THEORY OF SOME DISCRETE SEIR MODELS AND THEIR APPLICATION TO COVID-19.

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ABSTRACT

In this thesis we explore the mathematical theory of some epidemiological models that represent infectious disease and try to establish the mathematical properties of the differential equations representing the models. We describe the SEIR models we study with time varying transmission and recovery coefficients and constant latency and vaccination rates. We prove that the models satisfy the requirements such as existence and uniqueness of solutions and the continuous dependence of solutions on initial conditions. Using these properties we derive the long term behavior and the condition for an outbreak to occur of the solutions. This helps us to understand the biological implications and the control measures that can be applied. We also develop an implicit discrete formulation for the numerical algorithms to use data and verify that the model can be used on the COVID-19 data.

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CHAPTER 1

INTRODUCTION

Mathematical modeling has been increasingly recognized as an important research tool for infectious diseases control. The objective of a mathematical model of an infectious disease is to describe the transmission process of the disease. When the disease spreads quickly to many people, it is an epidemic. When a disease spreads quickly, mathematical modeling tries to help the public health authorities to answer the following questions.[1]

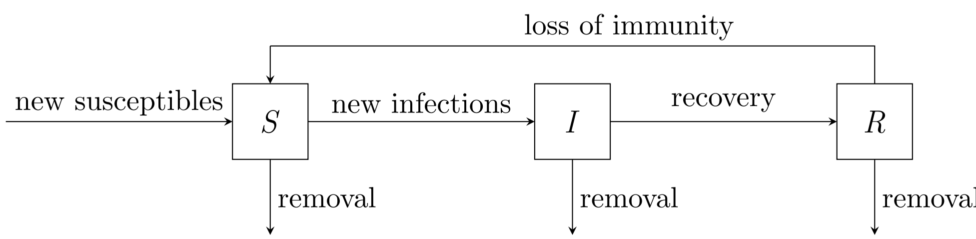
1. How severe will the epidemic be?
2. How long will it last? When will it peak? What will be its time course?
3. How effective will quarantine or vaccination be?
4. What could be the control measures?
5. What amount of vaccines and drugs are required?

Since the onset of COVID-19 global pandemic there has been an increased interest in comparing the data of infection to the various mathematical models to find which model the data matches well. Therefore, we try to apply the theory of the models that we develop to the data published by the state of Tennessee and CDC of the United States.

There have been various models proposed ever since Kermack and McKendrick have introduced their epidemiological SIR model [7] in 1927. However the main questions each model tries to answer remains the above. We aim to study the compartmental epidemic models similar to the SIR model in this thesis. We describe models with time dependent parameters in their implicit discrete form for numerical calculations.

1.1 Compartmental Epidemic Models

In the compartmental epidemic models, the host population is divided into compartments which are mutually exclusive groups of population. For a simple infectious disease, a simple model assumes the the compartments to be, S : susceptible hosts, I : infectious hosts and R : recovered hosts. The disease transmission process can be illustrated using *transfer diagrams* to show the movement of individuals from a compartment to another. Mathematical models track the number of individuals in a compartment as a function of time.



The arrows in the diagram indicate the direction of movement of individuals. The transfer diagram given above can be modeled by equations as follows. At time t, let $S(t), I(t)$ and $R(t)$ be the number of individuals in each compartment. Supposing that new individuals are being added to the susceptible compartment and on recovery some of the individuals become susceptible again, and deaths or removals happen from all the three compartments , we consider a small interval of time $[t, t + \Delta t]$ and write the change in the number of individuals in the compartments during this interval as,

$\Delta S(t) =$ new susceptibles + transfer from R - new infections - removal from S .

$\Delta I(t) =$ new infections - transfer into R - removal from I .

$\Delta R(t) =$ transfer from I - transfer into S - removal from R .

The model then proceeds to derive the differential equations by taking the limit

$\Delta t \to 0$, of the change in the number of individuals in each compartment.

$$\left.\begin{aligned} \lim_{\Delta t \to 0} \frac{S\left(t+\Delta t\right)}{\Delta t} &= S^{'}\left(t\right). \\ \lim_{\Delta t \to 0} \frac{I\left(t+\Delta t\right)}{\Delta t} &= I^{'}\left(t\right). \\ \lim_{\Delta t \to 0} \frac{R\left(t+\Delta t\right)}{\Delta t} &= R^{'}\left(t\right). \end{aligned}\right\} \tag{1.1}$$

The differential equations representing this simple mathematical model consists of a system of equations representing $S^{'}\left(t\right), I^{'}\left(t\right)$ and $R^{'}\left(t\right)$ in terms of the populations in the compartments.

1.1.1 Exponential Probability Distribution

$$\boxed{N\left(t\right)} \xrightarrow{\quad rN\left(t\right) \quad}$$

The proportional transfer rates from a general compartment as described in [1], says that if the total population in a compartment C at time t is $N\left(t\right)$ and the proportional rate of transfer from the compartment is $rN\left(t\right)$, the change of population satisfies the differential equation

$$\frac{dN\left(t\right)}{dt} = -rN\left(t\right). \tag{1.2}$$

If N_0 is the population in the compartment at time $t=0$,

$$N\left(t\right) = N_0 \mathbf{e}^{-rt}. \tag{1.3}$$

The survival function defined as the probability of surviving t units of time which gives the fraction of individuals remaining in the compartment is, $\mathbf{e}^{-rt}$. Therefore, the associated probability distribution function is given by $F\left(t\right) = 1 - \mathbf{e}^{-rt} \quad ; \quad t \geq 0$.

The probability density function of the distribution of $F(t)$ is,

$$f(t) = \frac{dF(t)}{dt} = r\mathbf{e}^{-rt} \quad ; \quad t \geq 0. \tag{1.4}$$

The mean residence time in the compartment is given by,

$$\int_{-\infty}^{\infty} tf(t)\,dt = 1/r. \tag{1.5}$$

1.1.2 SIR model

$$\boxed{S} \xrightarrow{\quad \lambda IS \quad} \boxed{I} \xrightarrow{\quad \gamma I \quad} \boxed{R}$$

A simplistic SIR model was first described by Kermack and McKendrick in 1927 . The basic assumptions of the Kermack-McKendrick Model as stated in [1] are,

1. Transmission occurs horizontally through direct contact between hosts.
2. Mixing of individual hosts is homogeneous and thus the Law of Mass Action holds.
3. There is no latency period after getting infected and becoming infectious.
4. The rate of transfer from a compartment is proportional to the total population in the compartment at the time.
5. There is no loss of immunity and no possibility of reinfection.
6. There is no input of new susceptibles and no removal from any compartments. Therefore the total population remains a constant.

Based on these assumptions, the change of population equations in each compartment as,

$\Delta S(t) =$ - new infections

$\Delta I(t) =$ new infections - transfer into R

$\Delta R\left(t\right)$ = transfer from I

In addition to these assumptions, we also state that the coefficients of transmission, λ and γ, in the differential equations we derive shortly, are constants in this model. This means that the recovery rate as well as the mean time spend as infectious is a constant for the disease at all time. Similarly, the transmission rate is also a constant for the disease at all time. We note that a modified model with time dependent transmission coefficients is presented in [2].

1.1.3 SEIR model

For some diseases, there is a latency period after a host becomes infected and before the individual can be contagious or infectious. In such diseases, we include a latent compartment to which the individuals move from the susceptible compartment upon infection. Once the latent period is over, the individuals could spread the infection and they move to the infectious compartment. [1]

$$\boxed{S} \xrightarrow{\lambda IS} \boxed{E} \xrightarrow{\epsilon E} \boxed{I} \xrightarrow{\gamma I} \boxed{R}$$

The change of population equations in this case when there is no removal from any compartment and no influx of susceptibles are written as,

$\Delta S\left(t\right)$ = - new infections

$\Delta E\left(t\right)$ = new infections - latency completed individuals

$\Delta I\left(t\right)$ = latency completed individuals - transfer into R

$\Delta R\left(t\right)$ = transfer from I

1.2 Deriving Model Equations

1.2.1 Rate of incidence

The incidence term is derived based on the Law of Mass Action for chemical kinetics. If M and N are the quantities of the two substances that interact, the substitution force is equal to $\alpha M^a N^b$, where α, a, b are constants that depend only on the nature of the substances.

Incidence term modeled after mass action, called simple mass-action incidence or bilinear incidence is derived as follows[1]. The disease spreads due to the contact of infectious individuals and susceptible individuals. The rate of incidence, same as the rate of change of susceptible individuals, $S^{'}(t)$ is then proportional to the product of the total number of susceptible individuals and the total number of infectious individuals . Let $S(t)$ be susceptible and $I(t)$ be the number of infectious populations at time t . Then,

$$S^{'}(t) = -\lambda \, S(t) \, I(t) \, . \tag{1.6}$$

We consider the incidence form with λ being a positive constant. The basic assumptions for law of mass action according to [1] are, homogeneous mixing, law reactant densities and conservation of total mass. In the case of compartment models, these corresponds to homogeneous contact of individuals, lower population density and the total population being constant.

1.2.2 Rate of Infection and Rate of Recovery

We now proceed to derive the differential equations for the Infected and Recovered compartments and the mean residence time in these compartments. The incidence rate or the rate at which individuals move out of the susceptible compartment into the infectious compartment. Assuming the conditions for the Law of Mass Action, for the interaction between a susceptible individuals and infectious individuals, the rate of change of susceptibles does not get modified by probability.

$$\lambda\, S\,(t)\, I\,(t)\, G\,(t)\,. \tag{1.7}$$

Suppose that the residence time in the compartment I follows a general probability distribution, $P\,(t)$. Then, the associated survival function in the I compartment is given by, $G\,(t) = 1 - P\,(t)$. For any given time $\tau > 0, G\,(t-\tau)$ is the fraction of individuals who become infected at time $\tau > 0$ and are still infectious at time $t > \tau$. Therefore, the population infected at time τ and remain infectious at time t is,[1]

$$\lambda\, S\,(t)\, I\,(t)\, G\,(t)\,. \tag{1.8}$$

When $P\,(t) = 1 - \mathbf{e}^{-\gamma t}$, for $t \geq 0$ and $P\,(t) = 0$,for $t < 0$, the survival function follows the exponential distribution, $\mathbf{e}^{-\gamma t}$. Let $I\,(0)$ be the number of individuals in the I compartment at time $\tau = 0$ and let $I_0\,(t)$ be the number of individuals infected at time $\tau = 0$ and remain infected at time t. Since the residence time in the compartment follows the exponential probability distribution, $G\,(t)$, $I_0\,(t) = I\,(0)\,\mathbf{e}^{-\gamma t}$. The number of individuals accumulated in the I compartment at time t since $\tau = 0$ is ,

$$I\,(t) = I\,(0)\,\mathbf{e}^{-\gamma t} + \int_0^t \lambda\, S\,(\tau)\, I\,(\tau)\,\mathbf{e}^{-\gamma(t-\tau)} d\tau. \tag{1.9}$$

Differentiating with respect to t,

$$\begin{aligned} I^{'}\,(t) &= -\gamma I_0\,(t) + \lambda\, S\,(t)\, I\,(t) - \gamma \int_0^t \lambda\, S\,(\tau)\, I\,(\tau)\,\mathbf{e}^{-\gamma(\tau)} d\tau \\ &= -\gamma I_0\,(t) + \lambda\, S\,(t)\, I\,(t) - \gamma\,(I\,(t) - I_0\,(t)) = \lambda\, S\,(t)\, I\,(t) - \gamma\,(I\,(t))\,. \end{aligned} \tag{1.10}$$

Thus, $I^{'}\,(t) = \lambda\, S\,(t)\, I\,(t) - \gamma I\,(t)$ is the differential equation corresponding to the transfer to and from the I compartment when the survival function in the compartment follows the exponential probability distribution.

Since the residence time in the compartment I follows the probability distribution, $P\,(t) = 1 - \mathbf{e}^{-\gamma t}$, The mean residence time in the compartment is given by,

$$\int_0^\infty t\, \frac{dP\,(t)}{dt} dt = 1/\gamma. \tag{1.11}$$

To derive the equation for the rate of change of recovered individuals, we note that, when $G(t)$ is the survival function in the I compartment, $1 - G(t)$ is the survival function in the recovered compartment because the individuals get transferred from I to R compartment upon recovery. Then the equation for $R(t)$, the number of recovered individuals at time t is,

$$R(t) = R(0)\left(1 - \mathbf{e}^{-\gamma t}\right) + \int_0^t \lambda S(\tau) I(\tau)\left(1 - \mathbf{e}^{-\gamma(t-\tau)}\right) d\tau. \tag{1.12}$$

Differentiating with respect to t,

$$\begin{aligned} R^{'}(t) &= R(0)\left(1 - \mathbf{e}^{-\gamma t}\right) - \gamma R_0(t) + \lambda S(t) I(t) - \gamma \int_0^t \lambda S(\tau) I(\tau) \mathbf{e}^{-\gamma(\tau)} d\tau \\ &= \gamma I(t). \end{aligned} \tag{1.13}$$

Thus, $R^{'}(t) = \gamma I(t)$ is the differential equation for the transfer to the R compartment. The SIR model that assumes exponentially distributed infectious period, can be thus described in terms of the ODEs,

$$\left.\begin{aligned} S^{'}(t) &= -\lambda S(t) I(t), \\ I^{'}(t) &= \lambda S(t) I(t) - \gamma I(t), \\ R^{'}(t) &= \gamma I(t) \end{aligned}\right\} \tag{1.14}$$

In the case of SEIR [1] model, the differential equation for the rate of change of susceptibles remain the same. That is, $S^{'}(t) = -\lambda S(t) I(t)$. The susceptible individuals, on infection move to the latent compartment instead of being infectious immediately. Suppose we denote the latency period by ϵ. Then general probability distribution representing the residence time in the E compartment is $P(t) = 1 - \mathbf{e}^{-\epsilon t}$, for $t \geq 0$ and $P(t) = 0$, for $t < 0$, the survival function follows the exponential distribution, $\mathbf{e}^{-\epsilon t}$ in the E compartment. Let $E(0)$ be the number of individuals in the E compartment at time $\tau = 0$ and let $E_0(t)$ be the number of individuals infected at time $\tau = 0$ and remain infected at time t. Then, $E_0(t) = E(0)\,\mathbf{e}^{-\epsilon t}$. The number of individuals accumulated in the E compartment at time t since $\tau = 0$ is,

$$E(t) = E(0)\,\mathbf{e}^{-\epsilon t} + \int_0^t \lambda\, S(\tau)\, I(\tau)\,\mathbf{e}^{-\epsilon(t-\tau)} d\tau. \tag{1.15}$$

Differentiating with respect to t,

$$\begin{aligned} E^{'}(t) &= -\epsilon E_0(t) + \lambda\, S(t)\, I(t) - \epsilon \int_0^t \lambda\, S(\tau)\, I(\tau)\,\mathbf{e}^{-\epsilon(\tau)} d\tau \\ &= -\epsilon E_0(t) + \lambda\, S(t)\, I(t) - \epsilon\left(E(t) - E_0(t)\right) = \lambda\, S(t)\, I(t) - \epsilon E(t)\,. \end{aligned} \tag{1.16}$$

The rate of change of individuals in the I compartment could be similarly derived,

$$I^{'}(t) = \epsilon E(t) - \gamma\left(I(t)\right). \tag{1.17}$$

The differential equation representing the rate of change in the R compartment is the same as that in the SIR model,

$$R^{'}(t) = \gamma I(t)\,. \tag{1.18}$$

The system of ODEs representing the SEIR model [1] therefore is,

$$\left.\begin{aligned} S^{'}(t) &= -\lambda(t)\; S(t)\, I(t)\,, \\ E^{'}(t) &= \lambda(t)\; S(t)\, I(t) - \epsilon\, E(t)\,, \\ I^{'}(t) &= \epsilon\, E(t) - \gamma(t)\, I(t)\,, \\ R^{'}(t) &= \gamma(t)\, I(t) \end{aligned}\right\} \tag{1.19}$$

1.3 Mathematical Background

The theory developed in this thesis tries to find the following properties in the models.

1. Boundedness and nonnegativity : Whether the solutions to the system of ODE remain bounded in the region of existence? Are the solutions non negative since the solutions are number of individuals and need to be non negative.

2. Existence and uniqueness of solutions in the positive axes.

3. What is the long term behavior of the system.

4. Is the system well-posed?

To develop the theory given in this thesis, we depended on the following mathematical background; already known definitions and theorems.

Definition 1.1 ([2]) *The supremum norm of a continuous function* $f : [a, b] \to \mathbb{R}$ *in an arbitrary time interval* $[a, b]$ *is defined in [2] as* $\| f(t) \|_\infty := \sup_{t\in[a,b]} | f(t) |$

Definition 1.2 ([1])

Let E *be an open set in* $\mathbb{R}^d$*. A function* $f(y, z) = f\left(y^1, ..., y^d, z^1, ..., z^e\right)$ *defined on the* (y, z) *set* E*, where* $y \in \mathbb{R}^d$*,is said to be uniformly Lipschitz continuous on* E *with respect to* y *if there exists a constant* L *satisfying*

$$| f(\boldsymbol{x}, \boldsymbol{z}) - f(\boldsymbol{y}, \boldsymbol{z}) | \leq L | \boldsymbol{x} - \boldsymbol{y} |$$

for all $(\boldsymbol{x}, \boldsymbol{z}), (\boldsymbol{y}, \boldsymbol{z}) \in E$*. Any constant* L *satisfying this condition is called a Lipschitz constant for* f *on* E*. Further, we could chose a suitable norm on the corresponding Euclidean space* $\mathbb{R}^d$ *, so that for the Lipschitz constant for* f *on* E*.*

$$\| f(\boldsymbol{x}, \boldsymbol{z}) - f(\boldsymbol{y}, \boldsymbol{z}) \|_{\mathbb{R}^{d_2}} \leq L \| \boldsymbol{x} - \boldsymbol{y} \|_{\mathbb{R}^d}$$

for all $(\boldsymbol{x}, \boldsymbol{z}), (\boldsymbol{y}, \boldsymbol{z}) \in E$*.*

Definition 1.3 ([2]) *Let* $U \subset \mathbb{R}^{d_1}$ *be open. Let* $\boldsymbol{F} : U \to \mathbb{R}^{d_2}$ *.*$\boldsymbol{F}$ *is locally Lipschitz continuous if for every point* $\boldsymbol{x_0} \in U$*, there exists a neighborhood* V *of* $\boldsymbol{x_0}$ *such that the restriction of* $\boldsymbol{F}$ *to* V *is Lipschitz continuous on* V*.*

We state the Gronwall's Inequality in $\mathbb{R}$ here.

Lemma 1.4 *Gronwall's Inequality :* ([4]) *Suppose f is non-negative and continuous on $\mathbb{R}$, and suppose there exist positive constants C and K such that ,*

$$f(t) \leq C + K \int_0^t f(s)\ ds$$

for all $t \in [0, a]$. Then $f(t) \leq C\mathbf{e}^{Kt}$ for all $t \in [0, a]$.

We apply the following theorem [2] to prove the global existence of solutions for the differential equations, that is a direct consequence of Gronwall's Inequality.

Theorem 1.5 ([2]) *If $\boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, and if there exist nonnegative real constants B and K such that,*

$$|| \ G(t, \boldsymbol{z}(t)) \ ||_{\mathbb{R}^n} \leq \ K \ || \ \boldsymbol{z}(t) \ ||_{\mathbb{R}^n} + B$$

holds for all $\boldsymbol{z}(t) \in \mathbb{R}^n$, then the solution of the initial value problem (2) exists for all time $t \in \mathbb{R}$ and, moreover, it holds,

$$|| \ \boldsymbol{z}(t) \ ||_{\mathbb{R}^n} \leq || \ \boldsymbol{z_0}(t) \ ||_{\mathbb{R}^n} + \frac{B}{K}\left(\mathbf{e}^{K|t|} - 1\right).$$

To prove the dependence on initial conditions, we use the inequality named after Gronwall as stated below.

Theorem 1.6 ([2, Theorem 3]) Let $I := [a, b]$. Let $u, f : I \to [0, \infty]$ be continuous and nonnegative functions. Let $g : I \to (0, \infty)$ be a continuous, positive, and nondecreasing function. If the inequality

$$u \leq g(t) + \int_a^t f(s)\, u(s)\, ds \tag{1.20}$$

holds for all $t \in I$, then we obtain

$$u \leq g(t)\, \mathbf{e}^{\int_a^t f(s) ds} \tag{1.21}$$

for all $t \in I$.

We first state a simple definition of well-posedness given in [4]

Definition 1.7 ([4]) *An initial value problem* $x^{'} = f(x) \quad x(0) = x_0$ *is called well-posed if each of the following criteria is satisfied.*
i) Existence: The problem has at least one solution.
ii) Uniqueness: The problem has at most one solution.
iii) Dependence on Initial Conditions: A slight change in initial conditions does not profoundly impact the solution.

To prove global uniqueness in time, we need Banach's fixed point theorem which we recall below.

Theorem 1.8 *[2, Theorem 2] Let* (X, Q) *be a complete metric space with the metric mapping,* $Q : X \times X \to [0, \infty)$*. Let* $T : X \to X$ *be a contraction , that is, there exists a constant* $K \in [0, 1)$ *such that* $Q(Tx, Ty) \leq Q(x, y)$ *holds for all* $x, y \in X$*. Then the mapping* T *has a unique fixed point.*

1.3.1 Basic Reproduction Number

Based on the monotonicity properties and the boundedness of S, I and R, we can draw the following biological conclusions for the SIR model.[1]

1. Some number of susceptibles always escape the infection at the end of the epidemic.
2. The epidemic ends not because the susceptibles are exhausted.
3. The disease eventually dies out and the infectious population tends to zero after a long period of time.
4. Since the infectious population increases initially, there must be a period of time after which the change in I becomes negative. In other words, the epidemic first rises, the declines after reaching the maximum.

We further explore properties 1 and 2 here. The basic reproduction Number, denoted by $\mathcal{R}_0$ [1] measures the average number of secondary infections caused by a single infectious individual in an entirely susceptible population during the mean infectious period.

We can interpret, $\mathcal{R}_0$ as, [average number of effective contacts of a single infectious host] . [initial susceptible population]. [mean infectious period]

In the context of Kermack - McKendrik model, the basic reproduction number, at the beginning of the infection,

$$\mathcal{R}_0 = \lambda\, S_0\, \frac{1}{\gamma}.$$

The significance of $\mathcal{R}_0$ is in determining whether an epidemic outbreak will occur or not, given the constants λ and γ and the total susceptible population at the beginning of the infection. If the parameters λ and γ are constants, the basic reproduction number depends only on the value of S_0. From the assumptions $\lambda, \gamma \;\geq\; 0$, since $S_0 \;\geq\; 0$, we observe that, $\mathcal{R}_0 := \lambda\, S_0\, \frac{1}{\gamma} \;\geq\; 0$ for all $t \geq 0$.

Theorem 1.9 ([2]) *$I\,(t)$ steadily decreases with t when $\mathcal{R}_0 < 1$. In other words,an epidemic does not occur if the basic reproduction number $\mathcal{R}_0 < 1$.*

Proof. From equation 1.14, $I\,(t) = I_0 \mathbf{e}^{\lambda \int_0^t S(\tau) d\tau} \mathbf{e}^{\int_0^t -\gamma d\tau}$.
From the monotonicity of $S\,(t)$, we know that $\lambda \int_0^t S\,(\tau)\, d\tau \;\leq\; \lambda S_0 t$.
$I\,(t) \;<\; I_0$ if $\lambda \int_0^t S\,(\tau)\, d\tau \;<\; \int_0^t \gamma d\tau$. That is, when $\lambda S_0 \;<\; \gamma$ or, when $\mathcal{R}_0 \;<\; 1$.
An epidemic occurs when, $\mathcal{R}_0 > 1$, since $I\,(t) = I_0 \mathbf{e}^{\lambda \int_0^t S(\tau) d\tau}\, \mathbf{e}^{\int_0^t -\gamma d\tau} > I_0$ in this case.

CHAPTER 2

PROPERTIES OF THE MODELS

In this chapter, we discuss the properties of the differential equations associated with the SEIR model. We discuss the well-posedness property in detail and also examine the effect of perturbations on the model equations. We discuss the modified SEIR model with time continuous transmission coefficient and recovery rate and the time discrete model as given in [2]. Since we consider that the total population remains a constant over all time $t \geq 0$, we have ,

$$N(t) = S(t) + E(t) + I(t) + R(t). \tag{2.1}$$

2.1 SEIR model with time varying coefficients

In this section we examine the SEIR model with time dependent transmission, and recovery coefficients. We first state the assumptions of the model below.

1. The total population N is fixed over time.
2. The total population is divided into homogeneous groups, with $S(t)$ representing Susceptible population, $E(t)$ Latent population, $I(t)$ Infectious population and $R(t)$ Recovered population at time t. Since the total population is a constant, $N(t) = S(t) + E(t) + I(t) + R(t)$ for all time $t \in [0, \infty)$.
3. the time varying transmission coefficient $\lambda(t)$ is Lipschitz continuous and is continuously differentiable for all $t \in [0, \infty)$. It holds that $0 < \lambda_{min} \leq \lambda(t) \leq \lambda_{max}$, for all $t \in [0, \infty)$, where λ_{min} is the lowest and λ_{max} is the highest value λ achieves.
4. the latency period remains a constant and is independent of time.

5. the time varying recovery coefficient $\gamma(t)$ is Lipschitz continuous and is continuously differentiable for all $t \in [0, \infty)$. It holds that $0 < \gamma_{min} \leq \gamma(t) \leq \gamma_{max}$, for all $t \in [0, \infty)$, where γ_{min} is the lowest and γ_{max} is the highest value γ achieves.

