PROSPECTIVE TEACHERS' BELIEFS ABOUT MATHEMATICAL MISTAKES: AN EXPLORATORY CASE STUDY

by

Matthew D. Duncan

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics and Science Education

Middle Tennessee State University
August 2018

Dissertation Committee:

Dr. Angela Barlow, Co-Chair

Dr. Alyson Lischka, Co-Chair

Dr. Jennifer Lovett

Dr. Ryan Jones

Dr. Chris Stephens

This work is dedicated to my parents, Donald and Gwen Duncan, my wife, Lacey Duncan, and my daughter, Parker Duncan. You have inspired me more than you will ever know, and it would be impossible to do anything without you.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my family for all of their love and support and for always telling me that I can do and be anything. Without you, I would have never even imagined being able to do something like this. To Lacey, thank you for being there throughout this entire process, especially during times of frustration, and always supporting to me, even when I was wrong. I would not have wanted to do this with anyone else. To Parker, you inspired me to be better, and I look forward to being there to see the awesome person you become.

I would like to thank Dr. Angela Barlow and Dr. Alyson Lischka for their support and guidance in preparing this dissertation as well as during my entire time at MTSU. Your feedback and critiques have always pushed me to be a better scholar. To Dr. B, you have inspired me more than you will ever know. Thank you for everything. To Dr. L, you have always challenged and supported me, and I could not have done this without you. Thank you.

To my committee members, Dr. Jennifer Lovett, Dr. Ryan Jones, and Dr. Chris Stephens, I want to thank you for all of your ideas and feedback that went into this dissertation. It is better and I am better because of the hours that you put into helping me prepare this dissertation.

To my peers from MTSU, thank you for the countless hours of discussion, laughing, and complaining. You have all inspired me, and you have all helped shape me into the scholar and person that I am today. This is especially true of Dr. Chris Willingham, Dr. Natasha Gerstenschlager Wilson, Dr. Amber Matuszewski, Dr. Jeff Pair and soon-to-be Drs. Lucy Watson and Kristin Hartland.

Finally, to Dr. Heart, thank you for allowing me to spend hours in your classroom during the study. Observing your class and how you interact with prospective teachers was my pleasure. To Cindy and Harley, thank you for the extensive amount of time that you allowed me borrow and the insights you provided me. I hope that I get the opportunity to see the powerful teachers that you become.

ABSTRACT

Mathematics education reform documents consider mathematical mistakes to be vital for the teaching and learning of mathematics. Additionally, mathematical mistakes are viewed as opportunities for learning and catalysts for discussion. However, research has demonstrated that some teachers do not see mathematical mistakes as productive for learning and often avoid students' mathematical mistakes. Seminal educational research demonstrated that teachers' beliefs influence the ways in which teachers teach, and thus, teachers' beliefs concerning mathematical mistakes influence the ways in which they use mathematical mistakes. Researchers have found that in-service teachers' (ISTs) beliefs are not aligned with reform documents' ways of treating mistakes; however, there is a dearth of research concerning the beliefs of prospective teachers (PTs).

The purpose of this study was to investigate the beliefs of PTs concerning mathematical mistakes and answer two research questions:

1. What beliefs do PTs hold about mathematical mistakes?
2. What are the noticeable changes in PTs' beliefs systems concerning mathematical mistakes, if any, given the classroom context?

An exploratory multiple-case study design was implemented to examine the beliefs of two PTs while enrolled in a content course for teachers designed specifically to use mathematical mistakes in ways aligned with mathematics reform documents.

The study produced findings that were significant in at least five ways. First, the study produced results that demonstrated a change in PTs' beliefs while enrolled in a content course for teachers. This adds to a larger body of seminal research concerning PTs' beliefs and how they can change. Additionally, a content course being the context in

which this study took place adds to the significance that content courses have in teacher preparation. Second, the findings of this study contribute to a growing body of research on beliefs concerning mathematical mistakes. There is a scarcity of research on PTs' beliefs concerning mathematical mistakes, but the findings of this study complement research on ISTs' beliefs concerning mathematical mistakes. Third, the study revealed that the PTs' beliefs changed during a course with a positive error climate. This finding has potential applications for teacher educators and teacher preparation course designers including a need to create positive error climates to reduce the influence of affective qualities on when PTs share their mathematical mistakes. Fourth, the findings of the study suggested that mechanisms that promote changes in PTs' beliefs should include the error climate of the classroom. This has potential applications for teacher educators and teacher preparation course designers as well including a need to create positive error climates to support alignment of beliefs concerning mathematical mistakes with mathematics education reform documents. Finally, the findings of this study revealed the importance of PTs' teacher identities in how they interpreted their experiences with mathematical mistakes and, subsequently, their beliefs concerning mathematical mistakes. Future research with PTs' beliefs concerning mathematical mistakes should account for PTs' teacher identities as they played a substantial role in the findings of this study.

TABLE OF CONTENTS

LIST OF TABLES ... xiii

LIST OF FIGURES ... xiv

CHAPTER I: INTRODUCTION ... 1

Introduction ... 1

Background for the Study ... 1

The Role of Mistakes in Mathematics Teaching and Learning ... 2

The Role of Teachers' Beliefs in Mathematics ... 3

Beliefs of Prospective Teachers ... 4

Theoretical Framework ... 6

The Problem Statement ... 8

Statement of Purpose ... 9

Significance of the Study ... 9

Definitions ... 10

Beliefs ... 10

Error Climate ... 11

Positive Error Climate ... 11

Reform Documents ... 11

Traditional Teaching Methods ... 11

Mathematical Mistakes ... 11

Chapter Summary ... 12

CHAPTER II: REVIEW OF THE LITERATURE ... 14

Introduction ... 14

Influences on the Use of Mistakes ... 15
Classroom Error Climate ... 16
Significance of the Research ... 25
Beliefs ... 25
Importance of Beliefs in Mathematics ... 26
Operationalizing Beliefs ... 26
Significance of the Research ... 35
Teacher Education ... 36
General Overview of Teacher Preparation ... 36
Content Courses' Impact on PTs' Teacher Identity ... 37
Significance of Research ... 39
Implicit Theories ... 40
Research on Implicit Theories ... 40
Significance of Research ... 43
Conceptual Framework ... 44
Chapter Summary ... 46
CHAPTER III: METHODOLOGY ... 47
Introduction ... 47
Research Overview ... 48
Research Context ... 49
University ... 49
Course ... 49
Instructor ... 50

Participant Selection ... 52
Data Sources and Instruments ... 54
Implicit Theory Survey ... 54
Error Climate Survey ... 55
Semi-structured Interviews ... 57
Classroom Observations ... 58
Reflective Journal ... 59
Exit Tickets and In-class Reflections ... 59
The Researcher as an Instrument ... 60
Summary ... 61
Data Collection Procedures ... 62
Participant Selection ... 62
Prior Error Climate Description ... 62
Classroom Meetings and Reflective Journals ... 63
Current Error Climate Description ... 64
Data Analysis ... 65
Within-case Analysis ... 65
Cross-case Analysis ... 69
Issues of Trustworthiness ... 72
Limitations of the Study ... 74
Delimitations of the Study ... 75
Chapter Summary ... 76
CHAPTER IV: RESULTS ... 77

