Student Understanding of Conditional Probability

By

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ABSTRACT

With increased availability of data, the need to be statistically literate has become crucial for a person to consume news. One major branch of statistics is probability which describes the likelihood of an event to happen. This uncertain nature of probability can often cause confusion. Specifically, conditional probability has been shown to be difficult to comprehend for students.

This paper examines the importance of probability literacy. It highlights cognitive issues in learning conditional probability as well as difficulties teaching the concept. Research from the past few decades discussing student misconceptions, student understanding frameworks, and teaching strategies to overcome cognitive issues is reviewed and utilized to construct a unit plan for teaching conditional probability. The unit plan is presented in full and justified using research-based suggestions in aims to improve the teaching and learning of conditional probability.

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Chapter 1 – Introduction

In today's time, the need for statistical literacy has never been more important. We constantly take in information ranging from things like chance of rain, winning the lottery, or betting on your favorite sports team. Our lives are surrounded by data, and thus, a person should be able to make informed decisions on current issues (Bargagliotti et al., 2020). A recent event, the COVID-19 pandemic, shows just exactly how important understanding statistical information is. As coronavirus spread, our society consumed data on the rising number of cases, preventative measures, and creation of effective vaccines (Franklin, 2021). Additionally, we have access to this data and more daily in the palm of our hands with the use of smartphones. This continued growth of data analytics can ensure that most future jobs will require statistical literacy from applicants (Bargagliotti et al., 2020). As a result, "it should no longer be optional for our school-age students to develop statistical reasoning skills" (Franklin, 2021).

To ensure statistical literacy, statistics education should begin in elementary grades and continue to grow through the high school level (Bargagliotti et al., 2020). One branch of statistics is probability, and Bargagliotti et al. (2020) said, "probability is the foundation for how we make predictions in statistics." Thus, developing probabilistic reasoning is crucial to ensure statistical literacy. The focus of this thesis will highlight probabilistic thinking over statistical though they are coherent.

National Council of Teachers of Mathematics (NCTM) *Principles and Standards* (2000) also emphasized the importance of students' development of data analysis including probabilistic concepts. The need for incorporating opportunities to learn such basic concepts of probability should start in PreK through grade 2 and develop into more sophisticated content by grades 9-12 (NCTM, 2000). For example, younger students can begin to use probability informally to predict the likelihood of an event (Bargagliotti et al., 2020). The foundation of probability is understanding randomness and chance processes, yet people are poor judges in distinguishing random events (Shaughnessy, 2003). Thus, students in the 2nd grade and below should be introduced to using chance language like "unlikely" and "probably" to describe events (NCTM, 2000). Then, in elementary grades, third through fifth, students can further explore these chance processes through experiments (NCTM, 2000). These experiments can be using dice or spinners to model simple probability and finding the ratio of favorable outcomes to total outcomes of an event. Then, repeated repetitions using these chance devices can model a true random sequence (Shaughnessy, 2003). Some other big ideas in probability are understanding sample space, using simulation, comparing theoretical and experimental probability, and computing compound probabilities. These are recommended to be seen in the middle school grades (NCTM, 2000). However, students in high school should grow to understand conditional and independent events (NCTM, 2000). In fact, the high school *Common Core State Standards for Mathematics* highlights the following:

1) HSS-CP.A.3 – Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.
2) HSS-CP.A.5 – Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. (Common Core State Standards, 2010, p. 82)

Shaughnessy (2003) summarized research in student understanding of probability which ranged from exploring student conceptions of randomness to investigations in growing understanding through instruction. Many early studies assessed young students' intuitions of probability and discussed the increase of cognition students have when exposed to compound and conditional events (Fischbein & Gazit, 1984; Jones et al., 1999). According to Hogg & Tanis (1993, as cited in Tarr & Jones, 1997), the conditional probability of an event A, given that event B has occurred, P(A|B), can be defined as the probability where the only events of interest are those in subset A that can be found in subset B. To put plainly, it can be described as the probability of "A within B" or "A in B" (Ancker, 2006). Conditional probability involves updating probabilities as more information is known (Bargagliotti et al., 2020). "The sophistication of today's world demands students be educated to be able to cope in a conditional world" (Watson & Kelly, 2007). However, conditional probability is one of the most challenging strands of probability for students to learn (e.g., Falk, 1986; Pollastek et al., 1987; Prodromou, 2016; Tomlinson & Quinn, 1997; Watson & Kelly, 2007), and thus, it is the highlight of this thesis. The purpose of this thesis is to review research on issues of student understanding of conditional probability, and to use such research to inform and design a unit plan of the concept.

Overview of Thesis

This thesis focuses on student understanding of conditional probability. In Chapter 2, the literature review will have two main ideas. First, it will synthesize research on probabilistic thinking as a whole and then, progressively narrow into conditional probability understanding. The review of literature will guide the decisions of the unit plan development. In Chapter 3, the complete lesson makeup for the unit plan will be presented. It will include tasks created to help negate cognitive issues of learning conditional probability using best practices from research. Lastly, in Chapter 4, an overall discussion justifying unit plan decisions grounded in the literature is presented for implication and future use.

Definitions

Throughout this thesis, some mathematical terms will be frequently used. For ease, those terms will be defined here.

Complement of an event A^C – Event “not A”

Conditional Probability – Probability that one event happens given that another event is already known to have happened. Suppose we know that event A has happened. Then the probability that event B happens given that event A has happened is denoted $P(B|A)$. We find using $P(B|A) = \frac{P(A \cap B)}{P(B)}$

Event – Any collection of outcomes from some chance process. An event is a subset of the sample space.

General Addition Rule – If A and B are any two events resulting from some chance process, then the probability that event A or event B (or both) occurs is P(A or B) = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

General Multiplication Rule – The probability that events A and B both occur can be found using the formula $P(A \cap B) = P(A) \ * P(B|A)$ where $P(B|A)$ is the conditional probability that event B occurs given that event A has already occurred

Independent Events – Two events are independent if the occurrence of one event does not change the probability that the other event will happen. In other words, events A and B are independent if $P(A|B) = P(A)$ and $P(B|A) = P(B)$

Intersection – The intersection of events A and B, denoted by $A \cap B$, refers to the occurrence of both the two events at the same time.

Probability – A number between 0 and 1 that describes the proportion of times an outcome of a chance process would occur in a very long series of repetitions.

Sample Space S – Set of all possible outcomes of a chance process.

Tree Diagram – Diagram used to display the sample space for a chance process that involves a sequence of outcomes.

Two-Way Table – Table of counts that organizes data about two categorical variables.

Union – The union of events A and B, denoted by $A \cup B$, consists of all outcomes in A or B or both.

Venn diagrams – Used to display the sample space for a chance process.

Chapter 2 – Literature Review

This chapter will focus on the research behind cognitive issues of the learning and teaching of conditional probability. In the first part of this chapter, research concerning general probabilistic thinking will be synthesized. This will transition into the use of such thinking when understanding conditional probability. Specifically, the four levels of understanding conditional probability will be presented. The remainder of the chapter will focus on common student misconceptions when learning conditional probability, and it close with research-based teaching strategies to overcome those problems.

Probabilistic Thinking

Though the demand for understanding probability has increased over the last few decades, it remains that the average adult lacks probability literacy. A main cause of this is due to people having preconceived, incorrect notions of what the outcome of such an event means (e.g., Gal, 2005; Falk & Konold, 1992; Konold, 1991; Tversky & Kahneman, 1974). There are two general points of view when it comes to answering a problem about probability: the subjective or objective approach. An objective approach, commonly seen in probability textbooks, takes continued repetition of an event into consideration meaning the more times an event is repeated the closer it gets to approaching a predicted value (Kvatinsky & Even, 2002). However, the most common approach taken by the average person is the subjective viewpoint. A subjective outlook interprets a probability in relative terms to one's own beliefs. An example of such would be the opinion that you will land on a 3 when rolling a regular die, because the number 3 is your favorite number (Kvatinsky & Even, 2002). Konold (1991) also described a subjectivist as someone who becomes more confident in a predicted percentage as a single event nears that percent.

Since bias of one's beliefs impact probabilistic thinking so frequently, Tversky & Kahneman (1974) described representativeness, availability, and adjustment and anchoring as three heuristics people use when assessing predicted values. These methods reduce the complexity of probability into self-judgement. Representativeness occurs through determining whether event A belongs to some class B. Availability is used when frequency of a class is assessed. Lastly, adjustment from an anchor is present when an available value is used in the prediction of probabilities. Though the use of these heuristics can lead to a correct answer, there can be several underlying errors.

An individual who uses a representativeness judgement would select the combination of heads versus tails out of six-coin flips to be H-T-H-T-T-H over H-H-H-T-T-T and H-H-H-H-H-H because of both the appearance of randomness and equivalence to the true proportion 50-50. Also, when using this heuristic, a person may predict an individual's occupation by their description of character and not the numerical evidence that there are 70 engineers and 30 lawyers in a group of people. In either case, an individual would be looking at how well an outcome *represents* an event rather than considering any numerical descriptors (Tversky & Kahneman, 1974).

The availability heuristic is when a probability is determined by recalling event outcomes. A person may think an outcome is significant in comparison to the alternative based on personal recollection of that outcome happening, because we are more inclined to believe an outcome will happen knowing it has happened previously. Tversky & Kahneman (1974) gave the availability example, "one may assess the risk of a heart attack among middle-aged people by recalling such occurrences among one's acquaintances."

The last heuristic is adjustment and anchoring. This is when a person predicts a value based on a previous or given quantity; one may modify, manipulate, or *adjust* the given to create a possible futuristic value. The inherently subjective nature of probability can lead students and even experts to intertwine internal consistency and judged probabilities (Konold,1991; Tversky & Kahneman, 1974). These heuristics can lead to a correct conclusion, but this does not imply probabilistic literacy.

Rather than heuristic models on how people *form* a judgment, Konold (1991) described how people *interpret* probabilistic questions and values. The author described another approach as the "outcome approach". This is seen when the outcome of the next trial is the only thing being considered in a probability question (Konold, 1991). Students interpreting probability think of values 0%, 50%, and 100% to be not happening, unsure, and will happen respectively. Meaning, if there is a 70% chance of rain, it is safe to say, "it will rain tomorrow", because that value is close to 100% (Konold, 1991). Students learning probability show same knowledge and heuristics discussed here (e.g., Gal, 2005; Falk & Konold, 1992; Konold, 1991; Kvatinsky & Even, 2002; Tversky & Kahneman, 1974).

As probability has continued to surge in the 21st century, it is imperative that probability literacy is emphasized and development starts in the classroom (e.g., Gal, 2005; Jones et al., 1997; Konold, 1991; NCTM, 2000). Konold argued that one of the biggest goals in the classroom to strengthen probabilistic thinking should be understanding and meeting how students think before and during probability instruction

(1991). Further, Gal (2005) defines “the knowledge and dispositions that students may need to develop to be considered literate regarding real-world probabilistic matters” as probabilistic literacy. To help students achieve this, Gal (2005) listed the five knowledge elements as: big ideas, figuring probabilities, language, context, and critical questions. Teachers should engage students in the following elements to advance probabilistic literacy. The first element is gaining understanding for *big ideas* such as randomness, independence, and variation. These elements are stated to be the building blocks for understanding predictability and uncertainty. The second is *figuring probabilities* which includes basic computations of finding the probability of an event being between 0 – 1 and generating estimates of likelihood. Thirdly, students should understand the *language* of chance. Often, terms of probability are abstract and have different meanings in and out of the classroom. The fourth knowledge element is *context* which acknowledges the role probability has in a real-world sense and gives students awareness of the need to learn probability. Lastly, students should have opportunities to ask *critical questions* when finding a solution to probabilistic scenarios (Gal, 2005). Additionally, Jones et al. (1997) was one of the first to design a framework to assess students probabilistic thinking with a group of third graders. Hence, the next sections will highlight what students learn in schools about probability followed by the levels of understanding framework.