2.1.1 SEIR model equations

Since the total population remains a constant over all time $t \geq 0$,
we have , $N(t) = S(t) + E(t) + I(t) + R(t)$.
The differential equations for the SEIR model with time varying transmission and recovery rates are,

$$\left.\begin{aligned} S^{'}(t) &= -\lambda(t)\ S(t)\,I(t)\,, \\ E^{'}(t) &= \lambda(t)\,,\ S(t)\,I(t) - \epsilon\,E(t) \\ I^{'}(t) &= \epsilon\,E(t) - \gamma(t)\,I(t)\,, \\ R^{'}(t) &= \gamma(t)\,I(t)\,, \\ N(t) &= S(t) + E(t) + I(t) + R(t) \end{aligned}\right\} \tag{2.2}$$

The initial conditions for the system of equations are,

$$N(t) := N_0 = N$$

$$S(0) := S_0 > 0,\ I(0) := I_0 > 0,\ E(0) := E_0 \geq 0,\ R(0) := R_0 \geq 0$$

2.1.2 Nonnegativity and boundedness

We first find the feasible region for the solutions of the system of differential equations 2.2. Our aim is to prove that each of S, E, I, R are non negative functions of t.

Lemma 2.10 *Each solution of the system given by* 2.2 is bounded below by 0.

Proof

1. We use separation of variables and write the S equations as,

$$\frac{S^{'}(t)}{S(t)} = -\lambda(t)\ I(t) \tag{2.3}$$

Integrating,

$$\ln\left(\frac{S(t)}{S(0)}\right) = -\int_0^t \lambda(\tau)\, I(\tau)\, d\tau$$

$$S(t) = S(0)\, \mathbf{e}^{-\int_0^t \lambda(\tau) I(\tau) d\tau}. \tag{2.4}$$

Where $S(0)$ is the initial condition on $S(t)$, corresponding to the initial population in the S compartment. Since $S(0) \geq 0$, we see that $S(t) \geq 0$ at all time $t \geq 0$.

2. Now we divide the E equation by $E(t)$ and write,

$$\frac{E^{'}(t)}{E(t)} = \frac{\lambda(t)\ S(t)\, I(t)}{E(t)} - \epsilon \tag{2.5}$$

We replace $E(t)$ with $N-(S(t)+I(t)+R(t))$ on the right hand side to obtain,

$$\frac{E^{'}(t)}{E(t)} = \frac{\lambda(t)\ S(t)\, I(t)}{N-(S(t)+I(t)+R(t))} - \ \epsilon$$

Integration yields,

$$\ln\left(\frac{E(t)}{E(0)}\right) = \int_0^t \left(\frac{\lambda(\tau)\ S(\tau)\, I(\tau)}{N-(S(\tau)+I(\tau)+R(\tau))} - \ \epsilon\right) d\tau$$

Therefore,

$$E(t) = E_0 . \exp . \left[\int_0^t \left(\frac{\lambda(\tau)\ S(\tau)\, I(\tau)}{N-(S(\tau)+I(\tau)+R(\tau))} - \ \epsilon\right) d\tau\right]. \tag{2.6}$$

Since $E_0 \geq 0$, $E(t) \geq 0$ for all $t \geq 0$.

3. We divide the I equation by $I(t)$ and write,

$$\frac{I'(t)}{I(t)} = \frac{\epsilon E(t)}{I(t)} - \gamma(t) \tag{2.7}$$

We replace $I(t)$ with $N-(S(t)+E(t)+R(t))$ on the right hand side to obtain,

$$\frac{I'(t)}{I(t)} = \frac{\epsilon\, E(t)}{N-(S(t)+E(t)+R(t))} - \gamma(t)$$

Integration yields,

$$\ln\left(\frac{I(t)}{I(0)}\right) = \int_0^t \left(\frac{\epsilon E(t)}{N-(S(\tau)+E(\tau)+R(\tau))} - \gamma(t)\right) d\tau$$

Therefore,

$$I(t) = I_0 . \exp . \left[\int_0^t \left(\frac{\epsilon E(t)}{N-(S(\tau)+E(\tau)+R(\tau))} - \gamma(t)\right) d\tau\right]. \tag{2.8}$$

Since $I_0 \geq 0$, $I(t) \geq 0$ for all $t \geq 0$.

4. Since $R'(t) = \gamma(t) I(t)$, integration yields,

$$R(t) = R_0 + \int_0^t \gamma(\tau) I(\tau)\, d\tau. \tag{2.9}$$

Since$I(t) \geq 0$ and $\gamma(t) > 0$, $R(t) \geq 0$ for all $t \geq 0$.

We now state the boundedness theorem for the solutions.

Theorem 2.11 *For each solution of the system given by 1.19,*

1. $0 \leq S(t) \leq N$

2. $0 \leq E(t) \leq N$

3. $0 \leq I(t) \leq N$

4. $0 \leq R(t) \leq N$

For all $t \geq 0$.

2.1.3 Global existence of solutions

We define the solution vector,

$$\mathbf{z}\left(t\right) = \left(S\left(t\right), E\left(t\right), I\left(t\right), R\left(t\right)\right)$$

$$\mathbf{z}\left(0\right) = \mathbf{z_0} = \left(S\left(0\right), E\left(0\right), I\left(0\right), R\left(0\right)\right)$$

and $\mathbf{z}^{'}\left(t\right) = G\left(t, \mathbf{z}\left(t\right)\right)$.

We state the global existence of solutions as follows.

Theorem 2.12 ([1], [2]) *At least one solution to the differential equations 2.2 exist for all time $t \in [0, \infty)$.*

Proof.

We define a function $G\left(t, \mathbf{z}\left(t\right)\right) : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ as $\begin{pmatrix} -\lambda\left(t\right)\ S\left(t\right) I\left(t\right) \\ \lambda\left(t\right)\ S\left(t\right) I\left(t\right) - \epsilon E\left(t\right) \\ \epsilon E\left(t\right) - \gamma\left(t\right) I\left(t\right) \\ \gamma\left(t\right) I\left(t\right) \end{pmatrix}$.

We first prove that G is Lipschitz continuouous using the property that the upper bounds of S, E, I and R are N and $\lambda_{max}, \gamma_{max}$ exist.

Consider $\mathbf{z}_1\left(t\right) = \left(S_1\left(t\right), E_1\left(t\right), I_1\left(t\right), R_1\left(t\right)\right)$ and $\mathbf{z}_2\left(t\right) = \left(S_2\left(t\right), E_2\left(t\right), I_2\left(t\right), R_2\left(t\right)\right)$

$$
\begin{aligned}
&\mid G\left(t, \mathbf{z}_1(t)\right)-G\left(t, \mathbf{z}_2(t)\right) \mid \\
&= \begin{pmatrix}
\left|\lambda_2(t) S_2(t) I_2(t)-\lambda_1(t) S_1(t) I_1(t)\right| \\
\left|\lambda_1(t) S_1(t) I_1(t)-\lambda_2(t) S_2(t) I_2(t)-\left(\epsilon E_2(t)-\epsilon E_1(t)\right)\right| \\
\left|\epsilon E_1(t)-\epsilon E_2(t)-\left(\gamma_1(t) I_1(t)-\gamma_2(t) I_2(t)\right)\right| \\
\left|\gamma_1(t) I_1(t)-\gamma_2(t) I_2(t)\right|
\end{pmatrix} \\
&\leq \begin{pmatrix}
\left|\lambda_2(t) S_2(t) I_2(t)-\lambda_1(t) S_1(t) I_1(t)\right| \\
\left|\lambda_1(t) S_1(t) I_1(t) \frac{E_1(t)}{E_1(t)}-\lambda_2(t) S_2(t) I_2(t) \frac{E_2(t)}{E_2(t)}\right|-\left|\epsilon E_1(t)-\epsilon E_2(t)\right| \\
\left|\epsilon E_1(t) \frac{I_1(t)}{I_1(t)}-\epsilon E_2(t) \frac{I_2(t)}{I_2(t)}\right|-\left|\gamma_1(t) I_1(t)-\gamma_2(t) I_2(t)\right| \\
\left|\gamma_1(t) I_2(t)-\gamma_2(t) I_2(t)\right|
\end{pmatrix} \\
&\leq \mid \lambda_{max}+\gamma_{max}+\epsilon \mid \begin{pmatrix}
\mid N \mid \mid S_1(t)-S_2(t) \mid \\
\mid N \mid \mid E_1(t)-E_2(t) \mid \\
\mid N \mid \mid I_1(t)-I_2(t) \mid \\
\mid N \mid \mid R_1(t)-R_2(t) \mid
\end{pmatrix} \\
&= N\left(\lambda_{max}+\gamma_{max}+\epsilon\right) \begin{pmatrix}
\mid S_1(t)-S_2(t) \mid \\
\mid E_1(t)-E_2(t) \mid \\
\mid I_1(t)-I_2(t) \mid \\
\mid R_1(t)-R_2(t) \mid
\end{pmatrix} \\
&= N\left(\lambda_{max}+\gamma_{max}+\epsilon\right) \mid \mathbf{z}_2(t)-\mathbf{z}_1(t) . \mid \qquad (2.10)
\end{aligned}
$$

Thus G is Lipschitz continuouous in $\mathbf{z}(t)$.

Now we consider the supremum norm,

$$
\begin{aligned}
& \| G(t, \mathbf{z}(t)) \|_{\infty} \\
& = \sup_{t \in [0,\infty)} \{ | -\lambda(t)\ S(t) I(t) |, | \lambda(t)\ S(t) I(t) - \epsilon E(t) |, | \epsilon E(t) - \gamma(t) I(t) |, | \gamma(t) I(t) | \} \\
& \leq \sup_{t \in [0,\infty)} \{ \lambda_{max} \ | S(t) I(t) |,\ \lambda_{max} \ | S(t) I(t) | + \epsilon \ | E(t) |, \\
& \qquad \epsilon \ | E(t) | \ + \gamma_{max} \ | I(t) |, \gamma_{max} \ | I(t) | \} \\
& \leq \sup_{t \in [0,\infty)} \{ \lambda_{max} N \ | S(t) |,\ (\lambda_{max} N + \epsilon) \ | E(t) |,\ (\epsilon + \gamma_{max}) \ | I(t) |,\ \gamma_{max} \ | I(t) | \} \\
& \leq \sup_{t \in [0,\infty)} \{ \lambda_{max} N \ | S(t) |,\ \lambda_{max} N \ | E(t) | + \epsilon \ | E(t) |,\ \epsilon \ | I(t) | \ + \gamma_{max} \ | I(t) |, \\
& \qquad \gamma_{max} \ | R(t) | \} \\
& \leq N (\lambda_{max} + \epsilon + \gamma_{max}) \sup_{t \in [0,\infty)} \{ | S(t) |, | E(t) |, | I(t) |, | R(t) | \} \\
& \leq \ N (\lambda_{max} + \gamma_{max} + \epsilon) \ \| \mathbf{z}(t) \|_{\infty} \ . \qquad (2.11)
\end{aligned}
$$

by the boundedness of $S(t)$, $E(t)$, $I(t)$ and $R(t)$ and the transmission and recovery coefficients. Thus the conditions for Theorem 1.5 are satisfied and the proof is complete.

2.1.4 Global uniqueness of solutions

Theorem 2.13 ([1] , [2]) *There exists a unique solution for all time $t \in [0, \infty)$, for the initial value problem given by the differential equations 2.2*

Proof.

First choose an interval $[0, \tau]$ on which the Banach's fixed point theorem is applicable.

Now, suppose that there exist two different solutions in the interval $[0,\infty)$, say, $\mathbf{z}(t) = (S(t), E(t), I(t), R(t))$ and $\widetilde{\mathbf{z}}(t) = \left(\widetilde{S}(t), \widetilde{E}(t), \widetilde{I}(t), \widetilde{R}(t)\right)$.
Then,

$$
\begin{aligned}
&\sup_{t\in[0,\tau]}\left|S(t)-\widetilde{S}(t)\right| \\
&= \sup_{t\in[0,\tau]}\left|\int_0^t \left(\lambda(z)\,\widetilde{S(z)}\widetilde{I(z)} - \lambda(z)\,S(z)\,I(z)\right)dz\right| \\
&\leq \sup_{t\in[0,\tau]} \lambda_{max}\int_0^t \left|\widetilde{S(z)}\right|\left|\widetilde{I(z)}-I(z)\right| + |I(z)|\left|S(z)-\widetilde{S(z)}\right| dz \\
&\leq \sup_{t\in[0,\tau]} \lambda_{max}\int_0^t N\left|\widetilde{I(z)}-I(z)\right| + N\left|S(z)-\widetilde{S(z)}\right| dz \\
&\leq \sup_{t\in[0,\tau]} \lambda_{max}\,N\,t\,\left\{\left|\widetilde{I(t)}-I(t)\right| + \left|S(t)-\widetilde{S(t)}\right|\right\} \\
&\leq 2\lambda_{max}\,N\,\tau\,\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty .
\end{aligned} \tag{2.12}
$$

Similarly, we obtain,

$$
\begin{aligned}
&\sup_{t\in[0,\tau]}\left|E(t)-\widetilde{E}(t)\right| \\
&= \sup_{t\in[0,\tau]}\left|\int_0^t \lambda(z)\left(S(z)\,I(z)-\widetilde{S(z)}\widetilde{I(z)}\right)dz\right| + \sup_{t\in[0,\tau]}\left|\int_0^t \epsilon\left(E(z)-\widetilde{E(z)}\right)dz\right| \\
&\leq \sup_{t\in[0,\tau]} \lambda_{max}\,t\,N\left\{\left|\widetilde{I(t)}-I(t)\right| + \left|S(t)-\widetilde{S(t)}\right|\right\} + \sup_{t\in[0,\tau]} \epsilon\,t\,\left\{\left|\widetilde{E(t)}-E(t)\right|\right\} \\
&\leq (\lambda_{max}\,N+\epsilon)\,\tau\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \\
&\leq (2\lambda_{max}+\epsilon)\,N\tau\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
& \sup_{t\in[0,\,\tau]} \left| I\left(t\right) - \widetilde{I}\left(t\right) \right| \\
& = \sup_{t\in[0,\,\tau]} \left| \int_0^t \epsilon \left(E\left(z\right) - \widetilde{E\left(z\right)} \right) dz \right| + \sup_{t\in[0,\,\tau]} \left| \int_0^t \gamma\left(z\right) \left(I\left(z\right) - \widetilde{I\left(z\right)} \right) dz \right| \\
& \leq \sup_{t\in[0,\,\tau]} \epsilon\, t \left| \widetilde{E\left(t\right)} - E\left(t\right) \right| + \sup_{t\in[0,\,\tau]} \gamma_{max}\, t \left| \widetilde{I\left(t\right)} - I\left(t\right) \right| \\
& \leq \left(\gamma_{max} + \epsilon \right) \tau \left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty} \\
& \leq \left(\gamma_{max} + \epsilon \right) N\tau \left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty}. \qquad (2.14)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sup_{t\in[0,\,\tau]} \left| R\left(t\right) - \widetilde{R}\left(t\right) \right| &= \sup_{t\in[0,\,\tau]} \left| \int_0^t \gamma\left(z\right) \left(I\left(z\right) - \widetilde{I\left(z\right)} \right) dz \right| \\
&\leq \gamma_{max} N\tau \left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty}. \qquad (2.15)
\end{aligned}
$$

Summarizing the steps above, we obtain,

$$
\left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty} \leq \left(2\,\lambda_{max} + \epsilon + \gamma_{max} \right) N\tau \left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty}. \qquad (2.16)
$$

By choosing the interval $\tau = \frac{1}{2(2\,\lambda_{max}+\epsilon+\gamma_{max})N}$, we obtain the necessary contraction.

$$
\left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty} \leq \frac{1}{2} \left|\left| \mathbf{z}\left(t\right) - \widetilde{\mathbf{z}\left(t\right)} \right|\right|_{\infty}. \qquad (2.17)
$$

This proves the uniqueness of the solution in the interval, $[0, \tau]$.

We can now inductively derive the contraction for each interval, $[k\tau, (k+1)\,\tau]$ for all $k \in \mathbb{N}$. This proves the uniqueness of the solutions for all time $t \geq 0$.

2.1.5 Continuous dependence on initial conditions

We have stated a simple definition of well posedness in the previous section. todo quote the section.We now prove the dependence on initial conditions.

To prove the well posedness, we consider the perturbed initial value problem,

$$\left.\begin{aligned} S_a^{'}(t) &= -\lambda_a(t)\ S_a(t)\, I_a(t)\,, \\ E_a^{'}(t) &= \lambda_a(t)\ S_a(t)\, I_a(t) - \epsilon E_a(t)\,, \\ I_a^{'}(t) &= \epsilon E_a(t) - \gamma_a(t)\, I_a(t)\,, \\ R_a^{'}(t) &= \gamma_a I_a(t) \end{aligned}\right\} \tag{2.18}$$

with initial conditions, $S_a(0) = S_{a,0} > 0, E_a(0) = E_{a,0} \geq 0, I_a(0) = I_{a,0} > 0, R_a(0) = R_{a,0} \geq 0$

and

$$\left.\begin{aligned} S_b^{'}(t) &= -\lambda_b(t)\ S_b(t)\, I_b(t) \\ E_b^{'}(t) &= \lambda_b(t)\ S_b(t)\, I_b(t) - \epsilon E_b(t) \\ I_b^{'}(t) &= \epsilon E_b(t) - \gamma_b(t)\, I_b(t) \\ R_b^{'}(t) &= \gamma_b I_b(t) \end{aligned}\right\} \tag{2.19}$$

with initial conditions, $S_b(0) = S_{b,0} > 0, E_b(0) = E_{b,0} \geq 0, I_b(0) = I_{b,0} > 0, R_b(0) = R_{b,0} \geq 0$.

Here we consider the transmission rate λ_a and λ_b and the recovery rate, γ_a and γ_b have small differences, as well as the initial conditions have small perturbations. We proceed to prove that this lead to solutions that have small differences in short intervals of time $[0, T]$ as in [2].

Theorem 2.14 ([2](Theorem 3)) *Let* $\boldsymbol{z}_a(t) = \begin{pmatrix} S_a(t) \\ E_a(t) \\ I_a(t) \\ R_a(t) \end{pmatrix}$ *and* $\boldsymbol{z}_b(t) = \begin{pmatrix} S_b(t) \\ E_b(t) \\ I_b(t) \\ R_b(t) \end{pmatrix}$ *be the solutions of 2.18 and 2.19.*

Define a function

$g(t) := ||\ \boldsymbol{z}_a(0) - \boldsymbol{z}_b(0)\ ||_\infty + N_a^2\, t\ ||\ \lambda_a(t) - \lambda_b(t)\ ||_\infty + N_a\, t\ ||\ \gamma_a(t) - \gamma_b(t)\ ||_\infty$

and the constant, $K_{GB} := \{\max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_a\, N_b + \max\{\gamma_{max,b}, \gamma_{max,a}\} + \epsilon\}$

Then, $||\ \boldsymbol{z}_a(t) - \boldsymbol{z}_b(t)\ ||_\infty \leq\ g(t)\, \mathrm{e}^{K_{GB}\, t}$ *holds for arbitrary* $t \in [0, T]$ *with given* $T \geq 0$.

Proof.

We see from 2.2 that, $N_a = S_a(0) + E_a(0) + I_a(0) + R_a(0)$ and $N_b = S_b(0) + E_b(0) + I_b(0) + R_b(0)$ holds for all $t \in [0, T]$.

We recall the inequality,

$$|x_1 y_1 - x_2 y_2| \leq |x_1|\, |y_1 - y_2| + |y_2|\, |x_1 - x_2| \tag{2.20}$$

In all the derivations below, we apply the triangle inequality, the inequality 2.20 and the boundedness of the functions,$S(t), E(t), I(t)$ and $R(t)$.

1. First we estimate $|S_a(t) - S_b(t)|$ and obtain,

$$
\begin{aligned}
&|S_a(t) - S_b(t)| \\
&\quad \le\ |S_a(0) - S_b(0)| + \int_0^t |\lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)|\, d\tau \\
&\quad \le |\ S_a(0) - S_b(0)\ | + \int_0^t |\ \lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_a(\tau) I_a(\tau)\ |\, d\tau \\
&\qquad + \int_0^t |\ \lambda_b(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_a(\tau)\ |\, d\tau \\
&\qquad + \int_0^t |\ \lambda_b(\tau) S_b(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)\ |\, d\tau \\
&\quad \le\ |S_a(0) - S_b(0)| + N_a^2\, t\ ||\lambda_a(t) - \lambda_b(t)||_\infty \\
&\qquad + \max\{\lambda_{max,ax}, \lambda_{max,b}\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\
&\qquad + \max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau
\end{aligned} \tag{2.21}
$$

for any $t \in [0, T]$.

2. For $|E_a(t) - E_b(t)|$ we have,

$$
\begin{aligned}
&|E_a(t) - E_b(t)| \\
&\quad \le\ |E_a(0) - E_b(0)| + \int_0^t |\lambda_a(\tau) E_a(\tau) I_a(\tau) - \lambda_b(\tau) E_b(\tau) I_b(\tau)|\, d\tau \\
&\qquad + \int_0^t |\epsilon E_a(\tau) - \epsilon E_b(\tau)|\, d\tau
\end{aligned}
$$

Define equation
$I := \int_0^t |\lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)|\, d\tau$
and

We see that,

$$\begin{aligned} I \ \le \ & |I_a(0) - I_b(0)| + N_a^2\, t \ ||\lambda_a(t) - \lambda_b(t)||_\infty \\ & + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\ & + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau \end{aligned}$$

and Thus we obtain

$$\begin{aligned} & |E_a(t) - E_b(t)| \\ & \le \ |E_a(0) - E_b(0)| + N_a^2\, t \ ||\lambda_a(t) - \lambda_b(t)||_\infty \\ & \quad + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\ & \quad + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau \\ & \quad + \epsilon \int_0^t \,|\,E_a(\tau) - E_b(\tau)\,|\, d\tau \end{aligned} \tag{2.22}$$

for any $t \in [0, T]$.

3. Now we estimate $|I_a(t) - I_b(t)|$.

$$\begin{aligned} & |I_a(t) - I_b(t)| \\ & \quad \le \ |I_a(0) - I_b(0)| + \int_0^t |\epsilon_a E_a(\tau) - \epsilon_b E_b(\tau)|\, d\tau \\ & \quad\ + \int_0^t |\gamma_a I_a(\tau) - \gamma_b I_b(\tau)|\, d\tau \end{aligned}$$

Define equation

$I := \int_0^t \,|\,\gamma_a I_a(\tau) - \gamma_b I_b(\tau)\,|\, d\tau.$

We see that,

$$
\begin{aligned}
II \;\le\; & \int_0^t \left|\, \gamma_a\left(\tau\right) I_a\left(\tau\right) - \gamma_b\left(\tau\right) I_a\left(\tau\right) \right| d\tau + \int_0^t \left|\, \gamma_b\left(\tau\right) I_a\left(\tau\right) - \gamma_{b(\tau)} I_b\left(\tau\right) \right| d\tau \\
\le\; & N_a \, t \, \left|\left|\, \gamma_a\left(t\right) - \gamma_b\left(t\right) \right|\right|_\infty + \max\left\{\gamma_{max,b}, \gamma_{max,a}\right\} \int_0^t \left|\, I_a\left(\tau\right) - I_b\left(\tau\right) \right| d\tau
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
& \left|I_a\left(t\right) - I_b\left(t\right)\right| \\
& \quad \le\; \left|I_a\left(0\right) - I_b\left(0\right)\right| + \epsilon \int_0^t \left|\, E_a\left(\tau\right) - E_b\left(\tau\right) \right| d\tau \\
& \quad\quad + N_a \, t \, \left|\, \gamma_a - \gamma_b \,\right| + \max\left\{\gamma_b, \gamma_a\right\} \int_0^t \left|\, I_a\left(\tau\right) - I_b\left(\tau\right) \right| d\tau
\end{aligned}
\tag{2.23}
$$

for any $t \in [0, T]$.

4. Similarly, we estimate, $\left|R_a\left(t\right) - R_b\left(t\right)\right|$

$$
\begin{aligned}
& \left|R_a\left(t\right) - R_b\left(t\right)\right| \\
& \le\; \left|R_a\left(0\right) - R_b\left(0\right)\right| + \int_0^t \left|\gamma_a\left(\tau\right) I_a\left(\tau\right) - \gamma_b\left(\tau\right) I_b\left(\tau\right)\right| d\tau \\
& \le\; \left|R_a\left(0\right) - R_b\left(0\right)\right| + \int_0^t \left|\, \gamma_a\left(\tau\right) I_a\left(\tau\right) - \gamma_b\left(\tau\right) I_a\left(\tau\right) \right| d\tau \\
& \quad + \int_0^t \left|\, \gamma_b\left(\tau\right) I_a\left(\tau\right) - \gamma_b\left(\tau\right) I_b\left(\tau\right) \right| d\tau \\
& \le\; \left|R_a\left(0\right) - R_b\left(0\right)\right| \\
& \quad + N_a \, t \, \left|\left|\, \gamma_a\left(t\right) - \gamma_b\left(t\right) \right|\right|_\infty \\
& \quad + \max\left\{\gamma_{max,b}, \gamma_{max,a}\right\} \int_0^t \left|\, I_a\left(\tau\right) - I_b\left(\tau\right) \right| d\tau
\end{aligned}
\tag{2.24}
$$

for any $t \in [0, T]$.

Finally from $|S_a(t) - S_b(t)|, |E_a(t) - E_b(t)| |I_a(t) - I_b(t)|, |R_a(t) - R_b(t)|$ we find, $||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty$

$$
\begin{aligned}
& ||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty \\
\leq\ & ||\mathbf{z}_a(0) - \mathbf{z}_b(0)||_\infty + N_a^2\, t\ ||\lambda_a(t) - \lambda_b(t)||_\infty \\
& + N_a\, t\ ||\,\gamma_a(t) - \gamma_b(t)\,||_\infty \\
& + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a\, N_b \int_0^t ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau \\
& + \epsilon \int_0^t ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau \\
& + \max\{\gamma_{max,b}, \gamma_{max,a}\} \int_0^t ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau
\end{aligned} \tag{2.25}
$$

for any $t \in [0, T]$.