Introduction 77

The Case of Cindy 78

Implicit Theory 78

Prior Mathematics Experiences 82

Perceived Error Climate for Previous Mathematics Classes 87

Cindy's Central Beliefs Concerning Mathematical Mistakes 90

Belief Clusters 99

Noticeable Changes in Beliefs System 132

Summary 144

The Case of Harley 147

Implicit Theory 148

Prior Mathematics Experiences 151

Perceived Error Climate for Previous Mathematics Classes 154

Harley's Central Beliefs Concerning Mathematical Mistakes 157

Belief Clusters 168

Noticeable Changes in Beliefs System 197

Summary 210

Cross-Case Analysis of Cindy and Harley 213

Implicit Theory 214

Mathematics Background 216

Central Beliefs Concerning Mathematical Mistakes 218

Belief Clusters Concerning Mathematical Mistakes 223

Changes in Beliefs Systems Concerning Mathematical Mistakes 228

Summary 232
Chapter Summary 232
CHAPTER V: SUMMARY AND DISCUSSION 233
Introduction 233
The Research Problem 234
Review of Methodology 234
Summary of Results 235
Implicit Theory 235
Prior Mathematical Experiences 236
Central Beliefs Concerning Mathematical Mistakes 237
Belief Clusters Concerning Mathematical Mistakes 238
Noticeable Changes in Beliefs System 239
Discussion of Results 240
Connections to Prior Research 241
Theoretical Implications 247
Suggestions for Practice 248
Recommendations for Future Research 252
Chapter Summary 254
REFERENCES 256
APPENDICES 270
APPENDIX A: IMPLICIT THEORY SURVEY 271
APPENDIX B: PREVIOUS MATHEMATICS CLASSROOM ERROR CLIMATE .. 272
APPENDIX C: FIRST POST ERROR CLIMATE INTERVIEW PROTOCOL 274

APPENDIX D: CLASSROOM OBSERVATION PROTOCOL 277

APPENDIX E: POST CLASSROOM OBSERVATION INTERVIEW PROTOCOL . 278

APPENDIX F: REFLECTION JOURNAL PROMPTS 279

APPENDIX G: CURRENT CLASSROOM ERROR CLIMATE 280

APPENDIX H: FINAL ERROR CLIMATE INTERVIEW PROTOCOL 282

APPENDIX I: IMPLICIT THEORY OBSERVATION PROTOCOL 284

APPENDIX J: EXAMPLE EXIT TICKETS.. 286

APPENDIX K: EXAMPLE IN-CLASS REFLECTIONS 287

APPENDIX L: INSTITUTIONAL REVIEW BOARD APPROVAL 288

APPENDIX M: FIRST SET OF CODES AND THEMES 290

APPENDIX N: FINAL CODES LIST 291

APPENDIX O: COPYRIGHT PERMISSION FOR FIGURE 11 AND FIGURE 20.... 292

LIST OF TABLES

Table 1. Types of Mistakes 12

Table 2. Correlations Between Error Climate Dimensions and Student Achievement.. 23

Table 3. Research Questions in Relation to Data Sources 61

Table 4. Initial Codes List Conceptualized Prior to Open Coding................................ 67

Table 5. Research Quality and Strategies Employed 74

Table 6. Cindy's Prior Classrooms Error Climate Results.. 89

Table 7. Cindy's Prior and Current Error Climate Results 133

Table 8. Harley's Prior Classrooms Error Climate Results... 157

Table 9. Harley's Prior and Current Error Climate Results 199

Table 10. Cindy and Harley's Implicit Theory Averages 215

Table 11. Central Beliefs Comparison of Cindy and Harley... 223

Table 12. Comparison of Cindy's and Harley's Clusters of Beliefs 228

Table 13. Cindy's and Harley's Perceived Error Climate Averages............................. 230

LIST OF FIGURES

Figure 1. Expected effects of perceived classroom error climate 21

Figure 2. Process of beliefs changing 31

Figure 3. Grading and evaluation expectations for Dr. Heart’s class 51

Figure 4. Network of codes created on ATLAS.ti. 71

Figure 5. Cindy’s cooccurrence table for codes related to learning from mistakes and the types of mistakes 72

Figure 6. Examples of obtuse angles 81

Figure 7. Which triangles exist table 82

Figure 8. Finding area of triangle on grid paper 92

Figure 9. A recreation of Dr. Heart's incorrect strategy of finding the area of the triangle by doubling the shape 93

Figure 10. Quadrilateral properties checklist 104

Figure 11. In-class task with two mistakes in finding the perimeter 107

Figure 12. Cindy’s replication of a student’s mathematical mistake presented by Dr. Heart 109

Figure 13. Cindy’s in-class reflection work 110

Figure 14. Venn diagrams of squares and rectangles 116

Figure 15. Hidden shape maker activity 118

Figure 16. Recreated non-procedural mistake provided by Dr. Heart. 124

Figure 17. Cindy's central beliefs concerning mathematical mistakes 145

Figure 18. Cindy's beliefs clusters concerning mathematical mistakes 146

Figure 19. Cindy's beliefs system at the end of the study 147

Figure 20. In-class activity with two mistakes in finding the perimeter...................... 161

Figure 21. Example of a student’s conceptual mistake provided by Dr. Heart............ 190

Figure 22. Angles provided in an exit slip given to Harley on August 30, 2017 202

Figure 23. Harley's central beliefs concerning mathematical mistakes........................ 211

Figure 24. Harley's belief clusters.. 212

Figure 25. Harley’s beliefs systems at the end of the study 213

CHAPTER I: INTRODUCTION

Introduction

This study is a description of an exploratory case study that investigated the beliefs of prospective teachers (PTs) concerning mistakes in the mathematics classroom. Specifically, the study took place in a content course for elementary PTs and focused on the PTs' beliefs concerning mathematical mistakes in light of a classroom environment that closely aligned with mathematics reform documents' description of the utility of mathematical mistakes. This chapter contains an introduction to the study including a brief review of its background. The background focuses on significant literature concerning the role that mathematical mistakes should play in teaching and learning and then focuses on beliefs concerning mathematical mistakes. The background then concentrates on the salient literature concerning PTs' beliefs in general. This is followed by a description of the nature of the problem and its significance. The study's theoretical framework, purpose, and definitions of key terms are also presented.

