$ and $B \subseteq C$, then by the transitivity of $\subseteq$ we have that $A \subseteq C$.

Thus, for any non-empty set X, $(\wp(X), \subseteq)$ is a partial order.
Now, if we consider $\downarrow A$ for arbitrary $A \in \wp(X)$, we have that $\downarrow A = \{B \in \wp : B \subset A\}$. So, the lowerset generated by A is precisely the set of subsets of X which are subsets of A; or in more interesting terms, we can say that $\downarrow A$ is precisely $\wp(A)$.
On the other hand, the principal upperset generated by arbitrary A is given by $\uparrow A = \{B \in \wp(X) : A \subseteq B\}$. We can see that this is the collection of subsets of X that contain A. Also, we can write that $\wp(X) - \uparrow A = \wp(X - A)$.

Remark. Above we defined a principal lowerset as a lowerset generated by a single element. We can easily extend this to consider the lowerset generated by some subset of our set X.

If $Y \subseteq X$, the lower-set generated by Y is given by $\downarrow Y = \{z \in X : z \leq y, \text{for some } y \in Y\}$. The upperset is generated similarly.

As of now in our investigations into order, we have only considered subsets of our partial order. If we begin to consider the elements of the partial order, a few properties become apparent.

In the example above, $\emptyset$ has a unique position in $(\wp(X), \subseteq)$ in that for every $A \in \wp(X)$, we have that $\emptyset \subseteq A$. That is, $\forall A \in \wp(X)$, $\emptyset \in \downarrow A$.s Thus, every element is "above" the empty set. On the other hand, the element X is unique in that every element in $\wp(X)$ is "below" X. We now formalize these intuitions.

Definition 1.3.6. *Suppose $(X, \leq)$ is a poset. We say that $x \in X$ is minimal if $\downarrow x = \{x\}$. Dually, we say that $y \in X$ is maximal if $\uparrow y = \{y\}$.*

In our previous example, $\emptyset$ was the only minimal element while X was the only maximal element. It might not be the case that there are unique minimal and maximal elements in a poset. To see an example where this is not the case consider the following poset.

Let $\mathbb{N}$ represent the set of natural numbers and let $X = \{\mathbb{N} \times \{0\}\} \cup \{\mathbb{N} \times \{1\}\}$ and define an order $\leq_2$ on X as the following:

$$\text{If } (a, b), (a', b') \in X, \text{ we say } (a, b) \leq_2 (a'b') \text{ if } b = b' \text{ and } a \leq a'.$$

Where $a \leq a'$ is from the natural ordering on $\mathbb{N}$. This is sometimes referred to as the "lexicographic" or "dictionary" ordering on $\mathbb{N} \times \mathbb{N}$.

Now, if we consider the elements $(1, 0)$ and $(1, 1)$ we see that both are minimal, yet

$(1,0)||(1,1)$. Thus, the existence of a minimal element does not guarantee the existence of a unique minimal element. Notice here also that $(X, \leq_2)$ has no maximal elements.

In the case that we do have a unique maximal or minimal element, we give them special designations.

Definition 1.3.7. *If $(X, \leq)$ is a poset, we say that $(X, \leq)$ has a least element, denoted $\perp$, if it has a unique minimal element. If $(X, \leq)$ has a unique maximal element, we call it $\top$ and say it is the greatest element of $(X, \leq)$.*

A poset with a least element is said to be lower-bounded, and a poset with a greatest element is said to be upper-bounded. We can go further and extend this concept of boundedness to sub-posets of X.

Definition 1.3.8. *Given a poset $(X, \leq)$, a sub-poset Y is said to be lower-bounded if there exists an $x \in X$ such that $x \in \downarrow y$ for all $y \in Y$. Similarly, Y is upper-bounded if there exists an $x \in X$ such that $x \in \uparrow y$ for all $y \in Y$.*

From our discussion of potential incomparability of elements of $(X, \leq)$, we know that we aren't guaranteed to have a unique lower bound for an arbitrary $Y \subset X$. Thus, we say that $m(Y)$ is the collection of lower-bounds of a subset Y. Likewise, we denote the collection of upper bounds of Y by $j(Y)$. Further, for a given $x \in X$, we denote the set of elements that are incomparable to x as $\mathbf{Inc}(x)$.
For example, let X be a non-empty set and consider the partial order $(\wp(X), \subseteq)$. If we let $Y = \{A \in \wp(X) : |A| = 1\}$, then $j(Y) = \{B \in \wp(X) : |B| \geq 2\}$ while $m(Y) = \{\emptyset\}$.

When working in a poset $(X, \leq)$ we might find that a given subset $Y \subset X$ not only has a non-empty set of upper-bounds, but also that this set of upper-bounds $j(Y)$ has a least element. When this is the case, we say that the least element of

$j(Y)$ is the **supremum** or **join** of the set Y and we denote this (necessarily unique) element as $\bigvee Y$.

Similarly, if $Y \subset X$ and $m(Y)$ has a greatest element, we call this the **infimum** or **meet** of Y and denote it as $\bigwedge Y$.

If we take the smallest possible interesting non-empty subset of X, namely a two element subset $\{x_1, x_2\}$, we write $x_1 \wedge x_2$ and $x_1 \vee x_2$ for the meet and join of $\{x_1, x_2\}$, respectively. Thus, if $Y \subset X$ is a countable set we can define $\wedge Y$ inductively. That is, if $Y = \{x_1, x_2, ..., x_n\}$ we have

$$\bigwedge Y = x_1 \wedge x_2 \wedge ... \wedge x_n.$$

We can define the join of a countable subset inductively in the same manner.

If given an arbitrary poset, we might wish to know if each pair of elements has a join or meet. This situation arises frequently enough that we give a few pieces of terminology to talk more clearly about it.

Definition 1.3.9. *If $(L, \leq)$ is a non-empty poset where for every $x_1, x_2 \in L$ the element $x_1 \wedge x_2$ exists, we say that $(L, \leq)$ is a meet-semilattice. Dually, if $x_1 \vee x_2$ exists for each pair of elements, then we say $(L, \leq)$ is a join-semilattice.*

To see an example of a poset that is a join-semilattice but not a meet-semilattice, consider the poset $(\Omega^+(X), \subseteq)$ where X is any non-empty set and $\Omega^+(X)$ is the collection of all non-empty subsets of X.. We have that if $Y_1, Y_2 \in \Omega^+(X)$, then $Y_1 \cup Y_2$ serves as the join of the two elements in $\Omega^+(X)$. The intersection operator could work as the meet of two elements, but if we choose two disjoint subsets their intersection would be empty. Thus, they would have no greatest lower bound in $\Omega^+(X)$.

If the poset $(X, \leq)$ happens to be both a join-semilattice and a meet-semilattice, we say that $(X, \leq)$ is a **lattice**.

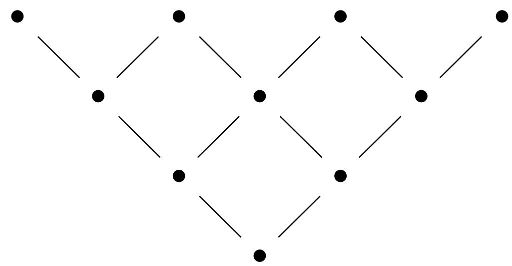

Figure 3: A Complete Meet Semi-lattice

We can see quite quickly that for a given non-empty set X, the poset $(\wp(X), \subseteq)$ is a lattice where the meet of two elements is given by their intersection and the join is given by their union.

When working in a lattice, we might be interested in more than the join (or meet) of just two elements, instead we might want to find an element that serves as the join of some arbitrary subset of elements of our given lattice. This is a much more stringent requirement for our lattice.

Definition 1.3.10. *Let $\mathcal{L}$ be a lattice. If every non-empty proper subset of $\mathcal{L}$ has a meet, we call $\mathcal{L}$ a complete meet semi-lattice. Similarly, if every non-empty proper subset of $\mathcal{L}$ has a join, we call $\mathcal{L}$ a complete join semi-lattice.*

We impose the condition on every non-empty subset to sidestep the issue of boundedness. Taking the meet of the empty set gives the top element and taking the join of the empty set gives the bottom element. So we expand the typical definition slightly so as to not require our lattice to be bounded both above and below. Consider the partial Hasse diagram given in Figure 3. The diagram continues in this fashion upward, forming an infinite lattice.

By considering the poset given in Figure 3, it is easy to see that every pair of elements has both a meet and a join, thus this is a lattice. Also, taking any subset, we can always find the meet for this subset. This makes it a complete meet semi-lattice.

However, if we take the whole lattice, it has no upper bound and thus it is not a complete join semi-lattice. This observation leads us to the next theorem.

Theorem 1.3.11. *Let $\mathcal{L}$ be a lattice. If $\mathcal{L}$ is an upper-bounded complete meet semi-lattice, then it is a join semi-lattice. Dually, if $\mathcal{L}$ is a lower-bounded complete join semi-lattice, then it is a meet semi-lattice.*

Proof. Suppose $\mathcal{L}$ is a lattice and an upper-bounded complete meet semi-lattice. Let $X \subset \mathcal{L}$. Consider the set $j(X)$ of all upper-bounds of X. Since $\mathcal{L}$ is upper-bounded, we know that $j(X)$ is non-empty. Thus, $\bigwedge j(X)$ exists in $\mathcal{L}$ and hence $\bigvee X$ exists in $\mathcal{L}$ □

As with any algebraic structure, when working in a lattice we want find what sort of properties the meet and join obey. The following list of properties can be shown by applying the definitions of meet and join.

Theorem 1.3.12. *Let $\mathcal{L} = (L, \leq)$ be a lattice and $x, y, z \in L$. We have that the following identities hold:*

1. $x \vee y = y \vee x$
2. $(x \vee y) \vee z = x \vee (y \vee z)$
3. $x \vee x = x$
4. $x \vee (x \wedge y) = x$

Each of the above identities also holds if each instance of $\vee$ is replaced with $\wedge$.

Theorem 1.3.12 suggest that $\vee$ and $\wedge$ can be thought of as binary relations on the set L that satisfies the commutative law (1), associative law (2), idempotence law (3), and the absorption law (4). The validity of each of these laws holds based upon the order induced by $\leq$. Thus, we might naturally ask: if we have two binary operations satisfying these laws, do they induce a lattice structure? The following theorem answers this question in the affirmative.

Theorem 1.3.13. *Let L be a set equipped with binary operations $m(-,-)$ and $j(-,-)$ which are commutative, associative, idempotent and satisfy the absorption laws. Define binary relations as follows:*

1. $x \leq y \iff m(x,y) = x$
2. $x \sqsubseteq y \iff j(x,y) = y$

Then $\leq = \sqsubseteq$; further, L is a lattice with $x \vee y = j(x,y)$ and $x \wedge y = m(x,y)$.

Proof. We begin by showing $\leq = \sqsubseteq$. First, suppose $x \leq y$. Then we have $m(x,y) = x$. The absorption law tells us $y = j(y, m(y,x)) = j(y,x) = j(x,y)$. Thus, the relation $\leq$ is a subset of the relation $\sqsubseteq$. Next, suppose $x \sqsubseteq y$. Then we have $j(x,y) = y$. Again, the absorption law tells us $x = m(x, j(x,y)) = m(x,y)$. Thus, the relation $\sqsubseteq$ is a subset of the relation $\leq$. Therefore the relation $\leq, \sqsubseteq$ are equal as sets.

Next, we show that the join of two elements corresponds to the binary operation $j(-,-)$. First, we can see that $j(x,y)$ serves as an upper bound for the set $\{x,y\}$ by considering the absorption law $x = m(x, j(x,y))$. From the definition of $m(-,-)$ this means that $x \leq j(x,y)$. Similarly, we have that $y \leq j(x,y)$.

We must now show that $j(x,y)$ is truly the least upper bound for the set $\{x,y\}$. Suppose $\exists w \in L$ such that $x \leq w$ and $y \leq w$. Consider $j(w, j(x,y))$. By the idempotent law we have $j(w, j(x,y)) = j(j(w,w), j(x,y))$. By associativity we have $j(j(w,w), j(x,y)) = j(w, j(w,x), y)$; then by the assumption that $x \leq w$ we can say that $j(w, j(w,x), y) = j(w,w,y) = j(w,y) = w$. Thus, we know that $j(w, j(x,y)) = w$. So, in particular we can say that $j(x,y) \leq w$. Thus, $j(x,y)$ is the least upper bound for $\{x,y\}$. $\square$

We give two final interesting properties that a lattice might have. In a sense these capture the idea of finiteness in a more local sense as opposed to a global sense.

Theorem 1.3.14. *If a poset* **P** *is finitary (or upward-finitary), then it is locally finite.*

Proof. Suppose **P** is a finitary poset. Let $[\alpha, \beta] \subset P$. Then, $[\alpha, \beta] \subset \downarrow \beta$. Thus, $[\alpha, \beta]$ is finite.
Similarly, $[\alpha, \beta] \subset \uparrow \alpha$, hence it $[\alpha, \beta]$ is finite. □

Theorem 1.3.15. *Every finitary lattice satisfies the Descending Chain Condition.*

Proof. Let **L** be a finitary lattice and $S = x_1 < x_2 < x_3 < \ldots$ be a descending chain of elements of **L**. Then we have that $S \subseteq \downarrow x_1$. Since $\downarrow x_1$ is finite, so is S. Thus, S has only finitely many distinct members.

□

So far we have only focused on the relations between single elements or pairs of elements. Because a poset is a type of algebraic structure, we might also ask if there are interesting sub-objects of a given poset. Take as analogy the sub-group or sub-ring construction. Furthermore, we extend the concept of an ideal in Ring Theory to an ideal in Order Theory.

Definition 1.3.16. *Let $(P, \leq)$ be a poset and D a sub-poset of $(P, \leq)$. We say D is directed if every finite subset of D has an upper bound in D. Further, if D is a directed lowerset of $(P, \leq)$, we call D an ideal of $(P, \leq)$. We denote the set of ideals of $(P, \leq)$ as* ***Idl***(P).

Given a poset **P**, the set **Idl(P)** is always a poset where the order is given by subset inclusion. Dually, we say that a directed upper set is a filter, and if we denote the set of all filters of **P** by **Fil(P)**, we see that **Fil(P)** is also a poset where the order is given by reverse subset inclusion.

Note that in the definition of a directed subset of $\mathbf{P}$, we only require that every finite subset has an upper bound, the reason for this becomes apparent when we begin to apply Zorn's lemma to constructions on directed posets. Taking this idea of directedness to its logical next step, we define a directed complete poset.

Definition 1.3.17. *A poset* $\mathbf{P}$ *is called directed complete (DCPO) if every directed subset of* P *has a join in* $\mathbf{P}$*.*

This leads us to the following theorem whereby we show that given a poset $\mathbf{P}$, we always have that the set of ideals of $\mathbf{P}$ is a DCPO. This strong statement tells us that if we wish to look at some structure in $\mathbf{P}$ requiring the directed completeness property, we can get an approximation to that structure by considering the corresponding set of ideals.

Theorem 1.3.18. *The union of a directed family of ideal is an ideal and* $\mathbf{Idl}(P)$ *is a DCPO.*

Proof. Let $\{V_i\}$ be a directed family of ideals of P. To show $\bigcup V_i$ is an ideal we first show it is a downset. Let $x \in \bigcup V_i$ and $y \in P$ such that $y \leq x$. Because $x \in \bigcup V_i$ we know there is some j so that $x \in V_j$, and since V_j is an ideal we have that $y \in V_j$. Thus, $y \in \bigcup V_i$ allowing us to say $\bigcup V_i$ is a downset.

Next, we show that $\bigcup V_i$ is directed. Let $x, y \in \bigcup V_i$. If there exists an index j such that $x, y \in V_j$ then because V_j is an ideal we would have that the set $\{x, y\}$ has an upper bound in $V_j \subseteq \bigcup V_i$. Otherwise, there must exist m, n such that $x \in V_m$ and $y \in V_n$. Now, since the family $\{V_i\}$ is directed, we have that there is an upper bound for the set $\{V_m, V_n\}$ call it V_u. Thus, we would have that $x \in V_m \subseteq V_u$ and

$y \in V_n \subseteq V_u$ implying that $x, y \in V_u$. Lastly, since V_u is an ideal we have that the set $\{x, y\}$ has an upper bound in $V_u \subseteq \bigcup V_i$ since V_u is an ideal.
We conclude that $\mathbf{Idl}(P)$ is a DCPO since the join of any directed family of ideals is again an ideal. □

Checking that a given subset is an ideal is not always straightforward. Under some simple conditions there is an easier condition for a given subset to be an ideal.

Theorem 1.3.19. *Let $\mathcal{L}$ be a join semilattice and let $I \subseteq L$ be nonempty, then I is an ideal of $\mathcal{L}$ if and only if I is a downset of $\mathcal{L}$ with the property that $x \vee y \in I$ whenever $x, y \in I$.*

Proof. ($\Rightarrow$)
Suppose I is an ideal of $\mathcal{L}$. By definition I is a downset and thus we need only show that the join of any two elements of I is in I. Let $x, y \in I$. Because $\mathcal{L}$ is a join semi-lattice, we know that $x \vee y \in \mathcal{L}$. Thus, we need only show that $x \vee y \in I$. Since I is directed we know that the set $\{x, y\}$ has an upper-bound, call it z, in I. However, $x \vee y \leq z$ since $x \vee y$ is the least upper bound for the set $\{x, y\}$, and since I is a downset it must be the case that $x \vee y \in I$. Hence, I is a downset closed under joins.
($\Leftarrow$)
Suppose I is a downset with the property that $x \vee y \in I$ for all $x, y \in I$. I is a downset, thus we need only show it is directed. Let V be a nonempty finite subset of I. We can write $V = \{v_1, v_2, ..., v_n\}$. To show that V has an upper-bound in I, we induct on the cardinality of V.
Base Case($n = 2$): If $V = \{v_1, v_2\}$ then $v_1 \vee v_2$ is an upper bound that, by hypothesis, is an element of I.
Induction Step: Suppose $V_n = \{v_1, v_2, ..., v_n\}$ has an upper bound in I, call it x. Consider the set $V_{n+1} = V_n \cup \{v_{n+1}\}$. We wish to show that V_{n+1} has an upper-bound in I. We observe that $x \vee v_{n+1}$ serves as an upper-bound for the set V_{n+1}. Also, since

$x, v_{n+1} \in I$ we have by hypothesis that $x \vee v_{n+1} \in I$. Thus, V_{n+1} has an upper bound in I. $\square$

From the preceding discussion, we have seen that the set of ideals on a given poset typically has more structure than the poset. We saw in Theorem 1.3.18 that for any poset, the ideal poset is always a DCPO. This tells us we should be investigating the ideal poset to determine what structures are hidden in our original poset. We shall see that the ideal poset generated from any lower-bounded join semi-lattice is always a complete lattice.

Theorem 1.3.20. *Let $\mathcal{L}$ be a lower-bounded join semi-lattice. If $I, J \in \mathbf{Idl}(\mathcal{L})$, then $I \cap J \in \mathbf{Idl}(\mathcal{L})$. Further, $\mathbf{Idl}(\mathcal{L})$ is a complete lattice.*

Proof. Let $K = I \cap J$. First, we show K is a downset. Suppose $x \in K, y \in \mathcal{L}$ such that $y \leq x$. Since $x \in K \subseteq I \cap J \subseteq I$ we have that $x \in I$ and $y \leq x$ implies $y \in I$. Similarly, we have that $y \in J$. Thus, $y \in K$ allowing us to conclude that K is indeed a downset. Next, we show that for any $x, y \in K$, their join $x \vee y \in K$. Since $x, y \in K \subseteq I$, and I is an ideal we know that $x \vee y \in I$. Likewise, since $x, y \in K \subseteq J$ and J is an ideal we know that $x \vee y \in J$. Thus, $x \vee y \in K$. Hence, $K = I \cap J$ is an ideal.

To see that $\mathbf{Idl}(\mathcal{L})$ is a complete lattice, observe that we have already shown that the union of any directed family of ideals is an ideal and thus $\mathbf{Idl}(\mathcal{L})$ is a join semi-lattice since the union of two ideals serves as their join. Also, we have just shown that $\mathbf{Idl}(\mathcal{L})$ is a DCPO. We also know that $\mathbf{Idl}(\mathcal{L})$ is lower-bounded, because the ideal generated by $\perp \in \mathcal{L}$ is the least element of $\mathbf{Idl}(\mathcal{L})$. Hence, since $\mathbf{Idl}(\mathcal{L})$ is a lower-bounded join semilattice and a DCPO, we know it is a complete lattice.

$\square$

Now we move from general posets to a specific class of partially ordered sets called total orders. A total order is a poset in which the set of incomparable elements is empty. That is, in a total order each element is comparable to each other element.

Total orders are also called **chains** and we will use both terms freely. The dual of a chain is called an **antichain**. If $\mathbf{P}$ is an antichain, then the set of incomparables in $\mathbf{P}$ is all of $\mathbf{P}$. That is, *no* two elements in an antichain are comparable.

For instance, if we take the natural numbers under their natural ordering, we have that this is a chain. Alternatively, if we take any non-empty set X and let $S(X)$ be the subset of $\wp(X)$ that contains only singletons, we see that $S(X)$ along with the order induced by subset inclusion is an antichain.

One of the most powerful tools in Order Theory is Zorn's Lemma. Depending on the mathematician one speaks to, Zorn's Lemma is either entirely plausible or outright absurd. Much of this stems from the fact that Zorn's Lemma is logically equivalent to the Axiom of Choice. Regardless, we take no issue with the non-constructive proofs Zorn's Lemma allows us to make.

Lemma 1.3.21. ***Zorn's Lemma:*** *If every chain in a non-empty poset* $\mathbf{P}$ *has an upper bound in* $\mathbf{P}$*, then* $\mathbf{P}$ *has a maximal element.*

We can apply Zorn's Lemma in a large number of circumstances. To give an idea the flavor and utility of Zorn's Lemma we exhibit the following theorem.

Theorem 1.3.22. *Every nonempty poset contains a maximal antichain.*

Proof. Let P be a nonempty poset. Let M be the set of all antichains in P, and form the poset $\mathbb{M} = (M, \subseteq)$ of antichains ordered by subset inclusion. We know that $\mathbb{M}$ is nonempty, because if $x \in P$, then $\{x\} \in M$. Next, consider a chain $C \subseteq M$. We wish to show that C has an upper bound in M. Clearly, $\bigcup C$ is this required element. Each antichain $M_1 \in \bigcup C$, thus $\bigcup C$ is an upper-bound. We need also show that $\bigcup C$ is an antichain. Let $x, y \in \bigcup C$. If x, y are in the same antichain of C, then they are incomparable. However, if x, y are in two different antichains of C, then one must be a subset of the other (since C is a chain), therefore they are incomparable. Hence, $\bigcup C$ is an antichain. We have created a nonempty poset where every chain has an upper bound in $\mathbb{M}$, hence, by Zorn's Lemma 1.3.21, $\mathbb{M}$ must have a greatest element. Thus, P contains a maximal antichain. $\square$

Intrestingly enough, Zorn's Lemma is equivalent to another semi-controversial tool useds in mathematics: the Well-Ordering Principle.

Definition 1.3.23. *A poset* $\mathbf{P}$ *is set to be well-ordered if every non-empty subset has a least element.*

The inspiration for well-ordering comes from the integers. Given a set of any amount of natural numbers, there is always a smallest one. We know that this fact does not hold if one were to consider the rationals or the real numbers, it even breaks down when considering all of the integers. Thus, if a poset is well-ordered, we can think of it as having some similarities with the integers under their natural ordering. However, this analogy isn't perfect. A great deal of care must be made here because the ordering imposed upon our given poset might not be as simple as the ordering of the natural numbers. Given a well-ordered set, we obtain a few nice properties, one of which is proven below.

Theorem 1.3.24. *In a well-ordered set, every subset which is bounded above has a least upper bound.*

Proof. Suppose that P is a well-ordered poset and A is a subset that is bounded above. Thus, $\mathrm{j}(A) \neq \emptyset$. Since $\mathrm{j}(A) \subseteq P$ it must have a least element since P is well-ordered. Hence, A has a least upper bound. $\square$

We have seen that total orders emulate one aspect of the natural numbers, namely that of comparability. We can also observe that the natural numbers obey the distributive law, in that multiplication distributes over addition. This concept is familiar to those who have studied some Ring Theory. When defining two operations on a given set, the distributive law is the link between the ring multiplication and addition. We can extend this idea to our study of lattices.

Definition 1.3.25. *Let $\mathcal{L}$ be a lattice. We say that $\mathcal{L}$ is a distributive lattice if*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{15}$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{16}$$

for all $x, y, z \in \mathcal{L}$

As is the case with commutative rings, since the operations $\vee$ and $\wedge$ are commutative, we need only have one distributive law for each operation. We can actually simplify it even more. If a lattice satisfies one of the distributive properties above, it will automatically satisfy the other. The next lemma exhibits this claim.

Lemma 1.3.26. *Let $\mathcal{L}$ be a lattice and $x, y, z \in \mathcal{L}$. Then, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Proof. Assume that the first equation holds, then apply the absorption law to see

$$(x \vee y) \wedge (x \vee z) = [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z]$$
$$(x \vee y) \wedge (x \vee z) = x \vee [(x \vee y) \wedge z]$$
$$(x \vee y) \wedge (x \vee z) = x \vee [(x \wedge z) \vee (y \wedge z)]$$
$$(x \vee y) \wedge (x \vee z) = [x \vee (x \wedge z)] \vee (y \wedge z)$$
$$(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$$

Hence, the first equation implies the second. The reverse statement follows by duality. $\square$

Distributive lattices form a large, highly interesting class of lattices with many nice properties. We have seen that to show a lattice is distributive we need only show one of the identities holds in the definition above. But, we can go even further than that.

In fact, in any lattice, we always have that the identity $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ holds, as can be seen by using the absorption law. Thus, to show a lattice is distributive, we need only verify the reverse inequality for each triple x, y, z.