Define the functions,

$$
\begin{aligned}
u(t) &:= ||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty \\
g(t) &:= ||\mathbf{z}_a(0) - \mathbf{z}_b(0)||_\infty + N_a^2\, t\, |\lambda_a - \lambda_b| + N_a\, t\, |\,\gamma_a - \gamma_b\,| \\
f(t) &:= \{\,\max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a\, N_b + \max\{\gamma_{max,b}, \gamma_{max,a}\} + \epsilon\} := K_{GB}
\end{aligned}
$$

We see that the assumptions of Theorem 3 are fulfilled and

$$
||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty \ \leq\ g(t)\ \mathbf{e}^{K_{GB}\, t} \tag{2.26}
$$

for any $t \in [0, T]$.

2.1.6 Long time behavior of solutions

In this section, we derive some monotonicity properties and the long time behaviour of our solutions.

Theorem 2.15 ([2]) We see the following behavior for $S(t)$, $E(t)$, $I(t)$ and $R(t)$

1. $S(t)$ decreases monotonically from $S(0) > 0$. There exists $S^* \geq 0$ such that $\lim_{t\to\infty} S(t) = S^*$. Further, it holds that $S^* > 0$.
2. $R(t)$ increases monotonically from $R(0) \geq 0$. There exists $R^* \geq 0$ such that $\lim_{t\to\infty} R(t) = R^*$.
3. I is Lebesgue-integrable on $[0,\infty)$ and $\lim_{t\to\infty} I(t) = 0$.
4. $\lim_{t\to\infty} E(t) = 0$.

Proof. [2]

1. $S'(t) = -\lambda(t) S(t) I(t)$, and by Lemma 2.10, we know that since $S'(t) < 0$ since $\lambda(t) > 0$. By the boundedness of $S(t)$, therefore, $0 \leq S(t) \leq S_0$. Thus, $S : [0,\infty) \to [0,\infty)$ monotonically decreases and is bounded below by 0.
   To prove that $\lim_{t\to\infty} S(t)$ exists , we divide the S' equation by R' ,
   $$\frac{S'(t)}{R'(t)} = -\frac{\lambda(t)}{\gamma(t)} S(t) \;\geq\; -\frac{\lambda_{max}}{\gamma_{min}} S(t)\,.$$
   By separation of variables,
   $$\frac{S'(t)}{S(0)} \geq -\frac{\lambda_{max}}{\gamma_{min}} R'(t)$$
   Integrating, we obtain,
   $$S(t) \geq S(0)\,\mathbf{e}^{-\frac{\lambda_{max}}{\gamma_{min}}(R(t)-R(0))} \geq S_0 \mathbf{e}^{-\frac{\lambda_{max}}{\gamma_{min}}(R(t)-R_0)} \geq 0 \tag{2.27}$$
   for all $t \geq 0$.
   Thus, $S^* := \lim_{t\to\infty} S(t) \geq 0$ exists for all $t \geq 0$. Since $S_0 \mathbf{e}^{-\frac{\lambda_{max}}{\gamma_{min}}(R(t)-R_0)} \geq S_0 \mathbf{e}^{-\frac{\lambda_{max}}{\gamma_{min}}N} > 0$, we have, $S^* > 0$.

2. $R^{'}(t) = \gamma(t) I(t) \geq 0$ for all $t \geq 0$ and $0 \leq R(t) \leq N$ by Lemma 2.10. Therefore, $R : [0, \infty) \rightarrow [0, \infty)$ is monotonically increasing and is bounded above by N. Therefore, it follows that $R^{*} := \lim_{t\rightarrow\infty} R(t) \geq 0$ exists.

3. By assumption, $\lambda(t) \geq 0$ and from Lemma 2.10. , $S, I : [0, \infty) \rightarrow [0, \infty) \geq 0$ and are bounded above by N.

   Therefore,

$$\int_0^{\infty} S^{'}(\tau)\, d\tau = S^{*} - S_0 = \int_0^{\infty} -\lambda(\tau) S(\tau) I(\tau)\, d\tau$$

$$\int_0^{\infty} \lambda(\tau) S(\tau) I(\tau)\, d\tau = S_0 - S^{*} \geq \lambda_{min} S^{*} \int_0^{t} I(\tau)\, d\tau \qquad (2.28)$$

   Therefore, I is Lebesgue-integrable on $[0, \infty)$ and $\lim_{t\rightarrow\infty} I(t) = 0$.

4. From property 3 and from $I^{'}(t) = \epsilon E(t) - \gamma(t) I(t)$, we see that, $\lim_{t\rightarrow\infty} E(t) = 0$.

Based on the monotonicity properties and the boundedness of S, I and R, we can draw the following biological conclusions.[1]

1. Some number of susceptibles always escape the infection at the end of the epidemic.

2. The epidemic ends not because the suseptibles are exhausted.

3. The disease eventually dies out and the infectious population tends to zero after a long period of time.

4. Since the infectious population increases initially, there must be a period of time after which the change in I becomes negative. In other words, the epidemic first rises, the declines after reaching the maximum.

2.1.7 Basic Reproduction Number

The basic reproduction number, $\mathcal{R}_0$ [1] in the SEIR model, is defined by,

$$\mathcal{R}_0(t) := \lambda(t)\,\epsilon(t)\,\frac{1}{\gamma(t)\,\epsilon(t)} \cdot S(t) = \frac{\lambda(t)}{\gamma(t)} \cdot S(t).$$

From the non negativity and boundedness of $\lambda(t)$ and $\gamma(t)$, we see that ,

$$0 \ \leq \ \frac{\lambda_{min}}{\gamma_{max}} \cdot S_0 \ \leq \ \mathcal{R}_0(t) \ \leq \ \frac{\lambda_{max}}{\gamma_{min}} \cdot S_0 \tag{2.29}$$

2.2 Discrete SEIR model with time dependent coefficients

We now examine the time discrete SEIR model with time dependent transmission and recovery rate . We assume that a time interval $[0, T)$ has been divided into a strictly increasing sequence $\{t_j\}_{j=1}^{j=M}$ such that $t_1 = 0$ and $t_M = T$. We represent $f(t_j)$ by f_j for any time dependent function f.

In this model we state a fully explicit form ,

$$\left.\begin{aligned} \frac{\Delta S_j}{\Delta t_j} &= \frac{S_{j+1} - S_j}{t_{j+1} - tj} = -\lambda_{j+1} S_j I_j, \\ \frac{\Delta E_j}{\Delta t_j} &= \frac{E_{j+1} - E_j}{t_{j+1} - tj} = \lambda_{j+1} S_j I_j - \epsilon E_j \\ \frac{\Delta I_j}{\Delta t_j} &= \frac{I_{j+1} - Ij,}{t_{j+1} - tj} = \epsilon E_j - \gamma_{j+1} I_j, \\ \frac{\Delta R_j}{\Delta t_j} &= \frac{R_{j+1} - Rj}{t_{j+1} - tj} = \gamma_{j+1} I_j \end{aligned}\right\} \tag{2.30}$$

and a fully implicit form as in [1]

$$\left.\begin{aligned} \frac{\Delta S_j}{\Delta t_j} &= \frac{S_{j+1} - S_j}{t_{j+1} - tj} = -\lambda_{j+1} S_{j+1} I_{j+1}, \\ \frac{\Delta E_j}{\Delta t_j} &= \frac{E_{j+1} - E_j}{t_{j+1} - tj} = \lambda_{j+1} S_{j+1} I_{j+1} - \epsilon E_{j+1}, \\ \frac{\Delta I_j}{\Delta t_j} &= \frac{I_{j+1} - Ij}{t_{j+1} - tj} = \epsilon E_{j+1} - \gamma_{j+1} I_{j+1}, \\ \frac{\Delta R_j}{\Delta t_j} &= \frac{R_{j+1} - Rj}{t_{j+1} - tj} = \gamma_{j+1} I_{j+1} \end{aligned}\right\} \tag{2.31}$$

for all $j \in \{1, \ldots, M-1\}$. Both of these satisfy,

$$N = S_{j+1} + E_{j+1} + I_{j+1} + R_{j+1} = S_j + E_j + I_j + R_j. \tag{2.32}$$

We first show that the fully explicit form reduces to a set of linear equations and the fully implicit form has non-linear nature. From equation 2.30, the fully explicit form,

$$\left.\begin{aligned}
S_{j+1} &= S_j - \lambda_{j+1} S_j I_j \Delta t_j = S_j - \lambda_{j+1} S_j I_j \left(t_{j+1} - t_j\right), \\
E_{j+1} &= E_j + \left(\lambda_{j+1} S_j I_j + \epsilon E_j\right) \Delta t_j = E_j + \left(\lambda_{j+1} S_j I_j + \epsilon E_j\right)\left(t_{j+1} - t_j\right), \\
I_{j+1} &= I_j + \left(\epsilon E_j - \gamma_{j+1} I_j\right) \Delta t_j = I_j + \left(\epsilon E_j - \gamma_{j+1} I_j\right)\left(t_{j+1} - t_j\right), \\
R_{j+1} &= R_j + \gamma_{j+1} I_j \Delta t_j = R_j + \gamma_{j+1} I_j \left(t_{j+1} - t_j\right)
\end{aligned}\right\} \tag{2.33}$$

is linear. From equation 2.31, the fully implicit form, appears to be in the nonlinear form below.

$$\left.\begin{aligned}
S_{j+1} &= \frac{S_j}{\left(1 + \lambda_{j+1} I_{j+1} \left(t_{j+1} - t_j\right)\right)}, \\
E_{j+1} &= \frac{E_j + \lambda_{j+1} S_{j+1} I_{j+1} \left(t_{j+1} - t_j\right)}{\left(1 + \epsilon \left(t_{j+1} - t_j\right)\right)}, \\
I_{j+1} &= \frac{I_j + \epsilon E_{j+1} \left(t_{j+1} - t_j\right)}{\left(1 + \gamma_{j+1} \left(t_{j+1} - t_j\right)\right)}, \\
R_{j+1} &= R_j + \gamma_{j+1} I_{j+1} \left(t_{j+1} - t_j\right)
\end{aligned}\right\} \tag{2.34}$$

for $j \in \{1, \ldots, M-1\}$

2.2.1 Unique Solvability

The time discrete implicit form given in 2.34 is uniquely solvable for every $j \in \{1, \ldots, M-1\}$. If we replace E_{j+1}in the I_{j+1} equation with we find that,

$$I_{j+1} = \frac{I_j + \epsilon E_{j+1}\left(t_{j+1} - t_j\right)}{\left(1 + \gamma_{j+1}\left(t_{j+1} - t_j\right)\right)}$$
$$= \frac{I_j + \epsilon\left(\dfrac{E_j + \lambda_{j+1} S_{j+1} I_{j+1}\left(t_{j+1} - t_j\right)}{\left(1 + \epsilon\left(t_{j+1} - t_j\right)\right)}\right)\left(t_{j+1} - t_j\right)}{\left(1 + \gamma_{j+1}\left(t_{j+1} - t_j\right)\right)}$$
$$= \frac{I_j\left(1 + \epsilon\left(t_{j+1} - t_j\right)\right) + \epsilon\left(E_j + \lambda_{j+1} S_{j+1} I_{j+1}\left(t_{j+1} - t_j\right)\right)\left(t_{j+1} - t_j\right)}{\left(1 + \epsilon\left(t_{j+1} - t_j\right)\right)\left(1 + \gamma_{j+1}\left(t_{j+1} - t_j\right)\right)}.$$

We now replace S_{j+1} using equation 2.70

$$I_{j+1} = \frac{I_j\left(1 + \epsilon \Delta t_{j+1}\right) + \epsilon \Delta t_{j+1}\left(E_j + \lambda_{j+1}\dfrac{S_j}{\left(1 + \lambda_{j+1} I_{j+1} \Delta t_{j+1}\right)} I_{j+1} \Delta t_{j+1}\right)}{\left(1 + \epsilon \Delta t_{j+1}\right)\left(1 + \gamma_{j+1} \Delta t_{j+1}\right)}. \tag{2.35}$$

Rearranging, we have,

$$I_{j+1}\left[1 + \lambda_{j+1} I_{j+1} \Delta t_{j+1}\right]\left(1 + \epsilon \Delta t_{j+1}\right)\left(1 + \gamma_{j+1} \Delta t_{j+1}\right)$$
$$= I_{j+1}\left(I_j \lambda_{j+1} \Delta t_{j+1} + \epsilon \lambda_{j+1} I_j \Delta^2 t_{j+1} + \epsilon E_j \lambda_{j+1} \Delta^2 t_{j+1} + \epsilon S_j \lambda_{j+1} \Delta^2 t_{j+1}\right)$$
$$+ I_j\left(1 + \epsilon \Delta t_{j+1}\right) + \epsilon \Delta t_{j+1} E_j.$$

$$\begin{aligned}
&I_{j+1}^2\left[\lambda_{j+1} \Delta t_{j+1} + \gamma_{j+1} \lambda_{j+1} \Delta^2 t_{j+1} + \lambda_{j+1} \epsilon \Delta^2 t_{j+1} + \lambda_{j+1} \gamma_{j+1} \epsilon \Delta^3 t_{j+1}\right] \\
&+ I_{j+1}\left[1 + \gamma_{j+1} \Delta t_{j+1} + \epsilon \Delta t_{j+1} + \epsilon \gamma_{j+1} \Delta^2 t_{j+1}\right] \\
&- I_{j+1}\left[\left(I_j \lambda_{j+1} \Delta t_{j+1} + \epsilon \lambda_{j+1} I_j \Delta^2 t_{j+1} + \epsilon E_j \lambda_{j+1} \Delta^2 t_{j+1} + \epsilon S_j \lambda_{j+1} \Delta^2 t_{j+1}\right)\right] \\
&= I_j\left(1 + \epsilon \Delta t_{j+1}\right) + \epsilon \Delta t_{j+1} E_j.
\end{aligned} \tag{2.36}$$

Now we define,

$$\left.\begin{aligned}
A_{j+1} &:= \left[\lambda_{j+1} \Delta t_{j+1} + \gamma_{j+1} \lambda_{j+1} \Delta^2 t_{j+1} + \lambda_{j+1} \epsilon \Delta^2 t_{j+1} + \lambda_{j+1} \gamma_{j+1} \epsilon \Delta^3 t_{j+1}\right], \\
B_{j+1} &:= \frac{1}{2}\left[1 + \Delta t_{j+1}\left(\gamma_{j+1} + \epsilon - I_j \lambda_{j+1}\right) + \Delta^2 t_{j+1}\left(\epsilon \gamma_{j+1} - \epsilon \lambda_{j+1} I_j - \epsilon \lambda_{j+1} E_j - \epsilon \lambda_{j+1} S_j\right)\right]
\end{aligned}\right\} \tag{2.37}$$

Thus Equation 2.36 becomes,

$$A_{j+1}I_{j+1}^2 + 2B_{j+1}I_{j+1} - I_j\left(1+\epsilon\Delta t_{j+1}\right) - \epsilon\Delta t_{j+1}E_j = 0 \tag{2.38}$$

So that,

$$I_{j+1} = \frac{-B_{j+1}}{A_{j+1}} + \sqrt{\frac{B_{j+1}^2}{A_{j+1}^2} + \frac{I_j\left(1+\epsilon\Delta t_{j+1}\right) + \epsilon\Delta t_{j+1}E_j}{A_{j+1}}} \tag{2.39}$$

Since $I_{j+1} \geq 0$ for all $j \in \{1, \ldots, M-1\}$

2.2.2 Monotonicity and Long Time Behavior

Theorem 2.16 *For the time discrete implicit form given in equation 2.31, the following hold*

1. $0 \leq I_j \leq N$ *for all* $j \in \{1, \ldots, M-1\}$.
2. $0 \leq E_j \leq N$ *for all* $j \in \{1, \ldots, M-1\}$.
3. $0 \leq S_j \leq N$ *for all* $j \in \{1, \ldots, M-1\}$ *and* $S_{j+1} \leq S_j$ *for all* $j \in \{1, \ldots, M-1\}$.
4. $0 \leq R_j \leq N$ *for all* $j \in \{1, \ldots, M-1\}$ *and* $R_{j+1} \geq R_j$ *for all* $j \in \{1, \ldots, M-1\}$.
5. $\lim_{j\to\infty} I_j = 0$.
6. $\lim_{j\to\infty} E_j = 0$.

Proof

1. By assumption, $I_j > 0$ for all $j \in \{1, \ldots, M-1\}$.Thus, by 2.31, $I_j \geq 0$ and from 2.32 , $I_j \leq N$ for all $j \in \{1, \ldots, M-1\}$.
2. By property 1 and by equation 2.32 , $E_j \geq 0$ and , $E_j \leq N$ for all $j \in \{1, \ldots, M-1\}$.

3. By assumption, $S_j > 0$ for all $j \in \{1, \ldots, M-1\}$. Thus, $S_j \geq 0$ and by Property 1) and from 2.32 , $S_j \leq N$ for all $j \in \{1, \ldots, M-1\}$. Further , from 2.32, $S_{j+1} = \dfrac{S_j}{1 + \lambda_{j+1} I_{j+1} \left(t_{j+1} - t_j\right)} \leq S_j$, since $I_{j+1} \geq 0, \lambda_{j+1} > 0$ and $t_{j+1} - t_j > 0$.

4. By properties 1, 2 and from equation 2.32 , $0 \leq R_j \leq N$ for all $j \in \{1, \ldots, M-1\}$. Further, from equation 2.32, $R_{j+1} = R_j + \gamma_{j+1} I_{j+1} \left(t_{j+1} - t_j\right) \geq R_j$, since γ_{j+1} , $t_{j+1} - t_j > 0$ by assumption.

5. Since R_{j+1} is monotonically increasing and bounded above by N . there exists a positive R^* , such that $\lim_{j \to \infty} R_j = R^*$.
   Further, $R_{j+1} - R_j = \gamma_{j+1} I_{j+1} \left(t_{j+1} - t_j\right)$ implies, $I_{j+1} = \dfrac{R_{j+1} - R_j}{\gamma_{j+1} \left(t_{j+1} - t_j\right)}$.
   Therefore, $\lim_{j \to \infty} I_j = \lim_{j \to \infty} \frac{R_{j+1} - R_j}{\gamma_{j+1} \left(t_{j+1} - t_j\right)} = 0$.

6. From property 5 and equation 2.32, we can conclude that $\lim_{j \to \infty} E_j = 0$.

2.2.3 Basic Reproduction Number

The basic reproduction number, $\mathcal{R}_0 \left(t_k\right)$ [1] in the time-discrete SEIR model, is defined by,

$$\mathcal{R}_0 \left(t_k\right) = \lambda \left(t_k\right) \epsilon \left(t_k\right) \frac{1}{\gamma \left(t_k\right) \epsilon \left(t_k\right)} \cdot S \left(t_k\right) = \frac{\lambda \left(t_k\right)}{\gamma \left(t_k\right)} \cdot S \left(t_k\right). \tag{2.40}$$

for arbitrary $k \in \{1, \cdots, M\}$. From the non negativity and boundedness of $\lambda \left(t_k\right)$ and $\gamma \left(t_k\right)$, we see that ,

$$0 \leq \frac{\lambda_{min}}{\gamma_{max}} \cdot S_0 \leq \mathcal{R}_0 \left(t_k\right) \leq \frac{\lambda_{max}}{\gamma_{min}} \cdot S_0 \tag{2.41}$$

2.2.4 Final Size Formula

The final size formula gives the fraction of susceptibles that escaped the infection at the end of the epidemic. Since ,

$$S(t) = S(0)\,\mathbf{e}^{-\frac{\lambda}{\gamma}(R(t)-R(0))} = S_0\mathbf{e}^{-\frac{\lambda}{\gamma}(R(t)-R_0)}$$

$$S^* := \lim_{t\to\infty} S(t) = \lim_{t\to\infty} S_0\mathbf{e}^{-\frac{\lambda}{\gamma}(R(t)-R_0)}.$$

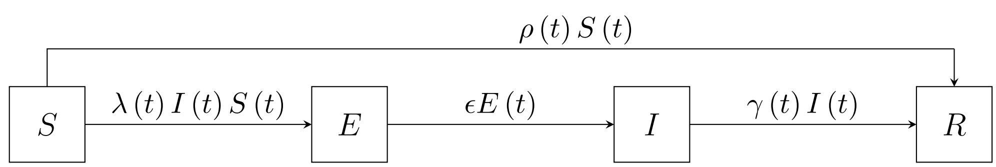
Therefore,

$$\frac{S^*}{S_0} = \lim_{t\to\infty} \mathbf{e}^{-\frac{\lambda}{\gamma}(R(t)-R_0)}.$$

2.3 SEIR model with vaccination

2.3.1 Assumptions

In this model, the compartments remain the same as in the SEIR model. A fraction of the susceptibles are directly placed to the recovered compartment on vaccination. We do not assume a loss of immunity as in the previous cases. So, all vaccianted individuals are moved to the R compartment and remain there for all time. The fraction of individuals vaccinated is also considered a function of time in our model. Since the total population remains a constant over all time $t \geq 0$,
we have , $N(t) = S(t) + E(t) + I(t) + R(t)$.



We now state the assumptions of the model below.

1. The total population N is fixed over time.

2. The total population is divided into homogeneous groups, with $S(t)$ representing Susceptible population, $E(t)$ Latent population, $I(t)$ Infectious population and $R(t)$ Recovered population at time t. Since the total population is a constant, $N(t) = S(t) + E(t) + I(t) + R(t)$ for all time $t \in [0, \infty)$.

3. The time varying transmission coefficient $\lambda(t)$ is Lipschitz continuous and is continuously differentiable for all $t \in [0, \infty)$. It holds that $0 < \lambda_{min} \leq \lambda(t) \leq \lambda_{max}$, for all $t \in [0, \infty)$, where λ_{min} is the lowest and λ_{max} is the highest value λ achieves.

4. The latency period remains a constant and is independent of time.

5. We consider a time varying vaccination coefficient $\rho(t)$ which is also Lipschitz continuous and is continuously differentiable for all $t \in [0, \infty)$. It holds that $0 \leq \rho_{min} \leq \rho(t) \leq \rho_{max}$, for all $t \in [0, \infty)$, where ρ_{min} is the lowest and ρ_{max} is the highest value ρ achieves.

6. The time varying recovery coefficient $\gamma(t)$ is Lipschitz continuous and is continuously differentiable for all $t \in [0, \infty)$. It holds that $0 < \gamma_{min} \leq \gamma(t) \leq \gamma_{max}$, for all $t \in [0, \infty)$, where γ_{min} is the lowest and γ_{max} is the highest value γ achieves.

The differential equations for the SEIR model with vaccination are,

$$\left.\begin{aligned}
S^{'}(t) &= -\lambda(t)\, S(t)\, I(t) - \rho(t)\, S(t)\,, \\
E^{'}(t) &= \lambda(t)\, S(t)\, I(t) - \epsilon\, E(t)\,, \\
I^{'}(t) &= \epsilon\, E(t) - \gamma(t)\, I(t)\,, \\
R^{'}(t) &= \gamma(t)\, I(t) + \rho(t)\, S(t)\,, \\
N(t) &= S(t) + E(t) + I(t) + R(t)
\end{aligned}\right\} \tag{2.42}$$

The initial conditions for the system of equations are,

$$N(t) := N_0 = N$$

$$S(0) := S_0 > 0,\ I(0) := I_0 > 0,\ E(0) := E_0 \geq 0,\ R(0) := R_0 \geq 0$$

2.3.2 Nonnegativity and boundedness

We first find the feasible region for the solutions of the system of differential equations 2.42. Since the equation for E and I are same as the previous model, we prove that S and R are non negative functions of t.

Lemma 2.17 *Each solution of the system given by* 2.42 is bounded below by 0.

Proof

1. We use separation of variables and write the S equations as,

$$\frac{S'(t)}{S(t)} = -\lambda(t)\ I(t) - \rho(t)\,. \tag{2.43}$$

   Integrating,

$$\ln\left(\frac{S(t)}{S(0)}\right) = -\left(\int_0^t \lambda(\tau)\, I(\tau)\, d\tau + \int_0^t \rho(\tau)\, d\tau\right)$$

$$S(t) = S(0)\, \mathbf{e}^{-\int_0^t (\lambda(\tau) I(\tau) + \rho(\tau)) d\tau}. \tag{2.44}$$

   Where $S(0)$ is the initial condition on $S(t)$, corresponding to the initial population in the S compartment. Since $S(0) \geq 0$, we see that $S(t) \geq 0$ at all time $t \geq 0$.

2. Since $R'(t) = \gamma(t)\, I(t) + \rho(t)\ S(t)$, integration yields,

$$R(t) = R_0 + \int_0^t \gamma(\tau)\, I(\tau)\, d\tau\ + \int_0^t \rho(t)\ S(t)\, d\tau. \tag{2.45}$$

   Since$I(t) \geq 0\, S(t) \geq 0$ and $\gamma(t) > 0\, \rho(t) \geq 0$, we know $R(t) \geq 0$ for all $t \geq 0$.

We now state the boundedness theorem for the solutions.

Theorem 2.18 *For each solution of the system given by* 2.42,

1. $0 \leq S(t) \leq N$
2. $0 \leq E(t) \leq N$
3. $0 \leq I(t) \leq N$
4. $0 \leq R(t) \leq N$

For all $t \geq 0$.

2.3.3 Global existence of solutions

We define the solution vector,

$$\mathbf{z}(t) = (S(t), E(t), I(t), R(t))$$

$$\mathbf{z}(0) = \mathbf{z_0} = (S(0), E(0), I(0), R(0))$$

and $\mathbf{z}'(t) = G(t, \mathbf{z}(t))$.

We state the global existence of solutions as follows.

