Collaborative Creativity in Undergraduate Mathematics:

Exploring Student Experiences of In-Class Collaborative Proving

By

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"Now to Him who is able to do far more abundantly than all that we ask or think, according to the power at work within us, to Him be glory in the church and in Christ Jesus throughout all generations, forever and ever. Amen." Ephesians 3:20-21

ABSTRACT

Creativity and collaboration are key components of a mathematician's work, and thus in the preparation of future mathematicians, undergraduate mathematics courses should aim to develop students who can effectively work collaboratively and creatively with one another. Despite this need, research on mathematical creativity has primarily investigated creativity as an individual, rather than collaborative, construct. At the undergraduate level, there is even less research on collaborative creativity and how students experience it in mathematics. Undergraduate introduction-to-proof courses provide a context in which students transition from computational-based mathematics to abstract, proof-oriented mathematics and are potentially challenged with becoming practitioners of their own mathematical ideas for the first time. This course provides a context ripe for exploration of experiences in collaborative creativity in mathematics. Throughout this dissertation, I respond to the overarching research question: How do students experience creativity in collaborative proving? This research is presented in the form of three research studies and corresponding manuscripts, each investigating student experiences of collaborative creativity in proving from a different perspective.

Each of the three studies presented in this dissertation were conducted in the same context of an undergraduate introduction-to-proof course. This course was instructed through collaborative, inquiry-oriented methods, including regular engagement in small-group collaborative proving activities and whole-class discussions. Data in the form of in-class video recordings, student written reflections, and focus-group stimulated-recall interviews were collected over the course of a semester-long introduction-to-proof course.

Beginning in Manuscript 1: *Critical Moments in Creative Collaborative Proving,* I examined the creative processes a group of three students engaged in during three different

collaborative proving episodes. In this first study, I examined collaborative creativity in proving through a lens of intersubjectivity, viewing the group of students as both a whole unit as well as each student as a component part. The second manuscript, *In-the-Moment Experiences of Creativity in Collaborative Proving*, focused upon only the individual experience of creativity through collaborative proving and examined student reflections of how they felt, or did not feel, creative in their in-class collaborative proving activities. The third and final manuscript, *Student Experiences of Creativity in a Collaborative, Inquiry-Oriented Introduction-to-Proof Course*, de-emphasized the daily collaborative proving course activities, and rather "zoomed out" to examine what contexts students recalled throughout their course as having fostered their creativity and what within each of those contexts allowed them to feel creative.

Many of the findings presented throughout these manuscripts illustrate how collaborative creativity in proving is similar to individual creativity in proving, but others highlighted how a collaborative context may introduce unique experiences and complexities to be considered in conceptualizing creativity in proving. Specifically, it was observed that students recognize themselves as creative in collaborative contexts, and unique features of a collaborative setting, like social risk taking (i.e., making mistakes, seeking/receiving help, diverging from a teammate's idea), a shared sense of responsibility and ownership over creative ideas, and exposure to a variety of mathematical proving approaches from multiple peers. These results distinguish collaborative creativity in proving from extant findings regarding individual mathematical creativity.

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Introduction-to-proof (ITP)

Stimulated-Recall Interview (SRI)

Multiple-Solution Task (MST)

Retrospective Writing Prompt/Assignment (RW)

Creativity-In-Progress Reflection (CPR) on Proving

Data Collection (DC)

Mathematical Association of America (MAA)

Introduction

Both collaboration and creativity are central to the work of mathematicians (Karakok et al., 2015; Sriraman, 2004). Mathematics and problem solving inherently require creativity, and being able to creatively collaborate with others is an important 21st Century skill (Geisinger, 2016; Honey et al., 2014). In the field of mathematics, professionals have indicated that collaboration is an important feature of their work (Sriraman, 2004), and the Conference Board of Mathematical Sciences (2016) has called for university mathematics classrooms to incorporate more active, collaborative learning. Therefore, there is a substantial need for K-16 education to focus on developing creative mathematicians and problem solvers with strong communication and collaboration abilities.

In mathematics, as students develop from novice to expert (Lave & Wenger, 2001) they follow a journey in which there is an eventual transition from computational to proof-based mathematics (Civian & Schley, 1996). This transition typically occurs at the undergraduate level, and at this point students are expected to become creative producers of mathematical ideas on their own for perhaps the first time (Boyle et al., 2015). Furthermore, the transition to proof offers an opportunity to engage students in authentic mathematical disciplinary activities (Melhuish, Vroom, et al., 2022), including collaboration and communication of mathematical ideas. For this reason, it is critical to study mathematical creativity and collaboration within the context of undergraduate mathematics courses and proof in order to cultivate the expertise needed by future mathematicians.

Research has suggested the most effective strategies for fostering mathematical creativity in the classroom are some of those same strategies aligned with active or inquiry-oriented learning (e.g., Asahid & Lomibao, 2020; Leikin & Elgrably, 2020; Levav-Waynberg & Leikin,

2012). Although some proof-based courses have begun to incorporate more active learning strategies (e.g., Bleiler-Baxter & Pair, 2017), most advanced mathematics courses continue to be taught in a lecture style, and most mathematicians believe lecture is the best way to teach (Melhuish, Fukawa-Connelly, et al., 2022) despite the importance of collaboration in mathematicians' work and research (Burton, 1998; Grossman, 2002; Sriraman, 2004). Thus, researchers need to explore the nature of and students' experiences with collaborative creativity in proof-based mathematics courses taught using a student-centered approach.

There are further benefits of teaching to foster creativity, as it has been seen to build positive affect and mathematics identity (Regier & Savić, 2020; Tang et al., 2022) and promote equity in the mathematics classroom (Kozlowski & Si, 2019; Luria et al., 2017). Subsequently, affective gains such as confidence and self-efficacy have been seen to impact persistence in mathematics (Ellis & Cooper, 2016). Moreover, teaching that promotes equity by valuing risk taking and different approaches to ideas may specifically help students from underrepresented minority populations, such as women and people of color, develop a stronger sense of belonging in mathematics, which may promote retention (Rattan et al., 2015).

Although some studies have addressed teacher actions to foster creativity at the undergraduate level (e.g., Satyam et al., 2022; Savić, El Turkey, et al., 2017b), these investigations have not explicitly explored the role of collaboration, which is both an element of student-centered mathematics instruction and a necessary skill for all professionals in STEM, including mathematicians. Therefore, it is vital to investigate the nature of collaborative creativity in proving and how collaboration can be used in proof-based courses to foster creativity.

Statement of the Problem

Research on mathematical creativity beyond K-12 education has primarily focused on professional mathematicians' views and experiences with creativity (e.g., Karakok et al., 2015; Sriraman, 2004), the opportunities available for creative reasoning in undergraduate mathematics courses (e.g., Bergqvist, 2007; Mac an Bhaird et al., 2017; Selden & Selden, 2013), teaching strategies to foster creativity (e.g., Asahid & Lomibao, 2020; Leikin & Elgrably, 2020; Satyam et al., 2022; Savić, El Turkey, et al., 2017a, 2017b), and the impact teaching to foster creativity has on affect, self-efficacy, or equity (e.g., Regier & Savić, 2020; Satyam et al., 2022; Tang et al., 2022). As listed, there has been compelling research conducted on how to teach to foster mathematical creativity in undergraduate mathematics courses. Yet, only a few of these studies have investigated the creative process in the specific context of proof-based courses (e.g., Savić, El Turkey, et al., 2017a, 2017b; Savić, Karakok, et al., 2017), and none of these studies have addressed how collaboration may interact with creativity in proving. As a key feature of both professional life in mathematics and student-centered learning, it is essential to better understand the role of collaboration in creative proving.

Study Purpose

The purpose of this qualitative study is to gain an understanding of students' processes and experiences with creativity in collaborative proving in an introduction-to-proof (ITP) course setting. Therefore, I have formed an overarching research goal of investigating the question: **How do students experience creativity in collaborative proving?** This research goal will be addressed in the form of three separate manuscripts, each addressing student experiences of creativity through collaboration in an introduction-to-proof course at different organizational levels: (1) processes of a group, (2) in-the-moment experiences individual students report from

collaborative proving, and (3) holistic experiences of creativity throughout the introduction-to-proof course taught through collaborative, inquiry-oriented methods.

Definitions Used Throughout This Dissertation

Throughout this dissertation, there are three terms used frequently that should be addressed: mathematical creativity, proof, and collaboration. Each of these three terms have many different definitions used in mathematics education research, and how I frame and define these different constructs, and their intersections, will have a marked impact on my study design and methodology as well as how I interpret my results. In the following subsections, I outline the definitions I intend to use throughout my dissertation research. This includes definitions of mathematical creativity, proof, and collaboration as well as the intersections among creativity in proving, collaborative proving, collaborative creativity, and collaborative creativity in proving. These definitions may also be considered delimitations of my research, or decisions regarding how I frame these constructs that are in alignment with previous research, and thus they may limit the scope of my research.

Mathematical Creativity

Even within the discipline of research on mathematical creativity, the word has been notoriously difficult to define. Mann (2005) claimed there are over 100 existing definitions of creativity in the mathematics education literature, and this number has only grown with the volume of research on creativity published in the last 17 years (Savić et al., 2022). In this study, I adopt a perspective on mathematical creativity that describes creativity as *domain-specific*, meaning I am interested in creativity specific to the context of mathematics and mathematical proof, *a process*, meaning I will consider the motivation, perception, learning, thinking, and communicating involved in creativity rather than analyze an end result or product of creative

thinking, and *relative*, meaning I am considering creativity within the context of the knowledge, abilities, and experiences of an individual rather than requiring creativity to produce completely novel contributions to the field of mathematics. Put succinctly, ***mathematical creativity*** is defined to be the processes of creating, constructing, or implementing mathematical ideas, strategies, or processes, which are perceived as non-routine by the individual.

Proof

In this study, situated within a university introduction-to-proof course, I adopt the definition of ***proof*** given by A. J. Stylianides, (2007):

> Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification.
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291, emphasis in original)

This definition allows for flexibility in the growth of a classroom community and the tools (both accepted statements and modes of argumentation) valid within this community.

Collaboration in Proving

One way to highlight the inner workings of students' approaches to the proving process (i.e., proof initiation, construction, and validation) is to engage students in collaborative proving. Collaboration is a tool for revealing the deeper processes behind proving and reflects the work of mathematicians (Grossman, 2002). To define collaboration beyond social interaction (Sriraman, 2004), I emphasize two elements of collaboration as a process: (1) requiring contribution from two or more parties and (2) sharing a common goal. This definition of collaboration can be applied to a classroom setting to describe collaborative learning, wherein typically student

groups of two or more gain mutual understanding or create a product (Smith & MacGregor, 1992).

In this study I use the terms ***collaborative creativity in proving***, or creative collaborative proving, to describe the process of approaching a shared goal in proving with significant contributions from two or more people, in which the proving task is either (a) assumed to be non-routine for all group members based on context or (b) established to be non-routine for all group members by asking them for their perspective.

High-Level Framing (Situating Myself in the Study)

The high-level theoretical framing (St. Pierre, 2000 as cited by Stinson & Walshaw, 2017) I take in this study is interpretivism. Ontologically, I maintain there is a truth, yet I also admit human perception clouds the individual experience of truth. Thus, in this study I work to understand individual experiences, yet acknowledge these individual experiences may not perfectly represent the objective truth. As I seek to understand students' experiences of the phenomenon of *collaborative creativity* in the context of mathematical proof, I further frame my work through a phenomenological perspective.

A relative view of creativity necessitates considering the participant perspective and interpretation of events, yet most research on mathematical creativity, particularly collaborative mathematical creativity, has privileged the researcher interpretation of events (i.e., an *etic* perspective). The issue of prioritizing researcher interpretation of events and therefore tarnishing the relative view of creativity is resolved by taking a *phenomenological approach to research* on mathematical creativity as well as an *emic approach to qualitative research*. The goal of phenomenology is to capture and describe how people experience a phenomenon (van Manen, 1990). This frame of qualitative inquiry prioritizes the lived experiences of those participating in

a given phenomenon. In this respect, a phenomenological methodology in studying the collaborative creative proving process would inquire with participants about how and when they felt creative during a task and what was their perception of the collaborative creative process. Keeping with a phenomenological perspective, throughout this study the data I draw upon are primarily students' own written or spoken words regarding their experiences, rather than my interpretation of a situation from my perspective as a researcher. This study values students' perception of their experiences and growth as well as attempts to reconcile the multiple intersubjective experiences of a collaborative episode in order to best approximate an objective reality.

Aligning with a phenomenological approach to qualitative research is an *emic approach* to describing observations in data. Emic refers to an insider's view of and language surrounding a phenomenon and is contrasted with an etic approach, which refers to the outsider's (or researcher's) perspective and language. The data collection and analysis methods used in the following three studies capitalize upon either in-depth interviews or detailed written narratives as well as inductive, in-vivo qualitative coding strategies in order to honor participant language and voice.

I adopt a perspective on learning grounded in Situated Learning Theory (Lave & Wenger, 2001). In this sense, I take a macroscopic view of learning in which a newcomer, or novice, to a community of practice learns by increasingly participating in the practices of the community. In mathematics, this means the goal of learning on a large scale is for students to become more legitimate practitioners of mathematics by enacting the practices experts effectively use in the profession. With respect to proof, situated learning theory helps distinguish the ITP course as a critical transition point for students in which they learn the practice of proving, a practice central

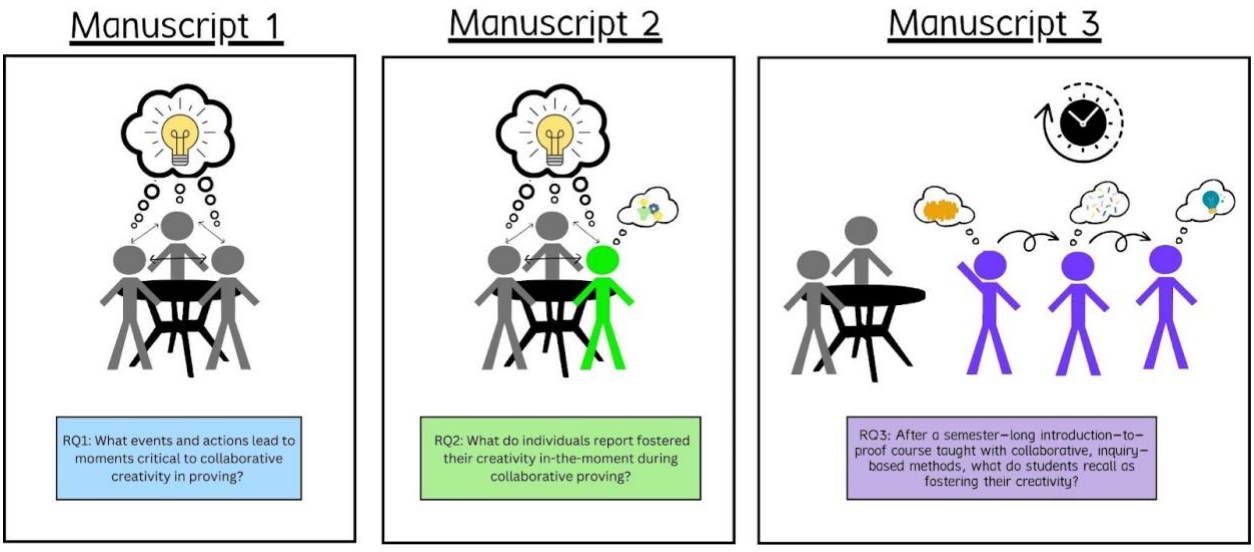
to the mathematics community. Throughout this dissertation, I supplement my high-level framing with mid-level and ground-level theories within each research manuscript, namely participatory intersubjectivity, residue, and definitions of creativity, collaboration, and proof.

Research Questions and Organization

I have formed an overarching research goal of investigating the question: **How do students experience creativity in collaborative proving?** This overarching research goal was addressed by separating the question into smaller, more manageable sub-questions, guided by inquiring (1) about the process of creative collaborative proving for a small collaborative group as a *whole*, (2) how *individual students* experience creativity in-the-moment during collaborative proving, and (3) what *individual students* recall as having fostered their creativity throughout the duration of an introduction-to-proof course taught through inquiry-oriented, collaborative methods. Figure 0-1 illustrates how these three perspectives and goals are distinct.

Figure 0-1

Visual Outline of Three Manuscripts and Research Questions



In the three manuscripts following this introduction, I respond to three research questions:

RQ1: What events and actions lead to moments critical to collaborative creativity in proving?

RQ2: What do individuals report fostered their creativity in-the-moment during collaborative proving?

RQ3: After a semester-long introduction-to-proof course taught with collaborative, inquiry-based methods, what do students recall as fostering their creativity?

These three research questions guided me in studying what propels creativity for groups in proof, how collaboration initiates creativity for an individual in-the-moment, and what students recall as fostering their creativity in an ITP course with consistent engagement in collaborative proving. Throughout these studies I explored three avenues in my dissertation to contribute to theory on collaborative creativity in proving by including and highlighting the student perspective. All three studies were conducted within the same setting and data will be collected from a proof-based mathematics course for undergraduate students with an emphasis on collaboration and active learning.

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Manuscript 1: Critical Moments in Creative Collaborative Proving

ABSTRACT: In this study, I examined the creative processes a group of three students engaged in during three different collaborative proving episodes. A theoretical lens of intersubjectivity was used to view a focus group of students as both a whole unit as well as each student as a component part. Focus-group stimulated-recall interview data were used to identify moments during the collaboration critical to the group's creativity, and then an abductive analysis strategy was conducted to describe the activities of the student group that allowed the critical moment to occur. Across three episodes, the critical moments included realizing a mistake that allowed the team to further generalize their claim, one student diverging from the path originally selected to pursue by her team, and recalling the logical properties of an implication statement to determine a claim was vacuously true. The events and circumstances that allowed these critical moments to transpire included making mistakes, seeking counterexamples, responding to instructor questioning, team willingness to pursue multiple avenues for proving a statement, incubation time, and instructor validation of student concerns and ideas.

1. Introduction

Both collaboration and creativity are central to the work of mathematicians (Karakok et al., 2015; Sriraman, 2004). Mathematics and problem solving inherently require creativity, and being able to creatively collaborate with others is an important 21st Century skill (Geisinger, 2016; Honey et al., 2014). In the field of mathematics, professionals have indicated that collaboration is an important feature of their work (Sriraman, 2004), and the Conference Board of Mathematical Sciences (2016) has called for university mathematics classrooms to incorporate more active, collaborative learning. Therefore, there is a substantial need for K-16 education to focus on developing creative mathematicians and problem solvers with strong communication and collaboration abilities. In the trajectory from novice to expert (Lave & Wenger, 2001) in the field of mathematics, students follow a journey in which there is an eventual transition from computational to proof-based mathematics (Civian & Schley, 1996). This transition typically occurs at the undergraduate level, and at this point students are expected to become producers of mathematical ideas on their own for perhaps the first time (Boyle et al., 2015). For this reason, it is critical to study mathematical creativity and collaboration within the context of undergraduate mathematics courses and proof in order to cultivate the expertise needed by future mathematicians.

Alan Schoenfeld (2009) noted, "If problem solving is the 'heart of mathematics,' then proof is its soul" (p. xii). Proof is central to the field of mathematics; in fact, it is considered so important to the undergraduate degree in mathematics that 82% of United States universities with R1 or R2 Carnegie classifications offer a course dedicated to introducing students to formal proof (David & Zazkis, 2020); these courses are often referred to as introduction-to-proof (ITP) courses. ITP courses are typically required of undergraduate mathematics majors following their

calculus sequences and either as a prerequisite or corequisite to Abstract Algebra or Analysis (David & Zazkis, 2020). The goal of ITP courses is to ease students' transition from calculation-based courses to proof-centered, abstract mathematics courses. Despite the efforts of ITP courses, students encounter a number of difficulties during this transition, such as a decline in mathematical confidence (Civian & Schley, 1996; Rainey et al., 2018) or struggle with their sense of belonging in mathematics (Rainey et al., 2018). Moreover, when students take their ITP course, they become producers of mathematical ideas for what may be the first time (Boyle et al., 2015). Yet, teaching to foster creativity has been seen to build positive affect and identity (Regier & Savić, 2020; Tang et al., 2022) and promote equity in the mathematics classroom (Kozlowski & Si, 2019; Luria et al., 2017). If, however, instructors wish to promote *collaborative* creativity in their proof-based classrooms, research must first seek to understand how the collaborative creative proving process can be characterized and described as well as what events allow moments critical to team creativity to transpire in collaborative proving. Learning more about the intersection of collaboration and creativity in ITP courses will help instructors learn how to best foster collaborative creativity in proving.

Therefore, in this study I seek to investigate and characterize the nature of collaborative creativity in proving in a real undergraduate introduction-to-proof course. Specifically, I seek to answer, **how can the process of collaborative creative proving be characterized?** In particular, **what events and actions lead to moments critical to collaborative creativity in proving?**

2. Background: Mathematical Creativity as a Process - Introducing Collaboration

The foundation of research on mathematical creativity as a process is Wallas' (1926) four stages of creativity: preparation, incubation, illumination, and verification. Hadamard (1954)

popularized this model in the study of mathematical creativity. Because Wallas' (1926) model was designed with domain-general, long-term, individual creativity in mind, it is reasonable to ask if (a) the model applies to the mathematical creative process, (b) the model could possibly be extended to describe the process of collaborative creativity, and (c) what other research has been conducted to describe collaborative creativity in mathematics. The first two of these questions have been explored, yet research regarding how to describe the collaborative creative process is sparse.

2.1 Applicability of Wallas' (1926) Model to Mathematicians and Mathematics Students

Studies have investigated the applicability of Wallas' (1926) four-stage model to the processes involved in the creative work of mathematicians (e.g., Savić, 2016; Sriraman, 2004) and mathematics students (e.g., Bicer & Bicer, 2022; Prusak, 2015; Schindler & Lilienthal, 2020). The results of these studies are mixed. Both Sriraman (2004) and Prusak (2015) concluded that the processes followed by mathematicians and mathematics students can be appropriately described by Wallas' (1926) process of preparation (understanding a problem), incubation (allowing the mind to consider a problem subconsciously), illumination (generating an idea or having an "Aha!" moment), and verification (determining if the idea was correct or viable). Both of these studies, however, drew upon the recollection of participants' processes in interviews. Because Wallas' (1926) model is a psychodynamic view of creativity, it submits to the influence of both conscious and unconscious activity. Savić (2016) recommended research on the mathematical creativity process employ neuroscientific methodologies to better reveal the un/subconscious dimension of the creative process.

Schindler and Lilienthal (2020) and Bicer and Bicer (2022) both responded to this call and conducted research on the subconscious activity within the individual mathematical

creativity process using neuroscientific eye-tracking technology. This line of research has primarily modified Wallas' (1926) model to give a more detailed description of the creative process observed in mathematics students. For example, Schindler and Lilienthal (2020) introduced the phrase "mini-incubation" (p. 1581) to distinguish their participant's experience working on several approaches at once and suddenly having a revelation or insight about one approach while working on another for the same task. Wallas' (1926) incubation is distinct from this idea because it is used to describe abandoning the task altogether and occupying the mind with things not relevant to the task.

Another primary adaptation to Wallas' model through these eye tracking studies with mathematics students was to characterize the process as a cycle in which ideas may be discarded and the student returns to the initial stage of preparation (Bicer & Bicer, 2022) rather than a linear process. These adaptations to Wallas' (1926) model are valuable for considering the work of students in a classroom because students are often not given opportunities to experience true incubation (Savić, 2016); moreover, the redesign of the process as cyclic is valuable to both students and mathematicians as they both often experience the abandonment of one idea to start the task again from the beginning and develop a new strategy.

2.2 Applicability of Wallas' (1926) Model to *Collaborative* Creativity in Mathematics

Both Schindler and Lilienthal (2020) and Bicer and Bicer (2022) answered Savić's (2016) call to explore the mathematical creative process using neuroscience to refine Wallas' (1926) model and include more of the subconscious activity within the process through eye tracking technology. Both of these studies, however, are still restricted to describing the creative process of an individual and do not account for how the creative process may be different when people collaborate on mathematical tasks.

Recently, Schindler and Lilienthal (2022) expanded their research to investigate the collaborative creative process of two graduate students working together on a geometric multiple solution task (MST) while wearing eye tracking technology. Further, Schindler and Lilienthal (2022) conducted stimulated recall interviews (SRIs) using the eye tracking videos in which the participants were invited to explain their thinking process as they observed their own gaze-overlaid video data during the MST. Transcribed data from the MSTs and SRIs were analyzed inductively to determine phases throughout the collaborative creative work. This study resulted in a tentative model of collaborative creative process (Figure 1-1) with two branches: (A) one student has an idea and presents this approach to the other, and (B) one or both students have an intuition for an approach, but they develop the approach and solution together. These branches had both similarities and differences from previous models of the individual creative process. Similar to the individual process, the authors identified a "looking for a start" (p.172) phase, which often led to ideas or intuitions, and although the students experienced "mini-incubation" (the revelation of a new idea while doing something else related to the problem), in the collaborative space, this often occurred while the other participant shared his/her thoughts or approach. Differing from individual creative processes, the collaboration added complexity to the development and presentation of ideas as well as a more complicated idea-verification process rooted in the students' differing values and priorities in the task.

Figure 1-1

Schindler and Lilienthal's (2022, p. 174) Two Branches of the Collaborative Process

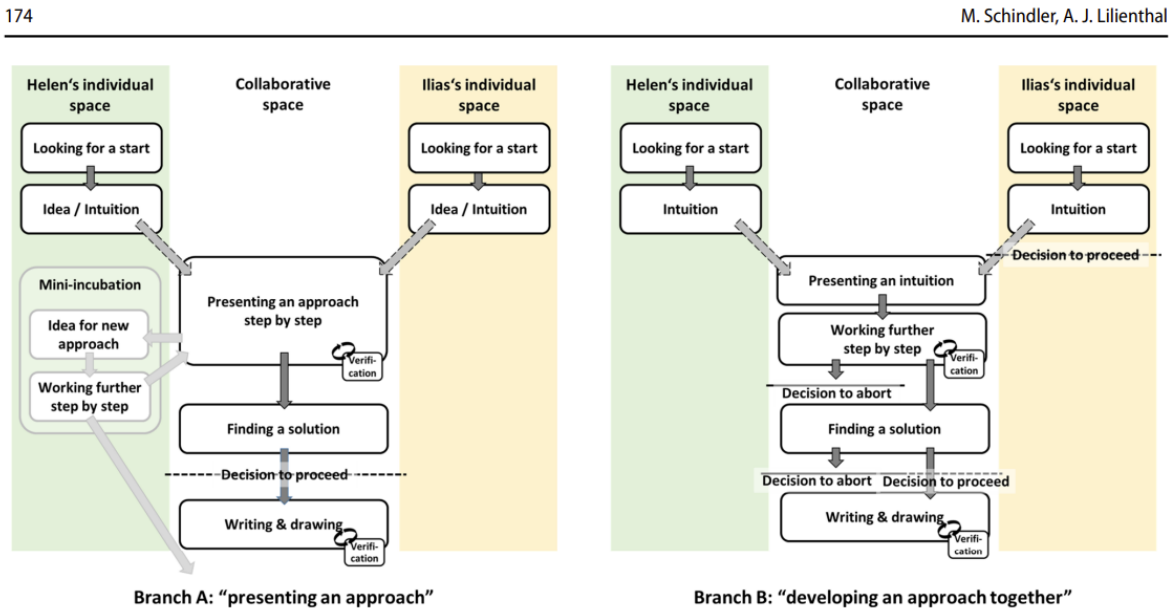


Fig. 9 Helen's and Ilias's collaborative creative processes

Note: From Schindler, M., & Lilienthal, A. J. (2022). Students' collaborative creative process and its phases in mathematics: An explorative study using dual eye tracking and stimulated recall interviews. ZDM – Mathematics Education, 54(1), 163–178. https://doi.org/10.1007/s11858-022-01327-9

2.3 Other Studies Concerning the Collaborative Mathematical Creativity Process

There have been some additional studies conducted regarding the collaborative creative process in mathematics that have not centered around Wallas' (1926) model (e.g., Adiredja & Zandieh, 2020; Aljarrah, 2020; Levenson, 2011; Sarmiento & Stahl, 2008). Yet, none of these studies capture the activities involved in an undergraduate small-group classroom proving setting. For example, Adiredja and Zandieh (2020) conducted interviews with linear algebra students and found that a creative product could be spurred by simply asking for a new example, shifting the audience, or asking a student how they would explain the idea to a friend. Although

this may be applicable to a small group setting, their claims were results of an interviewer-participant interaction, which does not reflect authentic collaboration in a real classroom.

Others have attempted to characterize interactions within a group as they work creatively. Most of the groups investigated were small groups of three to four people, yet Levenson (2011) investigated the creative process in the context of a whole-class discussion. Aljarrah (2020) drew upon pre-existing literature to develop four collective creative acts: summing forces, expanding possibilities, divergent thinking, and assembling things in new ways. Using a design-based research methodology, Aljarrah observed sixth grade students working on mathematics in small groups and presented one case to demonstrate how collective creativity can be described according to these four themes and engaging in these collective creative acts is an interwoven process which spurs the emergence of new and crucial ideas.

Similarly, Sarimiento and Stahl (2008) examined group synchronic interactions (i.e., in parallel and simultaneously) as well as diachronic exchanges (i.e., over long spans of time) using ethnomethodology. Teams of middle-school students were selected to engage in online math discussions, and analysis of these groups resulted in observing three processes fundamental to the creative engagement of the teams: indexical referencing, group remembering, and bridging across discontinuities. This study was very context-dependent on the technology used and thus may not be generalizable to face-to-face group creativity.

2.4 Summary: Research on the Mathematical Creativity Process

When searching the literature, readers will find some research on the mathematical creativity process, but this research is heavily weighted toward characterizing the process of individual creativity. Such studies suggest that Wallas' (1926) four stage model for creativity may be appropriate for the work of individual mathematicians (Sriraman, 2004), and other

studies have adapted the model to better describe the creative process of individual students by using eye-tracking technology to understand the subconscious element of creativity (Bicer & Bicer, 2022; Schindler & Lilienthal, 2020). There have only been a few studies exploring the process of collaborative creativity in mathematics (e.g., Aljarrah, 2020; Sarimiento & Stahl 2008; Schindler & Lilienthal, 2022), which have mixed results and characterizations of collaboration. Further, these studies do not focus on undergraduate mathematics students nor on proving. Therefore, there needs to be supplemental research, specifically with undergraduate students as they make the transition from computational to proof-based mathematics, on the collaborative mathematical creative process and its impact on both group creativity and individual growth.

3. Theoretical Framing

This study is framed by two overarching theoretical perspectives: mathematical creativity as defined by the Creativity-in-Progress Rubric on Proving (CPR; Savić et al., 2017) and participatory intersubjectivity (Matusov, 1996; Sawyer, 2019). The CPR on Proving provided a guiding definition for mathematical creativity for both the researcher and the participants. Participatory intersubjectivity as a guiding theory is used to explain how individuals function as component parts of a cohesive, collaborative group.

3.1 Defining Mathematical Creativity: The CPR on Proving

In this study, I adopt a perspective on mathematical creativity that describes creativity as *domain-specific*, meaning I am interested in creativity specific to the context of mathematics and mathematical proof, *a process*, meaning I will consider the motivation, perception, learning, thinking, and communicating involved in creativity rather than analyze an end result or product of creative thinking, and *relative*, meaning I am considering creativity within the context of the

knowledge, abilities, and experiences of an individual rather than requiring creativity to produce completely novel contributions to the field of mathematics. Put succinctly, ***mathematical creativity*** is defined to be the processes of creating, constructing, or implementing mathematical ideas, strategies, or processes, which are perceived as non-routine by the individual or assumed to be non-routine for a group.

Mathematical creativity as a domain-specific, relative, process is theoretically aligned with the assumptions of the Creativity-In-Progress Rubric (CPR) on proving (Savić et al., 2017), which was used to guide participants in identifying moments relevant to their creativity throughout the collaborative proving process. The CPR on Proving was developed by Savić et al. (2014; 2017) as a formative assessment and metacognitive tool for individual students in mathematical proving contexts. The CPR describes creativity in proving according to two major categories: making connections and taking risks. Each category of the CPR has more detailed subcategories. The making connections subcategories are: between definitions/theorems, between examples, and between representations. The taking risks subcategories are: tools and tricks, flexibility, posing questions, and evaluation of the proof attempt. These categories and subcategories guided the definition of mathematical creativity in proving adopted by the researcher and participants in the current study, particularly in the discussion of collaborative proving during stimulated-recall interviews (see methods section).

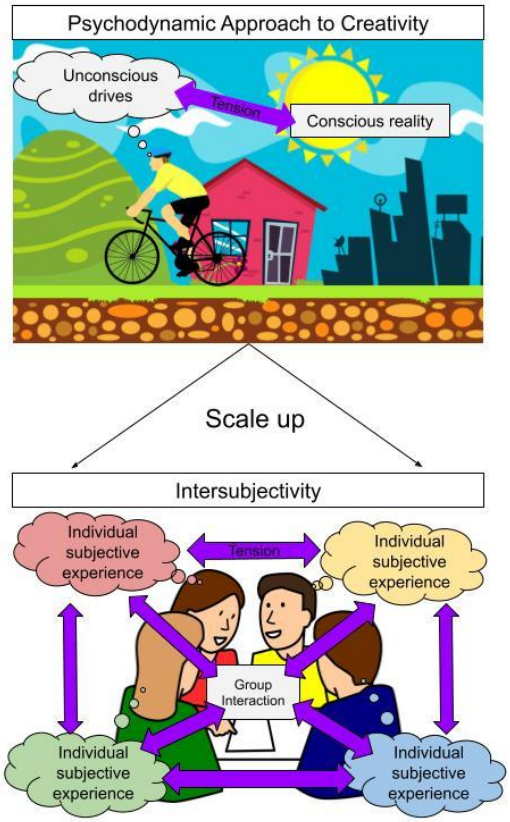
3.2 Psychodynamic Creativity and Participatory Intersubjectivity

In studying creativity as an individual construct, the psychodynamic approach investigates creativity as the tension between unconscious drives and conscious reality (Sriraman, 2004). One example of a psychodynamic approach to creativity is Wallas' (1926) four stage process of creativity (see section 2.1; Savić, 2016). Psychodynamic approaches to

creativity are more concerned with the *processes* involved in creativity, rather than the products. I contend that in a group setting, the psychodynamic perspective can be scaled up to a multi-party scenario to investigate the tension between individual subjective experience and the objective reality of actions within a group. This interplay between multiple parties' subjective experiences is known as intersubjectivity (Matusov, 1996; Sawyer, 2019).

Figure 1-2

Psychodynamic Approach to Creativity Scaled to Intersubjectivity



A traditional definition of intersubjectivity may describe intersubjectivity as an overlapping set of individual subjectivities or when more than one individual shares the same subjective perspective of reality (Matusov, 1996). Matusov (1996) introduced an alternative

definition of intersubjectivity called participatory intersubjectivity, which is defined as "process of coordination of individual contributions to joint activity rather than as a state of agreement" (Matusov, 1996, p. 34), and I adopt the participatory approach to intersubjectivity for this study. This view enables one to account for how something new could be created by group interaction (Sawyer, 2019). This definition of intersubjectivity has been used to describe the creativity of a group engaging in improvisational theatre (Sawyer, 2019). In this way, intersubjectivity explains how, "Although each actor may have a rather different interpretation of what is going on and where the scene might be going, they can nonetheless proceed to collectively create a coherent dramatic frame" (Sawyer, 2019, p. 578). A group of students engaging in mathematical proving is quite similar to a troupe of improvisational actors. Each student may have a different interpretation of what should happen within the proof of a given statement and a different idea of what the end product of a proof may look like, but these perspectives and ideas evolve as they act and react to their peers during the collaborative process. Eventually, students create a coherent proof despite the differences among their subjective experiences. Sawyer (2019) refers to the process in which collaborating groups of individuals collectively generate a shared creative product as collaborative emergence, and participatory intersubjectivity is one of the characteristics of such collaborative emergence.

4. Methods

This study was conducted using qualitative methods and a phenomenological research perspective, in which the goal was to understand the essence of the experience of collaborative creativity in proving as informed by the lived experiences of the participants. In this section I describe the study context, participants, data collection and data analysis methods used in this study.

4.1 Study Context

This study was conducted over the duration of a semester-long (15 week) introduction-to-proof course at a large public university in the southeastern United States. Eleven students were enrolled in the course, which met twice a week for 85 minutes each. The introduction-to-proof course was taught using active and collaborative learning strategies. The course was segmented into six course modules: (1) Introduction to Upper-Level Mathematics and Proof, (2) Logic and the Grammar of Mathematics, (3) Direct and Indirect Proof Methods, (4) Proof by Mathematical Induction, (5) Set Theory, (6) Relations and Functions. The course was taught by Dr. Hadley (pseudonym), a white, female professor of mathematics with a PhD in Mathematics Education and 11 years of experience teaching this course.

Most class periods were structured by first engaging in a brief instructor-led reflection upon the previous class meeting and an instructor-led review of definitions or concepts relevant to the lesson and module, then students, in assigned groups of three or four, were assigned a proof-related task to work on collaboratively. After the collaborative group work, the instructor facilitated presentations of selected groups' work and led a whole-class discussion regarding the different assigned tasks.

4.1.1 Structure of Collaborative Group Work

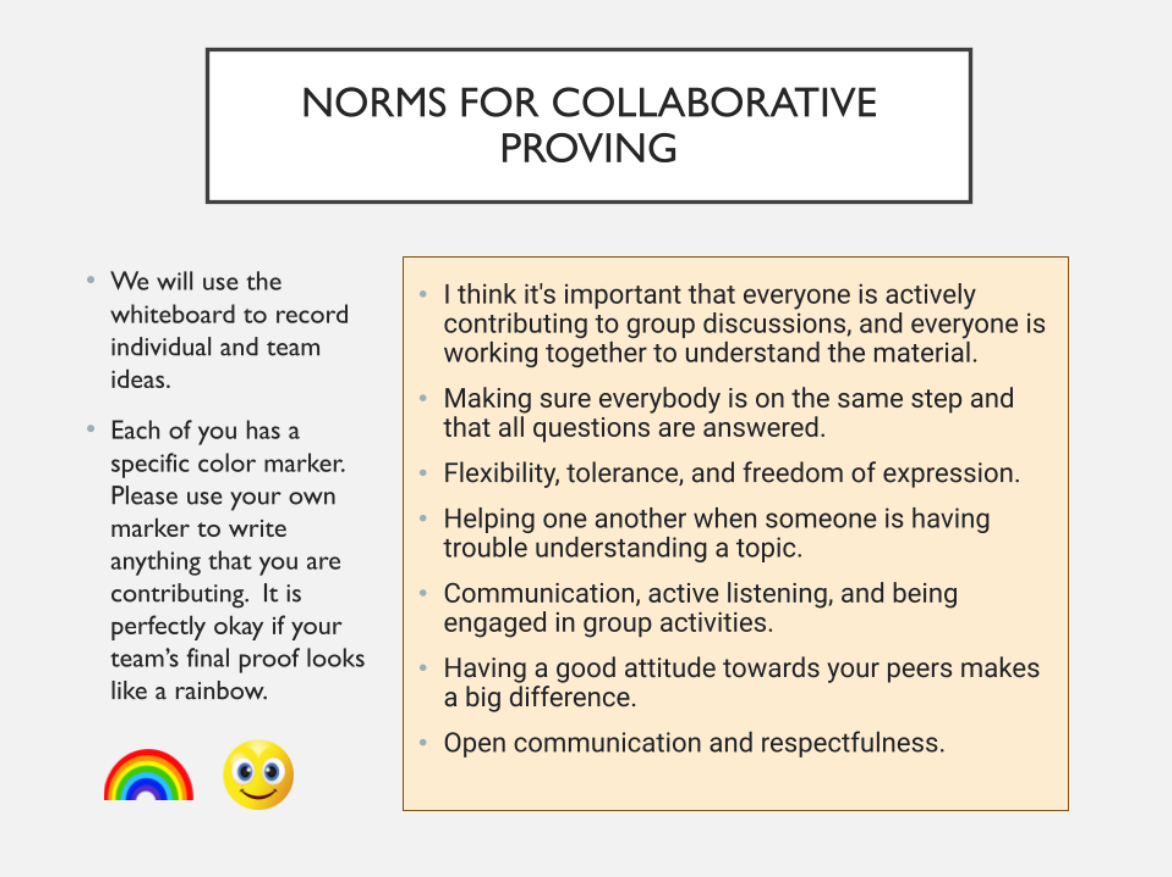
Each time students in the course were directed to work collaboratively on a proving task, they were in instructor-assigned groups of three or four students each. Groups were seated at desk clumps around the classroom. When assigned a proving task, groups were instructed to take three to five minutes of individual think-time about the task before collaborating. Following individual think time, groups were instructed to stand and move to dry-erase boards mounted on the walls around the classroom to work on their tasks together. Each student in a group was

given a different colored dry-erase marker to use throughout the collaboration. The different colored markers allowed both the course instructor and the researcher to track the contributions of each group member to the groups' written work. Groups were typically given 20-40 minutes to work together on their proving tasks. During some class sessions all student groups were given the same proving task, and, in others, each group was assigned a different proving task.

During the first week of the course, the instructor introduced classroom norms for collaborative proving. In an initial course getting-to-know-you survey, the instructor asked students in the course, "In this course, we will be working together with peers collaboratively on most days. What are some standards/norms for collaborative work that you believe will be important for our classroom?" The course instructor compiled the student responses into a list (Figure 1-3) and displayed the set of norms to the class before groups engaged in their collaborative proving the first time students were tasked with working together.

Figure 1-3

PowerPoint Slide of Class-Created List of Norms for Collaborative Proving



4.1.2 Participants

Eleven students were enrolled in the introduction-to-proof course, and ten students consented to participate in the research study. From the ten research participants in the larger dissertation study, three participants were purposefully sampled to compose a focus group for this study. These three participants were selected based on an initial course writing assignment in which they described their perceptions of mathematical creativity and previous experiences feeling creative in mathematics. The participants were selected for this study because they demonstrated an ability to articulate their thoughts, feelings, and experiences about mathematical

creativity well to others. Although not part of the selection criteria, all participants in the focus group ended the course with similar passing course averages, indicating the group was homogeneous with respect to ability and achievement in the course. Across the entire course section, two students received As, one received a B+, three received Bs, one received a C-, two received Ds, and two received Fs.

In the following sections, I describe each participant in the focus group in detail. These descriptions were compiled from a course getting-to-know-you survey, participant introductions during an initial focus group interview, and a final research questionnaire in which participants were invited to choose their research pseudonym and self-describe identities important to their depiction in research. All pseudonyms were self-selected by participants.

Brad Junior. Brad Junior is a white, male, sophomore student majoring in general, or "pure," mathematics. Brad Junior also identified himself as a Christian and the oldest of four brothers. Brad Junior had not taken any mathematics courses involving proof before enrolling in the introduction-to-proof course. He described himself as having always been "one of those advanced kids, especially when it comes to math." He said being advanced in mathematics and working ahead of his grade level strengthened his enjoyment of mathematics. Brad Junior experienced his first real challenge in mathematics when he took Calculus III because it was "confusing" and "took a lot more work," but he stated he was able to pass the class despite these challenges.

Ashley. Ashley is a white, female, junior majoring in mathematics with a concentration in business mathematics. Before enrolling in the introduction-to-proof course, Ashley previously took a course in discrete mathematics, which introduced some formal logic and proof techniques in addition to matrices, graphs, formal grammars, finite state machines, Turing machines, and

binary coding schemes. When asked about her relationship with mathematics, Ashley said “I can’t do stats, but I am doing stats. It’s not my thing, but give me Calc I, II, or III any day. I don’t know why.”

Participant B. Participant B is an African American, male, junior majoring in mathematics with a concentration in statistics. Participant B was a transfer student to the university where this study took place, and this semester was his first semester at the new university. Participant B said, “I love statistics. I just use Microsoft Excel like nobody’s business.” He described his relationship with mathematics in general as “hit or miss,” saying he enjoyed Calculus I, hated Calculus II, and loved Calculus III.

4.2 Data Collection

Data used in this study were collected from three separate episodes of in-class collaborative creative proving. For each in-class proving episode, or round of data collection (DC), two forms of data were collected for use in this study: (1) audio and video recordings of the focus group throughout their in-class collaborative proving and (2) focus-group stimulated recall interviews, which were recorded with audio and video. The three in-class episodes selected for data collection occurred during weeks 1, 5, and 11 of the 15-week semester-long course. The tasks assigned to the class during these class meetings are described in Table 1-1, with the tasks in bold assigned to the focus group.