Conditional Probability’s Place in School Curriculum

During the 1990s, studies began to advocate for curriculum reforms to broaden the scope of probability in school mathematics (e.g., Jones et al., 1999; Shaughnessy, 1992; Tarr & Jones, 1997; Watson, 1995). Shaughnessy (1992) states that although there has not been a count of how much probability is being taught in school, the upbringing of probability concepts such as conditional probability in elementary and middle schools has only just begun. Watson (1995) also argues that "it would be a disservice to save conditional probability only for advanced students in the final years of high school" (p.16). These two thoughts agree with introducing probabilistic ideas earlier in school curriculum rather than restricting them to grades 9-12 (Bargagliotti et al., 2020; NCTM, 2000; Tarr, 2002). By consistently teaching probability throughout students' school career, they can leave high school with a stronger foundation of these ideas (Watson, 1995). Even though researchers agree with progressive development many states have not acted upon such recommendations. For example, in Tennessee, basic probability is not introduced until the 7th grade, compound probability can be seen in 8th grade, and conditional probability is shown in Algebra II or an equivalent Mathematics course. Students may have another opportunity to see such probability standards again in their final year(s) of high school if enrolled in Bridge Mathematics or Statistics, which is the opposite of research and policy document recommendations.

Even when probability is taught, it is not given importance in the US curriculum (Tarr, 2002). In fact, Tarr points out that probability concepts are pushed to the end of curriculum and rarely shown on standardized tests in comparison to other mathematical strands. NCTM *Principles and Standards* (2000) described a fluid sequence of probability from introducing vocabulary as simple as the terms "probably" and "unlikely" with PreK to 2nd grade to chance experiments with devices such as coins and dice in grades 3 through 5, and then with middle grades using appropriate terminology and computation of simple compound events to finally understanding conditional and independent events in high school. However, Tarr (2002) stated that the probability section of *Principles and Standards* was still minimal in comparison to the other standards. Therefore, it becomes more important to know how students understand conditional probability. Tarr & Jones (1997) were the first to put a framework together describing levels of student understanding.

Levels of Understanding Framework

Though older investigations of conditional probability assessed students' thinking with things such as questionnaires and surveys (e.g., Fischbein & Gazit, 1984; Pollatsek et al., 1987), a coherent framework was not implemented in the studies. About ten years later, other researchers filled this gap by creating a framework that describes how students understand conditional probability problems and independence (Jones et al., 1997; Jones et al., 1999; Tarr & Jones, 1997). Jones et al. (1997) first developed a four-level framework covering general probability with a group of third grade students. This was then further revamped with 15 middle school students focusing on conditional probability and independence (Tarr & Jones, 1997). The purpose of the four levels was to describe and predict students' thinking when answering conditional probability problems. The levels range from subjective thinking to using numerical reasoning: Level 1 is related to subjective thinking, Level 2 transitions between subjective and naïve, Level 3 implies informal quantitative thinking, and Level 4 uses numerical reasoning. This framework for describing students' reasoning set forth a way future researchers could coherently assess students' understanding of conditional probability (e.g., Prodromou, 2016; Reaburn, 2013; Watson & Kelly, 2007; Watson & Mortiz, 1999).

Level 1

Students displaying Level 1 understanding do not focus on conditional probability in a meaningful way. Instead, these students use subjective judgements, relate to personal experiences, or create their own system of regularity when problem solving (Tarr & Jones, 1997). They tend to believe that they can control the outcome of an event. For example, a fifth-grade student showed Level 1 thinking when she justified her prediction of drawing a red chip out of 4 blue, 3 green, 2 red, and 1 yellow chip by stating “it is my favorite color” (Tarr & Jones, 1997). This type of response indicates a lack of numerical reasoning and instead shows a personal relation to her answer choice. Students at Level 1 tend to ignore the relevance of quantities when making probabilistic judgements and believe previous outcomes affect future ones.

Level 2

Students demonstrating Level 2 thinking begin to transition from subjective to informal quantitative judgements. These students sometimes make use of the quantitative information presented, but they are still likely to divert into making subjective judgements and use representativeness strategies (Shaughnessy, 1992; Tversky & Kahneman, 1974). This was shown by Tarr & Jones (1997) when a seventh grader was attempting to guess a certain make up of a 3-digit number; he believed that he had a 50-50 chance of being correct or incorrect. Even after selecting the correct first digit, the student persisted with his odds being 50-50 though the probability had changed. Another case of Level 2 thinking was shown by a fifth grader thinking the likelihood of a spinner to land on yellow or red was the same, but after two trials, he changed his response to red being more likely due to the previous outcomes landing on red. Responses like these exhibit the tendency of Level 2 students to make predictions based solely on previous outcomes and irrelevant features.

Level 3

Students exhibiting Level 3 understanding can make conditional probability judgements by considering the role quantities play in the problem. Numerical probabilities and reasoning are still not often demonstrated, but these students are able to use some form of odds in an appropriate way to determine outcomes of both “with” and “without” replacements (Tarr & Jones, 1997). They can recognize the sample space, and they can keep track of the composition as events vary. For instance, a sixth grader with Level 3 thinking used a “competition strategy” when predicting the results of a class election; after the president was chosen, she compared her chance at president, ¼, with her chance at vice president, 1/3, by stating her number of opponents had decreased from four to three strengthening her likelihood of vice president (Tarr & Jones, 1997). In this case, the student was able to compare the number of her opponents with the chance of winning.

Level 4

Students demonstrating Level 4 thinking use numerical reasoning when interpreting probability situations. These students form probability judgements while recognizing the role numbers play. They know the importance of sample space and use it to assign numerical probabilities with valid explanations; they can also acknowledge the difference of independent and dependent events (Tarr & Jones, 1997). Unlike the seventh grader showing Level 2 thinking, an eighth-grade student was able to recognize the probability of guessing the 3-digit number combination increased when the first digit was given. The student gave the following reasoning, “Now I have a better chance [of getting the three-digit number] because I got to eliminate four choices so now there’s only four for me to choose from [out of a total of 8]” (Tarr & Jones, 1997, p.54). Furthermore, the student explained that the chance of then guessing the second digit correctly had not changed. It’s important to notice that this student compared the change in quantities as the conditional event was continuously evaluated.

Use of Framework

The development of this framework for assessing students’ conditional probability understanding led to future research on discovering how students’ levels change and develop (e.g., Prodromou, 2016; Reaburn, 2013; Watson & Kelly, 2007; Watson & Mortiz, 1999). In both Watson & Mortiz (1999) and Watson & Kelly (2007), the question of whether student levels changed with age was investigated. Watson & Mortiz (1999) found that older students begin to numerically express probability, and that younger students may struggle more in indicating distinctive events. In Watson & Kelly (2007), it was determined that Grade 11 students were able to recognize when to implement the use of tree diagrams over younger grades. Both studies found that grade level or age has little effect on thinking levels. However, Fischbein & Schnarch (1997) found that some misconceptions in the case of conditional probabilities increased with age (except for college level students). This may be due to the strengthening of an individual’s theoretical decisions over time. Moreover, to combat such issues students could and should be introduced with lower-level conditional problems like marbles in an urn, and as they progress, then they should be introduced to the social contextual problems like medical diagnositics (Rossman & Short, 1995; Watson & Kelly, 2007).

Other researchers studied the effects on thinking levels for naïve versus experienced learners (e.g., Prodromou, 2016; Reaburn, 2013). Similar to Watson & Kelly (2007), the difficulties between naïve and experienced students weren't with lower-level questions but the wordier social context problems that steer away from frequency, straight-forward language. A popular example of such a problem is Tversky and Kahneman's taxicab problem (1974). The problem states that there has been a night-time hit and run accident. 85% of cabs are green and 15% of cabs are blue and there is some witness who claimed to see the cab and identified it as blue. The witness withgoes a test to identify cabs (half blue, half green) under mock night-time conditions, and the witness correctly identifies 80% of the cases shown. The question prompt is "What is the probability that the cab in the accident was blue rather than green?" (Tversky & Kahneman, 1974). In the context of "wordy" problems, Tversky & Kahneman's question gives a lot of information in which a student must validate and decompose. Student errors and misconceptions can begin with getting lost in translation of a probability problem, and this may progress into other issues.

Student Misconceptions

As research started to unfold the levels of student thinking, common misconceptions of conditional probability started being detected (e.g., Budgett & Pfannkuch, 2019; Chong & Shahrill, 2014; Diaz, 2007; Falk,1986; Watson & Kelly, 2007). The most prevalent underlying cause of misconceptions is the use of language seen in conditional probability problems (e.g., Ancker, 2006; Chong & Shahrill, 2014; Pollastek et al., 1987; Watson & Kelly, 2007). The misinterpretation of language can then develop into other errors such as confusing conditional with causality (e.g., Falk, 1986;

Pollastek et al, 1987; Watson & Mortiz, 2002), mistaking what the conditional event is (Diaz, 2007; Rossman & Short, 1995; Watson & Kelly, 1995), and interchanging the inverse (Falk, 1986; Reaburn, 2013; Watson & Kelly, 2007).

Language

Language interpretation is significant in the development of conditional probability understanding, and without allowing time for students to get a foundation of probabilistic terminology, other difficulties will arise (Ancker, 2006; Pollatsek et al., 1987; Watson & Kelly, 2007). Ancker (2006) explains the difficulty in using statistical language can be due to terms meaning something different in common English. For example, a statistician will use the word “significant” to describe an unlikely event under the null hypothesis, but in casual conversation, “significant” means something of importance (Ancker, 2006). This complexity in language can lead students into making other errors due to misunderstanding when appropriate terms are being used.

Joint versus Conditional Probability

One example of misuse of language was seen when the following prompt was posed to 86 undergraduate students:

> “Indicate which of the two events is more probable. If you think the two events are equally probable indicate by circling (c).
>
> (a) The percentage of green-eyed people that have brown hair
>
> (b) The percentage of brown-haired people that have green eyes
>
> (c) The two percentages are equal” (Pollastek et al., 1987)

About half of the student participants decided with answer choice (c). This showed that one major translation error was the confusion between conditional probability, $P(A|B)$, and joint probability, $p(\mathrm{A} \cap \mathrm{B})$ (Pollastek et al., 1987). In a similar study with 414 students, 31% of students confused conditional with a joint probability confirming that the verbal ambiguity of conditional probability can cause difficulty in distinguishing between the two (Diaz, 2007; Falk, 1989).

Interpreting Conditionality as Causality

Apart from language, Falk (1986) lent insight into three of the biggest issues involving conditional probability. The first issue was interpreting conditionality as causality. In other words, students use cause and effect relationships to form probability conclusions. For example, Falk (1986) posed the following urn problem:

> "An urn contains two white balls and two black balls. We blindly draw two balls, one after the other, without replacement from that urn. First, what is the probability that the second ball is white given the first one is white ($P(W_{II}|W_I)$? Second, what is the probability the first ball is white given the second ball is white ($P(W_I|W_{II})$?"

The first question is easily answered by students giving 1/3. However, the latter begins to cause issues for students due to thinking the second draw has no effect on the first draw. Similarly in a contextual setting, Pollastek et al. presented the problem:

> "Which of the following events are more probable?
>
> (A) That as girl has blue eyes if her mother has blue eyes
>
> (B) That a mother has blue eyes if her daughter has blue eyes
>
> (C) The two are equally probable (N = 86)" (1987, p. 258).

Most of the student participants selected (C) $p(A|B) = p(B|A)$, but if that was not selected, more students selected (A) $p(effect|cause) > p(cause|effect)$ over the alternative (Pollastek et al., 1987). This is due to students believing that an event B cannot condition another event that occurs before it (Falk, 1986; Fischbein & Schnarch, 1997; Reaburn, 2013; Watson & Mortiz, 2002). Falk described this as the time axis fallacy (1986). While Gras & Totohasina (1995) described these cognitive issues as chronological and causal conception. In their study with 172 secondary and post-secondary students, 63% of subjects had chronological conception or thought that the conditioning event B must precede event A, and 28% of students had causal conception or thought that the conditioning event B is the "cause" of event A (Gras & Totohasina, 1995). Both studies show students refusal to accept the information presented in a problem when indifferent from temporal order (Falk, 1986; Gras & Tothasina, 1995). Watson & Kelly (2007) indicated that using partial contextual information can cause students to think of causality over conditionality.