Theorem 2.19 ([1], [2]) *At least one solution to the differential equations* 2.42 exist for all time $t \in [0, \infty)$.

Proof.

We define a function $G(t, \mathbf{z}(t)) : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ as $\begin{pmatrix} -\lambda(t)\ S(t)\,I(t) - \rho(t)\ S(t) \\ \lambda(t)\ S(t)\,I(t) - \epsilon E(t) \\ \epsilon E(t) - \gamma(t)\,I(t) \\ \gamma(t)\,I(t) + \rho(t)\ S(t) \end{pmatrix}$.

We first prove that G is Lipschitz continuouous using the property that the upper bounds of the coefficients exist amd the upper bounds of S, E, I and R are N.

Consider $\mathbf{z}_1(t) = (S_1(t), E_1(t), I_1(t), R_1(t))$ and $\mathbf{z}_2(t) = (S_2(t), E_2(t), I_2(t), R_2(t))$

$$
\begin{aligned}
& \mid G(t, \mathbf{z}_1(t)) - G(t, \mathbf{z}_2(t)) \mid \\
& = \begin{pmatrix} \mid \lambda_2(t) S_2(t) I_2(t) - \lambda_1(t) S_1(t) I_1(t) + \rho_2(t) S_2(t) - \rho_1(t) S_1(t) \mid \\ \mid \lambda_1(t) S_1(t) I_1(t) - \lambda_2(t) S_2(t) I_2(t) - \epsilon E_1(t) + \epsilon E_2(t) \mid \\ \mid \epsilon E_1(t) - \epsilon E_2(t) - \gamma_1(t) I_1(t) + \gamma_2(t) I_2(t) \mid \\ \mid \gamma_1(t) I_1(t) - \gamma_2(t) I_2(t) + \rho_1(t)\ S_1(t) - \rho_2(t)\ S_2(t) \mid \end{pmatrix} \\
& \leq \begin{pmatrix} \mid \lambda_2(t) S_2(t) I_2(t) - \lambda_1(t) S_1(t) I_1(t) \mid + \mid \rho_2(t) S_2(t) - \rho_1(t) S_1(t) \mid \\ \mid \lambda_1(t) S_1(t) I_1(t) \dfrac{E_1(t)}{E_1(t)} - \lambda_2(t) S_2(t) I_2(t) \dfrac{E_2(t)}{E_2(t)} \mid - \mid \epsilon E_1(t) - \epsilon E_2(t) \mid \\ \mid \epsilon E_1(t) \dfrac{I_1(t)}{I_1(t)} - \epsilon E_2(t) \dfrac{I_2(t)}{I_2(t)} \mid - \mid \gamma_1(t) I_1(t) - \gamma_2(t) I_2(t) \mid \\ \mid \gamma_1(t) I_1(t) - \gamma_2(t) I_2(t) \mid + \mid \rho_1(t) S_1(t) - \rho_2(t) S_2(t) \mid \end{pmatrix} \\
& \leq \mid \lambda_{max} + \gamma_{max} + \epsilon + \rho_{max} \mid \begin{pmatrix} \mid N \mid \mid S_1(t) - S_2(t) \mid \\ \mid N \mid \mid E_1(t) - E_1(t) \mid \\ \mid N \mid \mid I_1(t) - I_2(t) \mid \\ \mid N \mid \mid R_1(t) - R_2(t) \mid \end{pmatrix} \\
& = N(\lambda_{max} + \gamma_{max} + \epsilon + \rho_{max}) \begin{pmatrix} \mid S_1(t) - S_2(t) \mid \\ \mid E_1(t) - E_2(t) \mid \\ \mid I_1(t) - I_2(t) \mid \\ \mid R_1(t) - R_2(t) \mid \end{pmatrix} \\
& = N(\lambda_{max} + \gamma_{max} + \epsilon + \rho_{max}) \mid \mathbf{z}_1(t) - \mathbf{z}_2(t) \mid . \qquad (2.46)
\end{aligned}
$$

Thus G is Lipschitz continuouous in $\mathbf{z}(t)$.

Now we consider the supremum norm,

$$
\begin{aligned}
& \| G(t, \mathbf{z}(t)) \|_\infty \\
&= \sup_{t\in[0,\infty)} \{ | -\lambda(t)\, S(t) I(t) - \rho(t)\, S(t) |, | \lambda(t)\, S(t) I(t) - \epsilon E(t) |, \\
&\qquad | \epsilon E(t) - \gamma(t) I(t) |, | \gamma(t) I(t) + \rho(t) S(t) | \} \\
&\le \sup_{t\in[0,\infty)} \{ \lambda_{max} \, | S(t) I(t) | + \rho_{max} \, | S(t) |, \lambda_{max} \, | S(t) I(t) | + \epsilon \, | E(t) |, \\
&\qquad \epsilon \, | E(t) | + \gamma_{max} \, | I(t) |, \gamma_{max} \, | I(t) | + \rho_{max} \, | S(t) | \} \\
&\le \sup_{t\in[0,\infty)} \{ \lambda_{max} N \, | S(t) | , \ (\lambda_{max} N + \epsilon) \, | E(t) | , \ (\epsilon + \gamma_{max}) \, | I(t) | , \\
&\qquad (\gamma_{max} + \rho_{max}) \, | R(t) | \} \\
&\le N(\lambda_{max} + \epsilon + \gamma_{max} + \rho_{max}) \sup_{t\in[0,\infty)} \{ | S(t) |, | E(t) |, | I(t) |, | R(t) | \} \\
&\le N(\lambda_{max} + \epsilon + \gamma_{max} + \rho_{max}) \, \| \mathbf{z}(t) \|_\infty \, .
\end{aligned} \tag{2.47}
$$

by the boundedness of $S(t)$, $E(t)$, $I(t)$ and $R(t)$ and the transmission and recovery coefficients. Thus the conditions for Theorem 1.5 are satisfied and the proof is complete.

2.3.4 Global uniqueness of solutions

Theorem 2.20 ([1] , [2]) *There exists a unique solution for all time* $t \in [0, \infty)$*, for the initial value problem given by the differential equations* 2.42

Proof.

First choose an interval $[0, \tau]$ on which the Banach's fixed point theorem is applicable.

Now, suppose that there exist two different solutions in the interval $[0,\infty)$, say, $\mathbf{z}(t)=(S(t),E(t),I(t),R(t))$ and $\widetilde{\mathbf{z}}(t)=\left(\widetilde{S}(t),\widetilde{E}(t),\widetilde{I}(t),\widetilde{R}(t)\right)$. Then,

$$
\begin{aligned}
&\sup_{t\in[0,\tau]}\left|S(t)-\widetilde{S}(t)\right| \\
&= \sup_{t\in[0,\tau]}\left|\int_0^t\left(\lambda(z)\widetilde{S(z)}\widetilde{I(z)}-\lambda(z)S(z)I(z)\right)dz\right| \\
&\quad+\sup_{t\in[0,\tau]}\left|\int_0^t\left(\rho(z)\widetilde{S(z)}-\rho(z)S(z)\right)dz\right| \\
&\le \sup_{t\in[0,\tau]}\lambda_{max}\int_0^t\left|\widetilde{S(z)}\right|\left|\widetilde{I(z)}-I(z)\right|+|I(z)|\left|S(z)-\widetilde{S(z)}\right|dz \\
&\quad+\sup_{t\in[0,\tau]}\rho_{max}\int_0^t\left|\widetilde{S(z)}-S(z)\right|dz \\
&\le \sup_{t\in[0,\tau]}\lambda_{max}\int_0^t N\left|\widetilde{I(z)}-I(z)\right|+N\left|S(z)-\widetilde{S(z)}\right|dz \\
&\quad+\sup_{t\in[0,\tau]}\rho_{max}\int_0^t\left|\widetilde{S(z)}-S(z)\right|dz \\
&\le \sup_{t\in[0,\tau]}\lambda_{max}\,N\,t\left\{\left|\widetilde{I(t)}-I(t)\right|+\left|S(t)-\widetilde{S(t)}\right|\right\}+\sup_{t\in[0,\tau]}\rho_{max}\,t\left\{\left|\widetilde{S(z)}-S(z)\right|\right\} \\
&\le (2\lambda_{max}+\rho_{max})\,N\,\tau\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty . \qquad (2.48)
\end{aligned}
$$

Similarly, we obtain,

$$
\begin{aligned}
&\sup_{t\in[0,\tau]}\left|E(t)-\widetilde{E}(t)\right| \\
&= \sup_{t\in[0,\tau]}\left|\int_0^t\lambda(z)\left(S(z)I(z)-\widetilde{S(z)}\widetilde{I(z)}\right)dz\right|+\sup_{t\in[0,\tau]}\left|\int_0^t\epsilon\left(E(z)-\widetilde{E(z)}\right)dz\right| \\
&\le \sup_{t\in[0,\tau]}\lambda_{max}\,t\,N\left\{\left|\widetilde{I(t)}-I(t)\right|+\left|S(t)-\widetilde{S(t)}\right|\right\}+\sup_{t\in[0,\tau]}\epsilon\,t\left\{\left|\widetilde{E(t)}-E(t)\right|\right\} \\
&\le (\lambda_{max}\,N+\epsilon)\,\tau\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \\
&\le (2\lambda_{max}+\epsilon)\,N\tau\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \qquad (2.49)
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{t\in[0,\tau]} \left|I(t)-\widetilde{I}(t)\right| \\
&= \sup_{t\in[0,\tau]} \left|\int_0^t \epsilon\left(E(z)-\widetilde{E(z)}\right)dz\right| + \sup_{t\in[0,\tau]} \left|\int_0^t \gamma(z)\left(I(z)-\widetilde{I(z)}\right)dz\right| \\
&\leq \sup_{t\in[0,\tau]} \epsilon\, t \left|\widetilde{E(t)}-E(t)\right| + \sup_{t\in[0,\tau]} \gamma_{max}\, t \left|\widetilde{I(t)}-I(t)\right| \\
&\leq (\gamma_{max} + \epsilon)\,\tau \left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \\
&\leq (\gamma_{max} + \epsilon)\,N\tau \left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty . \qquad (2.50)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sup_{t\in[0,\tau]} \left|R(t)-\widetilde{R}(t)\right| &= \sup_{t\in[0,\tau]} \left|\int_0^t \gamma(z)\left(I(z)-\widetilde{I(z)}\right)dz\right| \\
&\quad + \sup_{t\in[0,\tau]} \left|\int_0^t \rho(z)\left(S(z)\,\widetilde{S(z)}\right)dz\right| \\
&\leq (\gamma_{max}+\rho_{max})\,N\tau \left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty . \qquad (2.51)
\end{aligned}
$$

Summarizing the steps above, we obtain,

$$
\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \leq (2\,\lambda_{max}+\epsilon+\gamma_{max}+\rho_{max})\,N\tau \left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty . \qquad (2.52)
$$

By choosing the interval $\tau = \frac{1}{2(2\,\lambda_{max}+\epsilon+\gamma_{max}+\rho_{max})N}$, we obtain the necessary contraction.

$$
\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty \leq \frac{1}{2}\left|\left|\mathbf{z}(t)-\widetilde{\mathbf{z}(t)}\right|\right|_\infty . \qquad (2.53)
$$

This proves the uniqueness of the solution in the interval, $[0, \tau]$.
We can now inductively derive the contraction for each interval, $[k\tau, (k+1)\tau]$ for all $k \in \mathbb{N}$. This proves the uniqueness of the solutions for all time $t \geq 0$.

2.3.5 Continuous dependence on initial conditions

We have stated a simple definition of well posedness in the previous section. We now prove the dependence on initial conditions for this model.
To prove the well posedness, we consider the perturbed initial value problem,

$$\left.\begin{aligned} S_a^{'}(t) &= -\lambda_a(t)\; S_a(t)\, I_a(t) - \rho_a\, S_a(t)\,, \\ E_a^{'}(t) &= \lambda_a(t)\; S_a(t)\, I_a(t) - \epsilon E_a(t)\,, \\ I_a^{'}(t) &= \epsilon E_a(t) - \gamma a(t)\, I_a(t)\,, \\ R_a^{'}(t) &= \gamma_a I_a(t) + \rho_a\, S_a(t) \end{aligned}\right\} \tag{2.54}$$

with initial conditions, $S_a(0) = S_{a,0} > 0, E_a(0) = E_{a,0} \geq 0, I_a(0) = I_{a,0} > 0, R_a(0) = R_{a,0} \geq 0$ and

$$\left.\begin{aligned} S_b^{'}(t) &= -\lambda_b(t)\; S_b(t)\, I_b(t) - \rho_b\, S_b(t)\,, \\ E_b^{'}(t) &= \lambda_b(t)\; S_b(t)\, I_b(t) - \epsilon E_b(t)\,, \\ I_b^{'}(t) &= \epsilon E_b(t) - \gamma b(t)\, I_b(t)\,, \\ R_b^{'}(t) &= \gamma_b I_b(t) + \rho_b\, S_b(t) \end{aligned}\right\} \tag{2.55}$$

with initial conditions, $S_b(0) = S_{b,0} > 0, E_b(0) = E_{b,0} \geq 0, I_b(0) = I_{b,0} > 0, R_b(0) = R_{b,0} \geq 0$.
Here we consider the transmission rates λ_a and λ_b , the vaccination rates ρ_a and ρ_b

and the recovery rates, γ_a and γ_b have small differences, as well as the initial conditions have small perturbations. We proceed to prove that this lead to solutions that have small differences in short intervals of time $[0, T]$.

Theorem 2.21 ([2](Theorem 3)) *Let* $\boldsymbol{z}_a(t) = \begin{pmatrix} S_a(t) \\ E_a(t) \\ I_a(t) \\ R_a(t) \end{pmatrix}$ *and* $\boldsymbol{z}_b(t) = \begin{pmatrix} S_b(t) \\ E_b(t) \\ I_b(t) \\ R_b(t) \end{pmatrix}$ *be the solutions of* 2.54 and 2.55 .

Define a function

$g(t) := \| \mathbf{z}_a(0) - \mathbf{z}_b(0) \|_\infty + N_a^2 t \; \| \lambda_a(t) - \lambda_b(t) \|_\infty + N_a t \, \| \gamma_a(t) - \gamma_b(t) \|_\infty$ $+ N_a t \, \| \rho_a - \rho_b \|_\infty$ and the constant,

$K_{GB} := \{\max\{ \lambda_{max,a}, \lambda_{max,b}\} N_a N_b + \max\{\gamma_{max,b}, \gamma_{max,a}\} + \epsilon + \max\{ \rho_{max,a}, \rho_{max,b} \}\}$

Then, $\| \mathbf{z}_a(t) - \mathbf{z}_b(t) \|_\infty \leq g(t) \, \mathbf{e}^{K_{GB} t}$ holds for arbitrary $t \in [0, T]$ with given $T \geq 0$.

Proof.

We see from 2.42 that, $N_a = S_a(0) + E_a(0) + I_a(0) + R_a(0)$ and $N_b = S_b(0) + E_b(0) + I_b(0) + R_b(0)$ holds for all $t \in [0, T]$.

We recall the inequality,

$$|x_1 y_1 - x_2 y_2| \leq |x_1| \, |y_1 - y_2| + |y_2| \, |x_1 - x_2| \tag{2.56}$$

In all the derivations below, we apply the triangle inequality, the inequality 2.56 and the boundedness of the functions, $S(t), E(t), I(t)$ and $R(t)$.

1. First we estimate $|S_a(t) - S_b(t)|$ and obtain,

$$\begin{aligned} & |S_a(t) - S_b(t)| \\ & \quad \leq |S_a(0) - S_b(0)| + \int_0^t |\lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)| \, d\tau \\ & \quad + \int_0^t |\rho_a(\tau) S_a(\tau) - \rho_b(\tau) S_b(\tau)| \, d\tau \end{aligned} \tag{2.57}$$

Define equation

$I := \int_0^t \left|\lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)\right| d\tau$

and

$II := \int_0^t \mid \rho_a S_a(\tau) - \rho_b S_b(\tau) \mid d\tau.$

We see that,

$$\begin{aligned} I \;\leq\; & \int_0^t \mid \lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_a(\tau) I_a(\tau) \mid d\tau \\ & + \int_0^t \mid \lambda_b(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_a(\tau) \mid d\tau \\ & + \int_0^t \mid \lambda_b(\tau) S_b(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau) \mid d\tau \\ \leq\; & N_a^2 \, t \, ||\lambda_a(t) - \lambda_b(t)||_\infty \\ & + \max\{\lambda_{max,a}, \lambda_{max,b}\} N_a \int_0^t |S_a(\tau) - S_b(\tau)| \, d\tau \\ & + \max\{\lambda_{max,a}, \lambda_{max,b}\} N_b \int_0^t |I_a(\tau) - I_b(\tau)| \, d\tau \end{aligned} \tag{2.58}$$

for any $t \in [0, T]$.

$$\begin{aligned} II \;\leq\; & \int_0^t \mid \rho_a(\tau) S_a(\tau) - \rho_b(\tau) S_a(\tau) \mid d\tau + \int_0^t \Big| \rho_b(\tau) S_a(\tau) - \rho_{b(\tau)} S_b(\tau) \Big| \, d\tau \\ \leq\; & N_a \, t \, || \rho_a(t) - \rho_b(t) ||_\infty + \max\{\rho_{max,b}, \rho_{max,a}\} \int_0^t \mid S_a(\tau) - S_b(\tau) \mid d\tau \end{aligned}$$

Thus, adding I and II , we obtain,

$$\begin{aligned}
&|S_a(t) - S_b(t)| \\
&\quad\le |S_a(0) - S_b(0)| + N_a^2\, t\, ||\lambda_a(t) - \lambda_b(t)||_\infty \\
&\quad + \max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\
&\quad + \max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau \\
&\quad + N_a\, t\, ||\rho_a(t) - \rho_b(t)||_\infty + \max\{\rho_{max,b}, \rho_{max,a}\} \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau
\end{aligned} \tag{2.59}$$

for any $t \in [0, T]$.

2. For $|E_a(t) - E_b(t)|$ we have,

$$\begin{aligned}
&|E_a(t) - E_b(t)| \\
&\quad\le |E_a(0) - E_b(0)| + \int_0^t |\lambda_a(\tau) E_a(\tau) I_a(\tau) - \lambda_b(\tau) E_b(\tau) I_b(\tau)|\, d\tau \\
&\quad + \int_0^t |\epsilon E_a(\tau) - \epsilon E_b(\tau)|\, d\tau
\end{aligned}$$

Define equation
$I := \int_0^t |\lambda_a(\tau) S_a(\tau) I_a(\tau) - \lambda_b(\tau) S_b(\tau) I_b(\tau)|\, d\tau$
We see that,

$$\begin{aligned}
I &\le |I_a(0) - I_b(0)| + N_a^2\, t\, ||\lambda_a(t) - \lambda_b(t)||_\infty \\
&\quad + \max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\
&\quad + \max\{\lambda_{max,a}, \lambda_{max,b}\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau
\end{aligned}$$

We obtain,

$$
\begin{aligned}
&|E_a(t) - E_b(t)| \\
&\le\ |E_a(0) - E_b(0)| + N_a^2\, t\ ||\lambda_a(t) - \lambda_b(t)||_\infty \\
&\quad + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a \int_0^t |S_a(\tau) - S_b(\tau)|\, d\tau \\
&\quad + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_b \int_0^t |I_a(\tau) - I_b(\tau)|\, d\tau \\
&\quad + \epsilon \int_0^t |\, E_a(\tau) - E_b(\tau)\,|\, d\tau
\end{aligned}
\tag{2.60}
$$

for any $t \in [0, T]$.

3. Now we estimate $|I_a(t) - I_b(t)|$.

$$
\begin{aligned}
&|I_a(t) - I_b(t)| \\
&\quad \le\ |I_a(0) - I_b(0)| + \int_0^t |\epsilon E_a(\tau) - \epsilon E_b(\tau)|\, d\tau \\
&\quad\ + \int_0^t |\gamma_a I_a(\tau) - \gamma_b I_b(\tau)|\, d\tau
\end{aligned}
$$

Define equation
$I := \int_0^t |\,\gamma_a I_a(\tau) - \gamma_b I_b(\tau)\,|\, d\tau.$
We see that,

$$
\begin{aligned}
I &\le \int_0^t |\,\gamma_a(\tau)\, I_a(\tau) - \gamma_b(\tau)\, I_a(\tau)\,|\, d\tau + \int_0^t \left|\,\gamma_b(\tau)\, I_a(\tau) - \gamma_{b(\tau)} I_b(\tau)\,\right| d\tau \\
&\le\ N_a\, t\ ||\,\gamma_a(t) - \gamma_b(t)\,||_\infty + \max\{\gamma_{max,b}, \gamma_{max,a}\} \int_0^t |\, I_a(\tau) - I_b(\tau)\,|\, d\tau
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
& |I_a(t) - I_b(t)| \\
& \quad \leq \ |I_a(0) - I_b(0)| + \epsilon \int_0^t |E_a(\tau) - E_b(\tau)| \, d\tau \\
& \qquad + N_a \, t \, | \, \gamma_a - \gamma_b \, | + \max\{\gamma_b, \gamma_a\} \int_0^t | \, I_a(\tau) - I_b(\tau) \, | \, d\tau
\end{aligned} \tag{2.61}
$$

for any $t \in [0, T]$.

4. Similarly we estimate $|R_a(t) - R_b(t)|$

$$
\begin{aligned}
& |R_a(t) - R_b(t)| \\
& \leq \ |R_a(0) - R_b(0)| + \int_0^t |\gamma_a(\tau) \, I_a(\tau) - \gamma_b(\tau) \, I_b(\tau)| \, d\tau \\
& \quad + \ \int_0^t |\rho_a(\tau) \, S_a(\tau) - \rho_b(\tau) \, S_b(\tau)| \, d\tau \\
& \leq \ |R_a(0) - R_b(0)| + \int_0^t | \, \gamma_a(\tau) \, I_a(\tau) - \gamma_b(\tau) \, I_a(\tau) \, | \, d\tau \\
& \quad + \int_0^t | \, \gamma_b(\tau) \, I_a(\tau) - \gamma_b(\tau) \, I_b(\tau) \, | \, d\tau \\
& \quad + \int_0^t | \, \rho_a(\tau) \, S_a(\tau) - \rho_b(\tau) \, S_a(\tau)| \, d\tau + \int_0^t \Big| \, \rho_b(\tau) \, S_a(\tau) - \rho_{b(\tau)} S_b(\tau) \, \Big| \, d\tau \\
& \leq \ |R_a(0) - R_b(0)| \\
& \quad + N_a \, t \, || \, \gamma_a(t) - \gamma_b(t) \, ||_\infty \\
& \quad + \max\{\gamma_{max,b}, \gamma_{max,a}\} \int_0^t | \, I_a(\tau) - I_b(\tau) \, | \, d\tau \\
& \quad + N_a \, t \, || \, \rho_a(t) - \rho_b(t) \, ||_\infty \\
& \quad + \max\{\rho_{max,b}, \rho_{max,a}\} \int_0^t | \, S_a(\tau) - S_b(\tau) \, | \, d\tau
\end{aligned} \tag{2.62}
$$

for any $t \in [0,T]$.

Finally from $|S_a(t) - S_b(t)|, |E_a(t) - E_b(t)|\, |I_a(t) - I_b(t)|, |R_a(t) - R_b(t)|$ we find, $||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty$

$$
\begin{aligned}
&||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty \\
&\leq\ ||\mathbf{z}_a(0) - \mathbf{z}_b(0)||_\infty + N_a^2\, t\ ||\lambda_a(t) - \lambda_b(t)||_\infty \\
&\quad + N_a\, t\ ||\,\gamma_a(t) - \gamma_b(t)\,||_\infty + N_a\, t\ ||\,\rho_a(t) - \rho_b(t)\,||_\infty \\
&\quad + \max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a\, N_b \int_0^t ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau \\
&\quad + \epsilon \int_0^t ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau \\
&\quad + \max\{\gamma_{max,b}, \gamma_{max,a}\} \int_0^t\ ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau \\
&\quad + \max\{\rho_{max,b}, \rho_{max,a}\} \int_0^t\ ||\mathbf{z}_a(\tau) - \mathbf{z}_b(\tau)||_\infty\, d\tau
\end{aligned} \tag{2.63}
$$

for any $t \in [0,T]$.

Define the functions,

$$
\begin{aligned}
u(t)\ &:=\ ||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty \\
g(t)\ &:=\ ||\mathbf{z}_a(0) - \mathbf{z}_b(0)||_\infty + N_a^2\, t\ |\lambda_a - \lambda_b| + N_a\, t\ |\,\gamma_a - \gamma_b\,| + N_a\, t\ |\,\rho_a - \rho_b\,| \\
f(t)\ &:=\ \{\max\{\,\lambda_{max,a}, \lambda_{max,b}\,\}\, N_a\, N_b + \max\{\gamma_{max,b}, \gamma_{max,a}\} \\
&\qquad + \epsilon + \max\{\,\rho_{max,a}, \rho_{max,b}\,\}\}\ :=\ K_{GB}
\end{aligned}
$$

We see that the assumptions of Theorem 3 are fulfilled and

$$
||\mathbf{z}_a(t) - \mathbf{z}_b(t)||_\infty\ \leq\ g(t)\ \mathbf{e}^{K_{GB}\, t} \tag{2.64}
$$

for any $t \in [0, T]$.

2.3.6 Long time behavior of solutions

In this section, we derive some monotonicity properties and the long time behaviour of our solutions.

Theorem 2.22 [2] We see the following behavior for $S(t)$, $E(t)$, $I(t)$ and $R(t)$

1. $S(t)$ decreases monotonically from $S(0) > 0$. There exists $S^* \geq 0$ such that $\lim_{t\to\infty} S(t) = S^*$. Further, it holds that $S^* > 0$.

2. $R(t)$ increases monotonically from $R(0) \geq 0$. There exists $R^* \geq 0$ such that $\lim_{t\to\infty} R(t) = R^*$.