Background for the Study

Mathematics reform documents (e.g., Association of Mathematics Teacher Educators [AMTE], 2017; Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000, 2014; National Research Council [NRC], 2001) encourage teachers to teach in ways that focus on student thinking and leverage students' experiences with mathematics content. As NCTM (2000) stated, "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p. 15). Additionally, teaching in the spirit of these documents involves engaging students in meaningful tasks as well as encouraging student interaction and discussion centered on the students' work (NCTM, 2014). Many of these discussions focus on students' mistakes and provide opportunities for students to reflect on those mistakes (NCTM, 2014). However, the way reform documents describe teaching is in stark contrast to traditional ways of teaching that remain pervasive in today's mathematics classrooms (Bahr, Monroe, & Eggett, 2014; Barlow & Reddish, 2006; Philipp, 2007). Whereas reform documents describe teaching that focuses on students' prior knowledge and how to build upon it, traditional methods of teaching are dominated by demonstration, repeated practice, memorization, and emphasis on correct answers (Boaler, 2002; College Board of Mathematical Societies [CBMS], 2010; NCTM, 2014). The next section will provide a more detailed description of the role of mathematical mistakes in teaching and learning.

The Role of Mistakes in Mathematics Teaching and Learning

Although mathematical mistakes are commonly avoided in U.S. classrooms (Bray & Santagata, 2014), mistakes play an important role in teaching mathematics and are a natural part of learning (Boaler, 2016; Lampert, 1992). For teachers, students' mathematical mistakes serve as opportunities for learning and places to advance students' mathematical understanding (Borasi, 1994; Kazemi & Stipek, 2001; Santagata, 2005). Additionally, mistakes provide a unique occasion for teachers to support students in being persistent and enduring productive struggle when problem solving (Bray & Santagata, 2014). Therefore, mathematical mistakes should not be avoided or seen as deficits in students' learning but rather as opportunities to learn.

The previously mentioned reform documents (i.e., CCSSI, 2010; NCTM 2000, 2014) also support highlighting mistakes in the mathematics classroom. For example, as stated in *Principles and Standards for School Mathematics* (NCTM, 2000), “Mistakes are not seen as dead ends but rather as potential avenues of learning” (p. 144). In the Standards for Mathematical Practice (CCSSI, 2010), teachers are encouraged to support students in critiquing the reasoning of others to gain understanding of mathematical topics. Additionally, CCSSI (2010) stated that mathematically proficient students can distinguish between correct and flawed logic. Finally, NCTM (2014) stated that when supporting students in productive struggle teachers should facilitate discussions around students’ mathematical mistakes. Mistakes were also seen as evidence of student thinking, and students should be encouraged to reflect on their mistakes to advance their mathematical understanding (NCTM, 2014). In attending to mathematical mistakes, though, there is work to be done by teachers, as teachers are responsible for seeing the value in mistakes and using mistakes as catalysts for learning. However, unproductive beliefs held by teachers can limit those opportunities and “limit student access to important mathematical content and practices” (NCTM, 2014, p. 11).

Most teachers’ mathematical experiences are shaped by traditional methods of teaching (Boaler, 2002; Cooney, Shealy, & Arvold, 1998), which helps explain the tenacity of traditional teaching methods in mathematics classrooms today and the minimization of students’ mistakes. However, there are other mitigating factors that influence the methods and strategies teachers employ in their classrooms. One of those factors is teachers’ beliefs about mathematical content (Thompson, 1992).

The Role of Teachers’ Beliefs in Mathematics

In mathematics, teachers’ beliefs play a significant role in teaching and learning (Pajares, 1992; Thompson, 1992). Their beliefs influence not only how they teach mathematics but what mathematics is, what it means to do mathematics, and how they envision the roles of the teacher and the students in a mathematics classroom (AMTE, 2017; Barlow & Reddish, 2006; Franke, Fennema, & Carpenter, 1997; Thompson, 1992). Within a mathematics classroom environment where mistakes are not of value (Bray, 2011; Stigler & Hiebert, 1999; Wang & Murphy, 2004), teachers' beliefs concerning mistakes are that of a sign of "failure in learning" (Stevenson & Stigler, 1992, p. 192). If mathematical mistakes are to play the role that research and reform documents advocate, then teachers' beliefs concerning mistakes need to be in alignment with that role (CBMS, 2010; Hart, 2002, 2004; Jensen, 1993), or mathematics education's progress will remain compromised (NCTM, 2014).

Research, mostly conducted by Santagata, Bray, and Brodie (e.g., Bray, 2011; Bray & Santagata, 2014; Brodie, 2014; Santagata, 2004, 2005; Santagata & Bray, 2016), focused on in-service teachers' [ISTs] beliefs concerning mathematical mistakes suggested that ISTs' beliefs are not aligned with reform documents. Although, these researchers are conducting professional developments (PDs) and further research with ISTs to encourage alignment, there is a lack of literature addressing PTs' beliefs about mistakes. Therefore, the next section will provide an overview of the literature that is available concerning PTs' beliefs concerning mathematics and how they can change.

Beliefs of Prospective Teachers

Teachers' beliefs influence instructional practices (Pajares, 1992; Philipp, 2007; Thompson, 1992) and subsequent minimization of mathematical mistakes (Bray, 2011; Brodie, 2014; Santagata, 2004, 2005; Santagata & Bray, 2016). Their beliefs are shaped through a variety of experiences. These include experiences during their teacher preparation programs, where PTs have the opportunity to interact with other PTs and teacher educators. Thus, it is necessary to provide background on PTs' beliefs. With limited available research specifically on PTs' beliefs concerning mathematical mistakes, this section will provide insight into PTs' beliefs on a global level and how they can change.

When PTs enter their teacher preparation programs, their beliefs and belief structures are already well established (Hart, 2004; Pajares, 1992). Their experiences before entering college and during college shape their beliefs and belief structures. These experiences include the PTs' years of being a student in primary and secondary mathematics classrooms (Cooney et al., 1998; Hart, 2002, 2004; Jensen, 1993; Thompson, 1992). However, PTs' beliefs concerning mathematics are also influenced by personal attitudes and experiences in other classes (Maher, Bailey, Etheridge, & Warby, 2013).

Although PTs' beliefs are difficult to change (Cooney et al., 1998; Hart, 2002, 2004; Jensen, 1993; Pajares, 1992), research suggests that PTs' beliefs can be developed, and teacher preparation programs "play a crucial role" (AMTE, 2017, p. 54) in that process. Additionally, research shows there are a variety of factors that can influence PTs' beliefs (cf. Joram & Gabriele, 1998; Ng, Nicholas, & Williams, 2009; Teo & Noyes, 2011), but one influential factor relevant to this study is how PTs' beliefs can change in response to pressure from and interactions with others. As PTs have experiences with teachers and peers (Aston & Hyle, 1997; Goos, 2005), beliefs can be restructured and become more consistent with productive beliefs (see NCTM, 2014). However, those experiences must be reoccurring to nurture desired belief structures (Aston & Hyle, 1997;

Hart, 2004) in addition to reflection on those experiences (Hart, 2004; Maher et al., 2013). These experiences and reflection on those experiences do not guarantee that PTs' beliefs will change; however, they do provide opportunities for change (Hart 2004), especially in instances where PTs are given occasion to challenge their traditional beliefs (Aston & Hyle, 1997). The change in beliefs is further promoted if PTs are provided exchanges with teachers and peers, with reflection, in both their content and methods courses (Hart, 2002).