Table 1-1

Collaborative Proving Tasks

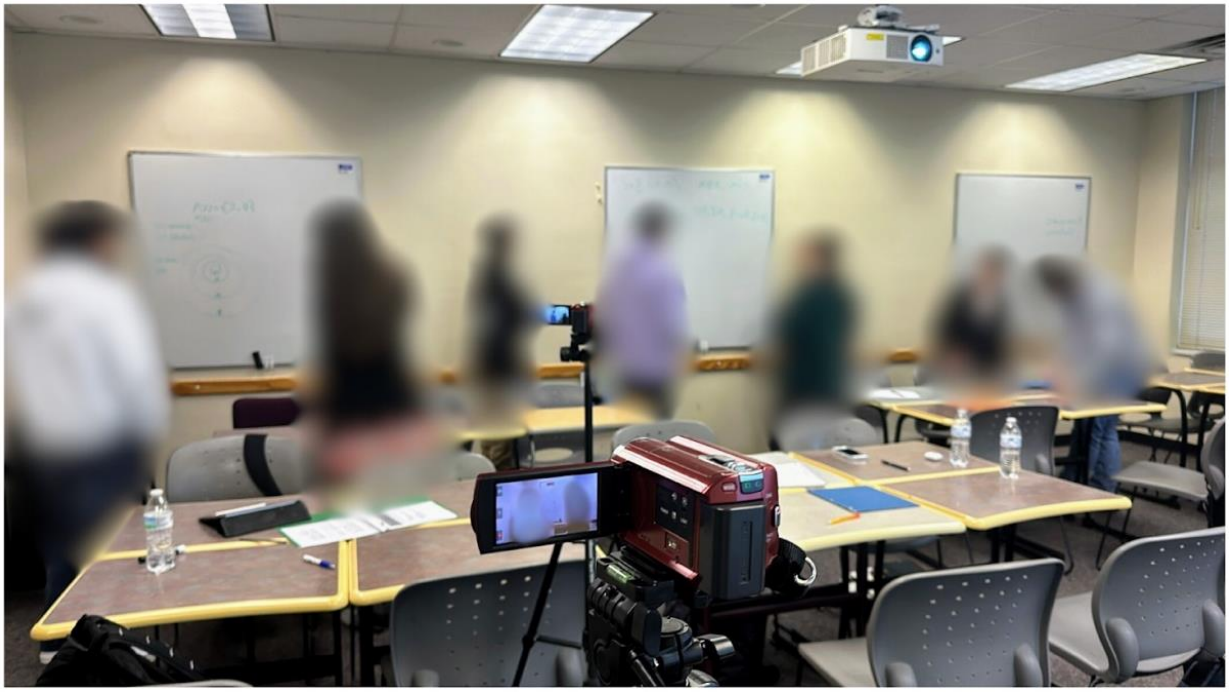
	Course Module (Week)	Task (Tasks in bold were assigned to the focus group)
DC1	Module 1: Introduction to Upper-Level Mathematics and Proof (Week 1)	**Prove: The product of consecutive twin primes is one less than a perfect square.**
DC2	Module 3: Direct and Indirect Proof Methods (Week 5)	**Either (a) Prove, or (b) Disprove and Salvage If Possible.** Team 1: An integer is even if and only if its square is even. Team 2: If *l* and *m* are odd integers, then so is *lm*. **Team 3: If p is an integer, then $p^2 + 3p + 2$ is even.**
DC3	Module 5: Set Theory (Week 11)	**Either (a) Prove, or (b) Disprove and Salvage If Possible.** Team 1: (The Transitive Property) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. **Team 2: If A is a subset of the universal set U, then $\emptyset \subseteq A$.** Team 3: If $A \subseteq B$ then $B^C \subseteq A^C$.

4.2.1 Audio/Video Recordings of In-Class Collaborative Proving

Videos of in-class collaborative proving episodes were captured using camcorders on tripods placed around the classroom. Cameras were positioned to capture the group as they worked at their desks and follow the participants as they moved to the dry-erase boards mounted on the classroom walls. Thus, the cameras captured both the participants and the work recorded on the dry-erase boards throughout the collaborative proving. Audio was captured using small handheld digital voice recorders. These recorders were mounted to the bottom of each classroom dry-erase board to ensure the participant voices were captured when they faced the boards (and away from the camera). Figure 1-4 provides a depiction of the in-class data collection and classroom environment.

Figure 1-4

Image of In-Class Data Collection and Classroom Setup



4.2.2 Focus Group Stimulated-Recall Interviews

Following each of the three in-class proving episodes, I conducted stimulated-recall interviews with the focus group participants. Before each interview, I edited the audio and video recordings collected in class to compile one cohesive recording of the group's in-class proving. The recordings from the three episodes were 31 minutes (DC1), 37 minutes (DC2), and 43 minutes (DC3). The focus group participants, as a group, participated in stimulated-recall interviews (SRIs) in the days following each proving episode.

During the SRIs, the three participants (two participants in Episode 3) were seated in a conference room at a table with the interviewer facing a television screen, which displayed the recording of their in-class proving episode, with the interviewer in control of the video. Participants viewed the recording of their in-class collaborative proving and were instructed to

ask for the recording to be paused when they observed a moment that they thought was important to their group's creative process during the task. The key motivation behind carrying out SRIs is the promise that "cognitive processes can be investigated by inviting subjects to recall, when prompted by a video sequence, their concurrent thinking during that event" (Schindler & Lilienthal, 2022, p. 167). To mitigate potential conflict among the three participants' perceptions of and definitions of creativity, participants were instructed to follow the categories of the CPR on Proving (Savić et al., 2017) to guide their identification of moments important to their group's creativity. Figure 1-5 depicts the instructions provided to the participants during the SRIs.

Figure 1-5

Instructions Provided to Stimulated-Recall Interview Participants

I would like for you to tell me to pause the video when you see your group…

1) **making connections** to definitions, theorems, different representations, or examples,
2) **taking risks** by attempting to use a new tool or trick, switching strategies, asking questions, making suggestions, or evaluating your proof attempt, OR
3) **engaging in any other actions or behaviors you consider creative**.

After a participant asked for the video to be paused, I paused the recording and asked the participants about the moment paused upon. Participants were instructed that I could rewind the recording upon request at any time. Participants were also ensured that there were no right or wrong times to pause the recording, and there were no right or wrong answers when discussing the video. A complete interview protocol is provided in Appendix 1-A. At the conclusion of the recording, participants were asked what moments they thought were most important to their team's creativity. The SRIs took 90-100 minutes each including initial instructions, watching the

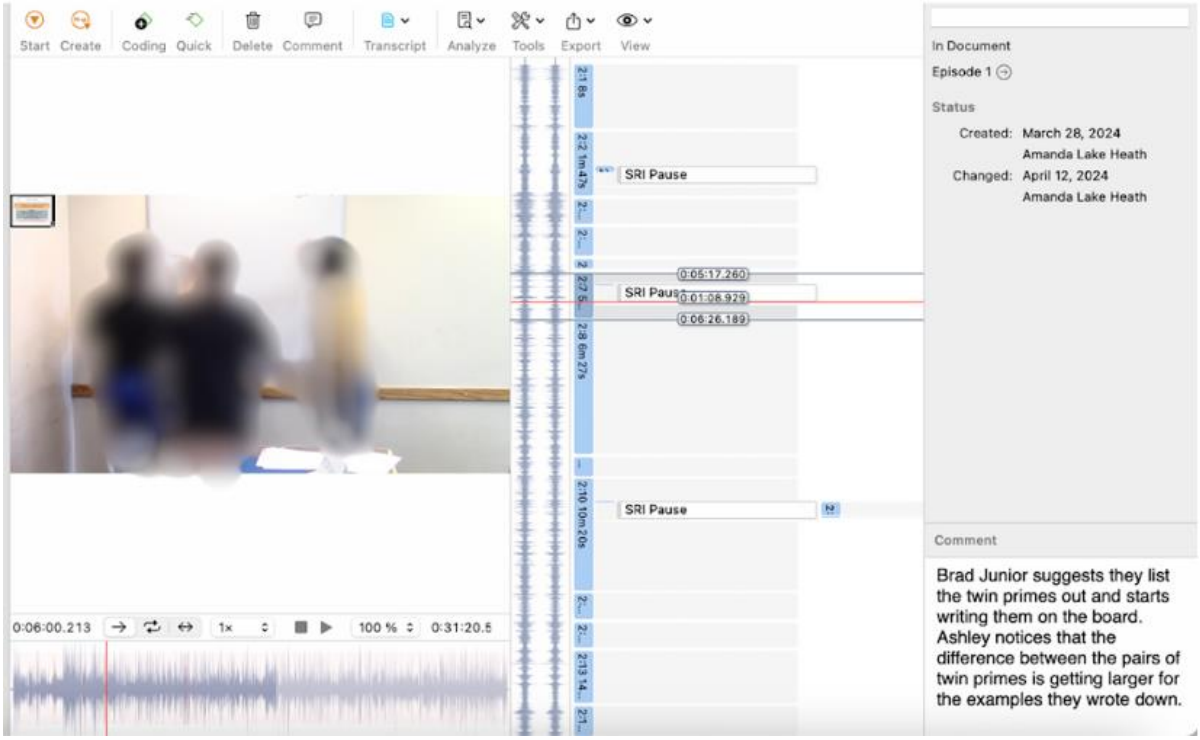
recordings of the proving episodes, discussions of paused moments, and concluding discussions. All three participants were present for the first two SRIs, and due to unforeseen circumstances Ashley was unable to attend the third SRI, so the final interview was conducted with only Brad Junior and Participant B present. The three SRIs were audio and video recorded and the audio data were transcribed using Otter.ai transcription software. The three focus group participants were modestly compensated for their time with Amazon gift cards for each of the three interviews, and there was an additional gift card bonus incentive for completing all three interviews.

4.3 Data Analysis

Data for this study included audio and video recordings of three in-class proving episodes and three corresponding SRIs. Analysis included creating content logs for each in-class proving episode video, labeling moments in the proving episode videos paused and discussed during SRIs and writing analytical memos for each “pause” discussion during the SRIs. All analysis was conducted using Atlas.ti qualitative data software. Figure 1-6 provides an example of how this analysis appeared in Atlas.ti for a portion of Episode 1.

Figure 1-6

Example of Data Analysis in Atlas.ti



4.3.1 Proving Episode Content Log

The first step of analysis was to segment the video data for each episode into identifiable chunks that defined group activities or actions. For each chunk, I wrote a one to two sentence description of what was occurring in the group proving. Across the three episodes, the chunks receiving descriptions were an average of one minute and three seconds and median 50.9 seconds, with the longest chunk being five minutes and one second long, from Episode 2, and the shortest chunk being 7.2 seconds, from Episode 3. The descriptions written in the content log were compiled to summarize larger chunks of the proving process and this collapsing process

was informed by literature on creative processes (e.g., Schindler & Lilienthal, 2020; Wallas, 1926) and proving process (Otten et al., 2017).

4.3.2 SRI Pause Analytical Memos

Participants were instructed to pause the video when they noticed a moment they thought was important to their creative proving. Moments during the stimulated recall interviews (SRIs) when participants paused the episode recording were transcribed and their timing was marked throughout the proving episode recordings in Atlas.ti. For each pause during the SRI, I wrote a summary and analytical memo as a comment in Atlas.ti. The goal of these summaries and analytical memos was to have quick reference of the topics of discussion during the pause moments in the SRI.

4.3.3 Identifying Critical Moments and Abductive Analysis

At the conclusion of each SRI, I asked the interview participants “Now that we've been able to watch and discuss the entire recording, what moments do you think were the most important to your team's creativity?” This portion of the interview was consulted to establish an overarching storyline and identify the most critical moments of the proving episode for the team’s creativity. After these critical moments were identified, I carried out an abductive analysis (Patton, 2016) to work backwards from these moments to reconstruct the events that led to that moment. I analyzed both portions of the video recording of the proving episode and the SRI transcript relevant to the critical moment. Finally, I compiled a timeline for each proving episode and the SRI pause moments to aid in storytelling and event reconstruction.

5. Findings

In this section, for each in-class proving episode, I provide a summary of the process of the collaborative group as they worked to complete their assigned proving task. Then, I present

the critical moments identified by the focus group as most important for their creativity during the episode and provide detailed description of the events that occurred to allow the critical moment to transpire during the proving process.

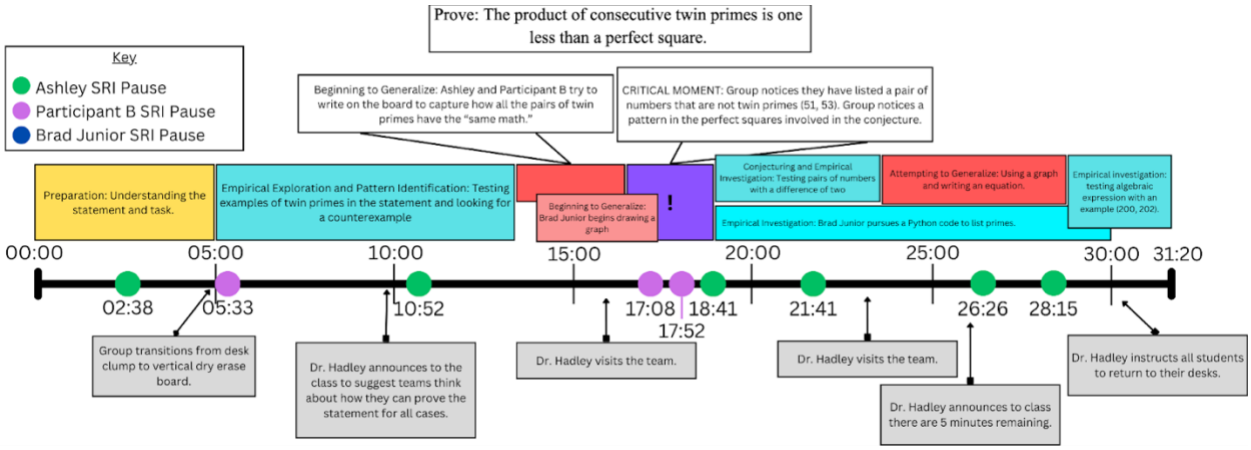
5.1 Episode 1: "The Mistake was the Most Important Part"

In Episode 1, Ashley, Brad Junior, and Participant B were tasked with proving *The product of consecutive twin primes is one less than a perfect square.* This task was chosen by the course instructor because research has illustrated students find many different ways to approach and complete this task (Bleiler-Baxter et al., 2023; Smith, 2006). This task is also valuable to teaching students the practices of mathematicians as it requires students to consider how to further generalize the statement. A mathematician completing this proving task may take a possible proving route of first noticing the statement is actually true for any pair of integers with a difference of two, not only pairs of twin primes, possibly by exploring a few examples. Then a direct proof would proceed by illustrating that one less than any integer squared can be represented as $n^2 - 1$ and this expression can be factored using the *difference of squares* factorization and show $n^2 - 1 = (n - 1)(n + 1)$ Then, a mathematician would explain that $n - 1$ and $n + 1$ are integers with a difference of two and thus, the product of any two integers with a difference of two is one less than a perfect square. The original statement would then follow as the set of prime numbers, and therefore also all twin prime numbers, is a subset of the integers. Ashley, Brad Junior, and Participant B's approach to this task was, as expected, different from what an experienced mathematician might do to complete the task as all three students were new to proving. Figure 1-7 provides a visual depiction of the process the focus group engaged in throughout Episode 1 as well as the moments participants paused the recording during the SRI of

Episode 1. I summarize the activities of the group during the in-class collaboration in the next section.

Figure 1-7

Episode 1 Timeline and SRI Pauses



5.1.1 Episode 1 Summary

Proving episode one occurred during the first week of the introduction-to-proof course. This was the first time Brad Junior, Ashley, and Participant B worked together as a group. All of the student groups in the class were given the proving task to *Prove: The product of consecutive twin primes is one less than a perfect square.* All groups were instructed to take five minutes of individual think-time before beginning to share their initial thoughts with their teammates; however, Brad Junior began talking with his teammates immediately, beginning to dissect the statement and understand the component parts (i.e., twin primes, perfect square). Dr. Hadley noticed that students had not followed the instruction to take individual think-time, so she instructed students to move to their dry erase board spaces to work collaboratively if they did not need independent time. Upon this announcement, the group realized their mistake in following

Dr. Hadley's instruction and worked silently and independently for a couple minutes, until Ashley suggested the team move to their dry erase board.

At their board, Brad Junior suggested they start by listing pairs of twin primes to identify numbers for which the statement was relevant. As the team worked together to write out the first pairs of twin primes (3 and 5, 5 and 7, 11 and 13, 17 and 19), Ashley noticed that the difference between the pairs appeared to be getting larger. Participant B suggested to the team that they use the pairs they listed to test the statement. Ashley and Brad Junior agreed this would be a good approach. The team then worked together to test several pairs of twin primes with the statement. Participant B took responsibility for finding the product of the twin primes, Brad Junior calculated the square root of one more than this product, and Ashley kept a log of the twin primes and the square roots on the board. The team worked together through pairs of twin primes up to 29 and 31 when Dr. Hadley interrupted the class to say,

> I see a lot of groups trying some examples. That's really great. Maybe a next thing that you could think about is what – maybe talk in your groups about what a proof might look like. First, are you convinced that this is true for *all* cases? And if you are, what might a proof look like? And if you're not, how could you convince us otherwise?

Following this prompt, Ashley suggested to her team that they look for cases which would disprove the statement. She expressed that she did not think the team had yet proved the statement because, "All of [the examples we've shown] are true, but it's not every one. [The statement] doesn't say just some of them, it says–" and Brad Junior completed her thought, "all of them." Upon this suggestion to look for a counterexample to the statement, the team worked together to find the next pair of twin primes to test if it could show the statement was false. Ashley wrote 7 and 9 on the board to test the statement, but upon realizing 9 is not prime, she erased the numbers from the board. The team then (incorrectly) determined the next pair of twin primes to be 51 and 53, not noticing that 51 is not prime (51 is divisible by 3 and 17).

After testing 51 and 53 in the statement and determining the claim still holds for that pair of numbers, Ashley began trying to reason about a pattern across the examples they tested the statement against. Ashley noted, “The same math applies to all these. If they’re next to each other and they’re prime, they’re going to equal [gestures to her square roots] this. So, we can reasonably assume, since they’re all based on the same math, it’s going to continue.” Ashley then suggested the team try to write out this idea on their board. Following this suggestion, Participant B and Ashley worked together to capture Ashley’s more general idea on the board. Ashley wrote on the board, “Every example we checked proved the proof. Therefore, we can assume that this will continue.” At the same time, Brad Junior began to draw a coordinate plane on the board, attempting to capture a pattern from their examples.

During this time, Dr. Hadley visited the group. She asked, “How are we doing over here?” Ashley explained, “We know it works, but we don’t know how to prove it works forever.” Participant B added they know the statement is true because it has worked for all of their examples. Dr. Hadley then inquired about the example with 51 and 53 and asked if the statement still worked. Ashley and Participant B recalculated the product and square root and Participant B remarked, “Yeah, it’s 52.” Upon this statement, Brad Junior noticed a pattern between the twin primes and the square roots. He said, “Oh, is that the trend?” and explained that the square root found for each example is the number in between the twin primes being tested.

After Ashely and Participant B understood Brad Junior’s observation, Dr. Hadley asked the group if 51 was prime. Participant B exclaimed, “Oh! It’s not!” Ashley quickly reached for the eraser to erase their example using 51 and 53 as “twin primes,” but Dr. Hadley instructed Ashley to leave the example on the board. The team then debated whether 51 was prime. Brad Junior recalled he wrote a Python code in a previous course that could list prime numbers for

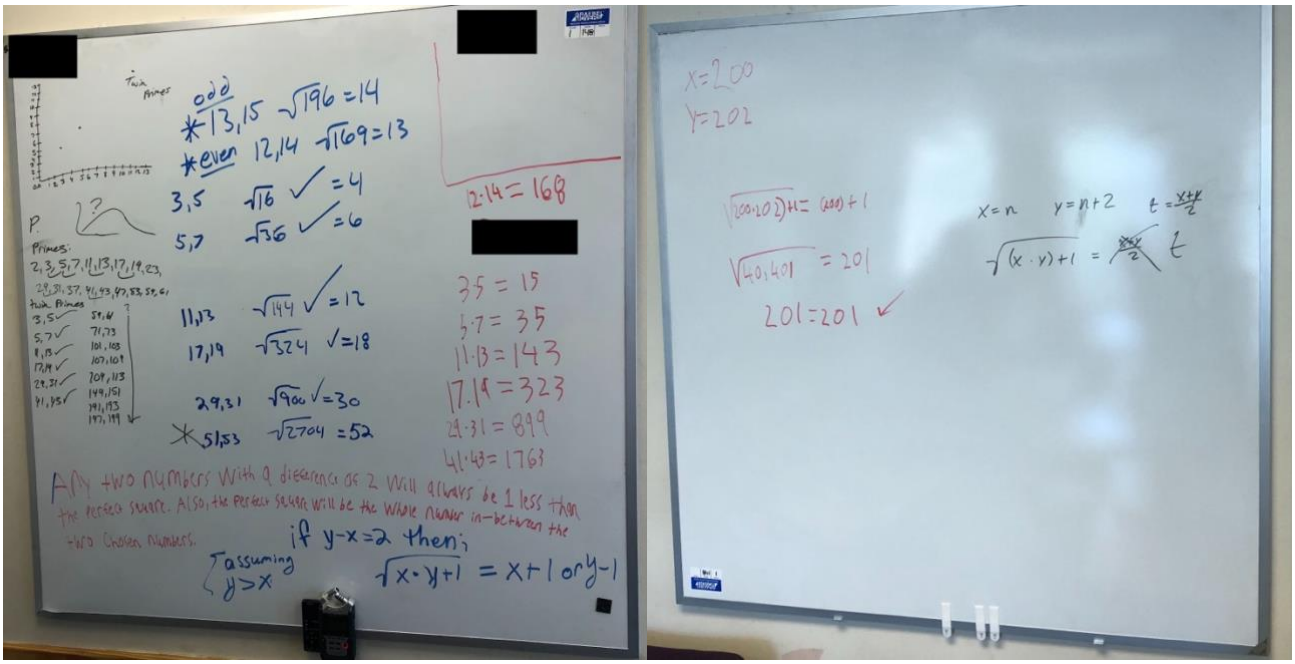
him, and upon Dr. Hadley's permission he began trying to pull up this Python code on his computer. Meanwhile, Participant B explained to Ashley that 51 is 3 times 17, so it is indeed not prime. Ashley said, "by golly" and again reached for the eraser to remove the example using 51 and 53 from the board. Again, Dr. Hadley instructed Ashley to keep the example on the board. Dr. Hadley said, "I think it's interesting that 51, according to Participant B, is not a prime, yet this still seems to work." After making this statement, Dr. Hadley left the group, and Ashley and Participant B began conjecturing about what numbers the statement may hold true for. While Brad Junior continued looking for his Python code, Ashley and Participant B began testing pairs of non-primes with a difference of two against the statement. After testing 13 and 15, Ashley said, "Oh yeah, no, it's always going to work. It's every number. It's every number." After making this conjecture, Ashley erased the claim she wrote on the board regarding twin primes and Participant B wrote on the board, "Any two numbers with a difference of 2 will always be 1 less than the perfect square."

Ashley then suggested the team "try some more numbers," and Participant B suggested they test a pair of even numbers. After testing 12 and 14 and again observing the statement held, Participant B added to their written conjecture, "Also, the perfect square will be the whole number in-between the two numbers. At this point, Dr. Hadley visited the team again and asked what the team had found. Ashley explained to Dr. Hadley they found the statement works for all pairs of numbers with a difference of two and "It works for odd and even… Maybe. For now." Dr. Hadley asked the team if they could prove their conjecture works in general. Participant B brought up Brad Junior's previous idea of using a graph on a coordinate plane to assist them in proving the statement in general. Dr. Hadley told the team she liked the idea of graphing and left the group to continue working.

After Dr. Hadley left the group, the three students try to develop ideas for how they might represent their conjecture using a graph. Participant B noted they could graph something if they had an equation to represent the conjecture. Ashley asked if plotting the points from their example would just form a line. At this point, Dr. Hadley interrupted the class to announce groups could use a second dry erase board if they would like to use the space for a more formal write up and informed the class they had five more minutes to work as a team. Ashley asked Participant B if he would rather pursue the graph or writing a formula, and he said he would rather pursue a formula. Meanwhile, Brad Junior was still working on his computer to find his Python code. Participant B and Ashley worked together to write an equation using variables x and y. Ashley wrote on the board, "If $y - x = 2$ then $\sqrt{x \cdot y + 1} = x + 1$ or $y + 1$, assuming $y > x$. Participant B suggested they test out their equation with a pair of values they had not previously tested. He and Ashley choose 200 and 202. Participant B wrote this example on their second board using the format of their equation to verify it worked. Meanwhile, Brad Junior explained to Ashley he was able to have his Python program generate a list of primes from 1 to 1000. Dr. Hadley then called the class to return to their desk clumps and the team did so. Figure 1-8 provides images of the group's final work on their dry-erase boards.

Figure 1-8

Dry-Erase Board Work from Episode 1



Note: In Episode 1, Ashley used a blue marker, Brad Junior used a black marker, and Participant B used a red marker. Participant names were redacted from the image.

5.1.2 Episode 1 Critical Moment

At the conclusion of the SRI, the participants collectively determined the moment when they discovered they had incorrectly presumed 51 to be a prime number as the most important to their group's creativity. Participant B noted how noticing this mistake allowed their team to go "more in depth" with their proof than other teams in the class were. Brad Junior emphasized the importance of this moment to their creativity despite having help from Dr. Hadley in noticing their mistake. He said, "If it wasn't pointed out to us we probably would have kept on going with the twin primes and trying to find twin primes and not even recognize that 51 wasn't prime." Ashley expanded on the importance of their mistake and considering how they could capitalize on their mistake rather than dismiss it as irrelevant,

> It was more beneficial for us to make the mistake than it was doing any of the actual, like, solutions that we were doing. Like, the mistake was the most important part, I think of the entire exercise. And I think the fact that [Dr. Hadley] said "keep it on the board." We were like... "we need to look at this." But once we were okay with leaving our mistake on the board, it really opened up the possibility for us to, like, discuss everything about it, look at the similarities, look at the differences and really come to a conclusion. I think that was really important for us to just say, "This is a mistake, what can we learn from that mistake?" Instead of erase it.

The critical moment of noticing their mistake allowed the group to construct a more general version of the statement they were asked to prove, which they perceived as crucial to their creativity during the proving process. In the following section, I provide the findings of the abductive analysis which reconstructed the events that allowed this critical moment to occur.

5.1.2 Episode 1 Events Leading to Critical Moment

The events leading to this critical moment of noticing the team's mistake in presuming 51 was prime began when the team transitioned to their dry erase board. The team began their work empirically by listing primes, identifying pairs of twin primes, and testing the statement held true for the pairs of twin primes. Dr. Hadley encouraged the class to think of how they could prove the statement true for all twin prime pairs or disprove the statement. During the SRI, Ashley noted that the push to prove the statement true generally prompted her to want to try to disprove the statement using a counterexample. She said, "I've taken a class like discrete math, and we did proofs like induction, direct proofs, indirect. And so, I was just thinking about that. [...] That's the only thing I could keep thinking about is like it doesn't matter if you can prove it if you can disprove it." Here, she noted that it would be simpler to disprove the statement by finding a counterexample than to prove the statement in general.

In seeking a counterexample, the team decided to continue their empirical approach of testing pairs of twin primes against the statement. In seeking a counterexample, the team made their mistake and presumed 51 was a prime number. The next catalyst for noticing their mistake

was Dr. Hadley's visit to the group during which she asked the students to explain their work so far and she asked them to check their example using 51 and 53. In response to this suggestion, the team re-tested the conjecture using this pair of values and Dr. Hadley asked, "Is 51 a prime?" Upon this question, the team assumed this question meant they had made a mistake, but when Dr. Hadley insisted the team leave the example on the board, they further investigated whether or not 51 was prime.

During the SRI, the team reflected on this moment and remarked that 51 "sound[ed] prime" (Participant B) and they "really want[ed] it to be prime" (Ashley). Brad Junior reflected on this moment saying,

> I mean, as Participant B said, like 51 sounds like a prime number. So like, we were just-- especially after the theorem worked on it, too. We were just like, "oh, yeah, there's no way that's not prime." Even after he realized it was 3 and 17. I was like, "There's no way." That's why I checked on the calculator. I was just like… I don't know. That realization really is what I think spurred us on.

Finally, participants said that making this mistake and then being told to leave the mistake on the board was contrary to their previous experiences in math classes. During the SRI (18:41), Ashley and Participant B remarked:

> Ashley: Something interesting about that is that I really-- she's like, "that's not a prime." My immediate reaction is "get rid of it." Like, anything that's not supposed to be there, get rid of. We would have actually realized it earlier, because I kept thinking nine was a prime. And I tried to do seven and nine. If we had continued that, we would have noticed it even earlier, but [we] kept checking ourselves, like not allowing ourselves to make mistakes. And when we did make a mistake, we immediately wanted to get rid of the evidence. Just get rid of the evidence instead of exploring it. So, I did notice that about myself. I was very quick to like cross- erase, erase our creativity. So–
>
> Participant B: I think that's like how we are with all math classes. We're, like, so focused on making sure that everything we're doing is right that we don't really learn from our mistakes. Because the moment we make a mistake we just...
>
> Ashley: Get rid of the evidence!

In this conversation, Ashley highlighted how she had previously almost made a similar mistake in testing seven and nine as a pair of twin primes, but that her insistence on this being a mistake and not within the bounds of their statement delayed their revelation that the statement could be generalized to all integers with a difference of two.

In summary, the team's efforts in empirically testing the conjecture, seeking a counterexample, mistaking 51 to be a prime number, being prodded by Dr. Hadley about this example, and allowing their mistake to stay on the board allowed them to realize the importance of the mistake they made and further generalize their conjecture.

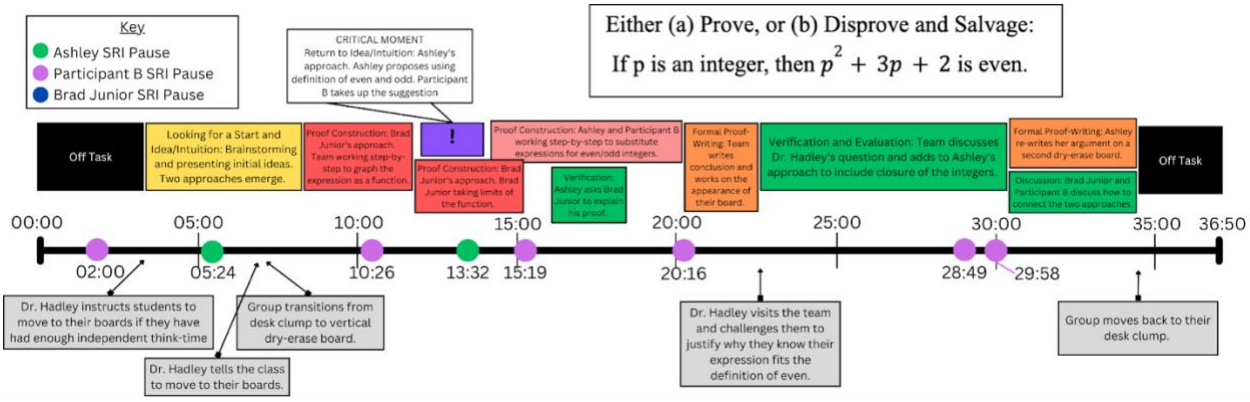
5.2 Episode 2: Two Approaches from "Different Hemispheres" and One "Middleman"

In Episode 2, Ashley, Brad Junior, and Participant B were tasked with either proving or disproving and salvaging the claim *If p is an integer, then* $p^2 + 3p + 2$ *is even.* The challenge of uncertainty in whether the given claim should be proved or disproved added to the opportunity for students to explore the truth of the claim. This statement is indeed true, and a conventional route to prove this statement would be to substitute an expression for an odd integer $2n + 1$ and an even integer $2n$ into the expression $p^2 + 3p + 2$ for *p* and rearrange the expressions algebraically to show the resulting expression will be of the form 2 multiplied by some integer expression. An alternative, but still not surprising, route a mathematician might take to prove this statement would be to factor the expression $p^2 + 3p + 2$ to show $p^2 + 3p + 2 = (p + 1)(p + 2)$ and then note $p + 1$ and $p + 2$ are consecutive integers, which would imply one must be even and the other odd. The proof via this route would conclude by claiming, or showing, that the product of an even integer and an odd integer is always even and thus $p^2 + 3p + 2$. In Ashley, Brad Junior, and Participant B's attempt to prove or disprove and salvage this claim, there are some elements that point to a more typical or expected proving route, but there

are also some approaches which would be considered non-conventional by the broader mathematical community. Figure 1-9 illustrates the sequence of events during Episode 2 and the following section summarizes the focus group's activities during this task.

Figure 1-9

Episode 2 Timeline and SRI Pauses



5.2.1 Episode 2 Summary

Similar to Episode 1, Episode 2 began with Dr. Hadley introducing the collaborative proving tasks to the class and instructing students to take individual think-time before collaborating. In this episode, however, the three student groups in the class were tasked with different statements to prove or disprove and salvage. Brad Junior, Ashley, and Participant B were tasked with *Prove or disprove and salvage: If p is an integer, then* $p^2 + 3p + 2$ *is even.* Students were instructed that they would need to reference their problem sets to view their assigned problems. After the task was introduced, during assigned individual think-time, the team began pulling up their problem sets and having an off-task conversation about Ashley's cellphone, which was an older model iPhone. Eventually, Brad Junior directed the team's discussion to their assigned task. After a few minutes, Dr. Hadley announced across the

classroom that she noticed teams had begun discussing their tasks and if they have had enough independent think-time they should move to their dry erase boards.

Upon this announcement, the team did not move to their board immediately, but rather continued discussing their initial ideas and intuitions at their desk clump. From the beginning of their collaboration, two distinct approaches emerged from the group. First, Ashley presented her idea of splitting the integers into odd and even and considering what happens when p is odd and when p is even. Brad Junior presented his idea to graph the expression as a function and then perform a limit test to show the function is continuous.

Noticing groups had still not transitioned to their dry erase boards, Dr. Hadley instructed students to leave their desks and begin working on their boards. As the team transitioned to their dry erase board, Participant B remarked that he graphed the expression on Desmos. Brad Junior began graphing the expression on a coordinate plane on their board, but Ashley asked "Does it matter? It doesn't matter if it's continuous, it just matters if it's even." Brad Junior explained to Ashley that "continuous shows that the pattern holds true." Although Ashley did not understand Brad Junior's approach, she encouraged him to go ahead with his idea.

The team then began attempting to construct a proof using Brad Junior's approach, with Ashley making a table of values for the function $f(x) = x^2 + 3x + 2$ and Participant B plotting the points on a coordinate plane. After the table of values for x= -3, -2, -1, 0, 1, 2, 3 was complete and the graph was plotted on the coordinate plane, Brad Junior proceeded to take limits of the function as x approached positive and negative infinity. While Brad Junior worked on his limits, Ashley stepped away from the board to take a vitamin and returned and wrote "odd = 2m+1" and "even = 2m" on the board and said, "I think there's something we can do with odd and even and plug it in so we can find absolutely." Although Participant B was working with

Brad Junior when Ashley suggested this, Participant B listened to Ashley's suggestion and began writing on the board the expression with *2m+1* and *2m* substituted for *p*. Participant B and Ashley worked together to expand, simplify, and factor the expressions to eventually show the expression will result in two multiplied by a number in either case, making the result even.

While Ashley and Participant B worked together on the cases when p was either odd or even, Brad Junior completed his work involving limits. Ashley asked Brad Junior to explain his work, engaging in proof validation and verification. Brad Junior explained his work to Ashley.

When both approaches had, in the opinion of the participants, reached a conclusion and sufficiently proved the claim, Ashley remarked that their argument was difficult to follow, so Participant B suggested writing a formal conclusion on their board. Ashley wrote on the board "Conclusion: Given any even or odd # we have proved it will always be an even #." At this point, Participant B moved to their second dry-erase board and rewrote the original claim "If *p* is an integer, then $p^2 + 3p + 2$ is even." Ashley continued trying to annotate their original board to make their argument easier to follow.

Dr. Hadley then approached the team to ask about their work so far. Participant B and Ashley explained their work using cases to Dr. Hadley. Brad Junior also explained his work to Dr. Hadley, and Dr. Hadley challenged him to explain how his work showed the expression would always be even. Brad Junior explained that it was supposed to be shown in the y-values of the function. Dr. Hadley then shifted their discussion back to Ashley's approach and challenged the team to justify their claim that the expressions they have multiplied by two ($2m^2 + 5m + 3$ and $2m^2 + 3m + 1$) would be integers and ensure their result would in fact be even.

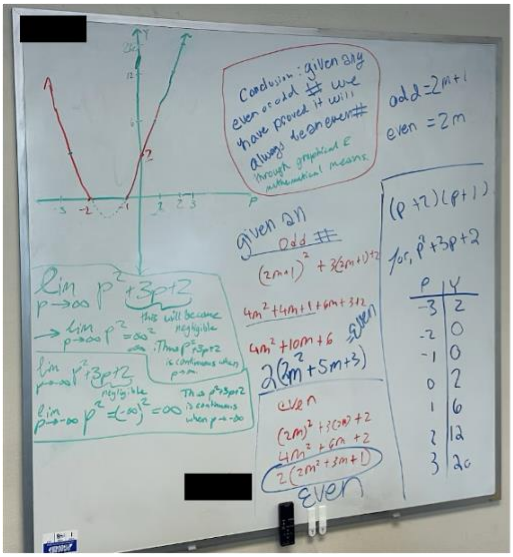
Dr. Hadley left the group to discuss her question, and the team spent some time discussing and eventually determined these expressions would be integers because *m* was

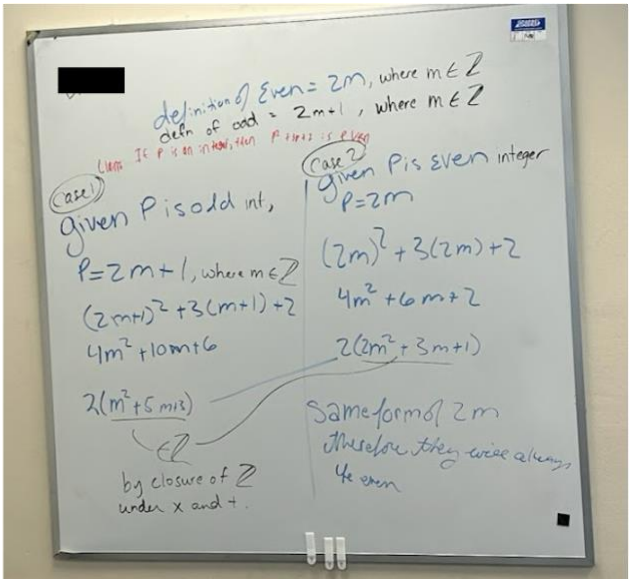
assumed to be an integer, from the definition of odd and even, and the only way to combine integers together and result in a non-integer was through division and that the integers are closed under multiplication and addition.

Ashley, still dissatisfied with their board organization, then rewrote the work from her argument on the team's second dry-erase board. Meanwhile, Participant B and Brad Junior discussed how they might present their work to the class and align the two approaches initiated by Brad Junior and Ashley. Brad Junior and Participant B discussed the possibility of factoring and/or graphing the expressions that resulted from Ashley's approach, but the two eventually gave up. When the whole team reconvened, Brad Junior added "through graphical and mathematical means" to the team's proof conclusion to articulate the two approaches the team used. The team moved back to their desk clump and discussed their graphing calculators until Dr. Hadley directed the other teams in the class to move back to their desks for a whole-class discussion. Figure 1-10 includes images of the group's board work at the conclusion of the whole-class discussion for Episode 2.

Figure 1-10

Dry-Erase Board Work from Episode 2





Note: In Episode 2, Ashley used a blue marker, Brad Junior used a green marker, and Participant B used a red marker. Dr. Hadley also wrote on these boards in a black marker during the subsequent whole-class discussion. Participant names were redacted from the image.

5.2.2 Episode 2 Critical Moment

When asked about the most important moments in the team's creativity, the participants emphasized the importance of following through with both Brad Junior and Ashley's ideas and having multiple approaches to the proof. In particular, the participants noted the importance of Ashley's divergence from Brad Junior's approach to prove the statement using the definitions of even and odd. Participant B said at the conclusion of the SRI,

> I would say the most important moments of like, the creativity was when like, we decided the multiple ways we could think about how we could prove this, like, with graphs and factoring in definitions of odd and even. Because I think that's what the main thing that makes mathematical problems so creative is that you're thinking of multiple ways you can solve the problem. That's like, the main reason why people think mathematical creativity is so important. Because if there's just, like, one way to solve a problem then you're not going to have any sort of like depth or understanding of it.

Moreover, both Ashley and Brad Junior commented on the importance of Participant B's flexibility in working on both approaches with them. Ashley said,

> Yeah, we both worked out both of our ideas. And I really liked– my favorite part was the fact that Participant B was so able to, like switch between them, you know, because he was like, "Okay, let's do Brad Junior's", but then when I was like, "let's do this." He was-- it was so easy for him to jump and say, "Okay, let's work on this now." And so that was my favorite part, seeing how flexible he was in trying it. I think it really brought the group together. Because me and Brad Junior were on, like, different hemispheres. And there's Participant B, like, working between us, you know, so I really liked that.

Brad Junior agreed that Participant B's flexibility helped bring the group together, and Ashley's pursuit of the "mathematical route" was a crucial moment in their team's creativity.

> Like Ashley was saying, Participant B, and being that middleman. And when she was starting to work out using the definitions of odd and even numbers to show it, I feel like that was probably better for the class's understanding, because it seemed like most of them didn't like my graph, which I'm perfectly fine with because I-- I understand that. So, they definitely were seeming to grasp the mathematical expression of it all much easier. And so, I think it showed it to be maybe the more optimal route to take. And so, the creativity of it was very essential. Like them recognizing those definitions and then using them-- finding the way to use them was very, very crucial to it.

In this quote, Brad Junior recognized that the class did not understand his route using graphs and limits and submitted that Ashley's approach was probably the better one, especially considering their audience. When asked about what moment she found most important to the team's creativity, Ashley said,

> I'm glad we went and we started off with the graph, you know, we were all open to trying that out. And then once we'd gotten far enough into that, we were kind of done, I'm glad we kind of took a pivot, you know? I was like, "let me write these definitions down." And just try to like, try that. So, I'm glad we had time to try both. And I'm glad that we were able to know when to kind of switch it up and try something new and see if that would get us a little bit further for everyone's understanding.

Given these reflections at the conclusion of the SRI, the critical moment in Episode 2 was determined to be when Ashley wrote "odd = 2m+1" and "even = 2m" on the board and suggested the team could use the definitions of odd and even to prove the statement was true for all integers

and Participant B began engaging with this suggested route, diverging from the method originally suggested and pursued by Brad Junior.

5.2.3 Episode 2 Events Leading to Critical Moment

The events leading to Ashley's suggestion to utilize the definitions of even and odd integers in their proof began during the group's initial discussion when both Ashley and Brad Junior presented their intuitions for approaching the proof. It is evident Ashley had the idea to use the definitions for even and odd provided in the problem set from the beginning of the proving episode. Ashley remarked on the emergence of these two approaches in the SRI (5:24),

> So, like during this time, when we're first, like, starting to talk about it, you can kind of see we take two like completely different approaches. I'm like "What is even, what is odd" and Brad Junior's thinking more of like, "Let's find a limit." And so, kind of like talking about how we can approach both of those from different areas and just exploring our options. And it's interesting to see how we took completely different approaches.

Although these two approaches emerged from the outset of the episode, the team decided to pursue Brad Junior's idea first, even though Ashley had challenged the idea that continuity was relevant to the proof. In the SRI (5:24), Brad Junior described the team's decision to pursue his idea,

> When it came to Ashley's, like, way of processing it, I think-- it was very evident to me that like the differentiation was mine was like a graphical analysis, and hers was the mathematical analysis. Like, she was using the known mathematical terminology for like even and odd numbers to show that the equation would produce an even number for both. And I liked it a lot, especially when I realized that's what they were doing. Because I typically am fairly vocal I was like, "okay, well I'm going to do this first."

The above quote indicates that Brad Junior attributed the team pursuing his idea first to him being more "vocal." Later in the SRI (13:32), Brad Junior again noted that his "loud"-ness caused the group to follow through with his idea first, "The both of us had ideas almost instantly when reading it [the task]. And just because I'm a little bit louder more times than not, I think I kind of got my way first."

Ashley reflected on the moment when the group decided to pursue Brad Junior's approach first during the SRI (5:24) as well,

> Yeah, Brad Junior definitely went and did his thing again, which I support 100%, and I really liked-- he got to like express himself and work the problem out the way he wanted. I wasn't understanding it, you'll see, I'm very confused, but probably rewatching. I'll get a grasp. His level of math is just above me.

Ashley's comment during this portion of the SRI provides context for why Ashley agreed to pursue Brad Junior's idea even though she did not understand it. She deferred to Brad Junior's idea because she perceived his mathematical abilities as stronger than hers.

When the group pursued Brad Junior's approach, they delegated different components of the task. Ashley created a table of values to find points for Participant B to place on the graph, and then Brad Junior assisted with graphing and then took limits of the function. Ashley's role in creating the table of values to aid in graphing the expression as a function and lack of involvement in plotting the points on the graph allowed her contribution to Brad Junior's approach to reach a natural stopping point before Participant B's. When Ashley had completed her table of values, Participant B and Brad Junior were still discussing the appropriate ways to represent the points on the graph (e.g., discussing if they should allow negative y-values on the graph, discussing if the graph should be a solid or dotted curve), but Ashley was not involved in this conversation.