3. I is Lebesgue-integrable on $[0, \infty)$ and $\lim_{t\to\infty} I(t) = 0$.

4. $\lim_{t\to\infty} E(t) = 0$.

Proof.

1. We know that , $S'(t) = -\lambda(t) S(t) I(t) - \rho(t) S(t)$. By Lemma 2.17, we know that $S'(t) < 0$ since $\lambda(t) > 0$ and $\rho(t) \geq 0$. By the boundedness of $S(t)$ therefore, $0 \leq S(t) \leq S_0$. Thus, $S : [0, \infty) \to [0, \infty)$ monotonically decreases and is bounded below by 0.
   $S'(t) = -\lambda(t) S(t) I(t) - \rho(t) S(t)$.
   So, $S'(t) = -(\lambda(t) I(t) + \rho(t)) S(t) \geq -(\lambda_{max} I(t) + \rho_{max}) S(t)$.

   Therefore, $\dfrac{S'(t)}{S(t)} \geq -(\lambda_{max} I(t) + \rho_{max})$.
   Integrating,
   $S(t) \geq S_0 \cdot \exp\left(-\left(\lambda_{max} \int_0^t I(t)\, dt + \rho_{max} t\right)\right) > 0$ for all $t \geq 0$. Hence it holds that $S^* > 0$. Note that the monotonicity of $S(t)$ ensures that $S(t) < S_0$ for all $t > 0$. Hence $S^* < S_0$.

2. $R^{'}(t) = \gamma(t) I(t) + \rho(t) S(t) \geq 0$ for all $t \geq 0$ and $0 \leq R(t) \leq N$ by Lemma 2.17. Therefore, $R : [0, \infty) \to [0, \infty)$ is monotonically increasing and is bounded above by N. Therefore, it follows that $R^* := \lim_{t\to\infty} R(t) \geq 0$ exists.

3. By assumption, $\lambda(t) \geq 0$ and from Lemma 2.17, $S, I : [0, \infty) \to [0, \infty) \geq 0$ and are bounded above by N.
   Therefore,
   $$\int_0^\infty S^{'}(\tau)\, d\tau = S^* - S_0 = \int_0^\infty -\lambda(\tau) S(\tau) I(\tau)\, d\tau + \int_0^\infty -\rho(\tau) S(\tau)\, d\tau$$
   $$\int_0^\infty \lambda(\tau) S(\tau) I(\tau)\, d\tau = S_0 - S^* - \int_0^\infty \rho(\tau) S(\tau)\, d\tau$$
   $$\geq \lambda_{min} S^* \int_0^t I(\tau)\, d\tau$$
   Since $\int_0^\infty \rho(\tau) S(\tau)\, d\tau \leq \rho_{max} S_0$, $S_0 - S^* - \int_0^\infty \rho(\tau) S(\tau)\, d\tau > 0$. Therefore, I is Lebesgue-integrable on $[0, \infty)$ and $\lim_{t\to\infty} I(t) = 0$.

4. From property 3 and from $I^{'}(t) = \epsilon E(t) - \gamma(t) I(t)$, we see that, $\lim_{t\to\infty} E(t) = 0$.

Based on the monotonicity properties and the boundedness of S, I and R, we can draw the following biological conclusions.[1]

1. Some number of susceptibles always escape the infection at the end of the epidemic.

2. The epidemic ends not because the suseptibles are exhausted.

3. The disease eventually dies out and the infectious population tends to zero after a long period of time otherwise.

4. Vaccination can reduce the number of susceptibles but not stop the infection as long as there are infectious individuals.

5. Since the infectiousfect population increases initially, there must be a period of time after which the change in I becomes negative. In other words, the epidemic first rises, then declines after reaching the maximum.

2.3.7 Basic Reproduction Number

The basic reproduction number, $\mathcal{R}_0$ [1] in the SEIR model, is defined by,

$$\mathcal{R}_0\left(t\right) := \lambda\left(t\right)\epsilon\left(t\right)\frac{1}{\gamma\left(t\right)\epsilon\left(t\right)}\cdot S\left(t_k\right) = \frac{\lambda\left(t\right)}{\gamma\left(t\right)}\cdot S\left(t\right).$$

From the non negativity and boundedness of $\lambda\left(t\right)$ and $\gamma\left(t\right)$, we see that as in the previous case,

$$0 \;\leq\; \frac{\lambda_{min}}{\gamma_{max}}\cdot S_0 \;\leq\; \mathcal{R}_0\left(t\right) \;\leq\; \frac{\lambda_{max}}{\gamma_{min}}\cdot S_0 \tag{2.65}$$

2.4 Discrete SEIR model with vaccination

We now examine the time discrete SEIR model with vaccination and time dependent transmission and recovery rate. We assume that a time interval $[0,T)$ has been divided into a strictly increasing sequence $\{t_j\}_{j=1}^{j=M}$ such that $t_1 = 0$ and $t_M = T$. We represent $f\left(t_j\right)$ by f_j for any time dependent function f.

In this model we state a fully explicit form ,

$$\left.\begin{aligned}
\frac{\Delta S_j}{\Delta t_j} = \frac{S_{j+1}-S_j}{t_{j+1}-tj} &= -\lambda_{j+1}S_jI_j - \rho_{j+1}S_j,\\
\frac{\Delta E_j}{\Delta t_j} = \frac{E_{j+1}-E_j}{t_{j+1}-tj} &= \lambda_{j+1}S_jI_j - \epsilon E_j,\\
\frac{\Delta I_j}{\Delta t_j} = \frac{I_{j+1}-Ij}{t_{j+1}-tj} &= \epsilon E_j - \gamma_{j+1}I_j,\\
\frac{\Delta R_j}{\Delta t_j} = \frac{R_{j+1}-Rj}{t_{j+1}-tj} &= \gamma_{j+1}I_j + \rho_{j+1}S_j
\end{aligned}\right\} \tag{2.66}$$

and a fully implicit form as in [1]

$$\left.\begin{aligned} \frac{\Delta S_j}{\Delta t_j} = \frac{S_{j+1} - S_j}{t_{j+1} - tj} &= -\lambda_{j+1} S_{j+1} I_{j+1} - \rho_{j+1} S_{j+1}, \\ \frac{\Delta E_j}{\Delta t_j} = \frac{E_{j+1} - E_j}{t_{j+1} - tj} &= \lambda_{j+1} S_{j+1} I_{j+1} - \epsilon E_{j+1}, \\ \frac{\Delta I_j}{\Delta t_j} = \frac{I_{j+1} - Ij}{t_{j+1} - tj} &= \epsilon E_{j+1} - \gamma_{j+1} I_{j+1}, \\ \frac{\Delta R_j}{\Delta t_j} = \frac{R_{j+1} - Rj}{t_{j+1} - tj} &= \gamma_{j+1} I_{j+1} + \rho_{j+1} S_{j+1} \end{aligned}\right\} \tag{2.67}$$

for all $j \in \{1, \ldots, M-1\}$. Both of these satisfy,

$$N = S_{j+1} + E_{j+1} + I_{j+1} + R_{j+1} = S_j + E_j + I_j + R_j. \tag{2.68}$$

We first show that the fully explicit form reduces to a set of linear equations and the fully implicit form has non-linear nature. From equation 2.66, the fully explicit form,

$$\left.\begin{aligned} &S_{j+1} = S_j - (\lambda_{j+1} S_j I_j + \rho_{j+1} S_j)\,\Delta t_j = S_j - (\lambda_{j+1} S_j I_j + \rho_{j+1} S_j)\,(t_{j+1} - t_j)\,, \\ &E_{j+1} = E_j + (\lambda_{j+1} S_j I_j + \epsilon E_j)\,\Delta t_j = E_j + (\lambda_{j+1} S_j I_j + \epsilon E_j)\,(t_{j+1} - t_j)\,, \\ &I_{j+1} = I_j + (\epsilon E_j - \gamma_{j+1} I_j)\,\Delta t_j = I_j + (\epsilon E_j - \gamma_{j+1} I_j)\,(t_{j+1} - t_j)\,, \\ &R_{j+1} = R_j + (\gamma_{j+1} I_j + \rho_{j+1} S_j)\,\Delta t_j = R_j + (\gamma_{j+1} I_j + \rho_{j+1} S_j)\,(t_{j+1} - t_j) \end{aligned}\right\} \tag{2.69}$$

is linear. From equation 2.67, the fully implicit form, appears to be in the nonlinear form below.

$$\left.\begin{aligned} S_{j+1} &= \frac{S_j}{(1+(\lambda_{j+1}I_{j+1}+\rho_{j+1})(t_{j+1}-t_j))}, \\ E_{j+1} &= \frac{E_j+\lambda_{j+1}S_{j+1}I_{j+1}(t_{j+1}-t_j)}{(1+\epsilon(t_{j+1}-t_j))}, \\ I_{j+1} &= \frac{I_j+\epsilon E_{j+1}(t_{j+1}-t_j)}{(1+\gamma_{j+1}(t_{j+1}-t_j))}, \\ R_{j+1} &= R_j+(\gamma_{j+1}I_{j+1}+\rho_{j+1}S_{j+1})(t_{j+1}-t_j) \end{aligned}\right\} \tag{2.70}$$

for $j \in \{1,\ldots,M-1\}$

2.4.1 Unique Solvability

The time discrete implicit form given in 2.70 is uniquely solvable for every $j \in \{1,\ldots,M-1\}$. If we replace E_{j+1}in the I_{j+1} equation with we find that,

$$\begin{aligned} I_{j+1} &= \frac{I_j+\epsilon E_{j+1}(t_{j+1}-t_j)}{(1+\gamma_{j+1}(t_{j+1}-t_j))} \\ &= \frac{I_j+\epsilon\left(\dfrac{E_j+\lambda_{j+1}S_{j+1}I_{j+1}(t_{j+1}-t_j)}{(1+\epsilon(t_{j+1}-t_j))}\right)(t_{j+1}-t_j)}{(1+\gamma_{j+1}(t_{j+1}-t_j))} \\ &= \frac{I_j(1+\epsilon(t_{j+1}-t_j))+\epsilon(E_j+\lambda_{j+1}S_{j+1}I_{j+1}(t_{j+1}-t_j))(t_{j+1}-t_j)}{(1+\epsilon(t_{j+1}-t_j))(1+\gamma_{j+1}(t_{j+1}-t_j))}. \end{aligned}$$

We now replace S_{j+1} using equation 2.70

$$I_{j+1} = \frac{I_j(1+\epsilon\Delta t_{j+1})+\epsilon\Delta t_{j+1}\left(E_j+\lambda_{j+1}\dfrac{S_j}{(1+(\lambda_{j+1}I_{j+1}+\rho_{j+1})\Delta t_{j+1})}I_{j+1}\Delta t_{j+1}\right)}{(1+\epsilon\Delta t_{j+1})(1+\gamma_{j+1}\Delta t_{j+1})}. \tag{2.71}$$

Rearranging, we have,

$$\begin{aligned} &I_{j+1}\left[1+(\lambda_{j+1}I_{j+1}+\rho_{j+1})\Delta t_{j+1}\right](1+\epsilon\Delta t_{j+1})(1+\gamma_{j+1}\Delta t_{j+1}) \\ &= I_{j+1}\left(I_j\lambda_{j+1}\Delta t_{j+1}+\epsilon\lambda_{j+1}I_j\Delta^2 t_{j+1}+\epsilon E_j\lambda_{j+1}\Delta^2 t_{j+1}+\epsilon S_j\lambda_{j+1}\Delta^2 t_{j+1}\right) \quad \cdot \\ &+ I_j\left(1+\epsilon\Delta t_{j+1}+\rho_{j+1}\Delta t_{j+1}+\epsilon\rho_{j+1}\Delta^2 t_{j+1}\right)+\epsilon\Delta t_{j+1}E_j+\epsilon\rho_{j+1}\Delta^2 t_{j+1}E_j \end{aligned}$$

That is,

$$I_{j+1}^2\left[\lambda_{j+1}\Delta t_{j+1}+\gamma_{j+1}\lambda_{j+1}\Delta^2 t_{j+1}+\lambda_{j+1}\epsilon\Delta^2 t_{j+1}+\lambda_{j+1}\gamma_{j+1}\epsilon\Delta^3 t_{j+1}+\rho_{j+1}\gamma_{j+1}\epsilon\Delta^3 t_{j+1}\right]$$
$$+\ I_{j+1}\left[\,1+\gamma_{j+1}\Delta t_{j+1}+\epsilon\Delta t_{j+1}+\rho_{j+1}\Delta t_{j+1}+\rho_{j+1}\gamma_{j+1}\Delta^2 t_{j+1}\right.$$
$$\left.+\,\epsilon\gamma_{j+1}\Delta^2 t_{j+1}+\rho_{j+1}\epsilon\Delta^2 t_{j+1}+\rho_{j+1}\gamma_{j+1}\epsilon\Delta^3 t_{j+1}\,\right]$$
$$-\,I_{j+1}\ \left[\left(I_j\lambda_{j+1}\Delta t_{j+1}+\epsilon\lambda_{j+1}I_j\Delta^2 t_{j+1}+\epsilon E_j\lambda_{j+1}\Delta^2 t_{j+1}+\epsilon S_j\lambda_{j+1}\Delta^2 t_{j+1}\right)\,\right]$$
$$=I_j\left(1+\epsilon\Delta t_{j+1}+\rho_{j+1}\Delta t_{j+1}+\epsilon\rho_{j+1}\Delta^2 t_{j+1}\right)+\epsilon\rho_{j+1}\Delta^2 t_{j+1}E_j+\epsilon\Delta t_{j+1}E_j. \quad (2.72)$$

Now we define,

$$\left.\begin{aligned}
A_{j+1} &:= \left[\lambda_{j+1}\Delta t_{j+1}+\gamma_{j+1}\lambda_{j+1}\Delta^2 t_{j+1}+\lambda_{j+1}\epsilon\Delta^2 t_{j+1}+\lambda_{j+1}\gamma_{j+1}\epsilon\Delta^3 t_{j+1}+\rho_{j+1}\gamma_{j+1}\epsilon\Delta^3 t_{j+1}\right],\\
D_{j+1} &:= \epsilon\gamma_{j+1}+\epsilon\rho_{j+1}+\rho_{j+1}\gamma_{j+1}-\epsilon\lambda_{j+1}I_j-\epsilon\lambda_{j+1}E_j-\epsilon\lambda_{j+1}S_j,\\
B_{j+1} &:= \frac{1}{2}\left[1+\Delta t_{j+1}\left(\gamma_{j+1}+\rho_{j+1}+\epsilon-I_j\lambda_{j+1}\right)+\Delta^2 t_{j+1}D_{j+1}+\Delta^3 t_{j+1}\rho_{j+1}\gamma_{j+1}\epsilon\right],\\
C_{j+1} &:= -\left(1+\epsilon\Delta t_{j+1}+\rho_{j+1}\Delta t_{j+1}+\epsilon\rho_{j+1}\Delta^2 t_{j+1}\right)I_j-\epsilon\Delta t_{j+1}+\epsilon\rho_{j+1}\Delta^2 t_{j+1}E_j
\end{aligned}\right\} \quad (2.73)$$

Then Equation 2.72 becomes,

$$A_{j+1}I_{j+1}^2+2B_{j+1}I_{j+1}-C_{j+1}=0. \quad (2.74)$$

So that,

$$I_{j+1}=\frac{-B_{j+1}}{A_{j+1}}\ +\ \sqrt{\frac{B_{j+1}^2}{A_{j+1}^2}+\frac{I_j\left(1+\epsilon\Delta^2 t_{j+1}\right)+\epsilon\Delta t_{j+1}E_j}{A_{j+1}}} \quad (2.75)$$

Since $I_{j+1}\geq 0$ for all $j\in\{1,\ldots,M-1\}$

2.4.2 Monotonicity and Long Time Behavior

Theorem 2.23 [2] For the time discrete implicit form given in equation 2.67 , the following hold

1. $0 \leq I_j \leq N$ for all $j \in \{1, \dots, M-1\}$.

2. $0 \leq E_j \leq N$ for all $j \in \{1, \dots, M-1\}$.

3. $0 \leq S_j \leq N$ for all $j \in \{1, \dots, M-1\}$ and $S_{j+1} \leq S_j$ for all $j \in \{1, \dots, M-1\}$.

4. $0 \leq R_j \leq N$ for all $j \in \{1, \dots, M-1\}$ and $R_{j+1} \geq R_j$ for all $j \in \{1, \dots, M-1\}$.

5. $\lim_{j \to \infty} I_j = 0$.

6. $\lim_{j \to \infty} E_j = 0$.

Proof

1. By assumption, $I_j > 0$ for all $j \in \{1, \dots, M-1\}$.Thus, by 2.67, $I_j \geq 0$ and from 2.68 , $I_j \leq N$ for all $j \in \{1, \dots, M-1\}$.

2. By property 1 ,2 and by equation 2.68 , $E_j \geq 0$ and , $E_j \leq N$ for all $j \in \{1, \dots, M-1\}$.

3. By assumption, $S_j > 0$ for all $j \in \{1, \dots, M-1\}$. Thus, $S_j \geq 0$ and by Property 1) and from 2.68 , $S_j \leq N$ for all $j \in \{1, \dots, M-1\}$. Further , from 2.70, $S_{j+1} = \dfrac{S_j}{1 + (\lambda_{j+1} I_{j+1} + \rho_{j+1})(t_{j+1} - t_j)} \leq S_j$, since $I_{j+1} \geq 0, \lambda_{j+1} > 0$ and $t_{j+1} - t_j > 0$.

4. By properties 1, 2 and 3 and from equation 2.68 , $0 \leq R_j \leq N$ for all $j \in \{1, \dots, M-1\}$. Further, from equation 2.70,
$R_{j+1} = R_j + (\gamma_{j+1} I_{j+1} + \rho_{j+1} S_{j+1})(t_{j+1} - t_j) \geq R_j$, since γ_{j+1}, ρ_{j+1}, $t_{j+1} - t_j > 0$ by assumption.

5. Since R_{j+1} is monotonically increasing and bounded above by N . there exists a positive R^* , such that $\lim_{j \to \infty} R_j = R^*$.

Further, $R_{j+1} - R_j = (\gamma_{j+1} I_{j+1} + \rho_{j+1} S_{j+1})(t_{j+1} - t_j)$ implies,

$$I_{j+1} = \frac{R_{j+1} - R_j}{\gamma_{j+1}(t_{j+1} - t_j)} - \frac{\rho_{j+1}}{\gamma_{j+1}} S_{j+1}.$$

Therefore, $\lim_{j\to\infty} I_j = 0$ due to the non negative property of I_j.

6. From property 5 and equation 2.70, we can conclude that $\lim_{j\to\infty} E_j = 0$.

2.4.3 Basic Reproduction Number

The basic reproduction number, $\mathcal{R}_0(t_k)$ [1] in the time-discrete SEIR model, is defined by,

$$\mathcal{R}_0(t_k) = \lambda(t_k)\,\epsilon(t_k)\,\frac{1}{\gamma(t_k)\,\epsilon(t_k)} \cdot S(t_k) = \frac{\lambda(t_k)}{\gamma(t_k)} \cdot S(t_k)\,. \tag{2.76}$$

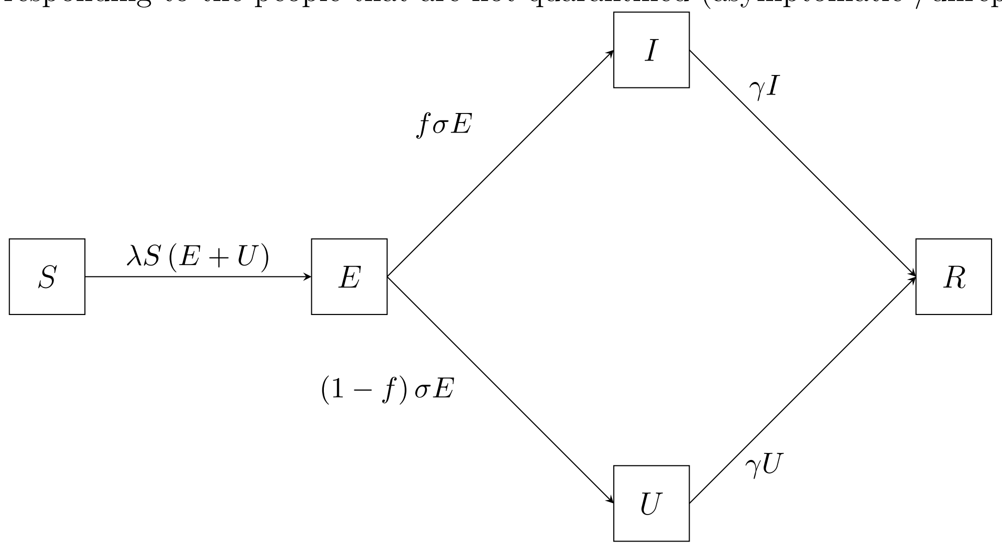
for arbitrary $k \in \{1, \cdots, M\}$. From the non negativity and boundedness of $\lambda(t_k)$ and $\gamma(t_k)$, we see that ,

$$0 \leq \frac{\lambda_{min}}{\gamma_{max}} \cdot S_0 \leq \mathcal{R}_0(t_k) \leq \frac{\lambda_{max}}{\gamma_{min}} \cdot S_0 \tag{2.77}$$

2.5 A model with quarantine and unreported infections

A model with quarantine is studied in [3]. There is no latency for the disease in this model. However there is incubation period for the disease, the disease symptoms are not immediately apparent in the individuals exposed to the virus. However, such individuals could also transmit the disease to others. The reported infectious compartment represent those individuals who are quarantined on confirmation of infection. Those individuals that are asymptomatic or who do not report infection are in the unreported compartment.

A diagram representing the model is given below.[3]. Note that the E compartment here stands for the "Exposed" to infection compartment containing infectious individuals and the I compartment stands for the "Infected and Reported" (Quarantined and so not infectious) compartment. The U compartment contains infectious individuals corresponding to the people that are not quarantined (asymptomatic /unreported).



Compartment S Susceptible, E Exposed , I Reported infected, U Unreported infected and R Recovered. We here replace the constant coefficients with time dependent coefficients. The system of differential equations for this model is [3].

$$\left.\begin{aligned} S^{'}(t) &= -\lambda(t)\ S(t)\left[E(t)+U(t)\right], \\ E^{'}(t) &= \lambda(t)\ S(t)\left[E(t)+U(t)\right]-\sigma(t)\,E(t), \\ I^{'}(t) &= \sigma(t)\ f\,E(t)-\gamma(t)\,I(t), \\ U^{'}(t) &= \sigma(t)\ (1-f)\ E(t)-\gamma(t)\,U(t), \\ R^{'}(t) &= \gamma(t)\left[E(t)+U(t)\right], \\ N(t) &= S(t)+E(t)+I(t)+U(t)+R(t) \end{aligned}\right\} \tag{2.78}$$

The initial conditions for the system of equations are,

$$N(t) := N_0 = N$$

$$S(0) := S_0 > 0,\ I(0) := I_0 \geq 0,\ E(0) := E_0 > 0,\ U(0) := U_0 \geq 0,\ R(0) := R_0 \geq 0$$

We can prove the non negativity property and boundedness of this model using the fact that the total population N does not change during all time and the total population in each of the compartment is less than or equal to N. However here we wish to state the conditions for an outbreak to occur assuming the non-negativity of meaningful solutions.

The condition for an outbreak given in [3] is that $E^{'}(t) + U^{'}(t) + I^{'}(t) > 0$. So assuming that there exists a σ_{max} and a σ_{min} such that $\sigma_{min} \leq \sigma(t) \leq \sigma_{max}$, an outbreak occurs if, $\lambda(t)\ S(t)\,[E(t) + U(t)] - \gamma(t)\,[I(t) + U(t)] > 0$. So an outbreak does not occur when $\lambda(t)\ S(t)\,[E(t) + U(t)] \leq \gamma(t)\,[I(t) + U(t)]$.

1. This could be the case when $E(t) = U(t) = I(t) = 0$, that is the trivial solution where there are no infections.
2. When there are infections, $E(t) + U(t) \leq (\,I(t) + U(t)\,)\,\dfrac{\gamma(t)}{\lambda(t)} \cdot \dfrac{1}{S(t)}$ ensures that there is no outbreak.

2.5.1 Basic Reproduction Number

The condition two above stated differently gives the basic reproduction number of this model. When $\dfrac{\gamma(t)}{\lambda(t)} \cdot \dfrac{1}{S(t)} = 1$, $E(t) \leq I(t)$, gives the condition for no outbreak. This implies that when more infected individuals are quarantined than that are exposed to the infection, an outbreak does not happen. Thus, the basic reproduction number can be estimated by,

$$0 \leq \frac{\lambda_{min}}{\gamma_{max}} \cdot S_0 \leq \mathcal{R}_0(t) := \frac{\lambda(t)}{\gamma(t)} \cdot S(t) \leq \frac{\lambda_{max}}{\gamma_{min}} \cdot S_0 \tag{2.79}$$

Then, in terms of the basic reproduction number $\mathcal{R}_0(t)$, the condition for no outbreak is,

$$\mathcal{R}_0(t)\, E(t) + \left(\mathcal{R}_0(t) - 1\right) U(t) \leq I(t)\,. \tag{2.80}$$

CHAPTER 3

ANALYSIS OF THE MODELS

3.1 Analysis of discrete and continuous SEIR models

In this section we formulate the difference between the solutions of the time discrete and the time continuous problem formulations. We state the assumptions below.[2]

1. Let the considered time interval be $[0, T]$ with $t_1 = 0$ and $t_M = T$.
2. Let the initial conditions of the time continuous and the time discrete models coincide.
3. Let the solution functions $S, E, I, R : [0, T] \to [0, \infty)$ be twice continuously differentiable.
4. Let the time-varying transmission rate $\lambda : [0, T] \to [0, \infty)$ and the time-varying recovery rate $\gamma : [0, T] \to [0, \infty)$ be once continuously differentiable.
5. Let the time-varying transmission and recovery rates be bounded, i.e., there are nonnegative constants, $\lambda_{min}, \lambda_{max}, \gamma_{min}, \gamma_{max}$, such that, $0 < \lambda_{min} \leq \lambda(t) \leq \lambda_{max} < 1$ and $0 < \gamma_{min} \leq \gamma(t) \leq \gamma_{max} < 1$ for all $t \in [0, T]$.
6. The latent period ϵ is a nonnegative constant measured relative to the time period Δ_p.
7. Choose $\Delta_p < \min\left\{\dfrac{1}{4(\lambda_{max} + \gamma_{max})}, 1\right\} \leq 1$ for all $p \in \mathbb{N}$ and set $\Delta := \max_{p\in\mathbb{N}} \Delta_p$.