One of the most striking elementary results on distributive lattices is outlined below. It says that if one is given a distributive lattice, then the poset formed by the ideals of that lattice is itself a distributive lattice. We can go even further and show that a lattice is distributive if and only if its ideal lattice is distributive.

Theorem 1.3.27. *If $\mathcal{L}$ be a lower-bounded distributive lattice, then Idl($\mathcal{L}$) is a distributive lattice.*

Proof. Given $I, J, K \in \mathrm{Idl}(\mathcal{L})$ we show the distributive property holds for $\mathrm{Idl}(\mathcal{L})$. That is, we want to show $I \cap (J \sqcup K) \subseteq (I \cap J) \sqcup (I \cap K)$.

Suppose $x \in I \cap (J \sqcup K)$. We know then $x \in I$ and $x \in J \sqcup K$. Thus, we can find elements $j \in J$ and $k \in K$ such that $x \leq j \vee k$. This allows us to say that $x = x \wedge (j \vee k)$ and since $\mathcal{L}$ is distributive we have $x = (x \wedge j) \vee (x \wedge k)$.

Observe that $x \wedge j \leq x \in I$, thus $x \wedge j \in I$, and that $x \wedge j \leq j \in J$, thus $x \wedge j \in J$. Since I and J are ideals they are in particular downsets, so we can say $x \wedge j \in I \cap J$. We also can say that since $\mathcal{L}$ is lower-bounded, each of $(I \cap J)$ and $(I \cap K)$ are non-empty. Likewise, we can say $x \wedge k \in I \cap K$. Thus, because $(x \wedge j), (x \wedge k) \in (I \cap J) \cup (I \cap K)$ we have that $(x \wedge j) \vee (x \wedge k) \in (I \cap J) \sqcup (I \cap K)$. Finally, since $(x \wedge j) \vee (x \wedge k) = x$ we can conclude that $I \cap (J \sqcup K) \subseteq (I \cap J) \sqcup (I \cap K)$.

□

To conclude our discussion of elementary order theory, we give a few consequences of a lattice being distributive. The first of these facts is similar in flavor to an important result in Ring Theory, while the second is more similar to something one might see in Group Theory. We first need a few pieces of terminology concerning specific types of ideals.

Definition 1.3.28. *Let $\mathcal{L}$ be a lattice and $I \in \boldsymbol{Idl}(\mathcal{L}) - \mathcal{L}$. We say that I is a prime ideal of $\mathcal{L}$ if whenever $x \wedge y \in I$ then either $x \in I$ or $y \in I$.*

The intuition behind a prime ideal is analogous to the concept of a prime number. A number p is prime if, given any integers n, m such that $nm = p$ we always have that either $n = p$ or $m = p$. This turns out to be as powerful a statement for lattices as it does for the natural numbers.

Definition 1.3.29. *Let $\mathcal{L}$ be a lattice and $I \in \boldsymbol{Idl}(\mathcal{L}) - \mathcal{L}$. We say that I is a maximal ideal if, whenever $J \in \boldsymbol{Idl}(\mathcal{L})$ such that $I \subsetneq J$ we have that $J = \mathcal{L}$.*

Maximality of an ideal captures exactly what one would expect: an ideal is maximal if the only ideal properly containing it is the entire lattice itself. Now we can give a clear connection between these two concepts. The next theorem has a clear analogue in Ring Theory: if R is a commutative ring with unity and I is a maximal ideal of R, then I is a prime ideal of R as well.

Theorem 1.3.30. *If $\mathcal{L}$ be a distributive lattice, then every maximal ideal of $\mathcal{L}$ is a prime ideal.*

Proof. Suppose M is a maximal ideal of $\mathcal{L}$ that is not prime. Further, suppose $x \wedge y \in M$. Since M is not prime, then $x \notin M$ and $y \notin M$.
Next, consider the ideal generated by the set $M \cup \{x\}$. Since M was a maximal ideal,

we must have that $(M \cup \{x\}] = L$. Hence, there exists $a \in M$ such that $y \leq a \vee x$. So, by the definition of meet and distributivity we have $y = y \wedge (a \vee x) = (y \wedge a) \vee (y \wedge x)$. Because $a \in M$ and $y \leq x$ we must have that $y \in M$. However, this contradicts our assumption that M is not prime. Hence, we conclude that in a distributive lattice every maximal ideal is prime. □

When beginning a study of groups, one is told that the group concept captures one of the simpler types of algebraic relations that can be put on a given non-empty set. For a given set and operation to satisfies the group axioms, we need to be closure under the operation, an inverse for each element, and an identity element. One shows in a first class on Group Theory that in any group, no element can have more than one inverse. That is, inverses of group elements are necessarily unique.

While we don't exactly have the same level of structure in a lattice as a group, we can have something similar to an inverse. The next few definitions set up this idea, and Theorem 1.3.32 is a lattice analogue to the uniqueness of inverses.

Definition 1.3.31. *Let $\mathcal{L}$ be a bounded lattice and $a \in \mathcal{L}$. We say that b is the complement of a in $\mathcal{L}$ if*

$$a \wedge b = \bot$$

and

$$a \vee b = \top.$$

More generally, for any interval $[\alpha, \beta] \subset \mathcal{L}$ and $a, b \in [\alpha, \beta]$, we say that b is the relative complement of a in $[\alpha, \beta]$ if

$$a \wedge b = \alpha$$

and

$$a \vee b = \beta.$$

Theorem 1.3.32. *Let $\mathcal{L}$ be a distributive lattice, and let $a, b \in L$. An element of $[a, b]$ can have at most one relative complement in $[a, b]$.*

Proof. Suppose to the contrary that $x \in [a, b]$ has two distinct complements y, z. Thus, we have that $x \wedge y = a = x \wedge z$ and $x \vee y = b = x \vee z$. By using the first equality and then joining y to both sides, we have:
$x \wedge y = x \wedge z \Rightarrow y \vee (x \wedge y) = y \vee (x \wedge z) \Rightarrow y = (y \vee x) \wedge (y \vee z) = b \wedge (y \vee z)$.
Next, by considering the same equality and by joining z to both sides, we have:
$x \wedge y = x \wedge z \Rightarrow z \vee (x \wedge y) = z \vee (x \wedge z) \Rightarrow z = (z \vee x) \wedge (z \vee y) = b \wedge (z \vee y)$.
Since it must be the case that $b \wedge (z \vee y) = b \wedge (z \vee y)$, we have that $y = z$. Ergo, an element in $[a, b]$ can have at most one complement. □

As with many algebraic structures, we might want to extract more information from a poset than simply the order. For instance, we might wish to count the number of elements either above or below a given element, or we might want to count the number of paths from the least element to a given element. To answer these questions, and more like them, we introduce the concept of a graded poset.

Definition 1.3.33 (Stanley)**.** *We say that a lower-bounded poset $\mathbf{P} = (P, \leq)$ is graded if it can be written as a disjoint union of maximal antichains $\{P_i\}$ such that for any maximal chain of elements $p_0 \prec p_1 \prec ...$ we have that $p_i \in P_i$ for all i.*

Working in a graded poset gives us access to a rank function, which in turn gives us more information about a specific element. The rank function, usually denoted ρ, is a function from the graded poset $\mathbf{P}$ to the natural numbers (with their natural ordering), typically defined by

$$\rho : \bigcup_{i \in I} P_i \to \mathbb{N}$$

such that if $x \in P_i$, then $\rho(x) = i$.

Based on our indexing of the disjoint, maximal antichains we then have that the least element of $\mathbf{P}$ has rank 0.
There is a rich theory concering graded posets and more details can be found in [23].

As an example consider the poset $\wp(X)$ defined by looking at the powerset of some finite and non-empty set X, ordered under subset inclusion. If $|X| = n$, then $|\wp(X)| = 2^n$. If we take as $P_i = \{U \in \wp(X) : |U| = i\}$ for each $i \in \{0, 1, .., n\}$ we see that $\wp(X)$ is clearly a graded poset with rank function returning the size of a subset of X.

We can say a bit more about graded posets. An equivalent formulation of graded posets is to define the rank function by its properties.

Definition 1.3.34. *The poset* $\mathbf{P}$ *is said to be a graded poset if it is a poset equipped with a rank function* $\rho : \mathbf{P} \to \mathbb{N}$ *such that the following properties hold:*

1. *ρ is constant on all minimal elements.*
2. *ρ is an isotone map. i.e. $a \leq b \Rightarrow \rho(a) \leq \rho(b)$.*
3. *ρ is cover-preserving. i.e. $a \prec b \Rightarrow \rho(b) = \rho(a) + 1$*

We can think of these two different definitions as two different views of the rank of an element. In the first definition, we think of an element as sitting in a particular piece of the partition $\{P_i\}$, whereas in the second we can generate the pieces of the partition by looking at the pre-image of each natural number, similar to the way in which a partition generates an equivalence relation and vice-versa.

We should clarify here that the induced rank function defined above is not the only rank function that may be put on a poset. Any choice of function from $\mathbf{P}$ to $\mathbb{N}$ with all of the properties from Definition 1.3.34 will make $\mathbf{P}$ into a graded poset.

Further, we can extend the idea of a graded poset to that of a graded lattice. Much work has been done on graded lattices by Stanley [23]. For a taste of this type of thinking, we give one Proposition from [23] without proof. The proof is slightly technical and not needed for our future work so is omitted.

Theorem 1.3.35. *(Stanley) Let $\mathcal{L}$ be a finite lattice. The following two conditions are equivalent:*

1. *$\mathcal{L}$ is graded, and the rank function ρ of $\mathcal{L}$ satisfies*

$$\rho(s) + \rho(t) \geq \rho(s \vee t) + \rho(s \wedge t)$$

   for all $s, t \in \mathcal{L}$.

2. *If s and t both cover $s \wedge t$, then $s \vee t$ covers both $s \wedge t$.*

This concludes our brief introduction to order theory. The author would point an interested reader to [12], [7], [23] to further their study of lattices and order theory.

CHAPTER 2

Generalized Arithmetic Triangles

2.1 Alternating Products

From our earlier investigations into the properties of Pascal's Triangle ($\mathbf{\Delta}$) we have seen that there are a multitude of interesting number patterns in this infinite number array. The majority of the constructions used in Chapter 1 were formed either by looking at a strategically defined subset of $\mathbf{\Delta}$ or by summing some obvious subset of $\mathbf{\Delta}$. But now, we consider a varaint on the construction from observation 1.1.3.

Suppose we have a row of $\mathbf{\Delta}$, and we wish to investigate what happens when we alternately multiply and divide the elements, i.e we wish to emulate the alternating sum construction on $\mathbf{\Delta}$. We have seen in two distinct ways that the alternating sum of any row of $\mathbf{\Delta}$ (except the top element) is identically 0, so we have hopes that there might be something to be gained from looking at the alternating product of these same elements.

By first calculating the first few alternating products we might look to see if a pattern is apparent. To that end, let $a_n = \prod_{j=0}^{n} [\mathbf{\Delta}(n;j)]^{(-1)^j}$ where a_n records the alternating product of the elements in the nth row of $\mathbf{\Delta}$. The first fifteen values for a_n are given by

$$\left\{1, \frac{1}{2}, 1, \frac{3}{8}, 1, \frac{5}{16}, 1, \frac{35}{128}, 1, \frac{63}{256}, 1, \frac{231}{1024}, 1, \frac{429}{2048}, 1\right\}$$

Immediately, we notice a few striking things. First, it seems as the the rows in $\mathbf{\Delta}$ with odd-index have alternating product identically 1. That is, it looks as though $a_{2k-1} = 1$ for all $k \in \mathbb{N}$. However, the rows of even index do not seem to follow an obvious pattern. We notice that the denominator of each even-indexed term is 2^m for some positive integer m, but it is also not apparent here what form this sub-sequence of powers of two takes. We make the following conjectures from these observations.

Conjecture 2.1.1. *For all non-negative integers k, we have that*

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}(2k+1; j)]^{(-1)^j} = 1 \tag{17}$$

Conjecture 2.1.2. *For all non-negative integers k, we have*

$$\prod_{j=0}^{2k} [\mathbf{\Delta}(2k; j)]^{(-1)^j} = \frac{(2k-1)!!}{(2k)!!} \tag{18}$$

We can prove Conjecture 2.1.1 right away as a simple application of Observation 1.1.2.

Theorem 2.1.3. *For all non-negative integers k, we have that*

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}(2k+1; j)]^{(-1)^j} = 1.$$

Proof. For any $\mathbf{\Delta}(n; k) \in \mathbf{\Delta}$, we have shown that $\mathbf{\Delta}(n; k) = \mathbf{\Delta}(n; n-k)$. Thus, if we collect the terms of our product in the following fashion:

$$\Pi_1 = \prod_{j \text{ even}}^{2k+1} [\mathbf{\Delta}(2k+1; j)]$$

and

$$\Pi_2 = \prod_{j \text{ odd}}^{2k+1} [\mathbf{\Delta}(2k+1; j)].$$

We see that for each factor of Π_1 there is a corresponding factor of Π_2, by the symmetry of $\mathbf{\Delta}$. Thus, $\Pi_1 = \Pi_2$. Further, we have that $\prod_{j=0}^{2k+1} [\mathbf{\Delta}(2k+1; j)]^{(-1)^j} = \frac{\Pi_1}{\Pi_2}$. Hence, the result.

$\square$

To show the validity of Conjecture 2.1.2 is more difficult, because in Conjecture 2.1.2 we have given a connection between two combinatorial objects that are not defined in terms of one another. To show the validity of Conjecture 2.1.2, we first exhibit lemma 1.2.4. This will allow us to reduce the binomial coefficient interior to each term in a way amenable to our study of alternating products. This term should look familiar - it is precisely the reducing term pulled out in Lemma 1.2.4.

Lemma 2.1.4. *For any positive integer k, we have*

$$\prod_{j=0}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j} = 2k+1$$

Proof. We begin by noting that this large product can be decomposed as a product over even values of j and a product over odd values of j. Thus, we have:

$$\prod_{j=0}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j} =$$

$$\left(\prod_{j \text{ even}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j}\right)\left(\prod_{j \text{ odd}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j}\right) =$$

$$\Pi_1\Pi_2$$

Lets examine each of these products individually. Beginning with the product over the even values of j, we note a few things. The total number of terms of the first product in Equation 2.1.4 is $k+1$ and the total number of terms of the second product in Equation 2.1.4 is k. Next, since each index in Π_1 is even, the exponent in each term is positive. Likewise, the exponent in each term of Π_2 is -1, so we really have:

$$\prod_{j=0}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j} = \frac{\left(\prod_{j \text{ even}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]\right)}{\left(\prod_{j \text{ odd}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]\right)}$$

Now, looking more closely at the product Π_1. Using properties of the product operator we have

$$\prod_{j \text{ even}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right] = \frac{\prod_{j \text{ even}}^{2k}(2k+2)(2k+1)}{\prod_{j \text{ even}}^{2k}(2k+2-j)(2k+1-j)}$$

Because the top product does not rely on j, we can say:

$$\prod_{j\text{ even}}^{2k}(2k+2)(2k+1)=[(2k+2)(2k+1)]^{k+1}$$

The bottom product is less obvious to work with. If we expand the terms we notice something.

$$\prod_{j\text{ even}}^{2k}(2k+2-j)(2k+1-j)=(2k+2)(2k+1)(2k)(2k-1)(2k-1)(2k-2)...(3)(2)(1)$$

$$\prod_{j\text{ even}}^{2k}(2k+2-j)(2k+1-j)=(2k+2)!$$

Thus, we can write a simple expression for Π_1.

$$\prod_{j\text{ even}}^{2k}=\frac{[(2k+2)(2k+1)]^{k+1}}{(2k+2)!}$$

By following a nearly identical path of reasoning, we arrive at an analogous expression for the product over the odd values of j.

$$\prod_{j\text{ odd}}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]=\frac{[(2k+2)(2k+1)]^{k}}{(2k+1)!}$$

Thus, if we divide these terms we arrive at an easier to work with expression.

$$\prod_{j=0}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j}=\frac{[(2k+2)(2k+1)]^{k+1}}{[(2k+2)(2k+1)]^{k}}\frac{(2k+1)!}{(2k+2)!}$$

$$\frac{[(2k+2)(2k+1)]^{k+1}}{[(2k+2)(2k+1)]^{k}}\frac{(2k+1)!}{(2k+2)!}=\frac{(2k+2)(2k+1)}{(2k+2)}=2k+1$$

Hence, we have arrived at our result. Ergo,

$$\prod_{j=0}^{2k}\left[\frac{(2k+2)(2k+1)}{(2k+2-j)(2k+1-j)}\right]^{(-1)^j}=2k+1$$

$\square$

Note, that in the proof of Conjecture 2 we use binomial coefficient notation to make more apparent some of the reducing formulae employed.

Theorem 2.1.5. *For all positive integers, k, we have*

$$\prod_{j=0}^{2k}[\mathbf{\Delta}(2k;j)]^{(-1)^j} = \prod_{j=0}^{2k}\left[\binom{2k}{j}\right]^{(-1)^j} = \frac{(2k-1)!!}{(2k)!!}$$

Proof. We prove the statement using induction on the value of k. First, consider the case $k = 1$.

$$\prod_{j=0}^{2}\binom{2}{j}^{(-1)^j} = \binom{2}{0}\binom{2}{1}^{-1}\binom{2}{2} = 1 * \frac{1}{2} * 1 = \frac{1}{2}$$

$$\frac{(2-1)!!}{2!!} = \frac{1!!}{2!!} = \frac{1}{2}.$$

Thus, the result holds for $k = 1$.

Next, we assume the result holds for $k = n$ and we endeavor to show it holds for $k = n + 1$.

Consider the product:

$$\prod_{j=0}^{2(n+1)}\binom{2(n+1)}{j}^{(-1)^j} = \prod_{j=0}^{2n+2}\binom{2n+2}{j}^{(-1)^j}$$

To reduce this product, first we pull out the final two terms to get:

$$\prod_{j=0}^{2(n+1)}\binom{2(n+1)}{j}^{(-1)^j} = \binom{2n+2}{2n+2}\binom{2n+2}{2n+1}^{-1}\prod_{j=0}^{2n}\binom{2n+2}{j}^{(-1)^j}$$

$$\prod_{j=0}^{2(n+1)}\binom{2(n+1)}{j}^{(-1)^j} = \frac{1}{2n+2}\prod_{j=0}^{2n}\binom{2n+2}{j}^{(-1)^j}.$$

Now we'd like to simplify the binomial coefficient inside this product. Using lemma 1.2.4 we have:

$$\prod_{j=0}^{2k}\binom{2n+2}{j}^{(-1)^j} = \prod_{j=0}^{2n}\left[\frac{(2n+2)(2n+1)}{(2n+2-j)(2n+1-j)}\right]^{(-1)^j}\prod_{j=0}^{2n}\left[\binom{2n}{j}\right]^{(-1)^j}$$

By utilizing lemma 2.1.4 we can simplify this even further.

$$\prod_{j=0}^{2n}\binom{2n+2}{j}^{(-1)^j} = (2n+1)\prod_{j=0}^{2n}\left[\binom{2n}{j}\right]^{(-1)^j}.$$

Now, we put everything together.

$$\prod_{j=0}^{2n+2} \binom{2n+2}{j}^{(-1)^j} = \frac{(2n+1)}{(2n+2)} \prod_{j=0}^{2n} \binom{2n}{j}^{(-1)^j}.$$

Finally, using the assumption that our result holds for $k = 2n$ we receive:

$$\prod_{j=0}^{2n+2} \binom{2n+2}{j}^{(-1)^j} = \frac{(2n+1)}{(2n+2)} \frac{(2n-1)!!}{(2n)!!}$$

Last, we use some properties of the double factorial to see:

$$\frac{2n+1}{2n+2}\frac{(2n-1)!!}{(2n)!!} = \frac{(2n+1)(2n-1)!!}{(2n+2)(2n)!!} = \frac{(2n+1)!!}{(2n+2)!!} = \frac{(2(n+1)-1)!!}{((2(n+1))!!}$$

Hence, we have the result:

$$\prod_{j=0}^{2n+2} \binom{2n+2}{j}^{(-1)^j} = \frac{(2(n+1)-1)!!}{(2(n+1))!!}$$

□

Theorem 2.1.5 tells us that if given an even-indexed row of $\boldsymbol{\Delta}$, we can write the alternating product of the elements of that row as a ratio of the double factorials of consecutive integers. This fact explains our observation above, that the denominators of the terms a_{2n} were always powers of two, since we have seen from Lemma 1.2.7 that $(2k)!! = 2^k(k!)$.

We can make one peculiar connection to this theorem. The Wallis integrals [8] are well known in many branches of mathematics, combinatorics being only one. If we let $W_n = \int_0^{\frac{\pi}{2}} \cos^n(x)dx$, then we know that $W_{2n} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$ [5]. It is known then that for large values of n we have $\frac{(2n)!!}{(2n-1)!!} \approx \sqrt{\pi n}$

Corollary 2.1.6. *For large values of n, we have* $\prod_{j=0}^{2n} \binom{2n}{j}^{(-1)^j} \approx \frac{1}{\sqrt{\pi n}}$.

2.2 The Lucas Triangle

Our investigations into The Arithmetic Triangle have given us a complete characterization of the alternating products of the rows; however, The Arithmetic Triangle is not the only interesting triangular arrangement of numbers to consider. In this section we give a new treatment of the Lucas Triangle following the same scheme laid out in Chapter 1.

We begin by looking at the Lucas Triangle, a variant of Pascal's triangle where the second entry in the first row is 2 instead of 1.

$n=0$									2								
$n=1$								1		2							
$n=2$							1		3		2						
$n=3$						1		4		5		2					
$n=4$					1		5		9		7		2				
$n=5$				1		6		14		16		9		2			
$n=6$			1		7		20		30		25		11		2		
$n=7$		1		8		27		50		55		36		13		2	
$n=8$	1		9		35		77		105		91		49		15		2

Figure 4: Lucas Triangle

There are a whole host of interesting patterns hiding in this triangle. For instance, the odd integers are listed along one diagonal, while the integers greater than 2 are listed on the complimentary diagonal. Also, the next diagonal down on the right has the squares listed in order, while its complimentary diagonal counts the number of diagonals in an $n-$gon.

Just as the numbers in Pascal's Triangle can be generated through binomial coefficients, we can generate the numbers inTthe Lucas Triangle with a variant of binomial coefficients.

Definition 2.2.1. *Right Arithmetic Triangle Coefficients*
For $\alpha \geq 1$ we define:

$$\mathbf{\Delta}_\alpha(n;k) = \begin{cases} 0 & \textit{if } n < |k| \\ 1 & \textit{if } k = 0 \\ \alpha & \textit{if } n = k \\ \mathbf{\Delta}_\alpha(n-1;k) + \mathbf{\Delta}_\alpha(n-1;k-1) & \textit{otherwise} \end{cases} \tag{19}$$

Remark. Notice that if we let $\alpha = 1$, then we simply get back $\mathbf{\Delta}$ and if we let $\alpha = 2$, then we generate the Lucas Triangle shown in Figure 4. It is important here to note that for a given α, the right arithmetic triangle coefficients $\mathbf{\Delta}_\alpha(n;k)$ are completely determined by the values of $\mathbf{\Delta}_\alpha(1;0)$ and $\mathbf{\Delta}_\alpha(1;1)$. Thus, we only need to specify these two values, henceforth referred to as 'seed' values for the right arithmetic triangle, to identify a specific right arithmetic triangle.

Before considering the alternating product construction over these right arithmetic triangles, it is worth spending some time investigating what properties from $\mathbf{\Delta}$ carry over, and what new properties we can discern. One obvious observation we can make is that no longer are we assured of the symmetry that was so useful in $\mathbf{\Delta}$. Now, we have explicitly broken that symmetry, so we must consider the ramifications.

2.2.1 Properties of the Lucas Triangle

First, we notice that in the Arithmetic Triangle $\mathbf{\Delta}_2$ the odd numbers appear on the first diagonal. More explicitly we have

Observation 2.2.2. *For all integers $n \geq 1$ we have*

$$\mathbf{\Delta}_2(n; n-1) = 2n - 1 \tag{20}$$

Proof. We prove by induction on n. For $n = 1$ we have $\mathbf{\Delta}_2(1; 0) = 1 = 2(1) - 1$
Next, suppose that the result holds for n. That is, our induction hypothesis states that $\mathbf{\Delta}_2(n; n-1) = 2n - 1$. So, we consider $\mathbf{\Delta}_2(n+1; n)$. Using the recurrence relation for $\mathbf{\Delta}_2$, we have

$$\mathbf{\Delta}_2(n+1; n) = \mathbf{\Delta}_2(n; n) + \mathbf{\Delta}_2(n; n-1).$$

Recall that for all $n \in \mathbb{Z}^+$, we have that $\mathbf{\Delta}_2(n; n) = 2$, and using our induction hypothesis we see:

$$\mathbf{\Delta}_2(n+1; n) = 2 + 2n - 1 = 2n + 1 = 2(n+1) - 1.$$

Hence, we have the result. $\square$

In Observation 1.1.5, we saw that the Triangular numbers appeared on one of the diagonals of the elements of $\mathbf{\Delta}$. An equivalent statement holds for $\mathbf{\Delta}_2$ and this leads us to the following observation.

