Student Understanding of the Limit Definition of the Derivative

by

Samuel Dolinger

A thesis presented to the Honors College of Middle Tennessee State University in partial fulfillment of the requirements for graduation from the University Honors College

Fall 2023

Thesis Committee:

Jeremy Strayer, Thesis Director

Philip Phillips, Thesis Committee Chair

Student Understanding of the Limit-Definition of the Derivative.

by Samuel Dolinger

APPROVED:

Dr. Jeremy Strayer, Thesis Director

Professor, Department of Mathematics

Dr. Philip E. Phillips, Thesis Committee Chair

Associate Dean, University Honors College

Abstract

The study of Calculus requires an understanding of the limit-definition of the derivative. This study examined students' concept images of functions, limits, and average rate of change, and how these relate to the limit-definition of the derivative. Four Calculus II students were given a task-based interview, and their responses were coded via a framework. The results suggest that students tend to rely on graphical representations, and that students require a symbolic understanding of average rate of change to be able to have a conceptual understanding of the limit-definition of the derivative.

Table of Contents

BACKGROUND 1
LITERATURE REVIEW 4
Metonymy 5
METHODS 5
Task-Based Interview 6
Framework 6
RESULTS 8
Abundance of Graphical Representations 9
Significance of Symbolic Representation 11
AROC-Symbolic 12
CONCLUSIONS 14
REFERENCES 15
APPENDIX 19
Appendix A: Student Task 19
Appendix B: GeoGebra Task Screenshots 22

LIST OF TABLES/FIGURES

Table 1: Zandieh's framework for the metonymies of the derivative. 4

Table 2: Framework for coding students' responses. 7

Table 3: Code matrices of all students. 9

Figure 1: Joseph's graph of the derivative. 11

Figure 2: Joseph's graph of the limit. 11

Figure 3: Tristan's graph of the derivative 11

Figure 4: Tristan's graph of AROC between two points. 11

Figure 5: Secant line (AROC) between two points on a parabola. 22

Figure 6: Secant line (AROC) and tangent line on a sine graph 22

Figure 7: Secant line and tangent line on the graph $f(x) = 1/x$. 23

Figure 8: Secant line and tangent line on a parabola. 23

LIST OF ABBREVIATIONS/TERMS

LIMIT-DEFINITION: Limit-definition of the derivative.

ASPECTS: Aspects refers to the three aspects of the derivative which are functions, limits, and average rate of change.

REPRESENTATIONS: Representations refers to symbolic, physical, verbal, and graphical representations

AROC: Abbreviation of average rate of change.

Background

Calculus is a study of change, and the derivative is one of the most important concepts. This project investigates how calculus students understand the definition of this foundational concept. The derivative is the instantaneous rate of change of a function, and it is often represented graphically by the slope of a tangent line (i.e., a line that locally goes through a single point on a graph). The mathematical definition of the derivative of a function, or the *limit-definition of the derivative*, is the average rate of change for the function between two infinitesimally close points, which results in the "instant" rate of change. The limit-definition of the derivative combines three mathematical ideas: function, average rate of change, and limit.

A function is a rule that relates an input quantity x to an output quantity y (Clement, 2001). The function is a foundational concept for the derivative, and understanding "covariability" in a function's relationship, or how the output changes with respect to the input, is key for understanding the derivative of that function. Average rate of change is a ratio of the change in the output y with respect to the input x and is graphically represented as the slope of a secant line (line that goes through two points on the graph of a function) (Avgerinos, 2021). More technically, the average rate of change for a function is the constant rate of change that would produce the amount of change in the function's output variable for a given change in the input variable. Understanding the average rate of change is vital because the derivative is the instant rate of change, so they are quite similar. The limit is the hardest concept of the three for some students because it involves the concept of infinity (Jones, 2015). A limit of a function can be informally described as the value that the output of a function approaches as the inputs get closer to a given input value (Swinyard, 2012). Because the limit "approaches" a value, it gets infinitesimally closer, but it never reaches it. Combining these ideas, we get the limit-definition of the derivative which is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Literature Review

How do students come to understand multilayered content like the derivative? We know that, in general, students do not typically understand mathematics by extracting meaning from a definition. Instead, students learn by developing their concept image of an idea (Clement, 2001; Kidron 2015). A concept image is "the total cognitive structure that is associated with the concept," or all the mental pictures surrounding an idea (Clement 2001; Tall, 1981). Because students learn this way, it is imperative that their concept image of an idea become cohesive with the mathematical definition of that idea. When a student has a strong or very developed concept image, this can be referred to as a conceptual understanding which means that a student understands the "big picture" of a particular concept (Delos Santos, 2003). Students sometimes only understand the processes of a concept through memory, and this leads to a weaker understanding known as procedural understanding (Juter, 2006). The derivative has many processes attached to it that students learn, but this can lead to students adopting a procedural understanding which is detrimental for understanding the limit-definition as a concept (Delos Santos, 2003).