Theorem 3.24 ([2]) If the assumptions given above are satisfied by the $SEIR$ model, and if the differences between the actual value and the approximating functions can be ignored, the difference between the solutions of the continuous problem formulation and the time discrete problem formulation satisfies ,

$$||\mathbf{z}\,(t_{p+1}) - \widetilde{\mathbf{z}_{p+1}}||_{\infty} \leq C_{\text{loc}}\,.\,\Delta_{p+1}^{2}. \tag{3.1}$$

Time-discrete solutions are denoted as S_p, E_p, I_p, R_p at time t_p and time-continuous solutions as $S\,(t_p)\,, E\,(t_p)\,, R\,(t_p)\,, S\,(t_p)$. We first estimate the local errors between time-continuous and time-discrete solutions. We then consider the propagation of the error in time and then find the cumulative error.

For examining the local differences, by the assumptions, $(t_p, S_p) = (t_p, S\,(t_p))$, $(t_p, E_p) = (t_p, E\,(t_p))$, $(t_p, I_p) = (t_p, I\,(t_p))$, $(t_p, R_p) = (t_p, R\,(t_p))$ hold for arbitrary $p \in \{1, ..., M-1\}$ on the time interval $[t_p, t_{p+1}]$.

We denote the time discrete solutions at $p+1$ by $\widetilde{S_{p+1}}$, $\widetilde{E_{p+1}}$, $\widetilde{I_{p+1}}$, $\widetilde{R_{p+1}}$ and $(t_{p+1} - t_p)$ by Δ_{p+1}.

Proof

1. First we evaluate the difference in S between the models. It holds that,

$$\widetilde{S_{p+1}} = \frac{S_p}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}} = S\,(t_p) - \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}S\,(t_p)}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}.$$

Therefore,

$$\begin{aligned}
&\left|S\,(t_{p+1}) - \widetilde{S_{p+1}}\right| \\
&= \left|S\,(t_{p+1}) - S\,(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}S\,(t_p)}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&= \left|\int_{t_p}^{t_{p+1}} S^{'}\,(\tau)\,d\tau + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}S\,(t_p)}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&= \left|\int_{t_p}^{t_{p+1}} S^{'}\,(\tau)\,d\tau + \Delta_{p+1}S^{'}\,(t_p) - \Delta_{p+1}S^{'}\,(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}S\,(t_p)}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&\leq \left|\int_{t_p}^{t_{p+1}} S^{'}\,(\tau)\,d\tau - \Delta_{p+1}S^{'}\,(t_p)\right| + \left|\Delta_{p+1}S^{'}\,(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}S\,(t_p)}{1 + \lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right|
\end{aligned}$$

by triangle inequality.

Let $I_{S,1} := \left|\int_{t_p}^{t_{p+1}} S^{'}(\tau)\, d\tau - \Delta_{p+1} S^{'}(t_p)\right|$ and $II_{S,1} := \left|\Delta_{p+1} S^{'}(t_p) + \dfrac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1} S(t_p)}{1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right|$.

We evaluate $I_{S,1} = \left|\int_{t_p}^{t_{p+1}} \left\{S^{'}(\tau) - S^{'}(t_p)\right\} d\tau\right| = \left|\int_{t_p}^{t_{p+1}} (\tau - t_p)\dfrac{S^{'}(\tau) - S^{'}(t_p)}{\tau - t_p}\, d\tau\right|$.

By the mean value theorem of calculus, there exists $\xi_{S,1} \in (t_p, t_{p+1})$ such that, $\left|S^{''}(\xi_{S,1})\right| = \left|\dfrac{S^{'}(\tau) - S^{'}(t_p)}{\tau - t_p}\right| \leq \left|\left|S^{''}(t)\right|\right|_\infty$. This yields,

$$I_{S,1} = \left|\left|S^{''}(t)\right|\right|_\infty \int_{t_p}^{t_{p+1}} (\tau - t_p)\, d\tau = \frac{1}{2}\Delta_{p+1}^2 \left|\left|S^{''}(t)\right|\right|_\infty. \tag{3.2}$$

Now we evaluate

$$\begin{aligned}
II_{S,1} &= \left|\Delta_{p+1} S^{'}(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1} S(t_p)}{1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&= \left|-\lambda_p \Delta_{p+1} I(t_p) S(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1} S(t_p)}{1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&= \left|\Delta_{p+1} S(t_p)\right| . \left|-\lambda_p I(t_p) + \frac{\lambda_{p+1}\widetilde{I_{p+1}}}{1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}}\right| \\
&= \Delta_{p+1}. \left|\frac{-\lambda_p I(t_p)\left(1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right) + \lambda_{p+1}\widetilde{I_{p+1}}}{1+\lambda_{p+1}\widetilde{I_{p+1}}}\right| \\
&= \Delta_{p+1}. \left|-\lambda_p I(t_p)\left(1+\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right) + \lambda_{p+1}\widetilde{I_{p+1}}\right| \\
&= \Delta_{p+1}. \left|\lambda_{p+1}\widetilde{I_{p+1}} - \lambda_p I(t_p) . \left\{1 + \Delta_{p+1}\lambda_{p+1}\widetilde{I_{p+1}}\right\}\right| \\
&= \Delta_{p+1}. \left|\lambda_{p+1}\widetilde{I_{p+1}} - \lambda_p I(t_p)\right| + \Delta_{p+1}. \left|\lambda_p I(t_p)\lambda_{p+1}\widetilde{I_{p+1}}\right| \\
&= \Delta_{p+1}. \left|\lambda_{p+1}\widetilde{I_{p+1}} - \lambda_p I(t_p)\right| + \Delta_{p+1}^2 . \lambda_{max}^2 N^2.
\end{aligned} \tag{3.3}$$

We now define, $III_{S,1} := \Delta_{p+1}. \left|\lambda_{p+1}\widetilde{I_{p+1}} - \lambda_p I(t_p)\right|$ and estimate it.

We plugin $\widetilde{I_{p+1}} = \dfrac{I_p + \epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}$ and obtain,

$$
\begin{aligned}
III_{S,1} &= \Delta_{p+1}.\left|\lambda_{p+1}\frac{I_p+\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}-\lambda_p I\left(t_p\right)\right| \\
&= \Delta_{p+1}\left|\lambda_p I\left(t_p\right)-\lambda_{p+1}\frac{I_p+\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right| \\
&= \Delta_{p+1}\left|I\left(t_p\right)\right|.\left|\lambda_p-\lambda_{p+1}.\frac{I\left(t_p\right)+\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})\, I\left(t_p\right)}\right| \\
&= \Delta_{p+1}\left|I\left(t_p\right)\right|.\left|\frac{\lambda_p.\left(1+\gamma_{p+1}\right)\Delta_{p+1}I\left(t_p\right)-\lambda_{p+1}.I\left(t_p\right)-\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})\, I\left(t_p\right)}\right| \\
&= \Delta_{p+1}\left|I\left(t_p\right)\right|.\left|\frac{\lambda_p\Delta_{p+1}I\left(t_p\right)+\lambda_p\gamma_{p+1}\Delta_{p+1}I\left(t_p\right)-\lambda_{p+1}.I\left(t_p\right)-\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})\, I\left(t_p\right)}\right| \\
&\le \Delta_{p+1}N.\left|\frac{\lambda_p\Delta_{p+1}I\left(t_p\right)+\lambda_p\gamma_{p+1}\Delta_{p+1}I\left(t_p\right)-\lambda_{p+1}.I\left(t_p\right)-\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})\, I\left(t_p\right)}\right| \\
&\le \Delta_{p+1}N.\left|\frac{\Delta_{p+1}\left\{\lambda_p.I\left(t_p\right)+\lambda_p.\gamma_{max}.I\left(t_p\right)-\lambda_{p+1}.I\left(t_p\right)\right\}-\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})\, I\left(t_p\right)}\right| \\
&\le \frac{1}{1+\gamma_{max}}.\left|\Delta_{p+1}\left\{\lambda_p.I\left(t_p\right)-\lambda_{p+1}.I\left(t_p\right)\right\}-\Delta_{p+1}N\left\{\epsilon-\lambda_{max}\gamma_{max}\right\}\right|.
\end{aligned}
$$

by the boundedness property of $\gamma(t), \lambda(t)$ and E, I. We now apply the triangle inequality to obtain,

$$
\begin{aligned}
III_{S,1} &\le \frac{\Delta_{p+1}^2}{1+\gamma_{max}}\left\{\left|I\left(t_p\right)\right|\left|\frac{\left\{\lambda_p-\lambda_{p+1}\right\}}{\Delta_{p+1}}\right|+\left|\frac{N\left\{\epsilon-\lambda_{max}\gamma_{max}\right\}}{\Delta_{p+1}}\right|\right\} \\
&\le \frac{\Delta_{p+1}^2}{1+\gamma_{max}}.N\left\{\left|\frac{\left\{\lambda_p-\lambda_{p+1}\right\}}{\Delta_{p+1}}\right|+\left|\frac{\left\{\epsilon-\lambda_{max}\gamma_{max}\right\}}{\Delta_{p+1}}\right|\right\}.
\end{aligned}
$$

By the mean value theorem of calculus, there exists $\xi_{\lambda,1}\in[t_p, t_{p+1}]$ such that, $\left|\lambda'\left(\xi_{\lambda,1}\right)\right| = \left|\frac{\lambda_{p+1}-\lambda_p}{t_{p+1}-t_p}\right| \le \left|\left|\lambda'(t)\right|\right|_\infty$ holds. Hence by the boundedness property and triangle inequality again,

$$III_{S,1} \leq \frac{\Delta_{p+1}^2 N}{1+\gamma_{max}} \left\{ \left|\left| \lambda^{'}(t) \right|\right|_\infty + \{\epsilon + \lambda_{max}\gamma_{max}\} \right\}. \tag{3.4}$$

Therefore,

$$\begin{aligned}
&\left| S(t_{p+1}) - \widetilde{S_{p+1}} \right| \\
&\leq \frac{1}{2}\Delta_{p+1}^2 \left|\left| S^{''}(t) \right|\right|_\infty + \Delta_{p+1}^2.\lambda_{max}^2 N^2 + \frac{\Delta_{p+1}^2 N}{1+\gamma_{max}} \left\{ \left|\left| \lambda^{'}(t) \right|\right|_\infty + \{\epsilon + \lambda_{max}\gamma_{max}\} \right\} \\
&= \Delta_{p+1}^2 \left\{ \frac{1}{2} \left|\left| S^{''}(t) \right|\right|_\infty + \lambda_{max}^2 N^2 + \frac{N}{1+\gamma_{max}} \left\{ \left|\left| \lambda^{'}(t) \right|\right|_\infty + \{\epsilon + \lambda_{max}\gamma_{max}\} \right\} \right\}.
\end{aligned} \tag{3.5}$$

We define $C_{\mathrm{loc,S}} := \frac{1}{2} \left|\left| S^{''}(t) \right|\right|_\infty + \lambda_{max}^2 N^2 + \dfrac{N}{1+\gamma_{max}} \left\{ \left|\left| \lambda^{'}(t) \right|\right|_\infty + \{\epsilon + \lambda_{max}\gamma_{max}\} \right\}$. Thus, $\left| S(t_{p+1}) - \widetilde{S_{p+1}} \right| \leq \Delta_{p+1}^2.C_{\mathrm{loc,S}}$ holds true.

2. We now estimate the difference in E between the two models.

$$\widetilde{E_{p+1}} = \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} - \epsilon\widetilde{E_{p+1}}\widetilde{E_{p+1}} = \frac{E_p + \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{(1+\epsilon\Delta_{p+1})} = \frac{E(t_p) + \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{(1+\epsilon\Delta_{p+1})}$$

$$= E(t_p) - \frac{E(t_p)\,\epsilon\Delta_{p+1}}{(1+\epsilon\Delta_{p+1})} + \frac{\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{(1+\epsilon\Delta_{p+1})}$$

By the application of triangle inequality and mean value theorem as in the case of the previous case, we have,

$$
\begin{aligned}
&\left|E\left(t_{p+1}\right)-\widetilde{E_{p+1}}\right| \\
&=\left|E\left(t_{p+1}\right)-E\left(t_p\right)+\frac{E\left(t_p\right)\epsilon\Delta_{p+1}-\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{\left(1+\epsilon\Delta_{p+1}\right)}\right| \\
&=\left|\int_{t_p}^{t_{p+1}} E^{'}\left(\tau\right)d\tau-\frac{E\left(t_p\right)\epsilon\Delta_{p+1}-\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{\left(1+\epsilon\Delta_{p+1}\right)}\right| \\
&\leq\left|\int_{t_p}^{t_{p+1}} E^{'}\left(\tau\right)d\tau-E^{'}\left(t_p\right)\Delta_{p+1}\right| \\
&\quad+\left|E^{'}\left(t_p\right)\Delta_{p+1}+\frac{E\left(t_p\right)\epsilon\Delta_{p+1}-\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{\left(1+\epsilon\Delta_{p+1}\right)}\right| \\
&\leq\frac{\Delta_{p+1}^2}{2}\left|\left|E^{''}\left(t\right)\right|\right|_{\infty}+\left|E^{'}\left(t_p\right)\Delta_{p+1}+\frac{E\left(t_p\right)\epsilon\Delta_{p+1}-\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{\left(1+\epsilon\Delta_{p+1}\right)}\right|.
\end{aligned} \tag{3.6}
$$

We define, $I_{E,1}:=\left|E^{'}\left(t_p\right)\Delta_{p+1}+\frac{E\left(t_p\right)\epsilon\Delta_{p+1}-\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{\left(1+\epsilon\Delta_{p+1}\right)}\right|.$

We plugin $E^{'}(t_p) = \lambda_p S(t_p) I(t_p) - \epsilon E(t_p)$. Thus, we further obtain,

$$
\begin{aligned}
& I_{E,1} \\
& = \left| \lambda_p S(t_p) I(t_p) \Delta_{p+1} - \epsilon E(t_p) \Delta_{p+1} + \frac{E(t_p)\epsilon\Delta_{p+1} - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}}{(1+\epsilon\Delta_{p+1})} \right| \\
& = \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \epsilon E(t_p) + \frac{E(t_p)\epsilon - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}}{(1+\epsilon\Delta_{p+1})} \right| \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \frac{\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}}{(1+\epsilon\Delta_{p+1})} \right| + \Delta_{p+1} \left| \frac{E(t_p)\epsilon}{(1+\epsilon\Delta_{p+1})} - \epsilon E(t_p) \right| \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) . (1+\epsilon\Delta_{p+1}) - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} \right| \\
& \quad + \Delta_{p+1} \left| \frac{E(t_p)\epsilon}{(1+\epsilon\Delta_{p+1})} - \epsilon E(t_p) \right| \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} \right| + \left| \lambda_p S(t_p) I(t_p) \epsilon . \Delta_{p+1}^2 \right| + \\
& \quad + \Delta_{p+1} \left| \epsilon^2 E(t_p) \Delta_{p+1} \right| \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} \right| + \lambda_{max}\epsilon N^2 \Delta_{p+1}^2 + N\Delta_{p+1}^2\epsilon^2. \qquad (3.7)
\end{aligned}
$$

We define, $I_{E,2} := \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}} \right|$ and plugin $\widetilde{I_{p+1}} = \dfrac{I(t_p) + \epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{1+\gamma_{p+1}\Delta_{p+1}}$

$$
\begin{aligned}
& I_{E,2} \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \lambda_{p+1}\widetilde{S_{p+1}} I(t_p) \right| + \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) \gamma_{p+1}\Delta_{p+1} \right| \\
& \quad + \Delta_{p+1} \left| \lambda_{p+1}\widetilde{S_{p+1}}\epsilon\widetilde{E_{p+1}}\Delta_{p+1} \right| \\
& \leq \Delta_{p+1} \left| \lambda_p S(t_p) I(t_p) - \lambda_{p+1}\widetilde{S_{p+1}} I(t_p) \right| + \lambda_{max}\gamma_{max} N^2 \Delta_{p+1}^2 + \lambda_{max}\epsilon N^2 \Delta_{p+1}^2.
\end{aligned}
\qquad (3.8)
$$

We now substitute for $\widetilde{S_{p+1}}$ and obtain,

$$
\begin{aligned}
&I_{E,2} \\
&\leq \Delta_{p+1}\left|\lambda_p S\left(t_p\right) I\left(t_p\right)-\lambda_{p+1}\widetilde{S_{p+1}} I\left(t_p\right)\right|+\lambda_{max}\gamma_{max}N^2\Delta_{p+1}^2+\lambda_{max}\epsilon N^2\Delta_{p+1}^2 \\
&\leq \Delta_{p+1}N^2\left|\lambda_p-\lambda_{p+1}\right|+\left|\lambda_p\lambda_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right|+\lambda_{max}\gamma_{max}N^2\Delta_{p+1}^2+\lambda_{max}\epsilon N^2\Delta_{p+1}^2 \\
&\leq \Delta_{p+1}N^2\left|\left|\lambda\left(t\right)^{'}\right|\right|_{\infty}+\lambda_{max}^2 N\Delta_{p+1}^2 \\
&\qquad +\lambda_{max}\gamma_{max}N^2\Delta_{p+1}^2+\lambda_{max}\epsilon N^2\Delta_{p+1}^2 \\
&=\Delta_{p+1}^2N^2\left|\left|\lambda^{'}\left(t\right)\right|\right|_{\infty}+N^2\Delta_{p+1}^2\left(\lambda_{max}\gamma_{max}+\lambda_{max}\epsilon\right)+N\lambda_{max}^2\Delta_{p+1}^2. \qquad (3.9)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|E\left(t_{p+1}\right)-\widetilde{E_{p+1}}\right| \\
&\leq \frac{\Delta_{p+1}^2}{2}\left|\left|E^{''}\left(t\right)\right|\right|_{\infty}+\Delta_{p+1}^2N^2\left|\left|\lambda\left(t\right)^{'}\right|\right|_{\infty}+N\Delta_{p+1}^2\left(\lambda_{max}^2+\epsilon^2\right) \\
&\qquad +N^2\Delta_{p+1}^2\left(\lambda_{max}\gamma_{max}+2\lambda_{max}\epsilon+\lambda_{max}\right). \qquad (3.10)
\end{aligned}
$$

Define

$$
C_{\mathrm{loc,E}}:=\frac{1}{2}\left|\left|E^{''}\left(t\right)\right|\right|_{\infty}+N^2\left|\left|\lambda\left(t\right)^{'}\right|\right|_{\infty}+N\left(\lambda_{max}^2+\epsilon^2\right)+N^2\left(\lambda_{max}\gamma_{max}+2\,\lambda_{max}\,\epsilon+\lambda_{max}\right)
$$

Thus,

$$
\left|E\left(t_{p+1}\right)-\widetilde{E_{p+1}}\right|\leq\Delta_{p+1}^2C_{\mathrm{loc,E}}. \qquad (3.11)
$$

3. Next we estimate the difference in I between the two models. We note that,

$$\widetilde{I_{p+1}} = \frac{I_p + \epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} = \frac{I(t_p)}{(1+\gamma_{p+1}\Delta_{p+1})} + \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}$$
$$= I(t_p) - \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} + \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}.$$

Hence,

$$\begin{aligned}
&\left|I(t_{p+1}) - \widetilde{I_{p+1}}\right| \\
&= \left|I(t_{p+1}) - I(t_p) + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right| \\
&= \left|\int_{t_p}^{t_{p+1}} I'(\tau)\,d\tau + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right| \\
&\le \left|\int_{t_p}^{t_{p+1}} I'(\tau)\,d\tau - I'(t_p)\Delta_{p+1}\right| + \left|I'(t_p)\Delta_{p+1} + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right| \\
&\le \frac{\Delta_{p+1}^2}{2}.\left\|I''(t)\right\|_\infty + \left|I'(t_p)\Delta_{p+1} + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right|.
\end{aligned} \tag{3.12}$$

By following the same argument as in the previous cases.
We now define, $I_{I,1} := \left|I'(t_p)\Delta_{p+1} + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})}\right|$ and

plugin, $I^{'}(t_p) = \epsilon E(t_p) - \gamma_p I(t_p)$. Hence,

$$
\begin{aligned}
& I_{I,1} \\
&= \left| \epsilon E(t_p)\Delta_{p+1} - \gamma_p I(t_p)\Delta_{p+1} + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} \right| \\
&= \left| \epsilon E(t_p)\Delta_{p+1} - \frac{\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} + \frac{I(t_p)\gamma_{p+1}\Delta_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \gamma_p I(t_p)\Delta_{p+1} \right| \\
&= \Delta_{p+1} \left| \epsilon E(t_p) - \frac{\epsilon\widetilde{E_{p+1}}}{(1+\gamma_{p+1}\Delta_{p+1})} + \frac{I(t_p)\gamma_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \gamma_p I(t_p) \right| \\
&= \Delta_{p+1} \left| \epsilon \left\{ E(t_p) - \frac{\widetilde{E_{p+1}}}{(1+\gamma_{p+1}\Delta_{p+1})} \right\} + I(t_p) \left\{ \frac{\gamma_{p+1}}{(1+\gamma_{p+1}\Delta_{p+1})} - \gamma_p \right\} \right| \\
&= \Delta_{p+1} \left| \epsilon \left\{ \frac{E(t_p).(1+\gamma_{p+1}\Delta_{p+1}) - \widetilde{E_{p+1}}}{(1+\gamma_{p+1}\Delta_{p+1})} \right\} + I(t_p) \left\{ \frac{\gamma_{p+1} - \gamma_p.(1+\gamma_{p+1}\Delta_{p+1})}{(1+\gamma_{p+1}\Delta_{p+1})} \right\} \right| \\
&\le \Delta_{p+1} \left\{ \left| \epsilon \left(E(t_p) - \widetilde{E_{p+1}} \right) \right| + \left| \epsilon E(t_p)\gamma_{p+1}\Delta_{p+1} \right| + \left| I(t_p)(\gamma_{p+1} - \gamma_p) \right| + \left| I(t_p)\gamma_{p+1}\Delta_{p+1} \right| \right\} \\
&\le \Delta_{p+1} \left| \epsilon \left(E(t_p) - \widetilde{E_{p+1}} \right) \right| + \epsilon\gamma_{max} N \Delta_{p+1}^2 + \Delta_{p+1} \left| I(t_p)(\gamma_{p+1} - \gamma_p) \right| + N\gamma_{max}\Delta_{p+1}^2 \\
&\le \Delta_{p+1} \left| \epsilon \left(E(t_p) - \widetilde{E_{p+1}} \right) \right| + \epsilon\gamma_{max} N \Delta_{p+1}^2 + \Delta_{p+1}^2 N \left|\left| \gamma^{'}(t) \right|\right|_\infty + N\gamma_{max}\Delta_{p+1}^2.
\end{aligned}
\tag{3.13}
$$

We now plugin, $\widetilde{E_{p+1}} = \dfrac{E(t_p) + \lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_p+1}\Delta_{p+1}}{1+\epsilon\Delta_{p+1}}$ into Equation 3.13, and obtain

$$
\begin{aligned}
& I_{I,1} \\
& \leq \Delta_{p+1}\left|\Delta_{p+1}\epsilon E\left(t_p\right)\right| + \Delta_{p+1}\left|\lambda_{p+1}\widetilde{S_{p+1}}\widetilde{I_{p+1}}\Delta_{p+1}\right| \\
& + \epsilon\gamma_{max}N\Delta_{p+1}^2 + \Delta_{p+1}^2 N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty} + N\gamma_{max}\Delta_{p+1}^2 \\
& = N\epsilon\Delta_{p+1}^2 + \lambda_{max}N^2\Delta_{p+1}^2 + \epsilon\gamma_{max}N\Delta_{p+1}^2 + \Delta_{p+1}^2 N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty} + N\gamma_{max}\Delta_{p+1}^2 \\
& = N\Delta_{p+1}^2\left(\epsilon + \gamma_{max} + \epsilon\gamma_{max}\right) + \lambda_{max}N^2\Delta_{p+1}^2 + \Delta_{p+1}^2 N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty}.
\end{aligned}
\tag{3.14}
$$

Thus,

$$
\begin{aligned}
& \left|I\left(t_{p+1}\right) - \widetilde{I_{p+1}}\right| \\
& \leq \frac{\Delta_{p+1}^2}{2}\left|\left|I^{''}\left(t\right)\right|\right|_{\infty} + \Delta_{p+1}^2 N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty} + N\Delta_{p+1}^2\left(\epsilon + \gamma_{max} + \epsilon\gamma_{max}\right) + \lambda_{max}N^2\Delta_{p+1}^2
\end{aligned}
\tag{3.15}
$$

Define $C_{\text{loc,I}} := \frac{1}{2}\left|\left|I^{''}\left(t\right)\right|\right|_{\infty} + N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty} + N\left(\epsilon + \gamma_{max} + \epsilon\gamma_{max}\right) + \lambda_{max}N^2$. Thus,

$$
\left|I\left(t_{p+1}\right) - \widetilde{I_{p+1}}\right| \leq \Delta_{p+1}^2 C_{\text{loc,I}}. \tag{3.16}
$$