In summary of the background of this study, reform documents state that teachers should treat mathematical mistakes as opportunities for learning, rather than minimized for highlighted correct answers. In-service teachers' beliefs concerning mathematical mistakes do not align with reform documents, and these strongly held beliefs play a significant role in how mistakes are utilized in the classroom. Although beliefs are difficult to change, there is an opportunity to explore and develop PTs' beliefs concerning mathematical mistakes while PTs are in their teacher preparation programs. The following section will provide an overview of the theoretical framework for the study.

Theoretical Framework

The background literature previously provided suggests components of a study of this kind. As a "reliance on theoretical concepts to guide the design and data collection" (Yin, 2003, p. 1) is necessary and the theoretical framework is the construct that guides research (Merriam, 2009), two theoretical constructs guided the study. The first of these was the *error climate* of the classroom (Steuer, Rosentritt-Brunn, & Dresel, 2013). This framework accounts for the "perception, evaluation and utilization of errors . . . [and considers them] integral elements of the learning process within the social context of the classroom” (Steuer & Dresel, 2015, p. 263). Steuer et al. (2013) assumed that although learning from mistakes depended on individual characteristics of learners the classroom context was a precursor to those. Thus, classroom climate facilitates learning from mistakes and the potential to transcend individual attributes. This framework helped to define the context of the study and account for both verbal and nonverbal interactions among PTs and their interactions with the teacher concerning mathematical mistakes.

The second construct was Leatham’s (2006) theoretical framework for viewing beliefs as a sensible system. The sensible system framework assumes that an individual, whether or not that person is able to articulate a belief, places beliefs into “organized systems that make sense to them” (p. 93). Additionally, the sensible system framework not only leads to a need to describe what an individual believes but how those beliefs are connected to other beliefs. Borrowing from Green (1971) and Rokeach (1968), Leatham’s (2006) framework consists of three dimensions that assist in visualizing an individual’s belief. The first dimension is the psychological strength of a belief (Green, 1971). This is a description of the relative importance that a belief is to an individual that can vary from central to peripheral. Where a belief situates on that continuum depends on the connectedness (Rokeach, 1968) of the belief. The connectedness of a belief describes the degree that a belief is existential, shared, derived, or a matter of taste. Simply stated, the more that a belief is associated with one’s identity, the more connected and thus more central it is. Similarly, the more a belief is shared with peers, the more connected and central it is. If a belief is derived from a group with which an individual is associated, the less connected and thus less centrally the belief is held by the individual. Finally, there are beliefs that are “matters of taste” (Rokeach, 1968, p. 5). These beliefs are essentially arbitrary and thus less connected and less central. The second dimension is the "quasi-logical relationship" (Leatham, 2006, p. 94) that exists between beliefs (Green, 1971). That is, a particular belief may be an antecedent or consequent of another belief. The relationship between two beliefs does not, however, correlate with the psychological strength of the beliefs. That is, a belief may be a consequent of another belief but may also be more central than the belief that was the antecedent. The final dimension of the sensible system framework is the extent to which beliefs are clustered in isolation from other beliefs (Green, 1971). This clustering of beliefs allows for the contextualization of beliefs and explains why a belief in one context may not be as central in another. This framework provided guidance in data collection and data analysis. Together, the error climate and beliefs as a sensible system frameworks provided a lens to examine what PTs' beliefs were concerning mathematical mistakes, how those beliefs were connected and clustered in relation to other beliefs, and any noticeable changes in their beliefs system, if at all, in the presence of their perceived error climate.

The Problem Statement

Mathematical mistakes play a crucial role in learning in the environment described by reform documents (see CCSSI, 2010; NCTM 2000, 2014). Additionally, these documents call for teachers to teach in drastically different ways compared to the classroom experiences they themselves had in their primary and secondary years. These experiences shape PTs' beliefs and eventually the way that they will teach in the classroom (Cooney et al., 1998; Hart, 2002, 2004; Thompson, 1992). Understanding those beliefs and how they change is a crucial step in changing how a teacher will teach and use mistakes in the classroom (Jensen, 1993; Thompson, 1992). As Thompson

(1992) stated, “It is not until we have a clearer picture of how teachers modify and reorganize their beliefs in the presence of classroom demands and problems . . . that we can claim to understand the relationship between beliefs and practice” (p. 135). This study addressed the lack of research-based evidence concerning PTs’ beliefs about mathematical mistakes.

Statement of Purpose

Understanding the beliefs of PTs is the first step in ensuring alignment of PTs’ beliefs concerning mistakes with mathematics education research and teaching documents (e.g., Barlow & Reddish, 2006; CCSSI, 2010; NCTM 2000, 2014; Santagata, 2005). Additionally, accounting for the context in which those beliefs are investigated is necessary. Therefore, the purpose of this study was to investigate PTs’ beliefs concerning mathematical mistakes while considering the error climate of the classroom, that is, the way that mistakes were perceived in the classroom from the perspective of the PTs. The two primary research questions of the study were:

3. What beliefs do PTs hold about mathematical mistakes?
4. What are the noticeable changes in PTs’ beliefs systems concerning mathematical mistakes, if any, given the classroom context?

Significance of the Study

This study was significant in at least three ways. First, the study contributed to a larger body of literature on PTs’ beliefs and how those beliefs can change (e.g., Aston & Hyle, 1997; Cooney et al., 1998; Hart, 2002, 2004; Jensen, 1993; Thompson, 1992). Second, it contributed to a larger body of research on teachers’ beliefs concerning mathematical mistakes (e.g., Barlow & Reddish, 2006; Borasi, 1994; Bray & Santagata, 2014; Kazemi & Stipek, 2001; Santagata, 2005). Finally, for teacher educators, this study contributed to understanding PTs' beliefs concerning mistakes and will inform teacher educator high-leverage practices (see Grossman, Hammerness, & McDonald, 2009) to support alignment of PTs' beliefs with reform documents and literature of how mistakes should be used in the mathematics classroom (Hart, 2002, 2004; Jensen, 1993).

Definitions

In the remaining sections, key terms will be referred to repeatedly. This section is intended to bring clarity of meaning to these terms.

Beliefs

The term belief has been used in a variety ways (see Pajares, 1992), and in many of those cases, the term belief is often conflated with the term knowledge. Even though beliefs and knowledge are complementary, this study used Leatham's (2006) distinction between the two:

> Of all the things we believe, there are some things that we "just believe" and other things that we "more than believe – we know." Those things we "more than believe" we refer to as knowledge and those things we "just believe" we refer to as beliefs. (p. 92)

This study used the term beliefs to include the goals of mathematics teaching, the nature of mathematics, what counts as mathematics and mathematical activity (Raymond, 1997; Thompson, 1992), and beliefs concerning mathematical mistakes.