Defining the Conditional Event

The use of social context problems is highly recommended by researchers (Rossman & Short, 1995; Watson & Kelly, 1995; Watson & Mortiz, 1999). Several examples of these types of problems have already been discussed and can be seen in the questionnaire used in the research study conducted by Pollatsek et al. (1987). However, incorporating these types of problems lead to many student challenges (Diaz, 2007; Pollatsek et al., 1987; Watson & Kelly, 2007). One of these challenges is Falk's second issue, "the definition of the conditioning event is often problematic" (1986). In other words, students often struggle with decomposing and defining what the conditioning event is in a conditional probability word problem. Bar-Hillel & Falk (1982) presented several teasers that were found to be difficult in doing such. A popular problem is the three cards problem: "Three cards are in a hat. One is blue on both sides, denoted BB. One is green on both sides, denoted GG. One is blue on one side and green on the other, denoted BG. We draw one card blindly and put it on the table. It shows blue face up, denoted B_u. What is the probability that the hidden side is also blue?" (Falk, 1986). It is common for the answer ½ to automatically be chosen eliminating the double-green card. In fact, when presenting a matching question to a group of students, 66% of them gave the probability ½, because the card is either BB or BG (Bar-Hillel & Falk, 1982). However, the correct answer is 2/3 since BB is twice as likely to be the card on the table with both sides being blue. The probability of the target event should be conditioned on the immediate event given the problem information and not on some inferred event, the double-green is out. The exact method by which we obtained the given data is crucial in determining our conditioning event (Falk, 1986). Further, students fail to make comparisons with the total number of outcomes and change of sample space (Fischbein & Gazit, 1984). Often students disregard vital information when making a conclusion (Watson & Kelly, 2007).

Confusion of Inverse

The last issue Falk (1986) addresses is the confusion of the inverse. Students lack the ability to distinguish between the two directions of conditional probability, $P(A|B)$ and $P(B|A)$ (Falk, 1986). This confusion is evident across all student and professional levels. It commonly occurs in medical contexts when interpreting test results. For instance, "there is a difference between the probability that I have measles given that I have rash, and the probability that I have a rash given that I have measles. The latter is much higher than the former" (Shaughnessy, 1992, p. 927). The same error can be seen in context of the probability of a disease given a positive diagnostic test being the same as a positive diagnostic test given the disease. The former is predicting the power of the test while the latter is referring to the sensitivity of the test (Ancker, 2006). The confusion of the inverse may also be a direct link to the time-axis fallacy as seen in the previous Falk (1986) urn problem. In some instances, students believe the inverse is not possible due to thinking events must happen chronologically (Falk, 1986; Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997; Reaburn, 2013). In both Watson & Kelly (2007) and Reaburn's (2013) studies, it was shown that students had issues deciphering the different probabilistic values of the following scenario:

"Which probability do you think is bigger?

A) The probability that a woman is a schoolteacher

B) The probability that a schoolteacher is a woman or

C) Both (A) and (B) are equally probable"

Let's denote option A as $P(W|S)$ and option B as $P(S|W)$. Of the students grades 3 through 13 in Watson & Kelly's (2007) study, 51% of the participants could not determine the difference. A reduction in proportions was seen in Reaburn's (2013) study with university students showing only 13% of participants not acknowledging the difference. Further, in a study to address the inverse misconception, three of six undergraduate students initially demonstrated the misconception in a problem related to medical testing (Budgett & Pfannkuch, 2019).

As stated, there are many obstacles in understanding conditional probability. These are just some of the most common student misconceptions in research. Though this concept can be challenging in the classroom, there has been research studies to help overcome such issues. Thus, in the next section, research-based teaching strategies for the development of conditional probability will be presented.

Teaching Strategies for Student Understanding

Success in the classroom begins with teacher understanding of conditional probability (Groth, 2010; Stohl, 2005). Not only should teachers have a strong understanding before instructing students, but also, they must be aware of students' preconceptions and beliefs on probability. An understanding of student knowledge allows teachers to meet students where they are and address or redirect any misconceptions (Castro, 1998; Groth, 2010; Konold, 1991; Stohl, 2005). To grow student understanding a teacher must first begin by setting up the appropriate learning environment (Castro, 1998; Groth 2010). For conditional probability, some strategies that may benefit students are the decomposition of language (Ancker, 2006) and the use of multiple representations (e.g., Prodromou, 2016; Rossman & Short, 1995; Reaburn, 2013; Watson, 1995).

Learning Environment

A first step in overcoming issues with conditional probability is setting up a positive, enriching learning environment. Castro (1998) led a study that compared student learning in a traditional environment versus a didactic environment. A traditional environment is one that is "teacher-centered" with a linear presentation without consideration of student conceptions. While a didactic environment is more "student-centered" and focuses on eliciting thinking and encourages self-reflection (Castro, 1998). The study found that the latter had significantly improved student probabilistic reasoning compared to traditional (Castro, 1998). The key aspects to a positive mathematics learning environment include providing opportunities for students to collaborate together, engaging students in rich, worthwhile activities, and encouraging reflection on problem solving (Castro, 1998; Groth, 2010; Henningsen & Stein, 1997; Tarr & Lannin, 2005). In fact, Groth (2010) states that, “metacognition is vital to independent problem solving”. Similarly, Tarr & Lannin discussed that instruction programs which allowed students to predict, collect data, reexamine, and renew understanding led to deeper understanding of key ideas (2005). Ways to incorporate this into the classroom include teacher facilitated discourse, consistent assessment of student learning throughout the lesson, and encouraging authentic problem-solving that goes beyond terms such as “replace” (Groth, 2010). Further, if teachers habitually use students existing knowledge, skills, and beliefs, then student attitudes toward the subject potentially could improve. Teaching for understanding has been shown to be an effective student motivator (Groth, 2010).

Overcoming Linguistic Challenges

Teaching for understanding suggests building concepts first, then attach the vocabulary to establish meaning (Thompson & Rubenstein, 2000). The language of conditional probability is one of the most common misconceptions. Students may not completely understand concepts at a deeper conceptual level if they don’t understand the language surrounding them, thus other misconceptions of conditional probability often stem from the misinterpretations of the language (Ancker, 2006; Dunn et al., 2016; Falk, 1986; Pollatsek et al., 1987; Reaburn, 2013; Watson & Kelly, 2007). Dunn et al. (2016) suggested 7 solutions for linguistic challenges:

1. Incidental learning versus explicit teaching
2. Managing synonyms
3. Iterative learning

4. Embracing ambiguity
5. Avoiding ambiguity
6. Verbal interaction
7. Recursive learning

Some major takeaways from Dunn et al. (2016) suggestions are maintaining consistency in language, engaging students with language through problems and exercises, emphasizing formal language, and reinforcing meaning numerous times by recycling vocabulary. Another way to overcome linguistic issues is to use informal definitions or substitute familiar terms when teaching beginners (Ancker, 2006; Thompson & Rubenstein, 2000). Ancker (2006) advocates for introducing the language in simpler terms like using "the probability of A *within* B" to describe conditional probability rather than the traditional "the probability of A *given* B". This description encourages the context of set theory, and it can then be represented with a familiar diagram such as a Venn diagram where students are visually able to see A *within* B for clarification (Ancker, 2006). Allowing students to compare and contrast statistical language with everyday language can also help gain a deeper understanding of terms (Ancker, 2006; Dunn et al., 2016; Groth et al., 2016; Lavy & Mashiach-Eizenberg, 2009). Once students are accustomed to the construction, they should be weaned off informal, everyday language to the use of formal, technical vocabulary (Ancker, 2006; Groth et al., 2016; Leung, 2005; Thompson & Rubenstein, 2000).

Use of Multiple Representations

Reaburn (2013) provides a suggestion of solving problems in a variety of ways using multiple representations to give students the visibility in understanding conditional probability terms. In fact, much of research indicates the importance of incorporating a variety of problems and representations for the learning of conditional probability (e.g., Diaz, 2007; Fast, 1999; Kvatinsky & Even, 2002; Prodromou, 2016; Reaburn, 2013; Watson & Kelly, 2007; Watson & Mortiz, 1999). Teachers should have a basic repertoire of examples and know how to use and interpret different models as well as when to use alternative approaches for problems (Kvatinsky & Even, 2002). Further, presenting a variety of applications to realistic problems, proposing interactive activities, and using valuable representations can facilitate student learning (Diaz, 2007). One best practice is the use of visual aids in conditional probability problems (Watson & Mortiz, 2002). There have been several visualization methods created for conditional probability such as the dot diagram (Wu et al., 2017), frequency nets (Binder et al., 2015) the turtleback diagram (Yan & Davis, 2018), and even interactive diagrams like pachinkograms (Budgett & Pfannkuch, 2019). However, the most common representations seen in research are two-way tables, Venn diagrams, and tree diagrams (e.g., Prodromou, 2016; Reaburn, 2013; Rossman & Short, 1995; Watson, 1995).

Two Way Tables

Specifically, the two-way table has shown to help students visualize the intersection (Prodromou, 2016) as well as distinguish between the inverse (Rossman & Short, 1995). Additionally, two-way tables can elicit student reasoning and conceptual understanding of the role of sample space for the conditional probability of an event (Prodromou, 2016). Students may also benefit from the implementation of two-way tables when given application problems rather than presenting the complex formula that is Baye’s Theorem (Rossman & Short, 1995). On the other hand, Tomlinson & Quinn

(1997) creatively integrated Venn diagrams and tree diagram into a hybrid model to further help students visualize multistage experiments. They used a pictorial Venn diagram for each branch of the tree to show what it represented, and they suggested that the Venn diagrams could be dropped and replaced with the probabilistic values once students become familiar with the methods (Tomlinson & Quinn, 1997).

Reconstructing Problems

In addition to the use of visual aids, it’s been shown that reconstruction of social contextual problems can benefit student understanding (Fast, 1999; Watson & Mortiz, 1999; Watson & Mortiz, 2002; Watson & Kelly, 2007). For instance, students struggled more with exercises involving probabilities than when expressing problems in frequency formats (Watson & Mortiz, 1999; Watson & Mortiz, 2002), thus Watson & Kelly (2007) supported changes of conditional probability problems to a frequency point for better understanding. On the other hand, Fast (1999) proposed the use of analogies to change conditional probability problems. Techniques like simpler, cues-removed situations, extreme cases, or different perspective changes were chosen, and this reconstruction had a 0.72 success rate in the study (Fast, 1999).

These various ways of solving conditional probability better assist students in learning fundamental principles. Ultimately, students should transition from representation problems like two-way tables to urn like problems to more challenging social contextual problems (Watson & Mortiz, 1999). The promotion of multiple representations allows students a resource toolbox from which they can pull a method of their choosing out (Reaburn, 2013). As students progress through conditional probability problems, it is important that the social contexts involve real life data (Neumann et al., 2013). Rossman & Short (1995) present a couple of real contexts with genuine data involving AIDS testing and legal evidence. This promotes statistical literacy relevant to interpreting everyday media statements such as medical, scientific, or legal scenarios, and it can add value or motivation to students' learning process (Watson & Kelly, 2007). Without the transfer from countable situations to more social settings, student understanding of conditional probability will not continue to develop (Watson & Mortiz, 1999).

Overall, as the need for conditional probability continues to grow in school mathematics curriculum, new challenges arise for both teachers and students. Though recent research shows that this topic can be introduced with middle school students, it is important to consider the common misconceptions that students have and for teachers to be able to react and adjust their teaching accordingly. This can be done by using research-based knowledge of students' thinking such as Tarr & Jones' framework to guide instruction ultimately providing students with an understanding of key conditional probability concepts (Jones et al., 1997; Jones et al., 1999; Tarr & Jones, 1997).