4. Finally, we estimate the difference in R between the two models. We note that,

$$
\widetilde{R_{p+1}} = R_p + \gamma_{p+1}\widetilde{I_{j+1}}\left(t_{p+1} - t_p\right) = R_p + \gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}.
$$

Hence,

$$
\begin{aligned}
&\left|R\left(t_{p+1}\right)-\widetilde{R_{p+1}}\right| \\
&=\left|R\left(t_{p+1}\right)-R\left(t_{p}\right)-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right| \\
&=\left|\int_{t_p}^{t_{p+1}} R^{'}\left(\tau\right)d\tau-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right| \\
&\leq\left|\int_{t_p}^{t_{p+1}} R^{'}\left(\tau\right)d\tau-R^{'}\left(t_p\right)\Delta_{p+1}\right|+\left|R^{'}\left(t_p\right)\Delta_{p+1}-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right| \\
&\leq\frac{\Delta_{p+1}^2}{2}\left|\left|R^{''}\left(t\right)\right|\right|_{\infty}+\left|R^{'}\left(t_p\right)\Delta_{p+1}-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right|. \qquad (3.17)
\end{aligned}
$$

Define $I_{R,1} := \left|R^{'}\left(t_p\right)\Delta_{p+1}-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right|$ and plugin $R^{'}\left(t_p\right)=\gamma_p I\left(t_p\right)$ and $\widetilde{I_{p+1}}=\dfrac{I\left(t_p\right)+\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{\left(1+\gamma_{p+1}\Delta_{p+1}\right)}$ to obtain,

$$
\begin{aligned}
&I_{R,1} \\
&=\left|\gamma_p I\left(t_p\right))\Delta_{p+1}-\gamma_{p+1}\widetilde{I_{p+1}}\Delta_{p+1}\right| \\
&=\Delta_{p+1}\left|\gamma_p I\left(t_p\right)-\gamma_{p+1}\widetilde{I_{p+1}}\right| \\
&=\Delta_{p+1}\left|\gamma_p I\left(t_p\right)-\gamma_{p+1}\frac{I\left(t_p\right)+\epsilon\widetilde{E_{p+1}}\Delta_{p+1}}{\left(1+\gamma_{p+1}\Delta_{p+1}\right)}\right| \\
&\leq\Delta_{p+1}\left|\gamma_p I\left(t_p\right)-\gamma_{p+1}I\left(t_p\right)\right|+\Delta_{p+1}^2\left|\gamma_p\gamma_{p+1}I\left(t_p\right)\right|+\Delta_{p+1}^2\left|\epsilon\,\gamma_{p+1}\widetilde{E_{p+1}}\right| \\
&\leq\Delta_{p+1}^2 N\left|\left|\gamma^{'}\left(t\right)\right|\right|_{\infty}+\Delta_{p+1}^2 N\gamma_{max}^2+\Delta_{p+1}^2 N\epsilon\,\gamma_{max}.
\end{aligned}
\qquad (3.18)
$$

Thus,

$$\begin{aligned}
&\left|R\left(t_{p+1}\right)-\widetilde{R_{p+1}}\right| \\
&\leq \frac{\Delta_{p+1}^2}{2}\left|\left|R''\left(t\right)\right|\right|_\infty + \Delta_{p+1}^2 N\left|\left|\gamma'\left(t\right)\right|\right|_\infty + \Delta_{p+1}^2 N\gamma_{max}^2 + \Delta_{p+1}^2 N\epsilon\,\gamma_{max} \\
&= \frac{\Delta_{p+1}^2}{2}\left|\left|R''\left(t\right)\right|\right|_\infty + \Delta_{p+1}^2 N\left|\left|\gamma'\left(t\right)\right|\right|_\infty + \Delta_{p+1}^2 N\left(\gamma_{max}^2 + \epsilon\,\gamma_{max}\right). \qquad (3.19)
\end{aligned}$$

Define $C_{\text{loc,R}} := \frac{1}{2}\left|\left|R''\left(t\right)\right|\right|_\infty + N\left|\left|\gamma'\left(t\right)\right|\right|_\infty + N\left(\gamma_{max}^2 + \epsilon\,\gamma_{max}\right)$.

Thus,

$$\left|R\left(t_{p+1}\right)-\widetilde{R_{p+1}}\right| \leq \Delta_{p+1}^2 C_{\text{loc,R}} \qquad (3.20)$$

Now define $C_{\text{loc}} := \max\left\{C_{\text{loc,S}},\ C_{\text{loc,E}},\ C_{\text{loc,I}},\ C_{\text{loc,R}}\right\}$.

Then it holds that,

$$\left|\left|\mathbf{z}\left(t_{p+1}\right)-\widetilde{\mathbf{z}_{p+1}}\right|\right|_\infty \leq C_{\text{loc}}\,.\,\Delta_{p+1}^2 \qquad (3.21)$$

for local errors in time intervals, $[t_p, t_{p+1}]$. If the same assumptions are valid, the difference between the time continuous and time discrete solutions can be determined in the same way for the SEIR model with vaccination.

3.1.1 Error due to approximation

We now proceed to modify Theorem 3.26 **??** to account for the errors from the discrete solution, S_p, E_p, I_p, R_p not coinciding exactly with the time-continuous solution. We therefore examine how the procedural errors such as $S\left(t_p\right)-S_p$, $E\left(t_p\right)-E_p$, $I\left(t_p\right)-I_p$, $R\left(t_p\right)-R_p$ propagate to the next step in time.

From Theorem 2.19 , Equation 61, we see that,

$\mathbf{z}_{p+1}-\mathbf{z}\left(t_{p+1}\right)=\mathbf{z}_p-\mathbf{z}\left(t_p\right)+\Delta_{p+1}.\left\{\mathbf{G}\left(t_{p+1},\mathbf{z}_{p+1}\right)-\mathbf{G}\left(t_{p+1},\mathbf{z}\left(t_p\right)\right)\right\}$ holds.

This implies,

$||\mathbf{z}_{p+1} - \mathbf{z}(t_{p+1})||_\infty \leq ||\mathbf{z}_p - \mathbf{z}(t_p)||_\infty + \Delta_{p+1}. ||\mathbf{G}(t_{p+1}, \mathbf{z}_{p+1}) - \mathbf{G}(t_{p+1}, \mathbf{z}(t_p))||_\infty$ holds. We now proceed to estimate, $||\mathbf{G}(t_{p+1}, \mathbf{z}_{p+1}) - \mathbf{G}(t_{p+1}, \mathbf{z}(t_p))||_\infty$.

$$||\mathbf{G}(t_{p+1}, \mathbf{z}_{p+1}) - \mathbf{G}(t_{p+1}, \mathbf{z}(t_p))||_\infty$$

$$= \left\| \begin{pmatrix} \lambda_{p+1} \{ S(t_{p+1}) I(t_{p+1}) - S_{p+1} I_{p+1} \} \\ \lambda_{p+1} \{ S(t_{p+1}) I(t_{p+1}) - S_{p+1} I_{p+1} \} + \epsilon \{ E_{p+1} - E(t_{p+1}) \} \\ \epsilon \{ E_{p+1} - E(t_{p+1}) \} + \gamma_{p+1} \{ I_{p+1} - I(t_{p+1}) \} \\ \gamma_{p+1} \{ I_{p+1} - I(t_{p+1}) \} \end{pmatrix} \right\|_\infty$$

$$\leq \left\| \begin{pmatrix} \lambda_{p+1} \left\{ ||I_{p+1} - I(t_{p+1})||_\infty + ||S_{p+1} - S(t_{p+1})||_\infty \right\} \\ \lambda_{p+1} \left\{ ||I_{p+1} - I(t_{p+1})||_\infty + ||S_{p+1} - S(t_{p+1})||_\infty \right\} + \epsilon \left\{ ||E_{p+1} - E(t_{p+1})||_\infty \right\} \\ \epsilon \left\{ ||E_{p+1} - E(t_{p+1})||_\infty \right\} + \gamma_{p+1} \left\| \{ I_{p+1} - I(t_{p+1}) ||_\infty \right\} \\ \gamma_{p+1} \left\| \{ I_{p+1} - I(t_{p+1}) ||_\infty \right\} \end{pmatrix} \right\|_\infty$$

$$\leq 2. (\lambda_{max} + \gamma_{max} + \epsilon) ||\mathbf{z}_{p+1} - \mathbf{z}(t_{p+1})||_\infty . \tag{3.22}$$

This implies that,

$$||\mathbf{z}_{p+1} - \mathbf{z}(t_{p+1})||_\infty \ \leq \ \mathbf{z}_p - \mathbf{z}(t_p) + \Delta_{p+1}.2. (\lambda_{max} + \gamma_{max} + \epsilon) ||\mathbf{z}_{p+1} - \mathbf{z}(t_{p+1})||_\infty.$$

Hence, we conclude that,

$$||\mathbf{z}_{p+1} - \mathbf{z}(t_{p+1})||_\infty \ \leq \ \frac{1}{1 - 2. (\lambda_{max} + \gamma_{max} + \epsilon) \Delta} ||\mathbf{z}_p - \mathbf{z}(t_p)||_\infty.$$

with $\Delta := \max_{p \in \{1, \cdots, M-1\}} \Delta_{p+1} \ < \ \dfrac{1}{4. (\lambda_{max} + \gamma_{max} + \epsilon)}$, by assumption.

Next we want to modify the upper error bound between the time discrete and time continuous solutions. We state the modified theorem here,

Theorem 3.25 [2] If the assumptions given above are satisfied by the $SEIR$ model, and including the differences between the actual value and the approximating functions, the difference between the solutions of the continuous problem formulation and the time discrete problem formulation satisfies ,

$$||\mathbf{z}\left(t_{p+1}\right)-\widetilde{\mathbf{z}_{p+1}}||_{\infty}\leq C_{\text{loc}}\,.\,\Delta.\left\{\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{p}-1\right\}. \tag{3.23}$$

We use Mathematical Induction to prove the theorem. At first we find that,

$$||\mathbf{z}_2-\mathbf{z}\left(t_2\right)||_{\infty}\leq ||\mathbf{z}_2-\widetilde{\mathbf{z}}\left(t_2\right)||_{\infty}+||\widetilde{\mathbf{z}}\left(t_2\right)-\mathbf{z}\left(t_2\right)||_{\infty}$$
$$\leq \frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\,||\mathbf{z}_1-\mathbf{z}\left(t_1\right)||_{\infty}+C_{\text{loc}}\,.\,\Delta^2=C_{\text{loc}}\,.\,\Delta^2.$$

holds for $p=1$, by the assumption that the initial conditions coincide for the time discrete and time continuous models. For $p=2$, we obtain,

$$||\mathbf{z}_3-\mathbf{z}\left(t_3\right)||_{\infty}\leq ||\mathbf{z}_3-\widetilde{\mathbf{z}}\left(t_3\right)||_{\infty}+||\widetilde{\mathbf{z}}\left(t_3\right)-\mathbf{z}\left(t_3\right)||_{\infty}$$
$$\leq \frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\,||\mathbf{z}_2-\mathbf{z}\left(t_2\right)||_{\infty}+C_{\text{loc}}\,.\,\Delta^2$$
$$\leq \frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}C_{\text{loc}}\,.\,\Delta^2+C_{\text{loc}}\,.\,\Delta^2$$
$$=C_{\text{loc}}\,.\,\Delta^2.\left\{\sum_{j=0}^{j=3-2}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j}\right\}.$$

Now we assume that for arbitrary $p\in\{1,\cdots,M-2\}$,
$||\mathbf{z}_p-\mathbf{z}\left(t_p\right)||_{\infty}\leq C_{\text{loc}}\,.\,\Delta^2.\left\{\sum_{j=0}^{p-1}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j}\right\}$ is valid.

This yields,

$$\begin{aligned}
&\left|\left|\mathbf{z}_{p+2}-\mathbf{z}\left(t_{p+2}\right)\right|\right|_{\infty} \leq \left|\left|\mathbf{z}_{p+2}-\widetilde{\mathbf{z}}\left(t_{p+2}\right)\right|\right|_{\infty} + \left|\left|\widetilde{\mathbf{z}}\left(t_{p+2}\right)-\mathbf{z}\left(t_{p+2}\right)\right|\right|_{\infty} \\
&\leq \frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\left|\left|\mathbf{z}_{p+1}-\mathbf{z}\left(t_{p+1}\right)\right|\right|_{\infty} + C_{\text{loc}}\,.\,\Delta^2 \\
&\leq \Delta^2.\left\{\sum_{j=0}^{p-1}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j}\right\} + C_{\text{loc}}\,.\,\Delta^2 \\
&\leq C_{\text{loc}}\,.\,\Delta^2.\left\{1+\sum_{j=0}^{p-1}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j+1}\right\} \\
&= C_{\text{loc}}\,.\,\Delta^2.\left\{\sum_{j=0}^{p}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j+1}\right\}.
\end{aligned}$$

We now apply the geometric series to the sum to obtain,

$$\begin{aligned}
&\left|\left|\mathbf{z}_{p+1}-\mathbf{z}\left(t_{p+1}\right)\right|\right|_{\infty} \leq C_{\text{loc}}\,.\,\Delta^2.\left\{\sum_{j=0}^{p}\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{j+1}\right\} \\
&= C_{\text{loc}}\,.\,\Delta^2\,.\,\frac{\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{p}-1}{\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)-1}
\end{aligned}$$

If we assume that $\Delta < \dfrac{1}{4.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)}$,

we may assume that, $\dfrac{\Delta}{\left(1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta\right)-1} \leq 1$ and hence it follows that,

$$\left|\left|\mathbf{z}\left(t_{p+1}\right)-\widetilde{\mathbf{z}_{p+1}}\right|\right|_{\infty} \leq C_{\text{loc}}\,.\,\Delta.\left\{\left(\frac{1}{1-2.\left(\lambda_{max}+\gamma_{max}+\epsilon\right)\Delta}\right)^{p}-1\right\}. \tag{3.24}$$

Thus, our proof is complete.

We can similarly find the errors due to approximation for the time discrete SEIR model with vaccination.

3.2 Numerical Examples

3.2.1 Data Preprocessing

We obtain the real world data for a time interval $j = 1$ to $j = M$. We preprocess the real world data and obtain the cumulative confirmed infected, susceptible and recovered people as follows. For our examples we use the COVID-19 data published by CDC and State of Tennessee The data of total cases , total death and total recovered are obtained from [5]. Let $\{I_j\}_{j=1}^{M}$ represent the cumulative number of confirmed infected individuals, $\{R_j\}_{j=1}^{M}$ the cumulative number of recovered individuals and $\{D_j\}_{j=1}^{M}$ the cumulative number of deaths. Let us represent the preprocessed data by, $\widetilde{S}_j$, $\widetilde{E_j}$, $\widetilde{I}_j$, $\widetilde{R_j}$.

Table 1: Numerical Algorithm for Data preprocessing

Inputs	-Population size N
	-Cumulative confirmed cases sequence ${I_j}_{j=1}^{M}$
	-Cumulative recovered individuals sequence ${R_j}_{j=1}^{M}$
	-Cumulative number of deaths sequence ${D_j}_{j=1}^{M}$
	-New confirmed cases sequence ${c_j}_{j=1}^{M}$
	-latent period n
Step 1	-Calculate $\widetilde{E_j}$ using equation 3.25 starting from day $n+1$.
Step 1	-Calculate $\widetilde{R_j}$,$\widetilde{I}_j$ and $\widetilde{S}_j$ using equation 3.26.
Outputs	- Sequences $\{\widetilde{E_j}\}_{j=n}^{M}$, $\{\widetilde{R_j}\}_{j=n}^{M}$, $\{\widetilde{I}_j\}_{j=n}^{M}$, $\{\widetilde{S}_j\}_{j=n}^{M}$

To begin with we may assume the disease latency period to be n units of time. Let c_i be the number of new confirmed cases at time period i. We calculate the cumulative number of individuals in the latent compartment E as follows,

$$\widetilde{E_j} = \sum_{i=j-n}^{j} c_i \tag{3.25}$$

Then,

$$\left.\begin{aligned} \widetilde{R_j} &= R_j + D_j \\ \widetilde{I_j} &= I_j - \widetilde{R_j} \\ \widetilde{S_j} &= N - \widetilde{E_j} - \widetilde{I_j} - \widetilde{R_j} \end{aligned}\right\} \tag{3.26}$$

The data we obtained is recorded every day. So the units of time in our case is number of days.

3.2.2 Calculating the time varying transmission and recovery coefficients

We now proceed to calculate the time varying transmission and recovery coefficients from the real world data, using the implicit discrete form given by equations 2.70. Assuming that $\widetilde{S_{j+1}}$, $\widetilde{I_{j+1}} \neq 0$, after the data preprocessing.

$$\widetilde{\lambda_{j+1}} = \frac{\dfrac{\widetilde{S_j}}{\widetilde{S_{j+1}}} - 1}{\widetilde{I_{j+1}}\,\Delta t_j} = \frac{\widetilde{S_j} - \widetilde{S_{j+1}}}{\widetilde{S_{j+1}}\widetilde{I_{j+1}}\,\Delta t_j} \tag{3.27}$$

$$\widetilde{\gamma_{j+1}} = \frac{\widetilde{R_{j+1}} - \widetilde{R_j}}{\widetilde{I_{j+1}}\,\Delta t_j} \tag{3.28}$$

Here the assumption is that the latent period input is converted into the same units of time as the other sequences input. Also the time sequences are more granular than the latent period.
We now present a numerical algorithm to calculate the sequences of transmission and recovery coefficients from the preprocessed real world data.

The figures below show the transmission and recovery coefficients calculated from real world data for Tennessee state from the data using the above given algorithm. We have used the mean latent period, 5 days in these calculations. Since we are not approximating these curves to any time continuous functions, we are using the data directly in the next step.

Table 2: Numerical Algorithm for Calculating the Coefficients

Inputs	-Population size N
	-Cumulative confirmed cases sequence $I_{j}{}_{j=1}^{M}$
	-Cumulative recovered individuals sequence $R_{j}{}_{j=1}^{M}$
	-Cumulative number of deaths sequence $D_{j}{}_{j=1}^{M}$
	-New confirmed cases sequence $c_{j}{}_{j=1}^{M}$
	-latent period n
Step 1	-Calculate all $\Delta_{j+1} = t_{j+1} - t_j$ for $j \in \{1, \cdots, M\}$
Step 2	-Use the previous algorithm to calculate the sequences, $\widetilde{E_j}$, $\widetilde{R_j}$,$\widetilde{I_j}$ and $\widetilde{S_j}$
Step 4	-Calculate $\widetilde{\lambda_{j+1}}$,$\widetilde{\gamma_{j+1}}$ using equations 3.27 and 3.28 for $j \in \{1, \cdots, M\}$
Outputs	- Sequences $\{\widetilde{\lambda_j}\}_{j=2}^{M}$, $\{\widetilde{\gamma_j}\}_{j=2}^{M}$

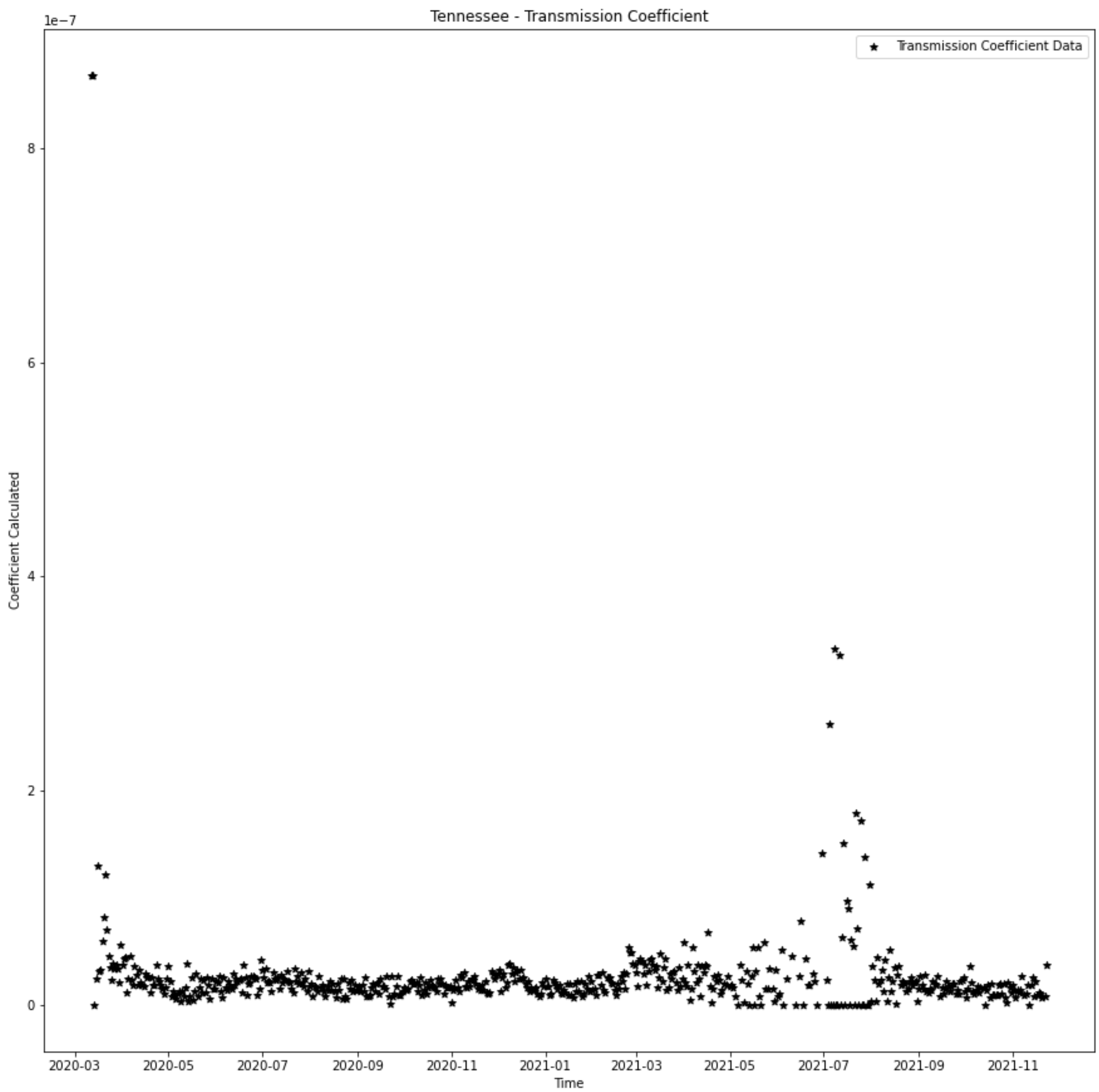


Figure 1: *Transmission coefficient - calculated from data*

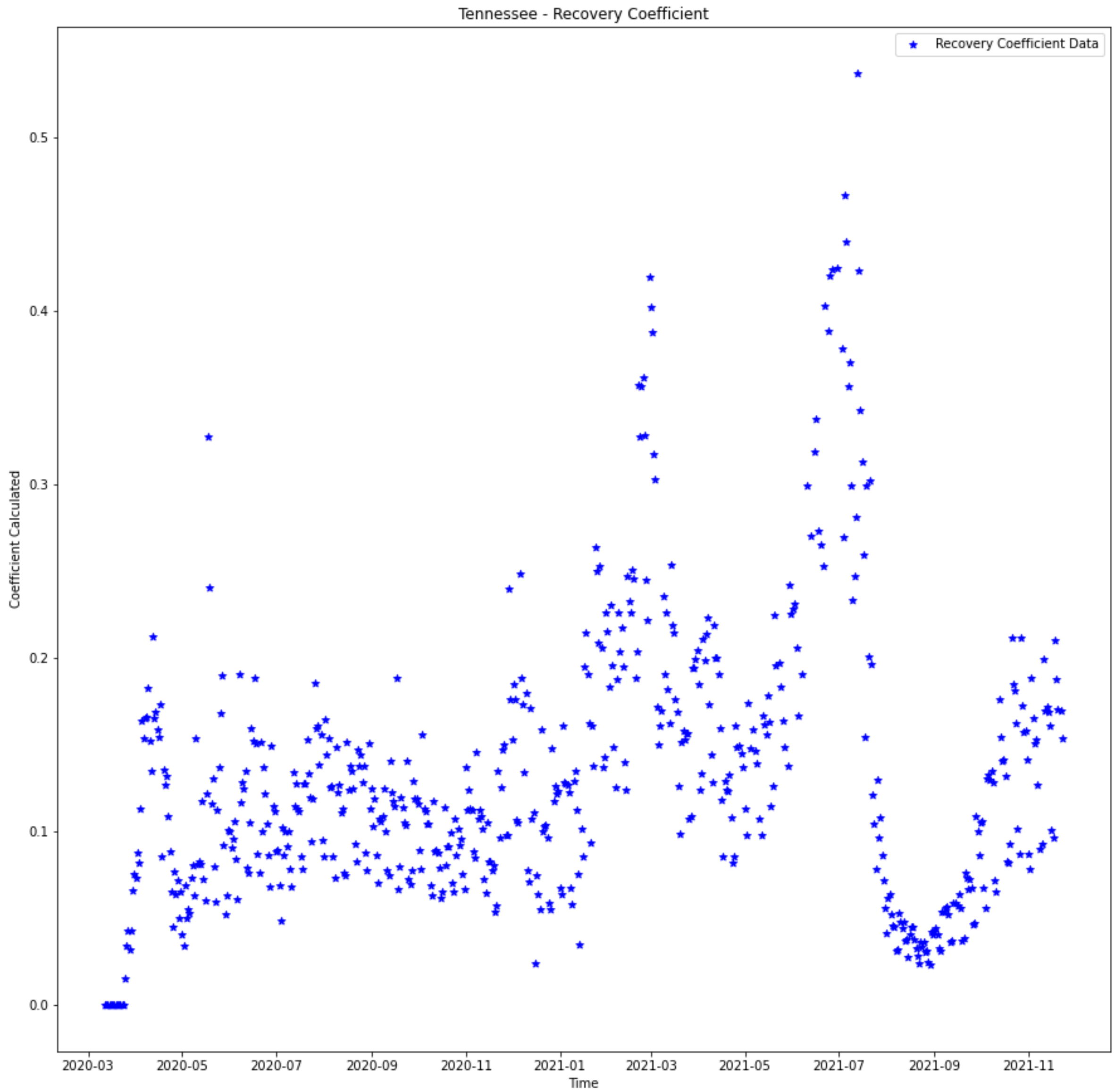


Figure 2: *Recovery coefficient - calculated from data*

3.2.3 Time discrete implicit SEIR form

We have added here plots below are for data from the Tennessee state [5] and from the Michigan state [6].