Error Climate

In general, error climate will refer to the perceptions of the value, consequence, or meaning of the mathematical mistakes learners make during learning tasks (Steuer et al., 2013).

Positive Error Climate

Throughout this report, the term positive error climate will refer to a classroom context where the "evaluation and use of errors [are] integral elements of the learning process in the social learning environment of the classroom" (Steuer et al., 2013, p. 198).

Reform Documents

The term reform documents will refer to a collection of documents that encourage a shift in teaching and learning from traditional teaching strategies to strategies that are student-centered and leverage students' prior knowledge in an effort to advance students' understanding of mathematics. Generally, the term reform documents will include, but not be limited to, AMTE (2017), CBMS (2012), CCSSI (2010), and NCTM (2000, 2014).

Traditional Teaching Methods

The term traditional teaching methods will refer to teaching through demonstration, repeated practice, memorization, and emphasis on correct answers (Boaler, 2002).

Mathematical Mistakes

Finally, the term mathematical mistakes will be used to refer to any error made in a mathematics environment. Additionally, the term mathematical mistakes will be used to include mistakes made by PTs in their content courses or that PTs choose to share throughout data collection. These will include, but not be limited to, conceptual, procedural, drawing, computational, distraction, and principle, property, or definition mistakes (Santagata, 2005). Table 1 provides an overview of these types of mistakes along with a description of the nature of each type of mistake.

Table 1

Types of Mistakes

Nature of Mathematical Mistake	Description of Mathematical Mistake
Conceptual	PT makes a mistake in solving a mathematical task or incorrectly answers a question that requires making connections between mathematical concepts.
Procedural	PT incorrectly executes an algorithm or application of a formula.
Drawing	PT makes a mistake in drawing a figure.
Computational	PT makes an arithmetic error.
Distraction	PT mistake is due to the PT being distracted.
Principle, Property, or Definition	PT does not recognize a mathematical principle or property, or PT defines a mathematical concept or property incorrectly.

Note. Adapted from Santagata (2005, p. 497).

Chapter Summary

This chapter included a brief introduction to a case study investigating PTs' beliefs concerning mistakes and the classroom context in which those mistakes are made.

Chapter II will provide a summary of the literature introduced in this chapter, and Chapter III will provide a detailed description of the research plan and methods that will guide the remainder of the study. Following, Chapter IV will provide the results from the study and describe each case in detail. Finally, Chapter V will provide a summary of the study as well as a discussion of the connections to literature and future directions from the results of the study.

CHAPTER II: REVIEW OF THE LITERATURE

Introduction

Teaching practices that leverage students' mathematical mistakes are well supported in mathematics education (e.g., AMTE, 2017; CBMS, 2012; CCSSI, 2010; NCTM, 2000, 2014). However, in-service teachers' (ISTs) use of mathematical mistakes do not align with those practices (Bray & Santagata, 2014; Santagata, 2005). Although there are many influences on why this is the case, beliefs are pivotal in teachers' practices. However, teachers' beliefs have the potential to be restructured during learning experiences, including, for prospective teachers' (PTs), teacher preparation programs (Cooney et al., 1998; Thompson, 1992). Therefore, the purpose of this study was to investigate PTs' beliefs concerning mathematical mistakes while considering the error climate of the classroom, that is, the way that mistakes were perceived in the classroom from the perspective of the PTs.

This chapter begins with a review of the literature concerning the influences on the use of mathematical mistakes in the classroom, including the classroom error climate. This will be followed by an examination of literature concerning beliefs of teachers, both in-service and prospective, which will include a review of literature concerning ISTs' beliefs concerning mathematical mistakes. Additionally, a review of the role that teacher preparation plays in developing practicing teachers is included along with the impact of content courses on PTs' identities. Finally, an examination of implicit theories is provided including its foundation and empirical research using implicit theories. Based on these reviews, the theoretical framework of the study will be examined again, and the connections between this literature base and the current study will be discussed.

This review of literature was not comprehensive or exhaustive of all available literature on these topics, especially PTs' beliefs. However, the literature review focuses on the underlying theoretical assumptions that are most central to this study (Yin, 2014). Additionally, the literature review includes empirical research that supported the research questions of the study. Furthermore, the literature review focused on works that are relevant to the justification, design, and theoretical framework of the study.

Influences on the Use of Mistakes

Learning from mistakes in any context is not a new idea, even in the mathematics classroom. For example, Gattegno (1954) stated:

> It is man's privilege to make mistakes; only through experience, experience that is often painful, does man learn and acquire some degree of wisdom. In the teaching of mathematics, the opportunity for gaining true understanding through experience is too often reduced to the minimum. (p. 11)

With behavioristic learning paradigms dominating mathematics education for decades (see Lambdin & Walcott, 2007), mathematical mistakes were avoided in favor of correct answers to prevent inhibiting the recall of learning procedures (e.g., Ayers & Reder, 1998). In contrast, contemporary mathematics education literature advocates learning from mathematical mistakes and using mathematical mistakes as starting points for classroom investigations and discussions (e.g., AMTE, 2017; CBMS, 2012; NCTM, 2000, 2014; NRC, 2001).

Additionally, empirical research indicates that mistakes initiate explanation and reflection processes in which learners compare and contrast competing concepts and thus have the opportunity to restructure mental models (e.g., Oser & Spychiger, 2005).

However, explaining and reflecting on mistakes does not guarantee a change in mental models. There are a host of mediating factors that can influence how a learner will act on experiences involving mistakes (Tulis, Steuer, & Dresel, 2016) including: the metacognitive activities that follow mistakes; motivation; implicit theory; attitude; the conception of the mistake; affect; and the teacher's attitude and beliefs concerning mistakes (cf. Demirdag, 2015; Matteucci, Corazza, & Santagata, 2015; Rybowiak, Garst, Frese, & Batinic, 1999). Other research has investigated the climate of the classroom and found it to be a determinant of learning (Fraser, 1989), one that can maximize the benefits of the previously mentioned factors (Steuer et al., 2013; Steuer & Dresel, 2015) and be an antecedent to these factors (Grassinger & Dresel, 2017). Particularly relevant to this study was the classroom error climate, conceptualized by Steuer et al. (2013). The following section will elaborate on the theoretical basis and empirical results of the classroom error climate.

Classroom Error Climate

This section will describe the background of the classroom error climate as well as provide empirical research supporting the use of the classroom error climate framework. The classroom error climate was a foundational element of the theoretical framework of this study.