At this point, Ashley perceived that she could not further contribute to Brad Junior's approach. In the SRI, Ashley paused the recording (13:32) to say,

> So, at this moment, I can see us all, like, trying to work on the limit thing. But after I make the table, I'm like, "What's next? What's next?" And so, Brad Junior is still working on proving the limit. But in my mind, I'm like, "what else can we do?" I guess this isn't leading us to anything like concrete, or it's showing the limit. But if we need more proof than that, going back to like what I thought of originally, when we were sitting down and talking about it. I was like, "let's just try it." You know? Let's get something, like,

> concrete, try it out. So, I go and I put the things [expressions to represent odd and even] on the board. And I started saying, "This is what we need to do."

When asked where her idea and initiative to write the expressions for odd and even on the board came from after she stepped away to take her vitamins, Ashley said,

> I just, I needed to take my vitamins, because I just ate before I went to class and then I was just like, "Okay, it's time." I don't know what I was really thinking I was just like, "Well, I'm not doing anything productive now." Like, I'm done with the table. I'm not doing anything. And I knew instinctively kind of the idea of odd and even because of the first problem set, like, the way you had to prove odd or even. What was it? Something... it was something to do with odd and even that you had to prove it. And so, I'd already worked out what an odd and even number was and how they react when you square them and stuff. So, in my head, I was like, "I'm just going to do the same thing."

In these two reflections, Ashley expressed that she felt like her contribution to Brad Junior's approach had reached a conclusion, and she was looking for ways she could continue to contribute to moving the group's proof forward ("what else can we do?;" "I'm not doing anything productive now;" "I'm done with the table"). Rather than find additional ways to contribute to Brad Junior's approach, Ashley returned to her original idea to use the definitions of odd and even provided to them in their problem set. She referenced both the fact that she had this idea from the outset ("what I thought of originally") and that this idea had emerged from her experience working similar problems in a previous problem set ("because of the first problem set").

In addition to needing Ashley's idea for her approach to the proof and her initiative to diverge from Brad Junior's approach, Ashley also recognized that this moment would not have been critical without Participant B's reception to also switching to Ashley's proposed approach. Ashley said during a pause of an SRI (13:32),

> And Participant B takes the initiative actually, to like work it out. So, I really liked that Participant B was really, like, open to switching from helping Brad Junior to doing something that I thought of so I really appreciated how flexible he was.

When asked about what he was thinking when he switched to Ashley's approach, Participant B said,

> I kind of understood what Brad Junior was coming from with, like, the graph and whatnot. But with Ashley, I see like that if we plug in the definition of an odd number for *p* in here and factor it so that it can be in the format of the definition of even, we can prove that- no matter what integer- that the expression that we've got will always be even for every single integer. So, like I kind of understood, like, where Brad Junior was coming from, but I feel like– [to Brad Junior] not to be rude to you – But I feel like Ashley was like more... I understood Ashley more in this moment.

Participant B's decision to switch to working on Ashely's approach originated from a sense of better understanding how Ashley's approach would lead to a proof of the statement ("we can prove that- no matter what integer…") compared to Brad Junior's approach. This was similar to Ashley's reasoning for returning to her original idea rather than seeking additional ways to contribute to Brad Junior's approach, as she said,

> I guess this [Brad Junior's approach] isn't leading us to anything like concrete, or it's showing the limit. But if we need more proof than that, going back to like what I thought of originally, when we were sitting down and talking about it. I was like, "let's just try it." You know? Let's get something, like, concrete, try it out.

Both Ashley and Participant B had an intuition that using the definitions of even and odd integers in their proof could lead to a more "concrete" proof compared to Brad Junior's graphical approach and using limits.

In summary, Ashley's reference to previous problem sets allowed her to develop her initial idea for using definitions of odd and even, and the team's decision to pursue Brad Junior's approach first along with Ashley's delegated task of creating a table of values provided space and the circumstances for Ashley to diverge from this approach. Finally, Participant B's intuition for Ashley's approach as more likely to produce a valid proof facilitated his flexibility in switching to Ashley's approach and supporting her suggestion when she diverged from Brad Junior's approach.

5.3 Episode 3: "We Fumbled the Ball on the One Yard Line"

In the third and final episode, Ashley, Brad Junior, and Participant B were asked to prove or disprove and salvage the statement *If A is a subset of the universal set U, then* $\emptyset \subseteq A$ *(where Ø represents the empty set)*. This task, like the one assigned in Episode 2, presented an additional challenge of also determining whether the statement was true or false; however, this task required additional abstract thinking. This statement is in fact true, but a typical proof actually relies on what mathematicians call *vacuous truth.* An implication statement $P \Longrightarrow Q$ is said to be vacuously true when the precedent of the statement, *P*, cannot be satisfied. The text used throughout this ITP course (Taylor, 2007), used the following example to explain vacuous truth:

> Suppose that your uncle Ted tells you that if you get an A in this class, then he will pay for your next semester's tuition. If you get an A and he pays, then he told you the truth. On the other hand, if you get an A and he does not pay, then you can conclude that he lied to you. But if you do not get an A, then whether he pays or not, in neither case can you conclude that he lied to you. (p. 5)

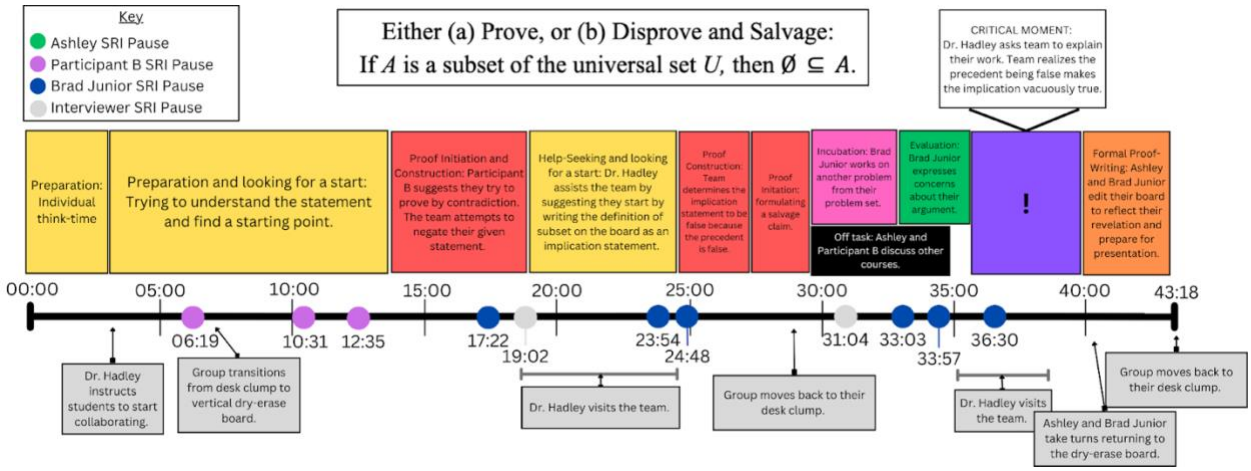
In the case that the precedent is not true, the entire implication statement is considered vacuously true because the precedent (or assumption of the conditions) was false, and it is impossible to make any claim about the truth value of the consequent. The *only* way to guarantee an entire implication statement is false is when the precedent is true, but the consequent is false (i.e., the student makes an A in the class, but Uncle Ted does not pay their tuition). Therefore, to be consistent, mathematicians say the implication statement is *vacuously true* when the precedent is false.

The claim that the empty set is a subset of an arbitrary set A can be rewritten using an implication statement using the definition of subset. Thus, $\emptyset \subseteq A$ is the same as "If $x \in \emptyset$ then $x \in A$" or "$x \in \emptyset \Longrightarrow x \in A$." To proceed in a direct proof of this implication statement, one would assume *x* was an element of the empty set Ø. However, by definition of the empty set, the

empty set has no elements, so $x \in \emptyset$ is always false which makes the entire claim vacuously true. Although Ashley, Brad Junior, and Participant B did eventually reach the conclusion their given statement is vacuously true, the team struggled to recall this mathematical convention throughout their proving attempt. Figure 1-11 depicts the timeline of the events of Episode 3 and the following section provides a summary of the group's proving attempt.

Figure 1-11

Episode 3 Timeline and SRI Pauses



5.2.1 Episode 3 Summary

Episode 3 occurred during week 11 of the 15 week semester in the first week of the module on set theory. During this class session, the three different student groups were tasked each with different proving tasks from their eighth problem set in the course. The focus group of Brad Junior, Ashley, and Participant B were tasked with *Prove or disprove and salvage: If A is a subset of the universal set U, then* $\emptyset \subseteq A$ *(where Ø represents the empty set).* Dr. Hadley instructed students to take three minutes of individual think-time before collaborating. Brad Junior, Ashley, and Participant B engaged in individual think-time for those three minutes and,

when instructed to do so, they began discussing their assigned statement. Ashley began their conversation by asking what the statement meant, and then the team engaged in a conversation preparing themselves to start their task and find a place to start their proving approach. The group's initial conversation was abstract and philosophical, and they contemplated if "nothing" can be inside of "something." When the group moved to their dry-erase board, Ashley began drawing a Venn diagram to represent their situation, labeling the universal set *U*, a set *A* as a subset of *U*, and the empty set as a subset of *A*. This led Ashley to conjecture that their given statement *If A is a subset of the universal set U, then* $\emptyset \subseteq A$ would also imply $\emptyset \subseteq U$.

Following Ashley's observation that their statement being true would mean $\emptyset \subseteq U$, Ashley and Participant B had different ideas for how they could proceed. Participant B wrote on the board an attempt to negate their given statement ($A \subseteq \mathrm{U} \wedge \emptyset \nsubseteq \mathrm{A}$) and proceed with a proof by contradiction. Ashley, still seeking to understand the claim, asked "what would be an example of an empty set in a set?" Trying to help Ashley think of an example, Brad Junior turned to Dr. Hadley and asked her if zero could be in an empty set, and Ashley asked, "what is an empty set?" Dr. Hadley prompted the team to go back to the definition provided in their problem set.

The team referenced their problem set to find the definition of both empty set and subset and they tried to reason with their claim $\emptyset \subseteq A$ using these definitions. The team danced back and forth for some time verbally applying these definitions to their statement and reasoning about "nothing" and "something." During the SRI, Brad Junior and Participant B remarked that during this time of the collaborative proving, they were trying to find a way to make the statement "operable" by trying to develop examples (Ashley's suggestion) and break down the component parts of their statement and work with the logic of proof by contradiction (Participant B's suggestion).

After Ashley, Brad Junior, and Participant B pursued Ashley's suggestion of working through an example, Brad Junior suggested they might think about how they could salvage the statement into something else they could prove. At this point, Participant B noted that he thought he could "find a contradiction." During this time, the students misused words like "contradiction" in place of what they meant as a logical negation; nonetheless, the team worked together for approximately five minutes to consider how they could pursue a proof by contradiction and negate their given statement appropriately. The team eventually reached $A \subseteq U \wedge (\emptyset \subseteq A)$' as the negation of their original statement, but the team struggled to rewrite $(\emptyset \subseteq A)$'. When they discussed what $(\emptyset \subseteq A)$' meant, the team spoke in mixed language regarding the mathematical objects they were working with. For example, Ashley wanted to apply De Morgan's law (of set complement) to the statement $(\emptyset \subseteq A)$' to "negate" the empty set and "negate" the set A, but she did not know what would happen to the "thingy," referring to the subset or equal to ($\subseteq$) symbol. During this time, Dr. Hadley concluded a conversation she was having with another student group and Ashley, Participant B, and Brad Junior looked to Dr. Hadley and asked her for help.

When Dr. Hadley arrived to the group, Ashley informed Dr. Hadley that they tried to look at definitions but they were "stuck" and Brad Junior told her they were trying to prove the empty set is a subset of A. Dr. Hadley noted that she heard Participant B earlier reading the definition of subset to the team and she encouraged them to write that definition of their board and Participant B wrote, "$A \subseteq B$ if and only if all elements of A are elements of B" on the board. After this, Dr. Hadley asked, "and what's the operable form of that definition? Let's keep going." Ashley read the definition from their problem set to Participant B, and along with

discussion with Dr. Hadley, eventually wrote "$A \subseteq B \Leftrightarrow x \in A \Rightarrow x \in B$" and "$(\emptyset \subseteq A \Leftrightarrow x \in \emptyset \Rightarrow x \in A$."

Once the group reached this point, Dr. Hadley prompted the team, "You know a lot about proving implication statements, so think about what you know about the logic behind implication statements and proving implication statements," and she left the team to discuss. Immediately Ashley turned to the team and said, "Can we ever find an x where it's an element of an empty set? No! So, this [pointing vaguely to the board] is always false! Implication statement for ya!" Participant B said, "Yeah! Now we salvage it," implying he understood the entire implication statement ($x \in \emptyset \Rightarrow x \in A$) to be false, rather than just the precedent ($x \in \emptyset$). At this point, however, Brad Junior interjected, "Also though, because it's an implication statement, it doesn't matter if the left side's [the precedent] false." Ashley jumped in to defend her claim, and Brad Junior quickly backed down from his comment.

The team had determined their given claim to be false, so they began looking for a way to salvage the claim. They considered one way to salvage the claim would be to state, "the empty set is not a part of A." Brad Junior then suggested they could use complements and salvage to the statement, "the complement of the empty set is a subset of A, which is a subset of U." Participant B wrote, "Salvage: $\emptyset^C \subseteq A \subseteq U$" on their board. Ashley then returned to their writing "$\emptyset \subseteq A \Leftrightarrow x \in \emptyset \Rightarrow x \in A$" and added to it, "always false" with an arrow pointing to $x \in \emptyset$, but circling the entire implication $x \in \emptyset \Rightarrow x \in A$. Dr. Hadley challenged the team by asking, "What's always false?" Participant B replied, "that x is a subset of the empty set," and Ashley gestured vaguely to the implication statement on the board and said, "this thingy." Participant B then continued to reason that if one side of a biconditional is false, then the "whole thing" is false. At this point, the team moved back to their desks.

When the team moved back to their desk clump, Ashley began discussing a different statistics course she and Participant B were enrolled together in with him. Meanwhile, Brad Junior worked on the problem set assigned for their introduction-to-proof course (which the problems assigned during class were drawn from). He worked on a different problem involving subsets, If $A \subseteq B$ then $B^C \subseteq A^C$. At two different points while the team sat at their desk clump, Brad Junior looked to Ashley and tried to express concerns about their proof. He returned to reasoning abstractly and philosophically about "nothing" in "something." Brad Junior concluded his comment by saying, "Since we know that A can be a subset of itself, then nothing can be a subset of something. But I don't know how to show it. So, I'm fine with this. I don't really even care." Ashley responded, "No, I agree with what you said. I think where we got, where we kind of drew the line is when we had it in another subset, you know? Because we couldn't find an example." Brad Junior looked back to his tablet and said, "Yeah, I don't know" and Ashley said, "I don't know. We'll see," and the team resumed silently sitting at their desks, with Brad Junior working on a proof on his tablet, and Ashley and Participant B looking at their phones. A few moments later, Brad Junior interrupted the silence again, reasoning again about "nothing" and "something," but this time thinking about logic and that "nothing can imply something." Conflating sets and statements, Brad Junior wanted to try to prove that "the empty set is implying the set of something." He said, "That's where I don't know where to go, so that's why I'm fine with doing this [gesturing to the board] and then letting the whole class work on it." Ashley agreed. Soon after, Dr. Hadley arrived at the team once again.

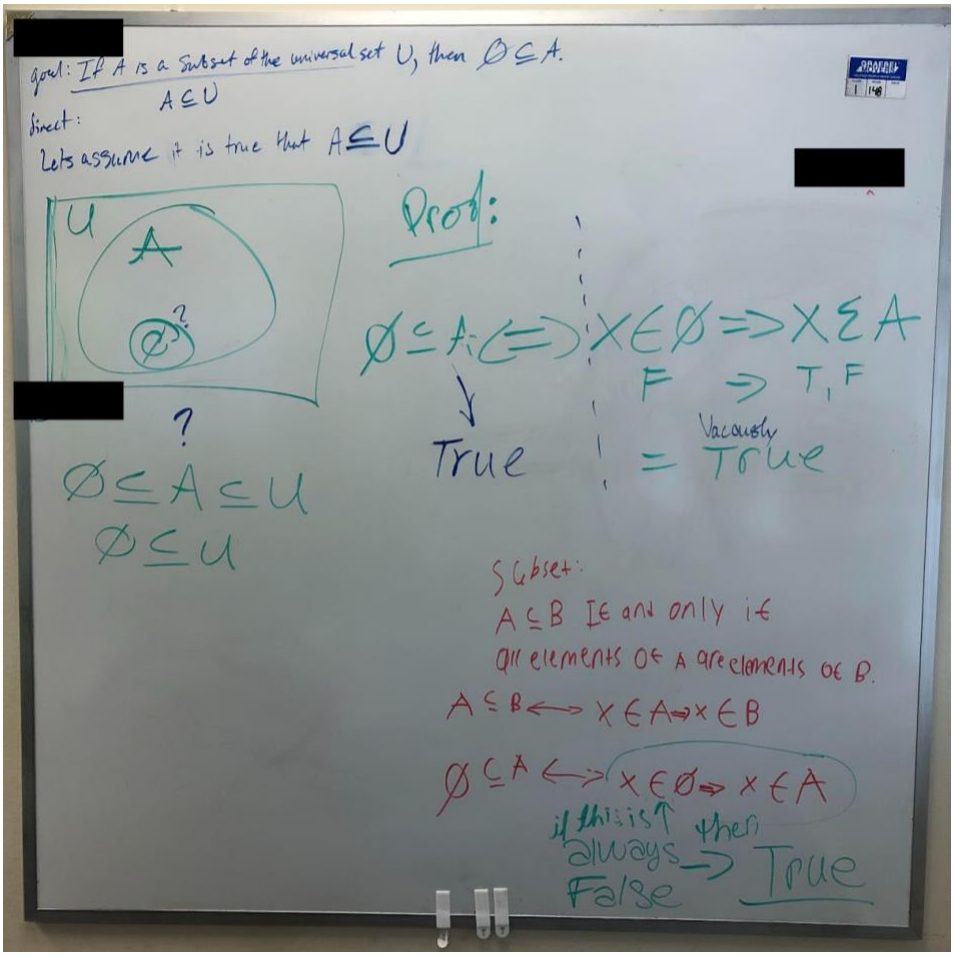
Dr. Hadley asked the team what they had, and Brad Junior expressed that he didn't necessarily agree with what they had. When he tried to express that he thought he needed to justify why "the set of nothing implies the set of something," Dr. Hadley pushed back on Brad

Junior's language and reference to object types. Dr. Hadley moved to the team's dry erase board and helped point out different object types (e.g., $x \in \emptyset$ is a statement and not a set). Dr. Hadley pointed out Ashley's note on the board that $x \in \emptyset$ (or, because of the ambiguity in the annotation, $x \in \emptyset \Rightarrow x \in A$) was "always false," but Brad Junior said he disagreed with that claim (referring to the claim that $x \in \emptyset \Rightarrow x \in A$ is false). Dr. Hadley facilitated a discussion about the component parts of the implication $x \in \emptyset \Rightarrow x \in A$ and that $x \in \emptyset$ must always be false. Brad Junior argued that $x \in \emptyset$ being false did not necessarily mean the entire implication would be false, and Dr. Hadley agreed with him. Brad Junior and Ashley reasoned and struggled to find the word "vacuously true" to describe when the precedent of an implication is false making the entire implication statement true, but with Dr. Hadley's help they found the phrase. She instructed the team to consider how they will communicate their argument to the class and left the team.

Astounded at their revelation, the team looked to their work on the board and Ashley moved to the board work and edited their writing to reflect their revelation $x \in \emptyset \Rightarrow x \in A$ was vacuously true. While editing their board, the team discussed how to follow the logic from their vacuously true claim to then determine $\emptyset \subseteq A$ is a true statement. Brad Junior and Ashley then worked together to continue editing their board to prepare to present their proof to the class. When they were satisfied, they all returned back to their desks. Figure 1-12 is a photo of the board work of the focus group at the conclusion of Episode 3.

Figure 1-12

Dry-Erase Board Work from Episode 3



Note: In Episode 3, Ashley used a green marker, Brad Junior used a blue marker, and Participant B used a red marker. Participant names were redacted from the image.

5.3.2 Episode 3 Critical Moment

During the SRI for proving episode three, only Brad Junior and Participant B were present. At the conclusion of the SRI when asked what moments were most important for the team's creativity, the participants agreed that the moment when they realized the implication $x \in \emptyset \Rightarrow x \in A$ was not false because $x \in \emptyset$ was false, but that the implication statement would be vacuously true. Participant B said, "The most important part was where we finally realized what

the implication statement actually meant. Like when we figured out it was vicariously [sic, vacuously] true, I feel like that was the most important part of it." In support of Participant B's identification of this critical moment, Brad Junior agreed,

> I think it was really important for us to recognize the vacuously true part of the implications. I mean, that's what the whole proof basically hinged on. It's like upon that recognition, we were able to say, "Okay, since this is true, then the other side of the biconditional's true, which makes the whole conjecture true." It's like a domino effect of trueness, and it allowed us to just kind of, like– the whole time we were doing it we would find something that we thought was a true thing, and it would be like trudging through mud to try to show that we thought it was true. But once we found the actual, like, true thing, it was just like water spilling out. It just like… it just flowed very smoothly and freely for us to just kind of articulate and recognize it.

In his comment, Brad Junior noted how recognizing the statement as vacuously true was important for their creativity because that statement being vacuously true allowed the team to seamlessly follow the logic backwards and show the claim $\emptyset \subseteq A$ was true.

In addition to recognizing the importance of when the team realized the implication was vacuously true, Brad Junior also emphasized the importance of using the definitions they were given and making the definitions operable. He said,

> I would have had to go to Dr. Hadley to even get started on this. And so, because of that, without the group I don't think I would have been able to do this because I wouldn't have even thought about checking the definitions. And now for every problem set I'm checking definitions seeing if they're useful. I knew definitions were important, and they had power, but I didn't realize how strong they were. Because in the past, most problem sets they didn't have, like operable things within the definition. Like the definition was basically just telling you what it was. And it kind of gave you operable things like for rational numbers or something, but not in the same way I don't feel as in these sets where like, the definition will tell you everything. And then at the very end, it's like "this is equal to this." It's like, oh, well, that makes it a lot easier.

This comment from Brad Junior highlights how realizing how important definitions could be to proving stuck with him and he now uses it when proving independently. When asked if there was a specific moment when finding and using definitions aided in their creativity, Brad Junior responded,

> Yeah, in fact, I felt like when we were doing it, I was wondering if we need to incorporate another definition we have for complement definition, I was contemplating trying to see if we need to incorporate. Oh, it was when we were doing the salvage if we needed to incorporate the definition of a complement, and we would incorporate that. But we didn't end up using that.

In this comment, Brad Junior brought up a moment during the collaborative proving where he had a thought about using the definition of complement when the team was constructing their salvage claim ($\emptyset^C \subseteq A \subseteq U$). He reflected that although he had this thought, he did not bring it up to the team. During the SRI, Brad Junior recognized that the complement of the empty set is the universal set, and this realization could have helped them recognize their salvage claim was not true.

Participant B also noted an important moment using definitions to recognize $x \in \emptyset$ must be false. He said,

> When we found it [the implication] was vacuously true. I think that was the most important part. It was either that or when we discovered that *x* can't be a part of-- it can't be an element of an empty set. Because that kind of led us to a step in the right direction to solving this.

Given these reflections from the conclusion of the SRI, the critical moment in Episode 3 was determined to be when the team used the logical properties of the implication statement to determine it was vacuously true. Given the additional importance of using definitions in the proving episode, special attention was given to the use of definitions in the abductive analysis for this critical moment.

5.3.3 Episode 3 Events Leading to Critical Moment

The critical moment in this episode occurred very late in the collaborative proving, so many of the actions and events leading to the critical moment can be traced back to very early events in the episode. Much of the group's discussion surrounding deciding whether or not they should prove or disprove their given statement was rooted in an initial search for intuition (e.g.,

reasoning about “nothing” in “something”) and looking for something tangible and operable they could use with respect to the empty set.

In seeking something tangible, Ashley suggested they think of an example of an empty set in a set, and this prompted Brad Junior to ask Dr. Hadley if zero could be in an empty set. She told the team they should reference the definition provided in their problem set. This moment is when definitions and their importance first appeared in the proving episode. During the SRI (12:35), Brad Junior and Participant B reflected on how they did not completely realize the power the definitions could have for them in proving their statement. Brad Junior said,

> I think the first time [we looked at definitions], I didn't really glean much of anything. [...] I just kind of skimmed over it and I was like, “okay, well, that means that there's no elements, I'm not really sure what that gives me.” I didn't even think about the fact that the definition gives us something operable. Like it gives us a way to express an empty set. And like how, not only-- because that definition gives something to express an empty set, but also like the definition of a subset gives us something operable. So like, when we looked at the definitions, I think it was just like, I kind of looked at it and I was like, "okay, I see this, but I'm not sure what to do with it.”

Here Brad Junior noted how they only consumed the definitions at surface level and did not realize how they could assist their team in finding something operable to use in their proof. Similarly, Participant B said at this point in the SRI,

> Yeah, that's kind of how I was the first time I actually read the problem set. Like, I just kind of skimmed over the definitions. And I was just looking at it like, “Well, I'm not really sure…What can I do with this?”

Soon after this moment, the team had spent some time working on a potential proof by contradiction but felt stuck and asked for help from Dr. Hadley. In Dr. Hadley’s first visit to the group, she instructed the team to physically write the definition of subset on their dry erase board in the operable form (using an implication). Brad Junior described this as a turning point in their proving (SRI pause at 23:54). He said,

> At this moment, it seems like we're just now kind of treading into new waters, and we're starting to realize the valuables within the waters. [...] It's like we were in an ocean of saltwater and it's like we started treading into fresh water that had like bits of gold in it.

This reflection from Brad Junior implies that he perceived this moment of using the definitions relevant to their statement as incredibly valuable, likening them to "bits of gold" in "new waters."

Dr. Hadley's prompt to write and utilize the definition of subset in their proof allowed the team to operationalize their claim $\emptyset \subseteq A$ as $x \in \emptyset \Rightarrow x \in A$. Before leaving the group to work further on their own, Dr. Hadley urged the team to recall what they knew about the logic behind implication statements to move forward. Yet, when the team was left to their own to work, they erroneously claimed the entire implication was false, rather than draw upon their previous work in the course constructing truth tables for implication statements. Brad Junior paused the recording during the SRI (24:48) and began the following conversation:

> Brad Junior: She alley-ooped us and we just threw it to the ground.
>
> Participant B: She actually stated to us like, "Well, if the first statement is false, what would that mean the second thing to be and for the whole implication statement to be?" And we just said, "Well, if the first thing is false then everything's false."
>
> Brad Junior: Yeah. "Here's the alley-oop," and we threw it out of bounds.
>
> Participant B: We just threw it in the stands. [...] We fumbled the ball on the one yard line.

This conversation exemplifies the frustration the participants felt with themselves during the SRI for not following through with what Dr. Hadley intended and set them up to do. Brad Junior attributed their eagerness to believe the implication was false to the pressure to be finished with their proof because other groups in the class had begun moving back to their desks. Although Brad Junior did briefly try to challenge Ashely's conclusion that the entire implication

statement was false, he stated he was not as confident and assertive when making this challenge because he did not have the appropriate vocabulary to articulate his thinking,

> I pointed out the vacuously true part, but I didn't use those words and I didn't... I was thinking about that, I think when I was typing my [reflection on this class], that like, when I was trying to express my concern for us, like, jumping so far ahead, I wasn't sure how to express that concern appropriately. And, like, when I did, I don't think I said it properly. So that's why eventually I was just like, "I feel like there's something wrong, and I think I know where it is, but I'm not really sure how to say it so I'm just gonna let it be." Literally I think that's exactly what I typed is that I kind of thought where I'm-- I thought I could recognize where it was wrong, but I wasn't sure how to articulate why it was wrong. And so, I didn't want to cause more turmoil by saying "this is wrong" so I'm just gonna let this be. (SRI, 25:14)

Moreover, Brad Junior attributed part of his resignation in challenging the conclusion his team made to the time pressure to be finished with their task,

> I wonder if I was even fully paying attention. Because I overlooked the implication, I think. I think I recognized that-- I agreed with everybody, because they were like, "Yeah, this side's false, like it's always false." And I was like, "Yeah, that's true." Because, you know, empty sets aren't supposed to have elements. But it's like, I feel like we kind of ignored the implication in some way. Like, we were just kind of jumping. And we were like, wanting to rush. I know I was kind of rushing because I was like… the other groups, I think they're sitting down. […]
>
> And so, like when they said that the whole thing was false. I was like-- you might hear me say it-- I eventually get to the point where we... I'm jumping ahead a little bit, but eventually it gets to the point where we're like, "Okay, this whole thing's false." The whole implication is false, because this is false. That means the whole biconditional is false. And so that means we need to salvage, because that means the whole thing is false." So, we kind of wrote down our brief salvage, which I think Participant B was saying we did incorrectly anyways. Which I mean, should have caught on to the fact that, hey, this isn't supposed to be salvaged. But when I was sitting down, I was like, "I feel off about something. I don't like the way I feel. I don't feel like it's complete. But I'm not sure and I'm okay with just being done and being corrected." Is the point I was at when we sat down. (SRI, 24:48)

In this reflection, Brad Junior noted that he was "overlooking" the implication; he was perhaps mistaking the logic behind an implication for that of a conjunction statement. He noted that other groups of students appeared to be completing their assigned task, so the external pressure to be finished with their task caused him to rush. Thus, when he did have the intuition to

challenge their claim that the implication statement was false, he was quick to resign and allow their proof to be incorrect.

This external pressure and perceived time constraint led the team to accept their erroneous reasoning and move on to trying to salvage the original claim. During this time, the team determined their salvage statement to be $\emptyset^C \subseteq A \subseteq U$ and moved back to their desks. Although the team perceived other teams in the class as finishing their tasks, other teams of students continued working after Brad Junior, Ashley, and Participant B returned to their desks. Brad Junior's decision to turn his mind to another problem relevant to proofs about sets was essential in his continued deliberation about the validity of his team's proof. During the SRI (31:09), Brad Junior explained his choice to work on a different problem from their assigned problem set during this time,

> So, I do this, just day in, day out, on tests, on all kinds of things. If I feel like I haven't finished a problem, or I'm kind of stumped on how to explain why I don't think I've finished the problem, or like, what's not clicking, I just take a step back. And I go work on something else, and I try to actually still work on something on a similar topic. Because I know some people that are like… they say, if you're working on it late at night, and you're able to just go to bed and work on it, the next day and it will kind of click because your subconscious will continue to work on it. So, if it's during the daytime or something, and I'm not obviously able to go and able to go take a nap or something and just take a break from it or... Some people, they go and they work on something completely different and kind of give that whole part of the brain a rest. I tend to work on the similar thing, something that's still in that alley, so that way, when it clicks for the subconscious part that it's working on, it can come to the forefront of my brain, and I know exactly what it's saying. And so that's kind of what I was doing here is I was working on another problem. And I was just kind of still in the back of my mind, while I was working on the problem, because it was more so I was just finalizing a [different] problem.

During this time, Brad Junior was working on the problem, if $A \subseteq B$ then $B^C \subseteq A^C$. He explained that he regularly takes a step away from problems he feels troubled or stuck on by working on a different problem with a similar topic because it would allow his subconscious to continue contemplating the problem while directing the forefront of his attention to something

else. While Brad Junior worked on this related problem, another student group was discussing the claim Ashley, Brad Junior, and Participant B had been assigned (If A is a subset of the universal set U, then $\emptyset \subseteq A$). Another student said, "you can add an empty set into another set, but you're essentially just going to be adding nothing to it." Overhearing this comment, Brad Junior said to his team, "Yeah, I don't know if you heard what he just said, but that's what I was thinking. I'm just not sure how to defend it." At this point, however, the team did not make an effort to revisit their logic, but rather decided to wait for the whole-class discussion. It was not until Dr. Hadley visited the team for a final time when Brad Junior's intuition that they did not have a correct proof was unpacked.

The final event facilitating the critical moment in this proving episode was Dr. Hadley's revisiting the team and giving credence to Brad Junior's argument that the precedent being false did not make the entire implication statement false. Brad Junior's attempts to challenge the team's conclusion had been previously disregarded by Ashley and Participant B, but Dr. Hadley told Brad Junior in front of Ashley and Participant B, "I agree with you. What does it do?" This invitation to pursue an alternate argument was the final ingredient needed for the team to realize the implication was vacuously true.

In summary, in order for the group to reach the critical moment in their proving of realizing the implication $x \in \emptyset \Rightarrow x \in A$ was vacuously true, Dr. Hadley instructed the team to write the operable form of the definition of subset on the board and write $\emptyset \subseteq A$ in the form of that definition, Dr. Hadley prompted the team to consider the logic behind implication statements, the team discovered $x \in \emptyset$ is always false and concluded the implication statement must be false, and finally Brad Junior engaged in incubation and contemplated his uneasiness

about their conclusion and his concerns were validated by Dr. Hadley in her final visit to the team.

6. Discussion

The three episodes described in this paper highlight the different behaviors and processes students might engage in during in-class creative collaborative proving. In the SRIs corresponding to the three episodes, participants identified distinct moments they perceived as most important to their creativity. In Episode 1, the critical moment was the identification of a mistake during the team's empirical exploration of their given task. In Episode 2, the critical moment was one team member's divergence from the team's original proving approach. Finally, in Episode 3, the critical moment was the team's recollection of properties of implication statements and the recognition of a statement as vacuously true. In each of these proving episodes, the team engaged in different mathematical behaviors and processes to lead them to these critical moments in their creativity. While viewing their proving after-the-fact during the SRI, the team also recognized decisions and behaviors that slowed their progress toward their eventual critical moment and the associated revelations.

In Episode 1, the focus team primarily engaged in empirical exploration by testing different pairs of numbers in their conjecture (the product of twin primes is one less than a perfect square). Throughout the episode, the team proposed different examples to test for different motivations. First, the team proposed examples of twin prime pairs to determine if they wanted to prove the claim or try to disprove the statement. This process is similar to what Otten et al. (2017) referred to as *proof initiation*, which is the process of choosing and/or refining a claim and determining whether or not it should be proved. Dr. Hadley's class-wide prompt to consider how teams could prove the statement for "all" cases caused Ashley to shift their

motivation for testing examples away from verifying and understanding their claim to seeking a counterexample that could disprove the claim. Seeking a counterexample is an attempt to shift the team from proof initiation to *proof construction*, or articulating an argument, yet the example chosen as a candidate for a counterexample (51 and 53) still fit the conjecture, so this led the team to shift to constructing an argument to prove their given claim true. Dr. Hadley's probing of the team regarding their example of 51 and 53 and insistence that the team leave their work relevant to this example on the board was instrumental in the team's noticing their mistake, which allowed the team to generalize their given claim and represent their new claim algebraically. In this episode's SRI, the participants emphasized that allowing themselves to consider examples that were not directly applicable to the claim they were assigned to prove (or disprove) would have the power to help them discover mathematics beyond the claim they were given. Participant B noted during the SRI, "I don't want to be conceited or something, [but finding our mistake that 51 wasn't a prime number let us go] more in depth with our proof than the other groups." In this way, there is evidence that instructors should allow students to make mistakes or erroneous assumptions, but also use these moments as an opportunity to highlight mathematical practices of refining and formulating mathematical conjectures.

The importance of making and realizing this mistake in Episode 1 relates to Aljarrah's (2020) notion of *expanding possibilities* in collective creativity in mathematics. Expanding possibilities entails being imaginative, asking questions, and playing. Had the participants in this study been more open to imagination and play in their proving and general approach to mathematics, they may have allowed themselves to knowingly explore if their claim could also apply to a more general set of numbers by testing pairs of numbers that were not twin primes (e.g., 7 and 9). The participants, however, viewed testing a pair of values not applicable to their

given claim as a mistake and instinctually wanted to remove the work from their board rather than play and explore what the anomaly could mean. Thus, expanding possibilities as a collective creative act in mathematics should possibly also include the willingness to make and learn from mistakes and/or take risks.

In Episode 2, the critical moment for the team was when Ashley diverged from Brad Junior's approach to their proof and Participant B shifted to take up Ashley's suggestion. This moment as important to the team's creativity is supported by the notion of *flexibility* (Guilford, 1950) as a component of creativity. Flexibility refers to the ability and process of changing directions or approaches during a task and has been previously linked to creativity in mathematical proof (e.g., Savić et al., 2017). Flexibility, however, functions differently in a collaborative setting compared to an individual setting. In a collaborative setting, particularly an undergraduate mathematics classroom, flexibility can apply to the team collectively deciding to change directions cohesively, but it can also apply to one individual diverging from the work their team has pursued up to a certain point. Although the former situation (the shared decision to collectively change directions or approaches) may imply a shared (and thus dispersed) sense of mathematical risk, the latter situation (in which one participant proposes diverging from the idea proposed by another) shifts the responsibility and weight of associated risk to the single participant who has proposed a change in direction. Moreover, diverging from a teammate's proposed solution path also involves a social risk. Literature related to collaborative creativity in domain-general settings might attribute Ashley's disposition to diverge from Brad Junior's approach to a sense of psychological safety, trust, reflexivity, and participation in their group and their class as a whole (Paulus et al., 2012). In Episode 2 of Ashley, Brad Junior, and Participant B, this team had worked together on previous occasions and had nearly a month of working

collaboratively in this course. Ashley contributed to Brad Junior's approach in the beginning of their time working together, and it was only when she felt she could no longer contribute to his idea when she diverged from their first approach. In other words, Ashley followed through with a teammate's suggestion before prioritizing her own. The class collaboration norms of being open-minded to others' ideas and respectful were highlighted in Episode 2 to facilitate an environment that allowed Ashley to feel comfortable asserting her ideas even when they may have been perceived in conflict with the route her team was pursuing previously.

Finally, the critical moment in Episode 3 for the team's creativity was the realization (or recollection) that an implication with a false precedent is vacuously true. This realization ultimately hinged upon Brad Junior developing an intuition that his team's work was not correct and voicing his concern for the team's proof. The first two times Brad Junior voiced his concerns were dismissed or unaddressed because (a) Brad Junior did not have the necessary language to express himself and (b) the team was not willing to expend the energy and time revisiting their existing argument and felt external pressure to be finished with their work. This episode was unique from the two others described in this paper because the team left their dry-erase boards and returned to their desks and seemingly were satisfied with their proof about 13 minutes before Dr. Hadley actually concluded the group-work portion of the class and directed students to a whole-class discussion. During this time, Brad Junior continued working on problems assigned to the class regarding sets and engaged in a blend of *incubation* and *proof validation*. Incubation (Wallas, 1926) is the second proposed phase of the creative process, and it is characterized by stepping away from a problem and allowing the unconscious mind to take over. Savić (2015) illustrated that mathematicians often engage in incubation and find this time helpful for overcoming an impasse in their work. In theorizing ways to promote the creative process in

undergraduate mathematics, Savić (2016) noted students may not have opportunities to engage in incubation in class. Similarly, Schindler and Lilienthal (2020) noted limited time, such as that of a class session or interview, may have limited incubation. In response to this issue, Schindler and Lilienthal (2020) proposed a phase of the creative process called *mini-incubation*. Mini-incubation was theorized in the context of multiple solution tasks (MSTs) and was defined as "a shortened incubation period that occurs while the student is working on another approach to the same MST" (p. 1583). Brad Junior's behavior in Episode 3 reflected neither true incubation (in which no work of the mind is consciously perceived) nor mini-incubation because Brad Junior was not working on a different approach to the same proving task. Thus, a contribution of this episode is that mini-incubation can be generalized to mathematical contexts beyond MSTs and include instances in which students or mathematicians intentionally work on a related, but distinct, problem to take a break from the original problem and allow the unconscious mind to work.

In Episode 3, the time allowed for Brad Junior's discontent with the team's argument to fester and grow spurred him on to continue voicing concerns about the proof. It was Dr. Hadley's validation of Brad Junior's concerns that finally facilitated the group's investment in returning to and considering if their argument could be flawed. Previous research on collaborative creativity in mathematics has primarily been conducted in a laboratory setting (e.g., Aljarrah, 2020; Schindler & Lilienthal, 2022) and has not considered the influence of the instructor's presence on collaborative creativity. Research on mathematical authority in proving (e.g., Bleiler-Baxter et al., 2023; Otten et al., 2017) may be relevant to why the team did not originally make effort to address Brad Junior's concerns about their argument and why Dr. Hadley's agreement with Brad Junior's concern did eventually probe the group to revisit and correct their argument. Although

Adiredija and Zandieh (2020) considered the influence an interviewer can have on the creativity of an interviewee in a task-based interview, their study was situated in an interview context rather than a classroom with a clear instructor-student dynamic. Moreover, Brad Junior, Ashley, and Participant B were influenced by other groups of students and their behaviors during their proving (e.g., hurrying to complete their work because other groups were finishing). Future research should expressly investigate the actions of an instructor in collaborative creative settings in mathematics in *real classrooms* (Heath & Bleiler-Baxter, In press) to highlight the intricacies of instructor interactions with collaborative groups as well as the influence the presence of other student groups has on creativity.

Across the three episodes, it is evident there is not a linear path this group took in their creative collaborative proving. Moreover, there is little evidence to claim there is a typical sequence of events the group experienced in the three episodes. Although the differences in processes the team engaged in across the three episodes could be attributed to the variation in tasks assigned, it is also possible to conjecture the paths the group took in the three episodes illustrate their growth across the semester. For example, in Episode 1, the primary activity the team engaged in was empirical investigation and testing examples against their conjecture, yet in Episode 2, the team did not explicitly test any examples with the motivation of testing their claim, but immediately began considering ways they could present a general proof for the statement. Interestingly, in Episode 3, when the group did not have initial ideas for how to approach their proof, they did revert to their instinct to find an example to help them understand the conjecture, but the statement they were tasked with proving or disproving in Episode 3 did not lend itself to example use in the same way the number-theoretic proofs in Episodes 1 and 2 did.

Although there is evidence of the group engaging in many of the phases of creativity set forth by Wallas (1926; e.g., preparation, incubation, verification), it is clear from observing Brad Junior, Ashley, and Participant B across these three episodes that the group did not follow the four step process linearly. This finding is similar to that of Schindler and Lilienthal (2020) who observed an individual's creative process as a complex iterative and cyclical process. The process observed in the group in this study more closely resembles what Lakatos (1976) described as a "zig-zag" from naive conjecture to examples and counterexamples, to conjectures, and finally to argumentation and proof. Although the investigation of these three episodes with a single group of students are not intended to provide generalizable evidence, it does provide a counterexample to extant models for the creative process and suggest there is room to further blend models for the proving process, collaborative process, and creative process to more deeply understand and model collaborative creative proving.

In summary, the three episodes and critical moments to the creative process of this focus group presented in this paper have highlighted how extant models for collaboration, creativity, and proving can be present in a group pursuing a non-routing proving task as well as provided insight into how the physical context of a real classroom and mathematical context of proving (compared to MSTs) might require adaptations to definitions of acts and processes currently theorized as a part of collaborative mathematical creativity.

7. Implications for Research and Teaching

Although the goal of this manuscript was to understand the essence of the experience of collaborative creativity in proving, rather than explain or draw conclusions about how to foster creativity through collaborative proving, there are still observations from each episode which lend themselves to possible implications for instruction to be investigated in future research.

For example, the importance of the moment in Episode 1 in which Ashley, Brad Junior, and Participant B discovered their error and were instructed not to erase it from the board suggests the potential importance of instructors designing and selecting tasks that lend themselves to students making and learning from errors, having flexible, impermanent workspaces for students that lower the stakes of making a written error, and allowing students to explore conjectures empirically.

In Episode 2, the critical moment of the focus group was Ashley's decision to break from her team and pursue an alternate proving strategy. Episode 2 and it's critical moment suggest instructors might consider how to set classroom and group work norms that allow students to disagree with one another and try multiple strategies for proving. Another instructional consideration from Episode 2 is to make sure all students have access and space to pursue their ideas; the instructional decision to give each student in each group their own dry-erase marker with a distinct color may have provided the opportunity for Ashley to diverge from her team that might not have been possible if the group was sharing a single marker.

Finally, the critical moment from Episode 3 highlighted the importance of an influence the actions of other peer groups can have on a team's decision to continue working or resign from the task and the instructor's validation of student ideas. Instructional implications from this episode to foster creativity through collaborative proving may include clear communication from the instructor regarding the length of time that will be allowed for the task as well as providing additional tasks for student groups who complete their assigned task early. In the case of Ashley, Brad Junior, and Participant B, had they known the group work time would continue for several more minutes and not felt pressured to finish their work due to other student groups returning to their seats, they may have invested more effort in Brad Junior's suggestion to reconsider the

team's work. Additionally, Episode 3 illustrated the power of an instructor's validation of student thinking. Ashley and Participant B did not attempt to engage with Brad Junior's intuition that their proof was incorrect until Dr. Hadley agreed with Brad Junior's thinking. This suggests instructors should pay close attention to when and how they publicly validate student thinking. Moreover, the power and authority illustrated in Dr. Hadley's validation in Episode 3 suggests there is potential for future research investigation into authority and power and its relationship to collaborative creativity, for example, in the spirit of Bleiler-Baxter, Kirby, and Reed (2023).