Since this study is focused on students' concept images of the limit-definition, it is important to distinguish how students understand mathematical definitions. Students can be categorized as formal and natural thinkers (Kabael, 2014). A formal thinker *extracts* meaning from the definition into their concept image. A natural thinker *gives* meaning to the definition by using related concept images. Kabael's study showed that natural thinkers could use dynamic reasoning to generate a sound concept image of a mathematical definition. Dynamic reasoning refers to how a student understands infinity by using "motion." For example, a student might imagine that infinity is an inexhaustible process as one can keep adding numbers and get "closer" to infinity (Jones, 2015). The studies by Kabael and Jones provide a foundation of how a student with natural thinking could develop a strong concept image that would align with the formal derivative. A student in Jones's study was asked to solve $\lim_{x \to \infty} \frac{\cos(x)}{x}$ and the student reasoned that the numerator *cos(x)* will always be between *1* and *-1* and that the denominator *x* will get increasingly large, thus the student says that the function will get smaller and smaller as it "goes throughout time." This co-variability is crucial for the limit-definition of the derivative.

Metonymy

When analyzing student responses to ascertain their understanding of an idea, several different strategies can be employed—specifically, the use of metaphor and metonymy. In mathematics, a metaphor is a way of explaining a mathematical idea by using a familiar idea (Rodríguez-Nieto, 2022; Zandieh, 2006). An example of a metaphor in this context would be how young students are taught fractions by imagining slices of a pizza. Students are familiar with the idea of a pizza having only half of its size, and this helps them understand fractions in a mathematical sense. The metonymy will be a more prevalent idea in this study. A metonymy is using a part of something to represent the whole (Zandieh, 2006). An example that Zandieh uses is how “a set of wheels” refers to a car. Furthermore, Zandieh’s study categorized different aspects, or metonymies, of the derivative which are limits, functions, and ratio, and related these three aspects to four representations: symbolic, physical, verbal, and graphical. In this study Zandieh showed how students operate with the three aspects and four representations, and she shows how students incorrectly refer to the derivative as only one of these three aspects. When a student was asked “What is the derivative?” The student replied, “It’s just a slope” (Zandieh 2006). This was a common occurrence in her study, and since students will be operating with multiple aspects of derivative, her framework is vital to differentiate students’ metonymies of derivative. Her framework shown in Table 1 was adapted for this study to classify when a student uses a particular aspect of derivative, and if their usage of certain aspects leads to a strong concept image of the limit-definition.

Table 1: Zandieh’s framework for the metonymies of the derivative

	Graphical	Verbal	Paradigmatic Physical	Symbolic
Process-object layer	Slope	Rate	Velocity	Difference Quotient
Ratio				
Limit				
Function				

Methods

This study aims to inform our understanding of how students understand the limit-definition of the derivative.

Specifically, the study asks these research questions:

1) How do students make connections between functions, limits, and average rate of change (specifically with respect to different representations of these concepts (symbolic, physical, verbal, and graphical), and how do these connections relate to their concept image of the limit-definition of the derivative when given a task that provides them the opportunity to make those connections?
2) What concept images do students draw on when thinking about these different representations for understanding the derivative?

Data was collected via four task-based interviews that covered functions, limits, average rate of change, and the limit-definition of the derivative. Four Calculus II students participated in the study, and each interview was completed in an hour. Calculus II students were chosen as these students have all taken a Calculus I course and have been exposed to the limit-definition in the prerequisite Calculus I course. Furthermore, students that take Calculus II have a STEM (Science, Technology, Engineering, and Mathematics) major, so they would have more of an interest in understanding mathematics. Potential participants were selected by Calculus II instructors and recruitment emails were sent to all potential participants. All participants were given pseudonyms. The interviews were transcribed, and the students' responses were coded from the framework adapted from Zandieh (see Table 2).

Task-Based Interview

The task-based interview consisted of four sections: a section for functions, a section for AROC, a section for limits, and a section for the limit-definition. Each section was designed for a student to be able to represent an aspect of the derivative using a graphical, verbal, symbolic, or physical representation. At the end of the interview, the students interacted with GeoGebra software that lets the students interact with the derivative via a graph. The GeoGebra software featured a graph with two points highlighted, and a student could alter one point and see the tangent line at one point and the AROC between the points. This software combined functions, limits, AROC, and the derivative in one task that allowed students to explore each aspect of the derivative and how they interact with one another. The task for the students and some screenshots of the GeoGebra software can be found in the Appendix (page 19).