Earlier descriptions of error climate. In education, research on the effects of the classroom error climate go back as far as the early 20th century (e.g., Wrightstone, 1933, 1951), with the classroom climate, in general, being a measure of the social climate of the classroom and a collective subjective measure of the classroom (Anderson, Hamilton, & Hattie, 2004). Researchers have since worked on ways to measure the classroom climate, accounting for cognitive, affective, and motivational factors in students (e.g., Anderson et al., 2004; Cheng, 1994; Church, Elliot, & Gable, 2001; Dweck & Leggett, 1988; Fraser, 1989). Other research, completed by Dresel and Ziegler (2002) and Tulis, Grassinger, and Dresel (2011), proposed a two-component model for the classroom climate that included one component accounting for affective and motivational factors and another component accounting for cognitive factors. Additionally, they stated that these components were interrelated, and influences on affective and motivational factors impacted cognitive activities as they were prerequisites for cognitive activities. Although these models attempted to capture the climate of the classroom, these studies instead focused on attributes of the students. Furthermore, no research had been conducted on the climate of the classroom as perceived by the students.

The term *error climate* originated in organizational settings and was labeled *error culture*. Much of the research concerning error culture stems from Cannon and Edmonson (2001) and Van Dyke, Frese, Baer, and Sonnentag (2005), where they examined the prevalent and exclusive use of mistake prevention in organizations. They "argue[d] that the exclusive emphasis on error prevention has its limits" (Van Dyke et al., p. 1229). Their research was based on the idea that mistakes are unavoidable in organizational settings and therefore should be utilized in some way. Additionally, they argued that mistake avoidance is in many ways in direct contradiction to organizational progress (e.g., experimentation). Their study of over 350 participants at 65 organizations found that routinely communicating about mistakes, sharing mistake knowledge, helping with mistakes, and coordinating mistake handling efforts (i.e., demonstrated positive error cultures) contributed significantly to firm performance and survivability. Van Dyke et al.'s (2005) study has been validated and replicated in several studies (see Keith & Frese, 2008). The significance and importance of error climate was established in organizational settings, but there exists research concerning error climate in educational settings, which is examined in the next section.

Conceptualizing error climate. The research that has been conducted on error climate in educational settings was started by Oser and Spychiger (2005) when they introduced the term *negative knowledge*, defined as:

> [Knowledge concerning] what something is not, (in contrast to what it is), and how something does not work, (in contrast to how it works), which strategies do not lead to the solution of complex problems (in contrast to those, that do so) and why certain connections do not add up (in contrast to why they add up). (p. 26)

In short, this was knowledge about what does not work, which is acquired by making mistakes (Oser & Spychiger, 2005). Further, Oser and Spychiger (2005) stated that through a culmination of experiences of making mistakes students develop "mental immune systems" (p. 42) that serve as a warning sign to guide subsequent actions and learning. However, development of this immune system is not done implicitly. Teacher actions foster these systems positively or negatively. For the latter, Oser and Spychiger (2005) introduced the phrase *Bermuda triangle of error correction,* referring to a common teacher practice that promotes these immune systems. This consists of another student being asked to answer the question correctly or correct the wrong answer of a first student who responded incorrectly. The first student is then left behind without having the chance to think about and reconsider their mistake. Teacher actions and activities that promote metacognition of mistakes is needed in order to foster this system to allow students to utilize mathematical mistakes as more than instances to be corrected.

Although organizational and educational researchers (cf. Oser & Spychiger, 2005; Van Dyke et al., 2005) showed that a positive error climate fosters "affective, motivational, cognitive, and behavioral reactions to errors" (Steuer et al., 2013, p. 196) and thus learning from mistakes, Steuer et al. (2013) noted that there was no consensus on a conceptualization of error climates in learning environments, and thus no consensus on the effect of the error climate. This led to their work on the perceived error climate of the classroom.

Building on research that accounted for the affective, motivational, and cognitive factors and research that accounted for the contextual factors that influence learning from mistakes, Steuer et al.'s (2013) work focused on a way to conceptualize the error climate of the classroom. They developed an instrument with eight sub-dimensions of the perceived error climate: error tolerance by the teacher, irrelevance of errors for assessment, teacher support following errors, absence of negative teacher reactions, absence of negative classroom reactions, taking the error risk, analysis of errors, and functionality for learning. Their initial study that used the instrument was conducted in 56 mathematics classrooms with 1,116 German students in sixth and seventh grades. Additionally, they surveyed the students concerning *affective adaptivity*, the perceived classroom goal structure, personal motivation, and cognitive engagement. Steuer et al. (2013) expected that the effects of the error climate were mediated through students' reactions to mistakes (see Figure 1). From Steuer et al.'s (2013) results, there were three major findings relevant to this study. First, the results showed that perceived classroom error climate across classes were highly varied, although details on the statistics of the individual classes were not provided. Second, the perceptions of the classroom goal structure and the perceived classroom climate were found to be interrelated but distinguishable. However, Steuer et al. (2013) argued the perceived classroom goal structure includes activities and interactions that are not related to mistakes in any way. Third, the handling of mistakes by students was dependent on both personal achievement motivation (i.e., implicit theory) and the perceived classroom error climate. These predictors significantly predicted students' handling of mistakes over the perceived classroom goal structure when accounting for students' personal achievement motivation. Finally, the results from this study offer researchers a way to "conceptualize and analyze the effects of the error climate in classrooms in a holistic" manner (Steuer et al., 2013, p. 208).

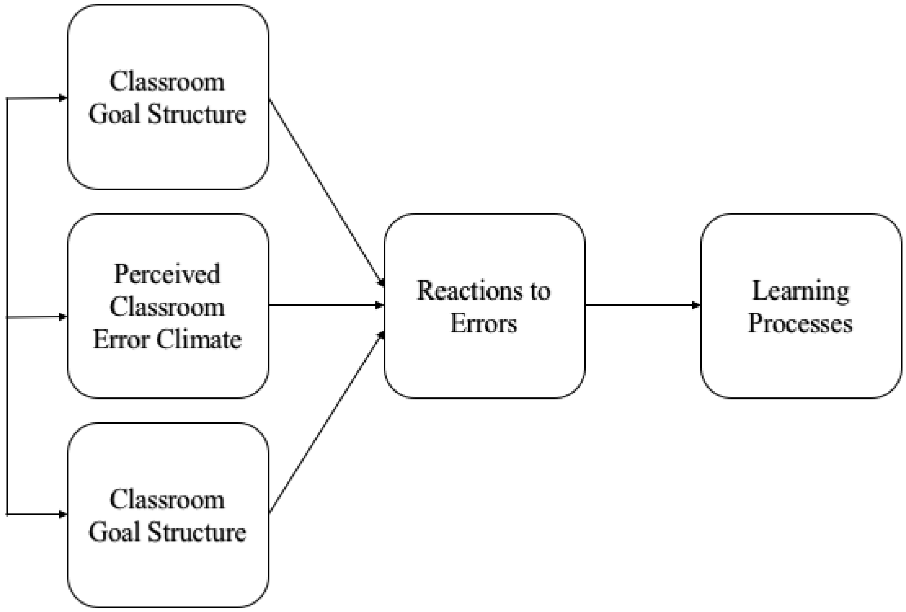


Figure 1. Expected effects of perceived classroom error climate. Expected effects of perceived classroom error climate. Adapted from “Dealing with errors in mathematics classrooms: Structure and relevance of perceived error climate,” by G. Steuer, G. Rosentritt-Brunn, & M. Dresel, 2013, *Contemporary Educational Psychology*, *38*, 196-210.