As previously mentioned, the purpose of this manuscript was not to make general claims or recommendations for instruction; however, I have provided some thoughts on the instructional elements relevant to the three episodes explored in this manuscript. All of these suggestions and speculations provide directions for future research in the realm of collaborative creativity in proving and mathematics more generally.

8. Limitations

The primary limitation of this study is the participants' ability to identify moments instrumental to their creativity with fidelity. Although participants were instructed to pause the recording of their group proving at moments relevant to their creativity, participants often paused the recording rather to comment on something that occurred or justify themselves (e.g., to remark they did not properly follow instructions for individual think time). Moreover, the SRI for Episode 3 and the corresponding data analysis was limited to the perspectives of Brad Junior and Participant B, as Ashley was not present to provide her thoughts on the recording of their proving. Finally, this research concerned three students who were purposefully sampled because of their demonstrated ability to communicate and reflect upon their experiences with creativity in

writing, so the selection of these participants may have introduced confounding variables and personality traits that do not reflect typical students in an introduction-to-proof course.

9. Conclusion

In this paper, I have described a study with the purpose of investigating how the process of collaborative creative proving can be characterized and what events and actions lead to moments critical to collaborative creativity in proving. By collecting video data from in-class collaborative proving episodes and conducting stimulated-recall interviews with a focus group of three students, I gained insight into what moments the students considered most critical to their creativity in their collaborative proving episodes. From these moments, I reconstructed the events that allowed the moment to transpire.

Across the three episodes investigated, the critical moments included realizing a mistake that allowed the team to further generalize their claim, one student diverging from the path originally selected to pursue by her team, and recalling the logical properties of an implication statement to determine a claim was vacuously true. The events and circumstances that allowed these critical moments to transpire included making mistakes, seeking counterexamples, responding to instructor questioning, team willingness to pursue multiple avenues for proving a statement, incubation time, and instructor validation of student concerns and ideas.

These findings mirror and connect to many different models for collaboration, creativity, and proving in extant literature (e.g., Aljarrah, 2020; Otten et al., 2017; Schindler & Lilienthal, 2020, 2022; Wallas, 1926), but this study offers unique contributions by both being conducted in a real classroom and relying upon participant perspectives and reflections on creativity. Future research on the collaborative, creative proving process should both expand investigations to consider multiple groups of students at a single point in time as well as more explicitly

investigate how a team's creative process might evolve over the duration of an introduction-to-proof course. The former will allow researchers to reach more general results with respect to identifying moments critical to creativity and events allowing such moments to transpire in collaborative proving. The latter suggestion will provide perspective on how mathematical context and experience with proving can affect the creative processes groups endure. Moreover, research should continue to be conducted in real classrooms and give special attention to the role of the instructor, their instructional decisions and actions, and how these interactions shape the creative process of a group proving together.

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Appendix 1-A

Focus Group Stimulated-Recall Interview Protocol

Introduction:

Moderator: Hello! Thank you all for joining me today. I'm Amanda Lake, and I am a doctoral candidate in mathematics education, and I'm currently working on my dissertation project. I greatly appreciate your time and energy during our interview today. First of all, I want to remind you all that everything you say during this interview is confidential and will not be shared with your course instructor until after the course has ended.

Moderator: Just to make sure we're all acquainted, I'd like to start by having everyone introduce themselves by saying their names and a little about your major and your history with mathematics.

(Participant introductions)

Moderator: Thank you all for sharing! To set your expectations for the rest of this interview, I will be moderating what is called a *stimulated recall interview*. This means that I will be playing an audio/video recording of a proving activity you all were a part of during class and asking you all to provide your input on the interactions. The goal of this activity is for me to learn about your experiences of creativity during the encounter. The video recording I have is about ___ minutes long (20-35 minutes). I anticipate this interview will take us about an hour and a half (90 minutes).

What I would like for you to do as I play the video is watch along, and then ask me to pause the video when you observe a moment you think was important to your group's creative process during the task. I would like for you to stop the video when you see your group (1) **making connections** to definitions, theorems, different representations, or examples, (2) **taking risks** by attempting to use a new tool or trick, switching strategies, asking questions, making suggestions, or evaluating your proof attempt, OR (3) engaging in any **other** actions or behaviors you consider creative. I've printed out these instructions for you to have as a reminder while we watch the recording together.

(Hand out instructions- Figure 1-3)

After someone asks for the video to be paused, I will pause the recording. I will ask you and the rest of the group about the moment that we paused upon. I can rewind the recording upon your request at any time. Once we are done discussing the part of the video we paused on, I will resume the video from that point. Throughout this session, I want to remind you there are no right or wrong times to pause the recording, and there are no right or wrong answers when we discuss the video. I just want to know what you think.

As a final note, it can feel a little funny to watch yourself on video, and that is to be expected. We want to make a sincere effort to be respectful and courteous when we refer to others in the video.

Are there any questions? Are you ready to start the video?

(Play video until participants indicate to pause).

Questions to ask during "pause" discussions.

1. Why did you ask to pause the video at this moment?
2. What were you all thinking during this moment?
3. What do you notice led up to this moment? How did you have the idea to do ____?
4. How do you think this moment was important or unimportant in your group's proving process?
5. Does anyone have a different perspective on this moment?
6. Do you have any other thoughts about this moment during your group's proving?

Conclusion:

Now that you have all reviewed and discussed the entire recording, what moments would you say were the most important to your team's creativity?

Does anyone have any concluding reflections or thoughts on the recording we watched?

Moderator: Thank you all again for participating in this interview today. I greatly appreciate your time and your thoughts!

Manuscript 2: In-the-Moment Experiences of Creativity in Collaborative Proving

ABSTRACT: In order to inform the instruction of undergraduate proof-based courses to foster collaborative creativity, it is necessary to first understand how students experience and perceive themselves as creative through collaborative proving activities. In this study, I investigate how students describe feeling and not feeling creative during in-class collaborative proving activities. Written reflection data was collected from 10 undergraduate student participants enrolled in an introduction-to-proof course, and these data were inductively analyzed to develop two overarching categories: Behaviors, actions, and/or moments and Circumstances. Findings of this study are presented through seven themes describing how students felt and did not feel creative relative to those themes. Major findings illustrate that experiences in collaborative creativity largely mirror those in individual creativity, including through making connections, taking risks, and having appropriate tasks; however, data in this study also indicate there are features of in-class collaboration that introduce complexities in the student experience of creativity, including social risk taking, time restraints, and expressing oneself in words.

1. Introduction

Both collaboration and creativity are central to the work of mathematicians (Karakok et al., 2015; Sriraman, 2004). Professional mathematicians have indicated collaboration is an important feature of their work (Sriraman, 2004), and the Conference Board of Mathematical Sciences (2016) has called for university mathematics classrooms to incorporate more active, collaborative learning. Therefore, there is a substantial need for K-16 education to focus on developing creative mathematicians and problem solvers with strong communication and collaboration abilities. In mathematics, students follow a journey in which there is an eventual transition from computational to proof-based mathematics (Civian & Schley, 1996). During this transition to proof, students are often expected to become producers of mathematical ideas on their own for perhaps the first time (Boyle et al., 2015). For this reason, it is critical to study mathematical creativity and collaboration within the context of undergraduate mathematics courses and proof to cultivate the expertise needed by future mathematicians.

The purpose of this study is to investigate the ways individual students experience creativity during collaborative proving. In Manuscripts 1 and 3 of this dissertation, I capture creativity as experienced by the group as a whole as well as experiences of creativity across an inquiry-oriented course in the transition to proof. In this manuscript, I respond to the research question, what do individuals report fostered their creativity in-the-moment during collaborative proving?

2. Background

2.1 Defining and Characterizing Mathematical Creativity

Creativity and mathematical creativity have been notoriously difficult to define, and there is no widely-agreed upon definition of mathematical creativity used in mathematics education

research (Mann, 2006). Moreover, extant frameworks for describing mathematical creativity are largely limited to considering creativity as an individual activity, as opposed to a collaborative activity.

One of the oldest characterizations of creativity comes from Guilford (1950). Guilford's (1950) characterization was created with the goal of providing testable factors of creativity and has been commonly used as a framework for measuring and defining creativity. The four aspects of creativity, as suggested by Guilford, are fluency, flexibility, originality, and elaboration. This characterization inspired the Torrance Tests of Creative Thinking (Torrance, 1966) to measure domain-general creativity, the similar Mathematical Creativity Test (Kattou et al., 2013) to measure domain-specific mathematical creativity, and many others (e.g., Leikin & Elgrably, 2020; Levav-Waynberg & Leikin, 2012; Siswono, 2010).

Guilford's (1950) description of creativity, as well as other domain-specific characterizations (e.g., the Creative Thinking Rubric from the Association for American Colleges and Universities (Rhodes, 2010) framed the Creativity Research Group's development of a rubric for creativity in mathematical proving, the Creativity-in-Progress Reflection on Proving (CPR; Savić et al., 2014, 2017). Savić et al. developed the CPR on proving by combining ideas from these extant domain-general characterizations of creativity with information gathered from interviews with research mathematicians regarding their perspectives on mathematical creativity (Karakok et al., 2015) as well as undergraduate student's experiences with creativity in proving (Tang et al., 2015). The CPR on proving assumes creativity to be domain-specific, a process, and relative. The CPR on proving (Savić et al., 2017) presents two categories: making connections and taking risks, which are split into subcategories: between definitions/theorems, between representations, and between examples (making connections

category); tools and tricks, flexibility, perseverance, posing questions, and evaluation of the proof attempt (taking risks category).

The CPR was designed to characterize the proving process during a single proof of an individual student. The CPR on Proving has been used as a formative assessment tool (e.g., El Turkey et al., 2018) and student reflection tool (e.g., Omar et al., 2019; Tang et al., 2017). The Creativity Research Group has also suggested future research on the influence of socialization and collaboration on creativity in proof (Savić, 2016) using the CPR. Although the goal of this study is not to create a rubric or assessment tool for collaborative creativity, the CPR's subcategories do provide a starting point for researchers and practitioners to anticipate how students might experience creativity in proving as an individual activity and thus compare these experiences to collaborative creativity in proving.

2.2 Developing and Fostering Mathematical Creativity

Research on mathematical creativity has grown immensely in the last ten years (Savić et al., 2022; Sriraman, 2017), yet this research has been slow to extend to the tertiary setting (Savić et al., 2022). In this study, I aim to describe what students experienced during in-class collaborative proving to foster creativity for either themselves or their group. Throughout my work, I adopt a developmental perspective on mathematical creativity, as many scholars generally accept that mathematical creativity can be developed and enhanced in students (Sriraman & Haavold, 2017; Zazkis & Holton, 2009), and mathematicians believe that mathematical creativity can and should be fostered in undergraduate mathematics courses (Karakok et al., 2015). Despite this consensus, there is little empirical research on what teaching strategies develop mathematical creativity among undergraduate students (Savić et al., 2022). No extant research has explicitly considered the impact collaborative work or collaborative proving

may have on creativity. Investigating student perceptions of how collaborating with their peers on proving tasks fosters (or does not foster) creativity for them can inform how to best facilitate collaborative activities to promote creativity in proving.

In extant literature, the teacher actions most supported as effectively promoting mathematical creativity are choosing appropriate tasks (Satyam et al., 2022; Savić, El Turkey, et al., 2017a; Tang et al., 2022), allowing time for incubation (e.g., Savić, El Turkey, et al., 2017b), demonstrating different ways to solve problems and illustrating the mathematical process for students (Satyam et al., 2022; Tang et al., 2022), making the classroom a safe place to take risks (Satyam et al., 2022; Savić, El Turkey, et al., 2017b; Tang et al., 2022), attending to the emotions of students (Satyam et al., 2022; Tang et al., 2022), and providing space for discussion and disagreement (Satyam et al., 2022; Savić, El Turkey, et al., 2017a, 2017b; Tang et al., 2022).

Both Cilli-Turner et al. (2023) and Satyam et al. (2022) investigated how to foster creativity in undergraduate settings by considering student perceptions of creativity. Both of these studies, conducted in the context of Calculus I, reported six themes of student views of creativity: Actions and Attitudes, Application, Different Ways, Originality, Against Authority, and Understanding. For example, a student who has an *application* view of creativity may consider it creative to apply mathematical ideas to other situations, whereas a student with an *originality* view of creativity may be concerned with the creation of novel mathematics (Satyam et al., 2022). Although these themes can help describe the ways in which students in an undergraduate setting may view creativity, I seek to understand the specific ways in which students perceive themselves (or their group) as creative and experience creativity in a collaborative proving setting. The present study will contribute to the growing body of knowledge concerning how to develop and foster undergraduate mathematical creativity by

specifically investigating how students experience creativity in-class small-group collaborative proving. These insights can provide a basis for future investigation into instructional choices to foster creativity through small-group collaboration in proof-based undergraduate course contexts.

3. Theoretical Framing

3.1 Phenomenological Perspective

I adopt a phenomenological perspective in this study, and my goal is to gain a deeper understanding of the nature of student experiences in collaborative mathematical creativity. Phenomenology is the study of lived experiences (van Manen, 1990), and although the goal of phenomenological research is to understand individuals' lived experiences, it accepts there is no way to achieve true introspection in-the-moment and all phenomenological reflection must be retrospective. In a phenomenological approach to research, the goal is to describe as accurately as possible the phenomenon of interest, “refraining from any pre-given framework” (Abakpa et al., 2017). The phenomenological perspective informed data collection and analysis in this study, namely the analysis of student retrospective writing assignments through open, inductive analysis. Throughout this study, however, I also compared findings in the data to extant characterizations of student experiences of creativity in order to make my work coherent with and informed by similar research in the domain of individual mathematical creativity.

3.2 Collaborative Mathematical Creativity as a Process

In this study and across my work broadly, I adopt a perspective on mathematical creativity that describes creativity as *domain-specific*, a *process*, and *relative*. This means I am interested in creativity specific to the context of mathematics and mathematical proof (domain specific), I will consider the motivation, perception, learning, thinking, and communicating involved in creativity rather than analyze an end result or product of creative thinking (process

rather than product), and I consider creativity within the context of the knowledge, abilities, and experiences of an individual rather than requiring creativity to produce completely novel contributions to the field of mathematics (relative rather than absolute). Put succinctly, ***mathematical creativity*** is defined to be the processes of creating, constructing, or implementing mathematical ideas, strategies, or processes, which are perceived as non-routine by the individual.

3.3 Residue

Defining mathematical creativity as a relative construct and taking a phenomenological perspective in this study require asking students what they recall as fostering creativity for them, and these recollections will describe "what actually comes to the fore of [students'] attention" (Marton et al., 2004) regarding their creativity during collaborative proving as well as other activities in the ITP course. Hiebert et al. (1996) described the importance of remembered experiences and influences as residue. Residue provides a way for thinking about what students take with them from classroom experiences. Bleiler-Baxter and Pair (2017) noted, however, that Hiebert et al. (1996) outlined the concept of residue from a cognitive perspective and this framework was designed to describe the mathematical content and lessons students carried with them from classroom experiences. In contrast to Hiebert et al.'s original use of residue, I reframe this construct to capture experiences more broadly. In this way, residue will also capture the elements of collaborative proving that foster students' creativity. As aligned with the phenomenological perspective and definition of mathematical creativity as relative, I consider a student's report and recollection of *feeling* creative as equivalent to *being* creative, as the experience of creativity as defined by the participant is the phenomenon of interest.

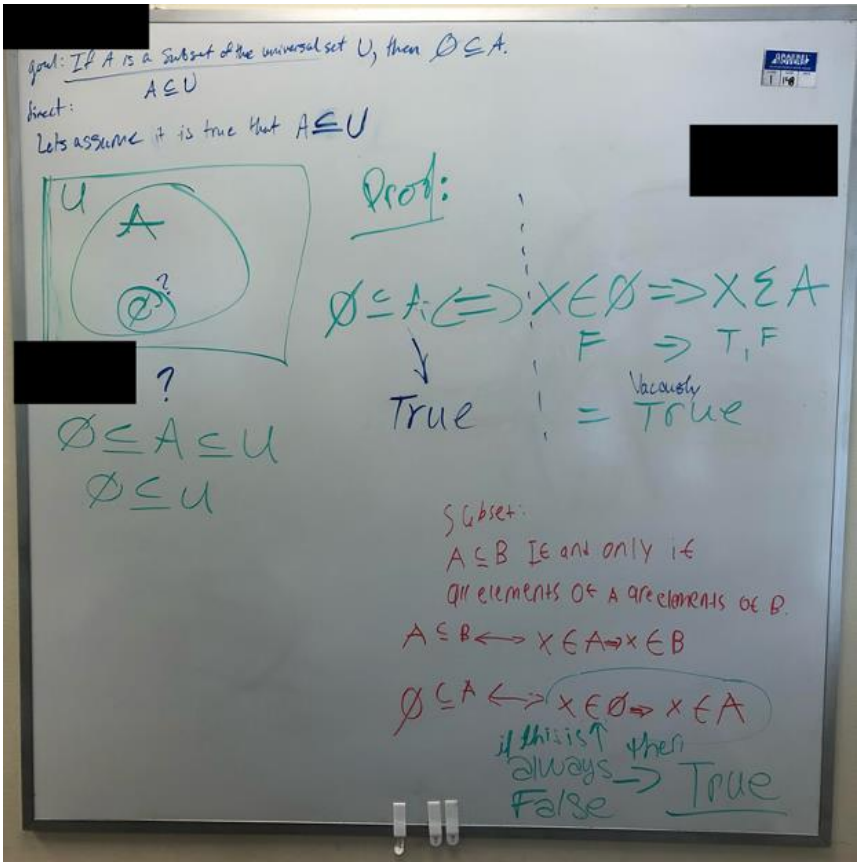
4. Methods

4.1 Context and Participants

Data were collected from an undergraduate introduction-to-proof course at a large public southeastern university in the United States. This course was facilitated in a collaborative, student-centered learning environment in which small groups of three or four students worked together to prove instructor-provided mathematical conjectures. Figure 2-1 provides an example of the typical collaborative workspace in the course. Each student group member was provided with a distinct colored dry-erase marker to encourage equitable participation in the group as well as provide a record of individual student contributions for instructional and research purposes. Table 2-1 provides the proving tasks assigned to students during the four points of data collection.

Figure 2-1

Sample Collaborative Proving Space



Note: Figure 2-1 provides an example of student collaborative proving work (Team 2, Week 11) at the conclusion of a collaborative activity. Each student used a different colored dry-erase marker during the task. Student names have been redacted from the image.

Table 2-1

Collaborative Proving Tasks

	Course Module (Week)	**Task**
DC1	Module 1: Introduction to Upper-Level Mathematics and Proof (Week 1)	Prove: The product of consecutive twin primes is one less than a perfect square.
DC2	Module 3: Direct and Indirect Proof Methods (Week 5)	Either (a) Prove, or (b) Disprove and Salvage If Possible. Team 1: An integer is even if and only if its square is even. Team 2: If l and m are odd integers, then so is lm. Team 3: If p is an integer, then $p^2 + 3p + 2$ is even.
DC3	Module 5: Set Theory (Week 11)	Either (a) Prove, or (b) Disprove and Salvage If Possible. Team 1: (The Transitive Property) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Team 2: If A is a subset of the universal set U, then $\emptyset \subseteq A$. Team 3: If $A \subseteq B$ then $B^C \subseteq A^C$.
DC4	Module 6: Power Sets and Set Products (Week 13)	Prove, or Disprove and Salvage if possible. Let A and B be sets. Teams 1 and 2: $\wp(A \cap B) = \wp(A) \cap \wp(B)$ Team 3: $\wp(A \cup B) = \wp(A) \cup \wp(B)$

Ten students consented to act as research participants. Participants were given the opportunity to choose their own pseudonym; any participants who did not choose their own pseudonym were assigned a pseudonym culturally similar to the name they used in class. A total of 30 Retrospective Writing (RW) responses were collected across the four class sessions in which data were collected. Table 2-2 summarizes the participants and their responses to the RW prompts across the four data collections (DC). Appendix 2-A provides a complete sample participant RW response.

Table 2-2

Participants, RW Responses, and Participant Groups

Participant	DC 1	DC 2	DC 3	DC 4	Total
Ashley	X	X	X	X	4
Brad Junior	X	X	X	X	4
Campbell	X	X	X	X	4
Participant B	X	X	X	X	4
Will	X	X	X	X	4
Tyler	X	X	X		3
Gabby		X	X	X	3
Sadie			X	X	2
Luis	X				1
Daniel	X				1
Total RWs	8	7	8	7	30

Note: An "X" denotes submission of the RW assignment available for study data. Shaded cells indicate presence in the class meeting during data collection, and matching colors across columns illustrate the assigned groups for that class session. A shaded cell without an "X" means a participant was present for the class meeting but did not submit their RW assignment.

4.2 Data Collection and Analysis

Students were instructed to remain seated at their group tables for five minutes of individual think-time and then move to a dry-erase board space to share their ideas with their peers and construct a proof. After each class session, students completed a retrospective writing (RW) assignment (Figure 2-2 and 2-3) in which they were tasked with reflecting upon their experience working collaboratively in class and describing the times in which they felt (or did not feel) creative during the activity. After the collection and preliminary analysis of DC1, prompt (II) was modified to inquire about both group creativity and individual creativity

separately, as it was observed in DC1 that participants primarily reflected on their group rather than individual moments of feeling (or not feeling) creative. These assignments were due at 11:59pm the day of the collaborative proving activity to mitigate delayed recall (Gass & Macky, 2000), in which a person creates a new mental process instead of recalling their original ideas (Lyle, 2003).

Figure 2-2

Retrospective Writing Prompt for DC1

Collaborative Proving
Retrospective Writing Prompt

Directions: Respond to each of the prompts in the space below. Type your responses and follow the length requirements specified below.

Reflect upon the collaborative proving task your group worked on during class today.

I. Describe what happened in your group proving today during class in a narrative form as if you were telling the story to a friend. What happened? Who did what? What made you think of certain ideas, and when? What different steps and ideas did your group go through in order to reach a final proof (or not)? Give as much detail as possible. *(At least 450 words)*

II. When during the collaboration did you feel like you or your group were creative? If you did not feel creative, explain why not. Include as many details as you can. *(At least 250 words)*

Figure 2-3

Retrospective Writing Prompt for DC 2, 3, 4

Collaborative Proving
Retrospective Writing Prompt

Directions: Respond to each of the prompts in the space below. Type your responses and follow the length requirements specified below.

Reflect upon the collaborative proving task your group worked on during class today.

I. Describe what happened in your group proving today during class in a narrative form as if you were telling the story to a friend. What happened? Who did what? What made you think of certain ideas, and when? What different steps and ideas did your group go through in order to reach a final proof (or not)? Give as much detail as possible. *(At least 450 words)*

II. When during the collaboration did you feel like your group was creative? If you did not feel like your group was creative, explain why not. Include as many details as you can. *(At least 150 words)*

III. When during the collaboration did you feel (personally) creative? If you did not feel creative, explain why not. Include as many details as you can. *(At least 150 words)*

To determine the actions or moments students perceived as influential on their individual and group creativity during the in-class activity, I narrowed my data source to only include written responses to task (II) for DC1 and tasks (II) and (III) for DC2, DC3, and DC4 of the writing prompts because these prompts included students' descriptions of their collaborative experiences with respect to when and why they felt creative, whereas task (I) only inquired about the general timeline of events. Data analysis followed the steps outlined in the following paragraphs as well as the figure provided in Appendix 2-B.

First, for each sentence of the student responses, I categorized the sentence, henceforth referred to as a quotation, as the student describing feeling creative or not feeling creative. Second, I conducted an inductive, in-vivo coding strategy (Saldaña, 2016). The in-vivo codes

were grouped into an initial 15 themes; however, in alignment with Saldaña's (2013, p. 21) recommendation that, "the final number of major themes or concepts should be held to a minimum to keep the analysis coherent," I further investigated each of the 15 themes and considered how they might be captured in broader categories. The 15 themes were split into 35 fine-grained concepts, and two overarching categories emerged: (1) participant description of behaviors, actions, or moments during the collaborative proving process and (2) participant holistic reflection on the overall circumstances affecting their feeling creative (or not). The 35 fine-grained categories were grouped into seven final themes within two branches (Behaviors, Actions, and/or Moments; Circumstances). Table 2-3 in the Findings section of this study provides the seven themes and their definitions. All quotations were then coded according to these seven themes, allowing quotations to be coded with as many themes as relevant. To present the data within each theme in the findings section, I considered all quotations in each theme and searched for commonalities in the details participants gave describing how they felt or did not feel creative in that theme.

5. Findings

The purpose of this study was to determine how students experience creativity through in-class collaborative proving. To respond to this goal, I first describe an overall summary of the frequency of participant reports of feeling creative or not feeling creative, and then I present and discuss the details of each of the seven themes, first attending to the themes in the Behaviors, Actions, and/or Moments category and then the themes in the Circumstances category.

5.1 Feeling Creative or Not Feeling Creative During Collaborative Proving

In this section, I report the frequencies of participants reporting feeling creative and not feeling creative across RW responses, quotations, and participants. There were 10 participants in

this study, yielding a total of 30 RW responses and 346 quotations. Across the 10 participants throughout the four points of data collection, all 10 participants expressed feeling creative in some way in at least one of their submitted RWs. Similarly, 9 of the 10 participants expressed not feeling creative in at least one of their RW responses at the four points of data collection. In the 30 RWs collected, however, only 19 included reflections of not feeling creative in some way, whereas all 30 included reflections of feeling creative in some way. This means 11 RW responses only included descriptions of feeling creative. Looking across the four points of data collection (DC), quotations describing not feeling creative were more prevalent in DC2 and DC4. Notably there was a higher prevalence of quotations describing not feeling creative in DC4. Table 2-3 summarizes the proportion of creative and not creative quotations in each of the data collection points.

Table 2-3

Proportion of Quotations Feeling/Not Feeling Creative Across Data Collection Points

	Creative	Not Creative
DC1	83.95%	16.05%
DC2	72.63%	27.37%
DC3	86.02%	13.98%
DC4	66.67%	33.33%

Looking across the 10 participants, some participants trended toward more often reporting feeling creative or not creative than others. For example, 100% of Luis's quotations described feeling creative; however, he only submitted one RW (for DC1). Yet, Ashley, who completed all four RWs, also reported feeling creative in 89.39% of her quotations. In contrast to

Luis and Ashley, Campbell had the highest proportion of quotations describing not feeling creative at 45.83% of his quotations.

Table 2-4

Proportion of Quotations Feeling/Not Feeling Creative Across Participants

	Creative		Not Creative	
Participant	**Quotations**	**Proportion**	**Quotations**	**Proportion**
Ashley **Quotations: 66; RWs: 4**	59	89.39%	7	10.61%
Brad Junior **Quotations: 53; RWs: 4**	42	79.25%	11	20.75%
Campbell **Quotations: 44; RWs: 4**	26	54.17%	22	45.83%
Participant B **Quotations: 50; RWs: 4**	40	78.43%	11	21.57%
Will **Quotations: 40; RWs: 4**	27	62.79%	16	37.21%
Tyler **Quotations: 22; RWs: 3**	18	81.82%	4	18.18%
Gabby **Quotations: 30; RWs: 3**	24	75.00%	8	25.00%
Sadie **Quotations: 24; RWs: 2**	23	95.83%	1	4.17%
Luis **Quotations: 13; RWs: 1**	13	100.00%	0	0.00%
Daniel **Quotations: 4; RWs: 1**	3	75.00%	1	25.00%
Total	275	79.48%	81	23.41%

Participant narratives describing when and why they felt creative (or did not feel creative) during the collaborative proving process fit into two broad categories: (1) behaviors, actions, and/or moments, and (2) circumstances. Within each of these categories, there are themes that describe in more detail how students experience creativity through collaborative proving. In the

following sections, I explain these categories and illustrate each theme and its complexities with examples from the data.

Table 2-3

Themes and Theme Definitions

Theme		Definition
Behaviors, Actions, and/or Moments	*Making Connections* (120; 105/17)	Making links between the proving task and available resources or ideas from the current or previous courses. This theme includes drawing upon definitions, proof techniques learned, properties of formal logic, and examples as well as making personally meaningful revelations to better understand a concept during the proving process.
	Taking Risks (77; 69/10)	Engaging in activities that have potential to result in failure, embarrassment, or rejection. These behaviors include pursuing a proving direction without certainty it will result in a successful proof, making and/or noticing mistakes or errors, asking questions, requesting and/or receiving help, and persevering through frustration or an impasse.
	Collaborating and Communicating (76; 66/13)	Efforts to collaborate with teammates throughout the proving task and communicate mathematical thoughts through written or verbal means. This theme captures both description of group social dynamics (e.g., listening to one another) as well as communication during the proving process and of the final proof.
	Creating Ideas (41; 33/8)	Developing ideas and proving strategies for the given task. Verifying the claim or discovering the claim is false. Making conjectures and/or salvaging a false statement.
	How I Felt (39; 21/22)	How the student felt during the proving process: feeling pleased/proud, confused, challenged, confident, or as if they have achieved a deeper understanding.
Circumstances	*The Task* (31; 20/13)	Features of the assigned task and its influence on creativity. This includes familiarity with the task (relating to previous course experience), the perceived ease or difficulty of the task, and whether or not the team completed the given task successfully.
	The Environment (24; 14/13)	The context and environment and their effect on creativity. This category includes student preparedness for class, the amount of class time provided for the proving activity, and the influence of whole-class interactions.

Note: The values in parentheses indicate the total number of quotations identified within that theme across the RWs, then the number of quotations in that theme co-occurring with feeling creative and not feeling creative. For example, in *Making Connections*, (120; 105/17) communicates there were 120 quotations in the Making Connections theme, and 105 co-occurred with feeling creative and 17 co-occurred with not feeling creative.

5.2 Behaviors, Actions, and/or Moments Reported

The overarching category of *Behaviors, Actions, and/or Moments* encompasses the descriptions of specific events that participants reported as having sparked creativity (or diminished creativity) during their collaborative proving. Throughout data analysis, several of the fine-grained categories resembled subcategories of the Creativity-In-Progress Reflection on Proving (CPR; Savić et al., 2014; 2017; Tang et al., 2015). The CPR provides an initial set of ways in which researchers and practitioners can expect students to experience creativity during individual proving. The data provide evidence that many of the existing subcategories of the CPR are relevant to student experiences in creative collaborative proving. Hence, two of the themes evident through my data analysis have identical titles and similar definitions to the categories of the CPR: **Making Connections** and **Taking Risks**. The data also suggest, however, different and/or additional ways in which students experience creativity when proving collaboratively; namely, by **Collaborating and Communicating** and **Creating Ideas**. The following sections provide detailed descriptions of each theme and the ways in which participants felt creative and did not feel creative through these behaviors, actions, and moments. Because the Making Connections and Taking Risks themes have a detailed and rich body of literature from research on individual mathematical creativity (Bicer et al., 2023; Boaler, 2016; Sternberg, 2017; Wan et al., 2021) which these data supported as relevant to collaborative proving; I provide a more comprehensive overview of the "Collaborating and Communicating" and "Creating Ideas" themes.

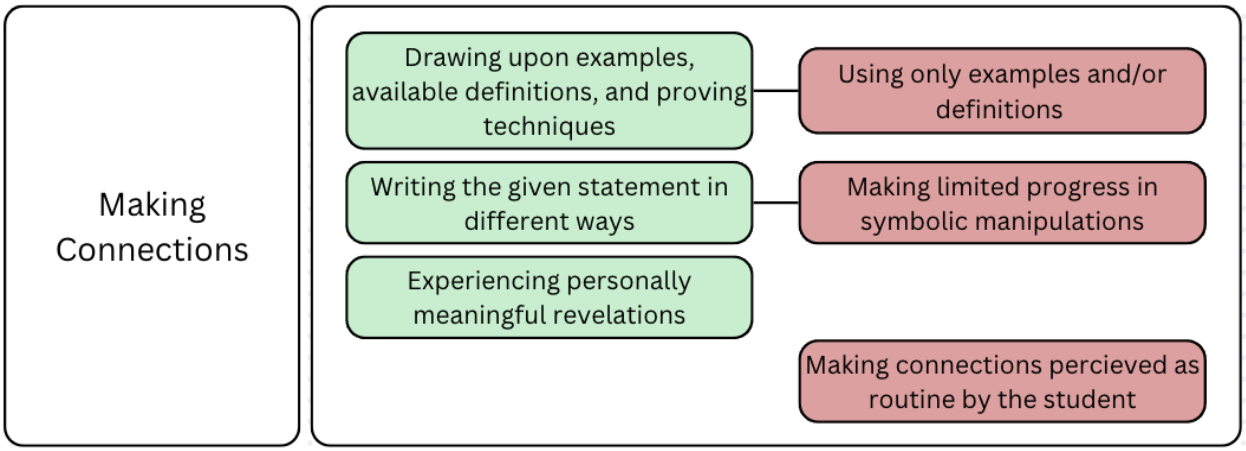
5.2.1 Making Connections

The ***Making Connections*** theme encompassed participant reflections upon how making links between the proving task and available resources or ideas influenced their creativity. This

theme includes descriptions of drawing upon definitions, proof techniques learned, properties of formal logic, and examples as well as making personally meaningful revelations to better understand a concept during the proving process. Making Connections was the most prevalent theme across all of the seven thematic codes, with 120 of the 346 total quotations (34.6%) being labeled as Making Connections and 27 out of the 30 RWs (90%) containing quotations labeled with the Making Connections code. Of these 120 quotations associated with making connections, 105 (87.5%) were in concert with feeling creative, and 17 (14.2%) were in concert with not feeling creative. Figure 2-4 provides a summary of the ways in which participants felt creative and did not feel creative through making connections in their collaborative proving.

Figure 2-4

Making Connections Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, and black lines connecting boxes illustrate matched pair observations.

5.2.1.1 Making Connections: Feeling Creative. Participants described feeling creative through making connections in their collaborative proving when they drew upon examples, available definitions, and proving techniques learned in the course, wrote the given statement in

different ways, and experienced personally meaningful revelations. In this section I provide several examples from participant writings to highlight these different observations. In the quotations provided in this section and elsewhere throughout the findings section, portions of a quotation highlighted in **bold** text are intended to bring attention to the content of the quotation relevant to the theme being discussed.

First, participants expressed feeling creative by making connections to examples, the definitions available and relevant to their given statement, and the proving techniques they learned throughout the course. Regarding example use, Will (DC1) wrote,

> **It was only after computing a few of the twin primes that we recognized this phenomenon, and it helped illustrate our formulas better,** especially the first formula with the single variable n, as the middle value would be understood to be n+1.

Similarly, in DC4, Participant B reflected upon making connections in their proving via examples,

> Me and Luis both **had a variety of ways we created examples in order for us to understand the definition of a power set**, as it was probably the biggest issue that we had with the problem, and it was the biggest issue with problem set in general.

These two quotations illustrate how drawing upon relevant examples allowed participants to feel creative by making connections to other representations ("illustrate our formulas better" (Will, DC1)) and better understand the mathematical content of the given statement ("understand the definition of a power set" (Participant B, DC4).

Participants also highlighted how they felt creative during their collaborative proving when they made connections to relevant definitions or specific proving techniques. Tyler (DC2) wrote,

> But it wasn't until I stopped when the thought of "these are still considered integers, **since the definition of an integer is any whole number ranging from negative infinity to positive infinity**," so I used them in my explanation on my part for proving.

Brad Junior (DC3) and Sadie (DC4) also echoed the sentiment that they felt creative when they made connections to definitions relevant to their statements:

> Proving a conjecture **by using other definitions to rewrite what we were doing** was a very creative and useful idea. (Brad Junior, DC3)

> The reason I felt creative from this was that **it made me think about all the definitions needed to solve this claim.** (Sadie, DC4)

Across Tyler (DC2), Brad Junior (DC3), and Sadie's (DC4) quotations, it is clear that their use of relevant definitions of concepts related to their given statements was instrumental in their feeling creative during the collaborative proving activities. Similarly, Participant B reflected upon how connecting their given statement to proving techniques learned in the course made him feel creative, "At first, I felt the most creative when I realized we could **try to prove the statement via proof by contradiction.**"

The second observation in the ways participants recalled feeling creative through making connections was in the ways they represented their given statements. For example, Will (DC2) wrote about how he and his teammate, Campbell, worked creatively in trying to express their given statement as a mathematical expression using variables, "Campbell made sure we had the claim written, and afterward, while working with the intricacies of the biconditional statement, **we gathered multiple ways to express a variable**." In a similar way, Participant B (DC3) also wrote about his experience realizing a subset relationship could be re-written by claiming an element x was an element of each set,

> However, perhaps the most creative all of us felt in this group was near the end of class, where **we all could have written each set in the statement in the form of writing it as x being an element in each set.**

These reflections from Will (DC2) and Participant B (DC3) show how participants perceived themselves and their groups as creative when they sought out alternative ways to represent the ideas within their statements.

The final observation among participant reflections of making connections and feeling creative was how participants felt creative when they experienced personally meaningful revelations during their collaborative proving. Consider the following quotations from Participant B (DC2), Brad Junior (DC3), and Gabby (DC4).

> **I realized** that the only way for integers to change into something that is not integers is if we divide integers together. (Participant B, DC2)
>
> **I was able to recognize the importance** and the why behind our what made it possible to prove the conjecture in the way we did. (Brad Junior, DC3)
>
> When we were happy with our problem we had asked for feedback and **that's when we had realized** that the step we had described as the distributive property of statements was not actually distributing statements and we had not proven that the step was true. (Gabby, DC4)

In each of these quotations the participants describe how they had a personally meaningful revelation ("I realized," "recognize the importance," and "we had realized"). In these three moments described, the participants made some kind of connection between their understanding of the mathematical constructs and the logic underlying their mathematical arguments. There is precedent for these revelations to be subsumed under the making connections category as this observation captures what Kaufman and Beghetto (2009) called mini-c creativity, which is, "novel and personally meaningful interpretation of experiences, actions, and events." These moments were considered a part of the *Making Connections* theme because participants were making connections between *ideas,* or having personally meaningful moments of learning; however, critics might argue these reflections should rather be captured by the *Creating Ideas* theme because they resulted in the creation of understanding or learning for the student.

5.2.1.2 Making Connections: Not Feeling Creative. Seventeen (17) quotations contained reflections in which participants expressed making connections and not feeling creative. Within these instances, participants described feeling a lack of creativity because their approaches to the proof were limited to only using examples and/or definitions, they did not make enough progress in symbolic manipulations, or they perceived the connection they made as routine for them. To illustrate how participants said they were not creative by restricted ways in which they made connections, Campbell (DC3) stated, "I also think we could have been more creative, but most of the time **we were sticking to the definitions and using examples**." Similarly, Daniel (DC1) described,

> I would say we were more on the side of methodical rather than creative. First, we broke down the problem by **looking at the definitions** for prime numbers, twin primes, and perfect squares. Then **we took each prime** that was adjacent to each other and multiplied them together to see if they were one less than a perfect square.

Here, it is evident the participants felt a lack of creativity when they restricted their behaviors and actions to drawing upon definitions and examples and did not make deeper connections between the examples and definitions to the structure and meaning of the proof.

Participants also expressed not feeling creative because they felt their representations and/or symbolic manipulation were deficient during proving. For example, Gabby wrote (DC3),

> I was unsure how to make the problem something I can prove with numbers or anything like that. **I had only written down actual words instead of the other form of sets** because I was unsure how to apply them to my proof.

Similarly, Campbell reflected (DC1),

> I felt that we could have been more creative though, because at the end of the class it was shown to us that **we could have gone further with our first equation**. [..] Looking back, I also can see how I could have been more creative on my own and **factoring the expression even further** and **even using another example** to show that the equation had worked to support the proof.

In both Gabby and Campbell's reflections, they described feeling as though their efforts in using representations (Gabby) and symbolically manipulating expressions and equations (Campbell) were deficient, causing them to feel as though they were not creative.

Finally, participants did not feel creative when making connections because they perceived the connection they made to a definition, theorem, or property as routine or formulaic. Campbell (DC4) was the primary source of this sentiment. He stated,

> Personally, I did not feel too creative when creating this proof. I knew exactly where we needed to start and finish, but had the most difficulty with the introduction of the power sets. I think that since **I knew that we had to use the distributive property** I was also fairly confident in what processes that I needed to use, such as **breaking down the set operators and sets into their logical operators** and then **use the distributive property** to prove each of the two logical statements.

Although Campbell explains that he made connections to and utilized different representations (sets and logical operators) and a property previously learned (left distributive property of implication over conjunction), he expressed feeling as though this was not creative. His language is nuanced here, and he suggested these connections were routine for him (e.g., "I knew…" and "I was also fairly confident…").

5.2.1.3 Summary and Discussion: Making Connections. The ***Making Connections*** theme was defined by descriptions of making links between the proving task and available resources or ideas and how these connections influenced student creativity. Participants recognized themselves as creative when they drew upon available definitions or proving techniques, writing the given statement in different ways, and experiencing personally meaningful revelations during the collaborative proving. Students who described not feeling creative described making connections differently. There was typically an attitude that the connections made were either deficient or routine for them. Participants still mentioned making connections through multiple relevant examples or definitions, but they did not consult other

potential resources or push themselves to build upon these connections in a meaningful way to advance their proof. Similarly, participants also did not recognize their making connections as creative in cases where they viewed that connection as routine. For example, Campbell (DC4) wrote about how his connection to the distributive property of implication over conjunction to his given statement did not feel creative because he had used this property previously in proving efforts.

The findings from the ***Making Connections*** theme support that the Making Connections category of the Creativity-In-Progress Rubric for proving (CPR; Savić et al., 2017) are relevant to creativity in collaborative proving as well as individual proving. The data collected in this study indicate that participants felt creative when they made connections to definitions, theorems/properties, representations and symbolic manipulation, and examples. These types of connections are mirrored in the CPR's subcategories of *Definitions/Theorems, Representations*, and *Examples*. The one type of connection participants noted that is not apparent in the CPR's Making Connection category is making connections to proving strategies (e.g., proof by contradiction). Although this could be subsumed under the *flexibility* subcategory, which describes how students might use different proof techniques, the data from this study indicate students may also be making connections to the logical underpinnings of a statement to the different proof techniques they have learned in the course.

The data from participants discussing making connections and not feeling creative also support the CPR's different levels of creativity. Specifically, the CPR describes students as "beginning" in their creativity when they may draw upon relevant examples or definitions, but they do not attempt to make connections or build upon these connections to advance their proving. Thus, student reflections that they were not creative because they "only" used examples

or definitions or “did not go far enough” with their symbolic representations mirror the CPR’s description of individual creativity in proving.

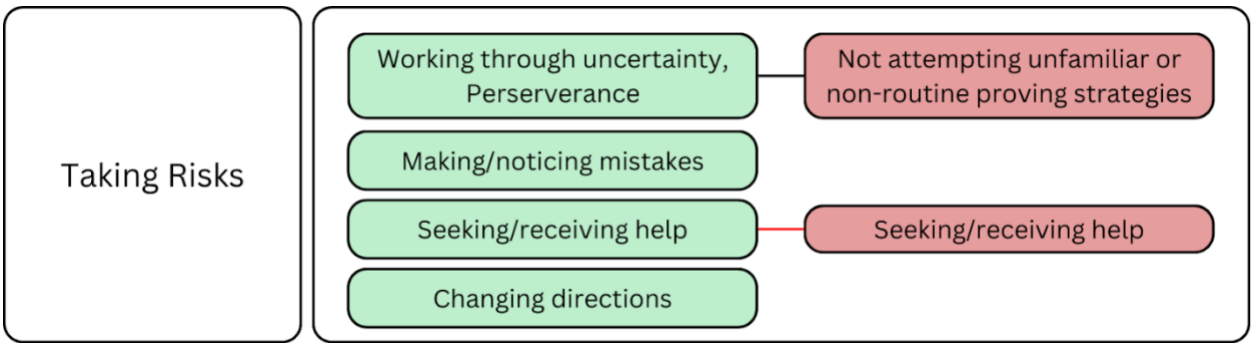
Altogether, the ***Making Connections*** theme suggests students feel creative in their collaborative proving activities when they are able to make meaningful progress in their proving by drawing upon examples, definitions, or properties, manipulating mathematical representations and expressions, and connecting their given statement to learned proving techniques. Findings from this theme also suggest students do not feel creative in collaborative proving when they limit their proving approaches to only connecting a given statement to examples and definitions, do not follow through with symbolic manipulations and representations as far as they could have, and perceive the connections they make in their proving (to resources, definitions, previous knowledge) as routine.

5.2.2 Taking Risks

The ***Taking Risks*** theme encompassed participant reflections upon how engaging in activities that have potential to result in failure, embarrassment, or rejection influenced their feeling or not feeling creative. These behaviors include pursuing a proving direction without certainty it will result in a successful proof, making and/or noticing mistakes or errors, changing directions during the proving process, asking questions, requesting and/or receiving help, and persevering through frustration or an impasse. Seventy-seven of the 346 total quotations (22.3%) were labeled as Taking Risks and 22 out of the 30 RWs (73.3%) contained quotations labeled with the Taking Risks code. Of these 77 quotations associated with taking risks, 69 (89.6%) were in concert with feeling creative, and 10 (13.0%) were in concert with not feeling creative. Figure 2-5 provides a summary of the ways in which participants felt creative and did not feel creative through taking risks in their collaborative proving.