In the next chapter, a four-day unit plan of conditional probability will be presented. This unit plan was created with consideration of the research provided in this literature review.

Chapter 3 – Lesson Plans

This chapter contains a 4-day unit plan on conditional probability. Specifically, this unit plan strives to combat cognitive issues students have when learning conditional probability. Thus, student exploration, use of multiple representations, and teaching conceptual understanding before procedural fluency is prioritized. The goal is to allow students to formulate and explain what it means to be conditional on their own through investigations. They will uncover both the concept of conditional probability and independent events by collecting and analyzing class data.

The first day introduces conditional probability without explicitly saying so. Students explore through a game show scenario which uses conditional probability to discover the likelihood of winning situations. The term conditional probability is not introduced until the second day where students collect categorical data, create representations, and calculate probabilities. At the end of the second day, symbolic notation for conditional probability is shown, but the formula remains unknown. By day three, students explore the common “with” and “without” replacement situations. Instead of using the classic urn problems, the students return to their class data collection from day two. The goal is to make a connection among the problem scenarios and independence definition. Finally, the fourth day uses multiple types of conditional probability examples for students to use their new range of diagrams as they choose.

Each day has its own lesson plan including standard and objective alignment, instruction pacing, and student assessment. All materials in reference to the plans can be found in the Appendices. One note to be made is that the unit plan was created under the assumption students have already covered basic probability and compound events such as intersection and union.

Day 1 Lesson

Lesson Title – Intro to Conditional Probability
Duration – 50 Minutes

Standards

HSS. CP. A. 5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

HSS. CP. A. 6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A and interpret the answer in terms of the model.

Learning objectives

1) Students will predict the likelihood of winning a conditional game.
2) Students will collect and analyze data from a simulation.
3) Students will explain the probability of winning a conditional game.

Materials

PowerPoint to display "Monty Hall" problem (See Appendix A). Students may need technology if using the online applet or a standard deck of playing cards if using hand-held manipulatives.

Instruction

Activation (10 minutes) – Teacher will present a version of the well-known "Monty Hall" problem on the board and/or a hard copy handout can be given to students (See Appendix B). Beginning the lesson prompting students with the following questions:

- What would you do if you were on the show? Explain your choice.

Students should submit their thoughts in writing or using technology, so teachers can evaluate initial thinking. This also holds students accountable to their first instinct.

Instruction (35 minutes) – Students will simulate playing the game. The teacher can use a couple of methods to do so. There is an applet on Rossman Chance website that students can use with access to internet

https://www.rossmanchance.com/applets/2021/montyhall/Monty.html.

Another method that could be implemented is the use of playing cards or some physical manipulative. For cards, each pair of students gets 3 cards: 2 black and 1 red. Black suits represent a door that has a goat and a red suit represents the prize door! Student 1 can shuffle the cards and lay them out for student 2 to then select a card for their chosen "door". Student 1 will then reveal one of the remaining two cards, and Student 2 will determine to stick with their original card or switch. Student pairs will record how many times the game is won in a series of repetitions, say 30-50. In either case, the teacher should begin by announcing, "Before we begin, let's have a show of hands of pairs who are going to choose to stick to their original card." The teacher can group pairs who have their hands raised with another pair that does not, so they can compare simulation winning results later. Continue by instructing, "If you think you have a better chance sticking with your card, then do so for every trial. If you think you should switch cards, then do so for every trial." If students do not choose one method to focus on, their ending probabilities will not demonstrate the Law of Large Numbers appropriately. Once students begin working, the teacher should circulate asking student pairs prompting questions such as:

- Do you think you have the same probability of winning regardless of a door being revealed? Is this the same as your original theory?
- What is the probability of winning if you keep/switch based on the trials you've done so far? Are you convinced one way or another yet? What will convince you?
- What do you see in your results so far? Is it surprising to you? Explain.

As groups begin to finish, the teacher should prompt them to work with their neighboring pair, the one with the opposite choice, and discuss what data they have collected. As these conversations happen, the teacher may ask:

- What do you notice between your two results?
- Explain what the results show.
- Are you convinced one way is better than the other? Why or why not?
- Are you thinking differently than at the start of class? Why or why not?

For groups that finish sooner, the teacher can prompt them to try the opposite choice. If they kept their original card, retry this time switching their card every time or vice versa. The goal is for all groups to eventually do this. Before trying the alternate approach, the teacher should bring the class together to discuss what simulation results have shown thus far. Teacher may say, "Now that we've ran some trials of the game, let's go back to our original question. Do you think you have the same probability of winning after one door has been revealed? Why or why not?" Teacher should call on several groups looking for responses that bring up switching cards has a higher probability of winning the game, though other responses are accepted at this point. For example, students may still see it as a 50/50 chance. Let them elaborate their views as well, but the teacher should ask questions to the rest of the groups that challenge them to form arguments for or against their peers:

- Does anyone think differently? Explain what you think.
- Does this conclusion make sense? What other conclusions are there?

Students should just be defending their argument. In which the teacher should question, “Why do you believe staying/switching has better/worse odds? What changes? I want you to think about this in your pairs as you complete another set of trials using the alternative approach. Again, record your winning results and then compare them with your original simulation. Once complete, each pair should record their results of both rounds on the board. By the end of the activity, we will have several data values to consider, so you and your partner can form an argument to fully convince someone whether they should stick with their original door or change!” Then, again, the students should simulate the game in the opposing direction keeping up with the number of times won. As they work, the teacher can ask questions similar to before:

- What is the probability of winning if you keep your card? What is the probability of winning if you switch?
- Are your results what you thought they would be? Why or why not?
- Explain your simulation results. What do they tell you?
- How do your results compare to those that have been put on the board?
- What does the class data suggest? Does it make you think differently than looking at your own? Why or why not?

- Based on your own data and the whole class results, how would you convince someone playing the game to switch or to say? Which provides you with more confidence for your choice, your group's data or the whole class? Explain.

The last question should be given to all students, and they should have a written response that can be collected. Teacher may say, "Okay, to finish this activity, I want you to make a convincing argument for or against switching doors to a new person playing the game. You want to convince this person that you know the true odds of winning so use what you've collected today to provide reasoning to why they should or shouldn't switch." The teacher will collect arguments.

Closure (5 minutes) – Once statements are collected, the teacher should have a closing discussion. Students may be called on to share their closing arguments. Teachers will want to be sure to do this following the collection of student work, so students do not change their own thinking as discussion is taking place. Teacher may call on a few volunteers to share their argument. The teacher will want to reiterate any solutions that highlight the fact that the probability changes after one door is revealed or the notion that the extra information provided by the reveal door benefits the player.

Assessment

Students will be assessed through discussion in the main portion of the activity. As the teacher is circulating, they should have conversations with student pairs using the questions provided to indicate student thinking. The teacher should prompt students to share the way they think as they work to help understand where they are at. Students will also be assessed by comparing their initial responses about the game from the activation to their closing arguments.

Day 2 Lesson

Lesson Title – What is Conditional Probability?
Duration – 50 Minutes

Standards

HSS. CP. A. 4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to approximate conditional probabilities.

HSS. CP. A. 5. Recognize and explain the concepts of conditional probability and independence in everyday language and every day situations.

HSS. CP. A. 6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A and interpret the answer in terms of the model.

Learning objectives

1) Students will find the probability of A within B (conditional probability).
2) Students will explain conditional probability values through representations.

Materials

PowerPoint or Padlet to display necessary questions discussed in Instruction section (See Appendix A).

Instruction

Activation (10 minutes) – Teacher should begin the lesson by summarizing the previous lesson's takeaways. For example, teacher may ask class, "Yesterday we played the Monty Hall three doors game. What was the conclusion that we made?" The teacher should call on a volunteer, then call a second volunteer to either generalize student 1's claim in their own words or form a rebuttal. The aim is to have a student recall that when we received more information, a revealed door, there was a shift in probability of winning. We found a trend for increasing your chance of winning when switching doors rather than staying with original choice. For the first time, the teacher should introduce conditional probability term. Say, "We have previously learned basic and compound probability. Can someone give me an example of each? [Look for responses like rolling a die or flipping a coin to satisfy basic and the combination of events like getting heads and rolling a 3 to represent compound.] We are now moving on to conditional probability. Yesterday's game was an introduction of conditional probability. Using what you know about the word 'conditional' and what we found from the three-door game, what might you expect conditional probability to be?" Give students about 1-minute alone think time and about 3-minute small group or partner discussion. For students who need an extra boost on what to consider, continue to ask:

- What does it mean if I say, "under one condition"? Or typically what do you think of when you hear something is conditional?
- What happened in yesterday's game that was important? What were we trying to find out?
- What did you learn from yesterday's game?

Teacher circulates to listen on student group discussion and use what's heard to decide who to call on when progressed into whole group discussion. When asking students questions, be sure to allow them to explain their thinking on conditional. As a class, the aim of the discussion is to present the idea that to be *conditional* something has to happen in order for a proceeding event to happen. Teacher should call on groups to share their thinking looking for responses that satisfy this goal. Teacher may then say, "In the three doors game, we learned that to increase our odds of winning we needed a revealed door. What is the condition in the scenario? What information do we know? [revealing what's behind one door] Today, we will answer more problems in which we are given information to help determine a probability, but first we want to collect and analyze our own data."

Instruction (35 minutes) – So, students will be posed with the following questions,

1) Do you have a sibling(s)?
2) Do you own a pet(s)?
3) Do you own a pet and also have a sibling?
4) Are you an only child with no pets?

Student responses can be collected in a variety of ways such as asking them aloud with show of hands, making a chart on the board for them to tally, or use of technology like a Padlet where students can submit their responses. In any case, students should keep track of the class data. The teacher will then instruct students by saying, "The first task for you today is to organize this data in a way that can represent the probability of events, both simple and compound like intersection. I am going to give you just a couple minutes to do so, and then you can compare your model to your partners." Students should have previously worked with basic and compound probability, and thus, they should have also seen Venn diagrams and two-way tables. The goal is for students to recall and create such representations for the data. If students struggle, teacher may prompt:

- Can you make a representation of the data?
- What value represents the intersection of having a sibling and owning a pet? How can you use that to build a diagram?

- How many total pet owners? How many total students with siblings? How might knowing this help you put something together?

Allow students time to consider diagrams on their own before presenting any ideas, say 3–5 minute think time. Once some students have begun, it can be beneficial to those struggling to pull the entire class in for discussion. Teacher can say, "We want to be sure our diagram can help represent probabilities, so what are some models we have created?" [If the teacher has seen a student begin a Venn diagram, that student may be called on to explain verbally what they have created. Then, if the teacher has seen another student use a two-way table, that student should also be selected to share multiple perspectives. Examples seen in Appendix C]. This gives options to those students who have not come up with a representation method yet. They may remember one diagram better than the other. As students work on creating their diagram, the next thing to pose to them is conditional probability problems. On the board or with a handout (See Appendix B), the teacher can prompt students with the following questions:

1) What is the probability of selecting a student who has a pet within the students who have a sibling? Explain your solution.
2) What is the probability of selecting a student who has a sibling within the students who have a pet? Explain your solution.
3) What is the probability of selecting an only child within the students who have a pet? Explain your solution.
4) What is the probability of selecting a student without a pet within the students who do not have a sibling? Explain your solution.

Without further instruction, teacher should allow students to explore these problems using their diagram. While circulating, the teacher could prompt students with some of the following questions:

- What does it mean to be “within”?
- What do you think the importance of including the “within” statement?
- What is the “extra information” in the problem?
- If you just looked at the first part of the sentence, how would you find the answer? How might you combine that procedure using the “within” statement?
- Explain your solution.
- How did your model help you determine the probability?
- Explain how your solution is represented within your diagram.

For students that finish early, the teacher should encourage them to try to create their own conditional probability problems using the data and that can be modeled with their diagram. If working in pairs, allow both students to create a problem and switch with their partner to calculate it. These other problems may include:

- What is the probability of selecting a student without a pet within students who are an only child?
- What is the probability of selecting a student who owns a pet within students who are an only child?
- What is the probability of selecting an only child within students who do not own a pet?