Framework

After the interview, students' responses were coded according to the framework, and they were given a code matrix. A code matrix counts every instance of a student using a particular concept image, which is defined by an *aspect-representation pair.* An aspect-representation pair is formed by one of the three aspects of the derivative, and one of the four representations. The code matrices gave an overview of their concept images of the limit-definition and allowed comparison to be made between students. Furthermore, key quotes were recorded from students that show a notable concept image or connections between aspects of the derivative. The interviews were then coded again after writing an analytic memo of each student's concept image to see if any patterns emerged from their quotes/code matrices.

The framework codes by relating the three aspects of the derivative, that is, functions, limits, and AROC to four representations: symbolic, verbal, physical, and graphical. Symbolic representation is having an understanding through mathematical language or symbols. Verbal representation is having an understanding through non-mathematical language. Physical representation is having an understanding through a real-world example or application. Graphical representation is having an understanding through a graph (an *x*-*y* coordinate plane).

Table 2. Framework for coding students' responses.

Theoretical Framework for Student Understanding of the Derivative				
	Symbolic	Verbal	Physical	Graphical
Rate of Change	Rate of change of points	"Rise over Run"	"Rate of Velocity"	Tangent/Secant Line
Limit	"Infinitesimally close"	"Approaching 0"	"Limit as Velocity..."	"Move towards a point"
Function	Y=MX+B	"Linear function"	"Velocity Function"	Image of function
(Adapted from Zandieh, 2000)				

Results

The results were guided by the research questions that were concerned with what concept images students draw upon when thinking of the limit-definition, how students make connections between the aspects of the derivative, and how these connections relate to their concept image of the limit-definition. The code matrices for all students are shown in Table 3. In this section, I will describe several observations that inform my research questions. After I describe these matrix observations, I will explore these patterns in greater depth by presenting data from the student interviews to illustrate how students make connections between the aspects of the limit definition of the derivative and the four representations used to conceive of these aspects.

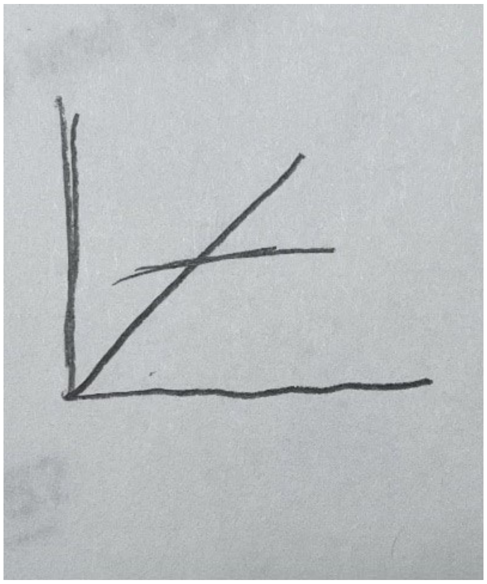
Abundance of Graphical Representations

The first significant observation in the code matrices is that all four students utilized graphical representations in their concept image of the limit definition of the derivative. Indeed, the code matrices in Table 3 showed that the number of graphical representations in each code matrix ranged from 38% to 50% (38% of Joseph's total entries are graphical and 50% of Thomas's and Tristan's total entries are graphical) of the total number of aspect-representation pairs in each code matrix.

Table 3. The code matrix of all study students.

Joseph	Symbolic	Physical	Verbal	Graphical
Rate of Change	1	0	4	0
Limit	0	0	4	3
Function	2	0	5	7
Thomas	Symbolic	Physical	Verbal	Graphical
Rate of Change	9	0	2	13
Limit	4	0	4	4
Function	2	2	0	6
Jack	Symbolic	Physical	Verbal	Graphical
Rate of Change	8	0	5	8
Limit	6	0	12	9
Function	3	0	3	12
Tristan	Symbolic	Physical	Verbal	Graphical
Rate of Change	3	0	7	9
Limit	0	0	5	4
Function	5	0	1	8

The data from the interviews reflect the abundance of graphical representations in the code matrices. When Thomas was asked “What is the derivative?” He replied, “So as for a derivative, I think of how a graph changes over time.” Likewise, every student at one point in the interview described the limit-definition with an explicit graphical concept image. Furthermore, every aspect of the derivative was described by the students with graphical concept images. Students described the aspects with phrases like: “I see a graph in my head”, “As being on a graph...” and many other phrases that indicate that their concept image is based on a graph. There are also multiple instances of the students drawing a graph to try and explain their understanding of a concept. For example, as Joseph drew the graph in Figure 3, he explained one of his concept images of the derivative: “You should think of a graph, a generic linear graph that looks like this, and then I think of the derivative as just the horizontal.” Joseph also drew a function to describe his concept image of the limit in Figure 4 and explained that you follow along the graph to find the limit. From Josephs’s graphs, it is evident he is thinking of the derivative and its aspects graphically. Likewise, Tristan also drew a graphical representation of the derivative, along with a graphical representation of AROC in Figure 5 and Figure 6 as to compare AROC and the derivative. Tristan, like Joseph, is thinking of the derivative and its aspects via a graphical representation. Because Tristan drew these graphs to compare, he may have a connection between the derivative and AROC via graphical representation. These graphs that Joseph and Tristan drew implies that their concept images for functions, AROC, the limit, and the derivative require a graph.