Figure 2-5

Taking Risks Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, black lines connecting boxes illustrate matched pair observations, and red lines connecting boxes illustrate opposing observations.

5.2.2.1 Taking Risks: Feeling Creative. Participants frequently discussed taking risks by working through uncertainty, making mistakes, changing directions, or being open to help during their proving. To better understand the complexity of how participants felt creative through risk taking, consider Ashley's reflection on her work with Brad Junior and Participant B in DC3 as they attempted to prove (or disprove and salvage) the statement, *if A is a subset of the universal set U, then the empty set is a subset of A.* The final work of this group is provided in Figure 2-1.

> **All of us had trouble fully grasping how to go about solving this problem.** In a way this made us more creative. During the process of proving our claim **we worked through many ideas** that all of us were having. In our **lack of understanding**, we were more accepting of listening to each other. **Many of our ideas didn't lead anywhere**, but they were important for our creative process. I feel we were the most creative because of our willingness to brainstorm ideas with each other. We all threw out ideas **even if we weren't sure they were right**. I think this was important because **we were ok with not being right,** so long as we were talking and **trying to figure the problem out**. I'm glad we tried really hard to figure it out. **We ended up needing help** to finally solve the problem but **because of our efforts** I think we learned a lot. [...]
>
> In the beginning of class **I was still having trouble completely understanding** sets and everything that went with them. I have done sets before in stats and such, but **I am not used to the language or symbols that we have been using**. I think it's going to take

> some time for me to get comfortable with all of the symbols, their meanings, and connotations. Besides this I think I was creative because **I tried to solve the proof and work through my confusions even though I didn't know everything yet**. I think this approach allowed me to be creative and still accept that I may be lacking some understanding. Since **I was ok with not knowing everything**, I was open to **asking my group for help**. This was good because they helped me understand things while cementing their understanding in the process. I think because **I was open to asking questions and listening to others** I was creative in my learning process today.

In Ashley's reflection she described feeling creative by taking risks in a variety of ways. She articulated that she felt creative because she attempted the problem and *put in effort despite feeling confused* about the concepts ("had trouble fully grasping…," "lack of understanding," "even though I didn't know everything yet," "I was ok with not knowing everything"). Ashley also demonstrated *perseverance* during the collaborative proving ("trying to figure the problem out," "because of our efforts," "I tried to solve the proof and work through my confusions"). Another way Ashley recognized feeling creative through the collaborative proving was her and her groupmate's *willingness to risk time, energy, and failure* and *try a variety of ways* to approach the proof ("our ideas didn't lead anywhere," "we weren't sure they were right," "we were ok with not being right"). Finally, Ashley perceived *seeking and receiving help,* which is a social risk, as promoting her creativity during the activity ("we ended up needing help," "asking my group for help," "open to asking questions").

In addition to the ways Ashley experienced creativity through risk taking, participants also described feeling creative when making (and discovering) errors and mistakes in their work. In DC4 Gabby and Brad Junior worked together to disprove and salvage $\wp(A \cup B) = \wp(A) \cup \wp(B)$. The pair was able to salvage the claim to $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$ and create a proof, but the students had a small logical jump that needed further explanation in their proof. Both Gabby and Brad Junior described receiving help and feedback as informing their creativity.

Gabby reflected upon this moment:

> I also felt our group was creative in recognizing object types when we were writing out our explanations of why we were able to **make the logic jumps** we did for our salvaged claim. When we were happy with our problem we had asked for feedback and that's when we had realized that the step we had described as the distributive property of statements **was not actually distributing statements** and **we had not proven that the step was true**.

Here, Gabby reflected upon how the process of resolving an error, a logical "jump" in their proof, allowed her team to be creative and recognize the importance of distinguishing object types in their proof. Regarding this moment, Brad Junior also wrote,

> I personally felt creative when I worked on the actual proof. That is, I was the one writing down each step that was needed under the definition of subset to show that our salvage was true. **Gabby was a lot of help in reassuring me** and making sure I was correct in what I was writing on the board. The moment I felt the most creative was at the end of class when **me and Dr. Hadley discussed** the importance of the lines that **filled in the gap** above. Specifically, when **Dr. Hadley was discussing** the idea that these lines were quite possibly what the proof hinged upon.

In his reflection, Brad Junior not only mentioned receiving help from his instructor, Dr. Hadley, to fill in the logical gap he did not previously notice as helping him feel creative, but he also claimed the feedback and support his teammate gave him while writing the proof contributed to him feeling creative. Gabby's acknowledgement of how an error and recovering that error facilitated her creativity and Brad Junior's credit to receiving help and support from others to his creativity illustrate how both participants' creativity was fostered through experiences that had potential to result in embarrassment and rejection.

5.2.2.2 Taking Risks: Feeling Not Creative. When participants referenced taking risks alongside not feeling creative, these quotations included reflections upon not taking enough mathematical risks as well as requiring help to move forward. Participants primarily discussed not feeling creative with respect to risk taking because they did not challenge themselves to think of and use approaches to the proof less familiar to them. In DC2, Brad Junior described, "I think

that for me to have been creative I would have needed to **think of a different method other than one based on my preconceived notions.**" Brad Junior's quotation illustrates how participants may not feel creative because they did not push themselves to take risks and go beyond their initial strategies in their proving.

A few participants also described feeling a lack of creativity because their group required help throughout the proving process. Campbell (DC2) noted,

> I also think we could have been more creative [...] **we were given a lot of help with what steps that we needed to take**, such as knowing to break each of the statements down into their definitions from the start.

Similarly, Will (DC4) expressed,

> During the collaboration, our group was not really creative as most of us were perplexed by the concept of power sets. [...] **With the help of Dr. Hadley**, she managed to rescue us from our lack of understanding and guided us toward a method to prove the conjecture.

Both Campbell (DC2) and Will (DC4) expressed not feeling creative during their groups' proving. Both men also described how their groups required receiving help from Dr. Hadley, a social risk, to find an initial route to proving their statements.

5.2.2.3 Summary and Discussion: Taking Risks. The second theme within the Behaviors, Actions, and/or Moments category was ***Taking Risks***, which was defined by engaging in activities that have potential to result in failure, embarrassment, or rejection. These activities included pursuing a proving direction without certainty it will result in a successful proof, making and/or noticing mistakes or errors, changing directions during the proving process, asking questions, requesting and/or receiving help, and persevering through frustration or an impasse. Participants felt creative when taking risks when they persevered through discomfort or confusion, pursued proving attempts without certainty it would be correct, and sought out help when needed. On the other hand, participants related not feeling creative with a *lack* of

attempting unfamiliar or non-routine proving strategies. This supports the notion that risk taking regarding mathematical methods may promote feeling creative during collaborative proving. In contrast to those participants who felt creative when they received help, some participants also felt receiving help meant they were less creative. The evidence that some students, however, may perceive seeking and receiving help differently with respect to feeling creative indicates that social risk taking may be more nuanced and complex.

When comparing the ***Taking Risks*** theme to extant literature from individual creativity (in mathematics and beyond), there is both evidence and theory that risk taking and uncertainty can be catalysts for creativity (Karakok et al. 2015; Savić et al., 2017; Sriraman, 2022; Tang et al. 2015; Wan et al. 2021). Specifically, regarding creativity in mathematical proving, Savić et al. (2017) described the Taking Risks category of the CPR according to five subcategories: tools and tricks, flexibility, perseverance, posing questions, and evaluation of the proof attempt. The findings from participant reflections in this study support that students may similarly experience these ways of taking risks as being creative in their collaborative proving. For example, the participants in this study also recognized how perseverance and flexibility positively related to their feeling creative. Moreover, participants reported that a lack of using tools or tricks and a lack of flexibility made them less creative in their collaborative proving. As expected, the emergence of social risk taking through seeking help and receiving help or feedback are not clearly evident in the literature regarding individual mathematical creativity. There is, however, some similarity in the ways participants discussed seeking and receiving help or feedback to the CPR's "posing questions" and "evaluation of the proof attempt" subcategories, as these categories describe the ways in which a student may question a detail or logical structure of their proof and or/ reflect upon the success or lack of success of their proving attempt. Because

participants in the present study both experienced receiving help as aiding and hindering their creativity, the collaborative setting and the nature of taking social risks when working creatively require a more detailed investigation.

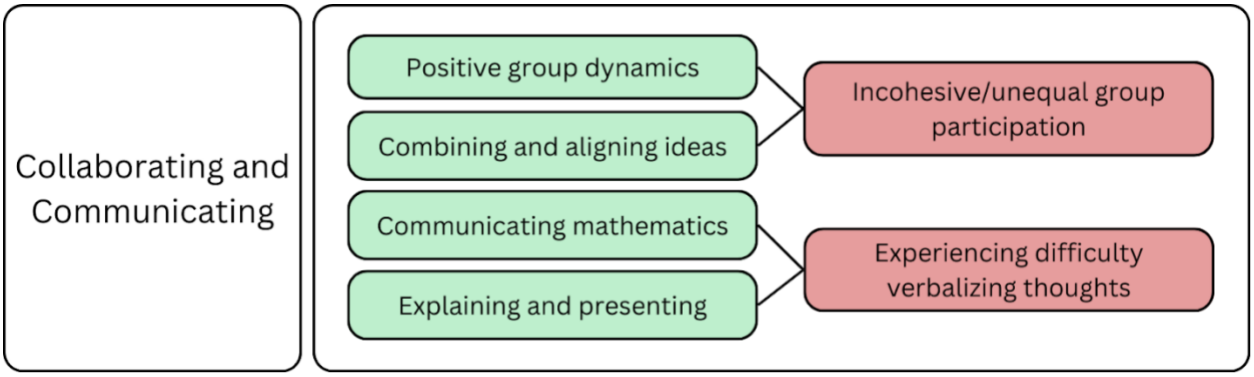
In summary the ***Taking Risks*** category is similar in many ways to existing descriptions of mathematical creativity at the individual level; however, the social setting of collaboration and the context of a real classroom with an instructor present may shift students' perceptions of how seeking and receiving help from peers and an instructor influence creativity in collaborative proving. Future research may investigate the specific conditions in which providing help and guidance will foster or impede creativity. Moreover, there is room for future investigation regarding if the social context of collaborative proving promotes student mathematical risk taking.

5.2.3 Collaborating and Communicating

The ***Collaborating and Communicating*** theme encompassed participant reflections on their efforts to collaborate with teammates throughout the proving task and communicate mathematical thoughts through written or verbal means. This theme captures both description of group social dynamics (e.g., listening to one another) as well as communication during the proving process and of the final proof. Seventy-six (76) of the 346 total quotations (22.0%) were labeled as Collaborating and Communicating and 24 out of the 30 RWs (80%) contained quotations labeled with the Collaborating and Communicating code. Of these 76 quotations associated with collaborating and communicating, 66 (86.8%) were in concert with feeling creative, and 13 (17.1%) were in concert with not feeling creative. Figure 2-6 provides a summary of the ways in which participants felt creative and did not feel creative through collaborating and communicating in their collaborative proving.

Figure 2-6

Collaborating and Communicating Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, and black lines connecting boxes illustrate matched pair observations.

5.2.3.1 Collaborating and Communicating: Feeling Creative. Participants expressed feeling creative during collaborative proving in the theme of Collaborating and Communicating because of their group dynamics, combining and aligning ideas in their groups, communicating mathematics, and explaining and presenting ideas. In the following paragraphs, I describe each of these observations in more detail and provide supporting examples from participant data.

Group Dynamics. With respect to group dynamics, participants attributed "working well together," listening to one another, and having trust and patience with their group mates to feeling creative. Ashley (DC1) emphasized the importance of working well with her teammates:

> I think my group and I were creative in our collaboration. I was very pleasantly surprised **how well my group and I worked together**. I think a vital part of collaboration is the group's overall **willingness to actively collaborate**. In this way **uncooperative group members stifle the creativity of the group**. I really appreciated that **no one in my group ignored the ideas of others**. This can be hard when everyone has individual ideas, and everyone wants to share theirs. In this case it is imperative that **the group allows everyone to get their points across** and then everyone can think through a variety of ideas and options.

In this reflection, Ashley attributed her groups' creativity to working well together as a group, including their willingness to actively collaborate and listen to one another's ideas ("no

one [...] ignored the ideas of others," "the group allows everyone to get their points across"). Ashley's writing also describes a group dynamic in which each teammate has a sense of contribution to the work, rather than following the ideas from one primary group member.

Other participants commented on trusting their teammates to correct them and appreciating the patience of their teammates. Brad Junior (DC4) described trusting his teammate, Gabby, in their work together:

> As a group I felt we were really creative. [...] **If it were not for Gabby** being in the group it would have likely meant that **I**, being the only other group member, **would not have realized the conjecture was false.** Meaning **I would have likely run in circles or wasted my time altogether** trying to figure out what I did wrong or how to even start.[...] **Gabby was a lot of help in reassuring me and making sure I was correct** in what I was writing on the board.

In this reflection, Brad Junior expressed his reliance upon (and trust in) his teammate, Gabby, to notice any mistakes ("I [...] would not have realized the conjecture was false") and affirm him in his proof writing ("Gabby was a lot of help in reassuring me…").

Finally, Ashley (DC4) noted the importance of groupmates having patience with one another and working to help everyone understand:

> The other member of our group was able to really help us better understand how to correctly write our ideas down and **he wasn't frustrated that we needed a bit more help understanding than he did.** Overall, I am very pleased with my group and I's creativity today.

Ashley recognized that her teammate's patience ("he wasn't frustrated…") with helping the other members of his team contributed to positive group dynamics and her feeling creative.

In summary, participants recognized that positive group dynamics contributed to their feeling creative throughout collaborative proving. These positive group dynamics included having teammates who were willing to actively collaborate and cooperate with their teams, listening to one another and making space for every team member to share their ideas, relying on

one another to correct and critique the group's work, and demonstrating patience with group members who needed help.

Combining and Aligning Ideas. Participants also described feeling creative through combining and aligning their ideas with their group mates. Within this thread of Collaborating and Communicating participants described putting individual ideas together within their teams to create a cohesive proof. For example, Ashley (DC2) said, "We all brought our own ideas into the mix and tried to solve the proof in a couple of different ways," Brad Junior (DC3) noted, "We each thought of different ways to approach the problem in the beginning, however, we were also able to align in our proving style in the end," and Sadie (DC4) described, "Once we were reaching the end of the class everyone was helping prove and putting our input in." These examples highlight how participants experienced creative collaborative proving when they built upon ideas each individual teammate had for the proof to create a proof the entire group could have ownership over.

Communicating Mathematics. Within this thread of Collaborating and Communicating, participants discussed how communicating mathematics (and about mathematics) in written and verbal forms related to their creativity during their collaborative proving. Several participants noted how they felt creative when writing their proofs and mathematical work on their dry-erase boards. Brad Junior (DC4) stated, "I personally felt creative when I worked on the actual proof. That is, **I was the one writing down each step** that was needed under the definition of subset to show that our salvage was true," and Daniel (DC1) mentioned, "[My teammate]'s idea to **write out his equation in words** made me feel like this was a great path to being more creative."

Explaining and Presenting. Several participants described feeling creative when they, or their groups, explained a concept or contributed to presenting their proofs to the entire class. Sadie (DC3) attributed her team's creativity to her groupmates' ability to explain their ideas:

> I always believe that my group is creative while working. **They are always really good at explaining how they did their proof or ideas that they have for proving something.** I always have a lot of questions when it comes to proving something, only because I want to completely understand everything that they are saying. I do not want to leave class and not understand something, so **my groups have done a really good job at helping me and others to understand everything** that is talked about in class. I believe that they did an amazing job on their proof of problem 4.17 this morning and **they had it laid out very nicely for the class to understand.**

Sadie recognized how her team's explaining their ideas and helping her understand the problem contributed to their creativity ("They are always really good at explaining…", "helping me and others to understand everything"). Moreover, Sadie noted how the presentation of her team's proof, as nicely laid out "for the class to understand" made her feel like her team had been creative. Sadie's teammate during this proving exercise, Will (DC3), also shared the sentiment that being able to deliver a proof to the whole class made him feel creative, "After finding a contradiction, **I was able to present it to the class, which made me feel very creative** even though I had the help of Dr. Hadley and my group members." Similarly, Brad Junior (DC3) described feeling creative because of his team's ability to present their proof well:

> Another moment within our collaboration **was in the presentation**, while I got voluntold, **Ashley and Participant B still jumped in to help** expound what they were trying to do and what they felt important to make extra clear. **This ability to have a cohesive presentation** really shows the creativity of our group."

Altogether, participants recognized explaining their thinking and then presenting their work to either their teammates or the whole class made them feel creative.

5.2.3.2 Collaborating and Communicating: Not Feeling Creative. Participants primarily discussed not feeling creative with respect to collaborating and communicating through

incohesive/unequal group participation or difficulty verbalizing thoughts. These two observations are expanded upon in the following paragraphs.

Incohesive/Unequal Group Participation. To exemplify how participants felt they were not creative through collaborating and communicating, consider the work of Ashley, Brad Junior, and Participant B in DC2. During this exercise, the group was tasked with proving *if p is an integer, then* $p^2 + 3p + 2$ *is even.* Brad Junior had one initial idea to approach this proof by graphing the expression as a function and taking limits of the function as *p* approaches infinity. The three group members participated in Brad Junior's approach at first, but eventually Ashley diverged from this approach and worked with Participant B to attempt to prove the statement by incorporating the definitions of even and odd numbers. Although Ashley expressed her appreciation for Participant B working with her on her ideas, "When me and Participant B started working through other ideas, I was very happy that he wanted to understand my thought process, and Participant B was willing to try some of my ideas out" (Ashley, DC2), both Participant B and Brad Junior reported not feeling creative because they perceived their contributions as minimal in the collaboration.

Participant B stated, "**I felt as though I was merely a spectator** while Ashley and Brad Junior did most of the work needed to help with the final proof. I basically plugged in the definitions into the problems" (Participant B, DC2). Although Ashley perceived Participant B's support and attention in pursuing her approach as important to the group's creativity, Participant B did not feel like his contribution was creative. Similarly, during Brad Junior's reflection on this episode, he stated he would have been more creative if he had been more involved in the approach Ashley and Participant B pursued after diverging from his original approach, "I think

that for me to have felt creative I would have **needed to involve myself in the work of my groupmates**."

In a separate episode (DC4), Participant B echoed the sentiment that he was not creative while collaborating with Luis on the task *Prove or Disprove and Salvage:* $\wp(A \cap B) = \wp(A) \cap \wp(B)$. In his retrospective writing response, Participant B wrote:

> I personally did not feel as though I had as much creative impact on the collaborative activity than Luis did. I felt as though **the main work that we did for the collaborative activity was with Luis and not me**. To be quite honest, I felt as though this is mostly my fault. [...] To be honest, **I commend Luis for working on this mostly by himself**, because I felt as though he was just as confused about this as I was.

Participant B described that he did not feel creative during the collaborative proving because he did not contribute as much to the proof as his teammate, Luis, did.

Difficulty Verbalizing Thoughts. The second way in which participants felt they were not creative with respect to collaborating and communicating was experiencing difficulty expressing their thoughts and mathematics in words. During DC2, Campbell described this sentiment:

> I did think that we could be more creative though, but most of the time we were stumped. While we knew that the logical statement was true, **we had a lot of difficulty putting our thoughts into words**. I think that **if we were better able to express ourselves**, I would have been more creative. I think that being creative with manipulating the variables did help though, **my only problem was just putting my thoughts into words.**

In this reflection, Campbell described how the struggle he and his teammates encountered in "expressing" themselves inhibited creativity for his team. Although Campbell expressed feeling creative in other ways ("manipulating variables"), his perceived ability to work creatively was blocked by trouble "putting [his] thoughts into words."

During DC3, Brad Junior offered a similar sentiment. Brad Junior recognized an error in his team's proof, but struggled to come up with the phrase *vacuously true* to describe what was actually happening in the logic of their statement:

> I personally felt creative when I was able to recognize, by gut feeling, that our supposed disproof was wrong. I however, did not feel creative in turn becaus**e I was not able to articulate or point towards what I believed was our mistake** in the disproof.

Just as Campbell (DC2) discussed, Brad Junior expressed that his inability to articulate his point inhibited his creativity.

5.2.3.3 Summary and Discussion: Collaborating and Communicating. Across the ***Collaborating and Communicating*** theme of student responses to when and why they felt creative during their collaborative proving, participants reflected on their efforts to collaborate with teammates throughout the proving task and communicate mathematical thoughts through written or verbal means. When students discussed feeling creative, they attributed, in part, their own and their team's creativity to positive group dynamics, efforts and ability to combine and align ideas across the individual members of their group, communicating mathematics, and explaining and/or presenting their proof or concepts within their proof to their group or the class. When students discussed not feeling creative, or space to have been more creative, they cited incohesive or unequal group participation and difficulty verbalizing their thoughts. The ways in which students described feeling or not feeling creative through collaborating and communicating mirror one another. Whereas students felt creative through positive group dynamics and combining and aligning their ideas as individuals, they also felt less creative due to incoherent or unequal participation in their groups. Similarly, students felt creative when they were able to communicate mathematics and/or explain and present their proofs, and students felt uncreative because of an inability to verbally express their thoughts.

In contrast to the ***Making Connections*** and ***Taking Risks*** themes, there is no corresponding category of the CPR on Proving (Savić et al., 2017) for the ***Collaborating and Communicating*** theme. There are, however, similar findings from extant research concerning collaborative creativity in other mathematical and non-mathematical contexts. Domain-general contexts have primarily explored the influence of group dynamics on collaborative creativity. Kurtzberg and Amabile (2001) summarized literature across domains to describe how cognitive diversity and conflict may influence group creativity. Diversity of perspectives was a factor mentioned by a few participants in this study in the thread of Combining and Aligning Ideas. For example, Brad Junior wrote, "We each thought of **different ways to approach the problem in the beginning,** however, we were also able to align in our proving style in the end." With respect to conflict, the findings from this study support the notion that relationship-based conflict might hinder creativity and having a sense of cooperation promotes collaborative creativity (Kurtzberg & Amabile, 2001), yet differences among group members' approaches for the problem at hand (task-based conflict) can enhance group creativity. In this way, participants' reflections that positive group dynamics (an absence of relationship-based conflict) and combining/aligning different perspectives promoted creativity in collaborative proving are consistent with the literature regarding domain-general collaborative creativity.

In mathematics, after observing sixth-grade students engage in a mathematical problem solving task, Aljarrah (2020) described four collective creative acts. Aljarrah's collective creative act "Summing Forces" encompasses many of the behaviors, actions, and moments described by participants in the present study regarding collaborating in the creative collaborative proving process. In particular, summing forces is similar to how participants described combining and aligning their ideas within their group. Aljarrah (2020) described

summing forces as describing when "the sum of forces is greater in magnitude and applied in a different direction from any individual force; the whole, however is not greater than the sum of the parts" (p. 3).

Although Kurtzberg and Amabile (2001) and Aljarrah's (2020) work relate the *collaborating* component of Collaborating and Communicating to related contexts in mathematical and domain-general creativity, *communicating* in both individual and collaborative proving requires a different perspective. One of the primary roles of proof (De Villiers, 1990) is proof as communication. When a student writes a proof individually, the student is communicating with some hypothetical reader. This hypothetical reader may be a course instructor, a peer, some ambiguous audience, or the student themselves. Sfard's (2007) commognition framework helps relate the *communication* component of this theme to existing mathematics education research. Commognition is, "the focal notion of the approach to learning grounded in the assumption that thinking can be usefully conceptualized as one's communication with oneself" (Sfard, 2020). Thus, as students individually construct and write a proof for a statement, they are engaging in commognition. However, when students engage in collaborative proving, the notion of commognition evolves into communication between peers and classmates. In this study, participants' reflections that experiencing difficulty verbalizing one's thoughts in collaborative proving hindered creativity and feeling an ability to communicate, explain, and present a proof enhanced creativity highlights how commognition might be revealed when students return to their individual proving outside of this collaboration-centric classroom.

In summary, the ***Collaborating and Communicating*** theme revealed in this study highlights how students engaging in collaborative proving may feel creative due to positive group dynamics, aligning their ideas with their teammates, and effectively communicating their

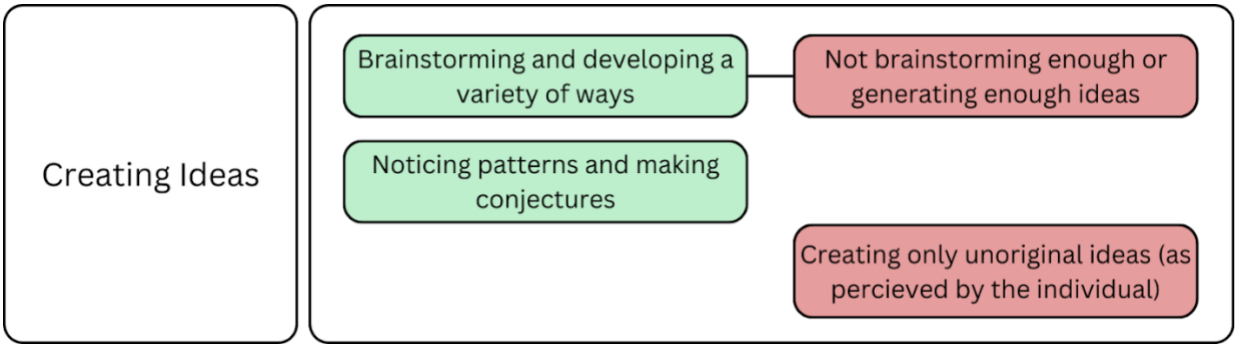
proofs. In a similar way, students may feel uncreative when they do not work cohesively as a team or when they experience difficulties expressing their thoughts in words. This theme is similar to some notions from domain-general collaborative creativity (Kurtzberg & Amabile, 2001), collective mathematical creativity (Aljarrah, 2020), and individual mathematics education (De Villiers, 1990; Sfard, 2007, 2020).

5.2.4 Creating Ideas

The ***Creating Ideas*** theme encompassed participant reflections on developing ideas and proving strategies for the given task. In this theme, participants discussed verifying the claim or discovering the claim is false, making conjectures, and salvaging a false statement. Forty-one (41) of the 346 total quotations (11.8%) were labeled as Creating Ideas and 20 out of the 30 RWs (66.7%) contained quotations labeled with the Creating Ideas code. Of these 41 quotations associated with taking risks, 33 (80.5%) were in concert with feeling creative, and 13 (19.5) were in concert with not feeling creative. Figure 2-7 provides a summary of the ways in which participants felt creative and did not feel creative through creating ideas in their collaborative proving.

Figure 2-7

Creating Ideas Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, and black lines connecting boxes illustrate matched pair observations.

5.2.4.1 Creating Ideas: Feeling Creative. Within the quotations in which participants described both feeling creative and creating ideas, the primary way in which participants described feeling creative was through brainstorming, or group discussion to develop ideas for approaching their proving task. Participants described this process using phrases like "bouncing/bounced ideas off of each other" (Gabby, DC2; Ashley, DC4), "coming up with ideas/a variety of ways" (Campbell, DC2; Participant B, DC4), "try to form an idea" (Sadie, DC4), and "thought of different ways to approach the problem" (Brad Junior, DC3). These many different phrases used to describe collectively generating ideas and proving approaches indicate the importance of the brainstorming process to students' creativity in collaborative proving. Heath (2024) provides a more in depth exploration of these data and of the role of brainstorming in collaborative creativity in proving.

In addition to brainstorming, participants also felt creative in Creating Ideas when they experienced a moment of illumination and were able to see a path to move forward or turning point in their proving through noticing patterns or making conjectures. For example, in DC1, while attempting to prove *the product of consecutive twin primes is one less than a perfect square*, Will reflected "Another instance of creativity is the **process of finding** that the square root of the perfect square would be the number in between the twin primes." Here, Will described experiencing illumination, or the noticing of a pattern, that helped his team move forward in proving. During the same class session, Brad Junior (DC1) offered a similar reflection,

> If it wasn't for Participant B and Ashley triple checking me on if 51 was prime or not, then **we would not have discovered this principle applying to any pair of numbers with a difference of 2 between them**. Without this teamwork our group would not have **recognized the pattern** that each of the perfect squares' square roots were the average between the selected numbers. Also, as a group **we developed the idea** that this equation could also apply to both positive to negative ends of the spectrum.

In this reflection, Brad Junior described how his team was able to make observations that helped his team move forward ("discovered this principle…"; "recognized the pattern") as well as made a conjecture about the values that would apply to the equation they developed ("we developed the ideas [...] could apply to both positive and negative [...]"). Across these sentiments Brad Junior and other participants illustrate the role of creating ideas by brainstorming, illumination, and discovery.

5.2.4.2 Creating Ideas: Not Creative. Participants expressed not feeling creative in the Creating Ideas theme because they perceived the ideas they created to be unoriginal (either with respect to their previous experiences or with respect to their peers) or because they felt their brainstorming and idea generation was deficient.

Brad Junior (DC2) was the primary source of quotations regarding not feeling creative because he perceived his ideas as unoriginal with respect to his previous experiences. He wrote,

> Some may have viewed what I did as creative, due to most people thinking in a mathematical way but since I have learned the importance of graphs and the uses of limit tests, I was able to use graphical analysis. Since **I was just using my prior knowledge**, I do not consider it creative. More so, I think that for me to have been creative I **would have needed to think of a different method** other than one based on my preconceived notions.

Here, Brad Junior expressed not feeling creative because his idea was not original or novel to him ("I was just using my prior knowledge"). Moreover, he described that he could have been more creative by developing a different or another idea ("I would have needed to think of a different method").

Other participants also discussed not feeling creative with respect to creating ideas because they viewed their ideas as unoriginal compared to their classmates. Daniel (DC1) stated, "I did not feel very creative at the beginning of the group exercise, as **I was sure that every**

group was thinking about writing out examples for the first step." Similarly, Campbell (DC1) described not feeling creative during his collaborative proving because:

> I had also noticed that **other groups had been creative in different ways than we were** and made completely new discoveries **going outside of the box**, such as creating their own conjectures, writing out their proofs in plain words, and even figuring out that the numbers did not have to be prime, just two odd numbers.

In these reflections, both participants described creating or having ideas during their collaborative proving, but they did not feel creative because they thought of their ideas as not original or novel enough compared to their peers.

The second way in which participants described not feeling creative in Creating Ideas was through deficient brainstorming. In this thread of reflections, Campbell (DC3) stated, "I also think that **we could have been more creative with coming up with ways to solve the actual proof** itself, instead of being confused with the definitions." Campbell's reflection indicates that he felt like he and his group were not creative because they did not brainstorm or develop enough ideas initially for how to approach their proving task. Ashley (DC4) also expressed struggling with brainstorming because she did not understand the mathematical content involved in the proof, "I do not think that I felt very personally creative today. [...] **I tried to come up with ideas and such,** but I really did not know what to do." In this way, Ashley did not feel creative in part due to not feeling able to develop enough ideas for how to prove the statement.

5.2.4.2 Summary and Discussion: Creating Ideas. The final theme in the overarching category of behaviors, actions, and/or moments is ***Creating Ideas***, in which participants described the processes involved with developing ideas and proving strategies for the task, discovering a given claim is false, noticing patterns and creating their own conjectures (salvaging), and assessing the originality of their ideas. Participants described feeling creative when they developed many different ideas to use in their attempt to prove their claim or

experienced a moment of illumination and were able to see a path to move forward or turning point in their proving either by noticing patterns or making conjectures. When describing not feeling creative, participants attributed this to deficient brainstorming or not attempting to generate many different ideas as well as creating only ideas they perceived as unoriginal.

When the ***Creating Ideas*** theme is situated among extant literature, there are connections to both Savić et al.'s (2015, 2017) CPR on Proving as well as domain-general creativity (e.g., Baruah & Paulus, 2019; Guilford, 1950; Hilliges et al., 2007; Ritter & Mostert, 2018; Zhou et al., 2019). An early publication of the CPR on proving included three different categories: making connections, taking risks, and creating ideas (Savić et al., 2015), yet the CPR was later collapsed into the two former categories and the creating ideas category was eliminated from the rubric. Figure 2-8 depicts the subcategories and levels of the Creating Ideas category in the CPR on Proving (Savić et al., 2015; Tang et al., 2015): Originality, posing questions, and conjectures. Both originality and conjectures are relevant to the ways in which participants in the present study described their creating ideas, however, the posing questions subcategory was subsumed under the Taking Risks theme (as well as the Taking Risks subcategory of the CPR in later editions (e.g., Savić et al., 2017). Notably, participants only cited a *lack* of originality as a reason for not feeling creative and cited making new conjectures as a reason for feeling creative. Brainstorming, however, was not included as a dimension of creating ideas in the CPR; rather, brainstorming was, in part, captured in the Taking Risks category under "proof technique flexibility," which described "thinking of different proving approaches" as a depiction of advancing in this category (Savić et al., 2015, p. 942). The findings from the present study, however, indicate there is a perceived difference between developing and generating a variety of ideas (which is captured in the ***Creating Ideas*** theme) and attempting to use a variety of different

proving approaches, or changing directions (which is captured in the ***Taking Risks*** theme). Creating many different ideas, or the brainstorming component of the ***Creating Ideas*** theme, is relevant to what many creativity scholars refer to as fluency.

Figure 2-8

Creating Ideas category of the CPR on Proving (Savić et al., 2015)

CREATING IDEAS:	**Beginner**	**Developing**	**Satisfactory**
Originality	Attempts to create original ideas for the proving attempt	Displays original ideas (for that student) that are somewhat expected but impressive	Creates a whole new idea never expected or unusual for the course
Posing Questions	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof	Poses questions that take account global understanding or modification of hypothesis of the theorem posed
Conjectures	Poses a trivial or incorrect conjecture, or rewords a previous theorem	Extends theorems or definitions in the form of corollaries or poses conjectures from patterns	Poses and attempts to prove a conjecture that leads to or indicates a generalization of prior ideas

Participants' descriptions of **developing many ideas** as relevant to their creativity strongly resembles Guilford's (1950) construct of fluency, the dimension of creativity characterized by the number of relevant responses generated in a problem-solving task. Moreover, students feeling creative when they brainstorm or develop many ideas for how to approach a proof is supported by literature encouraging multiple solution tasks to foster mathematical creativity (Levav-Waynberg & Leikin, 2012; Schindler & Lilienthal; 2020). Collaborative creativity literature at the domain-general level offers brainstorming strategies to optimize idea generation such as, encouraging participants to write down and record their ideas, alternate individual and group brainstorming, and keeping groups small (Baruah & Paulus, 2019).

Altogether, the ***Creating Ideas*** theme indicates that students may perceive developing multiplicity of ideas as more relevant to creative proving when working collaboratively than independently, as brainstorming was not pertinent in the CPR on Proving, even when creating ideas was identified as a category (Savić et al., 2015). Moreover, the findings from the present study reveal that behaviors such as conjecturing and pushing themselves to be original, which are evident in individual creativity in proving, can also be applicable to the collaborative proving context.

5.3 Circumstances Reported

The overarching category of *Circumstances* encompasses the descriptions of the set of conditions participants described as relevant to their creativity (or diminished creativity) during their collaborative proving. This overarching category emerged as distinct from the *Behaviors, Actions, and/or Moments* because it describes elements of the group proving process that were affecting the group from outside of the group and their actions. The themes that emerged in the *Circumstances* category are: **How I Felt**, **The Task**, and **The Environment**. The following sections provide detailed descriptions of each theme (How I Felt, The Task, and The Environment) and the ways in which participants felt creative and did not feel creative through these circumstances.

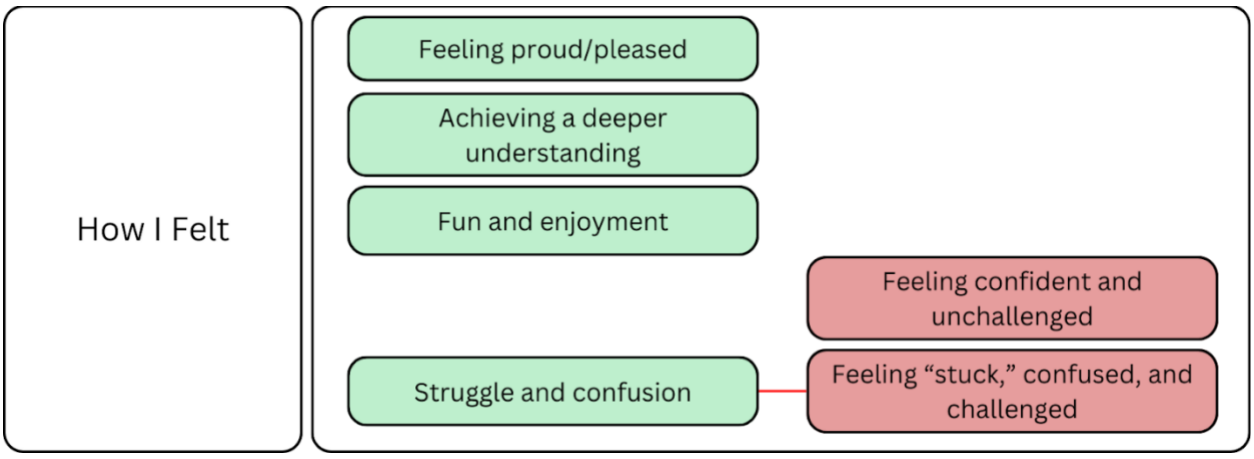
5.3.1 How I Felt

The theme of ***How I Felt*** encompasses student reflections upon how they felt during the collaborative proving process as related to their creativity. This theme includes both positive and negative emotional and affective states such as feeling pleased, proud, confused, challenged, confident, or as if they have achieved a deeper understanding. Thirty-nine (11.3%) of the 346 quotations were assigned the How I Felt code, and 15 of the 30 RW responses (50%) contained

quotations labeled with How I Felt. Participant reflections contained roughly an equal number of instances of describing How I Felt in concert with feeling creative and not feeling creative. Twenty-one (21) quotations of the 39 (53.8%) were labeled with How I Felt and Creative, 22 (56.4%) quotations were labeled with How I Felt and Not Creative, and 4 (10.2%) quotations received How I Felt along with both Creative and Not Creative. Figure 2-9 provides a summary of the observations in the How I Felt theme with respect to participants feeling creative and not feeling creative in their collaborative proving.

Figure 2-9

How I Felt Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, and red lines connecting boxes illustrate opposing observations.

5.3.1.1 How I Felt: Feeling Creative. Participants reflected upon feeling creative during their collaborative proving through a range of emotional and affective states, including both positive and negative emotions. The positive emotions participants described in conjunction with feeling creative were feeling pleased or proud of one's work, a sense of deeper understanding of

mathematical content, and fun or enjoyment of the process. For example, Brad Junior (DC4) reflected on feeling accomplished while working with Gabby,

> As a group I felt we were really creative. Since this was the first time, I approached a conjecture like this, and I believe Gabby had worked on the one before, it allowed us to both really think about what we were proving and how to do so. Lucky, as a group we were able to disprove and salvage the conjecture and then prove the salvage, **which is a great feeling.**

Participants also expressed feeling creative when they felt a sense of deeper understanding of either the course mathematical content (e.g., power sets) or the proving process. Consider Sadie's (DC4) reflection:

> The reason I felt creative from this was that it made me think about all the definitions needed to solve this claim. **I felt that I learned a lot from last week about my proving technique** and it **made me happy that I learned that skill**.

Sadie's reflection indicates that her sense of deeper understanding and gaining a new skill through the collaborative proving process contributed to her feeling creative during the exercise.

Participants did not exclusively discuss positive feelings and emotions with their feeling creative, but they also expressed feelings of struggle and confusion when feeling creative. Ashley, when working with Will, Campbell, and Sadie in DC4, was the primary source of instances in which participants discussed feeling both struggle or confusion along with feeling creative (but not expressing also feeling not creative). Ashley wrote,

> My group and I **struggled a good bit with understanding** the 'Grammar of the Math" as I like to call it. The complicated way that everything we have learned interacts has **made it hard to know when to use certain things**. This **frustration** did allow us to ask each other for help. We all collectively bounced ideas off of each other and tried to come to a general consensus. I personally was not able to provide much assistance in explaining things, and Sadie was a bit **confused** as well. The other member of our group was able to really help us better understand how to correctly write our ideas down and he wasn't frustrated that we needed a bit more help understanding than he did. Overall, **I am very pleased with my group and I's creativity today.**

Despite Ashley's feeling that she and her group mates were struggling to understand and felt confused about how to prove their statement, she still determined that her group had been creative.

In summary, participants described feeling creative when they felt proud or pleased, felt they had achieved a deeper understanding, experienced fun or enjoyment, or felt a sense of struggle or challenge.

5.3.1.2 How I Felt: Not Feeling Creative. Participants almost exclusively wrote about negative emotions and feelings in concert with not feeling creative; they described feeling stuck, confused, and challenged. The primary source of these reflections was DC4, in which students were tasked with proving statements regarding power sets. There were 18 quotations with How I Felt and Not Creative co-occurrences from DC4, which constitutes 81.8% of the 22 quotations in which participants described their emotions and affect alongside not being creative. Consider Will's (DC4) quotation to illustrate the ways in which participants did not feel creative in association with other emotions and feelings:

> During the collaboration, our group was not really creative as **most of us were perplexed by the concept of power sets**. I agree that **it was hard to understand** how to prove our prescribed conjecture, so it makes sense that **we struggled** as a team to make a valid proof. With the help of Dr. Hadley, she managed to **rescue us from our lack of understanding** and guided us toward a method to prove the conjecture. Still, though, **my team was mostly confused** to some degree, so we fell short of completing the proof. During the collaboration I felt only slightly creative.[...] I was able to understand what to do to complete the 1st section of the equality definition using subsets, but **my lack of understanding** how to fully prove the conjecture left me and my team without much time to finish the proof. Even writing this now, **our method of proving is a little confusing** due to parts I do not remember, but in due time I am sure to feel more adept with it after working on some more of the problems in Problem Set 9.

In this excerpt from Will's (DC4) reflection, he expressed feeling confused (e.g., "perplexed," "hard to understand," "lack of understanding," "mostly confused") as well as a sense of struggle or challenge ("we struggled as a team"). Will attributed this sense of confusion and struggle to

his, and his team's, lack of creativity (e.g., "our group was not really creative as …", "but my lack of understanding…"). Other participants' reflections on their negative emotional and affective states and not feeling creative during DC4 were similar to Will's.

There was only one instance in which a participant expressed a positive emotion or feeling contributing to not feeling creative. Campbell (DC4) described feeling both confident and a sense of struggle when proving a statement regarding power sets:

> Personally, I did not feel too creative when creating this proof. **I knew exactly where we needed to start and finish**, but **had the most difficulty** with the introduction of the power sets. I think that since I knew that we had to use the distributive property **I was also fairly confident** in what processes that I needed to use, such as breaking down the set operators and sets into their logical operators and then use the distributive property to prove each of the two logical statements. I think that **by knowing what steps I should take**, I was not as creative as I could have been, and did not try thinking of any other methods that we did not do in the past.

In this excerpt, Campbell emphasized how feeling confident in how to approach the proof ("I knew exactly…," "I was also fairly confident," "knowing what steps…") contributed to his not feeling creative during the collaborative proving. Although he acknowledged feeling challenged ("had the most difficulty") by the definition of power sets, his confidence in proving strategy inhibited his creativity during the task.

Overall, most quotations that indicated not feeling creative that also described other emotions or attitudes portrayed negative feelings and emotions of confusion, frustration, and struggle. There was one exception in which a participant expressed they were not creative because they had a sense of confidence and comfort with the task. Notably, most of these quotations emerged from DC4, in which the class was proving statements regarding power sets.

5.3.1.3 Summary and Discussion: How I Felt. The first theme among the circumstances related to feeling (or not feeling creative) was ***How I Felt***. The ***How I Felt*** theme encompassed student reflections upon how they felt during the collaborative proving process, including both

positive and negative emotions and affective states. Participants associated both positive and negative emotions with their feeling creative. They discussed feeling proud or pleased, a sense of deeper understanding, fun and enjoyment, and struggle and confusion in concert with feeling creative. On the other hand, participants primarily described negative emotions with not feeling creative; they discussed feeling stuck, confused, or challenged as well as feeling confident and unchallenged by the task.

Affect has been studied in relation to individual mathematical creativity at the tertiary level (e.g., Tang et al., 2022) and across K-16 education (e.g., Mann et al., 2017). In synthesizing research on affect and creativity, Mann et al. (2017) wrote, "A common premise in these studies is that affective states play a significant role in stimulating creative thinking and is a factor that can be influenced" (p. 4). Positive emotions are typically associated with promoting creativity (Eubanks et al., 2010; Leu & Chiu, 2015), and this notion is also applicable to collaborative creativity, as evidenced by the participants' reflections in this study. There is, however, one negative sentiment, struggling and feeling challenged, confused, or "stuck," that participants referred to in concert with both feeling and not feeling creative. Struggle as it relates to learning mathematics has a long history; in 1933, Dewey referred to the process of engaging students in "some perplexity, confusion or doubt" (p. 12) as essential for building deep understanding.