- What is the probability of selecting a student who has a sibling within the students who do not own pets?

Once groups have had sufficient time (15-20 minutes), to move through the problems, the teacher should call on students to share their solutions. Go over each problem calling on volunteers to come up to a document camera or board to share their solution. While students present their solution, the teacher should ask students how the probability can be seen in their diagram. Encourage them to explain through their representation. For example, on number 1, in a Venn diagram students may trace over the circle of students with siblings, then point to the intersection to show students who have a sibling and own a pet within the sibling circle. The teacher should purposely call on groups with different perspectives such as the two-way table versus Venn diagram. By doing so, the teacher can ask:

- How is conditional probability represented through a two-way table versus a Venn diagram?
- What's similar in the diagrams? What is different?
- Is there a preferred method? Why or why not?

Students can begin looking at how two-way tables indicate all intersections ($A \cap B, A^c \cap B, A \cap B^c, A^c \cap B^c, etc$) while in a Venn diagram $A \cap B$ is best seen. Then, if students who finished early successfully created their own problems, the teacher should have those students pose their question(s) to the class to answer in discussion.

Closure (5 minutes) – To close the lesson, the teacher can summarize the day's learning and introduce formal notation of conditional probability. Say, "We've learned that conditional probability is when we find a probability given some information. Like in the first problem we knew we were looking at students with siblings. I want to introduce the symbols to conditional probability as sometimes it can be easier to simplify our problems in this notation because wording is confusing. Conditional probability is written as $P(A|B)$ [write on board] and it can be read as probability of event A within event B. To conclude our lesson, I want you all to rewrite our problems using symbols, and you should give the probability value found. We will call having siblings event S and owning a pet even P. Follow up your solutions with an explanation on how you decided to rewrite the problems. This will be your exit slip." Teacher can post these instructions on the board or through Powerpoint as seen in Appendix A. It may be beneficial to do the first one together with students writing it as $P(P|S) = ?$ on the board. Otherwise, teacher could circulate and do one with those students struggling.

Assessment

Students will be assessed as they work through the problems. The teacher will ask students to explain their solution in reference to their representation. The teacher can use the provided questions to further prompt students for assessment. At the end of class, students will submit their exit slip where they are instructed to rewrite the problems using mathematical symbols and give the calculated value (Example key seen in Appendix C). They are also instructed to explain their symbolic representations.

Day 3 Lesson

Lesson Title – Conditional Probability & Independence
Duration – 50 minutes

Standards –

HSS. CP. A. 3. Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

HSS. CP. A. 4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.

HSS. CP. A. 6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

Learning objectives –

1) Students will calculate and compare conditional probabilities of with and without replacement events.
2) Students will explain conditional probability through representation.
3) Students will explain whether two events are independent or dependent.

Materials –

Teacher may want to present questions in a Powerpoint or handout (See Appendix A). Students will need butcher paper or poster for presenting answers, and they will need sticky notes for peer feedback.

Instruction –

Activation (5 minutes) – Teacher should begin with a Venn diagram of some events A and B on the board with the symbols of conditional probability $P(A|B)$ and $P(B|A)$ (See Appendix A). Teacher should begin by saying, "We have calculated conditional probability values. At the end of our last lesson, I introduced the following symbols [motion to the symbols on the board]. Let's recall what those symbols mean. There are no numbers in this Venn diagram, but we want to take a minute to remember what those symbols represent. Using the diagram, how would you describe the symbols in words? Take a moment on your own, then discuss with a partner." No numbers are in the Venn diagram, so students can truly explain the symbols without resorting to quantitative values. Teacher may call on students to share what the symbols represent in their own words. Students should recall that the symbols are read as "the probability of A within B / the probability of B within A". They may explain how this is represented in the Venn diagram by discussing how the probability of A within B could be found by finding those that are in both groups, the middle section, out of those that are in B. Next, the teacher should question students with, "Does $P(A|B)$ and $P(B|A)$ mean the same thing?" Again, allowing them to discuss among one another before sharing as a group. When sharing as a group, students may point out that what comes second is the specified event to look within. Using the Venn diagram, they may explain that the first looks within the B circle and the second looks within the A circle. They may also give the concrete example from yesterday indicating that the probability of a student who has a pet within those who have a sibling was not the same as saying the probability of a student having a sibling within those who have a pet.

Instruction (35 minutes) – Teacher say, “Let’s consider another type of conditional probability problem. Get out yesterday’s collected data and your diagram. This time, using our data, I want you to consider what would happen if I randomly selected two students one after another. Say I put all your names in a hat and drew one out followed by a second.” Then, pose the following problems on the board or in a handout (See Appendix B).

1) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who are an only child? Explain.
2) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who own pets? Explain.
3) If two students are randomly selected and the first student selected is not replaced before selecting the second, what is the probability of getting two students who are an only child? Explain.
4) If two students are randomly selected and the first student is not replaced before selecting the second, what is the probability of getting two students who own pets? Explain.
5) What’s different between #1/2 and #3/4? Explain.

As students work, the teacher can circulate through the groups asking them to explain how they came up with their answers to each. Students may need scaffolding to recall intersection. Teacher may ask groups:

- What is it called when we find the probability of two events happening at the same time? How do we find that value?

It may also be beneficial that after about 2–3-minute think time, the teacher discusses intersection as a whole group to ensure everyone makes progress through the questions. Ask aloud, “How do we find the probability of two events occurring at the same time?” Call on students to describe intersection formula as $P(A \cap B) = P(A) * P(B)$ or simply saying you multiply the two events. Then, release students back into working on the problems using this information. The teacher should discuss answers with each group as they work. Ask students what they are noticing with the questions and solutions. Looking for students to explain that in the first two problems the probability does not change, because you are replacing the first student. However, the without replacement problems do change. The teacher may also prompt students to make a representation that could model the scenario. Ask students:

- Is there some way you can represent/model the problem?
- Is there a model that can be made that shows all possible outcomes of selecting two students (in both cases – with and without)? Try to create one.

Students could create two-way tables using the intersection values to model the scenario. Ideally, this could also naturally bring in the incorporation of a tree diagram. The teacher should keep an eye out for students doing so. Otherwise, the teacher should use a “fake example student” work tree diagram to display during discussion (See Appendix C). As students have begun to make their representations and answer the questions, the teacher should ensure the diagrams make sense and redirect, if necessary, then hand out butcher paper or poster board for students to recreate their diagram on. Instruct students, “I have checked in on your diagrams and answers as you’ve worked. Your next thing to do is to recreate your diagram on your poster to share during discussion. As you do so, I will be coming around to tell you which problem your group will be presenting. For that problem, be sure to highlight where the probability is seen in your diagram and give a brief explanation of your methods." Depending on class size, each problem will have two groups responsible for presenting. The teacher should choose two groups with different solution methods whether that be a two-way table, tree diagram, pictorial, formula driven, or some other method. That way there are multiple representations for each. Then, there should be spots marked off around the classroom for where problems 1, 2, 3, and 4 should be hung. Further, numbers 1 and 3 should be placed near each other as well as numbers 2 and 4 for students to compare the results of the independent and dependent scenarios. As students finish their posters, with teacher approval, they should hang it up accordingly. The teacher should check that the posters have a clear probability solution as well as an explanation to reaching their solution. For students who finish early, prompt them with another task such as:

- Can you find another representation? Discuss how the multiple representations are the same. How are they different? Which do you prefer? Why?
- Create your own problem within the same context. What is the probability of such? How did you determine that?
- How would the probabilities change if 3 students were selected? What would be all possible combinations of students? How would your diagram change?

For ease, these tasks can be written on index cards and handed out to those that finish early. Once groups are finished with their posters, students will participate in a gallery walk around the room to each poster. The teacher will instruct, "You will be visiting other groups work of the problems. With sticky notes, I want you to make comments or ask questions on other groups posters. You should make at least one valuable contribution to each poster. Comments like 'good work!' are not valuable. Instead consider…" Students should be given the following prompts to consider as they walk around. These may be posted on the board for student reference as they circulate:

- What does the diagram represent well? What does it not represent? Can you give feedback on what may be missing?
- How does this model answer the question differently than what you did? How is it the same?
- Compare the two posters for problem. What do you notice? Give feedback on which method you prefer. Why?
- Is it clear how the probability value was determined? Do their methods make sense to you? Why or why not?
- In your own words, explain the representation.

As students circulate, the teacher should also see what is being discussed and written. The teacher should keep track of timing and instruct students to switch after 1-2 minutes at each question. Once all rotations are complete, groups should return to their own poster to reflect on the sticky note comments before discussion. Finally, as a whole class, the teacher should ask questions like:

- What did you notice across the posters?
- Were there methods you found to be better than your own? Why or why not?
- Were there methods that didn't make sense to you? Why or why not?
- What comments did you get on your own poster? Did that change your thinking?

- Were there any questions left on your poster that you can explain thoroughly now?

The teacher should go around the classroom calling on different groups to share their responses to such questions. The goal is for students to realize there are multiple representations that can be used for the same problem. They should compare which they prefer and explain why. For example, students may say, “Using a tree diagram made more sense to me because I could see all combinations of selecting two students by following the branches.” Students may recognize they like another group’s representation better than their own, and those changing views should also be discussed. For example, “I used the formulas because plugging things in was easy to me, but the two-way table gives a better visual of what those values actually represent.” Lastly, the fifth and final question should be discussed as a whole group. Teacher should say, “We have compared several representations and how they are useful for each problem. Our posters showed solutions to numbers 1-4 (examples in Appendix C), but we still have to discuss the final question. So, we noticed that numbers 1 and 3 were similar as well as 2 and 4, but how were they different?” Call on students to explain their thinking. The teacher should look for explanations that point out the “with” and “without” replacement phrases. Specifically, when we do not replace a student, our sample space changes; it is important students use such mathematical language like sample space to explain the change of probability values. On the other hand, when we replace a student, it is like we are starting over; nothing has been changed. Students may also refer to poster representations to explain such as saying, “In #1’s tree diagram, the probabilities stayed the same across the branches. In #3's tree diagram, the second set of branches had to change based on the first result."

Closure (10 minutes) – Teacher should formally introduce the definition of independence. Say, "Today we have compared multiple solution methods for the same probability problem. We also discussed the importance of the phrase "with" or "without" replacement. Finally, we have a new vocabulary term to discuss: Two events are independent if knowing whether one event has occurred does not change the probability that the other event will occur [this can also be displayed on the board for student reference.] As an exit slip, you will determine whether the events we have already explored this week are independent or dependent based on this definition and explain your thinking. In addition, you will be given one two-way table problem in which you will determine independence and use to help generalize a formula for such (see Appendix B)."

Assessment –

Students will be assessed through classroom discussion for much of the lesson. As the teacher walks around the class, they should have students explain their solutions to understand their thinking and redirect when necessary. The teacher can also evaluate student thinking as they participate in the gallery walk by reading what students are posting to other group posters. Lastly, students will be assessed through their exit slip in which they determine independence. Each problem asks students to explain their thinking, so the teacher can gauge their understanding of the new vocabulary term in reference to the day's work.

Day 4 Lesson

Lesson Title – Social Contexts
Duration – 50 Minutes

Standards

HSS. CP. A. 5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

HSS. CP. A. 6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A and interpret the answer in terms of the model.

Learning objectives

1) Students will solve social contextual conditional probability problems.
2) Students will create conditional probability models.
3) Students will compare conditional probability models.
4) Students will explain conditional probability values.

Materials

Instruction

Activation (5-8 minutes) – The teacher should begin the lesson with reviewing what the class has discovered over the last few class periods. This can be an opening discussion where the teacher asks questions like:

- What is conditional probability? Can you give me an example of conditional probability?
- How do you determine conditional probability values?
- What makes conditional probability different from simple or compound probability? Explain

- What does it mean to be an independent event? Can you give me an example of independent event?
- How did you generalize independence? [referring to yesterday's exit slip]
- What's different between independent and dependent? Explain.