Observation 2.2.3. *The square numbers appear as the subset $\{\mathbf{\Delta}_2(n; n-2) : n \geq 2\}$ of $\mathbf{\Delta}_2$.*

Proof. The square numbers $\{n^2 : n \in \mathbb{Z}\}$ are given by the recurrence relation $a_{n+1} = a_n + (2n+1)$ with $a_0 = 1$ for all $n \in \mathbb{Z}$.
Clearly, for the case of $n = 1$ we have that $\mathbf{\Delta}_2(2; 0) = 1 = 1^2$.
Now, assume the result holds for the case $n = k$. That is, we have that $\mathbf{\Delta}_2(n; n-2) = (n-1)^2$. We consider the element $\mathbf{\Delta}_2(n+1; n-1)$. Using the recurrence relation for $\mathbf{\Delta}_2$, we have

$$\mathbf{\Delta}_n(n+1; n-1) = \mathbf{\Delta}_2(n; n-1) + \mathbf{\Delta}_2(n; n-2).$$

Now, from our induction hypothesis we can say

$$\mathbf{\Delta}_2(n+1; n-1) = \mathbf{\Delta}_2(n; n-1) + (n-1)^2.$$

From property above we have that $\mathbf{\Delta}_2(n; n-1) = 2n-1$. Substituting and simplifying we have

$$\mathbf{\Delta}_2(n+1; 2) = (n-1)^2 + 2n - 1 = n^2 - 2n + 1 + 2n - 1 = n^2$$

Hence, by the Principle of Mathematical Induction we have the result holds for all $n \in \mathbb{Z}^+$. □

Observation 1.1.3 gives a nice closed form expression for the sum of the elements of a given row $\mathbf{\Delta}(n)$. Now we show that sums of rows and alternating sums of rows of elements of right arithmetic triangles have similarly well-behaved expressions.

Theorem 2.2.4. *For each positive integer n, we have*

$$\sum_{j=0}^{n} \mathbf{\Delta}_2(n; j) = 3 \cdot (2^{n-1}) \tag{21}$$

Proof. We show by induction on n. For the case $n = 1$, we have that $\mathbf{\Delta}_2(1; 0) + \mathbf{\Delta}_2(1; 1) = 1 + 2 = 3 = 3(1) = 3(2^0)$. Thus, the result holds for $n = 1$.
Now, suppose the result holds for some $n = k$ and consider the sum

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1; j).$$

We can decompose each element to see

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1; j) = \mathbf{\Delta}_2(n+1; n+1) + \mathbf{\Delta}_2(n+1; 0) + \sum_{j=1}^{n} (\mathbf{\Delta}_2(n; j) + \mathbf{\Delta}_2(n; j-1))$$

However, we know that for any n, m that $\mathbf{\Delta}_2(n; n) = \mathbf{\Delta}_2(m, m)$ and that $\mathbf{\Delta}_2(n; 0) = \mathbf{\Delta}_2(m; 0)$. So, we can make a small change to the above line to now write:

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1; j) = \mathbf{\Delta}_2(n; n) + \mathbf{\Delta}_2(n; 0) + \sum_{j=1}^{n} (\mathbf{\Delta}_2(n; j) + \mathbf{\Delta}_2(n; j-1))$$

Next, split the sum above into two and re-index, letting $k = j - 1$.

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1; j) = \mathbf{\Delta}_2(n; n) + \mathbf{\Delta}_2(n; 0) + \sum_{j=1}^{n} \mathbf{\Delta}_2(n; j) + \sum_{k=0}^{n-1} \mathbf{\Delta}(n; k).$$

Pulling the two extra terms in front back into the first and second sum, respectively, we have

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1;j) = \sum_{j=0}^{n} \mathbf{\Delta}_2(n;j) + \sum_{k=0}^{n} \mathbf{\Delta}(n;k).$$

From our induction hypothesis, we can say then that

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_2(n+1;j) = 2\left(\sum_{j=0}^{n} \mathbf{\Delta}_2(n;j)\right) = 2\left(3(2^{n-1}\right) = 3(2^n).$$

Hence, the result holds for all positive integers. □

Similarly, we can investigate alternating sums.

Theorem 2.2.5. *For any positive integer $n \geq 2$, we have*

$$\sum_{j=0}^{n} (-1)^j \mathbf{\Delta}_2(n;j) = 0 \tag{22}$$

Proof. First, we remark that for the 0th row, the alternating sum is 1 and for the first row of $\mathbf{\Delta}_2$, the alternating sum is -1. For the second row, where our induction begins, we have that

$$\mathbf{\Delta}_2(2;0) - \mathbf{\Delta}_2(2;1) + \mathbf{\Delta}_2(2;2) = 1 - 3 + 2 = 0.$$

Now, suppose the result holds for the nth row of $\mathbf{\Delta}_2$ and consider the alternating sum of the $n+1$st row. We can write

$$\sum_{j=0}^{n+1} (-1)^j \mathbf{\Delta}_2(n+1;j) = (-1)^{n+1}\mathbf{\Delta}_2(n+1;n+1) + \mathbf{\Delta}_2(n+1;0) + \sum_{j=1}^{n} (-1)^j \mathbf{\Delta}_2(n+1;j).$$

We can further decompose this sum using the Pascal-like recurrence relation for $\mathbf{\Delta}_2$. Namely:

$$\sum_{j=0}^{n+1} (-1)^j \mathbf{\Delta}_2(n+1;j) =$$

$$(-1)^{n+1}\mathbf{\Delta}_2(n+1;n+1) + \mathbf{\Delta}_2(n+1;0) + \sum_{j=1}^{n} (-1)^j [\mathbf{\Delta}_2(n;j) + \mathbf{\Delta}_2(n;j-1)]$$

Re-indexing the first two terms in a similar manner to Theorem 2.2.4, we can write

$$\sum_{j=0}^{n+1}(-1)^j\mathbf{\Delta}_2(n+1;j) =$$
$$(-1)(-1)^n\mathbf{\Delta}_2(n;n)+\mathbf{\Delta}_2(n;0)+\sum_{j=1}^{n}(-1)^j[\mathbf{\Delta}_2(n;j)+\mathbf{\Delta}_2(n;j-1)]$$

Next, split up the sum into two separate alternating sums to see

$$\sum_{j=0}^{n+1}(-1)^j\mathbf{\Delta}_2(n+1;j) =$$
$$(-1)(-1)^n\mathbf{\Delta}_2(n;n)+\mathbf{\Delta}_2(n;0)+\sum_{j=1}^{n}(-1)^j\mathbf{\Delta}_2(n;j)+\sum_{j=1}^{n}(-1)^j\mathbf{\Delta}_2(n;j-1).$$

Re-index the second of these sums by letting $k = j-1$ to see

$$\sum_{j=0}^{n+1}(-1)^j\mathbf{\Delta}_2(n+1;j) =$$
$$(-1)(-1)^n\mathbf{\Delta}_2(n;n)+\mathbf{\Delta}_2(n;0)+\sum_{j=1}^{n}(-1)^j\mathbf{\Delta}_2(n;j)-\sum_{k=0}^{n-1}(-1)^k\mathbf{\Delta}_2(n;k).$$

Finally, condensing terms into each sum as before we have

$$\sum_{j=0}^{n+1}(-1)^j\mathbf{\Delta}_2(n+1;j)=\sum_{j=0}^{n}(-1)^j\mathbf{\Delta}_2(n;j)-\sum_{k=0}^{n}(-1)^k\mathbf{\Delta}_2(n;k).$$

By our induction hypothesis each of these sums is exactly 0, so their difference is as well. Hence, the result holds for all positive integers greater than 1. □

In a surprising turn, we can see that the Hockey Stick identity from $\mathbf{\Delta}$ carries over. It is not obvious this property would be preserved since we broke the symmetry of $\mathbf{\Delta}$ when constructing $\mathbf{\Delta}_2$. The author had not noticed that the Lucas Triangle, despite the symmetry breaking, still satisfies this property. We thank Dr. Xiaoya Zha from MTSU and Dr. Mark Ellingham from Vanderbilt for noticing this.

Theorem 2.2.6. *For $n, k \in \mathbb{Z}^+$, we have*

$$\sum_{j=0}^{k}\mathbf{\Delta}_2(n+j;j)=\mathbf{\Delta}_2(n+k+1;k)$$

Proof. Fix $n \in \mathbb{Z}^+$. We prove by induction on k. For the case $k = 0$, the result is obvious since $\mathbf{\Delta}_2(n, 0) = \mathbf{\Delta}(n + 1, 0)$.

So, suppose the result for all positive integers less than or equal to k and consider the sum $\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n + j; j)$. From this, we can pull out the final term of our sum to see:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n + j; j) = \mathbf{\Delta}_2(n + k + 1; k + 1) + \sum_{j=0}^{k} \mathbf{\Delta}_2(n + j; j)$$

Observe that by applying our induction hypothesis we can say

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n + j; j) = \mathbf{\Delta}_2(n + k + 1; k + 1) + \mathbf{\Delta}_2(n + k + 1; k)$$

Finally, apply the Pascal-type rule for $\mathbf{\Delta}_2$ to achieve:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n + j; j) = \mathbf{\Delta}_2(n + k + 2; k + 1)$$

Hence, the result is true for all positive integers k. $\square$

We should take a moment here to note that the theorem above is not quite enough to claim that $\mathbf{\Delta}_2$ satisfies the Hockey Stick identity. In $\mathbf{\Delta}$, we only had to consider Hockey Sticks starting at either the left-most or right-most diagonals because of the symmetry of the elements of $\mathbf{\Delta}$. Now that the symmetry of $\mathbf{\Delta}$ is broken, to be thorough we must check that the Hockey Stick property holds even when we start on the right-most diagonal. Surprisingly, the following theorem tells us that we need not fret.

Theorem 2.2.7. *For $n, k \in \mathbb{Z}^+$, we have*

$$\sum_{j=0}^{k} \mathbf{\Delta}_2(n + j; n) = \mathbf{\Delta}_2(n + k + 1; n + 1)$$

Proof. Fix $n \in \mathbb{Z}^+$. We prove by induction on k. For the base case, when $k = 1$, we see that the result holds. Indeed, we have

$$\sum_{j=0}^{1} \mathbf{\Delta}_2(n + j; n) = \mathbf{\Delta}_2(n; n) + \mathbf{\Delta}_2(n + 1; n) = \mathbf{\Delta}_2(n + 2; n + 1).$$

Now, suppose the result holds for all positive integers up to k and consider the sum

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n+j;n)$$

We can reduce this sum by pulling out the final term to see

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n+j;n) = \mathbf{\Delta}_2(n+k+1;n) + \sum_{j=0}^{k} \mathbf{\Delta}_2(n+j;n)$$

Next, from the induction hypothesis we have:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n+j;n) = \mathbf{\Delta}_2(n+k+1;n) + \mathbf{\Delta}_2(n+k+1;n+1)$$

Last, we can apply the Pascal-Type rule for $\mathbf{\Delta}_2$ to achieve the desired result. That is:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_2(n+j;n) = \mathbf{\Delta}_2(n+k+1;n) + \mathbf{\Delta}_2(n+k+1;n+1) = \mathbf{\Delta}_2(n+k+2;n+1)$$

Hence, by the Principle of Mathematical Induction, the result holds for all non-negative integers k. □

From the previous two theorems, we see that the Lucas Triangle behaves in a way very similar to The Arithmetic Triangle. This shouldn't come as too much of a surprise since they are built from the same recurrence relation, only with different seed values. Last, before moving on to alternating products of these more general arithmetic triangle coefficients, we show that the Lucas sequence appears in the Lucas Triangle in the same way that the Fibonacci sequence appears in Pascal's Triangle.

We recall that the Lucas sequence is given by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{23}$$

for all $n > 2$ with initial values $L_2 = 1$ and $L_1 = 2$. The first few terms of the Lucas sequence are

$$\{2, 1, 3, 4, 7, 11, 18, 29, 47, 76\}.$$

Theorem 2.2.8. *The nth Lucas number, L_n is given the sum of the shallow diagonals of $\mathbf{\Delta}_2$. i.e.*

$$L_n = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \mathbf{\Delta}_2(n-(k+1);k). \tag{24}$$

for all $n > 1$

Proof. This proof follows the same strategy as that of Theorem 1.1.7.

We show by induction on n. For the cases $n = 1$ and $n = 2$, we have

$$\sum_{k=0}^{0} \mathbf{\Delta}_2(1-(k+1);k) = \mathbf{\Delta}_2(0;0) = 2 = L_1$$

$$\sum_{k=0}^{0} \mathbf{\Delta}_2(2-(k+1);k) = \mathbf{\Delta}_2(1;0) = 1 = L_2$$

Thus, the result holds for $n = 1$ and $n = 2$.

Now, suppose the result holds for some arbitrary integer n and $n+1$ with n odd. That is, we have

$$L_n = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \mathbf{\Delta}_2(n-(k+1);k)$$

and

$$L_{n+1} = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \mathbf{\Delta}_2(n+1-(k+1);k).$$

Now, consider the sum

$$\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \mathbf{\Delta}_2(n+2-(k+1);k) = \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \mathbf{\Delta}_2(n+1-k;k)$$

We can break this sum up in the same way done previously as

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) =$$

$$\mathbf{\Delta}_2\left(n+1-\left(\frac{n+1}{2}\right);\left(\frac{n+1}{2}\right)\right) + \mathbf{\Delta}_2(n+1;0) + \sum_{k=1}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n+1-k;k)$$

Simplifying we see

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) = \mathbf{\Delta}_2\left(\frac{n+1}{2};\frac{n+1}{2}\right) + \mathbf{\Delta}_2(n+1;0) + \sum_{k=1}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n+1-k;k)$$

Now, using the recurrence relation for $\mathbf{\Delta}_2$ we have

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) =$$

$$\mathbf{\Delta}_2\left(\frac{n+1}{2};\frac{n+1}{2}\right) + \mathbf{\Delta}_2(n+1;0) + \sum_{k=1}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n-k;k) + \sum_{k=1}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n-k;k-1)$$

Re-indexing the isolated terms in the way discussed previously, and re-indexing the second sum by letting $j = k-1$ we have

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) =$$

$$\mathbf{\Delta}_2\left(\frac{n-1}{2};\frac{n-1}{2}\right) + \mathbf{\Delta}_2(n;0) + \sum_{k=1}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n-k;k) + \sum_{j=0}^{\frac{n-1}{2}-1} \mathbf{\Delta}_2(n-j-1;j).$$

Finally, pulling the first two terms back in we have

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) = \sum_{k=0}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n-k;k) + \sum_{j=0}^{\frac{n-1}{2}} \mathbf{\Delta}_2(n-j-1;j).$$

We recognize these as precisely L_{n+1} and L_n respectively. To conclude we see

$$\sum_{k=0}^{\frac{n+1}{2}} \mathbf{\Delta}_2(n+1-k;k) = L_{n+1} + L_n = L_{n+2}.$$

Hence, the result holds for all positive integers n.

$\square$

Now, we turn our attention to alternating products of rows of the Lucas Triangle. As before, first we list the terms $a_n = \prod_{j=0}^{n} [\mathbf{\Delta}_2(n;j)^{(-1)^j}]$ to see if there are any obvious patterns. The first 10 values for a_n are given by

$$\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{5}{8}, \frac{18}{35}, \frac{21}{32}, \frac{100}{231}, \frac{429}{540}, \frac{490}{1287}, \frac{2431}{3584} \ldots\right\}$$

Immediately we notice a difference in these values and the corresponding values from $\mathbf{\Delta}$. No longer do we have that the alternating product of an odd-indexed row is identically 1. It seems that the broken symmetry of $\mathbf{\Delta}_2$ generates a new sequence of rational numbers to investigate. As before we consider the even-indexed and odd-indexed rows separately to look for patterns. .

Focusing our attention first on the rows with even index, we consider the alternating product of each row. That is, we wish to look at:

$$\prod_{j=0}^{2k} [\mathbf{\Delta}_2(2k; j)]^{(-1)^j} \tag{25}$$

Before diving in straight away, we need to make explicit the relationship between generalized right arithmetic triangle coefficients and the arithmetic triangle coefficients.

Lemma 2.2.9. *For any positive integer α, we have that*

$$\mathbf{\Delta}_\alpha(n+1; k) = \mathbf{\Delta}(n; k) + (\alpha - 1) \cdot \mathbf{\Delta}(n-1; k-1) \tag{26}$$

Proof. We show by using Pascal's rule and the Pascal-type relation for $\mathbf{\Delta}_\alpha$. Fix a positive integer α. We induct on the value of n for any k. For the base case, when $n = 0$, we have

$$\mathbf{\Delta}_\alpha(1; k) = \mathbf{\Delta}_\alpha(0; k) + \mathbf{\Delta}_\alpha(0; k-1)$$

If $k = 0$ then the result holds obviously. If $k = 1$, then we have

$$\mathbf{\Delta}_\alpha(1; 1) = \alpha = 1 + (\alpha - 1) = \mathbf{\Delta}(1; 1) + (\alpha - 1) \cdot \mathbf{\Delta}(1; 0)$$

Thus, the result holds for the $n = 0$ case.

Now, suppose the result holds for all integers less than or equal to n and consider the $n + 1$ case.

Using the Pascal-type rule for $\mathbf{\Delta}_\alpha$, we have

$$\mathbf{\Delta}_\alpha(n+1;k) = \mathbf{\Delta}_\alpha(n;k) + \mathbf{\Delta}_\alpha(n;k-1).$$

Applying our induction hypothesis we can see:

$$\mathbf{\Delta}_\alpha(n+1;k) = \mathbf{\Delta}(n;k)+(\alpha-1)\mathbf{\Delta}(n-1;k-1)+\mathbf{\Delta}(n;k-1)+(\alpha-1)\mathbf{\Delta}(n-1;k-2).$$

Now, if we collect terms and apply Pascal's Rule we see:

$$\mathbf{\Delta}_\alpha(n+1;k) = \mathbf{\Delta}(n+1;k) + (\alpha-1)\mathbf{\Delta}(n;k-1)$$

Hence, the result for all integers n. □

We give an even stronger characterization of the relationship between a right Arithmetic Triangle $\mathbf{\Delta}_\alpha$ and Pascal's Triangle $\mathbf{\Delta}$ than lemma 2.2.9. This will be very useful in our study of alternating products of more general Arithmetic Triangles, as it plays works better with products than does the characterization given in lemma 2.2.9.

Lemma 2.2.10. *For all positive integers α and integers $n, k > 1$ we have*

$$\mathbf{\Delta}_\alpha(n;k) = \left(\frac{n+k(\alpha-1)}{n}\right)\mathbf{\Delta}(n;k) \tag{27}$$

Proof. We show by direct computation using the properties of Generalized Right Arithmetic Triangle Coefficients. Recall the definition of Generalized Right Arithmetic Triangle coefficients from Definition 2.2.1. From lemma 2.2.9 we can break $\mathbf{\Delta}_\alpha(n;k)$ up as

$$\mathbf{\Delta}_\alpha(n;k) = \mathbf{\Delta}(n;k) + (\alpha-1)\cdot\mathbf{\Delta}(n-1;k-1)$$

Using standard properties of the binomial coefficients we have that

$$\mathbf{\Delta}(n-1;k-1) = \frac{k}{n}\cdot\mathbf{\Delta}(n;k)$$

Thus, we now have

$$\mathbf{\Delta}_\alpha(n;k) = \mathbf{\Delta}(n;k) + (\alpha-1)\cdot\frac{k}{n}\cdot\mathbf{\Delta}(n;k)$$

Simplifying we achieve the desired result:

$$\mathbf{\Delta}_\alpha(n;k) = \mathbf{\Delta}(n;k) \cdot \left(\frac{n - k(\alpha - 1)}{n}\right)$$

□

We develop one final tool before calculating the alternating product of the rows of the Lucas Triangle.

Lemma 2.2.11. *For all integers $k \geq 0$, we have:*

$$\prod_{j=0}^{2k} \left[\frac{2k+j}{2k}\right]^{(-1)^j} = \frac{(4k)!!}{(4k-1)!!} \frac{(2k-1)!!}{(2k)!!} \tag{28}$$

Proof. We prove by inducting on the value of k.

Base case $k = 1$. Computing on the left we have $\displaystyle\prod_{j=0}^{2k} \left[\frac{2k+j}{2k}\right]^{(-1)^j} = 1 \cdot \frac{2}{3} \cdot 2 = \frac{4}{3}$.

Computing on the right we have $\dfrac{8}{3} \cdot \dfrac{1}{2} = \dfrac{4}{3}$. Hence the result holds for $k = 1$.

Now, assume the result holds for $k = n$. Consider the product $\displaystyle\prod_{j=0}^{2(n+1)} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j}$.

First, we can decompose this product as:

$$\prod_{j=0}^{2(n+1)} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} =$$

$$\left[\frac{2(n+1)+2(n+1)}{2(n+1)}\right] \cdot \left[\frac{2(n+1)+2n+1}{2(n+1)}\right]^{(-1)} \cdot \prod_{j=0}^{2n} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j}$$

$$\prod_{j=0}^{2(n+1)} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} = \left[\frac{4n+4}{2n+2}\right] \cdot \left[\frac{2n+2}{4n+3}\right] \cdot \prod_{j=0}^{2n} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j}$$

$$\prod_{j=0}^{2(n+1)} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} = \left[\frac{4n+4}{4n+3}\right] \prod_{j=0}^{2n} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j}$$

Now, we focus our attention on the rightmost product. We'd like to simplify it in such a way so that it is some multiple of the sum in the $k = n$ case. Observe that we can make the following decomposition:

$$\prod_{j=0}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} = \frac{\prod_{j\text{ even}}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right]}{\prod_{j\text{ odd}}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right]}$$

This decomposition into even and odd indices of j gives a better idea of what is happening in this product. Focusing on the product over even j, we see:

$$\prod_{j\text{ even}}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right] = \frac{(4n+2)!!}{(2n+2)^{n+1}(2n)!!}$$

Likewise for the product over the odd values of j we have:

$$\prod_{j\text{ odd}}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right] = \frac{(4n+1)!!}{(2n+2)^{n}(2n+1)!!}$$

Thus, by combining these and simplifying we have:

$$\prod_{j=0}^{2n}\left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} = \frac{(4n+2)!!}{(4n+1)!!}\frac{(2n+1)!!}{(2n+2)!!}$$

Finally, combining this with the intermediate result above we obtain:

$$\prod_{j=0}^{2(n+1)}\left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j} = \left[\frac{4n+4}{4n+3}\right]\frac{(4n+2)!!}{(4n+1)!!}\frac{(2n+1)!!}{(2n+2)!!} =$$
$$\frac{(4(n+1))!!}{(4(n+1)-1)!!}\frac{(2(n+1)-1)!!}{(2(n+1))!!}$$

Ergo, the result is shown for all $k \in \mathbb{W}$. □

Now we are ready to consider the alternating product of of the rows of the Lucas Triangle.

Theorem 2.2.12. *For all non-negative integers of the form $2k$ we have:*

$$\prod_{j=0}^{2k}\left[\boldsymbol{\Delta}_2(2k;j)\right]^{(-1)^j} = \frac{(4k)!!}{(4k-1)!!}\left[\frac{(2k-1)!!}{(2k)!!}\right]^2 \tag{29}$$

Proof. We prove by inducting on the value of k. The result holds easily when $k=1$. Now, assume the result holds for $k=n$ and consider the product:

$$\prod_{j=0}^{2(n+1)}\left[\boldsymbol{\Delta}_2(2k;j)\right]^{(-1)^j}$$

We can decompose this product into two quantities for which we know the alternating product. First we can apply lemma 2.2.10 to rephrase the right arithmetic triangle coefficient as a product of a term with an element from $\mathbf{\Delta}$. Thus:

$$\prod_{j=0}^{2(n+1)} [\mathbf{\Delta}_2(2(n+1);j)]^{(-1)^j} = \prod_{j=0}^{2(n+1)} \left[\mathbf{\Delta}(2(n+1);j)\left(\frac{2(n+1)+j}{2(n+1)}\right)\right]^{(-1)^j}$$

$$\prod_{j=0}^{2(n+1)} [\mathbf{\Delta}_2(2(n+1);j)]^{(-1)^j} = \prod_{j=0}^{2(n+1)} [\mathbf{\Delta}(2(n+1);j)]^{(-1)^j} \cdot \prod_{j=0}^{2(n+1)} \left[\frac{2(n+1)+j}{2(n+1)}\right]^{(-1)^j}$$

Now, we can apply lemma 2.2.11 to the product on the right to see:

$$\prod_{j=0}^{2(n+1)} [\mathbf{\Delta}_2(2(n+1);j)]^{(-1)^j} =$$

$$\prod_{j=0}^{2(n+1)} [\mathbf{\Delta}(2(n+1);j)]^{(-1)^j} \cdot \left(\frac{(4(n+1))!!}{(4(n+1)-1)!!}\frac{(2(n+1)-1)!!}{(2(n+1))!!}\right)$$

From our previous result on alternating products of binomial coefficients, Theorem 2.1.5, we have:

$$\prod_{j=0}^{2(n+1)} [\mathbf{\Delta}_2(2(n+1);j)]^{(-1)^j} = \left(\frac{(2(n+1)-1)!!}{(2(n+1))!!}\right)\cdot\left(\frac{(4(n+1))!!}{(4(n+1)-1)!!}\frac{(2(n+1)-1)!!}{(2(n+1))!!}\right)$$

Ergo, our result is shown. □

In Pascal's Triangle we were only interested in the rows of even index, since the alternating product of any row with odd index is always 1. However, this is not the case for the more general Arithmetic Triangles we have just considered. By inspecting the odd-indexed rows we arrive at the following theorem.

Theorem 2.2.13. *For all integers of the form $2k+1$ we have:*

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}_2(2k+1;j)]^{(-1)^j} = \frac{(2k)!_2(4k+1)!_2}{(2k-1)!_2(4k+2)!_2} \tag{30}$$

Proof. We prove by induction on k. For the case when $k=0$, we have

$$\prod_{j=0}^{1} [\mathbf{\Delta}_2(1;j)]^{(-1)^j} = \mathbf{\Delta}_2(1;0)\left(\mathbf{\Delta}_2(1;1)\right)^{(-1)^1} = \frac{1}{2} = \frac{(2*0)!_2(4*0+1)!_2}{(2*0-1)!_2(4*0+2)!_2}.$$

This shows the result holds in the case $k = 0$.