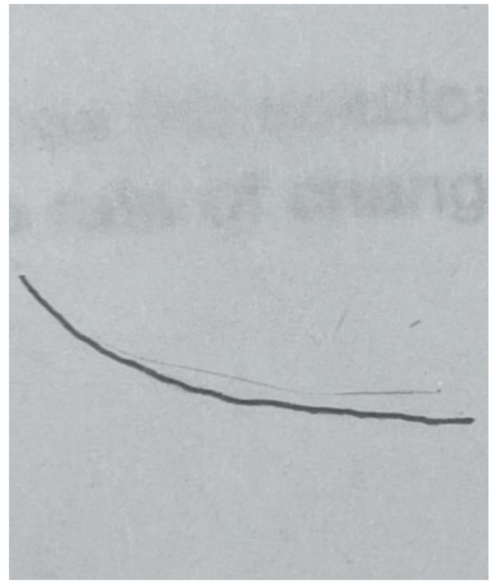
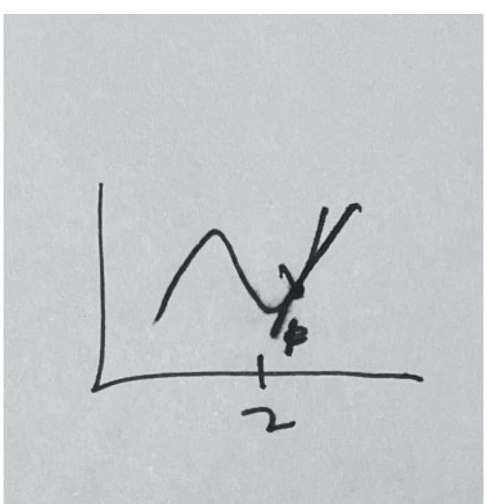
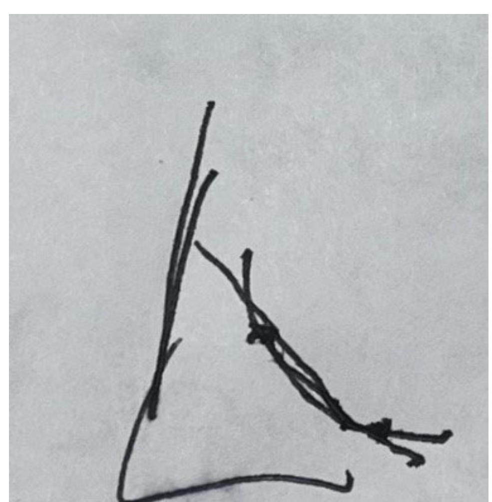


Figure 1: Joseph's graph of the derivative.

Figure 2: Joseph's graph of the limit.

From the matrices we see that all four students had a graphical understanding of the three aspects of the derivative. Additionally, the data from the interviews highlight the students' usage of graphical representations. I argue that it follows that these students rely on graphical representations to form their concept images that inform their understanding of the limit-definition of the derivative.





Symbolic Representation Significance

The symbolic representation seemed to be the most significant type of understanding for forming concept images. While the graphical representation is crucial to form most of the students' concept images, the symbolic understanding is crucial to form accurate concept images that align with the concept in question. The students who possessed a relatively large number of symbolic entries in their code matrices had accurate and rich concept images of the limit-definition and its aspects. The data from the interviews illustrates this further. Consider a quotation from Joseph who rarely used a symbolic representation: "Well, I know that the derivative is the tangent line." There is not much of a conceptual understanding that is evident in this statement, as Joseph is only using a verbal understanding. This statement is an example of an incorrect metonymy of the derivative and without any symbolic understanding, it appears that there is not a strong connection between a mathematical concept and the student's concept image which can lead to errors (Zandieh, 2006). Joseph also said about the derivative, "I understand the derivative as taking a degree from your function, and by doing that you get the slope of your initial function." Joseph is describing one of the many processes to compute the derivative, which is not strongly related to the concept of the limit-definition (Delos Santos, 2003). Furthermore, Joseph seems to have a procedural understanding of the derivative which hurts his understanding (Delos Santos, 2003; Juter, 2006). On the other hand, students who often used symbolic representation showed conceptual understandings of the aspects of the derivative. When Tristan was finding the AROC of the function $f(x) = 3x + 6$, he said: "The AROC is 3. Oh, I could have just looked at the function because m is the slope of a function $y = mx + b$." Tristan realized that since AROC is the slope, he could also just use the symbolic representation of a linear function, $y = mx + b$, to find the AROC. Because of this usage of symbolic representation, it seems that Tristan gained a conceptual understanding linking functions to AROC as Tristan did not just find the AROC, he found out *why* the AROC was 3 for this function. Similar to how graphical representations are useful for forming concept images, symbolic representations also form a concept image that the student can make accurate conclusions from. The next section will provide data that furthers the importance of symbolic representation.