The teacher will want to highlight any responses that describe conditional probability as when you are looking for an event "within" another event. Students may also state that conditional probability differs from simple and compound by stating that conditional includes extra information that changes the sample space. For independence, students should indicate that the extra information has no effect on the probability, or they may give the previous days example of with and without replacement. Specifically, the teacher should look for the generalized formula from the previous day that if an event is independent, then $P(A) = P(A|B) = P(A|B^c)$. The teacher may even pick out a student solution from the previous lesson's exit slips to discuss. Finally, the teacher will want to engage students in discovering a formula for conditional probability. Say, "At the end of the last lesson, you generalized independence. Now, I want you to now generalize conditional probability. What kind of formula can you create to demonstrate conditional probability, that is, $P(A|B)$ [written on board or displayed]?" Students may be given a few minutes to discuss in their groups. As they do, the teacher may prompt them with:

- What do our problems from the week all have in common?
- How do you find conditional probability looking at a representation? Explain.
- Can you write what you are thinking in symbols?

Then, students should share as a class to determine $P(A|B) = \frac{P(A \cap B)}{P(B)}$, and the teacher should post this formula on the board. Students should discuss the intersection of the two events is always the numerator and the denominator is always the extra information provided or *subset* that we look at. Further, the teacher should introduce the phrasing $P(A|B)$ = the probability of A "given" B by saying, "We have read conditional as probability of A *within* B to easily visualize the procedure this week. However, you formally read it as the probability of A *given* B." This is a heads up for any problem that may be worded as "given" in today's assignment.

Instruction (40 minutes) – Students will be working on the four problems from Watson & Kelly's study (2007) (in Appendix B). The questions have been modified some to provide students with multiple entry points and to require them to explain and compare solutions. Students can work through the problems as a group. The teacher should instruct, "Today's activity combines everything we have worked on the past few lessons. That is, conditional probability and its representations in addition to independent events. As groups finish through number 4, we'll review as a whole class. You will randomly be selected to present your solutions." Students should have plenty of time to collaborate and work through the problems. As they work, the teacher may ask students to explain their solutions in addition to questions such as:

For Q1:

- Would it make a difference if you added count values to the problem?
- Do you think these events the same? Why or why not?
- Why might one probability be bigger than the other?

- Can you give a counterexample to your solution?
- Explain your solution.

For Q2:

- What representation did you choose to create? Why?
- How would you describe the events given? How did you use that to create your representation?
- Explain how you arrived at your solution to (b) / (c)?
- What is the relevance of the “if” statement in the problem?

For Q3:

- What’s the difference between (a) and (b) probability? Why do you think that is?
- How did you determine part (b)? How does your diagram map out this procedure?
- Were your results as you expected or surprising for (a) and (b)?

For Q4:

- What are you given in this problem?
- How did you determine what diagram to use to help keep your information organized?
- What are you looking for in this problem? What is the “given” or conditional in the scenario?

For Q5:

- Ask students to explain their choice of diagrams.
- Compare their choice to another option with reasoning – For example: “You used a two-way table for #2 rather than a Venn diagram, why?”

This assignment allows students to work through a variety of conditional probability problems. It includes more common questions like an urn scenario, but it also includes a more social contextual problem where students investigate a hit and run scenario. As the students work, the teacher should circulate and ask questions requiring students to explain their thought process. As the teacher moves to each group, they may want to keep note of each group's solution method to strategically select the order in which groups present their work. For example, showing a student who used a tree diagram for Q3 before showing a student who just used the formula. This allows for connection of the tree diagram to the formula. Students should be called on to present to the whole class like on a document camera. After students present, the teacher may ask for other solution methods in which students can volunteer to share. When this happens, the class should discuss the similarities and differences, and students should be asked if there is a preferred method for a specific question type.

Closure (2-3 minutes) – Once problems 1-4 have been reviewed, the teacher can close the lesson by summarizing student solutions. Say, "We have now seen conditional probability in multiple ways: two-way table, Venn diagrams, tree diagram, and even a formula. On the last problem, number 5, you are asked to explain your choices for representations. Compare the multiple methods we have gone over and argue why your method makes sense to have used." Students will finish number five if they haven't already, and the activity page should be turned in for teacher review.

Assessment

Again, assessment of student understanding will mainly come from discussion as the teacher intervenes with group work. It will be important that as the teacher approaches a group working, they ask each student a prompting question relating to the work they are doing to indicate all understand not only one group member. This may even be done by asking one group member a question, and then following up with a second member to reiterate what they discussed in their own words. Students should be encouraged to explain their solution to the teacher and not only state the numerical value. Secondly, students will be assessed as they begin to present their solution strategies when reviewing answers. Again, the teacher may ask a second group member to reformulate their presented ideas to ensure understanding among all members, or the teacher may ask a nongroup member to restate what was said to ensure class participation within the presentations. Lastly, students will reflect on the conditional probability unit explaining what solution strategy works best for them and why. The teacher can obtain common themes from their replies, and it may help determine whether one method did not reach the class as anticipated. Moving forward, the teacher may present students with a summative assessment such as a quiz or test to assess individual achievement as opposed to the collaborative work done throughout the unit.

Chapter 4 – Unit Plan Justification

This chapter will provide justification for each day's lesson plan decisions using research cited in the literature review chapter. It will begin with an overall discussion for the unit plan sequence and structure. Then, each lesson will be broken down in its own section explaining the activity selection and instructional methods.

The Unit Plan

The unit plan is sequenced to explore and use multiple representations of conditional probability to solve problems. Reaburn (2013) suggests that giving students a variety of representations to visualize conditional probability can promote student understanding, so this is a focus throughout. Further, the unit plan highlights students understanding independence and conditional probability and use them to interpret data as stated in the *Common Core State Standards for Mathematics* (2010). It was also designed with the eight mathematical teaching practices in mind as stated in NCTM (2014):

1. Establish mathematics goals to focus learning.
2. Implement tasks that promote reasoning and problem solving.
3. Use and connect mathematical representations.
4. Facilitate meaningful mathematical discourse.
5. Pose purposeful questions.
6. Build procedural fluency from conceptual understanding.
7. Support productive struggle in learning mathematics.
8. Elicit and use evidence of student thinking.

Throughout the unit plan, it follows Castro's (1998) recommendation that the classroom be student-led rather than teacher-led. Thus, a major part of each day's lesson is eliciting student thinking through mathematical discourse. Students are often encouraged to collaborate with one another as they engage in each day’s activities. Further, the fourth mathematical teaching practice is used as the teacher will facilitate meaningful discourse in both small group and whole class discussions (NCTM, 2014).

For each lesson, the goal is to build mathematical concepts prior to attaching vocabulary terms (Thompson & Rubenstein, 2000). Thus, students are instructed to explore without heavy instruction or lecture. Instead, the teacher will build procedural fluency from conceptual understanding (NCTM, 2014). As students work, language remains informal substituting familiar terms in place as needed, and it isn’t until the closure of each lesson that formal language, definitions, symbolic notation, or formulas are introduced. This is aimed to get students more accustomed to the underlying concepts and provide a smooth transition to technical vocabulary and procedures (Leung, 2005).

Additionally, students are presented with a variety of ways to solve problems (Reaburn, 2013). As students solve problems through the unit, they are pushed to create representations to model the problems (Watson & Kelly, 2007). The unit should progress through representations such as Venn diagrams, two-way tables, and tree diagrams. While students present different methods, the teacher will facilitate mathematical discourse promoting the use of multiple representations (NCTM, 2014).

The unit plan is created to progress student thinking to that of Tarr & Jones’ Level 4 understanding. That is, students are pushed to use numerical reasoning, decipher independent and dependent events, and provide explanations for their probabilities. When solving each problem throughout the unit, students are always asked to explain their methods. This is to indicate what level of thinking they are at, so the teacher can progress them accordingly. Additionally, in each lesson, students are asked to make predictions, collect data, and reexamine solutions as Tarr & Lannin (2005) encouraged.

Day 1 Lesson – Intro to Probability: Monty Hall Problem

The first lesson plan introduces conditional probability in a way that students must first collect and analyze data following a game's guidelines. Tarr & Lannin (2005) support such self-generated data analysis, and Castro (1998) indicates that a student-led discovery lesson can promote learning. Therefore, students are instructed to explore the problem, but they are never formally introduced to the idea that they are finding and comparing conditional probability values. Again, this is aimed to follow Thompson & Rubenstein (2000) suggestion that the building of concepts should be prioritized first. Thus, students are only prompted with using self-judgement to predict the best outcome of the game at the start of the lesson. They then get to investigate using simulation to formalize a conclusion. This decision for the lesson follows NCTM recommendations that students "draw conclusions from data in order to answer questions or make informed decisions" (2000, p. 325).

Moreover, implementing simulation as a student exploration task aims to promote problem solving which is one of the eight mathematical teaching practices in NCTM (2014). The game provides a rich activity in which students must reason mathematically through as they discover the odds of winning. In addition, this task targets the sixth mathematical teaching practice, building procedural fluency from conceptual understanding, by allowing students to conceptualize conditional probability with the simulation before introducing its procedures (NCTM, 2014). Rossman & Short (1995) state that such interactive activities make learning conditional probability understandable and interesting.

In order to evaluate student learning as they investigate, it is recommended that the teacher is consistently assessing students. Hence, the lesson provides purposeful questions throughout to ask as students work (Groth, 2010; NCTM, 2014). Groth (2010) also emphasizes the importance of metacognition, thus one of the questions seen is to compare their initial prediction to the results they obtain. Then, as a whole class, students can discuss their findings, compare their predictions to their conclusion, and form an argument for how someone should play the game. In effort to having meaningful discourse and promote reasoning, students are asked to use mathematical reasoning from the simulation to justify their decision as the teacher facilitates class discussion (NCTM, 2014). Further, the goal of students collecting their own real data is to provide a natural transition into conditional probability in the following days (Watson, 1995).

Day 2 Lesson – What is Conditional Probability?

The second day of the unit plan builds from the previous day's simulation and introduces the term conditional probability. The teacher will start with facilitating a discussion of the previous day's game to transition into the meaning of conditional probability. Students are encouraged to use what they know about the word "conditional" in everyday language to connect a mathematical meaning to conditional probability (Lavy & Mashiach, 2009). In addition, students are told that the Monty Hall game was an example of conditional probability to help them build a definition. Then, they can use this conceptual knowledge for the day's problems (NCTM, 2014)

To begin the lesson, students are asked to collect data on two categorical variables in which they will later calculate probabilities from. To help visualize the scenarios, students are instructed to create some representation to model the data as encouraged by Watson & Mortiz (2002). Students are given the opportunity to use varied diagrams such as Venn diagrams and two-way tables to show the data. The goal of this lesson is to implement NCTM's (2014) mathematical practice to use and connect multiple representations. Thus, the problems offer varied entry points to organize the data, and a class discussion follows in which they compare student representation methods. Kvatinsky & Even (2002) stated that students should be introduced to a variety of representations and to know when to use multiple models, so when it is time to review student solutions, it is encouraged to select students with different perspectives. During discussion, questions are provided to emphasize having students explain their conditional probability through their representation. Questions in which students must compare similarities and differences are also provided to promote mathematical discourse (NCTM, 2014). Specifically, it is suggested for the teacher to have students share Venn diagrams and two-way table representations. Prodromou (2016) emphasizes the importance of two-way tables in understanding conditional probability, because it allows students to visualize the conditional sample space. Allowing students to share and compare multiple representations can provide alternate, preferred methods for them to turn to (Reaburn, 2013).