Now, assume the result holds for the case $k = n$ and consider the product

$$\prod_{j=0}^{2(k+1)+1} [\mathbf{\Delta}_2(2(k+1)+1;j)]^{(-1)^j} = \prod_{j=0}^{2k+3} [\mathbf{\Delta}_2(2k+3;j)]^{(-1)^j}$$

We can decompose this product in a similar way to the previous theorem. That is, we can say

$$\prod_{j=0}^{2k+3} [\mathbf{\Delta}_2(2k+3;j)]^{(-1)^j} = \prod_{j=0}^{2k+3} \left[\mathbf{\Delta}(2k+3;j)\left(\frac{2k+3+j}{2k+3}\right)\right]^{(-1)^j}$$

But, we recall from our investigation into $\mathbf{\Delta}$, that the alternating product of the odd-indexed rows is identically 1. Thus, we can reduce this further to see that

$$\prod_{j=0}^{2k+3} [\mathbf{\Delta}_2(2k+3;j)]^{(-1)^j} = \prod_{j=0}^{2k+3} \left[\frac{2k+3+j}{2k+3}\right]^{(-1)^j}$$

Now, following the same pattern of tricks as lemma 2.2.11, we can say that

$$\prod_{j=0}^{2k+3} \left[\frac{2k+3+j}{2k+3}\right]^{(-1)^j} = \frac{(2k+2)!!(4k+5)!!}{(2k+1)!!(4k+6)!!}$$

Thus, we have the result. So, we conclude that the alternating product of an odd-indexed row of $\mathbf{\Delta}_2$ is given by:

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}_2(2k+1;j)]^{(-1)^j} = \frac{(2k)!!(4k+1)!!}{(2k-1)!!(4k+2)!!}$$

□

Remark. The two previous theorems give a complete characterization for the alternating products of rows of $\mathbf{\Delta}_2$. We have seen is that there is a connection between the alternating products of $\mathbf{\Delta}$ and the alternating products of $\mathbf{\Delta}_2$. We might expect this since we have seen that the entries in $\mathbf{\Delta}_2$ are formed as weighted sums of entries from $\mathbf{\Delta}$, but it is not always clear how multiplication and addition play together.

To make this connection more explicit, notice that if we let $X_{2k} = \dfrac{(2k-1)!!}{(2k)!!}$, then we have that the alternating product of the $2k$ row of $\mathbf{\Delta}_2$ is a multiple of X_{2k}. Even more interesting, the alternating product of the $2k+1$ row of $\mathbf{\Delta}_2$ is a multiple of $\dfrac{1}{X_{2k}}$

When making the jump from Pascal's Triangle to the Lucas Triangle, we replaced the right-most entry in the first row with a 2. A very natural question to ask at this point would be, what happens if we replace the right-most entry with some other integer? We have already seen that this is possible in that the way Generalized Arithmetic Triangle Coefficients are defined lends itself well to considering arbitrary Arithmetic Triangles.

Thus, we are led to wonder if we can give a complete characterization for an arbitrary Arithmetic Triangle. That is, we wish to give a closed form expression for

$$\prod_{j=0}^{n} [\mathbf{\Delta}_\alpha(n;j)]^{(-1)^j} \tag{31}$$

For any positive integer α and non-negative integer n. Previously we have seen that we have to consider two cases: one when n is even and another when n is odd. Before considering this conjecture in its most general form, we exhibit one more example of the method used, namely in the cases where $\alpha = 3$. A few technical lemmas in the same vein as lemma 2.2.11 included in Appendix 1 are used throughout the following discussions.

First, we consider the case when $\alpha = 3$. We see that this generates the Arithmetic Triangle given in Figure 5.

By examining the Arithmetic Triangle generated in Figure 5, we can see that the same sorts of patterns are apparent in $\mathbf{\Delta}_3$ as those in both $\mathbf{\Delta}_2$ and $\mathbf{\Delta}$. As before, we give a listing of the sequence generated by taking alternating products of rows of $\mathbf{\Delta}_3$. The first 10 terms of this sequence, and all other alternating product sequences can be found in Appendix 2. Again, we notice the disparity between even-indexed rows

$n=0$										3								
$n=1$									1		3							
$n=2$								1		4		3						
$n=3$							1		5		7		3					
$n=4$						1		6		12		10		3				
$n=5$					1		7		18		22		13		3			
$n=6$				1		8		25		40		35		16		3		
$n=7$			1		9		33		65		75		51		19		3	
$n=8$		1		10		42		98		140		126		70		22		3

Figure 5: Arithmetic Triangle $\mathbf{\Delta}_3$

and odd-indexed rows. Thus, we have to separate cases, but each follows easily from the work we have already done.

Theorem 2.2.14. *For all positive integers of the form $2k$ we have:*

$$\prod_{j=0}^{2k} [\mathbf{\Delta}_3(2k;j)]^{(-1)^j} = \frac{(2k-1)!_2}{(2k)!_2} \frac{(3k)!_2}{(3k-1)!_2} \frac{(k-1)!_2}{(k)!_2}. \tag{32}$$

Proof. The result follows easily from Theorem 2.1.5 and lemma A.1.10. Observe:

$$\prod_{j=0}^{2k} [\mathbf{\Delta}_3(2k;j)]^{(-1)^j} = \prod_{j=0}^{2k} \left[\mathbf{\Delta}(2k;j)\left(\frac{2k+2j}{2k}\right)\right]^{(-1)^j}$$

Thus, simply decompose this product into two products and apply Theorem 2.1.5 and lemma A.1.10 to obtain the result. □

Theorem 2.2.15. *For all positive integers of the form $2k+1$, we have:*

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}_3(2k+1;j)]^{(-1)^j} = \frac{(6k+1)!_4}{(6k+3)!_4} \cdot \frac{(2k-1)!_4}{(2k-3)!_4} \tag{33}$$

Proof. The proof follows quickly from Theorem 2.1.5 and technical lemma A.1.11 after decomposing as:

$$\prod_{j=0}^{2k+1} [\mathbf{\Delta}_3(2k+1;j)]^{(-1)^j} = \prod_{j=0}^{2k+1} \left[\mathbf{\Delta}(2k+1;j) \cdot \left(\frac{2k+1+2j}{2k+1}\right)\right]^{(-1)^j}$$

□

2.3 Generalized Right Arithmetic Triangles

Now that we've seen a few examples of how the alternating product construction can be extended to more than simply $\mathbf{\Delta}$, we are ready to look at a very general case. Namely, instead of specifying a value for α, we treat it as any positive integer. Thus, we consider the Arithmetic Triangle given in Figure 6.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & \alpha & & & & \\
 & & & 1 & & 1+\alpha & & \alpha & & & \\
 & & 1 & & 2+\alpha & & 1+2\alpha & & \alpha & & \\
 & 1 & & 3+\alpha & & 3+3\alpha & & 1+3\alpha & & \alpha & \\
1 & & 4+\alpha & & 3+2\alpha & & 2+3\alpha & & 1+4\alpha & & \alpha
\end{array}
$$

Figure 6: Arithmetic Triangle $\mathbf{\Delta}_\alpha$

A great deal of insights may be gained by looking at $\mathbf{\Delta}_\alpha$; in Section 3.1 we study $\mathbf{\Delta}_\alpha$ from the perspective of order theory, and in Section 3.2 we study $\mathbf{\Delta}_\alpha$ from a purely algebraic point of view.

One observation we can make immediately is that while $\mathbf{\Delta}_\alpha$ does not have the same type of symmetry as $\mathbf{\Delta}$, it does have some sort of internal symmetry. Viewing $\mathbf{\Delta}_\alpha$ in this way makes apparent the observation that we can write each entry in $\mathbf{\Delta}_\alpha$ as a weighted sum of elements from $\mathbf{\Delta}$.

Before exhibiting the generalized alternating product formulae, we take a moment to show that those properties that make $\mathbf{\Delta}$ interesting are still valid in this general case.

2.3.1 Properties of Generalized Right Arithmetic Triangles

We saw that the sum of any row of elements of $\mathbf{\Delta}$ was always a power of two. In the triangle $\mathbf{\Delta}_\alpha$ we have a similar property, but it is modulated by a factor of $(1+\alpha)$.

Observation 2.3.1. *For all positive integers n, α, we have:*

$$\sum_{j=0}^{n} \mathbf{\Delta}_\alpha(n;j) = 2^{n-1}(1+\alpha) \tag{34}$$

Proof. As is typical, we show by induction. For the case $n=1$, we have

$$\sum_{j=0}^{1} \mathbf{\Delta}_\alpha(1;j) = \mathbf{\Delta}_\alpha(1;0) + \mathbf{\Delta}_\alpha(1;1) = 1+\alpha = 2^{1-1}(1+\alpha)$$

Thus, the result holds in the base case. Now, suppose the result holds for some positive integer n and consider the sum:

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_\alpha(n+1;j)$$

First, we can apply the Pascal-Type rule for $\mathbf{\Delta}_\alpha$ to see:

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_\alpha(n+1;j) = \sum_{j=0}^{n+1} \mathbf{\Delta}_\alpha(n;j) + \mathbf{\Delta}_\alpha(n;j-1)$$

Next, split up the sum and re-index the second sum to see

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_\alpha(n+1;j) = \sum_{j=0}^{n} \mathbf{\Delta}_\alpha(n;j) + \sum_{j=0}^{n} \mathbf{\Delta}_\alpha(n;j)$$

Last, apply our induction hypothesis on each of these sums separately to see:

$$\sum_{j=0}^{n+1} \mathbf{\Delta}_\alpha(n+1;j) = 2^{n-1}(1+\alpha) + 2^{n-1}(1+\alpha) = 2^{n}(1+\alpha)$$

Hence, the result holds for all positive integers. □

One of the surprising, yet useful, properties of $\mathbf{\Delta}$ concerns the alternating sum of a row of elements of $\mathbf{\Delta}$. Due to the internal symmetry of $\mathbf{\Delta}$, the alternating sum of any row except the first is identically zero. The following property of $\mathbf{\Delta}_\alpha$ is a direct analogue to this observation. However, note that in $\mathbf{\Delta}_\alpha$ all but the first two rows are identically zero. This might not be what one would expect since we broke the symmetry that was seemingly the key to showing this property holds in $\mathbf{\Delta}$.

Observation 2.3.2. *For all integers $n \geq 2$, we have:*

$$\sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j) = 0 \tag{35}$$

Proof. First, we show this claim holds when $n = 2$. Namely, we have

$$\sum_{j=0}^{2}(-1)^j \mathbf{\Delta}_\alpha(2;j) = \mathbf{\Delta}_\alpha(2;0) - \mathbf{\Delta}_\alpha(2;1) + \mathbf{\Delta}_\alpha(2;2)$$

If we write down explicitly these elements, we see:

$$\sum_{j=0}^{2}(-1)^j \mathbf{\Delta}_\alpha(2;j) = 1 - (1+\alpha) + \alpha = 0$$

Thus, the result holds for $n = 2$.

Now, suppose the claim holds for some positive integer $n > 2$ and consider the sum

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j)$$

To decompose this sum, we can pull out the last term and then apply the Pascal-Type rule to simplify our summand. That is:

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j) = (-1)^{n+1}\mathbf{\Delta}_\alpha(n;n+1) + \sum_{j=0}^{n}(-1)^j\left[\mathbf{\Delta}_\alpha(n;j) + \mathbf{\Delta}_\alpha(n;j-1)\right]$$

Remembering that if $k > n$, then $\mathbf{\Delta}_\alpha(n;k) = 0$ and splitting this sum up we now have:

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j) = \sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j) + \sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j-1)$$

By re-indexing the second sum we now see that

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j) = \sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j) + \sum_{j=0}^{n}(-1)^{j+1} \mathbf{\Delta}_\alpha(n;j)$$

Simplifying more we can then use our induction hypothesis.

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j) = \sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j) - \sum_{j=0}^{n}(-1)^j \mathbf{\Delta}_\alpha(n;j)$$

$$\sum_{j=0}^{n+1}(-1)^j \mathbf{\Delta}_\alpha(n+1;j) = 0 - 0 = 0$$

Hence, the claim holds for all positive integers $n > 2$. □

The final property of $\mathbf{\Delta}_\alpha$ we wish to show is the Hockey Stick Identity. Just as was the case for $\mathbf{\Delta}_2$ we must consider two cases: whether we begin on the left-most diagonal or the right-most diagonal. We begin by looking at a Hockey Stick beginning on the left-most diagonal.

Theorem 2.3.3. *For positive integers n, k, and α, we have:*

$$\sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n+k+1;k) \tag{36}$$

Proof. Fix $n \in \mathbb{Z}^+$. We show the statement holds by induction on k.
For the base case $k = 1$, we have:

$$\sum_{j=0}^{1} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n;0) + \mathbf{\Delta}_\alpha(n+1;1)$$

We know that for any $m, l \in \mathbb{Z}^+$ that $\mathbf{\Delta}_\alpha(m;0) = \mathbf{\Delta}_\alpha(l;0)$, thus:

$$\sum_{j=0}^{1} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n+1;0) + \mathbf{\Delta}_\alpha(n+1;1) = \mathbf{\Delta}(n+2;1)$$

Where we used the Pascal-Type rule for $\mathbf{\Delta}_\alpha$ in the last equality. Thus, the result holds in the base case.

Now, suppose the result holds for some integer k. That is, we assume for k that

$$\sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n+k+1;k).$$

Consider the $k+1$ case. We see that

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n+k+1;k+1) + \sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;j)$$

Finally, using the induction hypothesis and the Pascal-Type rule for $\mathbf{\Delta}_\alpha$ we have:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_\alpha(n+j;j) = \mathbf{\Delta}_\alpha(n+k+1;k+1) + \mathbf{\Delta}_\alpha(n+k+1;k) = \mathbf{\Delta}_\alpha(n+(k+1)+1;k+1)$$

Ergo, the result holds for all $k \in \mathbb{Z}^+$. $\square$

Next, we show the property holds when looking at a Hockey Stick beginning on the right-most diagonal.

Theorem 2.3.4. *For all positive integers n, k, and α we have:*

$$\sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;n) = \mathbf{\Delta}_\alpha(n+k+1;n+1) \tag{37}$$

Proof. We show this in an analogous manner as the left Hockey Stick identity.
Fix $n \in \mathbb{Z}^+$. We induct on k. For the case $k = 1$, we have

$$\sum_{j=0}^{1} \mathbf{\Delta}_\alpha(n+j;n) = \mathbf{\Delta}_\alpha(n;n) + \mathbf{\Delta}_\alpha(n+1;n) = \mathbf{\Delta}_\alpha(n+1;n+1) + \mathbf{\Delta}_\alpha(n+1;n) =$$
$$\mathbf{\Delta}_\alpha(n+2;n+1)$$

In the above calculation we used the Pascal-Type rule for $\mathbf{\Delta}_\alpha$ and the trick used in Theorem 2.3.3. We have then that the result holds for $k = 1$.

Now, suppose the result holds for some $k \in \mathbb{Z}^+$. That is, we take as induction hypothesis that

$$\sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;n) = \mathbf{\Delta}_\alpha(n+k+1;n+1)$$

Consider the $k+1$ case . We then have:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_\alpha(n+j;n) = \mathbf{\Delta}_\alpha(n+k+1;n) + \sum_{j=0}^{k} \mathbf{\Delta}_\alpha(n+j;n)$$

Applying our induction hypothesis we see that:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_{\alpha}(n+j;n) = \mathbf{\Delta}_{\alpha}(n+k+1;n) + \mathbf{\Delta}_{\alpha}(n+k+1;n+1)$$

Finally, applying the Pascal-Type rule we achieve the result:

$$\sum_{j=0}^{k+1} \mathbf{\Delta}_{\alpha}(n+j;n) = \mathbf{\Delta}_{\alpha}(n+(k+1)+1;n+1)$$

□

After spending some time with the object $\mathbf{\Delta}_{\alpha}$ we are ready to answer the big question: What do alternating products in $\mathbf{\Delta}_{\alpha}$ look like? As before, we can list the terms $a_n = \prod_{j=0}^{n} [\mathbf{\Delta}_{\alpha}(n;j)]^{(-1)^j}$ to discern a pattern. The first 7 values for a_n, starting with a_0, are

$$\left\{1, \frac{1}{\alpha}, \frac{\alpha}{1+\alpha}, \frac{1+2\alpha}{2\alpha+\alpha^2}, \frac{3\alpha+3\alpha^2}{3+10\alpha+3\alpha^2}, \frac{6+28\alpha+16\alpha^2}{16\alpha+28\alpha^2+6\alpha^3}, \frac{50\alpha+125\alpha^2+50\alpha^3}{50+310\alpha+310\alpha^2+50\alpha^3}\right\}$$

These values quickly become cumbersome to calculate by hand. There are a few interesting things we can notice simply with this small number of terms.

Note that if we consider each term a rational function in the variable α, then for each even term, the degree of the numerator and denominator are equal. However, the odd terms are rational functions where the numerator is one degree lower than the denominator.

Also, it is interesting to note that there seems to be some sort of internal symmetry to these terms. Take for instance $a_5 = \dfrac{6+28\alpha+16\alpha^2}{16\alpha+28\alpha^2+6\alpha^3}$. We see that the coefficients of the numerator and denominator are identical, just with the order reversed. That is, the leading term of the numerator becomes the constant term of the denominator.

With these few observations, we can now characterize alternating products of rows of the Arithmetic Triangle $\mathbf{\Delta}_{\alpha}$.

Theorem 2.3.5. *For all integers of the form $2k$ and values positive integers α, we have:*

$$\prod_{j=0}^{2k} [\boldsymbol{\Delta}_\alpha(2k;j)]^{(-1)^j} = \frac{(2k\alpha)!_{2(\alpha-1)}}{(2k\alpha-(\alpha-1))!_{2(\alpha-1)}} \frac{(2k-(\alpha-1))!_{2(\alpha-1)}}{(2k)!_{2(\alpha-1)}} \frac{(2k-1)!_2}{(2k)!_2} \tag{38}$$

Proof. To show this is rather simple in comparison to the original property for $\boldsymbol{\Delta}$. We show directly by utilizing Theorem 2.1.5 and lemma A.1.12.
First, note that we can perform the following decomposition.

$$\prod_{j=0}^{2k} [\boldsymbol{\Delta}_\alpha(2k;j)]^{(-1)^j} = \prod_{j=0}^{2k} \left[\left(\frac{2k+j(\alpha-1)}{2k}\right) \boldsymbol{\Delta}(2k;j)\right]^{(-1)^j}$$

We can now split this into two alternating products for which we already have closed form expressions. Namely,

$$\prod_{j=0}^{2k} [\boldsymbol{\Delta}_\alpha(2k;j)]^{(-1)^j} = \prod_{j=0}^{2k} \left[\frac{2k+j(\alpha-1)}{2k}\right]^{(-1)^j} \prod_{j=0}^{2k} [\boldsymbol{\Delta}(2k;j)]^{(-1)^j}$$

By applying lemma A.1.12 with $m = \alpha - 1$ and using Theorem 2.1.5 we have the desired result. $\square$

Similarly for the case when our row index is odd, we can give a closed-form expression. In this case, we employ the fact that for any row of odd-index in $\boldsymbol{\Delta}$ the alternating product is identically 1.

Theorem 2.3.6. *For all integers of the form $2k+1$ and positive integers α, we have:*

$$\prod_{j=0}^{2k+1} [\boldsymbol{\Delta}_\alpha(2k+1;j)]^{(-1)^j} = \frac{(2k\alpha+1))!_{2(\alpha-1)}(2k+1-(\alpha-1))!_{2(\alpha-1)}}{(2k+1-2(\alpha-1))!_{2(\alpha-1)}(\alpha(2k+1))!_{2(\alpha-1)}} \tag{39}$$

Proof. As with the case of even-indexed rows, we make short work of proving this gnarly claim. We prove directly by applying lemma A.1.13 and Theorem 2.1.3.
First, note that we can make the following decomposition.

$$\prod_{j=0}^{2k+1} [\boldsymbol{\Delta}_\alpha(2k+1;j)]^{(-1)^j} = \prod_{j=0}^{2k+1} \left[\frac{2k+1+j(\alpha-1)}{2k+1} \boldsymbol{\Delta}(2k+1;j)\right]^{(-1)^j}$$

As before, we can take advantage of properties of the product operator to split this into two more familiar alternating products. Namely:

$$\prod_{j=0}^{2k+1} [\boldsymbol{\Delta}_\alpha(2k+1;j)]^{(-1)^j} = \prod_{j=0}^{2k+1} \left[\frac{2k+1+j(\alpha-1)}{2k+1}\right]^{(-1)^j} \prod_{j=0}^{2k+1} [\boldsymbol{\Delta}(2k+1;j)]^{(-1)^j}$$

Last, apply lemma A.1.13 with $m = \alpha - 1$ and Theorem 2.1.3 to achieve the desired result. □

Thus, we have completely characterized the alternating product construction on any Right Arithmetic Triangle $\boldsymbol{\Delta}_\alpha$.

This leads us to another very natural question. What happens if we define Left Arithmetic Triangles also? Do the properties carry over in an obvious way? Is anything to be gained by considering the Left Arithmetic Triangle ${}_\alpha\boldsymbol{\Delta}$? We investigate these questions in the next section.

2.4 Generalized Left Arithmetic Triangles

First we define the Left Arithmetic Triangle coefficients. This way of defining the coefficients in ${}_\beta\boldsymbol{\Delta}$ is dual to how we defined coefficients in $\boldsymbol{\Delta}$.

Definition 2.4.1. *Left Arithmetic Triangle Coefficients*
For $\beta \geq 1$ we define:

$$ {}_\beta\boldsymbol{\Delta}(n;k) = \begin{cases} 0 & \text{if } n < |k| \\ \beta & \text{if } k = 0 \\ 1 & \text{if } n = k \\ {}_\beta\boldsymbol{\Delta}(n-1;k) + {}_\beta\boldsymbol{\Delta}(n-1;k-1) & \text{otherwise} \end{cases} \tag{40}$$

As in the case with the right Arithmetic Triangle, if we let $\beta = 1$, then we have $\mathbf{\Delta}$. All of the properties that we have proven hold for $\mathbf{\Delta}_\alpha$ also hold for ${}_\beta\mathbf{\Delta}$, however the indexing might be different. Instead of showing each property again, we can exhibit a bijection between $\mathbf{\Delta}_\alpha$ and ${}_\beta\mathbf{\Delta}$. Then, any property of $\mathbf{\Delta}_\alpha$ based on the Pascal-Type recurrence relation carries over naturally to ${}_\beta\mathbf{\Delta}$.

The bijection we shall define, ϕ, will also be an idempotent function. That is, for any value $\mathbf{\Delta}_\alpha(n;k)$ we have

$$\phi\left(\phi\left(\mathbf{\Delta}_\alpha(n;k)\right)\right) = \mathbf{\Delta}_\alpha(n;k) \tag{41}$$

This fact, combined with the observation that ϕ is bijective lets us call ϕ an isomorphism of recurrence relation. Not only does ϕ map a Left Arithmetic Triangle to a Right Arithmetic Triangle bijectively, but it also respects the structure of each triangle.

We shall say more about this function and functions similar to it in Section 3.2. There we take advantage of this symmetry to prove some properties related to The Arithmetic Triangle Group.

Definition 2.4.2. *We call ϕ the canonical isomorphism between the Left Arithmetic Triangle ${}_\alpha\mathbf{\Delta}$ and the Right Arithmetic Triangle $\mathbf{\Delta}_\alpha$. More specifically, we have:*

$$\phi : \mathbf{\Delta}_\alpha \to {}_\alpha\mathbf{\Delta} \tag{42}$$

$$\phi : \mathbf{\Delta}_\alpha(n;k) \mapsto {}_\alpha\mathbf{\Delta}(n;n-k) \tag{43}$$

Now, we show that ϕ respects the recurrence relation in each direction.

Theorem 2.4.3. *The function ϕ defined above is an isomorphism of recurrence relations. That is, for all $\mathbf{\Delta}_\alpha(n,k)$ we have*

$$\phi(\mathbf{\Delta}_\alpha(n;k)) = \phi(\mathbf{\Delta}_\alpha(n-1;k-1)) + \phi(\mathbf{\Delta}_\alpha(n-1;k)) \tag{44}$$

Proof. We show this from the definition of ϕ. First, note that

$$\phi(\mathbf{\Delta}_\alpha(1;0)) = {}_\alpha\mathbf{\Delta}(1;1)$$

and

$$\phi(\mathbf{\Delta}_\alpha(1;1)) = {}_\alpha\mathbf{\Delta}(1;0)$$

Thus, ϕ takes the seed values of $\mathbf{\Delta}_\alpha$ to the seed values of ${}_\alpha\mathbf{\Delta}$. So, we need only show that ϕ preserves the recurrence relation between elements of ${}_\alpha\mathbf{\Delta}$. That is, we wish to show

$$\phi(\mathbf{\Delta}_\alpha(n;k)) = \phi(\mathbf{\Delta}_\alpha(n-1;k-1)) + \phi(\mathbf{\Delta}_\alpha(n-1;k))$$

On the left we have:

$$\phi(\mathbf{\Delta}_\alpha(n;k)) = {}_\alpha\mathbf{\Delta}(n;n-k)$$

While on the right we have:

$$\phi(\mathbf{\Delta}_\alpha(n-1;k-1)) + \phi(\mathbf{\Delta}_\alpha(n-1;k)) = {}_\alpha\mathbf{\Delta}(n-1;n-k) + {}_\alpha\mathbf{\Delta}(n-1;n-k-1)$$

But, from the recurrence relation on ${}_\alpha\mathbf{\Delta}$, we have that

$${}_\alpha\mathbf{\Delta}(n-1;n-k) + {}_\alpha\mathbf{\Delta}(n-1;n-k-1) = {}_\alpha\mathbf{\Delta}(n;n-k)$$

Hence, ϕ preserves the recurrence relation on ${}_\alpha\mathbf{\Delta}$.
The same argument can be used to show that ϕ has a natural inverse that preserves the recurrence relation on $\mathbf{\Delta}_\alpha$ □

We can see that the canonical map ϕ preserves the Pascal-Type recurrence relation, and thus any property based on the recurrence relation is preserved under ϕ. In other words, ϕ preserves all positive properties of $\mathbf{\Delta}_\alpha$. This allows us to give the following corollaries. The proof for each is omitted, but can be seen as taking the corresponding proof for $\mathbf{\Delta}_\alpha$ and then looking at the image under ϕ.