Without symbolic representation, the students' concept image is more likely not to align with the concept, which can lead to errors. With symbolic representation, it seems the student has a good connection from their concept image to the concept itself, which results in a stronger concept image.

AROC-Symbolic Significance

Symbolic understanding is a critical element for a student's concept image of the limit-definition. Jack and Thomas had a remarkable number of symbolic representations for concept images that allowed them to form links between the aspects of the derivative and the limit-definition. Specifically, their code matrices in Table 3 show that the most common use of symbolic representation was on AROC. The interviews from Thomas and Jack highlight the importance of symbolic representations of AROC. Thomas said the following when he was asked about the limit-definition: "So whenever h is approaching 0, you are decreasing the distance between $x + h$ and x and basically what that means is as you get closer and closer to $h = 0$, you're getting closer to finding the instantaneous slope or the slope at one point**.**" Thomas essentially described *how* the limit-definition works in this quote. Thomas used a dynamic understanding of the limit to explain that the limit-definition acts like AROC between two points, x and $x+h$, and that these points are getting infinitesimally closer to one another which causes the AROC to be the slope at one point. (Jones, 2015; Kabael, 2014). The most significant part of this quote is the use of a symbolic understanding of AROC. The use of a symbolic understanding of AROC allowed Thomas to use his dynamic understanding of the limit to form his concept image of the limit-definition. Similarly, Jack had a symbolic understanding of the limit-definition, and he then used a dynamic understanding of the limit to form a concept image of the limit-definition, "As being on a graph, when h is approaching 0, the distance between x and $x + h$ becomes smaller." While Jack's quote is not as rich as Thomas's, Jack has a similar concept image. Jack's quote also features the covariability that is showcased by both Thomas and Jack, that is, the limit is changing the distance between x

and $x+h$ (Clement, 2001). In both quotations, the students' understanding of the limit is being applied to their symbolic understanding of AROC. From these key quotes, and from Thomas and Jack's rich code matrices, I argue that their concept images are dependent upon symbolic representations of AROC. Another indication of this fact comes from the code matrices in Table 3. A pattern that is noticeable is that if a student has a Limit-Symbolic in their code matrix, then they also have AROC-Symbolic. Indeed, this pattern in the matrices can be explained by the previous quote from Thomas and Jack. The only time in the interviews when a student uses a symbolic representation of a limit is when they are *already* using an AROC-Symbolic concept image. And like the quote before, the limit is being used dynamically *upon* their AROC-Symbolic concept image. These two representations together seem to be the core for a rich and complex concept image that is accurate with regards to the limit-definition.

The AROC-Symbolic concept image was the most common symbolic concept image for Thomas and Jack. AROC-Symbolic concept images seem to form a foundation on which a dynamic understanding of the limit can be used. The significance of the symbolic representation of AROC is that it appears to be the central, required concept image to have an accurate concept image of the limit-definition.

Conclusion

This study examined students' concept images of the three aspects of the derivative and how they relate to the limit-definition of the derivative. Four Calculus II students were given a task-based interview, and their responses were coded by a framework adapted by Zandieh. More importantly, the data indicates that an AROC-Symbolic concept image is central and necessary for a rich and accurate concept image of the limit-definition of the derivative. The results of this study echoed the results of the studies of Jones and Kabael where students could form a strong understanding of the limit with dynamic reasoning. Furthermore, the framework for Zandieh proved to highlight the applicability of dynamic reasoning. The study of concept images was vital to this research and is fundamental for studying student's understandings of any mathematical concept. Future research could be done on investigating students' formation of symbolic concept images to help foster an understanding of the limit-definition.

References

Avgerinos, E., & Remoundou, D. (2021). The Language of "Rate of Change" in Mathematics. European Journal of Investigation in Health, Psychology & Education (EJIHPE), 11(4), 1599–1609. https://doi-org.ezproxy.mtsu.edu/10.3390/ejihpe11040113

Clement, L. L. (2001). What Do Students Really Know about Functions? The Mathematics Teacher, 94(9), 745–748.