When creativity is taken at the relative levels of mini-c (novel and personally meaningful interpretations of experiences) and little-c (everyday) creativity (Kaufman & Beghetto, 2009), one can consider the process of building deep understanding as a component of mathematical creativity. Therefore, the emotions surrounding struggle and feeling/not feeling creative can be related to the concept of productive struggle (Warshauer, 2014). Mathematics students struggle productively when the student works through this struggle at the intended level of cognitive

demand and either completes the task or continues to engage in it; struggle is unproductive when students give up, continue to be confused and/or unable to figure out how to do the task or understand why something works (Warshauer, 2014). In the reflections provided above by student participants, it is evident that those students who reported a sense of struggle, yet felt creative, were engaged in productive struggle because they continued to engage with the task and began understanding. In contrast, students who reported struggle and not feeling creative were likely engaged in unproductive struggle, as they remained confused at the conclusion of the task or felt as though they had to lower the cognitive demand of the task to engage with it (e.g., Will, DC4).

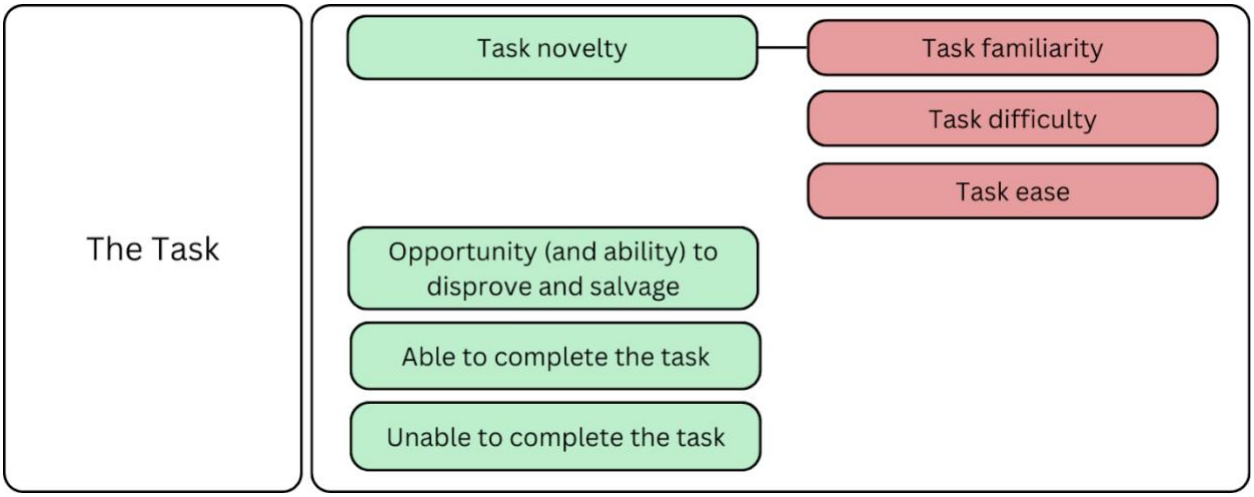
The only positive emotion associated with not feeling creative in the students' reflections was a sense of confidence, which perhaps indicates that the task was not a high enough cognitive demand for that student. The nature of the task is further discussed in section 5.3.2.

5.3.2 The Task

The theme of ***The Task*** captures participant reflections upon features of the assigned task and its perceived influence on creativity. This includes familiarity with the task (relating to previous course experience), the perceived ease or difficulty of the task, and whether or not the team completed the given task successfully. Thirty-one (9.0%) of the 346 quotations fell under The Task theme, and 13 of the 30 RW responses (43.3%) contained such quotations. Figure 2-10 provides a summary of the observations in The Task theme with respect to participants feeling creative and not feeling creative in their collaborative proving.

Figure 2-10

The Task Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, and black lines connecting boxes illustrate matched pair observations.

5.3.2.1 The Task: Feeling Creative. Twenty participant quotations included descriptions of how the task allowed for creativity in collaborative proving. Participants reported the task facilitating their creativity because it was a task unfamiliar to them or it reminded them of a previous proving task. Participants also reflected upon their ability to complete or not complete the proving task relative to their feeling creative.

Several students noted how the novelty of the task (to them) allowed them to work creatively. For example, Ashley in DC1 wrote:

> I think something that may have affected our creativeness was the unknown. **None of us had ever tried something like this before**, so we weren't sure what path to take. This allowed us to be creative because **we were doing something for the first time**. Anything we did **would have been new and worthwhile** because we would have been learning. It also may have impeded creativity because **we were unused to thinking about problems in this way**, so we are unaccustomed to finding answers this way. In most cases in school, we simply solve a problem with skills that were specifically taught to us. This situation **forced us to come up with our own set of rules and decide how to proceed**.

Ashley perceived the novelty of the task to both allow for more creativity ("this allowed us to be creative") as well as hindered their creativity ("it also may have impeded creativity…"). Later on in the course, during DC4 while tasked with either proving or disproving and salvaging the claim $\wp(A \cup B) = \wp(A) \cup \wp(B)$, Brad Junior also expressed how the novelty of a task allowed him to be creative:

> As a group I felt we were really creative. Since **this was the first time I approached a conjecture like this**, and I believe Gabby had worked on the one before, it allowed us to both really think about what we were proving and how to do so. Lucky, as a group we were able to disprove and salvage the conjecture and then prove the salvage, which is a great feeling.

Here, Brad Junior's (DC4) reflection supports Ashley's (DC1) attitude that the task's novelty promoted their creativity during the collaborative proving activity. Moreover, the excerpt from Brad Junior's DC4 reflection also provides evidence that participants associated their ability (or inability) to complete the given task ("we were able to disprove and salvage").

Several other students also mentioned their ability, or inability, to complete the assigned task along with their feeling creative. For example, when working together Brad Junior and Gabby (DC4) both mentioned feeling creative and being able to accomplish their given task and disprove the given statement:

> Today I had personally felt creative when **I was able to prove that the original claim could be disproven**. At first glance the claim looked like it could have been true, but because I wanted to double check to reassure myself, **I was able to prove otherwise.** (Gabby, DC4)

> As a group I felt we were really creative. Since this was the first time, I approached a conjecture like this, and I believe Gabby had worked on the one before, it allowed us to both really must think about what we were proving and how to do so. Lucky, as a group **we were able to disprove and salvage the conjecture and then prove the salvage**, which is a great feeling. (Brad Junior, DC4).

Gabby and Brad Junior both expressed that their ability to disprove their given statement and form and prove a salvaged conjecture helped them feel creative during the collaboration.

Although these participants discussed how completing the task facilitated their feeling creative, other participants described feeling creative despite not completing their task. Also, during DC4, Participant B wrote:

> The main parts where me and Luis felt the most creative during this exercise was the parts where we had to come up with a variety of ways get a better understanding of what a problem set was. Me and Luis both had a variety of ways we created examples in order for us to understand the definition of a power set, as it was probably the biggest issue that we had with the problem, and it was the biggest issue with problem set in general. Unfortunately, despite all of our efforts to using the examples in order to giving us a good idea for us to create our proof, **we were still unable to create an actual proof for the claim.** So, because **we were not able to create a proof for the claim that we were given**, we felt as though all of that creativity was for nothing at the time.

In this excerpt, Participant B described many of the ways he perceived he and Luis were creative (e.g., developing a variety of ways to approach the problem, creating examples), but also described how he and Luis were unable to complete the task of proving their given statement. Participant B stated that not being able to complete their proof made them feel like "all of that creativity was for nothing at the time." This statement indicates that it is possible for students, like Participant B, to still view their proving processes and attempts as creative even when they did not create a complete proof.

5.3.2.2 The Task: Not Feeling Creative. Participants wrote about not feeling creative with respect to how the assigned task in 13 quotations (16% of the Not Creative quotations). Participants referenced the difficulty (or ease) of the given task and the familiarity of the task when discussing not feeling creative.

Will (DC2) provides an example of participants reported the difficulty of a task inhibiting their creativity:

> I had a hard time being creative [...] since **I was not entirely sure how to prove the biconditional statement in full**. To be frank, I personally did not feel creative. When I worked with Campbell, I tried combining my ideas with his and hearing out what he had to say to **understand how to go about proving the biconditional statement**. Had I had

> time to look at our work so far and think about how to complete the proof, I maybe would have thought to have creativity in solving the proof, but that was not the case. This is not so bad, though, since we got relatively far, and **it was understood that the biconditional proof was the harder problem to tackle (I would assume)**.

This excerpt from Will's reflection illustrates how Will described his task as more difficult because it was to prove a biconditional statement, and the complexity of proving a biconditional statement (combined with a lack of class time) made it difficult for him to feel creative.

In contrast with Will's (DC4) reflection, participants also cited the ease of a task impeding their creativity. Gabby (RW3) wrote:

> During the collaboration I don't think we were that creative, but **I don't think that was necessarily our fault**. **The problem we had was very straight forward and relatively easy**. I did feel like we were all more creative when we were working on **the other group's problem**, proving an empty set is a subset of set A.

Gabby's reflection indicates she felt her group was not able to exercise as much creativity on their assigned task because the task was "straight forward and relatively easy." She did, however, express that another group's assigned task allowed her team to feel more creative ("when we were working on the other group's problem"), indicating her feeling creative (and not feeling creative) were related to the (perceived) ease of task she was assigned.

Another way in which participants discussed not feeling creative in conjunction with the familiarity of the task assigned to them. Campbell (DC4) wrote:

> During this process I felt like the group was creative in some aspects, but for the most part **was able to use our past knowledge of the distributive property** to be able to **relate this to the problems that we had done in class from before**. Though, we were introduced to a new concept in this problem, which is the idea of power sets. While this had its own unique challenges, we were able to work through them by using the definition of power sets, which **based off the previous problems was prominent in those**. I do think that we could have been more creative, though I am not sure where, since **we have done many problems with similar processes.** I also feel fairly confident when proving **other problems with similar patterns** as those that **we have done in the past couple of classes.**

In this reflection, Campbell recognized how his connection to previous use of the distributive property inhibited his ability to be creative. His inclination to approach the problem using a process similar to problems he had seen before contributed to his perception that this task was routine for him and did not require him to be creative.

5.3.2.2 Summary and Discussion: The Task. ***The Task*** theme encapsulated participant reflections upon features of their assigned task. Participants recognized task novelty, opportunities to disprove and salvage a claim, and both their ability and inability to successfully complete the task as relevant to their feeling creative. Participants who did not feel creative described their tasks as familiar to them, difficult, or too easy.

Amongst mathematics education and mathematical creativity literature, there is an abundance of research regarding how to provide high-cognitive demand and non-routine tasks (e.g., Beghetto, 2017; Boston & Smith, 2009; Stein & Smith, 1998). Being able to ensure the task is non-routine and cognitively demanding for students could resolve factors associated with feeling not creative, such as tasks being familiar or too easy for students. There is, however, a risk of making tasks too challenging for students in this effort, which is another relevant feature of the task students related to not feeling creative. Moreover, designing proving tasks for collaborative groups introduces the additional challenge of trying to make tasks cognitively demanding, non-routine, and not-too-challenging for all students who compose the group, which is extremely difficult when students composing a group have different mathematical backgrounds and abilities. A possible approach for resolving this challenge is to implement “low threshold, high ceiling” tasks (Myers et al., 2000, p. 6). These tasks are accessible to novices (low threshold) but offer the opportunity to create sophisticated solutions (high ceiling). Resnick et al., (2005) proposed tasks designed to foster creativity should also have “wide walls” (p. 3),

meaning they should offer a wide range of solution approaches and support many possible explorations.

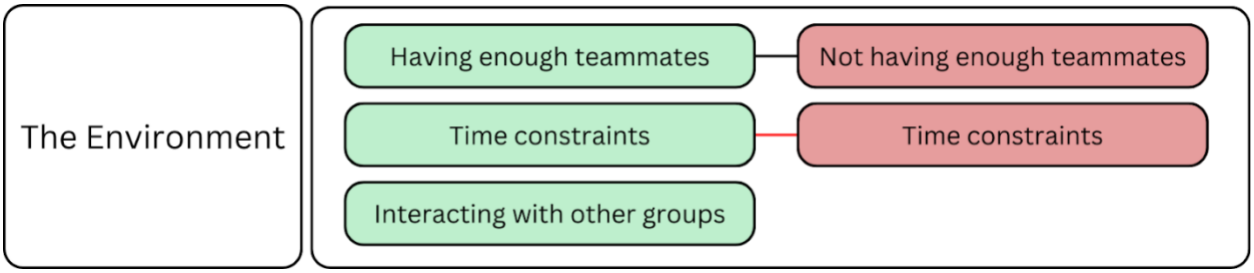
The data described within this theme illustrated when tasks were challenging for students, students could recognize themselves as creative even when unable to complete their assigned task in the time given. Similarly, participants noted feeling creative when a task allowed for the opportunity to discover a claim was false and create their own claim.

5.3.3 The Environment

The final theme of ***The Environment*** encompasses student reflections upon the classroom context and environment and their effect on creativity. This category includes student preparedness for class, the amount of class time provided for the proving activity, and the influence of whole-class interactions. Twenty-four (6.9%) of the quotations were assigned the Environment code, and 12 of the 30 RW responses (40%) contained quotations labeled with The Environment. Participant reflections contained roughly an equal number of instances of describing the Environment in conjunction with feeling creative and not feeling creative. Of the 24 *environment* quotations, 14 (58.3%) were labeled with The Environment and Creative, 13 (54.2) quotations were labeled with The Environment and Not Creative, and three quotations (12.5%) received The Environment along with both Creative and Not Creative. Figure 2-11 provides a summary of the observations in the Environment theme with respect to participants feeling creative and not feeling creative in their collaborative proving.

Figure 2-11

The Environment Theme and Feeling/Not Feeling Creative



Note: Green boxes describe the ways participants reported feeling creative, red boxes describe ways participants reported not feeling creative, black lines connecting boxes illustrate matched pair observations, and red lines connecting boxes illustrate opposing observations.

5.3.3.1 The Environment: Feeling Creative. Within this theme, participants cited elements of the environment such as group size, time restraints, and interactions with other student groups in relation to their feeling creative. In DC2, Gabby wrote about how working with others contributed to her creativity, "**Without someone else there to bounce ideas off of** I think both Tyler and I would have been stuck for a good while." In addition to attributing her team's progress to having a group of at least two students. Later in this same reflection, Gabby wrote,

> I felt personally creative today, but I also think **I could have been more creative with more people to bounce ideas off of**. When we were doing a class collaboration and critique, Will pointed out that while the equation that we had put down did cover some cases it does not cover all of them because by using the same variable in both odd number definitions we are implying we are multiplying the same odd integer by itself rather than any two odd integers. I**f we had even one more person in our group to bounce ideas off of**, I think I or one of my other group members would have come to the same conclusion.

In this second portion of her reflection, Gabby expressed that having a group size larger than two may have allowed her to be even more creative.

Participants also discussed the time constraint of a classroom environment along with their creativity. For example, in DC1 Will wrote about how working within a time constraint promoted his group's creativity:

> I thought my group was very creative throughout the whole process.[...] Yet another instance of our mathematical creativity was **understanding that disproving the conjecture would be rough with our time constraint**, so coming up with a formula that could withstand a few examples would be more applicable.

In his reflection, Will recognized that his team's consideration of the time they had to complete the given task offered a constraint that pushed them to move onto different approaches for their task. The pressure of a time constraint for in-class proving may motivate students to be practical and consider alternative approaches in their proving.

In addition to discussing how the size of their collaborative group and the time allowed for the task facilitated creativity, participants also cited interactions with and the presence of other student groups in the classroom as fostering their creativity. During DC3, both Sadie and Gabby, while working together with Will, described how they felt creative when building upon the work of another student group while working on a different task, *Prove or Disprove and Salvage:* If A is a subset of the universal set U, then $\emptyset \subseteq A$. Gabby wrote,

> I did feel like we were all more creative when we were working on the other group's problem, proving an empty set is a subset of set A. **When we looked at the other group's proof we disagreed with their findings, so we decided to try to approach it differently.** We all began to work independently, but once my groupmate was able to get a proof started, we were able to look at it and find how this approach proved the claim that we thought was true. Eventually **the other group was able to fix their direct proof and realized it actually proved the claim rather than disproving it**, but I felt both groups were creative because we were both able to prove the same claim, but from very different approaches.

In this reflection, Gabby noted how the opportunity for her team to critique ("we disagreed with their findings") another team's work and attempt to prove the statement themselves ("try to approach it a different way") allowed her team to feel creative. Gabby even mentioned how observing the other team continue to work on their task and address their error contributed to a classroom environment that facilitated creativity.

In this same collaborative proving episode, Sadie also wrote about this incident:

> I always believe that my group is creative while working.[...] I really liked being able to talk with my table about the problem 4.18 and **try to see if we could prove the same one that another team was already trying to prove to see if we could do it as well**. [...] **It was nice to help try and solve a problem that another team was having trouble with solving.**

Here Sadie's reflection that she enjoyed working on another team's task with her group in response to when and why she thought her team was creative indicates that she, similarly to Gabby, perceived the classroom environment in which other students were working on different tasks as fostering creativity in the classroom and for her group.

5.3.3.2 The Environment: Not Feeling Creative. There were 13 quotations in which participants described the environment in conjunction with not feeling creative, accounting for 16% of the not creative quotations. Similarly to the ways in which participants wrote about elements of the environment facilitating creativity, participants also described group size and time restraints diminishing their creativity.

In section 5.3.3.1, Gabby's DC2 reflection highlighted how having a teammate to work with helped her team be creative, but she also expressed that a larger group might have allowed her to be more creative. Campbell echoed this sentiment in his DC2 reflection when working in a pair with Will, "I think that **having more group members would have helped a lot more in being creative**, since most of the time we spent during this exercise was thinking about how we should have written out what we were thinking." Although Campbell recognized his team and himself as creative through the exercise, he thought their creativity could have been furthered with additional group members.

The time allowed for the task in class was also mentioned in concert with not feeling creative. Although in the previous section, 5.3.3.2, Will (DC1) noted how the pressure of

working in a time constraint forced his group to think more efficiently and therefore feel creative, in DC2 an DC4 Will expressed that a time restraint impeded his creativity:

> To be frank, I personally did not feel creative. When I worked with Campbell, I tried combining my ideas with his and hearing out what he had to say to understand how to go about proving the biconditional statement. **Had I had time to look at our work so far and think about how to complete the proof,** I maybe would have thought to have creativity in solving the proof, but that was not the case. (DC2)

> During the collaboration I felt only slightly creative. [...] I was able to understand what to do to complete the 1st section of the equality definition using subsets, but my lack of understanding how to fully prove the conjecture **left me and my team without much time to finish the proof.** (DC4)

Will's reflections from DC2 and DC4 highlight how in these situations he perceived that he was not as creative as he could have been because he did not have enough time.

5.3.3.3 Summary and Discussion: The Environment. ***The Environment*** theme encompassed student reflections upon the classroom context and environment and their effect on creativity. Participants described elements of the classroom environment and context as influencing their feeling creative and not feeling creative. Within this theme, I observed how group size and time restraint can both positively and negatively influence students feeling creative during collaborative proving.

With respect to group size, participants primarily noted working in teams promoted their creativity overall, but a group of only two students may be too small to reach their full creative potential. It is, however, impossible to know if participants had an optimal group size in mind when making these comments. The largest group size participants experienced during the course duration was four students, and no participants mentioned too many students in their group inhibiting their creativity. There is extant literature from business (e.g. Jung, 2001; Kurtzberg & Amabile, 2001) and organizational psychology (e.g., Paulus et al., 2012) that has addressed

optimal group size and composition for collaborative creativity, and the findings from this theme support those assertions.

When discussing time constraints necessarily placed on collaborative proving, due to the classroom context, Will discussed how this time constraint both encouraged him to be creative (DC1) and inhibited him from being creative (DC2, DC4). In contrast to when Will (DC1) described how a time restraint forced his team to think creatively, in DC2 and DC4 Will expressed that he would have needed more time in order to be creative. One observation in these two situations which may explain this dichotomy in Will's reflections is in both DC2 and DC4 Will also expressed feeling challenged or confused with the content of the task. In DC2, his confusion stemmed from having a biconditional statement to prove ("I had a hard time being creative outside of this since I was not entirely sure how to prove the biconditional statement in full." (Will, DC2)). In DC4, Will expressed confusion and frustration with power sets. See section 5.3.1.2 (How I Felt: Not feeling creative) for a more detailed discussion of Will's DC4 reflection. Thus, Will's contrasting reflections on working within a time constraint affecting his creativity indicate there may be certain conditions regarding the difficulty of the mathematical or logical content of the task that better lend themselves to more or less time allowed.

6. Discussion and Conclusion

6.1 Summary and Discussion

The goal of this study was to investigate the ways individual students experience creativity during collaborative proving. To achieve this goal, ten student participants enrolled in an introduction-to-proof course completed retrospective writing (RW) assignments following in-class collaborative proving activities. In the RW responses, participants detailed what occurred during their collaborative proving to help them feel, or not feel, creative. These written data were

analyzed using an inductive coding strategy to form themes that captured the ways in which the participants described feeling or not feeling creative.

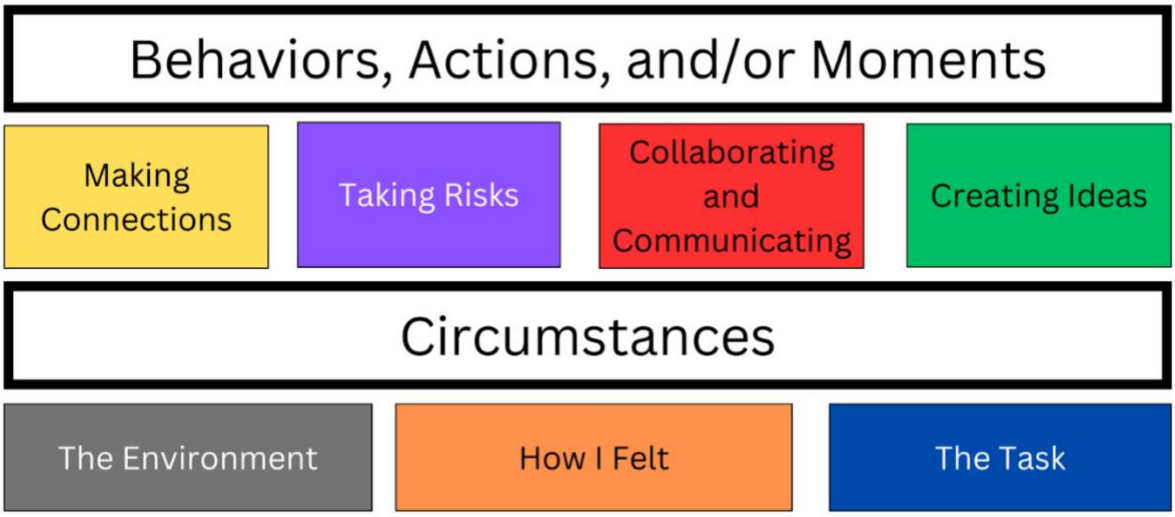
The findings of this study culminated in two overarching categories. The *Behaviors, Actions, and/or Moments* category captured finer-grained, in-the-moment, contingent activities during collaborative proving that facilitated creativity in the group. This first category included reflections on feeling/not feeling creative with respect to the themes of making connections, taking risks, collaborating and communicating, and creating ideas. In contrast, the *Circumstances* category captured larger-grained experiences during and components of the collaborative proving that influenced students' creativity. Participant reflections in the *Circumstances* category examined how their creativity was associated with the themes of how I felt, the task, and the environment. With respect to how these two categories relate to instruction in an introduction-to-proof course incorporating collaborative proving activities, the *Circumstances* category is more related to the design and pre-course preparation of the collaborative activity as well as classroom norms which influence student affect and emotion, as it concerns the choice of appropriate tasks, the physical classroom environment, time allowed for the task, and students' attitudes throughout the proving process, particularly with respect to struggle. The *Behaviors, Actions, and/or Moments* category is more related to instructor's decisions, and students' activities, during the class meeting and the collaborative proving activity as it concerns how the instructor engages, or does not engage, in help or guidance-giving, encouragement to make connections or take risks, and facilitation of team collaboration and presentations.

The findings of these two categories and seven themes (Figure 2-12) contribute to existing literature on student perceptions and experiences of mathematical creativity in proving and the collaborative mathematical creativity process. The findings of this study provide a

framework for how students experience creativity in proving through collaboration. In particular, this study illustrated how students recognize how both small moment-to-moment interactions and decisions as well as broader circumstances surrounding their proving influence their creativity.

Figure 2-12

Framework of Student Experiences of Collaborative Creativity in Proving



One large contribution of this study is the dissection of participant reflections in each theme according to the ways in which they not only felt creative but did not feel creative. For example, the *Making Connections* theme was present in 27/30 RWs and had the highest frequency of quotations associated with it. Yet, when compared with the *Taking Risks* theme, of the quotations in which participants described taking risks, there was a higher proportion (90%) describing feeling creative in taking risks than making connections (87.5% of quotations). The *Taking Risks* theme in this study further highlighted an interesting observation that participants may view the same phenomenon (e.g., receiving help) as fostering or not fostering their

creativity. The theme that highlighted the greatest potential for not feeling creative was the *How I Felt* theme, in which participants described how their affect, emotions, and attitudes inhibited their creativity, primarily through frustration and struggle. In the *How I Felt* theme, 56.4% of the quotations were simultaneously describing not feeling creative; however, fewer quotations and RWs included mention of *How I Felt* in comparison with the themes in the *Behaviors, Actions, and/or Moments* category.

In contrast to extant literature regarding student perceptions and experiences of mathematical creativity at the tertiary level (e.g., Satyam et al., 2022; Tang et al., 2015), this study offers a contribution of specifically investigating the activities involved in collaborative proving in-the-moment. Tang et al. (2015) as a part of the Creativity Research Group's efforts to develop the Creativity-in-Progress Reflection (CPR) on Proving, interviewed tertiary students and mathematicians regarding their perspectives on creativity in proving. Tang et al. (2015) found students' responses to more frequently emphasize the "creating ideas" category of the CPR compared to mathematicians. Although the Creativity Research Group later modified the CPR on Proving to no longer include a separate category for creating ideas (Savić et al., 2017), the re-emergence of this theme in the present study is aligned with their previous observation that students may view this as an important part of their creative proving process and still emphasize this feature of their proving in collaborative activities. With respect to the CPR, this study both validates many of the categories as relevant to student experiences of collaborative creativity in proving as well as highlights some ways student experiences of collaborative creativity are unique from general or individual creativity (e.g., collaborating and communicating, social risk taking).

Many of the themes observed in the present study are also relevant to Satyam et al.'s (2022) teacher actions to foster creativity in calculus, which emerged from interviews with undergraduate Calculus I students inquiring when and why they felt creative during their course. The connections between the themes in the *Circumstances* category and Satyam et al.'s (2022) framework for teacher actions provide evidence that there may be similarities in student experiences of creativity across course content (Calculus compared to introduction-to-proof) as well as classroom activities (e.g., lecture, whole class discussion, small group collaborations). The data also suggest nuanced differences between these settings. This study and Satyam et al.'s (2022) study should, however, be compared with caution as participants in the latter were asked specifically about their teachers' actions that fostered their creativity, and participants in the present study were asked about their experience more broadly during the collaborative proving activities. Moreover, this study offers a unique contribution, compared to Tang et al. (2015) and Satyam et al. (2022) by investigating student experiences immediately after specific collaborative proving activities conducted in class rather than inquiring about more general experiences with creativity in mathematics or across the duration of a course.

Looking at how the present study is situated in extant literature regarding the *collaborative* creative process of specific mathematical collaborative activities (e.g., Aljarrah, 2020; Schindler & Lilienthal, 2022), my work also provides a novel contribution by using an emic, or insider, perspective. Both Aljarrah (2020) and Schindler and Lilienthal (2022), like the present study, investigated the inner workings of the collaborative creative process situated by specific mathematical tasks (as opposed to a whole course or experiences more broadly); however, these studies relied upon an etic, or outsider, determination of what is considered creativity and what is not. Aljarrah (2020) used video recordings of task-based interviews of

students engaged in group problem solving sessions, and the researcher determined the most important moments throughout the collaborative process to analyze. Similarly, Schindler and Lilienthal (2022) analyzed a video recording of two participants working together on a multiple solution task as well as eye-tracking and stimulated-recall interviews with the participants regarding their thinking during the multiple solution task. Aljarrah (2020) determined, as the researcher and outside expert, what was creative throughout the group problem solving task, and Schindler and Lilienthal's (2022) study was based on the assumption that the nature of a multiple solution task would solicit creativity. Although Schindler and Lilienthal (2022) conducted stimulated-recall interviews with their participants, they did not use these interviews to determine how the participants experienced creativity, but rather used these interview data to triangulate and support claims made in their etic-oriented analysis. The emic perspective taken in the present study not only remains true to the conceptualization of creativity as relative, but it also structures the framework in the students' own words. The frequencies with which participants mentioned different aspects of their collaborative proving that made them feel creative also give an insider perspective, via residue, to what elements are the most influential on their creativity. Using an emic perspective, as in this study, can help prevent researchers from imposing their own perceptions and experiences of creativity upon their participants.

6.2 Limitations

Although this study has offered a novel framework to capture student experiences of collaborative creativity in proving and contributed to extant literature by (a) focusing specifically on collaborative proving and (b) taking an emic perspective to research, there are a few limitations of the study which should be considered. The primary limitation of this study is the small sample size of 10 students from a single introduction-to-proof course. All participants had

the same course instructor, and it is possible a different instructor, and different classmates, would have altered the experiences students had and therefore altered their reports of what allowed them to feel creative. Moreover, the nature of written responses as data do not offer the same opportunity to further inquire into participant descriptions of their experiences (as interviews might). The decision to only collect written reflection data for this study, however, was necessary to prevent over-burdening participants with frequent and potentially emotionally taxing interviews. Future research in this topic could consider using participant interviews to inquire into students' experiences in collaborative creative proving, but interviews conducted in real time may also influence participants' willingness to honestly share their experiences.

6.3 Future Directions

Although mathematical creativity is gaining traction as a research topic at the tertiary level, there is still much left to be explored, especially as it pertains to collaboration. In particular, this study has laid the foundation for future research in at least three distinct areas. First, the findings from the present study could be further supported, or refined, by collecting similar data from participants enrolled in various introduction-to-proof courses with varying instructors. Second, research should connect the two categories and seven themes observed (Figure 12-2) in this study to teaching moves that can help foster such experiences. For example, researchers could seek to answer the question "what teaching moves during collaborative activities can facilitate students' making connections or taking risks in their proving?" Lastly, there are very interesting findings from the present study that could be expounded upon in more detail. In particular, all of the findings that illustrate reports of how the same experience made some students feel creative, yet others uncreative. For example, in the *Taking Risks* theme, some participants reported seeking and receiving help allowed them to feel creative, yet others

reported this same activity made them less creative. Similarly in *The Environment* theme, some participants noted that time constraints during the collaborative proving forced them to be creative, but others said a time constraint prevented them from pursuing creative avenues. Research investigating what nuances make these kinds of experiences more likely to foster or not foster creativity could be valuable in determining how the social setting of collaboration and physical setting of a real classroom might influence current understandings of creativity in the field of mathematics.

6.4 Conclusion

Throughout this study, I have sought to understand the student experience of creativity through collaborative proving activities in an introduction-to-proof course setting. Participants in the study highlighted, through written reflection data, how they felt creative and did not feel creative during collaborative proving. Namely, participants recognized both small moment-to-moment experiences (Behaviors, Actions, and/or Moments) as well as broader influencing factors (Circumstances) either allowed them to feel creative or inhibited them from feeling creative. This study contributed to the field of research on mathematical creativity by focusing on collaborative proving, valuing the student (emic) perspective on creativity, and being conducted in a real classroom (as opposed to a research laboratory setting or interview). Future work can build on the findings of this study by validating or refining the framework presented here with additional data from different introduction-to-proof courses incorporating collaborative activities, connecting the framework presented here to instructional decisions to foster creativity, and further investigating the nuances of some seemingly contradictory observations found in the present study to contribute to the field's understanding of the influence of social dynamics and a classroom setting on mathematical creativity.

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Appendix 2-A
Complete Sample Participant RW Response

[Gabby – DC2]
Collaborative Proving
Retrospective Writing Prompt

Directions: Respond to each of the prompts in the space below. Type your responses and follow the length requirements specified below.

Reflect upon the collaborative proving task your group worked on during class today.

I. Describe what happened in your group proving today during class in a narrative form as if you were telling the story to a friend. What happened? Who did what? What made you think of certain ideas, and when? What different steps and ideas did your group go through in order to reach a final proof (or not)? Give as much detail as possible. *(At least 450 words)*

Today me and [Tyler] were trying to prove that if you multiply two odd integer together it will always equal an odd integer. Before we started out proving tasks for today, we discussed our class rubric for a good proof. One of the things we discussed in detail was examples and a proof being generalized to cover all cases. We decided that examples are not always necessary and do not necessarily prove that something does work for all cases. This thought stuck with me when we started looking at the claim we were trying to prove or disprove. We both initially thought that the claim was true because we remembered from other algebra classes that multiplying two odd integers together would give you an odd integer but multiplying an even integer by an even or an odd integer will result in another even integer, but we could not just put that idea out there without proving that it works. When we went to the board [Tyler] started to put examples down of multiplication between different odd integers as well as even and even, and even and odd because that's where both of our minds went first. I wrote down the claim and started to brainstorm on how we can definitively prove the claim with something more than examples. I began looking back at the problem set and noticed the definition of an odd and even integer at the top of the problems. I decided that to cover all cases we can take the definition of an odd integer and multiply it by itself, and it would still result in an odd integer proving our claim. When I went to do this, I had initially done the distribution incorrect and plugged-in examples and did not know why it was not working. [Dr. Hadley] then asked us about our process so far and I explained the thought process we were on, but explained it wasn't working but felt like it should. She verified that it should work and to write out the equation completely then distribute. We erased my initial equation and [Tyler] helped me to realize my mistake when distributing. I found that when looking at our statement that even integers multiplied by even or odd integers would make our first two integers we were adding even and then the "+1" would create an odd result. [Tyler] and I agreed that the "+1" refers to the definition of an odd number and was proof enough for our claim. What we failed to realize was that we had only proved that the same odd number multiplied with itself would equal an odd integer. We then proved this after we presented our proof to the class and to solve this issue, we did the same process but changed one of the variables for the odd integer to imply a different odd integer. After following the same process

another group pointed out for both equations we found if you factored out a 2 and left the "+1" out of the parenthesis you would have the definition of an odd integer as long as we proved the variables inside the parenthesis would still result in an integer.

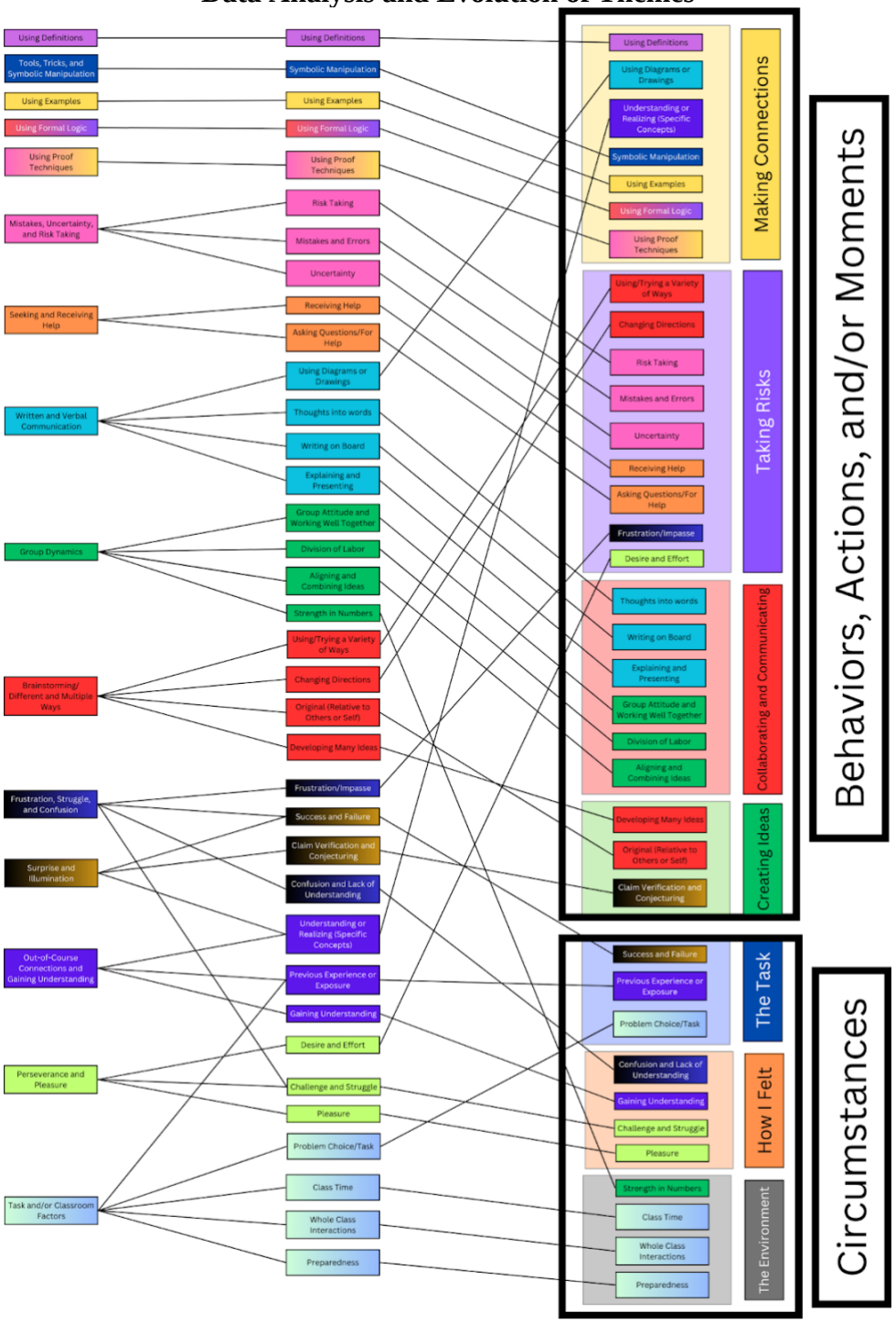
II. When during the collaboration did you feel like your group was creative? If you did not feel like your group was creative, explain why not. Include as many details as you can. *(At least 150 words)*

As a group I felt that we were creative. [Tyler] and I unfortunately had the same thought process at first with examples, so we went ahead at first with this idea and tried to see if it would spark any other ideas because we knew this alone would not be enough to be a complete proof. [Tyler] wrote down "odd*odd=odd, even*odd=even, even*even=even" which made me think of the definition I saw on the problem set when I was glancing over it this morning. I decided to write that down on the board because I figured that would be the only way we'd be able to cover all cases. [Tyler] and I then began bouncing ideas off of each other to try to figure out how the definition of an odd number could help us. From there we tried multiplying out the definitions for two odd numbers. Without someone else there to bounce ideas off of I think both [Tyler] and I would have been stuck for a good while before we would've thought to try the definition rather than proving every single example of two odd integers multiplied together.

III. When during the collaboration did you feel (personally) creative? If you did not feel creative, explain why not. Include as many details as you can. *(At least 150 words)*

I felt personally creative today, but I also think I could have been more creative with more people to bounce ideas off of. When we were doing a class collaboration and critique, [Will] pointed out that while the equation that we had put down did cover some cases it does not cover all of them because by using the same variable in both odd number definitions we are implying we are multiplying the same odd integer by itself rather than any two odd integers. If we had even one more person in our group to bounce ideas off of, I think I or one of my other group members would have come to the same conclusion. I was happy that I eventually broke free of just thinking in terms of examples and thought to use the resources that were provided for me rather than letting myself get stuck in this incomplete form of proving. The examples helped to verify our initial thoughts, but we would never be able to cover all cases in our given class time.

Appendix 2-B
Data Analysis and Evolution of Themes



Note: This figure depicts the qualitative data analysis process of this study. The leftmost column contains the initial 15 themes derived from grouping in-vivo codes together. The middle column contains the smaller grain-sized components of each of the initial themes, and the connectors from the left column to the middle column connect the initial themes to the pieces they were broken into. The rightmost column illustrates how the small grain-sized components of the initial themes were then regrouped to create the final two categories and seven themes. The lines connecting the middle and right columns show where the small grain-sized components were regrouped to in the final list of themes.

Manuscript 3: Student Experiences of Creativity in a Collaborative, Inquiry-Oriented Introduction-to-Proof Course

ABSTRACT: Although collaboration and creativity are key features of the work of professional mathematicians, undergraduate proof-based courses are still largely taught through lecture. Some instructors, however, have begun to incorporate more active learning in their proof-based courses, but there is little known regarding how these different active learning activities foster creativity for students. Specifically, there is little research on how instruction with a heavy emphasis on collaboration fosters mathematical creativity. In this study, I investigate how students report having experienced creativity in an introduction-to-proof course taught through collaborative, inquiry-oriented methods. At the conclusion of this semester-long introduction-to-proof course, seven student participants enrolled in the course responded to a written reflection prompt and described when and how they experienced creativity throughout the course activities. These written responses were analyzed inductively to first identify the contexts students described feeling creative. These contexts included: (1) creating proofs with specific mathematical content, (2) using specific proving strategies, (3) working in small groups, (4) interacting with the whole class, and (5) working individually on problem sets. Within each of these contexts, I unpack student responses to detail how students experienced creativity within each of those contexts. Primary findings of this study include a majority of students, five of seven, referencing both small group and whole class collaborations as contexts which fostered their creativity, and observations across contexts emphasize how students recognized opportunities to make connections to other mathematical content, whether learned in the course or elsewhere, fostered their creativity.

1. Introduction

In recent years, there have been initiatives to promote collaborative learning, such as small-group learning, flipped instruction, inquiry-based pedagogy, and modified Moore method in undergraduate mathematics courses (e.g., Conference Board of the Mathematical Sciences, 2016; Saxe & Braddy, 2015). For example, the Mathematical Association of America (MAA) put forth in their *Common Vision* that undergraduate programs in the mathematical sciences should incorporate pedagogies to "promote collaboration and provide opportunities to practice communicating ideas" (Saxe & Braddy, 2015, p. 19).

As students move from novice to expert (Lave & Wenger, 2001) in the field of mathematics, students follow a journey in which there is an eventual transition from computational to proof-based mathematics (Civian & Schley, 1996). Proof and proving are key practices in upper-level mathematics (Harel & Sowder, 2007; Stylianides et al., 2017). Collaboration and learning from peers are central to the proving process for professional mathematicians (e.g., Burton, 1998; Sriraman, 2004; Weber & Fukawa-Connelly, 2022), and hence is a practice students should learn and develop as they transition to become participants in the mathematical community of practice. Moreover, it is at this transition students are expected to become producers of mathematical ideas on their own for what may be the first time (Boyle et al., 2015), so it is critical to offer opportunities that foster mathematical creativity in this transition to proof. With respect to mathematics major programs, the MAA's Committee on Undergraduate Programs in Mathematics outlined a cognitive goal and recommendation that, "Major programs should include activities designed to promote students' progress in learning to [...] approach mathematical problems with curiosity and creativity and persist in the face of difficulties [and] work creatively and self-sufficiently with mathematics."

Despite initiatives put forth by professional organizations to promote collaboration and creativity in undergraduate mathematics, there are still few opportunities for students to engage in collaboration or creative thinking in undergraduate courses. Undergraduate proof-based mathematics courses are still predominantly taught using a lecture format (Johnson et al., 2018; 2019) and mathematicians believe lecture is the best way to teach (Fukawa-Connelly et al., 2016). Although lecture can be an effective way to teach mathematics and instructors can still incorporate student centered strategies (e.g., think-pair-share; Alcock, 2018; Johnson et al., 2018), students cite courses dominated by lecture-based instruction, lack of collaboration, and high emphasis on memorization as reasons for their disaffection with mathematics (Ward-Penny et al., 2011). Furthermore, students value collaborative learning environments (Hall et al., 2022). In addition to these student perspectives, teaching to foster mathematical creativity has been seen to positively influence students' mathematical identity (Regier et al., 2021), self-efficacy (Regier & Savić, 2020), and general enjoyment of mathematics (Tang et al., 2022), as well as promote equity (Luria et al., 2017).

In this study, I investigate students' perspectives on what activities they report fostered their creativity over the course of a semester-long introduction-to-proof (ITP) course taught through collaborative, inquiry-oriented pedagogy. Moreover, I present findings on how students attributed collaborating with their peers as having fostered (or not fostered) their creativity. I seek to answer the research question: After a semester-long introduction-to-proof course taught with collaborative, inquiry-based methods, what do students recall as fostering their creativity?

2. Background

It is generally accepted mathematical creativity can be developed and enhanced in students (Sriraman & Haavold, 2017; Zazkis & Holton, 2009), and mathematicians believe

mathematical creativity can and should be fostered in undergraduate mathematics courses (Karakok et al., 2015). Despite this consensus, there is little empirical research on what teaching strategies develop mathematical creativity among undergraduate students (Savić et al., 2022). None of this extant research has explicitly considered the impact collaborative work or collaborative proving may have on creativity.