2.4.1 Properties of Generalized Left Arithmetic Triangles

Corollary 2.4.4. *For positive integers n and β, we have:*

$$\sum_{j=0}^{n} {}_{\beta}\mathbf{\Delta}(n;j) = 2^{n-1}(\beta+1)$$

Corollary 2.4.5. *For all integers $n \geq 2$, we have:*

$$\sum_{j=0}^{n} (-1)^j {}_{\beta}\mathbf{\Delta}(n;j) = 0$$

Corollary 2.4.6. *For positive integers n, k and β, we have:*

$$\sum_{j=0}^{k} {}_{\beta}\mathbf{\Delta}(n+j;j) = {}_{\beta}\mathbf{\Delta}(n+k+1;k)$$

Corollary 2.4.7. *For positive integers n, k and β, we have:*

$$\sum_{j=0}^{k} {}_{\beta}\mathbf{\Delta}(n+j;n) = {}_{\beta}\mathbf{\Delta}(n+k+1;n+1)$$

We don't have it quite so easy if we wish to consider alternating products of the Left Arithmetic Triangle ${}_{\beta}\mathbf{\Delta}$ since there is no guarantee that the preservation of the Pascal-Type recurrence relation implies that the alternating sum formulae will be identical. Thus, we explicitly prove the two alternating sum formulae for ${}_{\beta}\mathbf{\Delta}$.

But, before doing so we need to make explicit the connection between the elements in ${}_{\beta}\mathbf{\Delta}$ and the elements in $\mathbf{\Delta}$. The following lemma has a familiar form, it serves the same purpose as lemma 2.2.9

Lemma 2.4.8. *For all positive integers n, k and β we have:*

$${}_{\beta}\mathbf{\Delta}(n;k) = \mathbf{\Delta}(n;k) + (\beta-1)\mathbf{\Delta}(n-1;k)$$

Proof. To show the veracity of this statement, we induct on the value n. When $n = 1$ and $k = 0$ we have:

$$_{\beta}\mathbf{\Delta}(1;0) = \beta = \mathbf{\Delta}(1;0) + (\beta-1)\mathbf{\Delta}(0;0)$$

Also, when $n = 1$ and $k = 0$ we have:

$$_{\beta}\mathbf{\Delta}(1;1) = 1 = \mathbf{\Delta}(1;1) + (\beta-1)\mathbf{\Delta}(0;1)$$

Hence, the result holds in the base case $n = 1$.

Next, suppose the result holds for all positive integers smaller than n and consider the $n+1$ case. Applying the Pascal-Type rule for $_{\beta}\mathbf{\Delta}$ we see:

$$_{\beta}\mathbf{\Delta}(n+1;k) = {}_{\beta}\mathbf{\Delta}(n;k) + {}_{\beta}\mathbf{\Delta}(n;k-1)$$

Now, applying the induction hypothesis twice we have:

$$_{\beta}\mathbf{\Delta}(n+1;k) = \mathbf{\Delta}(n;k) + (\beta-1)\mathbf{\Delta}(n-1;k) + \mathbf{\Delta}(n;k-1) + (\beta-1)\mathbf{\Delta}(n-1;k-1)$$

By rearranging we can say that

$$_{\beta}\mathbf{\Delta}(n+1;k) = \mathbf{\Delta}(n+1;k) + (\beta-1)\mathbf{\Delta}(n;k)$$

Ergo, the result holds for all integers n, k. $\square$

From this we obtain a very useful corollary in much the same way we obtained lemma 2.2.10 from lemma 2.2.9.

Corollary 2.4.9. *For all non-negative integers n, k and positive integers β we have:*

$$_{\beta}\mathbf{\Delta}(n;k) = \left(\frac{n + (n-k)(\beta-1)}{n}\right)\mathbf{\Delta}(n;k) \tag{45}$$

Now we are ready to look at the alternating product construction on $_{\beta}\mathbf{\Delta}$. As before, we must separate into even-indexed rows and odd-indexed rows. The major "beats" of these two theorems should be familiar by now.

Theorem 2.4.10. *For all integers of the form $2k$ we have that*

$$\prod_{j=0}^{2k} \left[{}_{\beta}\mathbf{\Delta}(2k;j)\right]^{(-1)^j} = \frac{(2k\beta)!_{2(\beta-1)}}{(2k)!_{2(\beta-1)}} \cdot \frac{(2k-(\beta-1))!_{2(\beta-1)}}{(2k\beta-(\beta-1))!_{2(\beta-1)}} \cdot \frac{(2k-1)!!}{(2k)!!} \tag{46}$$

Proof. To prove this claim, apply Corollary 2.4.9 to the interior of this product to obtain:

$$\prod_{j=0}^{2k}\left[\left(\frac{2k+(2k-j)(\beta-1)}{2k}\right)\mathbf{\Delta}(2k;j)\right]^{(-1)^j}$$

Then, split this product into two products. Apply Theorem 2.1.5 and lemma A.1.15 with $m=\beta-1$ to achieve the result. □

Similarly, we handle the odd case.

Theorem 2.4.11. *For all integers of the form $2k+1$ we have that*

$$\prod_{j=0}^{2k+1}\left[{}_{\beta}\mathbf{\Delta}(2k+1;j)\right]^{(-1)^j}=\frac{(\beta(2k+1))!_{2(\beta-1)}}{(2k\beta+1)!_{2(\beta-1)}}\cdot\frac{(2k+1-2(\beta-1))!_{2(\beta-1)}}{((2k+1)-(\beta-1))!_{2(\beta-1)}} \qquad (47)$$

Proof. We show the theorem holds by applying lemma A.1.16 with $m=\beta-1$ and recalling that the alternating product of the odd-indexed rows of $\mathbf{\Delta}$ is identically 1 (Theorem 2.1.3). □

We now have a complete characterization of the alternating product of any row of${}_{\beta}\mathbf{\Delta}$. Finally, before examining these objects from a more algebraic lens, we take one more step up the abstraction ladder.

2.5 Generalized Arithmetic Triangles

Previously we have considered Arithmetic Triangles where one of the seed values was chosen and the other was fixed at 1. We can ask the natural question then: what happens if we choose two distinct values for the seed values? To investigate this question we generalize definitions 2.2.1 and 2.4.1 to construct the Generalized Arithmetic Triangle Coefficients.

Definition 2.5.1. *Generalized Arithmetic Triangle Coefficients*
For positive integers α,β with $\alpha\leq\beta$, we define:

$$
{}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \begin{cases} 0 & \textit{if } n < |k| \\ 1 & \textit{if } n = 0 = k \\ \alpha & \textit{if } k = 0 \\ \beta & \textit{if } n = k \\ {}_{\alpha}\mathbf{\Delta}_{\beta}(n-1;k) + {}_{\alpha}\mathbf{\Delta}_{\beta}(n-1;k-1) & \textit{otherwise} \end{cases} \tag{48}
$$

This seems like a very natural generalization based upon our investigation so far. For instance, we can get back Pascal's Triangle by letting $\alpha = \beta = 1$. In fact, by letting $\alpha = \beta$ we always get a triangle isomorphic (up to a constant multiple) to Pascal's Triangle. We can also get back either the Left Arithmetic Triangles or Right Arithmetic Triangles by setting $\beta = 1$ or $\alpha = 1$, respectively. To get an idea of this most general triangle with which we would like to work, we include a portion in Figure 7.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & \alpha & & \beta & & & & \\
 & & & \alpha & & \alpha+\beta & & \beta & & & \\
 & & \alpha & & 2\alpha+\beta & & \alpha+2\beta & & \beta & & \\
 & \alpha & & 3\alpha+\beta & & 3\alpha+3\beta & & \alpha+3\beta & & \beta & \\
\alpha & & 4\alpha+\beta & & 3\alpha+2\beta & & 2\alpha+3\beta & & \alpha+4\beta & & \beta
\end{array}
$$

Figure 7: Generalized Arithmetic Triangle ${}_{\alpha}\mathbf{\Delta}_{\beta}$

As we can see from Figure 7, there is a great deal of structure embedded in ${}_{\alpha}\mathbf{\Delta}_{\beta}$. It is important to note here that we have not constructed a single triangle, but a two-indexed family of triangles. Considering this class of triangles will include our investigations of the strictly Left or Right Arithmetic Triangles. We first wish to make explicit the connection between ${}_{\alpha}\mathbf{\Delta}_{\beta}$ for arbitrary α, β and $\mathbf{\Delta}$. In doing this, we see that it really is the case that all triangles ${}_{\alpha}\mathbf{\Delta}_{\beta}$ are built from the generator triangle $\mathbf{\Delta}$. First we show that we can recast the recurrence relation underpinning ${}_{\alpha}\mathbf{\Delta}_{\beta}$ to a weighted sum of elements of $\mathbf{\Delta}$.

Lemma 2.5.2. *For positive integers $n, k, \alpha,$ and β we have:*

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \alpha\mathbf{\Delta}(n-1;k) + \beta\mathbf{\Delta}(n-1;k-1) \tag{49} $$

Proof. We show this claim holds by showing this generates the same recurrence relation as that already on ${}_{\alpha}\mathbf{\Delta}_{\beta}$.
First, for the seed values we have:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(1;0) = \alpha\mathbf{\Delta}(0;0) + \beta\mathbf{\Delta}(0;-1) = \alpha $$

and

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(1;1) = \alpha\mathbf{\Delta}(0;1) + \beta\mathbf{\Delta}(0;0) = \beta $$

Thus, this mapping preserves seed values. Now, suppose the relation holds for all values in ${}_{\alpha}\mathbf{\Delta}_{\beta}$ where $n < N$. Consider ${}_{\alpha}\mathbf{\Delta}_{\beta}(N+1;k)$. Using the Pascal-Type recurrence relation for ${}_{\alpha}\mathbf{\Delta}_{\beta}$, we have:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(N+1;k) = {}_{\alpha}\mathbf{\Delta}_{\beta}(N;k) + {}_{\alpha}\mathbf{\Delta}_{\beta}(N;k-1) $$

Using the induction hypothesis we can now write:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(N+1;k) = \alpha\mathbf{\Delta}(N-1;k) + \beta\mathbf{\Delta}(N-1;k-1) + \alpha\mathbf{\Delta}(N-1;k-1) + \beta\mathbf{\Delta}(N-1;k-2) $$

Combining like terms we have:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(N+1;k) = $$
$$ \alpha \cdot (\mathbf{\Delta}(N-1;k) + \mathbf{\Delta}(N-1;k-1)) + \beta \cdot (\mathbf{\Delta}(N-1;k-1) + \mathbf{\Delta}(N-1;k-2)) $$

Applying Pascal's Rule twice to the above sum we see:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(N+1;k) = \alpha\mathbf{\Delta}(N;k) + \beta\mathbf{\Delta}(N;k-1) $$

Hence, the result holds for all entries in ${}_{\alpha}\mathbf{\Delta}_{\beta}$. □

We now see that from any entry in ${}_{\alpha}\mathbf{\Delta}_{\beta}$ we can write it as a weighted sum of entries from $\mathbf{\Delta}$, where the weights come directly from the seed values α and β. Lemma 2.5.2 is a generalization and unification of lemmas 2.2.9 and 2.4.8. Similarly, we now recast this fact in terms of a corollary which will be used in the forthcoming investigations into the alternating products of ${}_{\alpha}\mathbf{\Delta}_{\beta}$.

Corollary 2.5.3. *For all positive integers n, k, α, and β we have:*

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \left(\frac{\alpha n + k(\beta - \alpha)}{n}\right)\mathbf{\Delta}(n;k) \tag{50} $$

Proof. The proof for this is strictly algebraic. We use known properties of the binomial coefficients and lemma 2.5.2. First, from lemma 2.5.2 we have:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \alpha\mathbf{\Delta}(n-1;k) + \beta\mathbf{\Delta}(n-1;k-1) $$

Expanding these as binomial coefficients we now see:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \alpha\frac{(n-1)!}{(k)!(n-1-k)!} + \beta\frac{(n-1)!}{(k-1)!(n-k)!} $$

By creating a common denominator and like numerators we can see that:

$$ {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = \alpha\left(\frac{n-k}{n}\right)\mathbf{\Delta}(n;k) + \beta\left(\frac{k}{n}\right)\mathbf{\Delta}(n;k) $$

Finally, by factoring out $\mathbf{\Delta}(n;k)$ and combining fractions we achieve the result. □

2.5.1 Properties of the Generalized Arithmetic Triangle

Just as before, for completeness, we show that the familiar properties of $\mathbf{\Delta}$ carry over to the less familiar object ${}_{\alpha}\mathbf{\Delta}_{\beta}$ with just a few modifications.

Corollary 2.5.4. *For positive integers n, α, and β we have:*

$$ \sum_{j=0}^{n} {}_{\alpha}\mathbf{\Delta}_{\beta}(n;j) = 2^{n-1}(\alpha + \beta) \tag{51} $$

Corollary 2.5.5. *For every positive integer $n > 1$ and positive integers α, β we have:*

$$\sum_{j=0}^{n} (-1)^j {}_{\alpha}\mathbf{\Delta}_{\beta}(n; j) = 0 \tag{52}$$

Corollary 2.5.6. *(Hockey Stick) For positive integers $n, k, \alpha,$ and β we have:*

$$\sum_{j=0}^{k} {}_{\alpha}\mathbf{\Delta}_{\beta}(n + j; j) = {}_{\alpha}\mathbf{\Delta}_{\beta}(n + k + 1; k) \tag{53}$$

Corollary 2.5.7. *(Hockey Stick) For all positive integers $n, k, \alpha,$ and β we have:*

$$\sum_{j=0}^{k} {}_{\alpha}\mathbf{\Delta}_{\beta}(n + j; n) = {}_{\alpha}\mathbf{\Delta}_{\beta}(n + k + 1; n + 1) \tag{54}$$

The proof for each of the above corollaries follows easily from the corresponding proof of elements of $\mathbf{\Delta}$. It should come as no surprise here that these nice properties of Pascal's Triangle are preserved by considering the larger class of Arithmetic Triangles.

Now, we can consider the alternating products of ${}_{\alpha}\mathbf{\Delta}_{\beta}$. As before, we must consider two cases: rows with even index and rows with odd index. Technical lemmas A.1.17 and A.1.18 will do all of the heavy lifting in the next two theorems. The first few terms of each of these alternating products are given in Appendix 2.

First, we consider the rows of even index.

Theorem 2.5.8. *For all positive integers of the form $2k$ and positive integers α, β with $\beta \geq \alpha$ we have that*

$$\prod_{j=0}^{2k} \left[{}_{\alpha}\mathbf{\Delta}_{\beta}(2k; j)\right]^{(-1)^j} = \alpha \cdot \frac{(2k\beta)!_{2(\beta-\alpha)}}{(2k\alpha)!_{2(\beta-\alpha)}} \frac{(2k\alpha - (\beta - \alpha))!_{2(\beta-\alpha)}}{(2k\beta - (\beta - \alpha))!_{2(\beta-\alpha)}} \frac{(2k-1)!_2}{(2k)!_2} \tag{55}$$

Proof. We show this holds by direct computation. Applying corollary 2.5.3 we can write:

$$\prod_{j=0}^{2k} \left[{}_{\alpha}\mathbf{\Delta}_{\beta}(2k; j)\right]^{(-1)^j} = \prod_{j=0}^{2k} \left[\frac{2k\alpha + j(\beta - \alpha)}{2k} \mathbf{\Delta}(2k; j)\right]^{(-1)^j}$$

By applying lemma A.1.17 and Theorem 2.1.5 we have the result. □

Next, consider those rows of odd-index.

Theorem 2.5.9. *For all positive integers of the form $2k+1$ and positive integers α, β with $\alpha \leq \beta$ we have that*

$$\prod_{j=0}^{2k+1} \left[{}_{\alpha}\mathbf{\Delta}_{\beta}(2k+1;j)\right]^{(-1)^j} =$$

$$\frac{((2k+1)\beta-(\beta-\alpha))!_{2(\beta-\alpha)}}{((2k+1)\beta)!_{2(\beta-\alpha)}} \cdot \frac{((2k+1)\alpha-(\beta-\alpha))!_{2(\beta-\alpha)}}{((2k+1)\alpha-2(\beta-\alpha))!_{2(\beta-\alpha)}} \tag{56}$$

Proof. To prove this claim, we first apply corollary 2.5.3 to rewrite the inner term as a product of two terms. That is,

$$\prod_{j=0}^{2k+1} \left[{}_{\alpha}\mathbf{\Delta}_{\beta}(2k+1;j)\right]^{(-1)^j} = \prod_{j=0}^{2k+1} \left[\left(\frac{(2k+1)\alpha+j(\beta-\alpha)}{2k+1}\right)\mathbf{\Delta}(2k+1;j)\right]^{(-1)^j}$$

Apply lemma A.1.18 to the term on the left and recall that the alternating product for any odd-indexed row of $\mathbf{\Delta}$ is identically one (Theorem 2.1.3) to obtain the result. □

CHAPTER 3

Generalized Arithmetic Triangle Structures

3.1 The Arithmetic Triangle Lattice

By viewing Pascal's Triangle in a new light, we can glean new insights from it. One way of looking at an array of numbers is to endow it with an ordering. In the coming discussion, we will be considering each binomial coefficient $\binom{n}{k}$ as an algebraic object with two components instead of as a numeric object with a definite value. Figure 8 gives a partial Hasse Diagram for our proposed lattice. Notice that Figure 8 only shows the first four rows of The Arithmetic Triangle Lattice.

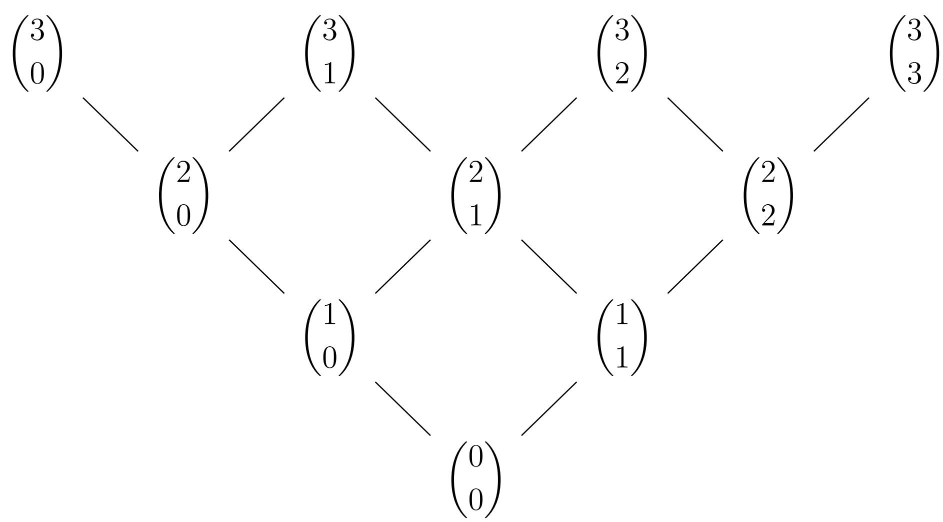

Figure 8: The Arithmetic Triangle Lattice (partial)

We can define an order on the set B by defining two binary operations on the binomial coefficient objects. We will show in Theorem 3.1.2 that these binary operations satisfy all the requirements to generate a partial order on B by appealing to Theorem 1.3.13.

Definition 3.1.1. *Let B denote the set of all binomial coefficients and define two binary operations on elements of this set. Given $\binom{n}{k}, \binom{m}{r} \in B$ we define:*

1. $\binom{n}{k} \wedge \binom{m}{r} = \binom{\min\{k,r\} + \min\{n-k, m-r\}}{\min\{k,r\}}$
2. $\binom{n}{k} \vee \binom{m}{r} = \binom{\max\{k,r\} + \max\{n-k, m-r\}}{\max\{k,r\}}$

With these operations of meet and join, respectively, we show that they satisfy the four requirements given in Theorem 1.3.13 to induce a partial order on B.

Theorem 3.1.2. *The operations $\vee$, $\wedge$ defined on the elements of B are commutative, associative, idempotent, and satisfy the absorption laws.*

Proof. We prove each law holds for $\wedge$ and the case for $\vee$ is proved similarly.

- **Commutative:** Clearly the operation $\wedge$ is commutative since both the max and min operators are commutative.

- **Associative:** Showing that $\wedge$ is associative takes a bit of cranking the wheel, but it works out. We have that

$$\binom{n}{k} \wedge \left[\binom{m}{r} \wedge \binom{p}{q}\right] = \binom{n}{k} \wedge \binom{\min\{r,q\} + \min\{m-r, p-q\}}{\min\{r,q\}} =$$

$$\binom{\min\{k,r,q\} + \min\{n-k, \min\{m-r, p-q\}\}}{\min\{k,r,q\}}$$

Considering the other side of the associative law we have:

$$\left[\binom{n}{k} \wedge \binom{m}{r}\right] \wedge \binom{p}{q} = \binom{\min\{k,r\} + \min\{n-k, m-r\}}{\min\{k,r\}} \wedge \binom{p}{q} =$$

$$\binom{\min\{k,q,r\} + \min\{p-q, \min\{n-k, m-r\}\}}{\min\{k,q,r\}}$$

Comparing these two binomial coefficients, and recalling that the min operator satisfies the associative property, we conclude that $\wedge$ is an associative binary operation.

- **Idempotent:** Obviously $\wedge$ is idempotent because both the max and min operators are idempotent.

- **Absorption:** We show by direct computation.

$$\binom{n}{k} \wedge \left[\binom{n}{k} \vee \binom{m}{r}\right] = \binom{n}{k} \wedge \binom{\max\{k,r\} + \max\{n-k, m-r\}}{\max\{k,r\}} =$$

$$\binom{\min\{k, \max\{k,r\}\} + \min\{n-k, \max\{n-k, m-r\}\}}{\min\{k, \max\{k,r\}\}} =$$

$$\binom{k + \min\{n-k, \max\{n-k, m-r\}\}}{k} = \binom{n}{k}$$

Thus, $\wedge$ satisfies the absorption law.

Hence $\wedge$ is commutative, associative, idempotent, and satisfies the absorption law. An analogous series of mini-proofs hold to show $\vee$ satisfies the same properties. $\square$

Alternatively, we can give another characterization of the same order. The order inherited from $\mathbf{\Delta}$ can be captured by declaring that the two elements which sum to form a given binomial coefficient are "below" said element. Unsurprisingly, this gives the same order as that generated by the binary operations $\wedge$ and $\vee$.

Definition 3.1.3. *Let B denote the set of all binomial coefficient objects, and define $\sqsubseteq$ on B in the following way:*
$\binom{n}{k} \sqsubseteq \binom{m}{r}$ *if (1) $n \leq m$ and $k = r$ or (2)$n < m$ and $k < r$*

For completeness, we show that this relation does indeed generate a partial order on the elements of B.

Theorem 3.1.4. *The set B along with order relation $\sqsubseteq$ forms a partial order.*

Proof. We must show this relation is reflexive, antisymmetric, and transitive.
First, it is clear that $\sqsubseteq$ is reflexive since $\binom{n}{k} \sqsubseteq \binom{n}{k}$ by condition (1).
Next, suppose $\binom{n}{k} \sqsubseteq \binom{m}{r}$ and $\binom{m}{r} \sqsubseteq \binom{n}{k}$. So, we would have that $n < m$ and $m < n$. Thus, $n = m$. If condition (1) is satisfied for both inequalities, then we are done. If we have mixed conditions in our inequalites, we are done. If both satisfy condition (2) we are also done because $k < r$ and $r < k$ implies $k = r$.
Last, we show that $\sqsubseteq$ is transitive. There are four possible cases, only three of which we consider here by symmetry. Suppose $\binom{n_1}{k_1} \sqsubseteq \binom{n_2}{k_2}$ and $\binom{n_2}{k_2} \sqsubseteq \binom{n_3}{k_3}$.

1. Case 1: Both inequalites satisfy (1). This is the easy case.

2. Case 2: The first inequality satisfies (1) and the second satisfies (2). So, we have $n_1 \leq n_2$ and $k_1 = k_2$ and $n_2 < n_3$ and $k_2 < k_3$. Thus, $n_1 < n_3$ and $k_1 < k_3$. Hence, $\binom{n_1}{k_1} \sqsubseteq \binom{n_3}{k_3}$ by condition (2).

3. Case 3: Both inequalities satisfy (2). This case is also easy.

□

We are now comfortable in saying that the order given by $\sqsubset$ or the order induced by $\wedge$ and $\vee$ does indeed generate a partial order on the set B. Henceforth we shall refer to this poset as $\mathbf{BC} = (B, \sqsubseteq) = (B, \wedge, \vee)$. However, we can go even a step further. Not only is $\mathbf{BC}$ a partial order, but it forms a lattice. To verify this claim we show that the operations $\wedge$ and $\vee$ serve as the meet and join, respectively, for $\mathbf{BC}$.

Theorem 3.1.5. *The binary operations $\wedge$ and $\vee$ defined above serve as the meet and join, respectively, for the poset* $\mathbf{BC}$.

Proof. Suppose $\binom{n}{k}, \binom{m}{r} \in B$ and let $p = \min\{k, r\}$ and $q = \min\{n-k, m-r\}$. First notice that $p \leq k$ and $p \leq q$. Also, observe that $p+q \leq n$ and $p+q \leq m$. Thus,

we have that $\binom{p+q}{q}$ is a lower-bound for the set $\left\{\binom{n}{k}, \binom{m}{r}\right\}$.
Next, suppose that $\binom{x}{y}$ is another lower-bound for this set. In particular, this means $x \leq n$ and $x \leq m$ and $y \leq k$ and $y \leq r$. Immediately this allows us to say that $y \leq p$. We also can glean from this that $x - k \leq n - k$ and $x - r \leq m - r$ which implies that $x - p \leq q$. Or that $x \leq p + q$. Hence, $\binom{p+q}{q}$ is the greatest lower-bound for the set $\left\{\binom{n}{k}, \binom{m}{r}\right\}$.
A similar argument suffices to show that $\vee$ serves as join. □

3.1.1 Properties of The Arithmetic Triangle Lattice

We now know that the set B of all binomial coefficient objects along with the operators $\wedge$ and $\vee$ form a lattice, with order given by Definition 3.1.1. To better understand this structure, we probe it a bit. First, we would like to know a bit more about what kind of lattice **BC** forms. Theorem 3.1.6 shows us that **BC** is actually a lower-bounded distributive lattice, one of the more well-studied objects in Order Theory.