Swinyard. (2012). Coming to Understand the Formal Definition of Limit: Insights Gained From Engaging Students in Reinvention. Journal for Research in Mathematics Education, 43(4), 465–493. https://doi-org.ezproxy.mtsu.edu/10.5951/jresematheduc.43.4.0465

Delos Santos, A. G., & Thomas, M. O. J. (2003). Representational Ability and Understanding of Derivative. International Group for the Psychology of Mathematics Education, 2, 325.

Jones, S. R. (2015). Calculus limits involving infinity: the role of students' informal dynamic reasoning. International Journal of Mathematical Education in Science and Technology, 46(1), 105-126–126. https://doi-org.ezproxy.mtsu.edu/10.1080/0020739X.2014.941427

Juter, K. (2006). Limits of functions: Students solving tasks. Australian Senior Mathematics Journal, 20(1), 15–30.

Kabael, T. (2014). Students' formalising process of the limit concept. Australian Senior Mathematics Journal, 28(2), 23.

Khairudin, Fauzan, A., & Armiati. (2022). Hypothetical Learning Trajectory of Limit and Derivative Based on Realistic Mathematics Education. Special Education, 1(43), 3608–3624.

Kidron, I. (2015). Is small, small enough? Students' understanding the need for the definition of the derivative as a limit. International Journal for Technology in

Mathematics Education, 22(1), 31-42–42. https://doi-org.ezproxy.mtsu.edu/10.1564/tme_v22.1.03.

Marie Jean Mendezabal, & Darin Jan Tindowen. (2018). Improving students' attitude, conceptual understanding and procedural skills in differential calculus through Microsoft mathematics. Journal of Technology and Science Education, 8(4), 385–397. https://doi-org.ezproxy.mtsu.edu/10.3926/jotse.356

Rodríguez-Nieto, C. A. (1), Rodríguez-Vásquez, F. M. (1), & Moll, V. F. (2). (2022). A new view about connections: the mathematical connections established by a teacher when teaching the derivative. International Journal of Mathematical Education in Science and Technology, 53(6), 1231-1256–1256. https://doi-org.ezproxy.mtsu.edu/10.1080/0020739X.2020.1799254

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. Educational Studies in Mathematics, 12(2), 151–169. http://www.jstor.org/stable/3482362

Yoon, H., & Thompson, P. W. (2020). Secondary teachers' meanings for function notation in the United States and South Korea. Journal of Mathematical Behavior, 60. https://doi-org.ezproxy.mtsu.edu/10.1016/j.jmathb.2020.100804

Zandieh, M. J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. Journal of Mathematical Behavior, 25(1), 1–17. https://doi-org.ezproxy.mtsu.edu/10.1016/j.jmathb.2005.11.002

APPENDIX

APPENDIX A: STUDENT TASK

The Function:

1. Let $f(x) = x^2 + x + 6$
    1. What does $(f(2))$ equal?
    2. What does $f(h)$ equal?
    3. What does $f(x + h)$ equal?

Average Rate of Change:

The formula for average rate of change of a function $f(x)$ is:

$\frac{f(b)-f(a)}{b-a}$ where a and b are x-values on a graph.

1. Let $f(x) = 3x - 6$.
    1. What is the average rate of change between the points $(1, -3)$ and $(4, 6)$?
    2. What is the average rate of change between $x = 2$ and $x = 5$?

1) Let's consider a general unknown function $f(x)$.
    a) What is the representation of the average rate of change for an input value of x and an input value of $(x + h)$?

Questions for Average Rate of Change

1) How do you view what the average rate of change is? Tell me about how you understand what the average rate of change is.
2) How do you understand "average" in AROC?

The Limit:

1) Consider $f(x) = 2x^2 + 5$.
   a) What does $\lim_{x \to 0} 2x^2 + 5$ equal?
   b) What does $\lim_{x \to \infty} 2x^2 + 5$ equal?

2) Consider $f(x) = \frac{x^2+x}{x}$
   a) What does $\lim_{x \to 0} \frac{x^2+x}{x}$ equal?

3) Consider $f = \frac{x}{a}$
   a) What is $\lim_{x \to \infty} \frac{x}{a}$?
   b) What is $\lim_{x \to 0^+} \frac{x}{a}$?
   c) What is $\lim_{a \to \infty} \frac{x}{a}$?
   d) What is $\lim_{a \to 0^+} \frac{x}{a}$?

4) What is $\lim_{h \to 0} x + h$?

Questions for The Limit

1) How do you view the limit?
   i) When you think of the limit, do you think of “motion” as x-values get closer and closer to the value at the limit?
   ii) Do you think of the “end behavior” as the limit?

The Formal Derivative.