Several types of classroom activities have been investigated quantitatively by comparing pre and post measures of creativity surrounding an intervention (e.g., Asahid & Lomibao, 2020; Leikin & Elgrably, 2020; Levav-Waynberg & Leikin, 2012). Some of the empirically-supported instructional tasks to foster creativity are problem posing (Yuan & Sriraman, 2011; Zazkis & Holton, 2009), model-eliciting activities and phenomenon-based learning (Asahid & Lomibao, 2020; Gilat & Amit, 2014), and multiple solution tasks (Levav-Waynberg & Leikin, 2012). Other instructional methods like the use of open-ended problems (Luria et al., 2017; Mann, 2006; Nadjafikhah et al., 2012; Regier & Savić, 2020; Savić, 2016; Silver, 1997), incubation (Regier & Savić, 2020; Savić, 2016; Sriraman, 2004), metacognitive tools (Savić, Karakok, et al., 2017), and risk taking (Luria et al., 2017; Mann, 2006; Regier & Savić, 2020; Savić, Karakok, et al., 2017; Sriraman, 2017) are primarily supported theoretically. There are, however, a few studies that qualitatively investigated the teacher actions that promote mathematical creativity.

Qualitative studies on teacher actions to foster mathematical creativity have been conducted with teachers of elementary students (e.g., Levenson, 2013), undergraduate Calculus I students (Satyam et al., 2022), advanced undergraduate mathematics students (Savić, El Turkey, et al., 2017b), and undergraduate mathematics instructors (Savić, El Turkey, et al., 2017a, 2017b). The teacher actions most supported by these studies as fostering mathematical creativity

are choosing appropriate tasks (Levenson, 2013; Satyam et al., 2022; Savić, El Turkey, et al., 2017a), allowing time for student incubation of ideas (e.g. Levenson, 2013; Savić, El Turkey, et al., 2017b), demonstrating a variety of ways to solve problems and illustrating the mathematical process (e.g., brainstorming, false starts, impasses) for students (Levenson, 2013; Satyam et al., 2022), making the classroom a safe place for students to take mathematical risks (Levenson, 2013, Satyam et al., 2022; Savić, El Turkey, et al., 2017b), attending to the emotions of students (Satyam et al., 2022), and providing space for classroom peer-to-peer discussion and disagreement (Satyam et al., 2022; Savić, El Turkey, et al., 2017a, 2017b).

Although some of the above research explores instructional methods utilizing collaboration, such as project-based learning or classroom discussion and discourse, none of this extant research explicitly investigated the effect incorporating collaboration on mathematical tasks has on creativity. Further, even of those studies conducted in an undergraduate setting, the representation of research in proof-oriented courses is sparse (e.g., Savić, El Turkey, et al., 2017a, 2017b; Savić, Karakok, et al., 2017). In order to inform instruction to foster creativity in transition-to-proof courses, particularly those taught with a focus on collaborative proving, it is important to understand how students perceive collaborative course activities as fostering their creativity. Therefore, more research in this specific context (transition to proof) with attention to collaborative proving as an intentional instructional choice is needed.

3. Theoretical Framing

3.1 Phenomenological Perspective

In this study, I adopt a phenomenological perspective, as I aim to gain a deeper understanding of the nature or meaning of everyday experiences in mathematical creativity. Phenomenology is the study of lived experiences (van Manen, 1990), and it “does not offer us

the possibility of effective theory with which we can now explain and/or control the world, but rather it offers us the possibility of plausible insights that bring us in more direct contact with the world" (van Manen, 1990, p. 10). Although the goal of phenomenological research is to understand individuals' lived experiences, it accepts that true introspection is impossible and that all phenomenological reflection is not introspective, but *retrospective*. The goal of the researcher in a phenomenological approach is to describe as accurately as possible the phenomenon, "refraining from any pre-given framework" (Abakpa et al., 2017). The phenomenological perspective shaped data collection and analysis in this study, namely the analysis of student retrospective writing assignments through open, inductive analysis.

3.2 Defining Mathematical Creativity

Creativity has been difficult to define across domains, including mathematical creativity. Mann (2005) claimed there are over 100 existing definitions of creativity in the mathematics education literature, and this number has only grown with the volume of research on creativity published in recent years (Savić et al., 2022). In this study, the definition of mathematical creativity is built on three major assumptions about the nature of creativity. First, creativity is *domain-specific*, meaning I am interested in creativity specific to the context of mathematics and mathematical proof. Second, mathematical creativity is *a process*, meaning I will consider the motivation, perception, learning, thinking, and communicating involved in creativity rather than analyze an end result or product of creative thinking. Finally, mathematical creativity is *relative*, meaning I am considering creativity within the context of the knowledge, abilities, and experiences of an individual rather than requiring creativity to produce completely novel contributions to the field of mathematics. Furthermore, I assume every student can be mathematically creative, which contrasts with the "genius" view of mathematical creativity

(Silver, 1997). Put succinctly, ***mathematical creativity*** is defined to be the processes of creating, constructing, or implementing mathematical ideas, strategies, or processes, which are perceived as non-routine by the individual. In a study with similar theoretical assumptions, Satyam et al. (2022) wrote,

> In keeping with the phenomenological approach, we view being and feeling creative as equivalent under our lens and as such is the student's internal judgement. This is inspired by asset-based approaches in mathematics education (e.g. Adiredja & Louie, 2020) and a positivist approach in psychological creativity research that 'creative potential exists within the individual' (Corazza & Glăveanu, 2020, p.82). (p. 153)

Taking a phenomenological perspective and conceptualizing mathematical creativity as relative require inquiry into student perceptions of their mathematical creativity. These theoretical frames shaped methodological decisions throughout this study and, as in Satyam et al. (2022), I equate being and feeling creative and determine this has occurred according to students' descriptions of their experiences.

3.2 Defining Inquiry-Oriented, Collaborative Learning

Inquiry-based learning has gained in popularity under a large umbrella of pedagogies coined *active learning* in undergraduate mathematics instruction, yet there is no clear agreed-upon definition of what it means to teach through inquiry (Cook et al., 2016). Techniques used in inquiry-based learning (IBL; e.g., Laursen et al., 2011) or inquiry-oriented instruction (IOI; Rasmussen & Kwon, 2007) vary from instructor to instructor and may include group work, think-pair-share, student presentations, worksheets, formative assessment response systems (e.g., clickers), project-based learning, and many other teaching techniques. In a review of recent uses of the phrase "inquiry," Cook et al. (2016) identified six major themes of courses taught through inquiry methods: student ownership of knowledge, new knowledge building on existing knowledge, students participating in mathematics, importance of the student-instructor

relationship, importance of student-to-student interaction, and increased student success. These themes, however, can be implemented in the classroom in a variety of ways, with or without an emphasis on small group peer-to-peer interaction. For this reason, I adopt the definition of inquiry-based learning described by Yoshinobu and Jones (2012):

> Inquiry-based learning (IBL) is an instructional paradigm in which the students interact, under the guidance of the instructor, with a curriculum typically consisting of a set of axioms, definitions, and problems. The core idea is that students are engaged in an apprenticeship into the practice of mathematics. Students actively participate in contributing their mathematical ideas to solve problems, rather than applying teacher-demonstrated techniques to similar exercises. The instructor facilitates student progress, ensuring that the class can move toward increasingly sophisticated ways of thinking. A key feature is that students present their ideas to small groups or to the entire class, and classmates peer review their work. Students construct their own ideas and evaluate the mathematics of their classmates for themselves, rather than rely on an external authority. In short, students do mathematics like research mathematicians do mathematics. (p. 307)

Thus, central to an ITP course taught through inquiry-oriented, collaborative instruction is *student interaction*, *instructor guidance*, *students contributing their mathematical ideas to solve problems,* and *instructor facilitation of student progress*. I specifically describe the classrooms I am interested in in this study as also emphasizing student collaboration, which entails *students working in small groups to achieve a common goal or reach mutual understanding* (Smith & MacGregor, 1992), as well as *students presenting ideas to the entire class* and *classmates peer reviewing each other's work.*

3.3 Residue

Defining mathematical creativity as a relative construct requires asking students what they recall as fostering creativity for them, and these recollections will describe "what actually comes to the fore of [students'] attention" (Marton et al., 2004) regarding their creativity during collaborative proving as well as other activities in the ITP course. Similar to Marton et al. (2004) in their description of *space of learning*, or interest in effectively arranging for learning to occur,

in this study I seek to understand the features of an ITP course that foster creativity or effectively arrange for creativity to occur. To understand these course contexts and experiences that allow students to experience creativity, I draw upon the framework of *residue* described by Hiebert et al. (1996). Hiebert et al. (1996) described residue as a way of "thinking about the understandings and skills that individual students take with them from classroom experiences" (p. 17-18). As noted by Bleiler-Baxter and Pair (2017), Hiebert et al. (1996) outlined the concept of residue from a cognitive perspective, designed to capture residue with respect to "insights into structure," "strategies for solving problems," and "dispositions toward mathematics." For the purpose of this study, I reframe residue to capture not only cognitive residue but also residue of experiences more broadly. In this way, residue will also capture the *contexts* and elements of the course activities students remember as impactful for their creative experiences. Residue for students with respect to creativity can give insight into the experiences, activities, and interactions that were meaningful for the student and inform how instructors can best shape classroom environments and activities to foster collaborative creativity in proving.

4. Methods

4.1 Study Context

This study was conducted as a portion of a larger study on collaboration and creativity in proving. This study occurred in the context of a semester-long introduction-to-proof course at a large public university in the southeastern United States. This course was taught using collaborative, inquiry-oriented methods, and students regularly engaged in collaborative proving as well as whole-class discussions. Eleven students were enrolled in the course, and 10 consented to participate in the research study. In addition to class meetings, students also completed 10

problem sets (as homework), 13 retrospective writing prompts (RWs), and took three in-class exams. In the following sections, I provide a detailed description of each of these activities.

4.1.1 Class Meetings

Each class session, students were assigned to groups of three or four students; groups changed frequently throughout the semester, usually every two to three class sessions. Typical class days involved a short introduction to the day's lesson, often including student reflections on the previous session or problem set or reminding students of relevant definitions or properties, then student groups were assigned collaborative proving tasks and worked together to complete their task. At the conclusion of collaborative work, the instructor typically facilitated group presentations and a discussion of the task. All class meetings were captured using the university lecture-capture software (audio, video, and screen capture) and posted for students to view on the course learning management system.

4.1.2 Problem Sets and Proving Journal

Throughout the semester, students were assigned 10 problem sets as out-of-class work. Course notes and problem sets were adapted from Taylor's (2007) *Introduction-to-proof* published in the Journal of Inquiry-Based Learning. Students were tasked with submitting their completed problem sets in the form of a *proving journal* (Figure 3-1). In contrast with traditional problem sets in which students submit only their final proofs or solutions, in the proving journal assignment students were expected to also submit their scratch work or *process* work for each problem as well as a written reflection upon a problem they found challenging or non-routine in the problem set. In the written reflections, students described their process for arriving at a solution, strategies that helped, impasses encountered, time spent on their different approaches,

and moments of illumination. Students were directed to date and label each page for the appropriate type of work (process, formal write-up, or reflection).

Figure 3-1

Problem Set and Proving Journal Assignment Description Provided in Course Syllabus

Problem Sets/Proving Journal

Throughout the semester, you'll maintain a proving journal to track your journey in solving problems from the course problem sets. This journal will help you plan, work on, and reflect upon the problems from the class assignments. Set up the journal using a 1.5-inch loose-leaf ring binder with paper and bring it to every class. **Remember to date each page.**

You can access the class problem sets on D2L. Print out each problem set and insert it into your proving journal. For "Exercises," write your responses directly on the printed copy. When dealing with a "Problem" that requires a proof, make sure your proving journal includes these two things on separate pages, clearly labeled with the problem number:

1. **Your Problem-Solving Process:** Keep a record of your attempts, including examples and false starts. Don't erase your work; you can cross things out lightly so the original attempts are still visible. Label these pages with a **P** for “process” in the upper-right corner.
2. **Formal Proof/Solution Write-up:** Present your final solution or proof separately. Mark these pages with a **W** for “write-up” in the upper-right corner.

For each problem set, choose one challenging or non-routine "Problem." Write a reflection on the problem, your proof, and the process you went through to arrive at a solution. Talk about strategies that helped, dead-ends you encountered, time spent on different approaches, and when new ideas clicked. Clearly mention the problem you're reflecting on and use **R** for “Reflection” in the upper-right corner of these pages. All reflections should be typed, double-spaced, and 1-3 pages in length.

The decision to collect out-of-class work in the form of a proving journal was motivated both by pedagogy and research. Pedagogically, the instructor sought a way to help students distinguish their informal process work when developing a proof from their formal proving. Recent research has indicated mathematicians and mathematics educators view example use as an important intuition-building and exploratory activity in proving, but do not think it is appropriate for a formal proof (Kirby, 2023). Moreover, the reflection component of the proving journal was added to promote metacognition and allow students to note habits and strategies they

found more, or less, effective in their proving. The proving journal also provided a wealth of data for researching students' individual proving process and their perspectives on what problems are challenging and/or non-routine. Proving journal submissions were graded based on comprehensive completion and the instructor provided thorough qualitative feedback on submitted proofs.

4.1.3 Retrospective Writing Prompts

In addition to out-of-class problem sets and proving journal assignments, students were also frequently assigned *retrospective writing* (RW) assignments. RWs asked students to recall, recount, and reflect upon the work they did in class while working collaboratively with their peers. Throughout the semester, students were assigned 12 RWs: two (one beginning-of-course and one end-of-course) regarding general experiences with mathematical creativity and collaboration, six regarding in-class collaborative proving and mathematical creativity, three regarding definition use in proving, and two regarding proof conceptions and criteria for proofs. As the RW prompts typically solicited reflection upon the in-class activities of the day, most RWs were due at 11:59pm the same day of class. Students were given five days to complete the beginning-of-course and end-of-course RWs to provide time for depth and length of reflection. Figure 3-2 depicts the end-of-course RW. RWs were graded based on completion, depth of reflection, and adherence to the given prompt.

Figure 3-2

End-of-Course Retrospective Writing (RW) Prompt

Mathematical Creativity and Collaboration
Retrospective Writing Prompt

Directions: Respond to each of the three prompts below. Type your responses and follow the length requirements specified below.

Reflect upon what you have learned and experienced throughout the Math 3460 course.

A. How do you define "mathematical creativity"? *(At least 100 words)*

B. Did you feel creative in your introduction-to-proofs course? Provide as many details as possible in this response. You might review our course PowerPoint slides and/or problem sets in order to jog your memory about particular instances when you felt creative. Describe clearly and completely your recollection of this event, for example, by providing background information on (a) the problem/proof on which you (or a group of you and your peers) were working, (b) how your work on that problem made you feel creative, (c) why you think that experience engaged you most in working creatively, and (d) any other information that would help me to understand how you see yourself and your peers engaging in creativity. If you did not feel creative during our course, please state so, and explain why. Then describe an activity that could be included in next semester's Foundations course to make sure that those students have the opportunity to work creatively. Please try to think of your own examples for each category, rather than discussing with peers before completing the assignment. *(At least 500 words.)*

C. What do you think you learned by collaborating with your peers during this course? *(At least 400 words.)*

Similar to the proving journal assignments, the RW assignments also served both a pedagogical and research purpose. Pedagogically, the RW assignments provided the instructor with insight into the students' classroom experiences and their development with respect to collaboration, proving, and conceptions of proof. The course instructor would regularly use the RW responses as an opportunity to highlight experiences among students at the beginning of the class following the submission of an RW assignment. The RW assignments functioned as an extended exit ticket in which students engaged in metacognition regarding their in-class learning

that day. For research purposes, the RW assignments provided a record of student experiences throughout the course as well as their changing perceptions from the beginning and end of the course regarding their definitions of and experiences with mathematical creativity. As further described in section 4.3, the end-of-course RW responses composed the primary source of data for this study.

4.2 Study Participants

Ten of the 11 students enrolled in the course consented to participate in the research study; however, data relevant to this study were available from only seven of the participants. Table 3-1 provides a summary of the seven study participants. Participants ranged from sophomore to senior. All names used in this study are pseudonyms and participants were given the opportunity to choose their own pseudonym.

Table 3-1

Study Participants

Participant	Major/Minor	School Year
Ashley	Mathematics (Business Concentration), Minor in Computer Science	Junior
Campbell	Computer Science	Junior
Gabby	Aerospace, Minor in Mathematics	Senior
Luis	Computer Science, Minor in Mathematics	Senior
Participant B	Mathematics (Statistics Concentration)	Junior
Sadie	Mathematics (Secondary Education Concentration)	Junior
Will	Mathematics	Sophomore

4.3 Data Collection

For this study, the primary data source was participant responses to the end-of-course RW prompt (Figure 3-2). Participants were given five days following the final course meeting to complete the assignment. For this study, only responses to part (B) of the end-of-course RW were used as data. This part of the RW assignment inquired if students felt creative during their introduction-to-proof course and asked them to explain when and why they felt creative. Secondary, supporting data were consulted in seeking out elucidating examples mentioned in the RW responses. These supporting data included proving journal and other RW course artifacts and video data of in-class collaborative proving episodes.

4.4 Data Analysis

To determine what in the ITP course students recalled as fostering their creativity, the end-of-course RW data were first open coded (Saldaña, 2016) to highlight students' words describing *when*, or in what context, they felt creative in the ITP course. These contextual open codes were compared and collapsed to make one list of contexts most frequently referenced by participants describing when they felt creative throughout the ITP course. The data were then revisited and re-coded using the new list of context codes. Finally, I restricted my analysis to the context codes applied to at least 40% of the participants' end-of-course RW narratives. Within each context, I explored the details participants reported as having fostered their creativity in the context. I consulted the secondary data sources (proving journals, in-class video data, and other RW responses) to find examples of the experiences highlighted in the end-of-course RW data.

5. Findings

The research question to address in this study is: *After a semester-long introduction-to-proof course taught with collaborative, inquiry-based methods, what do students recall as*

fostering their creativity? Seven contexts were identified in participant responses to part (B) of the end-of-course retrospective writing prompt, which inquired about whether or not students felt creative during the ITP course, what activities fostered their creativity, and why they felt creative through these activities (Figure 3-2 provides the full text of this assignment). These contexts were, from most to least prevalent: working on a specific problem, using specific strategies during proving, working in small groups, interacting with the whole class, working individually on problem sets, interacting with the course instructor, and creating a course rubric for proof writing. Table 3-2 provides the details of the definitions of and prevalence of these contexts across the participant writing. As further exploration of these contexts was limited to the contexts described in at least 40% of the participants' reflections, the remainder of this section directs attention to working on a specific problem, using specific strategies during proving, working in small groups, interacting with the whole class, and working individually on problem sets. In the following sections, I describe the ways in which students described these themes as fostering their creativity for each theme. Table 3-3 provides a summary of the findings across each context.

Table 3-2

Context Definitions and Prevalence Across Participants

Context	Definition	Number (Percentage) of Participants
Creating proofs with respect to specific mathematical content	Students identify a specific problem or problem set they worked on (individually or as groups) as fostering their creativity.	7 (100%)
Using specific strategies during proving	Students describe using specific strategies during their independent or group proving. These strategies may include rewriting their conjecture in a more operable form, relating their problem back to learned definitions, previous problems, or content from other courses.	5 (71.4%)

Working in small groups	Students describe working in small groups (in or out of class) or talking about problems with peers/friends.	5 (71.4%)
Interacting with the whole class	Students describe working with the whole class on a proof. This includes whole-class discussions and viewing and critiquing other groups' proofs.	3 (42.8%)
Working individually on problem sets	Students describe how working alone on their assigned problem sets fostered their creativity.	3 (42.8%)
Interacting with course instructor	Students describe how interactions with the instructor during class fostered their creativity.	2 (28.6%)
Creating the course rubric	Students discuss activities involved in creating and refining their course proof rubric.	1 (14.3%)

Note: The contexts in the bottom two rows shaded in gray were not examined for in-depth analysis because they were not mentioned by at least 40% of participants.

Table 3-3

Summary of Study Findings

Context	How students experienced creativity in that context
Creating proofs with respect to specific mathematical content	High level of comfort with mathematical content Had not previously attempted the given task Proving task allowed student to realize utility of proving Proving task allowed student to realize multiple valid ways of proving a statement Opportunity to connect content relevant to the statement to material learned previously in ITP course Opportunity to connect content relevant to the statement to material learned previously in other courses
Using specific strategies during proving	Using examples to determine the truth value of a statement Using definitions to aid their proving Rewriting given statement to make the statement more operable Rewriting given statement to increase understanding of the claim Connecting the mathematical content of their assigned task to previously learned mathematical content
Working in small groups	Shared sense of ownership over ideas and creativity Recalling collaborative efforts in when working independently to overcome impasses Witnessing how proofs could be completed using multiple approaches
Interacting with the whole-class	Presenting and critiquing work of small-groups Students positioned as responsible for gaining their own understanding Illustrated the multitude of approaches that can be used to prove a single statement Instructor guidance, questioning, and building on student thinking
Working individually on problem sets	Overcoming individual impasses via: Consulting recordings of in-class discussions Using proof skeletons Exploring ideas through process work

Throughout the following sections, participant quotations are presented to support and illustrate the findings observed, and portions of the quotations emphasized in **bold** text are intended to highlight the *ways* in which participants experienced creativity in a given context (cf., right column of Table 3-3), rather than show where the participant mentioned the context itself (cf., left column of Table 3-3).

5.1 Creating Proofs with Respect to Specific Mathematical Content

All of the seven participants described feeling creative when working on a specific problem. This context was defined as: students identify a specific problem or problem set they worked on (individually or as groups) as fostering their creativity**.** Across the reflections that referenced creating proofs within specific mathematical content, there were two content areas that were most often cited as fostering creativity. Participants described problems relating to parity of integers and integer expressions and set theory (subsets, set equality, and set cardinality). In this section for each mathematical content relevant to the specific problems I describe the circumstances surrounding the completion of these specific problems and the reasons students said these problems fostered their creativity.

5.1.1 Parity of Integers and Integer Expressions

Across the student reflections, four participants (Ashley, Gabby, Luis, and Sadie) cited specific problems from the fourth course problem set, and these problems instructed students to prove or disprove and salvage statements regarding even and odd integers. Figure 3-3 provides an image of the problems in Problem Set #4. The two problems specifically mentioned as having fostered creativity were Problem 2.3, *if l and m are odd integers, then so is lm,* and Problem 2.5, *if p is an integer then* $p^2 + 3p + 2$ *is even.* These problems were used in a group proving activity in addition to assigned for completion in students' proving journals. Students collaboratively

proved these statements during the fifth week of the course, students were assigned a retrospective writing prompt regarding this collaborative proving activity, and the corresponding problem set/proving journal entry was due eight days after the in-class collaborative proving activity. These problems, in addition to *an integer is even if and only if its square is even*, were discussed extensively in class over two class periods. During the second of these class periods, the researcher served as substitute instructor.

Figure 3-3

Problems assigned in Problem Set #4

DIRECTIONS. In order to gain some practice using the methods discussed above, we will start with a few statements involving numerical properties that you are familiar with. **For each of the problems below, you should either (a) Prove, or (b) Disprove and Salvage If Possible.** If you disprove a statement, you are expected to "salvage" the statement, meaning that you modify the statement in some way so that it is true, and then prove the true statement.

We begin with the definition for even and odd. Recall Alcock's suggestion to attend carefully to the definition and consider the operability of the definition. Note: Definitions are given. They do not need to be proven.

Definition 2.1. An integer n is **even** if and only if there exists an integer k such that $n = 2k$. An integer n is **odd** if and only if there exists an integer k such that $n = 2k+1$.

Problem 2.2. An integer is even if and only if its square is even.

Problem 2.3. If l and m are odd integers, then so is lm.

Problem 2.4. If l and m are odd integers, then $l + m$ is even.

Problem 2.5. If p is an integer, then $p^2 + 3p + 2$ is even.

Problem 2.6. If k, l, m are integers and $k + l$ is odd and $l + m$ is odd, then $k + m$ is odd.

Problem 2.7. Suppose p and q are distinct integers. Then $(p + 1)^2 = (q + 1)^2$ if and only if $p + q = -2$.

Problem 2.8. If p and q are distinct, positive, real numbers, then $\frac{p}{q} + \frac{q}{p} > 2$.

Problem 2.9. Use the knowledge that you have obtained through this problem set to conjecture and prove a new result (i.e., different from the results you proved above).

Both Ashley and Gabby reflected specifically upon the problems they were assigned to work collaboratively on in class. Ashley attributed her feeling creative through this problem to

her comfort and familiarity working with even and odd numbers. Bold font is used to emphasize parts of Ashley's reflection related to her comfort and confidence. She wrote,

> The proof that made me feel the most creative is one that I have previously spoken about. In our class time my group and I aimed to prove that an equation (p^2 + ?p + ?) [$p^2 + 3p + 2$] was even. This problem made me feel creative because I feel like I was really able to think this problem through. I feel like **I have a very good understanding** of even and odd numbers, so **I am most comfortable** working with them in problems. Since I **have a good understanding** of even and odd numbers, this proof was fun to think about **instead of frustrating**. I think the hardest part of this class is trying to prove something after only just being introduced to its definitions or concept. I have a hard time with learning something and then doing it. I normally tend to need time for this new information to stew a bit in my brain before I begin working with it. Normally I prefer to learn something and then see the teacher do examples to show how to do it. I like to follow along, and watching these examples familiarizes me with the concept and I usually end up understanding it much better after watching some examples being done. **Since, I was already comfortable with the meat of this problem**, it was much easier to think about it and talk with my peers about it. Because **we all understood the fundamentals** of this problem, we were all very engaged in solving it. This translated to us having really good discussions and eventually being able to solve the problem. Therefore, I simply love this problem and wish I could just keep solving it forever. (Ashley, End-of-course RW)

This reflection from Ashley indicates she felt she could be the most creative in the course when she was proving statements concerning content she felt confident and familiar with. In addition to this, she contrasted her previous learning experiences in more teacher-centered classrooms ("learn something and then see the teacher do examples to show how to do it") with how her comfort with the concepts of even and odd integers allowed her team to have a productive and creative collaboration ("we all understood the fundamentals," "us having really good discussions and [...] being able to solve the problem").

Although Ashley attributed this problem to fostering her creativity because she was familiar with the mathematical content, Gabby attributed the problem she referenced, *if l and m are odd integers, then so is lm,* as fostering her creativity because no one in her team had attempted the problem previously. Gabby wrote,

> Specifically, in our individual groups when my group was proving problem 2.3 from problem set #4, "if l and m are odd integers then so is lm." **My groupmates and I had not attempted the problem outside of class**, so we were all looking at the problem without looking much at it before. (Gabby, End-of-course RW)

Gabby's reflection indicates the novelty of the problem and having not yet attempted it outside of class allowed her and her team to work creatively.

Both Luis and Sadie also referenced problems from this in-class collaborative proving and problem set assignment, but Luis and Sadie were both absent from class during the collaborative proving activity. Luis referenced the same problem as Gabby, *if l and m are odd integers, then so is lm.* Luis attributed this problem as fostering his creativity because, "this problem in particular **opened my eye to the idea of proofs being actually useful** instead of being a somewhat annoying and tedious process like I once thought." Finally, Sadie attributed the problems from this problem set as fostering her creativity because they:

> Helped **show me that you can prove this in many different ways**, by using the mathematical techniques, and not only one. I always believed that you had to prove something a certain way, one way, to get a correct answer. This class has shown me that you can prove something in all different ways. (Sadie, End-of-course RW)

Thus, both Luis and Sadie highlighted how these problems involving parity allowed them to learn something about the nature of proof and its purpose. Luis stated his experience enlightened him to the utility proof can have ("proofs being actually useful"), and Sadie wrote that her experience showed her that there is not necessarily a single correct way to prove a statement ("you can prove something in all different ways").

5.1.2 Set Theory: Subsets, Set Equality, and Set Cardinality

Three of the seven participants referenced problems related to set theory, which were from the course problem sets #8 and #9. Campbell referenced the problem *If* $A \subseteq B$, *then* $B^C \subseteq A^C$ (where superscript C denotes the set complement), which was assigned for in-class

collaborative proving. Campbell was a part of the group assigned to this task in class, and the class was assigned a Retrospective Writing prompt concerning collaboration and mathematical creativity following this class. Participant B, rather, referenced the problem tasking students with proving one of DeMorgan's laws for set complement: $((A \cup B)^C = A^C \cap B^C$. Finally, Will referenced the problem: *Prove that if |S|=n, then |℘(S)| =*2^n (where ℘ denotes the power set). The problems referenced by Participant B and Will were not tasked for in-class collaborative proving, but all three problems were assigned in their problem sets/proving journal.

In class, Campbell's group was tasked with proving the statement he highlighted as fostering his creativity across the course, *If* $A \subseteq B$, *then* $B^C \subseteq A^C$. Campbell attributed this problem as having fostered his creativity to his initial uncertainty of how to approach the proof and his eventual connection between the set-theoretic operators of subset and complement and the logical operators of implication and negation. Boldface font is used to highlight Campbell's emphasis on connecting new and old knowledge. Campbell wrote,

> One specific example where I felt like I was creative was during module 5. In class we were given the problem "If $A \subseteq B$, then $B^C \subseteq A^C$. I think a big part of solving this problem was being able to **identify the relationship between set definitions and logical definitions**. When solving this proof I was the most creative when I realized that **the complement of a set is the set relation version of negation**, which is a definition used in logical statements. I recall that during solving that question, after breaking down the problem into the logical segments I was able to come up with the right answer without even knowing that I had it. I knew that I had the correct answer, when **I noticed that** *If* $B^C \subseteq A^C$ **is just the contrapositive of** $A \subseteq B$. I think that this problem made me think creatively, due to not knowing exactly how to answer the original problem but was **able to turn it into something that we had done previously**. I think that being able to **relate this question to something that I had previous knowledge of** allowed me to turn the problem into one that was operable. [...] Overall, I think that being able to create a balance between **using my past knowledge** to solve a problem directly and **using my past knowledge to link different ideas** that I have used in the past to solve a new problem to create new ideas. (Campbell, End-of-course RW)

Campbell's writing emphasizes the nature of proving something involving subsets and set complement allowed him to make connections back to his previous learning regarding logical

implication and negation, which fostered his creativity. Further, Campbell was able to generalize his realization that his proving could be aided by attempting to connect new content to content he had learned previously ("using my past knowledge to link different ideas").

Although the problem Participant B referenced in his end-of-course RW was not assigned for in-class collaborative proving, he offered a similar reflection to Campbell in why this problem fostered his creativity. Participant B also wrote about making connections between set theory and logic,

> Specifically, I felt the most creative working on the second portion of this problem. In this problem we had to prove the following statement; "$(A \cup B)^C = A^C$ Intersect B^C. When I looked at this problem, **I remembered about the definition of DeMorgan's Law that we learned about and proved all the way back in problem set #2**. The other thing that I noticed about this problem was that it also included **the equality sign, which we learned in this problem set seems to be the set theory equivalent of the biconditional statement in formal logic.** The part of this problem that made me feel the most creative was **using the knowledge that I already have about both these definitions, as well as the definitions that I learned from problem set 8**, to create my proof, and show that the statement that was given here was indeed true (Participant B, End-of-course RW).

Participant B expressed the opportunity to connect his prior knowledge from a previous problem set ("we learned about and proved all the way back in problem set #2") to his current task involving set equality fostered his creativity. Similar to Campbell, Participant B explicitly mentioned specific connections between set equality and biconditional statements ("the equality sign […] seems to be the set theory equivalent of the biconditional…").

Participant B and Campbell both reflected upon how the opportunity to connect set theory to formal logic fostered their creativity, but Will's reflection upon completing a problem involving the cardinality of a power set illustrated a different kind of connection-making. When writing about working on the problem *prove that if* $|S|=n$*, then* $|\wp(S)| = 2^n$, Will recognized this

problem as fostering his creativity because he was able to connect cardinality of the power set to the concept of the "choose" function he learned in a previous statistics course. He wrote,

> In Problem Set 9, there was a conjecture stating "If the cardinality of a set S is n, then the cardinality of the power set of the set S is 2 raised to the n." While reading this, something clicked in my head, as **I had remembered a method for statistics from the previous semester**. When working with cardinality, it is easily obtained by counting the elements in a set. Of course, with more sets it would become tedious to count how many elements, but an idea of how to make up all the elements in a power set came to mind. There is a method for counting the combinations of elements where order does not matter. This is called nCr, or "with n items, choose r items." If a set has a cardinality of 3, then there are 3 elements in it. Its power set is a set that contains all subsets of the particular set. Already this would include the **empty set (with 3 items choose 0)** and the **set itself (with 3 items choose 3)**. So also there are **3 sets that include 2 of the elements (with 3 items choose 2)** and 3 sets that include 1 of the elements (with 3 items choose 1). This, **for me after thinking and writing it out, was the sum of all nCr from r = 0 to r = n. A summation!** And although the rest of my proving was not as pretty as I had hoped, the idea that a summation of nCr was, to me at least, very creative (Will, End-of-course RW).

In contrast to Campbell and Participant B's reports of feeling creative because they were able to make connections between set theory and logic, a concept learned earlier in the introduction-to-proof course, Will reported feeling creative because he made a connection between the concepts in his ITP course to concepts learned in a statistics course. More explicitly, he recognized that he could calculate the cardinality of the power set of a set using the summation of choose functions.

5.1.3 Specific Mathematical Content: Summary

All seven participants made references to specific mathematical content in their end-of-course RW responses, and across these responses two types of mathematical content emerged as most prevalent: proofs regarding parity of integer expressions and proofs regarding subsets, set equality, and set cardinality. The participants who cited completing proofs about the parity of integer expressions explained that creating these proofs fostered their creativity because they had a high level of comfort regarding parity (Ashley), had not previously attempted these proofs (Gabby), realized utility in proving (Luis), or realized there are multiple ways to prove a single

statement (Sadie). On the other hand, the participants who said completing proofs regarding sets fostered their creativity attributed this creativity to the opportunities they had to connect the mathematical concepts involved in the proof to concepts learned previously in the course (Campbell and Participant B) or in other courses (Will).

Across the seven participants, five different specific proving tasks were referenced. Three of these tasks had been assigned for in-class collaborative proving exercises in two different class meetings, and three of the participants who mentioned these tasks were in groups assigned to the proof they referenced (Ashley, Gabby, and Campbell). For these collaborative proving exercises, students also completed RW prompts describing their in-class collaborative efforts and when they experienced creativity during this task. It is, therefore, possible these problems were particularly memorable for Ashley, Gabby, and Campbell because they not only worked on the problems in class in their collaborative groups and individually on their problem sets, but they also reflected upon their creativity on these problems within the formal write-ups of their RW prompts.

All five of the specific problems referenced were also assigned for completion in problem sets, specifically problem sets 4, 8, and 9 of the course. Yet, in their proving journals only two of the participants also chose to write about the problems they referenced in their end-of-course RW in the corresponding problem set (Ashley and Will) as the problems from the assignment they found non-routine and challenging. This indicates Ashley and Will's problems may have had a lasting impact on these students, particularly Ashley as the problem she referenced was completed much earlier in the course (problem set #4) compared to Will's (problem set #9). Moreover, the fact that other participants had not previously written about the problems they referenced in problem sets is notable, and the residue of these problems was likely derived from

the experience completing the proof rather than later writing about and reflecting upon the proof in the proving journal.

5.2 Using Specific Strategies During Proving

Five of the seven participants recalled using specific strategies in their proving when asked when during the course they felt mathematically creative. The strategies and actions students wrote about included using examples to test and/or verify the statement, referencing and using definitions, rewriting statements and mathematical expressions, and relating the given statement to previously learned mathematical content. In this section, I provide examples of student reflections that highlight each of these strategies.

Gabby's reflection illuminates many of the ways participants referenced using specific strategies in their proving as related to their creativity. In her end-of-course RW, Gabby highlighted how in the proof she felt most creative completing, she (and her team) used examples to determine the truth value of the given statement, referenced their given problem set for assistance and ideas, and used definitions to generalize their given statement. Gabby wrote:

> I felt especially creative in class when we were working with our groups on different proofs and as a whole class. Specifically, in our individual groups when my group was proving problem 2.3 from problem set #4, "if l and m are odd integers then so is lm." [...] We had started by **looking at examples to decide if we thought that the claim was true**, we came to the conclusion that we did but were currently unsure why or how we could prove it. I decided to **look through the problem set** to see if any information or exercises before the problem would nudge us in the right direction and my groupmate decided to look at other types of integers, two even or even and odd integers. What **I found was the definitions for even and odd integers**, my group member was then able to conjecture that the multiplication of two even integers would equal another even integer, and an even and odd integer would equal an even integer. **From that I had an eureka moment, we needed to be general, you cannot get more general than a given definition.** We can **use the general definition** to show that the multiplication will then produce another form of the definition by the closure property of integers under multiplication. (Gabby, End-of-Course RW)

In her writing, Gabby highlighted how examples provided her and her teammate with intuition and evidence that the claim they were given was indeed true ("looking at examples to decide if we thought that the claim was true, we came to the conclusion that we did"). Although examples convinced Gabby and her teammate the statement was true, she recognized examples alone did not prove the statement true and they needed to prove the statement true in general ("[we were] unsure why or how we could prove it," "we needed to be general"). In her attempt to help her group find a place to start in constructing a general argument, Gabby looked to her problem set, which provided definitions and notes regarding proving statements using direct proof, proof by contradiction, and proof by contrapositive. In the notes, she found the definition of even and odd (see Figure 3-3) and realized definitions could be used to generally represent the concepts in the given statement ("you cannot get more general than a given definition").

In addition to the sentiments offered in Gabby's reflection, other participants described rewriting their given statement or a mathematical expression as conducive to their creativity in proving. For example, Luis reflected upon his time individually proving the statement *if l and m are odd integers, then so is lm*:

> A lot of the problems I felt that I was the most creative in were problems from the beginning of the course. [...] In problem 2.3, we are given the statement 'if l and m are odd integers, then so is lm'. This problem implies that given two odd integers, l and m, their product will result in another odd integer. So, to prove this statement's validity, I **began by trying to convert the statement into simpler statements**. These statements helped to provide me with a better understanding of the problem. Then, **I assigned these statements to variable (i.e. A, B, C) to be able to rewrite these statements as an implication statement.** Once that was done, **I rewrote the problem as an implication statement**, which was set up as **(A ^ B) => C, where A is 'l is odd', B is 'm is odd', and C is 'lm is odd'.** Through this, I was able to understand the problem a little bit more. (Luis, End-of-Course RW)

In his reflection, Luis recalled how manipulating the given statement implication and breaking it into component parts allowed him to achieve a better understanding of the problem. Other

participants also wrote about how rewriting their given statement allowed them to be creative with their proving. Will wrote,

> Most of my personal creativity has come from working alone on Problem Sets, though. Digging through a Problem Set for an hour requires a lot of thinking, most of which is putting together a viable method to prove or disprove a conjecture within the Set. With what has been learned, definitions and proof methods, a lot of creativity is evoked for everything used in between learned definitions and proof methods. **Knowing how to reorient a conjecture's parts so that it can be more easily operated on, or substituting sections to make it easier to understand or read is creative.** In any conjecture **where a definition can be used to rewrite an expression** it is certainly creative in that it uniquely **redefines the conjecture into a more operable form.** (Will, End-of-Course RW)

In addition to acknowledging how rewriting a statement or mathematical expression allowed him to be creative ("reorient a conjecture's parts," "substituting sections to make it easier to understand," "rewrite an expression," "redefines the conjecture into a more operable form"), Will also referenced other strategies (highlighted above by Gabby's reflection). Will also recognized using definitions and proof methods (e.g., direct proof, proof by contradiction, proof by contrapositive, proof by mathematical induction) fostered creativity in the course ("a lot of creativity is evoked for everything used in between learned definitions and proof methods").

A final strategy participants referenced as allowing them to be creative in their proving was to connect the mathematical content of their given statement to mathematical content previously learned, either in the introduction-to-proof course or in other mathematics courses. As described in section 5.1.2, Campbell, Participant B, and Will all wrote about how specific proving tasks allowed them to make connections to other mathematics content previously learned. In addition to the evidence provided in section 5.1.2, Campbell also generalized his experience in the specific problem he referenced to working on other problems in the course. He wrote,

> By being able to see that not every question will be able to be solved straight on, **I then approached each of the next problems that we were given by trying to relate it to another question that I had done in the past before**. Looking back, I learned that being creative in solving problems was not just applying what I already know but manipulating the problem into an easier to digest state and connecting different ideas and using what I already know to figure out something new. (Campbell, End-of-course RW)

This quote illustrates how Campbell perceived himself as creative when he could connect the problem he was working on to what he had experienced and learned previously, regardless of the specific mathematical content of the problem.

5.2.1 Using Specific Strategies in Proving: Summary

In summary, participants recalled feeling creative in the introduction-to-proof course when they used examples to determine the truth value of a statement, used definitions to aid their proving, rewrote their given statement to make the statement more operable or increase their understanding of the claim, and connected the mathematical content of their assigned task to previously learned mathematical content. Participants recalled using these specific strategies in both their independent and collaborative proving, and there were no explicit mentions of the relationship between using these strategies and the context of working alone or in groups; however, many of these strategies were frequently observed and reflected upon in the in-class collaborative proving episodes (see Manuscript 2) as well as in the proving journals submitted by students throughout the semester. Moreover, throughout the course, the class regularly discussed the criteria for an argument to be a proof, and at certain points in the semester the class discussed how examples can contribute to the proving process as well as the criteria for a proof to build on previous knowledge (definitions and mathematical content). The criteria for proof, however, never included rewriting the statement, but students were often encouraged in class to consider how they might write a statement in a more operable form (for an example, reference Episode 3 in Manuscript 1). Thus, of all the specific strategies and approaches students used in their

proving in the course, participants highlighted that they experienced creativity when they proved statements by verifying the truth value of a statement using examples, using definitions, rewriting their given statement, and connecting the relevant mathematical content to their previous knowledge.

5.3 Working in Small Groups

Across the seven participants, five referenced working in small groups on proof-related tasks as a context in which they felt creative during the course. Across the reflections identified within this context, however, participants did not provide many details as to *why* working in small groups fostered their creativity. For example, Campbell only mentioned in-class collaborative proving as the context for working on a problem he felt particularly creative completing, "One specific example where I felt like I was creative was during module 5. In class we were given the problem "If A ⊆ B then B^C ⊆ A^C"."

Like Campbell, most of the other references to working in small, collaborative groups on proving tasks were referring to *in-class* collaborations with their designated group of peers. For example, Gabby wrote, "I felt especially creative in class when we were working with our groups on different proofs and as a whole class." Gabby, however, elaborated on how working in her small group fostered her creativity,

> Overall, problem solving in general makes me feel creative when working in a group or individually. **If I am not the one to have the original moment of clarity of the proof, when my group members are walking me through their thought process and the proof becomes clearer to me makes me feel creative as well.** (Gabby, End-of-Course RW)

Gabby's reflection provides insight that she not only felt creative when she could provide a "moment of clarity" to her proof for her team, but she also felt creative when she understood the ideas of her group mates and how their ideas could further their proving.

In addition to how working in collaborative groups during class fostered creativity for the participants, Ashley also described how collaborating in class helped her with individual proving. Ashley wrote,

> This was a very different class than what I am used to. I have never had a class that has been structured like this one. In all of my other classes to date, the majority of class is spent with the teacher lecturing and us listening. However, in this class, we spent most of our time working together to solve problems and coming to conjectures together. [...] **This process made us be creative in order to complete our problems in class and our problem sets outside of class.** I recall doing problem sets and simply getting stuck. In these cases, I would replay classes and listen to the professor and my peers talk about the topic. **Being able to rewatch our collaborations allowed me to recall everything that we had worked on and help guide myself through the proof.**

Ashley contrasted her experience in other mathematics courses ("spent with the teacher lecturing and us listening") to her experience in the inquiry-based, collaborative, ITP course ("we spent most of our time working together to solve problems and coming to conjectures together"). Ashley attributed the time spent working collaboratively to conjecture and prove statements as fostering creativity both in their collaborative groups and individually completing problem sets ("made us be creative in order to complete our problems in class and our problem sets outside of class"). Moreover, Ashley specifically referenced how revisiting and remembering (via class recordings posted on the course learning management system) what happened during in-class collaborations would help her face impasses in her individual proving.

Although Will cited primarily feeling creative in the course when working individually on his problem sets, "This introduction-to-proofs course has allowed a lot of creativeness to be elicited, both in my alone time and in groups. Most of my personal creativity has come from working alone on Problem Sets, though." He also mentioned working collaboratively with a friend outside of class as having fostered his creativity. Will wrote, "There are many instances as well where me and a friend talked about certain conjectures and came up with different methods

for solving it, coming to the same conclusion." It is unclear whether the friend Will referenced was a classmate from the course, but his comment does provide evidence he experienced feeling creative by discussing problems with others and seeing how different strategies or proving approaches could elicit the same valid conclusion.

5.3.1 Working in Small Groups: Summary

In their end-of-course RWs, five participants mentioned working in small groups as fostering their creativity in the course. Although not enough details were offered regarding why working in small groups fostered creativity to note general sentiments offered by the participants, there are examples available of how working in small groups promoted a shared sense of ownership over ideas and creativity (Gabby), helped with overcoming individual impasses (Ashley), or illustrated how proofs could be completed using multiple approaches (Will).