Theorem 3.1.6. *The poset* $\mathbf{BC} = (B, \sqsubseteq)$ *is a lower-bounded distributive lattice.*

Proof. Clearly the element $\binom{0}{0}$ serves as the lower bound for **BC** since for any $\binom{n}{k} \in B$, $\binom{0}{0} \wedge \binom{n}{k} = \binom{0}{0}$.

Also, the operators $\vee$ and $\wedge$ serve as join and meet respectively. Based on how we defined $\vee$ and $\wedge$ the join and meet of two element always exists.

To show that **BC** is a distributive lattice, we show that for any $\binom{n}{k}, \binom{m}{r}, \binom{p}{q}$ we have $\binom{n}{k} \wedge \left[\binom{m}{r} \vee \binom{p}{q}\right] = \left[\binom{n}{k} \wedge \binom{m}{r}\right] \vee \left[\binom{n}{k} \wedge \binom{p}{q}\right]$.

By direct computation, we have on the left hand side:

$$\binom{n}{k} \wedge \left[\binom{m}{r} \vee \binom{p}{q}\right] = \binom{\min\{n-k, \max\{m-r, p-q\}\} + \min\{k, \max\{r, q\}\}}{\min\{k, \max\{r, q\}\}}$$

Similarly, on the right, we have:

$$\left[\binom{n}{k} \wedge \binom{m}{r}\right] \vee \left[\binom{n}{k} \wedge \binom{p}{q}\right] =$$

$$\binom{\max\{\min\{n-k, m-r\}, \min\{n-k, p-q\}\} + \max\{\min\{k, r\}, \min\{k, q\}\}}{\max\{\min\{k, r\}, \min\{k, q\}\}}$$

Since the max and min operators satisfy the distributive law, these two quantities must be equal. Thus, **BC** is a distributive lattice. □

If we do a few computations with these operations a few patterns become apparent. Chief among these is the observation that every element in **BC** can be written as the join of two elements from specific subsets.

Definition 3.1.7. *In the lower bounded lattice* **BC**, *define the following subsets:*

- $B_0 = \left\{\binom{n}{0} | n \in \mathbb{Z}^+\right\}$
- $B_\Delta = \left\{\binom{n}{n} | n \in \mathbb{Z}^+\right\}$.

Theorem 3.1.8. *Every element in* $\mathbf{BC} - \binom{0}{0}$ *can be written uniquely as a join of an element from* B_0 *and an element from* B_Δ.

Proof. Suppose $\binom{m}{k} \in B$. Then, we can write $\binom{m}{k} = \binom{m-k}{0} \vee \binom{k}{k}$.

To show this decomposition is unique, suppose $n, r \in \mathbb{Z}^+$ such that $\binom{m}{k} = \binom{n}{0} \vee \binom{r}{r}$. Thus, from the definition of $\vee$ we have that $\binom{n}{0} \vee \binom{r}{r} = \binom{n+r}{r}$. This implies that $r = k$ and $n + r = m$. Hence, $n = m - r$ showing that this decomposition is indeed unique. □

Now that we have a lower-bounded, distributive lattice in **BC**, we would like to investigate the substructures living in **BC**. The first simple substructures to look at are principal lower-sets.

Lemma 3.1.9. *For each* $\binom{n}{k} \in \mathbf{BC}$*, we have*

$$\downarrow \binom{n}{k} = \left\{ \binom{p}{q} \in \mathbf{BC} \mid q \leq k \text{ and } p - q \leq n - k \right\} \tag{57}$$

Proof. First, observe that if $\binom{p}{q} \sqsubseteq \binom{n}{k}$, then $q \leq k$ and $p \leq n$. Thus, $\binom{p}{q} \in \downarrow \binom{n}{k}$. Next, suppose $\binom{p}{q} \in \downarrow \binom{n}{k}$. Thus, $q \leq k$ and $p - q \leq n - k$. This also means that $p \leq n - k + q$ and because $q \leq k$ we have $p \leq n$. Hence, $\binom{p}{q} \sqsubseteq \binom{n}{k}$. □

Upper-sets are constructed similarly.

Lemma 3.1.10. *For each* $\binom{n}{k} \in \mathbf{BC}$*, we have*

$$\uparrow \binom{n}{k} = \left\{ \binom{p}{q} \in \mathbf{BC} \mid k \leq q \text{ and } n - k \leq p - q \right\} \tag{58}$$

The lattice **BC** has a curious property which from the Pascal-Type construction. If we consider any principal lower set, we see that it must always be finite. Contrast this with dual property: for any $\binom{n}{k}$, we never have that $\uparrow \binom{n}{k}$ is finite. We give proofs for both of these claims below.

Lemma 3.1.11. *The lattice* **BC** *is finitary.*

Proof. We show by contradiction. Suppose $\downarrow \binom{n}{k}$ is infinite. Then, there would be infinitely many non-negative integers less than n. This is never the case. Thus, $\downarrow \binom{n}{k}$ is finite. □

Corollary 3.1.12. *The lattice* **BC** *satisfies the Descending Chain Condition.*

Corollary 3.1.13. *For each* $\binom{n}{k}$*, the set* $\uparrow\binom{n}{k}$ *is infinite.*
Alternatively, the lattice $\mathbf{BC}^{op}$ *is not finitary.*

Proof. We show by contradiction. Suppose $\uparrow\binom{n}{k}$ is finite. Then, there would be finitely many non-negative integers greater than n. This is never the case. Thus, $\uparrow\binom{n}{k}$ is infinite. □

By considering the lattice **BC** we notice something peculiar about upper-sets. Any principal upper-set 'looks' like the whole lattice **BC**. This leads us to the following theorem.

Theorem 3.1.14. *For any choice of* $\binom{p}{q} \in \mathbf{BC}$*, there exists an order isomorphism* $f : \mathbf{BC} \to \uparrow\binom{p}{q}$ *defined by:* $f : \binom{n}{k} \mapsto \binom{n+p}{k+q}$.

Proof. To show that f is an order isomorphism, we must show four things: f is a monomorphism, f is an epimorphism, f is order preserving, and f has an order preserving inverse. We show each in turn.

1. First, suppose $\binom{n_1}{k_1}, \binom{n_2}{k_2}$ such that $f\left(\binom{n_1}{k_1}\right) = f\left(\binom{n_2}{k_2}\right)s$. So, we have $\binom{n_1+p}{k_1+q} = \binom{n_2+p}{k_2+q}$. In particular, $\binom{n_1+p}{k_1+q} \sqsubseteq \binom{n_2+p}{k_2+q}$. It must be that $k_1 + q \leq k_2 + q$ and $n_1 + p \leq n_2 + p$. Cancelling on both sides we get that $k_1 \leq k_2$ and $n_1 \leq n_2$. Therefore, $\binom{n_1}{k_1} \sqsubseteq \binom{n_2}{k_2}$. A similar argument suffices to show the other inequality. Hence, f is a monomorphism.

2. Next, suppose $\binom{m}{p} \in \uparrow\binom{p}{q}$. We need to find an element $\binom{n}{k} \in \mathbf{BC}$ such that $\binom{n+p}{k+q} = \binom{m}{p}$. Let $n = m - p$ and $k = q - r$. We know that both n, k are non-zero since $\binom{p}{q} \sqsubseteq \binom{m}{p}$. Therefore, f is an epimorphism.

3. Last, suppose $\binom{n_1}{k_1} \sqsubseteq \binom{n_2}{k_2} \in \mathbf{BC}$. Then, $f\left(\binom{n_1}{k_1}\right) = \binom{n_1+p}{k_1+q}$ and $f\left(\binom{n_2}{k_2}\right) = \binom{n_2+p}{k_2+q}$. From our assumption we have $k_1 \leq k_2 \Rightarrow k_1 + q \leq$

$k_2 + q$ and $n_1 \leq n_2 \Rightarrow n_1 + p \leq n_2 + p$. Hence, $f\left(\binom{n_1}{k_1}\right) \sqsubseteq f\left(\binom{n_2}{k_2}\right)$ s.

4. To see that f has an order preserving inverse is easy to verify, simply form the inverse map in the natural way by subtracting the shift instead of adding.

Ergo, f is an order isomorphism. □

We know from Definition 3.1.1 that the meet and join of any two elements of **BC** always exists. This leads us to ask whether we can extend this to arbitrary subsets of **BC**. In one case, the finitary-ness of **BC** tells us that meets of arbitrary non-empty subsets always exists.

Theorem 3.1.15. *If $S \subseteq B$ is non-empty, then $\bigwedge S$ exists.*

Proof. Suppose $S \subseteq B$ is non-empty. Let $m(S)$ be the collection of lower-bounds for S. We know that $m(S) \neq \emptyset$ since **BC** is a lower-bounded lattice.
Because S is non-empty, there exists $\binom{n}{k} \in S$. If we consider $\downarrow\binom{n}{k}$, we see that it must be the case that $m(S) \subseteq \downarrow\binom{n}{k}$. Since $\downarrow\binom{n}{k}$ is finite, then S is also finite. Since **BC** has all finite joins, we conclude that $\bigwedge S$ exists. □

The case is not so simple for arbitrary joins. Since **BC** is lower-bounded, we know that the set of lower bounds for a given subset S is always non-empty, but this is not true for the set of upper bounds of S. Given a non-empty upper-bounded set S, we can say that S is finite. Thus, if we have an upper bounded subset, it will never be "too large".

Theorem 3.1.16. *If S is a non-empty upper-bounded subset of* **BC***, then S is finite.*

Proof. Let $S \subseteq BC$ be non-empty and suppose $\binom{n}{k}$ is an upper-bound for S. Then, we must have that $S \subseteq \downarrow\binom{n}{k}$. Because every principal lower set is finite, we must have that S is finite. □

Corollary 3.1.17. *The lattice* $\mathbf{BC}$ *admits arbitrary non-empty meets and all finite joins.*

In Chapter 2 we spent a great deal of time showing that the Arithmetic Triangle ${}_\alpha\mathbf{\Delta}_\beta$ is completely determined by its seed values. We can capture this same idea in terms of the order induced by $\wedge$ and $\vee$ by considering the atomicity of the lattice $\mathbf{BC}$.

Lemma 3.1.18. *Atom*$(\mathbf{BC}) = \left\{\binom{1}{0}, \binom{1}{1}\right\}$.

Proof. Clearly, we have for $k = 0, 1$ that $\perp = \binom{0}{0} \sqsubseteq \binom{1}{k}$ since $\binom{0}{0} \wedge \binom{1}{k} = \binom{0}{0}$. Since there does not exist $m \in \mathbb{Z}^+$ such that $0 < m < 1$, we have that $\binom{0}{0} \prec \binom{1}{k}$.

Suppose there exist some $\binom{p}{r} \in \mathrm{BC}$ such that $\binom{0}{0} \prec \binom{p}{r}$ and $p > 1$. Then we would have that $\binom{p}{r} \wedge \binom{1}{0} = \binom{1}{0}$, contradiction that $\binom{p}{r}$ covers $\binom{0}{0}$. □

Corollary 3.1.19. *The lattice* $\mathbf{BC}$ *is atomic.*

Proof. For any $\binom{p}{r} \in BC$ such that $p \neq r$, we have that $\binom{p}{r} \wedge \binom{1}{0} = \binom{1}{0}$. Hence, $\binom{1}{0} \in \downarrow\binom{p}{r}$.

If $p = r$, then we have $\binom{p}{r} \wedge \binom{1}{1} = \binom{1}{1}$. Hence, $\binom{1}{1} \in \downarrow\binom{p}{r}$. □

Now that we have a more full characterization of how the seed values impact the rest of the lattice $\mathbf{BC}$, we look more carefully at intervals in $\mathbf{BC}$ to see if any interesting conclusions may be drawn.

If we have two distinct elements in $\mathbf{BC}$ then the interval formed by them may take only a few different forms. Take for instance the elements $\binom{4}{3}$ and $\binom{5}{4}$. Then, the interval $\left[\binom{4}{3}, \binom{5}{4}\right]$ consists solely of those elements. More generally, if we have two binomial coefficient objects $\alpha := \binom{n}{k}$ and $\beta := \binom{n'}{k'}$ such that $\beta \in \mathrm{Cov}(\alpha)$, then $[\alpha, \beta] = \{\alpha, \beta\}$.

Next, consider the example $\alpha := \binom{2}{0}$ and $\beta := \binom{4}{3}$. By looking at $\uparrow \alpha$ and $\downarrow \beta$, we see that they have no elements in common. That is, $[\alpha, \beta] = \emptyset$. We can formalize this observation.

Observation 3.1.20. *Let* $\binom{n_1}{k_1}$, *and* $\binom{n_2}{k_2}$ *be distinct elements of the lattice* **BC** *such that* $n_2 - k_2 \leq n_1 - k_1$ *and* $k_2 < k_1$. *Then* $\left[\binom{n_1}{k_1}, \binom{n_2}{k_2}\right] = \emptyset$

Proof. We show by contradiction. Let $\binom{p}{q} \in \left[\binom{n_1}{k_1}, \binom{n_2}{k_2}\right]$. Thus,

$$\binom{n_1}{k_1} \wedge \binom{p}{q} = \binom{n_1}{k_1} \tag{59}$$

and

$$\binom{n_2}{k_2} \vee \binom{p}{q} = \binom{n_2}{k_2}. \tag{60}$$

From definition 3.1.1, we have that $k_1 \leq q$ and $q \leq k_2$. By transitivity then we also that $k_1 \leq k_2$. Since our chosen elements are distinct we must have then that $k_1 < k_2$. This contradicts our assumption. Thus, the interval $\left[\binom{n_1}{k_1}, \binom{n_2}{k_2}\right]$ is empty. $\square$

Last, we deal with the most general case: where a given interval is non-empty and not trivial.

Observation 3.1.21. *In the lattice* **BC**, *for any distinct* $\binom{n_1}{k_1}, \binom{n_2}{k_2} \in$ **BC**, *we have*

$$\left[\binom{n_1}{k_1}, \binom{n_2}{k_2}\right] = \left\{\binom{p}{q} : k_1 \leq q \leq k_2 \text{ and } n_1 - k_1 \leq p - q \leq n_2 - k_2\right\}. \tag{61}$$

When looking at non-trivial intervals in **BC**, we see that they typically look like diamonds or rhombi. In Figure 9 we label the interval $\left[\binom{1}{0}, \binom{5}{2}\right]$ in red. Notice that this interval forms a rhombus, where the "sides" are length two. The left-most and right-most elements of $\left[\binom{1}{0}, \binom{5}{2}\right]$ may be distinguished as the width, or girth, of the interval. These two elements defining the width of $\left[\binom{1}{0}, \binom{5}{2}\right]$ are special in another way: they are the only relatively complemented elements in this interval.

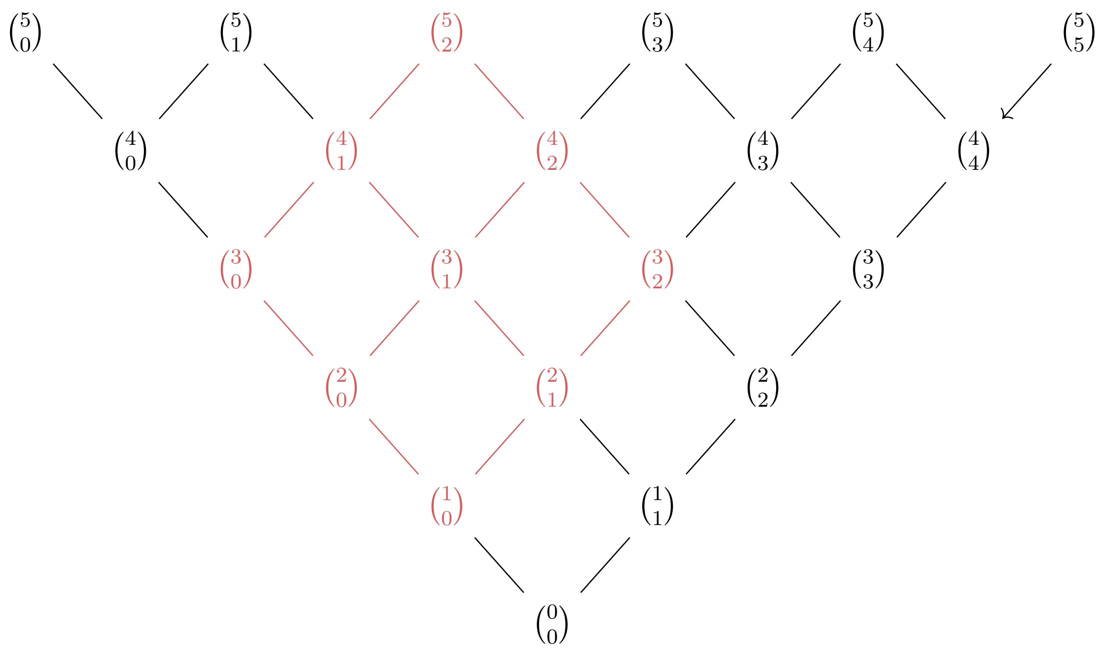

Figure 9: A typical interval in **BC**

Lemma 3.1.22. *For any* $\alpha = \binom{n_1}{k_1}, \beta = \binom{n_2}{k_2} \in \mathbf{BC}$*, the only relatively complemented elements in the interval* $[\alpha, \beta]$ *are*

$$x = \binom{\min\{k_1, k_2\} + \max\{n_1 - k_1, n_2 - k_2\}}{\min\{k_1, k_2\}} \quad \text{and} \quad z = \binom{\max\{k_1, k_2\} + \min\{n_1 - k_1, n_2 - k_2\}}{\max\{k_1, k_2\}}$$

Proof. First we show that x and z are relatively complemented in $[\alpha, \beta]$. Consider $x \wedge z$. We have $x \wedge z = \binom{\min\{k_1, k_2\} + \min\{n_1 - k_1, n_2 - k_2\}}{\min\{k_1, k_2\}} = \binom{n_1}{k_1}$ since we assumed that $n_1 - k_1 \leq n_2 - k_2$ and $k_1 \leq k_2$.

Similarly, we have that $x \vee z = \binom{n_2}{k_2}$. Thus, both x and z are relatively complemented in $[\alpha, \beta]$.

Next we show that for any other $w \in [\alpha, \beta]$ that w is not relatively complemented. We proceed by contradiction. Assume w is relatively complemented in $[\alpha, \beta]$, then $\exists w' \in [\alpha, \beta]$ such that $w \wedge w' = \alpha$ and $w \vee w' = \beta$. Without loss of generality, suppose that $w = \binom{p}{q}$ and $w' = \binom{m}{r}$. Then, we have:

$$\binom{p}{q} \wedge \binom{m}{r} = \binom{n_1}{k_1} \iff \min\{q, r\} = k_1 \text{ and } \min\{p - q, m - r\} = n_1 - k_1.$$

Assume, w.o.l.o.g that $\min\{q, r\} = r$ and $\min\{m-r, p-q\} = m-r$. Then, $w' = \binom{n_1}{k_1}$. However, $w \vee w' = \binom{p}{q} \vee \binom{n_1}{k_1} = \binom{p}{q} \neq \binom{n_2}{k_2}$. This contradicts our assumption that w is relatively complemented in $[\alpha, \beta]$. $\square$

To extend our investigation of the lattice **BC** from intervals to more general subsets, we make an observation. We have shown that **BC** is finitary: that every principal lower set is finite. This condition makes it so that any upper-bounded subset $S \subseteq \mathbf{BC}$ is also finite, as we saw in Theorem 3.1.16. The converse of this statement also holds, that if every upper-bounded set is finite, then each principal lower set is finite as well.

Theorem 3.1.23. *Every upper-bounded subset of* **BC** *is finite if and only if every principal lowerset is finite.*

Proof. We prove each direction in turn. First, assume that every upper-bounded subset of **BC** is finite. Let $x \in \mathbf{BC}$. Clearly, x acts as an upper-bound for $\downarrow x$, thus $\downarrow x$ is finite.

Now suppose that every principal lower set is finite and let $X \subseteq \mathbf{BC}s$ such that X is upper-bounded. Now, consider the set of uppper-bounds of X, $j(X)$. Since X is upper-bounded, we must have that $j(X) \neq \emptyset$. Choose some $z \in j(X)$. Because z is an upper-bound for X, we have that $X \subseteq \downarrow z$. Since every principal lower set is finite, we thuss have that X is finite. $\square$

The idea of primality shows up in nearly every sort of algebraic structure one wishes to investigate. In the realm of order theory, primality appears, in one guise, as meet-prime and join-prime elements of a poset. Thus, to continue our investigation into **BC**, we should ascertain whether it contains any meet-prime or join-prime elements. The next few statements provide answers in the affirmative. But first, we make two observations which will allow us to identify those elements which are meet-prime or join-prime.

Lemma 3.1.24. *For any finite subset $F \subseteq$ **BC**, if $\bigwedge F$ is of the form $\binom{n}{0}$, then there exists an element $\binom{p}{0} \in F$ where $p \leq n$.*

Proof. Without loss of generality, let $F = \left\{\binom{n_1}{k_1}, \binom{n_2}{k_2}, ..., \binom{n_r}{k_r}\right\}$. Then, it must be that

$$\bigwedge F = \binom{\min\{n_1 - k_1, n_2 - k_2, ..., n_r - k_r\} + \min\{k_1, k_2, ..., k_r\}}{\min\{k_1, k_2, ..., k_r\}} = \binom{n}{0}.$$

So, $\exists i \in \{1, 2, ..., r\}$ such that $k_i = 0$. Therefore, at least one of the elements of F is of the form $\binom{p}{0}$. $\square$

Lemma 3.1.25. *For any finite subset $F \subseteq$ **BC**, if $\bigwedge F$ is of the form $\binom{n}{n}$, then there exists an element $\binom{p}{p} \in F$ where $p \leq n$.*

Proof. Without loss of generality, let $F = \left\{\binom{n_1}{k_1}, \binom{n_2}{k_2}, ..., \binom{n_r}{k_r}\right\}$. Then our hypothesis tells us that

$$\bigwedge F = \binom{\min\{n_1 - k_1, n_2 - k_2, ...n_r - k_r\} + \min\{k_1, k_2, ..., k_r\}}{\min\{k_1, k_2, ..., k_r\}} = \binom{n}{n}.$$

Thus, we must have $\min\{k_1, k_2, ..., k_r\} = n$. This tells us that

$$\bigwedge F = \binom{\min\{n_1 - k_1, n_2 - k_2, ...n_r - k_r\} + n}{n} = \binom{n}{n}.$$

But this implies that $\exists i \in \{1, 2, ..., r\}$ such that $n_i - k_i = 0 \iff n_i = k_i$; or that for some element of F that it is of the form $\binom{p}{p}$. $\square$

We can now characterize the meet-prime elements of **BC**.

Theorem 3.1.26. *In the lattice* **BC** *every element of the form* $\binom{n}{0}$*, where* $n \in \mathbb{Z}^+$*, is meet-prime.*

Proof. Let $F \in Fin(\mathbf{BC})$ such that $\bigwedge F \sqsubseteq \binom{n}{0}$. Further, suppose $p, q \in \mathbb{W}$ such that $\bigwedge F = \binom{p}{q}$. Then, we have $\binom{p}{q} \vee \binom{n}{0} = \binom{n}{0}$. So, we can say that $q = 0$ and $p \leq n$. By lemma 3.1.24 there must be an element of the form $\binom{a}{0} \in F$ with $a \leq p$. Finally, by comparing this element with $\binom{n}{0}$ we see $\binom{n}{0} \vee \binom{a}{0} = \binom{n}{0}$. Thus $\binom{n}{0}$ is meet prime. $\square$

Theorem 3.1.27. *In the lattice* **BC***, every element of the form* $\binom{n}{n}$ *where* $n \in \mathbb{Z}^+$ *is meet prime.*

Proof. Suppose $F \in \text{Fin}(\mathbf{BC})$ such that $\bigwedge F \sqsubseteq \binom{n}{n}$ for some $n \in \mathbb{Z}^+$. Assume, without loss of generality, that $\bigwedge F = \binom{p}{q}$ for some $p, q \in \mathbb{Z}^+$. Then, by lemma 3.1.25 we have that there exists some $\binom{a}{a} \in F$ such that $a \leq n$. Thus, by comparing these two elements we have $\binom{a}{a} \vee \binom{n}{n} = \binom{n}{n}$. Hence, every element of the form $\binom{n}{n}$ is meet prime. $\square$

Theorems 3.1.26 and 3.1.27 tell us that sets B_0 and B_Δ from Definition 3.1.7 serve as the meet-prime elements for **BC**.

Digging a bit deeper into the structure of the Arithmetic Triangle Lattice **BC**, we spot something interesting. We have already shown that any element in **BC** has a unique decomposition into an element from B_0 and an element from B_Δ. Extending this idea of decomposition further, we can take any upper-bounded set and write it as a union of principal lower-sets.