Consider the rate of change between two points:

$(x, f(x))$ and $((x+h), f(x+h))$

Now consider the limit as h approaches 0 of the average rate of change between these points.

So $\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

1. What do you think $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ means as it relates to the average rate of change?

2. Let $f(x) = 3x + 6$.
    1. Solve for $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
    2. What does the solution to part (a) tell you about the function as it relates to the average rate of change?

1) Questions
    a) Talk about how you understand the derivative.
    b) Looking at the definition of the derivative, how is it related to average rate of change? How is it different?
    c) How does the definition of the derivative relate to a tangent line? Specifically, why does the definition “produce” a tangent line.

APPENDIX B: GEOGEBRA SOFTWARE

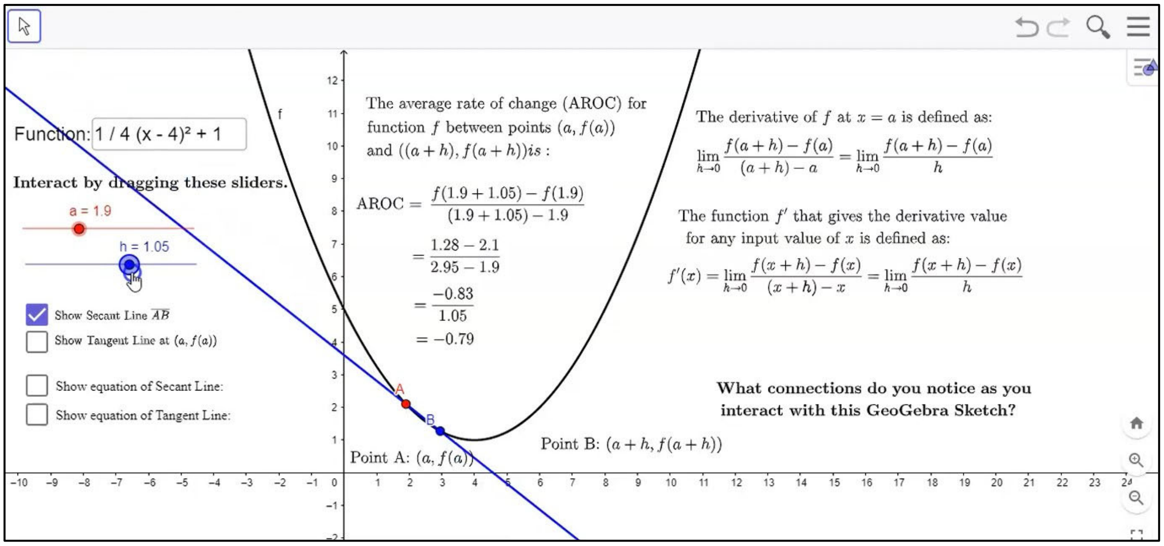


Figure 5: Secant line (AROC) between two points on a parabola.

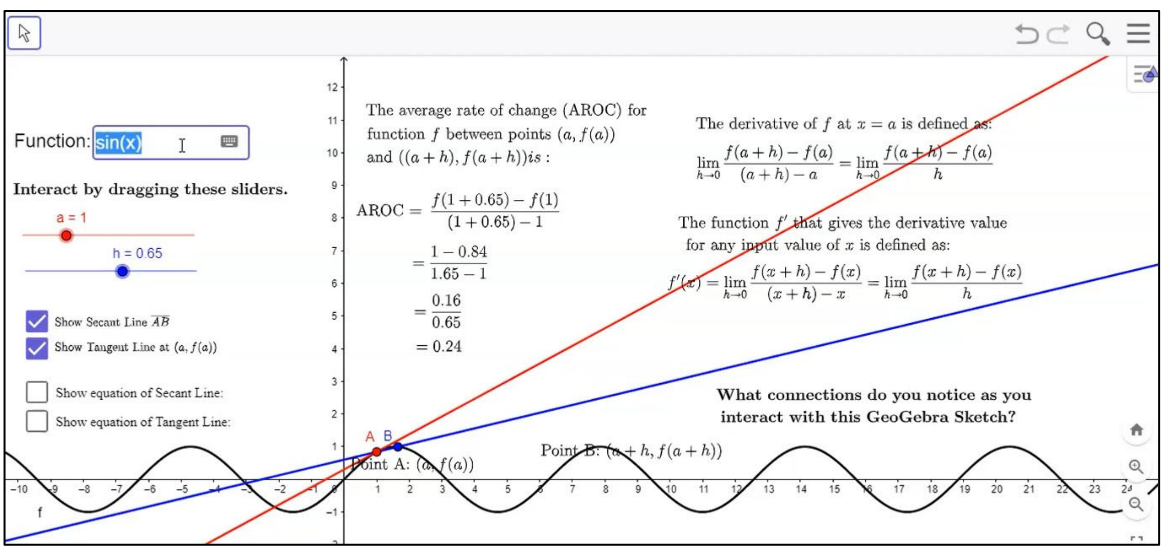


Figure 6: Secant line (AROC) and tangent line on a sine graph.