Problems in the second day's lesson were worded according to Ancker (2006) suggestion. It was stated that the phrase "given" can be problematic to student understanding, so the conditional probability questions in the lesson were asked for the probability of A “within” B. The relation to the word “within” and set theory helps provide a visual to students when used in correspondence to representations such as Venn diagrams (Ancker, 2006). Lastly, the closure of the lesson also aligns with Ancker’s (2006) suggestion by introducing the symbolic representation of conditional probability, $P(A|B)$, as the probability of A “within” B. Using what students have learned through the lesson, they are instructed to rewrite the problems using the symbolic notation. This provides another representation to compare with the diagrams, and it follows Leung (2005) that symbolic terms should be introduced after learning through exercises.

Day 3 Lesson – Conditional Probability & Independence

Day three’s goal is for students to compare two common types of conditional probability problems, “with” or “without” replacement scenarios, and introduce the meaning of independence. It continues to highlight the mathematical teaching practices: use and connect mathematical representations, meaningful mathematical discourse, as well as elicit and use evidence of student thinking seen in NCTM (2014). Instead of using typical urn problems, students are posed with questions referring to their class data collection from day two. Such data analysis is highlighted in NCTM (2000). Further, Watson & Mortiz (1999) supports implementing more data-driven, social context problems than marbles in an urn. As seen in day two, students are then pushed to create representations for the problems. Students may return to making two-way tables. However, the goal is for the creation of a tree diagram to come into discussion. In the case that no student makes a tree diagram, it is encouraged to have a “fake student work” prepared for the teacher to facilitate discourse around (NCTM, 2014). Tomlinson & Quinn (1997) indicate that tree diagrams are good representations for multistage experiments, so a conversation of comparing which models work best in given scenarios can be had. Further, including a tree diagram into discussion can encourage students to continue to solve problems in a variety of way. This demonstrates multiple representation methods for students to turn to when needed (Reaburn, 2013)

Class discussion is done differently in day three. All groups of students are required to explain a problem and their solution method thoroughly. The problem they present is strategically selected by the teacher in correspondence to Smith & Stein's five practices (Nabb et al., 2018). Additionally, the teacher is encouraged to select multiple representations of the same problems to be presented for students to compare (NCTM, 2014). Then, in each presentation students are to provide classmates with feedback on their solution strategies. This promotes student-to-student feedback and allows them to "talk mathematics" which can benefit understanding (Thompson & Rubenstein, 2000). Further, students are prompted to consider how their classmates representation solution differs to their own point of view which can enhance conditional probability understanding according to Reaburn (2013).

Finally, students engage in whole group discussion to compare the "with" and "without" replacement questions. The definition of independence is not introduced until the closing discussion per Leung (2005), and then, students are to grapple with the meaning of independence in relation to their previous work in the unit. For example, they are to recall the Monty Hall problem from day one and determine if it is independent or dependent. This is aimed to build new learning from prior knowledge and reexamine problems in new contexts (NCTM, 2000; Tarr & Lannin, 2005). Moreover, when students can decipher between independent and dependent events using mathematical reasoning, they indicate Tarr & Jones (1997) Level 4 understanding.

Day 4 Lesson – Social Contexts Variety

In the last day's lesson, students are presented with a variety of conditional probability problems. The included problems have been seen in several works of research including Watson & Kelly (2007), but they have been modified to incorporate student explanations for their solutions. These problems include a basic urn problem and progress to some realistic events like determining the probability of a hit-and-run (Watson & Mortiz, 1999). The latter describes a real-life data scenario which Neumann et al. (2013) recommends. Further, both the urn problem (Falk, 1986) and the hit-and-run (Tversky & Kahneman, 1974) date back to older research on conditional probability. As research promotes solving problems in a variety of ways (Reaburn, 2013), students can choose which solution strategy or representation they use to solve each problem given in the day's task. NCTM (2014) encourages such types of problems as they provide multiple entry points for student solutions. Additionally, this facilitates student learning by allowing them to use valuable representations and visual aids (Diaz, 2007; Watson & Kelly, 2002).

Due to the variation of solution strategies for the day's task, class discussion can be valuable in this lesson. As students work through the problems, it will be important for "the teacher to decide what approaches to share, the order in which they should be shared, and the questions that will help students make connections among the different strategies" (NCTM, 2014). As noted in the lesson plan, the teacher will want to strategically select the order of what student work is shown to elicit multiple perspectives.

The fourth day's activity concludes with a question that requires students to explicitly explain their solution strategy choices. This requires students to consider the alternative representations and provide justification for their best choice (Fast, 1999). Kvatinsky & Even (2002) state students should know how and when to use alternative ways of approaching problems. Thus, after discussion, students can reevaluate their solution strategies in comparison to their peers. NCTM (2000) says, "effective learners recognize the importance of reflecting on their thinking", so the final discussion prompt justifying students' methods allows for them to do so.

Conclusion

The purpose of this thesis was to provide teachers a unit plan on conditional probability using research-based suggestions to better student understanding. The literature followed that there have been several cognitive issues when learning probabilistic concepts. Further, it often stated how waiting until the later years of high school to teach conditional probability was a disservice to our students (Watson, 1995). Thus, many studies provided frameworks and strategies to improve student learning (Jones et al., 1997; Tarr & Jones, 1997). Using such studies, the unit plan in Chapter 3 was created. It considers common misconceptions and uses teaching strategies provided by the research to help combat such issues.

Throughout this thesis, it's been clear that the teaching and learning of conditional probability has been a challenge for several decades. The language of such statistical notions can be problematic for students, and it has also been shown that it is often problematic for the average adult (Ancker, 2006). Thus, as time has progressed, statistical and probabilistic literacy has become more and more important for our students. With students being surrounded by data, the incorporation of these concepts should be emphasized in the classroom, and the push for doing so has only just begun in the 21st century (Watson, 1995).

The unit plan provided in this thesis is only one step into educational reform. This lesson structure shows what can be implemented in the classroom to gain student understanding by allowing them the sense of control. Teachers of conditional probability should prioritize taking the time to meet their students where they are (Castro, 1998; Groth, 2010; Konold, 1991; Stohl, 2005). Further, they should encourage student discovery of concepts before lecturing procedures (Castro, 1998). As teachers introduce conditional probability, they should decompose language (Ancker, 2006; Leung, 2005), and also allow students the ability to choose from multiple representations to understand problems (e.g., Diaz, 2007; Fast, 1999; Kvatinsky & Even, 2002; Prodromou, 2016; Reaburn, 2013; Watson & Kelly, 2007; Watson & Mortiz, 1999). Thus, teachers should continue to advocate for statistics in mathematics courses, and in doing so, our students will benefit in the data-driven future.

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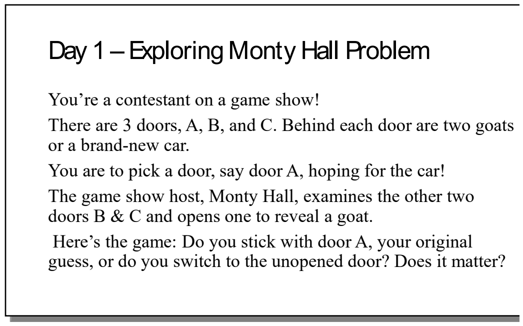
APPENDICES

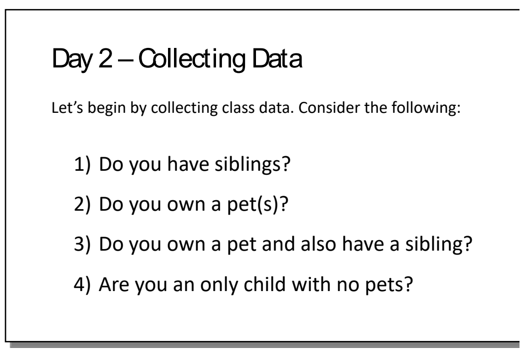
APPENDIX A

Unit Plan PowerPoint Example Slides

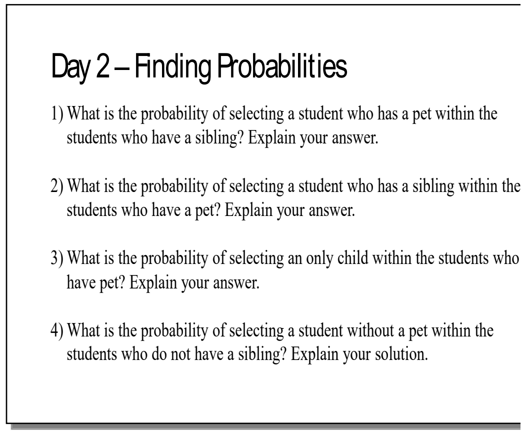
Slide 1



Slide 2



Slide 3



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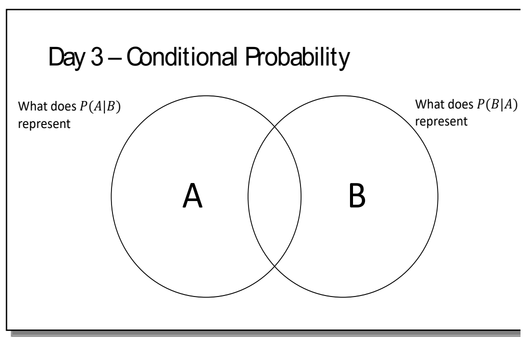

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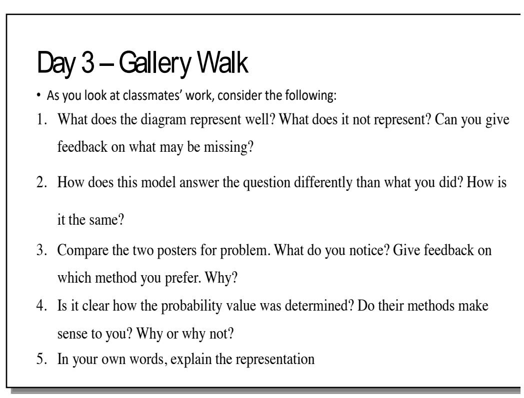


Slide 6

Day 3 – Independence

Two events are ***<u>independent</u>*** if knowing whether or not one event has occurred does not change the probability that the other event will occur.

Slide 7

Day 4 - Closure

On your index card, rank the solution strategies of conditional probability from your favorite to your least favorite. Explain your reasoning for each.

1. *Venn Diagram*
2. *Two Way Table*
3. *Tree Diagram*
4. *Formula*

APPENDIX B

Student Activities / Prompts

Day 1 – Monty Hall Problem

There are 3 doors, A, B, and C. Behind each door are two goats or a brand-new car. You are to pick a door, say door A, hoping for the car! The game show host, Monty Hall, examines the other two doors B & C and opens one to reveal a goat. Here's the game: Do you stick with door A, your original guess, or do you switch to the unopened door? Does it matter?

Day 2 – Conditional Probability Questions

1) What is the probability of selecting a student who has a pet within the students who have a sibling? Explain your answer.
2) What is the probability of selecting a student who has a sibling within the students who have a pet? Explain your answer.
3) What is the probability of selecting an only child within the students who have pet? Explain your answer.
4) What is the probability of selecting a student without a pet within the students who do not have a sibling? Explain your solution.

Day 3 – With/Without Replacement Data Questions

1) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who are an only child? Explain.
2) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who own pets? Explain.

3) If two students are randomly selected and the first student selected is not replaced before selecting the second, what is the probability of getting two students who are an only child? Explain.
4) If two students are randomly selected and the first student is not replaced before selecting the second, what is the probability of getting two students who own pets? Explain.
5) What's different between #1/2 and #3/4? Explain.

Day 3 Exit / Homework

1) Which of today's problems would you describe as independent? Explain your reasoning.

2) In our sibling versus pet owner data, are the two events independent? Why or why not?

3) Did playing the Monty Hall game have independent events?

4) The following two-way table shows data collection on 100 students' grade level and preferred subject:

	Juniors	Seniors	Total
English	22	33	55
Math	18	27	45
Total	40	60	100

a) Find $P(Juniors)$

b) Find $P(Junior \mid Math)$

c) Find $P(Junior \mid English)$

d) Are these independent or dependent events? Explain.

5) Given the above examples, generalize a formula to determine independence.

Day 4 Lesson Questions:

1. Which probability do you think is bigger? Explain your answer.

(a) The probability that a woman is a schoolteacher.