We use the sequences of coefficients and the preprocessed data from above sections in the time discrete implicit SEIR scheme given by equations 2.70. Here we give the numerical algorithm implement this scheme.

Table 3: Numerical Algorithm for Time discrete implicit SEIR model

Inputs	-Population size N
	- Initial values $S_0 = S_n$, $E_0 = E_n$, $I_0 = I_0$, $R_0 = I_n$
	- The sequences, $\widetilde{\lambda}_j$,$\widetilde{\gamma}_j$
	- Latency period n
Step 1	- Calculate all $\Delta_{j+1} = t_{j+1} - t_j$ for $j \in \{n, \cdots, M\}$
Step 2	- Compute I_{j+1} using the preprocessed data and equations 2.37, 2.38 and 2.39 for $j \in \{n, \cdots, M-1\}$
Step 4	- Compute S_{j+1} , E_{j+1} and R_{j+1} using equation 2.34
Outputs	- Sequences $\{S_j\}_{j=n}^M$, $\{E_j\}_{j=n}^M$, $\{I_j\}_{j=n}^M$, $\{R_j\}_{j=n}^M$

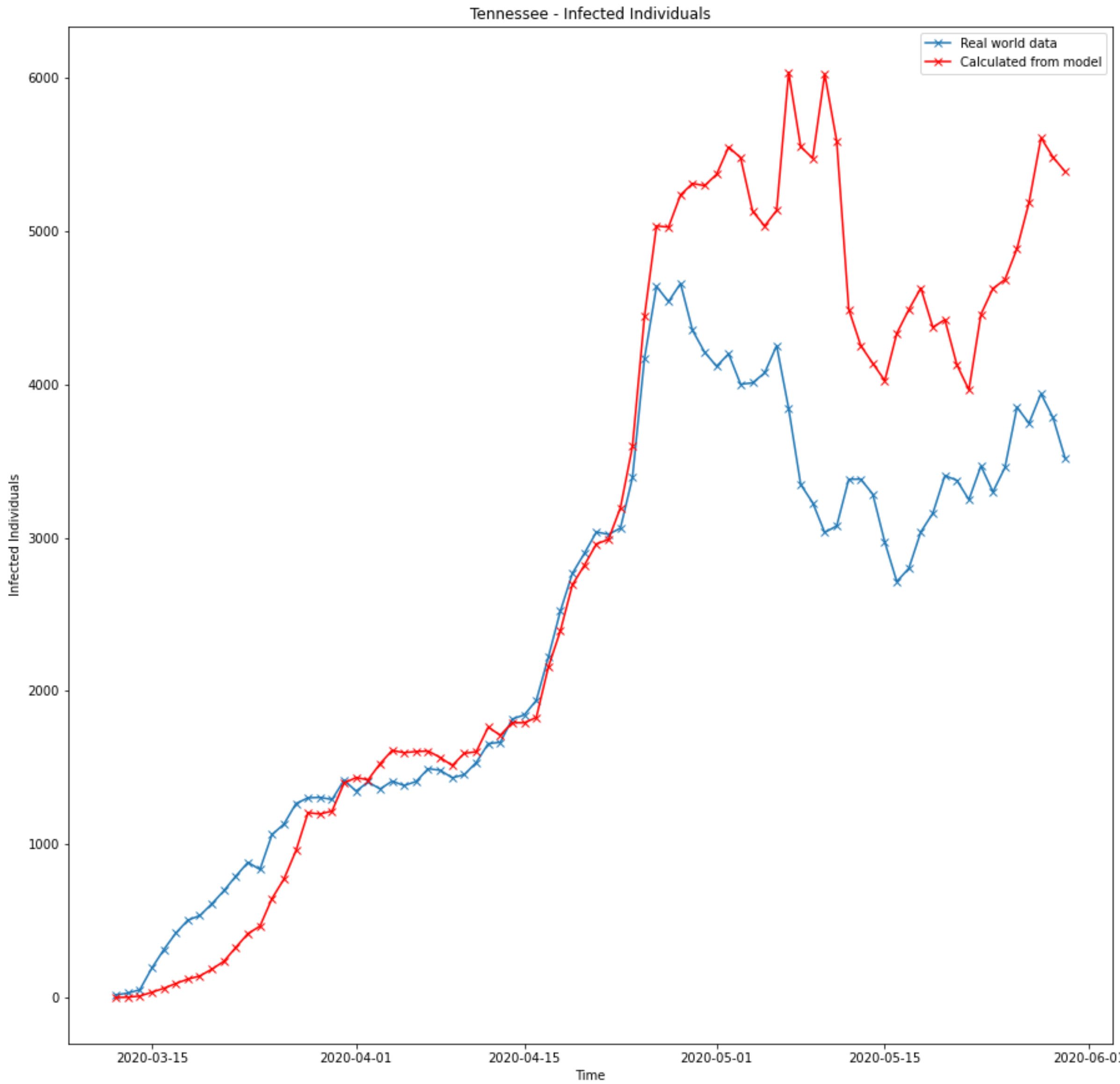


Figure 3: *Real world data and time discrete implicit form calculated infected population*

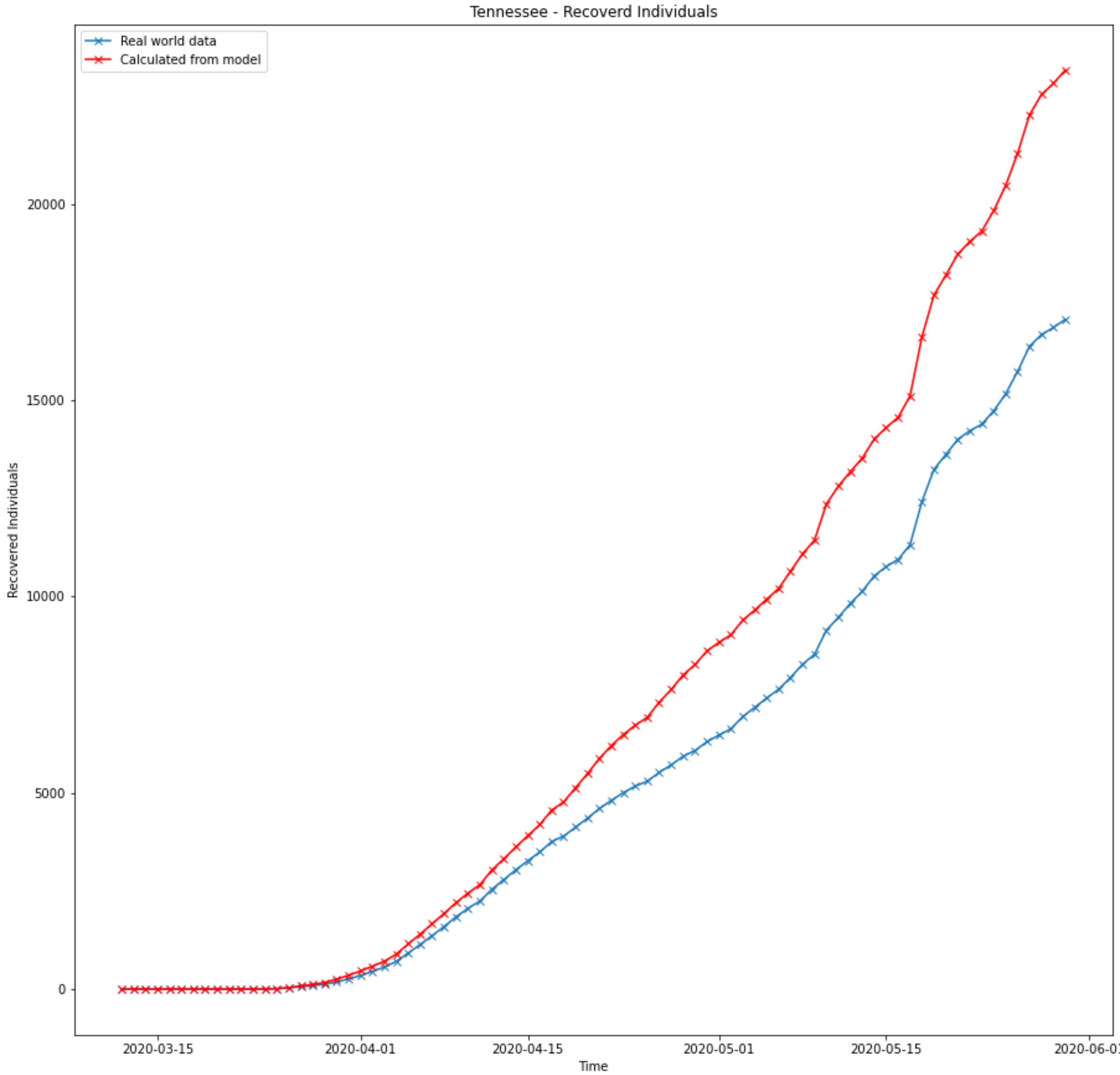


Figure 4: *Real world data and time discrete implicit form calculated recovered population*

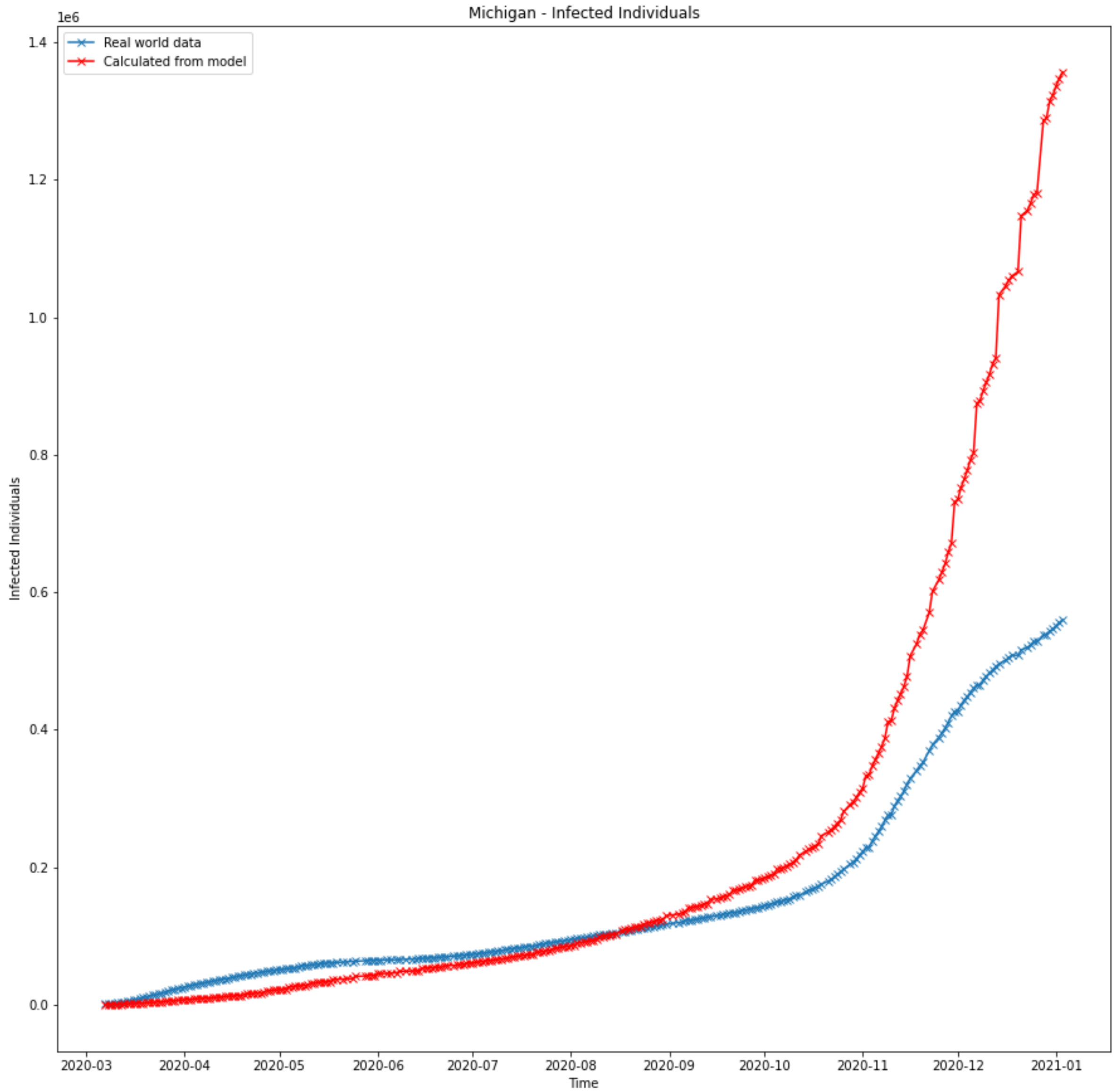


Figure 5: *Real world data and time discrete implicit form calculated infected population for Michigan*

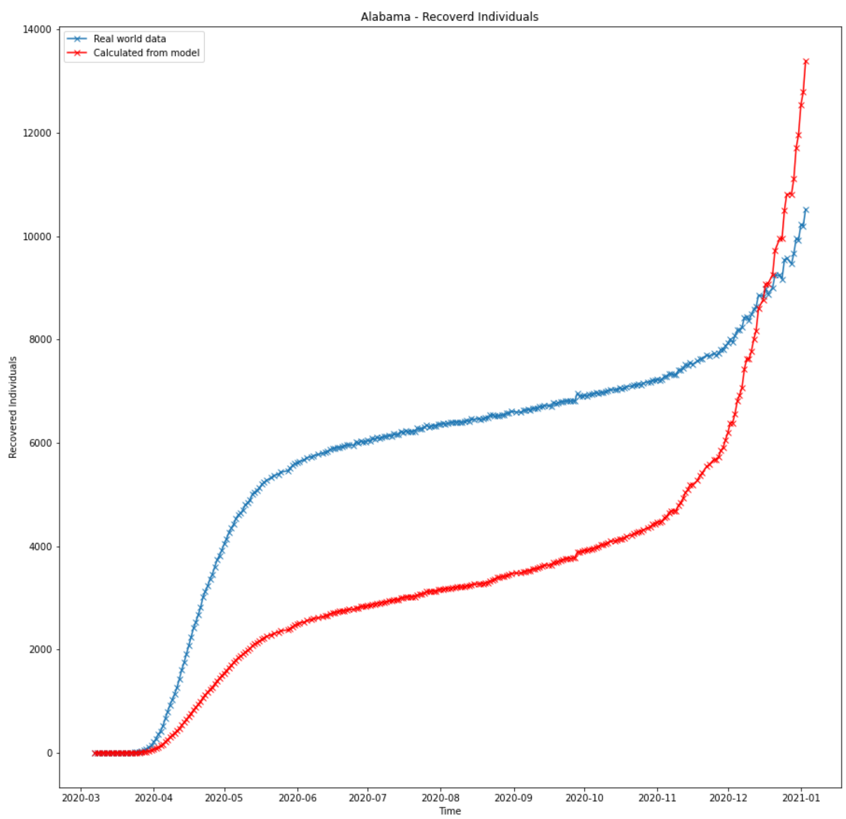


Figure 6: *Real world data and time discrete implicit form calculated recovered population for Michigan*

3.2.4 Calculation of basic reproduction number

We can numerically calculate the time discrete basic reproduction number $\mathcal{R}_0\left(t_j\right)$ readily from equation 2.76. Here we plot the time discrete basic reproduction number versus time.

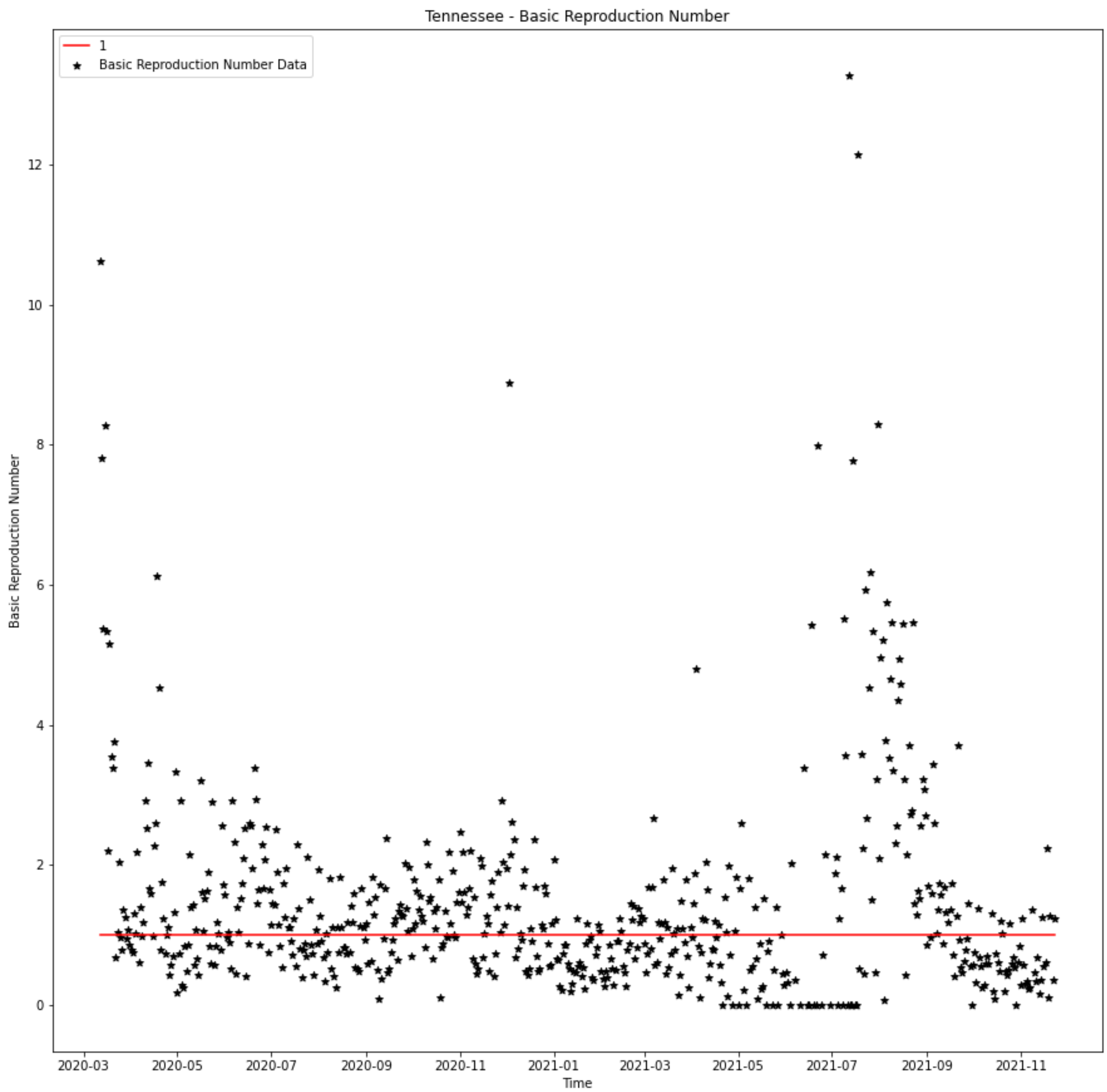


Figure 7: *Basic reproduction number - calculated from data - Tennessee*

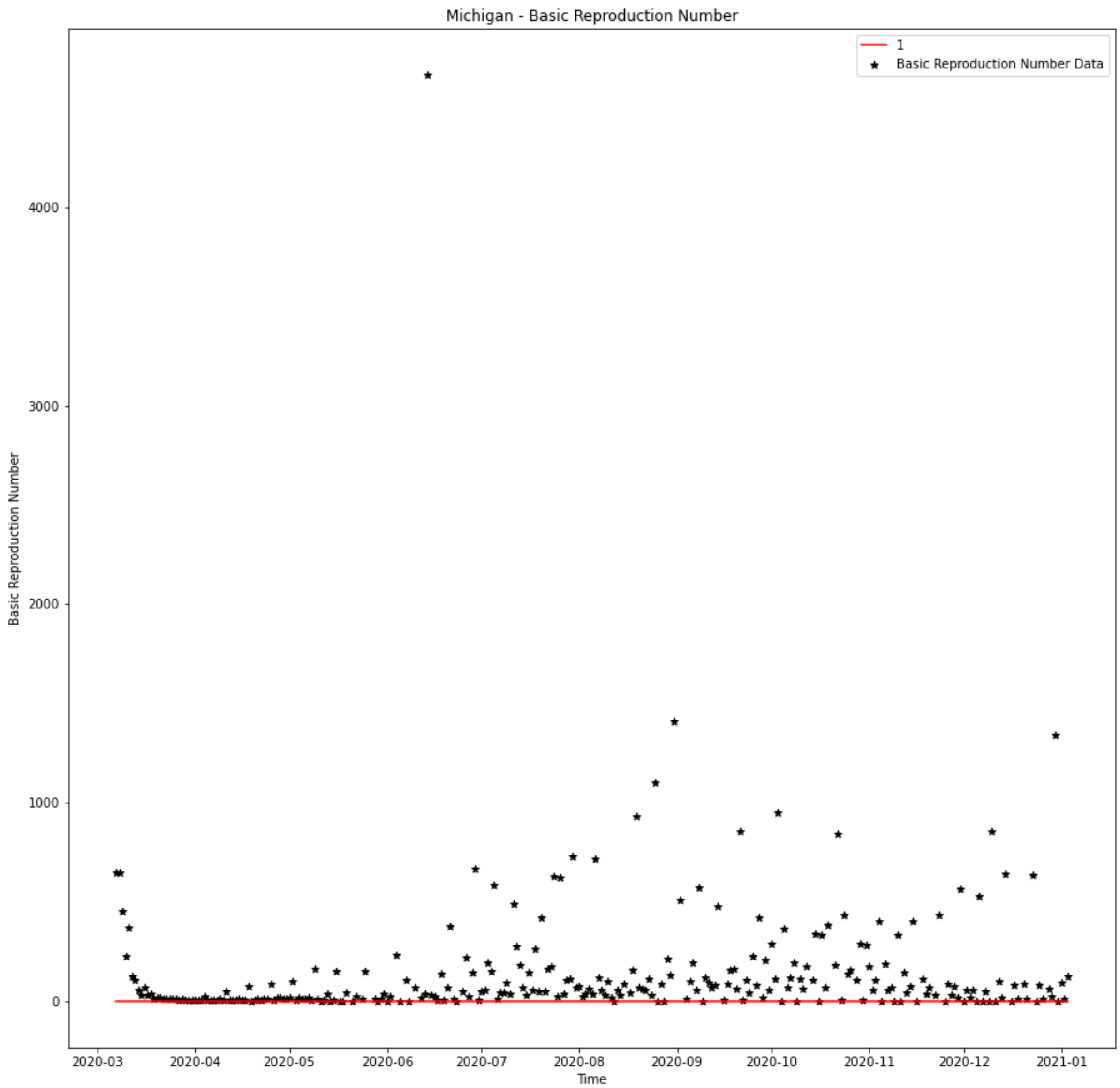


Figure 8: *Basic reproduction number - calculated from data - Michigan*

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

In the previous chapters we studied different mathematical models for epidemics. We arrived at some conclusions about the long time behavior of epidemic diseases. In this chapter we list the most important biological conclusions and also make further conclusions about the models and describe how the models are modified to suit various scenarios. We also list the future work that can be done in this area of discrete models.

The most important concept introduced in this thesis following [2] is that the time discrete implicit form of the differential equations representing the mathematical model can be used to understand the behavior of the disease with better accuracy. We also find the concept of coefficients being time dependent functions from [2]. We have developed the theory of an SEIR model and an SEIR model with vaccination introduced.

We have made the following biological conclusions about an epidemic in Chapter 2 , based on the long time behavior of the solutions.

1. Some number of susceptibles always escape the infection at the end of the epidemic.

2. The epidemic ends not because the suseptibles are exhausted.

3. The disease eventually dies out and the infectious population tends to zero after a long period of time.

4. The epidemic first rises, then declines after reaching the maximum.

We also estimated the basic reproduction number for the disease. From the model with vaccination, we concluded that, vaccination can reduce the number of susceptibles but not stop the infection as long as there are infectious individuals. We concluded that when more individuals are quarantined than exposed to infection, there will be no outbreak.

We developed the numerical algorithms that works on the published data and plot the curves to compare the actual data to that predicted by the implicit discrete SEIR model. Thus, using the model we can answer many questions that are usually expected to be answered from the mathematical models about the time line of the disease.

4.1 Future Work

It should be noted that we have depended on the actual data of the transmission and recovery coefficient and have not approximated the data to a mathematical function. The first important future work is to approximate the coefficients by Parameter Estimation [2] , [1].

We have also not developed the numerical algorithms for the models with vaccination or quarantine . We could study the conditions for outbreak [3] in detail for each model in detail. We could also try to answer the questions about the rate of vaccinations required and effectiveness of quarantine by developing the corresponding numerical algorithms for the models.

There are also more complex time continuous or piece-wise models involving the incubation period [8] denoting the asymptomatic infections used for COVID-19. However in most cases, the change of population is ignored since the change in population is negligible compared to total population. However we could still develop the theory of a model with change in population which could be applied when the change of population is comparable to total population.

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