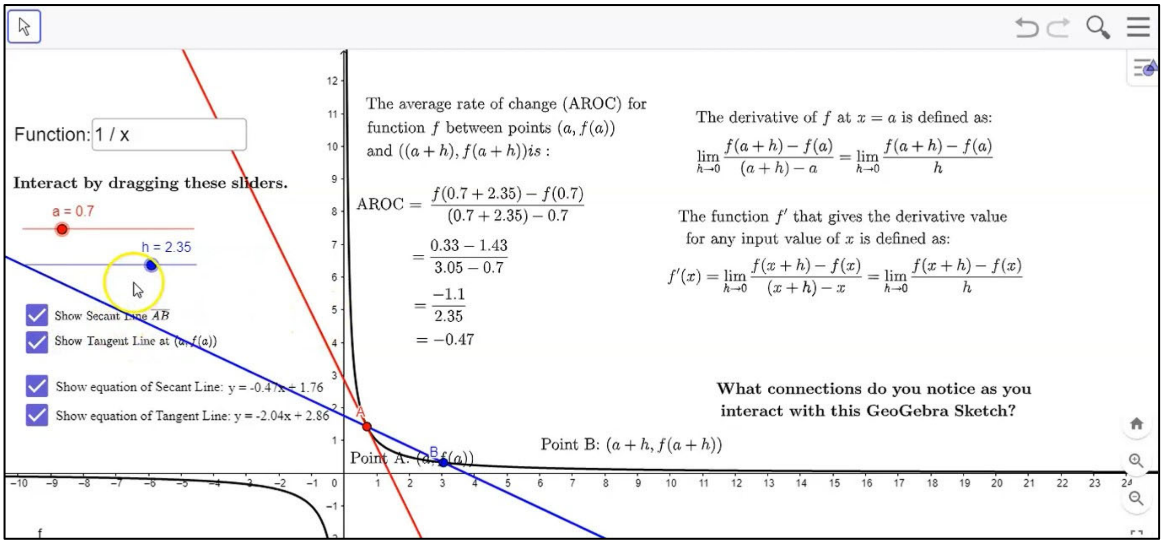


Figure 7: Secant line and tangent line on the graph $f(x) = 1/x$.

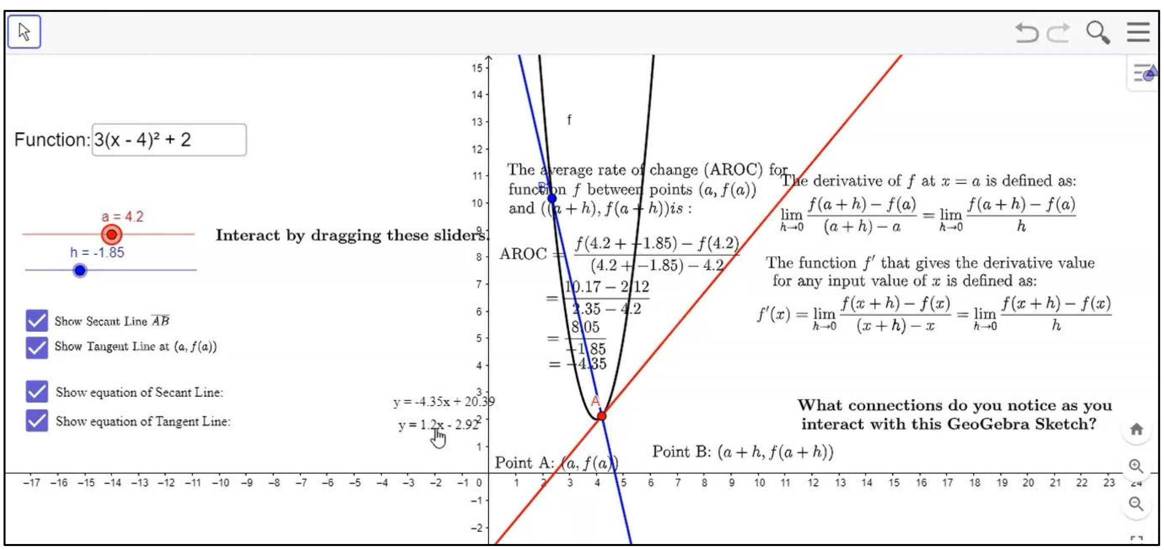


Figure 8: Secant line and tangent line on a parabola.