(b) The probability that a schoolteacher is a woman.

(c) Both (a) and (b) are equally probable.

2. The following data shows the number of defective TV's produced every week at two factories by the day shifts and by the night shifts.

    - 40 defective TVs by Factory A's Day Shift
    - 40 defective TVs by Factory A's Night Shift
    - 30 defective TVs by Factory B's Day Shift
    - 60 defective TVs by Factory B's Night Shift

    (a) Create a representation to model the data. Explain your choice. Is there another representation that could have been used?

    (b) If you were told that a defective TV was produced by a Day Shift, would it be more likely to have been made at Factory A or Factory B?

    (c) If you were told that a defective TV was produced at Factory A, what is the probability that it was produced by Day Shift?

3. An urn has two white balls and two black balls in it. Two balls are drawn out without replacing the first ball. Create a model for the scenario.

(a) What is the probability that the second ball is white, given that the first ball was white?

(b) What is the probability that the first ball was white, given that the second ball is white?

(c) Compare your results from (a) and (b). Were your methods the same? Why or why not?

(d) Are these events independent? Explain your thinking.

4. A taxi was involved in a hit and run accident late at night. Two taxi companies operate in the city: Green and Blue. This question asks you how likely it is that the taxi was Green or Blue. Of the taxis in the city, 90% are Green Taxis, and 10% are Blue Taxis. A witness identified the taxi involved in the accident as a Blue Taxi. On a different night, this witness was tested to identify the taxis going past in the street, under similar conditions to the night of the accident. For both the Green and the Blue Taxis, he identified the colour correctly for 80% of them, and incorrectly for 20% of them. What is the probability that the taxi involved in the accident was actually a Blue Taxi? Create a model.

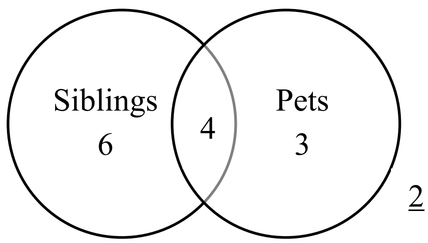
5. Did you use the same representation for each problem? Why or why not? Explain how you chose when to use each model.

APPENDIX C

Teacher Worked Solutions to Activities

Day 2 – Conditional Probability Questions: Mock Data

Example)	**Siblings (S)**	**No Siblings (S^c)**	**Total**
Pets (P)	4	3	7
No Pets (P^c)	6	2	8
Total	10	5	15



Note: Explanations will vary. Students should refer to their representation.

1) What is the probability of selecting a student who has a pet within the students who have a sibling? Explain your answer. $P(P|S) = \frac{4}{10} = \frac{2}{5} = 0.4$

2) What is the probability of selecting a student who has a sibling within the students who have a pet? Explain your answer. $P(S|P) = \frac{4}{7} = 0.51$

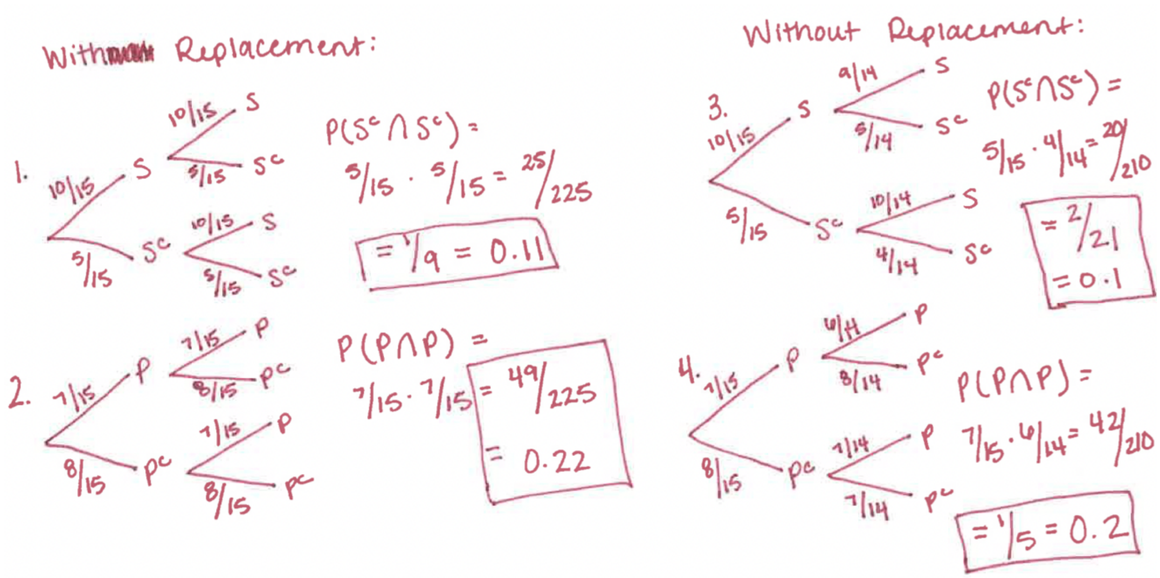
3) What is the probability of selecting an only child within the students who have pet? Explain your answer. $P(S^c|P) = \frac{3}{7} = 0.43$

4) What is the probability of selecting a student without a pet within the students who do not have a sibling? Explain your solution. $P(P^c|S^c) = \frac{2}{5} = 0.4$

Day 3 – With/Without Replacement Data Questions

Note: Referring to the data from previous example.

See the image below for worked solution and diagram for 1 – 4.



1) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who are an only child? Explain.
2) If two students are randomly selected, replacing the first student before selecting the second, what is the probability of getting two students who own pets? Explain.
3) If two students are randomly selected and the first student selected is not replaced before selecting the second, what is the probability of getting two students who are an only child? Explain.
4) If two students are randomly selected and the first student is not replaced before selecting the second, what is the probability of getting two students who own pets? Explain.
5) What's different between #1/2 and #3/4? Explain. Answers will vary.

Day 3 Exit / Homework Key

1) Which of today's problems would you describe as independent? Explain your reasoning.

   Without replacement scenarios – Explanations will vary. Student should mention the probabilities being unchanged.

2) In our sibling versus pet owner data, are the two events independent? Why or why not?

   $P(S) \neq P(S|P) \neq P(S|P^c)$ or $\frac{10}{15} \neq \frac{4}{7} \neq \frac{6}{8}$
   OR
   $P(P) \neq P(P|S) \neq P(P|S^c)$ or $\frac{7}{15} \neq \frac{4}{10} \neq \frac{3}{5}$

3) Did playing the Monty Hall game have independent events?

   No, because the probability of winning changes after revealing a door.

4) The following two-way table shows data collection on 100 students' grade level and preferred subject:

	Juniors	Seniors	Total
English	22	33	55
Math	18	27	45
Total	40	60	100

   e) Find $P(Juniors) = \frac{40}{100} = \frac{2}{5} = 0.4$

   f) Find $P(Junior \mid Math) = \frac{18}{45} = \frac{2}{5} = 0.4$

   g) Find $P(Junior \mid English) = \frac{22}{55} = \frac{2}{5} = 0.4$

h) Are these independent or dependent events? Explain. Independent.

Explanations will vary. Students should discuss having the same probabilities.

5) Given the above examples, generalize a formula to determine independence.

Answers will vary. Independent events have same probability regardless of the conditional. Further, $P(A) = P(A|B) = P(A|B^c)$

Day 4 Lesson Questions Key:

1. Which probability do you think is bigger? Explain your answer.

   (a) The probability that a woman is a schoolteacher. $P(T|W)$

   (b) The probability that a schoolteacher is a woman. $P(W|T)$

   (c) Both (a) and (b) are equally probable.

   $P(W|T) > P(T|W)$, because out of all teachers, you have limited genders within. However, out of all women, you have countless jobs they could have. Student responses will vary.

2. The following data shows the number of defective TV's produced every week at two factories by the day shifts and by the night shifts.

   - 40 defective TVs by Factory A's Day Shift
   - 40 defective TVs by Factory A's Night Shift
   - 30 defective TVs by Factory B's Day Shift
   - 60 defective TVs by Factory B's Night Shift

   (a) Create a representation to model the data. Explain your choice. Is there another representation that could have been used?

Example	**A**	**B**	**Total**
Day Shift	40	30	70
Night Shift	40	60	100
Total	80	90	170

(b) If you were told that a defective TV was produced by a Day Shift, would it be more likely to have been made at Factory A or Factory B?

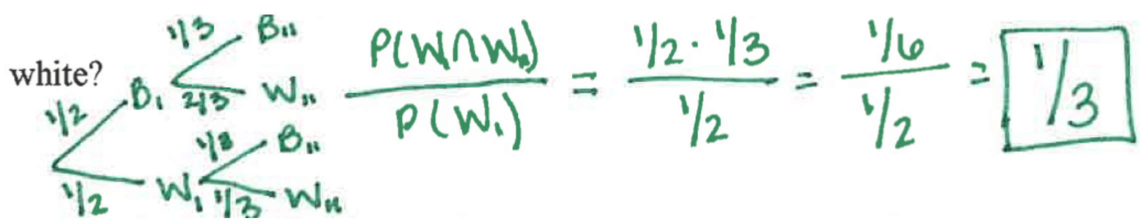
$P(A|D) = \frac{40}{70} = \frac{4}{7} = 0.57 > P(B|D) = \frac{30}{70} = \frac{3}{7} = 0.43$, so it would be more likely to have been made at Factory A. It has a higher probability.

(c) If you were told that a defective TV was produced at Factory A, what is the probability that it was produced by Day Shift?

$$P(D|A) = \frac{40}{80} = 0.5$$

3. An urn has two white balls and two black balls in it. Two balls are drawn out without replacing the first ball. Create a model for the scenario.

(a) What is the probability that the second ball is white, given that the first ball was white? See image below.



(b) What is the probability that the first ball was white, given that the second ball is white?

Using part (a) tree diagram: $\frac{P(W_I \cap W_{II})}{P(W_{II})} = \frac{\frac{1}{2}*\frac{1}{3}}{\left(\frac{1}{2}*\frac{2}{3}\right)+\left(\frac{1}{2}*\frac{1}{3}\right)}$

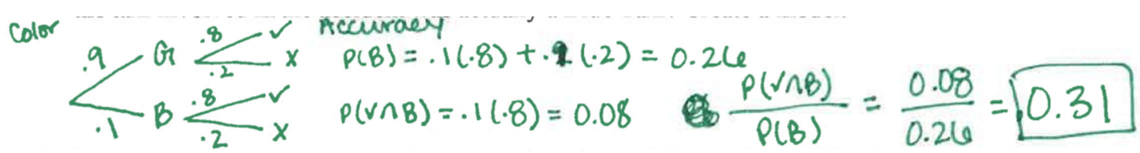
(c) Compare your results from (a) and (b). Were your methods the same? Why or why not?

Part (b) had to consider the two options of getting $W_{11} = B_1 \cap W_{11}$) and $P(W_1 \cap W_{11})$. The sum of these intersections is the denominator or sample space.

(d) Are these events independent? Explain your thinking.

No, because this is a without replacement situation. Two events are independent if knowing whether the conditional occurred does not change the probability of the event. Here, knowing the 1st draw affects the 2nd marble's probability.

4. A taxi was involved in a hit and run accident late at night. Two taxi companies operate in the city: Green and Blue. This question asks you how likely it is that the taxi was Green or Blue. Of the taxis in the city, 90% are Green Taxis, and 10% are Blue Taxis. A witness identified the taxi involved in the accident as a Blue Taxi. On a different night, this witness was tested to identify the taxis going past in the street, under similar conditions to the night of the accident. For both the Green and the Blue Taxis, he identified the colour correctly for 80% of them, and incorrectly for 20% of them. What is the probability that the taxi involved in the accident was actually a Blue Taxi? Create a model.
   See image below for model solution.



5. Did you use the same representation for each problem? Why or why not? Explain how you chose when to use each model.

   Answers will vary.