Empirical studies using classroom error climate. Following Steuer et al.’s (2013) study, other studies utilized the classroom error climate framework. First, Steuer and Dresel (2015) confirmed the findings from Steuer et al (2013). Their study included 1,525 students in 90 classrooms across three grade levels (i.e., seventh, eighth, and ninth grades). Steuer and Dresel (2015) also utilized this study to investigate the relationship between the classroom climate and student achievement. Student achievement was measured using an independent test, not associated with the students’ teacher evaluations, the authors described as “oriented on the scholastic curriculum of grades seven, eight, or nine” (Steuer & Dresel, 2015, p. 267). The researchers found significant positive correlations between student achievement and several perceived error climate dimensions on both the student and class level (see Table 2). This indicated that, for example, classrooms with a more positive error climate superordinate dimension scored higher on the achievement measure. Although the significant correlations were small in some cases (e.g., error tolerance by the teacher on the student level), the significant correlations on the classroom level were unexpected as the error climate only attended to how a mathematical mistake was handled in the classroom. The authors did not claim that a positive error climate led to higher achievement. However, the authors stated that the relationship between the influence of the classroom error climate and achievement was bidirectional. Although this focus on achievement is outside the scope of this study, this study illustrated the importance that researchers ascribe to the classroom error climate.

Table 2

Correlations Between Error Climate Dimensions and Student Achievement

Error Climate Dimension	Student Level[a]	Classroom Level[b]
Error Tolerance by Teacher	.06*	.11
Irrelevance of Errors for Assessment	.03	.15
Teacher Support Following Errors	.06*	.08
Absence of Negative Teacher Reactions to Errors	.01	.00
Absence of Negative Classroom Reactions to Errors	-.01	.17
Error-taking Risk	.07*	-.05
Analysis of Errors	.03	.23*
Functionality of Errors for Learning	.07*	.18*
Superordinate Score	.07*	.18*

[a]N = 1,525. [b]N = 90.
*$p < .05$

Tulis et al. (2016) emphasized the importance of the classroom error climate by incorporating it into their conception of a *process model of reactions and learning from mistakes* in their review of literature concerning *learning from errors*. In the process model, the classroom error climate was pivotal and was included in the *Person X Situation* part of the model. This part of the model encompassed the conditions of the classroom in which an activity or task took place and included the person (i.e., the learner) and the situation (i.e., the task and the classroom environment). Individual traits were included for the leaner (e.g., beliefs, prior knowledge), but the classroom error climate was conceptualized as the mediator for learners' direct reactions to mistakes, their affect and motivation following an error, the learning processes that occurred as a result of the error, and ultimately the learning from mistakes. Furthermore, their review indicated that classrooms with perceived positive error climates are more likely to have learning opportunities directly related to mistakes. Finally, the authors stated that the classroom error climate enabled the examination of contextual conditions on "error-related learning processes" (p. 22).

Grassinger and Dresel's (2017) study had similar findings to Steuer et al. (2013) and Steuer and Dresel (2015) when investigating the experiences that mathematics learners had after receiving a test back. Their study of 479 ninth-grade students verified that after students were given back a mathematics exam, their immediate reactions were aligned with Steuer et al.'s (2013) and Steuer and Dresel's (2015) expectations. That is, students that were in positive error climate classrooms had a significantly higher probability of showing affective and motivational adaptivity to their error(s) compared to students that were not. However, Grassinger and Dresel's (2017) study focused on students immediately following a multitude of mistakes on an exam as opposed to a culmination of experiences in a class. Additionally, the participants in this study were all enrolled in college preparatory high schools, whereas Steuer et al.'s (2013) and Steuer and Dresel's (2015) included students in middle grades where college preparation was not yet a focus. These studies, although few, illustrate the importance of the error climate framework when investigating other aspects of the mathematics classroom.

Significance of the Research

The research reviewed in this section supported the study in three ways. First, it provided insight into the complexity of factors that mediate learners' responses and reactions to mistakes (see Anderson et al., 2004; Cheng, 1994; Church et al., 2001; Dweck & Leggett, 1988; Fraser, 1989). Second, it explicated the importance of the classroom climate on those mediating factors and thus on learners. Finally, it helped to situate the findings of this study with its use of classroom error climate in other literature (e.g., Steuer & Dresel, 2015; Steuer et al., 2013; Tulis et al., 2016)

Beliefs

Research concerning mathematical beliefs has grown significantly in previous decades. That growth brought convolution to the definition of the term as educational researchers conceptualized beliefs differently (Pajares, 1992; Philipp, 2007; Thompson, 1992). There is consensus among researchers on at least two aspects concerning beliefs. Specifically, the literature suggested that researchers agreed there are beliefs that are difficult to change and some that are not (Green, 1971; Pajares, 1992; Philipp, 2007; Thompson, 1992). The underlying explanation for why some beliefs are harder to change than others was a derivative of the conceptualization used for beliefs. The other aspect on which there was consensus was the importance of understanding beliefs because of the influence beliefs have on practice (Cobb, Wood, & Yackel, 1990; Pajares, 1992; Philipp, 2007; Thompson, 1992). Therefore, the purpose of this section is to first provide an overview of why understanding teachers', both ISTs' and PTs', beliefs is important. Second, it will provide a description of the different ways the term *belief* has been used in the literature in an effort to provide clarity to how this study used the term. This will include a description of different conceptualizations available in the literature to again provide clarity as to how this study will conceptualize beliefs and thus how beliefs systems can change.

Importance of Beliefs in Mathematics

The importance of teachers’ beliefs is linked to the influence that they have on teachers’ practices and decisions that they make in the classroom (Cooney et al., 1998; Pajares, 1992; Philipp, 2007; Thompson, 1992). As Pajares (1992) stated, “Few would argue that the beliefs teachers hold influence their perceptions and judgments, which, in turn, affect their behavior in the classroom” (p. 307). Nevertheless, the influence of beliefs extends to teachers’ conception of knowing and understanding mathematics, which, therein, impacts their teaching (Cooney et al., 1998; Pajares, 1992). These conceptions come from a collection of experiences that teachers have throughout primary and secondary schooling as well as in their teacher preparation programs (Pajares, 1992; Thompson, 1992). As a result, the beliefs held by ISTs influence students’ beliefs and create a continuous cycle. The extent to which beliefs impact practices, however, has yielded mixed results (Thompson, 1992). This is expected due to the complexity of beliefs, inconsistent use of beliefs (Philipp, 2007), and how “messy” (Pajares, 1992, p. 329) the literature on beliefs appears. The next section will examine different definitions of beliefs and provide a description as to how this study operationalized beliefs.