Although the reasons participants found working in small groups to foster their creativity were not expanded upon in much detail in the end-of-course RW, it is notable most of the participants did reference working in small groups as influential to their creativity. Only one participant (Will) specifically articulated he felt more creative working alone than in groups in class. As working in small groups was nearly an everyday activity in class, it is not surprising these activities were meaningful to students. It is, however, notable that students related their in-class collaborative experiences to their individual out-of-class proving practices (Ashley and Will). More detail of how in-class collaborative proving fostered (or failed to foster) creativity in a short-term context is provided in Manuscript 2.

5.4 Interacting with the Whole Class

The previous section highlighted how participants experienced creativity in the context of working in small groups, but participants also noted whole-class discussions and collaborative

contexts in which they felt creative. Three participants (Ashley, Gabby, and Sadie) recalled feeling creative in the context of interacting with the whole class.

The primary whole-class activity these participants described as fostering their creativity was presenting, discussing, and critiquing the proofs presented by student groups following collaborative proving. Ashley wrote,

> I did really enjoy **the ends of class where we looked at everyone else's work and talked through it together.** In this way I think this class was a very collaborative experience. Everything we did we all worked towards together. To me this took a lot more brain power than just listening to a teacher lecture. I think **actually having to work for your understanding of a concept** instead of being given the entire process really makes the entire process harder but more rewarding as well. This process made us be creative in order to complete our problems in class and our problem sets outside of class. (Ashley, End-of-Course RW)

Ashley's reflection demonstrates how whole-class discussions in which they presented and critiqued proofs constructed by each student group allowed her to develop her own understanding of the mathematical content and practices of proving (e.g., "actually having to work for your understanding").

Although Ashley highlighted how interacting as a whole class positioned the students to take responsibility for building their own understanding, Sadie emphasized how whole-class discussions could be used to build upon small-group interactions and illustrate how statements can be proved using a multitude of strategies. Sadie wrote,

> One way to help students in next year's class feel more creative would be to have **students prove a certain problem in all different ways**. I am sure that we did that in class, however I think that having those conversations in small groups and then all together in the class, like we did, helps a lot with creativity.

Sadie's suggestion to incorporate more intentional activities in which students prove the same statement in different ways illustrates how moving conversations regarding proofs from small groups to the whole class can highlight different ways of thinking about a statement. Neither

Ashley nor Sadie, however, referenced a specific whole-class discussion as having been impactful on their creativity.

Gabby provided a specific example of a whole-class discussion she recalled as influential on her creativity. She wrote,

> When proving as a class, **I especially liked the pacing that [the instructor] had while walking us through the steps of proving a specific conjecture, usually [the instructor] started out with a proof skeleton to help us with a starting place but allowed us time to make suggestions or give ideas on how to move forward.** [The instructor] usually **did not just give us the answer** or if we were not understanding exactly what the next step was [the instructor] took us through the step and why we can take the step and why we are taking the step. Specifically, during our last class, **[the instructor] took over another group's proof skeleton to make sure we had time to understand the proof.**

Gabby's reflection on whole-class discussions emphasized the importance of the instructor's choices in these discussions (e.g., "[the instructor] started out with a proof skeleton," "[the instructor] usually did not just give us the answer,"). Gabby specifically referenced the final class meeting of the course as an example of how working as a whole class fostered her creativity.

This final class meeting was centered around understanding and proving statements about equivalence relations. This class meeting was carried out in the following sequence: the instructor first reminded the class of the definition of an equivalence relation, the instructor talked through a proof that a certain relation (Figure 3-4) was an equivalence relation with the students, the students worked in small groups to calculate equivalence classes for the assigned equivalence relation, the instructor facilitated student presentations of their examples, the instructor introduced two statements regarding equivalence relations (Figure 3-5), small groups worked collaboratively to prove the statement *Suppose ~ is an equivalence relation on a set S. If a ~ b for some a, b S, then [a] = [b],* and finally the instructor led a whole-class discussion of the

proof. This final portion of class, which Gabby referred to in her end-of-course RW, lasted approximately eight minutes.

Figure 3-4

Equivalence Relation Discussed in Final Course Meeting

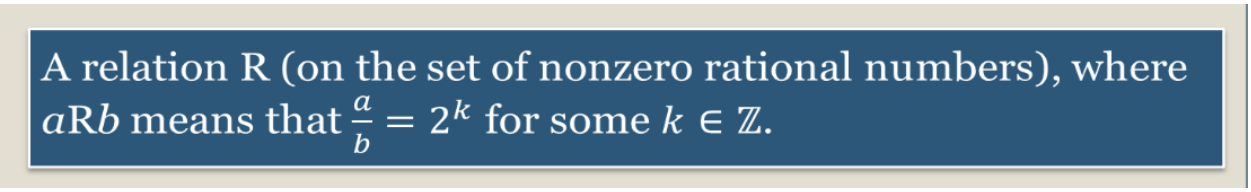
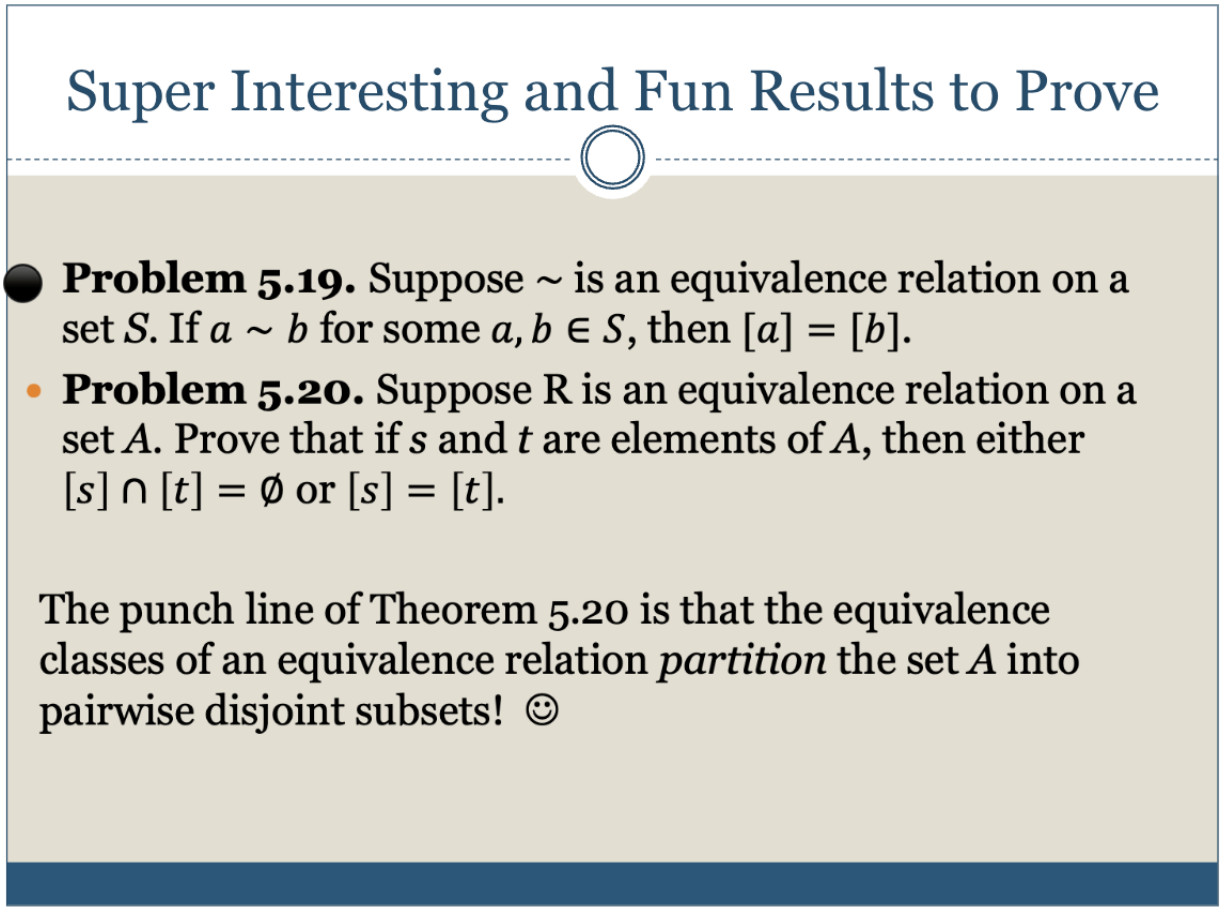


Figure 3-5

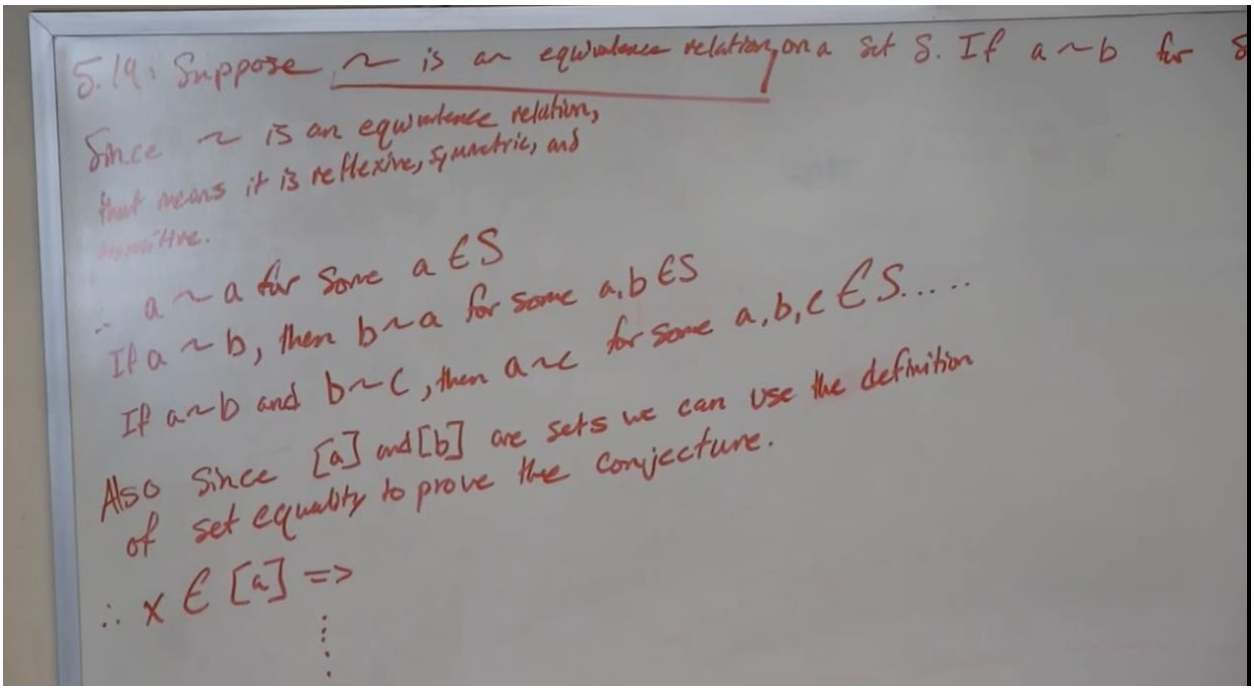
Statements Assigned to Prove during Final Course Meeting



The whole-class conversation began with the course instructor asking a student to present the work he had written on a classroom dry-erase board to begin the proof. This work is shown in Figure 3-6. The course instructor said, “You’ve done a really nice job outlining this. Do you mind sharing with us what you’ve done so far, and we’ll go from there?” This student proceeded to present his work to the class, explaining they know from the definition of equivalence relation that the relation must be reflexive, symmetric, and transitive. He concluded by saying he wanted to begin by assuming x[a] and eventually showing this would imply x[b], but he said “However, I do not know where to go from here.”

Figure 3-6

Student Work from Small-Group Proving Attempt



The course instructor then approached the board with the student work and began adding to it as she facilitated a whole-class discussion of the problem. She began the discussion by

saying “All right, I really, really, really, *really* love this” and affirming the students’ work to set up the problem in the form of a proof skeleton. She then framed their discussion as a whole-class effort by saying, “I really like how [the student] laid out our skeleton. [...] All right, well, **let's just kind of play around with it**.” The instructor proceeded to work from the students’ existing proof skeleton by working both forward (from the assumption x[a]) and backward (from the conclusion x[b]). She asked the class probing questions like, “If x is an element of the equivalence class of a, what do we know about it?,” “Now, what else do we know?,” “What are we assuming in the conjecture?” When students made an incorrect or premature assertion, rather than correct the student's thinking herself, she said to the class, “I don't think so. What's the problem there?” Eventually through this whole-class effort to prove the statement, guided by the course instructor, the class reached a complete proof for part of the claim, showing [a][b].

In her end-of-course RW, Gabby specifically noted how the instructor’s specific actions fostered her creativity through the whole-class discussion.

> [The instructor] began by moving down from step one and up from the last step using an arbitrary element as some suggested, from that point **[the instructor] suggested that the assumptions from the claim might give us a direction, this gave me that creative realization**, we could use symmetry and transitivity assumptions to manipulate the given assumptions that are more obvious in the claim. The problem was a lot simpler than it had originally looked. (Gabby, End-of-course RW)

Gabby’s reflection indicates this specific moment when the instructor asked students, “And what else is known in the... what are we assuming in the conjecture?,” she was given a “creative realization” of how to complete the proof.

5.4.1 Interacting with the Whole Class: Summary

Three participants referred to working on proving tasks as a whole class in their end-of-course RWs as having fostered their mathematical creativity throughout the course. These participants expressed the whole-class discussions both promoted long-term development of

mathematical creativity by positioning students as responsible for gaining their own understanding and illustrating the multitude of approaches that can be used to prove a single statement as well as short-term creativity elicited through specific instructor guidance and questioning through the whole-class discussions.

Only one of the participants (Gabby) referenced a specific whole-class discussion she recalled as fostering her creativity. In the whole-class proving episode Gabby described, the instructor guided the class through a proof by building on a student's existing proof skeleton and asking the students questions to require them to consider the assumptions being made in the statement and the utility of those assumptions. Gabby heavily emphasized the role of the instructor when working as a whole class on proving tasks. She noted the instructor did not give students the answer but rather allowed the students time to make suggestions on how to proceed with the proof.

5.5 Working Individually on Problem Sets

The final context at least 40% of the participants referenced feeling creative throughout the ITP course was working individually on their assigned problem sets. Three participants (Ashley, Participant B, and Will) noted how their creativity was fostered during their independent work on their problem sets. Participants said they felt creative when working independently on their problem sets because they overcame impasses on their own. Participant B and Will's reflections on problems they completed independently were discussed in section 5.1.2, however, in this section I discuss the conditions of the course context and problem set assignments that made it possible for these experiences to foster creativity, rather than the mathematical context that made their creativity possible.

Ashley noted feeling creative in the ITP course when working in small groups, working as a whole class, and working independently on her problem sets; however, her approach for overcoming impasses during her individual work was to revisit discussions that occurred during class. Ashley wrote:

> This process made us be creative in order to complete our problems in class and our problem sets outside of class. **I recall doing problem sets and simply getting stuck. In these cases, I would replay classes and listen to the professor and my peers talk about the topic.** Being able to rewatch our collaborations **allowed me to recall everything that we had worked on and help guide myself through the proof.** (Ashley, End-of-Course RW)

In her reflection, Ashley described how she would rewatch recordings of class meetings to listen to the course instructor and her peers talk about the topic of her problem set so she could work on a proof independently ("help guide myself through the proof"). After consulting Ashley's proving journals, however, there was only one mention of referencing recordings of class meetings in her 10th and final problem set submission.

During the final week of the course, there was one class meeting during which the class discussed equivalence relations and equivalence classes, but there was not enough dedicated class time to finish discussing functions, injectivity, and bijectivity. The course instructor, however, offered an optional study session during the university study day. This study session was recorded and posted to the course learning management system. Ashley, in her 10th and final problem set submission, wrote in her proving journal about a problem that was difficult and non-routine for her,

> The main topic of this problem set was applying the concepts that we have learned about relations, set relation equivalence, and briefly touching bijective/ surjective/ injective. We did not have enough time in class to cover these last topics (bijective/ surjective/ injective), **but I was able to watch the video from [the class study session] and get a better understanding of them**. Overall, I think this section of the course was so interesting because it touched on the part of math that us students have spent so much time on. For example, question 5.33 asked us about the range and codomain of a set

> relation. This was so similar to things that I and other students have spent a decade working on but never in this context. **I found it really helpful to take the professor's advice and map out the x and y values, then draw lines to show the connections of each listed pair.** This really helps the viewer look for whether or not the relation is a function, and it also makes it easier to find the range and codomain. (Ashley, Problem Set #10 Proving Journal)

This proving journal entry from Ashley illustrates one example of when she did not know how to proceed with her individual work on the assigned problem set, but she consulted recordings of in-class discussions to help her move forward. No other students in the course described referring to class recordings to assist their creativity in individual proving. Although there were no further mentions of overcoming impasses in individual work by referring back to recordings of in-class collaborations and discussions, this does not mean students did not do so, and they were not explicitly asked about their use of the recordings as a resource.

Participant B wrote in his end-of-class RW about feeling creative while working independently on problem set #8, specifically on a problem asking students to prove DeMorgan's law for set complement: $(A \cup B)^C = A^C \cap B^C$ (see section 5.1.2 for a thorough description of Participant B's reasoning for why this mathematical context fostered his creativity). When completing this problem, Participant B described he felt creative because he was able to use a proof skeleton to complete his proof. He wrote in his end-of-course RW, "Perhaps the main reason why I felt this proof made me feel mathematically creative was because **I felt as though this was the best problem that I could use a skeleton to create my proof**." Participant B, however, did not expand upon how using a proof skeleton aided his proving in his proving journal submission for problem set #8.

Finally, Will is the only participant who explicitly wrote about feeling more creative when working independently on problem sets than working collaboratively in class. Will wrote,

> Most of my personal creativity has come from working alone on Problem Sets, though. Digging through a Problem Set for an hour requires a lot of thinking, most of which is putting together a viable method to prove or disprove a conjecture within the Set. (Will, End-of-Course RW)

Specifically, Will noted how the process work, assigned to be turned in with each complete problem set as a part of the proving journal, fostered his creativity when working independently. He wrote,

> Without the need to make process work, not much creativity would happen within the Problem Sets. I recall in the first Problem Sets I would stray away from process work because I thought I knew how to solve a problem right away. Even if this were true, **process work still allows more ideas to come together by having it be posted on paper or onto a screen (for my case). Process work has helped me find creative ideas even!** (Will, End-of-Course RW)

In this reflection, Will noted how being required to complete process work in his proving journal assignments was crucial to the development of creative ideas on the course problem sets ("without the need to make process work, not much creativity would happen," "helped me find creative ideas even"). Although Will eventually recognized how process work could advance his creativity, he did acknowledge how engaging in process work was not immediately natural for him ("in the first problem sets I would stray away from process work"). He specifically noted the requirement to write down ideas during process work fostered his creativity ("allows more ideas to come together by having it be posted on paper"). In his description of how process work furthered his proving while working on a specific problem (see section 5.1.2 for a full description of this problem), Will wrote, "Without process work I would never have gone so far as to see if this method would work." Therefore, the course assignment to record process work and turn it in with the course proving journal fostered Will's creativity by creating the space for him to explore his preliminary ideas.

5.5.1 Working Independently on Problem Sets: Summary

The final way in which participants in this study expressed they felt creative throughout the course was when working independently on their assigned problem sets. Three participants noted how they felt creative in this context, and the three participants' reflection upon their independent work can be summarized by feeling creative when overcoming impasses in their individual work. These three participants each described different strategies that allowed them to work creatively on their problem sets. Ashley consulted video recordings of in-class collaborations and discussions about related problems, Participant B leaned on a proving strategy called a proof skeleton to piece together a mathematical argument, and Will explored preliminary ideas in his proving through the process work portion of the proving journal assignment.

6. Discussion and Implications

The purpose of this study was to understand what students who participated in a semester-long introduction-to-proof course taught through inquiry-oriented, collaborative instruction recalled as fostering their creativity throughout the course. Although there have been national efforts to promote collaborative and active learning strategies in undergraduate classrooms as well as develop undergraduate students into creative mathematicians capable of cooperative and collaborative work in their careers, there has been little research inquiring how collaborative, inquiry-oriented instruction during the transition to proof can foster creativity for students (Savić et al., 2022). Understanding the ways in which they feel creative in different contexts, as reported by students, can provide insight into how inquiry-oriented ITP courses can be structured to foster creativity over the duration of the course.

In order to understand the mechanisms of an inquiry-oriented, collaborative ITP course that foster creativity for students, students enrolled in a semester-long ITP course that

emphasized regular engagement in collaboration and exploration were prompted to write reflections at the end of the course describing if they felt creative during the course and what activities fostered their creativity. When students were asked if, when, and why they felt creative throughout this course, student reflective writing indicated several different activities that fostered their creativity. Namely, participants wrote about how the contexts of creating proofs of statements with specific mathematical content, using specific strategies to complete proofs, working in small groups, working as a whole class, and working individually on problem sets created circumstances that made them feel creative in the course. Asking the students what they remember from their ITP course as fostering their creativity highlighted the activities that resulted in residue for the students, meaning these contexts were most meaningful for students after their experiences, and it is more likely they will carry these experiences with them in their future learning and mathematical creativity. The findings of this study were not intended to (and would not be able to) provide causal evidence for certain activities promoting creative development, but this study can provide insight into the personally meaningful experiences of a small number of students and pave the way for continued future investigation. The findings of how students experienced creativity within different contexts are summarized in Table 3-3.

When viewing the findings of this study, it is not surprising all participants referenced specific mathematical content and a specific proving task as a context in their reflections, as this was assigned in the RW prompt, “Describe clearly and completely your recollection of this event, for example, by providing background information on (a) **the problem/proof on which you (or a group of you and your peers) were working**, (b) how your work on that problem made you feel creative…”. It is, however, notable that participants identified the same two content areas (parity of integers and set theory) and gave similar reasoning for why this content

allowed space for them to be creative. Namely, participants expressed that this content enabled them to make connections back to their previous knowledge, which aligns with Levenson's (2013) finding that elementary teachers view tasks that connect separate mathematical contexts as capable of fostering creativity and Satyam et al.'s (2022) finding that undergraduate students in Calculus I reported feeling creative because of high-cognitive tasks that require making connections. Now, however, the findings of the present study indicate undergraduate students in ITP courses can also experience creativity in this way. The ways in which participants discussed these specific tasks as having fostered their creativity does not directly align with other previous literature; none of the tasks assigned included problem posing (Yuan & Sriraman, 2011; Zazkis & Holton, 2009), model-eliciting activities or phenomenon-based learning (Asahid & Lomibao, 2020; Gilat & Amit, 2014), or multiple solution tasks (Levav-Waynberg & Leikin, 2012). The tasks identified by the participants were often open-ended (i.e., "prove or disprove and salvage") and offered a multitude of viable solution paths, which have been previously seen to foster creativity in mathematics (Luria et al., 2017; Mann, 2006; Nadjafikhah et al., 2012; Regier & Savić, 2020; Savić, 2016; Silver, 1997).

A more notable, and novel, contribution of this study, however, was the prevalence of participant reference to small-group and whole-class collaborations as contexts for feeling creative. Students were asked to reference a specific problem in their end-of-course RW, but they were not prompted to discuss collaboration in the question analyzed throughout this study, yet five of the seven study participants noted how collaborative work (both small group and whole class) positively influenced their creativity in the course. Only three student participants specifically referenced independent proving in their end-of-course reflections. Although Satyam et al. (2022) previously found students in Calculus I reported feeling creative in their course

when their instructor provided space for discussion and presentation, the present study adds to this previous finding by illustrating further the perceived impact of working in small groups in an ITP course context. For example, participants in the present study wrote about how working in small groups provided a sense of shared creativity as well as equipped them to work more creatively in their individual proving work. In addition, participants in the present study supported the findings from Satyam et al. (2022) that whole class discussions can expose students to different ways of approaching a problem and encourage mathematical behavior (i.e., conjecturing, critiquing). In contrast to Satyam et al.'s (2022) study in which instructors were participants in a professional development and asked to implement creativity-based tasks, the present study was situated in an instructional context intentionally designed around student collaboration (and not explicitly mathematical creativity). The findings from the present study indicate undergraduate students in an ITP course context may experience creativity similarly to a Calculus I context (Satyam et al., 2022). Moreover, the findings from this study illustrate instruction centered around student collaboration and inquiry can make an impression on students' creativity without explicit design centered around mathematical creativity.

Although this study offers deep insight into student experiences of mathematical creativity throughout an ITP course taught through inquiry-oriented, collaborative instruction, it is limited by the small number of participants. The frequencies with which participants expressed different activities fostered their creativity should be interpreted with caution as the sample is not large enough to draw conclusions that one activity was more effective for fostering creativity than another. Another potential limitation of this study was that the written reflections (as opposed to participant interviews) did not allow for the researcher to inquire for more detail into participant responses. Moreover, the written reflections were completed as a course assignment,

so participant responses could have been influenced by their knowledge that the instructor would be reading and assessing their responses. The written nature of the data, however, did allow participants more time to reflect upon their experiences and respond thoughtfully, compared to how an interview setting would require immediate responses and would be privy to interviewer influence.

Future research on the influence of inquiry-oriented, collaborative delivery of ITP courses on student experiences of mathematical creativity should incorporate more participants from a variety of sections of the course. Researchers should consider comparing student reported experiences of creativity in these inquiry-oriented ITP courses centered around student collaboration with other instructional approaches. For example, research could compare courses taught with an emphasis on student small-group collaboration (as in this study) with both traditional lecture-based ITP courses and active-learning (but not collaborative) ITP courses. Beyond understanding the dynamic between collaboration and mathematical creativity, this study has suggested that certain mathematical content created residue for students with respect to fostering their creativity. In this study, participants highlighted proving tasks regarding parity of integer expressions and set theory. Although not the primary data source for this study, the proving journals submitted by participants throughout the course are rich data for exploring the kinds of tasks students consider challenging and non-routine in their learning to prove. Moreover, these journals, both the process work and written reflections, can give insight into the process students engage in in their individual proving and how that work evolves throughout the duration of a course. These data will be explored further in future investigations of the individual creative proving process and its evolution throughout the transition to proof.

Considering implications for instruction, it is not the goal of this manuscript and the phenomenological perspective to make generalized claims or recommendations. However, these data provide insight into a specific group of students' experiences in an Introduction-to-Proof course taught with an emphasis on collaborative proving, and I can put forward some thoughts regarding what ITP course designed to foster creativity might look like in light of these data. Throughout this study I observed when an ITP course incorporated a variety of small group, whole class, and independent proving work, students reported feeling creative in different ways and through different activities. With respect to small group work, some participants reported small group work allowed them to appreciate how a task could be completed in multiple different ways. Thus, an instructor might consider how they can encourage groups to pursue a variety of approaches for completing their task. In whole-class interactions, students described how the opportunities to critique the arguments of other peer groups and witness different strategies groups across the class used for a task fostered their creativity. To facilitate these experiences, instructors might designate class discussion time to take inventory of the various strategies presented across groups and highlight the affordances and constraints of each. Finally, in independent work, the instructor in this study required students to submit both process work and final formal proofs with their homework assignments. One participant noted the importance of requiring "process work" in the proving journal, as it encouraged him to document his thinking and pursue routes he did not necessarily think would result in a successful proof. Instructors might follow this example and consider asking students to turn in both their process work and final proofs on independent homework assignments to encourage them to document their work and potentially explore novel proving routes. Across these course activities, participants also noted feeling creative when proving statements relevant to the parity of integer

expressions and set theory, and when implementing specific strategies during their proving. The underlying commonality of these contexts was students' opportunities to make connections to what they have learned in the ITP course or in previous settings. Therefore, these data would provide for speculation that instructors might encourage students to consider the various resources at their use (e.g., definitions, examples, theorems, previous proving tasks) and previous knowledge (from earlier in the course or from other course contexts) throughout a collaborative or independent proving task. All of these conjectures regarding an ideal ITP course to foster creativity in light of the data presented in this manuscript can provide additional directions for future research.

7. Conclusion

In this study, I described the contexts students recalled feeling creative in throughout an introduction-to-proof course taught through inquiry-oriented, collaborative methods. Participants described when and why they felt most creative throughout a course in an end-of-course retrospective writing assignment, and these data were analyzed through open and inductive coding. The findings were presented according to the different contexts students reported in their writing. These contexts included: proving results regarding specific mathematical content, using specific strategies while proving, working in small groups, interacting as a whole class, and working individually on problem sets. Each of these activities were described in detail, referencing both participant reflections upon these activities as well as secondary data sources to provide examples of and context for the activities described. All study participants reported they felt creative throughout the course, and the primary contribution of this study is that undergraduate students may experience creativity in the ITP course context similarly to a Calculus I context, including the perspective that tasks allowing or requiring students to connect

different mathematical concepts fosters their creativity and allowing for discussion and collaboration can foster creativity. Future research may more explicitly compare student reflections from collaboration-centered courses to those from more traditional lecture-oriented courses to understand the influence collaboration can have on creative growth throughout an ITP course.

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Conclusion

In this dissertation, I set out to understand how students in an introduction-to-proof course experience creativity in collaborative proving. Previous research in mathematics education has investigated the nature of individual mathematical creativity (e.g., Karakok et al., 2015; Satyam et al., 2022; Schindler & Lilienthal, 2020; Sriraman, 2004; Prusak, 2015), what teacher actions or instructional methods can foster mathematical creativity (e.g., Asahid & Lomibao, 2020; Leikin & Elgrably, 2020; Levav-Waynberg & Leikin, 2012; Levenson, 2013; Satyam et al., 2022; Savić, El Turkey, et al., 2017a, 2017b), and the impacts of teaching to promote mathematical creativity (Regier & Savić, 2020; Regier et al., 2021; Tang et al., 2022). Yet, very few studies have explored the process and nature of *collaborative* mathematical creativity. Aljarrah (2020) and Schindler and Lilienthal (2022) are two of the very few studies in this area that specifically investigated collaborative mathematical creativity as a process, wherein the collaborations were small-group student-to-student interactions. Both of these studies, however, took an etic, or outsider, perspective to describing the nature of collaborative mathematical creativity.

Thus, the three studies composing this dissertation contribute to the field of research on mathematical creativity by offering three different perspectives (from the three manuscripts) on the nature of collaborative creativity in mathematical proving based on the *student experience and perception.* By using an emic, or insider, perspective in this research the results of these three studies provide insight to practitioners and researchers alike into how a small number of students enrolled in an introduction-to-proof course experienced collaborative creativity in proving.

In the duration of this conclusion chapter, I provide summaries of each of the three manuscripts composing this dissertation, then highlight connections among the results of the three studies. Finally, I offer some personal reflections on my experience conducting these studies and make recommendations for future research in the domain of undergraduate collaborative mathematical creativity.

Introduction

In the first chapter, I introduced the motivation for this study. This motivation can be summarized by (1) the need to develop undergraduate students into creative and collaborative mathematicians during their transition to proof, (2) a lack of focus in extant research on mathematical creativity on collaboration, and (3) the need to consult the *student perspective* in describing collaborative creativity in mathematics, specifically in proving. In this chapter, I provided guiding definitions for three constructs central to this dissertation: mathematical creativity, collaboration, and proof. I further situated myself in the study by disclosing my high-level theoretical and ontological perspective of interpretivism. Finally, I outlined the three research questions to which I responded throughout the dissertation in the form of three corresponding research manuscripts.

Manuscript 1: Critical Moments in Creative Collaborative Proving

Manuscript 1 described the first research study I engaged in during this dissertation. The purpose of this study was to describe how interactions within a small group facilitate the emergence of critical moments to the group's creativity during collaborative proving. Framed by participatory intersubjectivity, this manuscript aimed to bring together the perspectives and experiences of three individual group members to achieve a holistic view of the collaborative creative process.

To address the goal of describing interactions that allow creative critical moments to occur, data were collected from a single focus group across three episodes of in-class collaborative proving. In-class collaborative proving episodes were video recorded, and then group stimulated-recall interviews (SRIs) were conducted in the days following the in-class episodes. In the SRIs, the three participants watched the video recording of their in-class proving and were instructed to pause the video when they witnessed moments related to their creativity. At the conclusion of the SRIs, participants were asked what moment or moments during the recording were most critical to their creativity. Data from these SRIs were analyzed using an abductive strategy, working backwards from the critical moments during the collaborative proving to retrace and uncover the preceding events and interactions that allowed the critical moment to occur.

The findings from this study highlighted the importance of making and learning from mistakes, diverging from a fellow teammate's proving approach, and recalling and referencing previously learned mathematical content in creative collaborative proving. The major observations from this study include that a group does not necessarily follow a linear process during creative collaborative proving, and the environment of a real classroom (as opposed to an interview or laboratory setting) including interactions with other student groups and the course instructor, can be incredibly influential in directing a collaborative group's proving. For example, in this manuscript, it was observed how the course instructor's direction to not erase a mistake from the board spurred the group forward in learning from and capitalizing upon their error. On the other hand, it was also illustrated how the external pressure of witnessing other student groups conclude their efforts deterred the focus group from reconsidering their work despite one of their teammates expressing he believed their work was incorrect.

Manuscript 2: In-the-Moment Experiences of Creativity in Collaborative Proving

The purpose of Manuscript 2 was to understand how individual students experienced creativity during in-class collaborative proving in-the-moment. To do this, data in the form of written reflections were collected from ten students enrolled in an introduction-to-proof course at four points throughout a semester. Each written reflection was collected immediately (within 24 hours) following an in-class collaborative proving activity, and in these reflections participants were asked to describe if, when, and why they felt they were or their team was creative during the collaboration. These written reflections were first analyzed, sentence by sentence, to determine if the sentence was describing feeling creative or not feeling creative. Then, the responses were further analyzed using an inductive, in-vivo coding strategy. The in-vivo codes underwent a thematic grouping process and eventually yielded two overarching categories and seven themes.

The findings of this study were presented by highlighting, for each of the seven themes, the ways in which participants described feeling creative or not feeling creative during their collaborative proving activities. In the category of *Behaviors, Actions, and/or Moments*, participants described making connections, taking risks, collaborating and communicating, and creating ideas as relevant to their feeling (or not feeling) creative. In the *Circumstances* category, participants referenced how they felt, the task, and the environment when discussing feeling or not feeling creative. The existence and prevalence of these themes among participant reflections highlighted how experiences in *collaborative* creative proving might align with experiences in individual mathematical creativity. The findings also suggested, however, some ways in which the experiences of collaborative creativity in proving may be distinct from existing models for individual creativity. For example, participants had mixed experiences regarding seeking and

receiving help during their collaborative proving, and this suggests *social* risk taking may be more nuanced and context-dependent than mathematical risk taking in how it is perceived as aiding in creativity.

Manuscript 3: Student Experiences of Creativity in a Collaborative, Inquiry-Oriented Introduction-to-Proof Course

The third and final manuscript presented in this dissertation aimed to capture how students experienced creativity more broadly throughout their introduction-to-proof course, which regularly engaged them in peer small group collaborative proving. To investigate what activities fostered creativity for students throughout the course, an end-of-course retrospective writing prompt was collected for seven student participants and these responses were inductively analyzed to identify the primary contexts participants described when recalling how they were creative throughout the course. Then, within each context I unpacked the features of the context participants described as having allowed them to be creative.

The contexts commonly referenced by participants as having fostered their creativity during the course were: creating proofs with specific mathematical content, using specific proving strategies, working in small groups, interacting with the whole class, and working individually on problem sets. Across these contexts, it was evident participants' creativity was fostered through opportunities to connect their proving back to prior learning and pursuing or witnessing multiple ways to approach a single proof. Five of the seven participants referenced small-group or whole-class interactions as context in which they felt creative, and only one participant expressed he felt more creative working independently than with others. Within these collaborative contexts, participants felt creative through a shared sense of ownership over

creative ideas, a multiplicity of ideas and proving approaches, feeling responsible for one's understanding and learning, and presentation and critique of one another's ideas.

Connections Among the Three Manuscripts

The three studies presented in this dissertation all examined the phenomenon of collaborative creativity in proving, yet each study examined this phenomenon at a different level. Beginning in Manuscript 1: *Critical Moments in Creative Collaborative Proving,* I examined the creative processes a group of three students engaged in during three different collaborative proving episodes. In this first study, I examined collaborative creativity in proving through a lens of intersubjectivity, viewing the group of students as both a whole unit as well as each student as a component part. The second manuscript, *In-the-Moment Experiences of Creativity in Collaborative Proving,* focused upon only the individual experience of creativity through collaborative proving and examined student reflections of how they felt, or did not feel, creative in their in-class collaborative proving activities. The third and final manuscript, *Student Experiences of Creativity in a Collaborative, Inquiry-Oriented Introduction-to-Proof Course,* de-emphasized the daily collaborative proving course activities, and rather "zoomed out" to examine what contexts students recalled throughout their course as having fostered their creativity and what within each of those contexts allowed them to feel creative. As the course was taught through near-daily collaborative activities, this study allowed me to examine how students perceive the role of their collaborative efforts on their creativity compared to other course activities.

Because each of these studies was conducted with a different perspective, the findings of each are distinct, but not contradictory. One major theme across the three studies is the importance of making connections to previously learned mathematics content. In Manuscript 1,

this was highlighted in both Episode 2, when Ashley proposed connecting their proof to the definition of even and odd numbers, and Episode 3, when the group recalled the concept of vacuous truth learned earlier in the course. In Manuscript 2, the Making Connections theme was the most prevalent across all of the participants RW responses with respect to also feeling creative, and participants noted they felt creative during collaborative proving when they drew upon course definitions, theorems, or used relevant examples, rewrote their given statement using different mathematical language or representations, and experienced personally meaningful revelations. Finally, in Manuscript 3, the importance of having opportunities to make connections back to previous learning was emphasized in the context of proving statements about parity of integer expressions and set theory.

Although it has been previously noted how making connections is relevant to individual creativity in proving (Savić et al., 2017), re-contextualizing the idea of making connections across these three manuscripts can highlight how a collaborative setting alters how students may attempt to make connections in their proving. For example, the findings in Manuscript 3 illustrated students recall, even at the end of the course, problems from early in the course they worked collaboratively on that allowed them to connect their proving to given definitions or previous learning. Manuscript 2 then highlighted the multitude of ways students might make connections during their collaborative proving, many of which were not highlighted in Manuscript 3 (e.g., drawing upon examples to better understand the given statement) as well as how student can feel they were not creative if their connections were limited to only relying on examples or definitions or perceiving their connections as routine. Then, looking to Manuscript 1, particularly episodes 2 and 3, illustrates how individual pursuits of making connections can or cannot be taken up by the entire group. For example, in Episode 2, Ashley had to diverge from

the work her group had attempted up to a certain point to confidently posit there was an alternative proving route using the definition of even and odd numbers. In Episode 3, however, Brad Junior attempted several times to help the group connect their work back to logic and the concept of vacuous truth, but his inability to express this concept using the words "vacuously true" as well as external perceived pressures of a time constraint led the group to not take up his suggestion until Dr. Hadley affirmed Brad Junior's suspicion that something was incorrect with their work.

Although the clearest connection across the three manuscripts is regarding opportunities to make connections throughout collaborative proving, there are also other corresponding themes, particularly among the findings of Manuscript 2 and Manuscript 3. These corresponding themes are highlighted in Table 5-1, wherein I have expanded upon the seven themes presented in Manuscript 2 and connected these to relevant findings in Manuscripts 3 and 1.

Table 4-1

Connecting Across the Three Manuscripts

	Manuscript 2 Theme	Relevant Manuscript 3 Finding	Manuscript 1 Example
Behaviors, Actions, and/or Moments	*Making Connections*	Students described feeling creative when they rewrote a conjecture in a more operable form, or related their problem back to learned definitions, previous problems, or content from other courses.	Ashley used the definition of even/odd to move her group's proof forward (Episode 2). Brad Junior connected the logic of their statement to vacuous truth (Episode 3).
	Taking Risks	"Process work" encouraged students, when working individually, to pursue ideas they were not sure would work out.	The team made an error in assuming 51 and 53 were twin primes. Recognizing this error allowed them to further generalize their claim (Episode 1). Ashley diverged from her team's original proving strategy (Episode 2).

			The team had help from Dr. Hadley but did not immediately capitalize upon her hints (Episode 3).
	Collaborating and Communicating	Students recalled feeling creative when talking with peers about problems (small group) and presenting and critiquing proofs (whole class).	The students worked together on one approach to their statement by each taking on a role (Episodes 1 and 2).
	Creating Ideas	Students felt creative when they saw or thought of multiple ways to approach a single proving task.	The team used two distinct approaches in their proof (Episode 2).
Circumstances	*How I Felt*	Certain mathematical contexts (parity of integers) allowed students to feel comfortable enough to work creatively.	Team felt "stuck" in how to approach their proof (Episode 3).
	The Task	Certain mathematical contexts (e.g., parity of integers or set theory) may foster creativity more than others.	Ashley felt confident working with even and odd numbers (Episode 2). Team was challenged by the abstract nature of the empty set (Episode 3).
	The Environment	Students recognized whole-class interactions as (positively) influential on their creativity.	The team felt pressure from other student groups finishing their work to not continue thinking critically about their own work (Episode 3).

Implications and Recommendations for Future Research

The findings across the three studies presented in this dissertation provide many implications for how researchers conceptualize collaborative creativity in proving as well as provide recommendations for students in how they engage in collaborative proving to experience creativity. Namely, students may feel creative when collaboratively proving if they seek out ways to make connections to what they have previously learned, allow themselves to make and learn from mistakes, and listen to their teammates and explore different approaches for proving a statement. Moreover, there are potential indirect implications for instruction of proof-based courses. For example, throughout the three studies, there were several examples of students

feeling creative when given a certain level of guidance and help from their instructor during their collaborative proving. Several students noted that not being immediately given an answer or immediately corrected or redirected allowed them to feel creative in their proving. These examples, however, should be further investigated with explicit attention given to instructor actions, as that was not the goal of these studies.

In addition to exploring the role of the instructor in fostering creativity through collaborative proving, future research on the topic of collaborative creativity in proving should build upon the findings of this study by adding to the number of participants across multiple sections of and instructional methods used in introduction-to-proof courses. There are also a few specific directions that were highlighted in Manuscript 2 that should be further investigated. Namely, researchers should explore the conditions under which students feel creative in risk taking, specifically social risk taking of making mistakes and receiving help. A second specific direction is to explore how to make mathematical proving tasks conducive to creativity for all members of a student group, as there were mixed reports of how struggle and/or confusion contributed to experiences of creativity during collaboration.

One observation to make regarding the work presented across the three manuscripts in this dissertation is the choice to approach research from a phenomenological, interpretivist perspective. A limitation, or delimitation, of this work is that I did not consider collaborative creativity in proving from a more critical perspective, and I did not explicitly consider the dynamics at play in collaborative groups with respect to race, gender, socioeconomics, or other factors relating to power and access. These dynamics are undoubtedly at play in the interactions I observed, and this is an area ripe for investigation in continued research on collaborative creativity in mathematics.

In line with how I have conducted this research, I strongly encourage researchers to continue exploring the phenomenon of collaborative creativity in proving through an emic perspective, consulting students and teachers about their experiences, and to continue conducting such research in the context of a real classroom. Although real classrooms pose many methodological challenges, the authentic interactions of a classroom, compared to a task-based interview or teaching experiment, highlight the complexities of students' classroom experiences and can allow for more actionable implications for students, teachers, and researchers.

Reflections Upon Dissertation Research

Planning, proposing, conducting, and writing the three studies presented in this dissertation has taken over 18 months of intense, independent research. When I set out to begin this lengthy process, I was advised by many mentors and peers to choose a dissertation topic I was passionate about, and I am grateful to have maintained my passion for understanding how collaboration interacts with creativity in undergraduate mathematical proving. Although I was prepared for the motivational challenges associated with maintaining passion for a single topic, I was not prepared for the social, emotional, and intellectual challenges of independent research. Carrying out dissertation research can feel very isolating and lonely, and it is intellectually challenging to make decisions regarding methodology, analysis, and presentation of research by yourself. I am immensely grateful to have a supportive advisor and peers who regularly engaged in scholarly conversations with me about my research. I encourage anyone engaging in independent research to surround themselves with other scholars and avoid working in isolation when possible. Research is creative work, and as evidenced throughout this dissertation, collaboration can have an incredible impact on one's work. In my continued research, I hope to work with colleagues in collaborative research, as I personally believe collaboration not only

improves the quality of my work, but also positively impacts my morale and lifts my spirits while conducting research. Finally, I am grateful to have learned from my participants throughout these studies: making mistakes can help us learn, sometimes it is necessary to diverge from an original plan, and although it is frustrating, it is also very satisfying to figure something out for yourself rather than simply being told the answer.

Conclusion

This dissertation, presented in the form of three qualitative research studies and three corresponding manuscripts, aimed to investigate how students in an introduction-to-proof course experience creativity in collaborative proving. Data in the form of in-class video recordings, student written reflections, and focus-group stimulated-recall interviews were collected over the course of a semester-long introduction-to-proof course. Each of the three manuscripts presented in this dissertation examined collaborative creativity in proving from a different perspective, but all three were framed by a phenomenological approach to research, in which the student-participant experience (an emic perspective) was highlighted, and the research was conducted in the context of a real university classroom in order to capture the complexities of in-class collaborative activities. Many of the findings presented throughout these manuscripts illustrate how collaborative creativity in proving is similar to individual creativity in proving, but others highlighted how a collaborative context may introduce unique experiences and complexities to be considered in conceptualizing creativity in proving.

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