Theorem 3.1.28. *Every upper-bounded subset $S \subset$ **BC** is a union of principal lower sets.*

Proof. Let $S \subset BC$ such that S is upper-bounded and non-empty and let $\max(S)$ denote the maximal elements of S.
Then, we wish to show that

$$S = \bigcup_{x \in \max(S)} \downarrow x. \tag{62}$$

First we show ($\subset$).
Let $y \in S$. Then, there exists $x_0 \in \max(S)$ such that $y \leq x_0$. Thus, $y \in \downarrow x_0$. So, we have that $S \subseteq \bigcup_{x \in \max(S)} \downarrow x$.
Next we show ($\supset$).
Let $z \in \bigcup_{x \in \max(S)} \downarrow x$. Then, $\exists x_1$ such that $z \in \downarrow x_1$. Since x_1 is an upper-bound for S, we have that $z \in S$. Thus, $\bigcup_{x \in \max(S)} \downarrow x \subset S$.
Hence, we have that $S = \bigcup_{x \in \max(S)} \downarrow x$.
The result still holds if $S = \emptyset$.

Suppose $S = \emptyset$. Then, $\emptyset \subseteq \downarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ serves as a maximal element of $\emptyset$. $\square$

This decomposition of upper-bounded sets in **BC** is reminiscent of graded posets introduced in Definition 1.3.33.

We now turn to considering elements of the ideal lattice Idl(**BC**).

Notice that we immediately have that Idl(**BC**) is distributive.

Theorem 3.1.29. *The ideal lattice Idl*(**BC**) *is distributive.*

Proof. In Theorem 3.1.6 we showed that **BC** is itself a distributive lattice. Theorem 1.3.27 tells us then that Idl(**BC**) is a distributive lattice as well. $\square$

The two special sets B_0 and B_Δ hold more significance here. These are the first two examples we have of infinite ideals in Idl(**BC**).

Lemma 3.1.30. *The set* $B_0 = \left\{ \binom{n}{0} : n \in \mathbb{W} \right\}$ *is an infinite ideal.*

Proof. We first show B_0 is a lowerset. Let $\binom{p}{q} \sqsubseteq \binom{n}{0}$ for some $n \in \mathbb{W}$. Then, $\binom{p}{q} \wedge \binom{n}{0} = \binom{p}{q} = \binom{\min\{p-q, n\} + \min\{0, q\}}{\min\{0, q\}} = \binom{n'}{0}$ for some $n' \in \mathbb{W}$. Thus, $\binom{p}{q} \in B_0$.

We show B_0 is directed. Suppose $S = \left\{ \binom{n_1}{0}, \binom{n_2}{0}, ..., \binom{n_k}{0} \right\}$. Then, $\bigvee S = \binom{\max\{n_1, n_2, ..., n_k\}}{0} \in S$. Thus, S is directed. So we conclude that B_0 is an ideal. $\square$

Lemma 3.1.31. *The set* $B_\Delta = \left\{ \binom{n}{n} : n \in \mathbb{W} \right\}$ *is an infinite ideal.*

Proof. First we show that B_Δ is a lowerset. Let $\binom{p}{q} \sqsubseteq \binom{n}{n}$ for some $p, q, n \in \mathbb{W}$. Then $\binom{p}{q} \wedge \binom{n}{n} = \binom{p}{q} = \binom{\min\{p-q, 0\} + \min\{q, n\}}{\min\{q, n\}} = \binom{r}{r}$ where $r = \min\{n, q\}$. Thus, $\binom{p}{q} \in B_\Delta$.

Next, we show that B_Δ is directed. Suppose $S = \left\{ \binom{n_1}{n_1}, \binom{n_2}{n_2}, ..., \binom{n_k}{n_k} \right\}$.

Then from our definition of the join of a finite set, we have $\bigvee S = \binom{\max\{n_1, n_2, ..., n_k\}}{\max\{n_1, n_2, ..., n_k\}} \in S$. Thus, S is directed. Ergo, we have that B_Δ is an ideal. $\square$

There are a host of finite ideals in Idl($\mathbf{BC}$); take for instance any principal lowerset. The sets B_0 and B_Δ are special in that they constitute our first examples of infinite ideals. We can say a bit more about these two distinguished ideals.

Theorem 3.1.32. *Every ideal of* $\mathbf{BC}$ *has non-empty intersection with* B_0 *and* B_Δ*. For every* $J \in Idl(\mathbf{BC})$*, we have that* $J \cap B_0 \neq \emptyset$ *and* $J \cap B_\Delta \neq \emptyset$*.*

Proof. Let $J \in \mathrm{Idl}(\mathbf{BC})$ and $\binom{n}{k} \in J$. Then by applying Theorem 3.1.8, we can uniquely decompose this element as

$$\binom{n}{k} = \binom{n-k}{0} \vee \binom{k}{k} \tag{63}$$

Since J is a lowerset, we have that $\binom{n-k}{0}, \binom{k}{k} \in J$. Thus, $J \cap B_0 \neq \emptyset$ and $J \cap B_\Delta \neq \emptyset$. $\square$

To wrap up our elementary investigation of the ideal lattice Idl($\mathbf{BC}$), we exhibit one more observation. Taking an arbitray subset of $\mathbf{BC}$ and looking at the filter generated by it does not always produce something nice. However, if we happen upon a situation when the extreme values of any row are included in our subset, then we regain the whole lattice.

Theorem 3.1.33. *Suppose* $S \subset \mathbf{BC}$*. If there exists an* $n \in \mathbb{Z}^+$ *such that* $\binom{n}{0}, \binom{n}{n} \in S$*, then* $[S) = \mathbf{BC}$*.*

Proof. Suppose $S \subset \mathbf{BC}$ and for some $n \in \mathbb{Z}^+$ that $\binom{n}{0}, \binom{n}{n} \in S$. Consider $[S)$. We already have that $[S) \subset \mathbf{BC}$. We also have $\binom{n}{0}, \binom{n}{n} \in [S)$. Because filters are closed with respect to meets, we must have $\binom{0}{0} = \binom{n}{0} \wedge \binom{n}{n} \in [S]$. Thus, $\uparrow \binom{0}{0} \subset [S]$. However, $\uparrow \binom{0}{0} = \mathbf{BC}$. Hence, $\mathbf{BC} \subset [S]$. $\square$

This concludes our investigation into the lattice **BC**. We have seen a host of interesting observations and patterns yet there are still a slew of unanswered questions regarding **BC**.

3.2 The Arithmetic Triangle Group

Last, we look at how the collection of all Arithmetic Triangles forms another kind of algebraic structure in a natural way: a group.

Motivating the forthcoming discussion is the observation that the collection of Arithmetic Triangles follows a similar sort of recursive structure as The Arithmetic Triangle $\mathbf{\Delta}$.

We saw in Section 2.5 that a Generalized Arithmetic Triangle is completely determined by its seed values. That is, to specify an Arithmetic Triangle we need only specify the values ${}_\alpha\mathbf{\Delta}_\beta(1;0)$ and ${}_\alpha\mathbf{\Delta}_\beta(1;1)$. However, we wish to have a bit more machinery to talk about the relationship between two Arithmetic Triangles. Before anything too fancy we first need to specify what equality of Arithmetic Triangles looks like. A first pass might look something like this

Given two Arithmetic Triangles, ${}_\alpha\mathbf{\Delta}_\beta$ and ${}_{\alpha'}\mathbf{\Delta}_{\beta'}$, we say that ${}_\alpha\mathbf{\Delta}_\beta \cong {}_{\alpha'}\mathbf{\Delta}_{\beta'}$ if they have the same seed values. Namely, if $\alpha = \alpha'$ and $\beta = \beta'$.
Equivalently, we say that ${}_\alpha\mathbf{\Delta}_\beta \cong {}_{\alpha'}\mathbf{\Delta}_{\beta'}$ if for each $n, k \in \mathbb{N}$ we have

$${}_\alpha\mathbf{\Delta}_\beta(n;k) = {}_{\alpha'}\mathbf{\Delta}_{\beta'}(n;k)$$

This is the obvious way to define equality of Arithmetic Triangles, but consider the case of the Arithmetic Triangle ${}_2\mathbf{\Delta}_2$. Clearly, this triangle is just $\mathbf{\Delta}$ where each entry has been multiplied by 2. We use this observation to give a more general, and more useful, notion of equality of two Arithmetic Triangles.

Given two Arithmetic Triangles, ${}_{\alpha}\mathbf{\Delta}_{\beta}$ and ${}_{\alpha'}\mathbf{\Delta}_{\beta'}$, we say that ${}_{\alpha}\mathbf{\Delta}_{\beta} \cong {}_{\alpha'}\mathbf{\Delta}_{\beta'}$ if there exists some constant Γ such that for all integers n, k we have:

$$\Gamma \cdot {}_{\alpha}\mathbf{\Delta}_{\beta}(n;k) = {}_{\alpha'}\mathbf{\Delta}_{\beta'}(n;k)$$

Notice that our second version of equality for Arithmetic Triangles includes, as special cases, our first notion of equality.

Now we can make the example above, that ${}_{2}\mathbf{\Delta}_{2}$ and $\mathbf{\Delta}$ are essentially the same, both more precise in statement and more general in conclusion.

Theorem 3.2.1. *Any arithmetic triangle with identical seed values is equal, in the sense of equality as defined above, to $\mathbf{\Delta}$. That is, if $\beta = \gamma$, then ${}_{\beta}\mathbf{\Delta}_{\gamma} \cong \mathbf{\Delta}$.*

Proof. Choose as constant $\Gamma := \dfrac{1}{\beta}$. Considering the seed values first, we have:

$$\Gamma \cdot {}_{\beta}\mathbf{\Delta}_{\gamma}(1;0) = \frac{1}{\beta} \cdot \beta = 1 = \mathbf{\Delta}(1;0)$$

and

$$\Gamma \cdot {}_{\beta}\mathbf{\Delta}_{\gamma}(1;1) = \frac{1}{\beta} \cdot \beta = 1 = \mathbf{\Delta}(1;1)$$

By considering an arbitrary element of ${}_{\beta}\mathbf{\Delta}_{\gamma}$ we see:

$$\Gamma \cdot {}_{\beta}\mathbf{\Delta}_{\gamma}(n;k) = \frac{1}{\beta} \cdot (\beta \cdot \mathbf{\Delta}(n;k)) = \mathbf{\Delta}(n;k)$$

□

Even though this seems like the most powerful statement of equality between Arithmetic Triangles, we go one step further.

Definition 3.2.2. *Let ${}_{\alpha}\mathbf{\Delta}_{\beta}$ and ${}_{\alpha'}\mathbf{\Delta}_{\beta'}$ be two Arithmetic Triangles. We say that φ is an Arithmetic Triangle homomorphism if:*

1. *φ takes the seed values of ${}_{\alpha}\mathbf{\Delta}_{\beta}$ to the seed values of ${}_{\alpha'}\mathbf{\Delta}_{\beta'}$.*

2. $\varphi\left({}_{\alpha}\mathbf{\Delta}_{\beta}(n+1;k)\right) = \varphi\left({}_{\alpha}\mathbf{\Delta}_{\beta}(n;k)\right) + \varphi\left({}_{\alpha}\mathbf{\Delta}_{\beta}(n;k-1)\right)$ *(Recursion Condition)*

The function φ is an isomorphism if there exists another Arithmetic Triangle homomorphism ψ such that $\varphi \circ \psi = 1_{\Delta}$ and $\psi \circ \varphi = 1_{\Delta}$, where 1_{Δ} is the identity Arithmetic Triangle homomorphism.

Remark. We shall see that Definition 3.2.2 does a tremendous amount of work for us. Not only have we given the most general way of defining equality of Arithmetic Triangle, but we have also defined structure preserving functions between Arithmetic Triangles. These Arithmetic Triangle homorphisms (AT-homomorphisms) preserve both the seed values and, more importantly, the Pascal-type rule inherent to each Arithmetic Triangle.

Clearly, the function defined in Theorem 3.2.1 is an isomorphism, as we can easily find its inverse. This leads us to think about other obvious Arithmetic Triangle homomorphisms. For instance, consider the Arithmetic Triangles given by $\mathbf{\Delta}_2$ and ${}_2\mathbf{\Delta}$. By examining their structure it seems as though they are identical, simply mirrored across the central column. We can formalize and generalize this observation.

Theorem 3.2.3. *Let α be some integer. The Arithmetic Triangles $\mathbf{\Delta}_{\alpha}$ and ${}_{\alpha}\mathbf{\Delta}$ are isomorphic.*

Proof. We show that the function

$$\varphi : \mathbf{\Delta}_{\alpha}(n;k) \mapsto {}_{\alpha}\mathbf{\Delta}(n;n-k) \tag{64}$$

is an Arithmetic Triangle isomorphism.

First, we check the seed value condition. Namely, we see:

$$\varphi\left(\mathbf{\Delta}_{\alpha}(1;0)\right) = {}_{\alpha}\mathbf{\Delta}(1;1)$$

and

$$\varphi\left(\mathbf{\Delta}_{\alpha}(1;1)\right) = {}_{\alpha}\mathbf{\Delta}(1;0)$$

So, φ preserves the seed values. Next, we check the recursion condition.

By direct computation on an arbitrary element, we have:

$$\varphi\left(\mathbf{\Delta}_{\alpha}(n;k)\right) = {}_{\alpha}\mathbf{\Delta}(n;n-k) = {}_{\alpha}\mathbf{\Delta}(n-1;n-k) + {}_{\alpha}\mathbf{\Delta}(n-1;n-k-1) =$$
$$\varphi\left(\mathbf{\Delta}_{\alpha}(n-1;k-1)\right) + \varphi\left(\mathbf{\Delta}_{\alpha}(n-1;k)\right)$$

Thus, φ satisfies the recursion condition. The function φ acts as its own inverse, hence it is an isomorphism. $\square$

The previous digression allows us to collect together a wide swath of all the Arithmetic Triangles. More clearly, when thinking about the properties of α-triangles, we need only consider three cases: $\mathbf{\Delta}$ itself will represent all Arithmetic Triangles where the seed values are identical, $\mathbf{\Delta}_{\alpha}$ will represent all one-sided α-triangles since we have shown that $\mathbf{\Delta}_{\alpha} \cong {}_{\alpha}\mathbf{\Delta}$, and lastly two-sided Arithmetic Triangles ${}_{\beta}\mathbf{\Delta}_{\gamma}$ where $\beta \neq \gamma$. However, we shall see that this last class of triangles can be thought of as a sum of α-triangles of the second kind.

To see an example of this, let's consider the Arithmetic Triangle ${}_2\mathbf{\Delta}_3$ shown in Figure 10.

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 2 & & 3 & & & & \\
n=2 & & & & 2 & & 5 & & 3 & & & \\
n=3 & & & 2 & & 7 & & 8 & & 3 & & \\
n=4 & & 2 & & 9 & & 15 & & 11 & & 3 & \\
n=5 & 2 & & 11 & & 24 & & 26 & & 14 & & 3
\end{array}
$$

Figure 10: Generalized Arithmetic Triangle ${}_2\mathbf{\Delta}_3$

We can see that the internal structure of ${}_2\mathbf{\Delta}_3$ looks to be much more complex than that of $\mathbf{\Delta}$, however upon further investigation we notice that each element of ${}_2\mathbf{\Delta}_3$ can be written as a sum of elements from $\mathbf{\Delta}$ and from the right α-triangle $\mathbf{\Delta}_2$. Informally, we have that

$$
{}_2\mathbf{\Delta}_3(n;k) = \mathbf{\Delta}(n;k) + \mathbf{\Delta}_2(n;k) \tag{65}
$$

So, this leads us to ask, can every Arithmetic Triangle of the form ${}_\beta\mathbf{\Delta}_\gamma$, $\beta \neq \gamma$ be decomposed as a sum of $\mathbf{\Delta}$ and a one-sided α-triangle?

Before definitively answering this question, we would look at one more concrete example to see if the observed relationship holds. To that end, consider the Arithmetic Triangle given by ${}_3\mathbf{\Delta}_7$ in the Figure 11.

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 3 & & 7 & & & & \\
n=2 & & & & 3 & & 10 & & 7 & & & \\
n=3 & & & 3 & & 13 & & 17 & & 7 & & \\
n=4 & & 3 & & 16 & & 30 & & 24 & & 7 & \\
n=5 & 3 & & 19 & & 46 & & 54 & & 31 & & 7
\end{array}
$$

Figure 11: Generalized Arithmetic Triangle ${}_3\mathbf{\Delta}_7$

We might try to decompose ${}_3\mathbf{\Delta}_7$ into $\mathbf{\Delta}$ and a right α-triangle since here $\gamma > \beta$. However, because $\beta > 2$ there would be no way to create the values in ${}_3\mathbf{\Delta}_7$. Thus, we have to use a two-sided α-triangle isomorphic to $\mathbf{\Delta}$. With this idea in mind, we observe that the elements in ${}_3\mathbf{\Delta}_7$ can be written as

$$ {}_3\mathbf{\Delta}_7(n;k) = {}_2\mathbf{\Delta}_2(n;k) + \mathbf{\Delta}_5(n;k) \tag{66}$$

This leads us to the following definition for addition of Arithmetic Triangles. One thing to note is that in each Arithmetic Triangle ${}_\beta\mathbf{\Delta}_\gamma$, we treat the element ${}_\beta\mathbf{\Delta}_\gamma(0;0)$ as a base-point; meaning that we don't consider it when defining a sum of Arithmetic Triangles.

Definition 3.2.4. *Given two Arithmetic Triangles ${}_{\beta_1}\mathbf{\Delta}_{\gamma_1}$ and ${}_{\beta_2}\mathbf{\Delta}_{\gamma_2}$, we define their sum as*

$$ {}_{\beta_1}\mathbf{\Delta}_{\gamma_1} \oplus {}_{\beta_2}\mathbf{\Delta}_{\gamma_2} = {}_{\beta_1+\beta_2}\mathbf{\Delta}_{\gamma_1+\gamma_2} \tag{67}$$

That is, for all integers $n \geq 1$, $k \geq 0$. we say

$$ {}_{\beta_1+\beta_2}\mathbf{\Delta}_{\gamma_1+\gamma_2}(n;k) = {}_{\beta_1}\mathbf{\Delta}_{\gamma_1}(n;k) + {}_{\beta_2}\mathbf{\Delta}_{\gamma_2}(n;k) \tag{68}$$

When defining a new binary operation, one of the first properties to check is associativity. We now show that the operation $\oplus$ defined above is associative.

Lemma 3.2.5. *Given three Arithmetic Triangles, ${}_{\beta_1}\mathbf{\Delta}_{\gamma_1}$, ${}_{\beta_2}\mathbf{\Delta}_{\gamma_2}$, and ${}_{\beta_3}\mathbf{\Delta}_{\gamma_3}$ we always have*

$$ {}_{\beta_1+(\beta_2+\beta_3)}\mathbf{\Delta}_{\gamma_1+(\gamma_2+\gamma_3)} = {}_{(\beta_1+\beta_2)+\beta_3}\mathbf{\Delta}_{(\gamma_1+\gamma_2)+\gamma_3} \tag{69}$$

Proof. The associativity of $\oplus$ is a consequence of the associativity of the natural numbers.

Suppose that $n \geq 1$ and $0 \leq k \leq n$. Starting on the left we have

$$
\begin{aligned}
{}_{\beta_1+(\beta_2+\beta_3)}\boldsymbol{\Delta}_{\gamma_1+(\gamma_2+\gamma_3)}(n;k) &= {}_{\beta_1}\boldsymbol{\Delta}_{\gamma_1}(n;k) + {}_{\beta_2+\beta_3}\boldsymbol{\Delta}_{\gamma_2+\gamma_3}(n;k) && (70)\\
&= {}_{\beta_1}\boldsymbol{\Delta}_{\gamma_1}(n;k) + ({}_{\beta_2}\boldsymbol{\Delta}_{\gamma_2}(n;k) + {}_{\beta_3}\boldsymbol{\Delta}_{\gamma_3}(n;k)) && (71)\\
&= ({}_{\beta_1}\boldsymbol{\Delta}_{\gamma_1}(n;k) + {}_{\beta_2}\boldsymbol{\Delta}_{\gamma_2}(n;k)) + {}_{\beta_3}\boldsymbol{\Delta}_{\gamma_3}(n;k) && (72)\\
&= {}_{(\beta_1+\beta_2)}\boldsymbol{\Delta}_{(\gamma_1+\gamma_2)}(n;k) + {}_{\beta_3}\boldsymbol{\Delta}_{\gamma_3}(n;k) && (73)\\
&= {}_{(\beta_1+\beta_2)+\beta_3}\boldsymbol{\Delta}_{(\gamma_1+\gamma_2+)\gamma_3}(n;k) && (74)
\end{aligned}
$$

Thus, we have that the binary operation $\oplus$ is associative. □

Using this definition of the sum of two Arithmetic Triangles, we can explicitly write down how to decompose any Arithmetic Triangle as a sum of a one-sided α-triangle and an isomorphic copy of $\boldsymbol{\Delta}$.

Theorem 3.2.6. *Given an Arithmetic Triangle of the form ${}_{\beta}\boldsymbol{\Delta}_{\gamma}$ where $\beta \leq \gamma$, we can write*

$$
{}_{\beta}\boldsymbol{\Delta}_{\gamma} = {}_{\beta-1}\boldsymbol{\Delta}_{\beta-1} \oplus \boldsymbol{\Delta}_{\gamma-(\beta-1)} \tag{75}
$$

Proof. We show that the equality holds for an arbitrary element in the array ${}_{\beta}\boldsymbol{\Delta}_{\gamma}$, thus showing equality holds for all elements.
Let $\beta, \gamma \in \mathbb{Z}^+$ such that $\beta \leq \gamma$ and further suppose $n, k \in \mathbb{Z}^+$ such that $0 \leq k \leq n$. Now, consider the element ${}_{\beta}\boldsymbol{\Delta}_{\gamma}(n;k)$. Applying lemma 2.2.9 we have

$$
{}_{\beta}\boldsymbol{\Delta}_{\gamma}(n;k) = \beta\boldsymbol{\Delta}(n;k) + \gamma\boldsymbol{\Delta}(n;k-1). \tag{76}
$$

On the other hand, if we start with the right hand-side we have

$$
{}_{\beta-1}\boldsymbol{\Delta}_{\beta-1}(n;k) + \boldsymbol{\Delta}_{\gamma-(\beta-1)}(n;k) =
$$
$$
(\beta-1)[\boldsymbol{\Delta}(n;k) + \boldsymbol{\Delta}(n;k-1)] + [\boldsymbol{\Delta}(n;k) + (\gamma-(\beta-1)\boldsymbol{\Delta}(n;k-1)].
$$

Now, by distributing and simplifying we see that

$$_{\beta-1}\mathbf{\Delta}_{\beta-1}(n;k) + \mathbf{\Delta}_{\gamma-(\beta-1)}(n;k) = (\beta-1)\mathbf{\Delta}(n;k) + \mathbf{\Delta}(n;k) + \gamma\mathbf{\Delta}(n;k-1) =$$
$$\beta\mathbf{\Delta}(n;k) + \gamma\mathbf{\Delta}(n;k-1).$$

Comparing this with the result above, we have that the two sides are equal for all n, k. Hence, the result holds for all elements of $_\beta\mathbf{\Delta}_\gamma$. □

The above decomposition theorem is very powerful, it lets us think of the whole class of Arithmetic Triangles as formed from the set of Arithmetic Triangles isomorphic to $\mathbf{\Delta}$ and the set of one-sided α-triangles. When encountering a class of objects and an operation on them, it is natural in mathematics to ask what sort of algebraic structures are inherent in said operation. By considering the operations above, we can see that the set of Arithmetic Triangles along with the operation $\oplus$ form an Abelian group, as the Theorem 3.2.7 makes explicit.

Theorem 3.2.7. *The set* $\{_\beta\mathbf{\Delta}_\gamma : \beta, \gamma \in \mathbb{Z}\}$ *along with the operation* $\oplus$ *defined above forms an Abelian group, henceforth known as* $\mathcal{AT} = \{_\beta\mathbf{\Delta}_\gamma, \oplus\}$

Proof. To show that $\mathcal{AT}$ forms an Abelian group, we show that it is closed with respect to $\oplus$, there is an identity element in $\mathcal{AT}$, and each element has an additive inverse. We have already shown that the operation $\oplus$ is associative.

To see that $\mathcal{AT}$ is closed with respect to $\oplus$, we note that based on how we have defined the operation $\oplus$ we always get an Arithmetic Triangle when summing two Arithmetic Triangles. Thus $\mathcal{AT}$ is closed with respect to $\oplus$.

Also, we see from the definition of $\oplus$ that it is a commutative operator, because at the element-wise level we use the standard $+$ operator which is obviously commutative.

Next, we show that the Arithmetic Triangle $_0\mathbf{\Delta}_0$ serves as the additive identity for $\mathcal{AT}$. Let $_\beta\mathbf{\Delta}_\gamma \in \mathcal{AT}$ such that $\beta, \gamma \neq 0$. Then, by the definition of $\oplus$, we have for each $n, k \geq 1$ that

$$ {}_{\beta}\mathbf{\Delta}_{\gamma}(n;k) + {}_{0}\mathbf{\Delta}_{0}(n;k) = {}_{\beta}\mathbf{\Delta}_{\gamma}(n;k) + 0 = {}_{\beta}\mathbf{\Delta}_{\gamma}(n;k) \tag{77} $$

Thus, we have that ${}_{0}\mathbf{\Delta}_{0}$ serves as the additive identity for $\mathcal{AT}$.

Last, we show that each ${}_{\beta}\mathbf{\Delta}_{\gamma}$ has an additive inverse. We claim that the Arithmetic Triangle ${}_{-\beta}\mathbf{\Delta}_{-\gamma}$ serves as this inverse. To see this, consider the sum ${}_{-\beta}\mathbf{\Delta}_{-\gamma} \oplus {}_{\beta}\mathbf{\Delta}_{\gamma}$. We have then for every pair of integers $n \geq 1$, $k \geq 0$ that

$$ {}_{-\beta}\mathbf{\Delta}_{-\gamma}(n;k) + {}_{\beta}\mathbf{\Delta}_{\gamma}(n;k) = 0 $$

Since each element in the sum is identically zero, we have that

$$ {}_{-\beta}\mathbf{\Delta}_{-\gamma} \oplus {}_{\beta}\mathbf{\Delta}_{\gamma} = {}_{0}\mathbf{\Delta}_{0} $$

Hence, each element of $\mathcal{AT}$ has an additive inverse.

Because we have shown that $\mathcal{AT}$ is closed with respect to $\oplus$, contains an additive identity and additive inverses, and that $\oplus$ is commutative, we conclude that $\mathcal{AT}$ is an Abelian group. $\square$

Theorem 3.2.7 tells us that the set of all Arithmetic Triangle forms a group under a perfectly natural operation $\oplus$. Now that we have an Abelian group, it is worthwhile to spend some time investigating its structure.

First, we formalize the conclusion of Theorem 3.2.1. Corollary 3.2.8 formalizes the observation that each Arithmetic Triangle with identical seed values is isomorphic to $\mathbf{\Delta}$.

Corollary 3.2.8. *The set* $\{{}_{\alpha}\mathbf{\Delta}_{\alpha} : \alpha \in \mathbb{Z}\}$ *forms a cyclic subgroup of* $\mathcal{AT}$.

Proof. If we let $\alpha = 0$, we see that the identity element of $\mathcal{AT}$ is a member of this set. Also, if we take two elements ${}_{\alpha_1}\mathbf{\Delta}_{\alpha_1}$ and ${}_{\alpha_2}\mathbf{\Delta}_{\alpha_2}$ we have that their sum ${}_{\alpha_1+\alpha_2}\mathbf{\Delta}_{\alpha_1+\alpha_2}$ is a member of this set, ensuring closure. Last, for any ${}_{\alpha}\mathbf{\Delta}_{\alpha}$ we know that its additive inverse, ${}_{-\alpha}\mathbf{\Delta}_{-\alpha}$, is a member of this set. So, we conclude that the set $\{{}_{\alpha}\mathbf{\Delta}_{\alpha} : \alpha \in \mathbb{Z}\}$ is a proper subgroup of $\mathcal{AT}$.

To see that this subgroup is cyclic, we note that any element ${}_{\alpha}\mathbf{\Delta}_{\alpha}$ can be written as a sum of α copies of $\mathbf{\Delta}$. If we have an element of the form ${}_{-\alpha}\mathbf{\Delta}_{-\alpha}$, we can write it as a sum of α copies of ${}_{-1}\mathbf{\Delta}_{-1} = -\mathbf{\Delta}$. Thus Pascal's Triangle is the generator for this cyclic subgroup. □

We finish this section by exhibiting one last striking result. If we denote the cyclic subgroup examined in Corollary 3.2.8 as the diagonal subgroup $\langle\mathbf{\Delta}\rangle$, we shall show that the quotient group formed by $\langle\mathbf{\Delta}\rangle$ is isomorphic to the group of integers.

Theorem 3.2.9. *Let $\mathbb{Z}$ represent the group of integers under the operation $+$, $\mathcal{AT}$ represent the group of Generalized Arithmetic Triangles under $\oplus$, and $\langle\mathbf{\Delta}\rangle$ be the cyclic subgroup of $\mathcal{AT}$ generated by $\mathbf{\Delta}$. Then,*

$$\mathcal{AT}/\langle\mathbf{\Delta}\rangle \cong \mathbb{Z} \tag{78}$$

Proof. To show Equation 78, we use the First Isomorphism Theorem for Groups. As a preliminary, observe that $\langle\mathbf{\Delta}\rangle$ is a normal subgroup of $\mathcal{AT}$ since $\mathcal{AT}$ is Abelian.

We claim that the function

$$f : \mathcal{AT} \to \mathbb{Z} \tag{79}$$

given by

$$f({}_{\alpha}\mathbf{\Delta}_{\beta}) = \beta - \alpha \tag{80}$$

is a homomorphism of groups. To see this, take two elements of $\mathcal{AT}$ and look at their images under f. For ${}_{\alpha}\mathbf{\Delta}_{\beta}$, ${}_{\alpha'}\mathbf{\Delta}_{\beta'} \in \mathcal{AT}$ We see:

$$f({}_{\alpha}\mathbf{\Delta}_{\beta} \oplus {}_{\alpha'}\mathbf{\Delta}_{\beta'}) = f({}_{\alpha+\alpha'}\mathbf{\Delta}_{\beta+\beta'}) = (\beta + \beta') - (\alpha + \alpha') \tag{81}$$

The value on the right of equation 81 is precisely the value of

$$f({}_{\alpha}\mathbf{\Delta}_{\beta}) + f({}_{\alpha'}\mathbf{\Delta}_{\beta'}) \tag{82}$$

Showing that f is indeed a group homomorphism.

Next we show that f is a surjective map. Let $n \in \mathbb{Z}$, and consider the element of $\mathcal{AT}$ given by ${}_{n+1}\mathbf{\Delta}$. Clearly, we have that $f({}_{n+1}\mathbf{\Delta}) = n$. Therefore, f is surjective. Finally, we show that $\langle \mathbf{\Delta} \rangle = \ker(f)$.

One subset inclusion follows immediately. That is, if ${}_{\alpha}\mathbf{\Delta}_{\alpha} \in \langle \mathbf{\Delta} \rangle$, then $f({}_{\alpha}\mathbf{\Delta}_{\alpha}) = \alpha - \alpha = 0$. Thus, ${}_{\alpha}\mathbf{\Delta}_{\alpha} \in \ker(f)$.

To see the other direction, suppose that ${}_{\alpha}\mathbf{\Delta}_{\beta} \in \ker(f)$. Then, we have that $\alpha - \beta = 0$ or that $\alpha = \beta$. Thus, $\ker(f) \subseteq \langle \mathbf{\Delta} \rangle$. Hence, we have that $\langle \mathbf{\Delta} \rangle = \ker(f)$.

By the First Isomorphism Theorem (Groups), we can now conclude that

$$\mathcal{AT}/\langle \mathbf{\Delta} \rangle \cong \mathbb{Z} \tag{83}$$

$\square$

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APPENDICES

APPENDIX A

TECHNICAL LEMMAS

Lemma A.1.10. *(Case for $\alpha = 3$) For all non-negative integers of the form $2k$ we have:*

$$\prod_{j=0}^{2k}\left[\frac{2k+2j}{2k}\right]^{(-1)^j} = \frac{(3k)!!}{(3k-1)!!}\frac{(k-1)!!}{(k)!!} = \frac{(6k)!_4}{(6k-2)!_4}\cdot\frac{(2k-2)!_4}{(2k)!_4}$$

Proof. We show by direct computation.

First, we split the product into even and odd indices letting $\Pi_1 = \prod_{j \text{ even}}^{2k}\left[\frac{2k+2j}{2k}\right]$ and $\Pi_1 = \prod_{j \text{ odd}}^{2k}\left[\frac{2k}{2k+2j}\right]$.

We know that the number of terms of $\prod_1$ is $k+1$ while the number of terms for $\prod_2$ is k, so we can simplify each to see:

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1}\prod_{j \text{ even}}^{2k}[2k+2j]$$

$$\Pi_2 = (2k)^k\prod_{j \text{ odd}}^{2k}\left[\frac{1}{2k+2j}\right]$$

By employing the multifactorial notation, we can write each of these products in the following way:

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1}\frac{(6k)!_4}{(2k-4)!_4}$$

$$\Pi_2 = (2k)^k\frac{(2k-2)!_4}{(6k-2)!_4}$$

Now, by combining these two products we have:

$$\prod_{j=0}^{2k}\left[\frac{2k+2j}{2k}\right]^{(-1)^j} = \frac{1}{2k}\cdot\frac{(6k)!_4}{(2k-4)!_4}\cdot\frac{(2k-2)!_4}{(6k-2)!_4} = \frac{(6k)!_4}{(6k-2)!_4}\cdot\frac{(2k-2)!_4}{(2k)!_4}$$

Hence the result holds.

□

Lemma A.1.11. *For all integers of the form $2k+1$, we have:*

$$\prod_{j=0}^{2k+1}\left[\frac{2k+1+2j}{2k+1}\right]^{(-1)^j} = \frac{(6k+1)!_4}{(6k+3)!_4}\frac{(2k-1)!_4}{(2k-3)!_4}$$

Proof. We show by direct computation.

First, we split the product into a product over even integers and over odd integers. Let $\Pi_1 = \prod_{j \text{ even}}^{2k+1}\left[\frac{2k+1+2j}{2k+1}\right]^{(-1)^j}$ and $\Pi_2 = \prod_{j \text{ odd}}^{2k+1}\left[\frac{2k+1+2j}{2k+1}\right]^{(-1)^j}$. We can simplify each of these to write:

$$\Pi_1 = \left(\frac{1}{2k+1}\right)^{k+1}\prod_{j \text{ even}}^{2k+1}\left[2k+1+2j\right]$$

and

$$\Pi_2 = (2k+1)^{k+1}\prod_{j \text{ odd}}^{2k+1}\left[\frac{1}{2k+1+2j}\right]$$

By recasting these products in multifactorial notation we can write:

$$\Pi_1 = \left(\frac{1}{2k+1}\right)^{k+1}\frac{(6k+1)!_4}{(2k-3)!_4}$$

and

$$\Pi_2 = (2k+1)^{k+1}\frac{(2k-1)!_4}{(6k+3)!_4}$$

So, if we put this all back together we achieve the result. That is:

$$\prod_{j=0}^{2k+1}\left[\frac{2k+1+2j}{2k+1}\right]^{(-1)^j} = \frac{(6k+1)!_4}{(6k+3)!_4}\frac{(2k-1)!_4}{(2k-3)!_4}$$

□

Lemma A.1.12. *For all integers of the form $2k$ and non-negative integers m we have:*

$$\prod_{j=0}^{2k}\left[\frac{2k+mj}{2k}\right]^{(-1)^j} = \frac{(2k(m+1))!_{2m}}{(2k(m+1)-m)!_{2m}}\frac{(2k-m)!_{2m}}{(2k)!_{2m}}$$

Proof. We show by direct computation. Fix some $m \in \mathbb{Z}^+$. Next, as in Lemma A.1.10 split this product into a product over even indices and a product over odd indices. Let $\Pi_1 = \prod_{j \text{ even}}^{2k}\left[\frac{2k+mj}{2k}\right]$ and $\Pi_2 = \prod_{j \text{ odd}}^{2k}\left[\frac{2k}{2k+mj}\right]$. We can simplify each of these in the following way.

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1}\prod_{j \text{ even}}^{2k}[2k+mj]$$

and

$$\Pi_2 = (2k)^k\prod_{j \text{ odd}}^{2k}\left[\frac{1}{2k+mj}\right]$$

Recasting each of these in terms of multi-factorials we have:

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1}\left(\frac{(2k(m+1))!_{2m}}{(2k-2m)!_{2m}}\right)$$

and

$$\Pi_2 = (2k)^k\left(\frac{(2k-m)!_{2m}}{(2k(m+1)-m)!_{2m}}\right)$$

Finally, put these pieces back together to achieve the result. That is, we have:

$$\prod_{j=0}^{2k}\left[\frac{2k+mj}{2k}\right]^{(-1)^j} = \left(\frac{1}{2k}\right)\cdot\left(\frac{(2k(m+1))!_{2m}}{(2k-2m)!_{2m}}\right)\cdot\left(\frac{(2k-m)!_{2m}}{(2k(m+1)-m)!_{2m}}\right) =$$
$$\frac{(2k(m+1))!_{2m}}{(2k(m+1)-m)!_{2m}}\frac{(2k-m)!_{2m}}{(2k)!_{2m}}$$

□

Lemma A.1.13. *For all integers of the form $2k+1$ and $m \geq 0$, we have:*

$$\prod_{j=0}^{2k+1}\left[\frac{2k+1+mj}{2k+1}\right]^{(-1)^j} = \frac{(2k(m+1)+1)!_{2m}(2k+1-m)!_{2m}}{(2k+1-2m)!_{2m}((2k+1)(m+1))!_{2m}}$$

Proof. We show by direct computation. Fix some $m \in \mathbb{Z}^+$. Let $\Pi_1 = \prod_{j \text{ even}}^{2k+1} \left[\frac{2k+1+mj}{2k+1}\right]^{(-1)^j}$ and $\Pi_2 = \prod_{j \text{ odd}}^{2k+1} \left[\frac{2k+1+mj}{2k+1}\right]^{(-1)^j}$. We can now simplify each of these to see:

$$\Pi_1 = \left(\frac{1}{2k+1}\right)^{k+1} \prod_{j \text{ even}}^{2k+1} [2k+1+mj]$$

and

$$\Pi_2 = (2k+1)^{k+1} \prod_{j \text{ odd}}^{2k+1} \left[\frac{1}{2k+1+mj}\right]$$

Recasting each of these products using multi-factorial notation, we have:

$$\Pi_1 = \left(\frac{1}{2k+1}\right)^{k+1} \frac{(2k(m+1)+1)!_{2m}}{(2k+1-2m)!_{2m}}$$

and

$$\Pi_2 = (2k+1)^{k+1} \frac{((2k+1)-m)!_{2m}}{((2k+1)(m+1))!_{2m}}$$

Finally, by combining these product back together we achieve the result. □

Lemma A.1.14. *For all integers of the form $2k$, we have:*

$$\prod_{j=0}^{2k} \left[\frac{2k+(2k-j)}{2k}\right]^{(-1)^j} = \frac{(2k-1)!!}{(2k)!!} \frac{(4k)!!}{(4k-1)!!}$$

Proof. We show by direct computation. We split this product into a product over even indices and a product over odd indices. Namely, let $\Pi_1 = \prod_{j \text{ even}} \left[\frac{2k+(2k-j)}{2k}\right]^{(-1)^j}$ and $\Pi_2 = \prod_{j \text{ odd}} \left[\frac{2k+(2k-j)}{2k}\right]^{(-1)^j}$. We can simplify each of these products by pulling out multiples of $2k$. Explicitly, we have:

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1} \prod_{j \text{ even}}^{2k} [2k+(2k-j)]$$

and

$$\Pi_2 = (2k)^k \prod_{j \text{ odd}}^{2k} \left[\frac{1}{2k + (2k - j)} \right]$$

By rewriting these products in multi-factorial notation we see:

$$\Pi_1 = \left(\frac{1}{2k}\right)^{k+1} \left(\frac{(4k)!!}{(2k-2)!!}\right) = \left(\frac{1}{2k}\right)^{k} \cdot \left(\frac{(4k)!!}{(2k)!!}\right)$$

and

$$\Pi_2 = (2k)^k \cdot \left(\frac{(2k-1)!!}{(4k)!!}\right)$$

Finally, by combining these two products we achieve the result.

$\square$

Lemma A.1.15. *For all positive integers of the form $2k$ and integers m, we have:*

$$\prod_{j=0}^{2k} \left[\frac{2k + (2k-j)m}{2k} \right]^{(-1)^j} = \frac{(2k(m+1))!_{2m}}{(2k)!_{2m}} \frac{(2k-m)!_{2m}}{(2k(m+1)-m)!_{2m}}$$

Proof. The proof for this statement follows the same argument as Lemma A.1.12 and Lemma A.1.14.

$\square$

Lemma A.1.16. *For all integers of the form $2k+1$ and positive integers m we have:*

$$\prod_{j=0}^{2k+1} \left[\frac{(2k+1) + (2k+1-j)m}{2k+1} \right]^{(-1)^j} = \frac{(2k+1) + (2k+1)m)!_{2m}}{((2k+1)+2km)!_{2m}} \cdot \frac{(2k+1-2m)!_{2m}}{((2k+1)-m)!_{2m}}$$

Proof. The proof for this statement follows similarly to Lemma A.1.15 $\square$

Lemma A.1.17. *For every positive integer of the form $2k$ and integers α, β where $\alpha \leq \beta$, we have:*

$$\prod_{j=0}^{2k} \left[\frac{2k\alpha + j(\beta - \alpha)}{2k} \right]^{(-1)^j} = \alpha \frac{(2k\beta)!_{2(\beta-\alpha)}}{(2k\alpha)!_{2(\beta-\alpha)}} \frac{(2k\alpha - (\beta-\alpha))!_{2(\beta-\alpha)}}{(2k\beta - (\beta-\alpha))!_{2(\beta-\alpha)}}$$

Proof. To prove this claim, we follow the same strategy as outlined in Lemmas A.1.12-A.1.16. Fix $\alpha, \beta \in \mathbb{Z}$ such that $\alpha \leq \beta$. We show by direct computation.

Let $\Pi_1 = \prod_{j \text{ even}}^{2k} \left[\frac{2k\alpha + j(\beta-\alpha)}{2k}\right]^{(-1)^j}$ and $\Pi_2 = \prod_{j \text{ odd}}^{2k} \left[\frac{2k\alpha + j(\beta-\alpha)}{2k}\right]^{(-1)^j}$.

We can simplify each of these by bring out the constant factor of $2k$. Namely, we can write:

$$\Pi_1 = \alpha \cdot \left(\frac{1}{2k}\right)^k \cdot \prod_{j \text{ even, positive}}^{2k} [2k\alpha + j(\beta-\alpha)]$$

and

$$\Pi_2 = (2k)^k \cdot \prod_{j \text{ odd}}^{2k} \left[\frac{1}{2k\alpha + j(\beta-\alpha)}\right]$$

Now, as in the previous lemmas, we rewrite using multi-factorial notation. Thus, we have:

$$\Pi_1 = \alpha \cdot \left(\frac{1}{2k}\right)^k \cdot \left(\frac{(2k\beta)!_{2(\beta-\alpha)}}{(2k\alpha)!_{2(\beta-\alpha)}}\right)$$

and

$$\Pi_2 = (2k)^k \cdot \left(\frac{(2k\alpha - (\beta-\alpha))!_{2(\beta-\alpha)}}{(2k\beta - (\beta-\alpha))!_{2(\beta-\alpha)}}\right)$$

Combining these two products together we have the result.

□

Lemma A.1.18. *For all positive integers of the form $2k+1$ and integers α, β where $\alpha \leq \beta$ we have:*

$$\prod_{j=0}^{2k+1} \left[\frac{(2k+1)\alpha + j(\beta-\alpha)}{2k+1}\right]^{(-1)^j} =$$
$$\frac{((2k+1)\beta - (\beta-\alpha))!_{2(\beta-\alpha)}}{((2k+1)\beta)!_{2(\beta-\alpha)}} \cdot \frac{((2k+1)\alpha - (\beta-\alpha))!_{2(\beta-\alpha)}}{((2k+1)\alpha - 2(\beta-\alpha))!_{2(\beta-\alpha)}}$$

Proof. The proof for this claim follows nearly identically from Lemmas A.1.17 and A.1.13

□

APPENDIX B

SEQUENCES AND CODE

B.2 Integer and Rational Sequences

Here we collect together some of the integer sequences that have been generated through the course of researching and writing this thesis. When applicable, a sequence will be tagged with its OEIS reference number.

Table B.1: $a_n = n!!$

a_1	1	a_6	48	a_{11}	10395
a_2	2	a_7	105	a_{12}	46080
a_3	3	a_8	384	a_{13}	135135
a_4	8	a_9	945	a_{14}	645120
a_5	15	a_{10}	3840	a_{15}	2027025

Table B.1 gives the first 15 values for $n!!$ and the associated OEIS sequence is given by reference number A006882. [22].

Table B.2: $a_n = n!_3$

a_1	1	a_6	18	a_{11}	880
a_2	2	a_7	28	a_{12}	1944
a_3	3	a_8	80	a_{13}	3640
a_4	4	a_9	162	a_{14}	12320
a_5	10	a_{10}	280	a_{15}	29160

Table B.2 gives the values of the modular factorial of n with $p = 3$. The corresponding OEIS sequence is given by reference code A007661. [22] Notice that for each $n < 4$, we have that $a_n = n$.

Table B.3: Alternating Products of $\mathbf{\Delta}$

a_1	1	a_6	$\frac{5}{16}$	a_{11}	1
a_2	$\frac{1}{2}$	a_7	1	a_{12}	$\frac{231}{1024}$
a_3	1	a_8	$\frac{35}{128}$	a_{13}	1
a_4	$\frac{3}{8}$	a_9	1	a_{14}	$\frac{429}{2048}$
a_5	1	a_{10}	$\frac{63}{256}$	a_{15}	1

Table B.3 gives the values of the alternating products of the rows of $\mathbf{\Delta}$. Notice that for every $k \in \mathbb{N}$, $a_{2k-1} = 1$. The value for even indices of a_n is given by Theorem 2.1.5 and Equation 18.

Table B.4: Alternating Products of $\mathbf{\Delta}_2$

a_1	$\frac{1}{2}$	a_6	$\frac{100}{231}$
a_2	$\frac{2}{3}$	a_7	$\frac{429}{640}$
a_3	$\frac{5}{8}$	a_8	$\frac{490}{1287}$
a_4	$\frac{18}{35}$	a_9	$\frac{2431}{3584}$
a_5	$\frac{21}{32}$	a_{10}	$\frac{15876}{46189}$

Table B.4 gives the values for

$$a_n = \prod_{j=0}^{n} [\mathbf{\Delta}_2(n;j)]^{(-1)^j}, \tag{B.84}$$

the alternating products for the Arithmetic Triangle given by seed values $\{1, 2\}$. Notice that we no longer have constant values for the odd indices a_{2k-1}.

Table B.5: Alternating Products of $\mathbf{\Delta}_3$

a_1	$\frac{1}{3}$	a_6	$\frac{525}{1024}$
a_2	$\frac{3}{4}$	a_7	$\frac{1045}{1989}$
a_3	$\frac{7}{15}$	a_8	$\frac{5}{11}$
a_4	$\frac{3}{5}$	a_9	$\frac{7735}{14421}$
a_5	$\frac{39}{77}$	a_{10}	$\frac{27027}{65536}$

Table B.5 gives the values for

$$a_n = \prod_{j=0}^{n} [\mathbf{\Delta}_3(n; j)]^{(-1)^j}, \tag{B.85}$$

the alternating products for the Arithmetic Triangle given by seed values $\{1, 3\}$. Notice that we no longer have constant values for the odd indices a_{2k-1}.

Table B.6: Alternating Products of $\mathbf{\Delta}_\alpha$

a_1	$\frac{1}{\alpha}$	a_4	$\frac{3\alpha + 3\alpha^2}{3 + 10\alpha + 3\alpha^2}$
a_2	$\frac{\alpha}{1+\alpha}$	a_5	$\frac{6 + 28\alpha + 16\alpha^2}{16\alpha + 28\alpha^2 + 6\alpha^3}$
a_3	$\frac{1 + 2\alpha}{2\alpha + \alpha^2}$	a_6	$\frac{50\alpha + 125\alpha^2 + 50\alpha^3}{50 + 310\alpha + 310\alpha^2 + 50\alpha^3}$

Table B.6 gives the first six values for the alternating products of the right Arithmetic Triangle $\mathbf{\Delta}_\alpha$ for a given $\alpha \in \mathbb{N}$. Notice that if $\alpha = 1$ we have a table identical to Table B.3. However, if we let $\alpha = 2$, then we have a sequence of rational numbers identical to those in Table B.4.

Table B.7: Alternating Products of ${}_\alpha\mathbf{\Delta}_\beta$

a_1	$\dfrac{\alpha}{\beta}$	a_4	$\dfrac{3\alpha^2\beta + 3\beta^2\alpha}{3\alpha^2 + 10\alpha\beta + 3\beta^2}$
a_2	$\dfrac{\alpha\beta}{\alpha + \beta}$	a_5	$\dfrac{6\alpha^3 + 10\alpha^2\beta + 16\beta^2\alpha}{16\alpha^2\beta + 28\alpha\beta^2 + 6\beta^3}$
a_3	$\dfrac{\alpha^2 + 2\alpha\beta}{2\alpha\beta + \beta^2}$	a_6	$\dfrac{50\alpha^3\beta + 125\alpha^2\beta^2 + 50\alpha\beta^3}{50\alpha^3 + 310\alpha^2\beta + 310\alpha\beta^2 + 50\beta^3}$

The last table of values we exhibit is Table B.7. These values give the general form for the alternating product of each row in the Generalized Arithmetic Triangle ${}_\alpha\mathbf{\Delta}_\beta$. Notice the symmetry of numerator and denominator. This gives us another explanation for why each odd entry in Table B.3 is identically 1.

B.2 Code

During the course of investigating $\mathbf{\Delta}$, its generalizations, and constructs placed upon it, we used a great deal of self-written computer code to generate arbitrary Generalized Arithmetic Triangles. We also used the capability of computers to perform a multitude of arithmetic operations in a fraction of the time required for a human to do the same tasks to calculate alternating products. Pieces of the code created are given below. All code was written in Python 3.8.10.

```
#Multifactorial function
def mult_fact(n,a):
    if n <= a:
        return n
    else:
        return n* mult_fact(n-a,a)

#Standard Binomial Coefficients
def binom_coeff(n,k):
    if n < abs(k):
        return 0
    elif n == k:
        return 1
    else:
        num = mult_fact(n,1)
        denom = mult_fact(k,1) * mult_fact(n-k,1)
        return num/denom

#Generates the elements of Alpha delta Beta
def gen_binom_coeff(alpha, beta, n,k):
    num = alpha * n + k * (beta - alpha)
    denom = n
```

```
    return (num * binom_coeff(n,k))/denom


    #Generates the elements of deltaS
def AGAT(S,n,k):
    if n == 0 and k == 0:
        return 1
    elif k == 0:
        return S[0]
    elif k == n :
        return S[len(S)-1]


    elif n == len(S)-1:
        return S[k]
    elif n < len(S) or n < abs(k):
        return 0
    else:
        return AGAT(S,n-1,k) + AGAT(S,n-1,k-1)

#Generates a single row of deltaS
def gen_row(S,n):
    out = []
    for ii in range(0,n+1):
        out.append(AGAT(S,n,ii))
    return out
```

```
#Gives Alternating Product of Row n
def alt_prod_row(S,n):
    out = 1
    row = gen_row(S,n)
    for ii in range(0,n+1):
        temp = (row[ii])**((-1)**ii)
        out *= temp
    return out
```