Operationalizing Beliefs

Beliefs are defined differently throughout mathematics education literature, including how beliefs are related to other constructs (cf. Philipp, 2007; Thompson, 1992). Pajares (1992) stated that “it will not be possible for researchers to come to grips with teachers' beliefs, however, without first deciding what they wish belief to mean and how this meaning will differ from that of similar constructs” (p. 308). Therefore, this section will focus on definitions of beliefs, factors related to beliefs, and the concept of beliefs as a system.

Definitions of beliefs. The word *beliefs* has attracted a myriad of definitions. In Pajares’s (1992) and Philipp’s (2007) reviews of beliefs, they offered comprehensive reviews of ways the term had been used. To offer a few, definitions of beliefs included: “mental constructions of experience” (Pajares, 1992, p. 313), dispositions to actions (Rokeach, 1968), an individual’s representation of reality, and reasonably explicit propositions. Additionally, Dewey (1933) defined beliefs as “something beyond itself by which its value is tested; it makes an assertion about some matter of fact or some principle or law” (p. 6). Further, he stated:

> [Beliefs cover] all the matters of which we have no sure knowledge and yet which we are sufficiently confident of to act upon and also the matters that we now accept as certainly true, as knowledge, but which nevertheless may be questioned in the future. (p. 6)

To distinguish beliefs from values, Philipp (2007) offered two competing views. First, beliefs were stated to be true or false statements about an object, whereas values are the favorable or unfavorable views about the object that are context dependent. Alternatively, values are a subset of beliefs or, otherwise stated, “values are enduring beliefs” (Philipp, 2007, p. 266). Although these definitions appeared to create an extremely complex view of beliefs, Philipp (2007) simplified this to some degree stating that values and beliefs were used interchangeably throughout the literature. Finally, Thompson (1992) operationalized beliefs as a subset of conceptions when defining conceptions as "more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130). Although these are not the only definitions used for beliefs in the literature, they provide insight into the inconsistency with which the term *beliefs* has been used.

Although there is disagreement on the precise definition of beliefs, there is more disparity in differentiating beliefs from knowledge (Pajares, 1992; Philipp, 2007). Pajares's (1992) contrasted beliefs with knowledge in two ways. First, beliefs and knowledge are different in affective and evaluative components. Beliefs were stated to either have a more affective component (e.g., feelings) than knowledge (see McLeod, 1992; Nespor, 1987) or a more evaluative component (i.e., based on judgement) than knowledge (see Nisbett & Ross, 1980). Second, how knowledge and beliefs were stored differentiated the two. Where knowledge was stored semantically, beliefs were stored episodically. Philipp (2007) also delineated beliefs and knowledge with knowledge's association with what is true. Summarizing his findings, he stated:

> If one takes the ontological view that truth exists and people have access to it, knowledge might be viewed as true belief. If one takes a view that truth, though it may exist, is not accessible to humans and instead the best one can hope for is viability, then knowledge is belief with certainty. (p. 268)

Leatham (2006) offered a more simplistic and understandable view of the relationship between beliefs and knowledge:

> Of all the things we believe, there are some things that we "just believe" and other things that we "more than believe – we know." Those things we "more than

> believe" we refer to as knowledge and those things we "just believe" we refer to as beliefs. (p. 92)

It is in this definition that both knowledge and beliefs are "complementary subsets" (Leatham, 2006, p. 92) of things that are believed. Additionally, this definition of beliefs aligned with Rokeach's (1968) definition of beliefs as dispositions to actions, and, although this definition of beliefs does not assume an objective truth exists, it incorporated the use of evidence of truths.

Beliefs as a system. To further conceptualize beliefs, belief systems have been used, which are especially useful "for examining and describing how an individual's beliefs are organized" (Thompson, 1992, p. 130). Belief systems are viewed as systems that contain all beliefs (Pajares, 1992) and are adaptive (Green, 1971; Leatham, 2006; Pajares, 1992). Much of the work on belief systems stems from Green's (1971) construct of belief systems. This included three dimensions of beliefs. The first was "psychological strength" (Green, 1971, p. 47) of a belief. This described the importance of a belief that can range from central to peripheral. Beliefs that are related to self-identity would be central, where beliefs that are "matters of taste" (p. 5) are peripheral. The more central a belief is, the harder it is to change. Additionally, the strength of a belief depends on how that belief coheres with the rest of the beliefs system (Leatham, 2006). These central beliefs are beliefs that had greater influence than other beliefs. The second dimension of the belief system was the "quasi-logical" (p. 44) relationship of beliefs. This consisted of the organization of beliefs in relation to another. This included a belief being a consequent of another belief. Green (1971) described these as the primary belief and the derivative belief. The third and final dimension was the clustering of beliefs and the extent to which a cluster is in isolation to other clusters. This final dimension is useful for explaining contextual variables in which a belief may be held in one instance or context but not in another. Additionally, this is useful for explaining why an observer may find contradictions in beliefs, especially in PTs' beliefs as they relate to practice (Leatham, 2006).

Leatham (2006) further defined beliefs as a sensible system, adopting Green's (1971) belief system and expanding it to include how a teacher makes sense of new experiences and information. An especially important note concerning sensible systems is that it described a beliefs system that offers no contradictions in beliefs. As Leatham (2007) stated, "Beliefs become viable for an individual when they make sense with respect to that individual's other beliefs. This viability via sense making assumes the desirability for the individual of an internally consistent organization of beliefs" (p. 187). This is especially true to the outside observer who might consider a PTs' actions inconsistent with assumed beliefs, if not considering Green's (1971) final dimension of how clusters of beliefs are in isolation to each other.

Changing beliefs systems. The process of changing beliefs in a beliefs systems consists of a person comparing new experiences and the beliefs held by others with current beliefs creating a continuous cycle of evaluating beliefs and making small changes in the beliefs system (Green, 1971; Stuart & Thurlow, 2000). If a new belief is adopted, then the belief will be situated within the beliefs system with a degree of psychological strength, a relationship with other beliefs, and in different clusters of beliefs. If a belief is not challenged and subsequently changed by experiences, then the belief would be reinforced and may increase in psychological strength because of its endurance through experiences. This process is illustrated in Figure 2. Two mechanisms for change in this system are offered by Kagan (1992) and Swars, Smith, Smith, and Hart (2009). These two mechanisms for change include offering PTs the opportunity to reflect on experiences grounded in the work of teaching and providing experiences that challenge current belief structures. However, Kagan (1992) and Swars et al. (2009) stated that these experiences could not be separated from the contexts in which these experiences took place. Therefore, per Figure 2, newly formed beliefs are subject to change with future experiences if not reinforced.

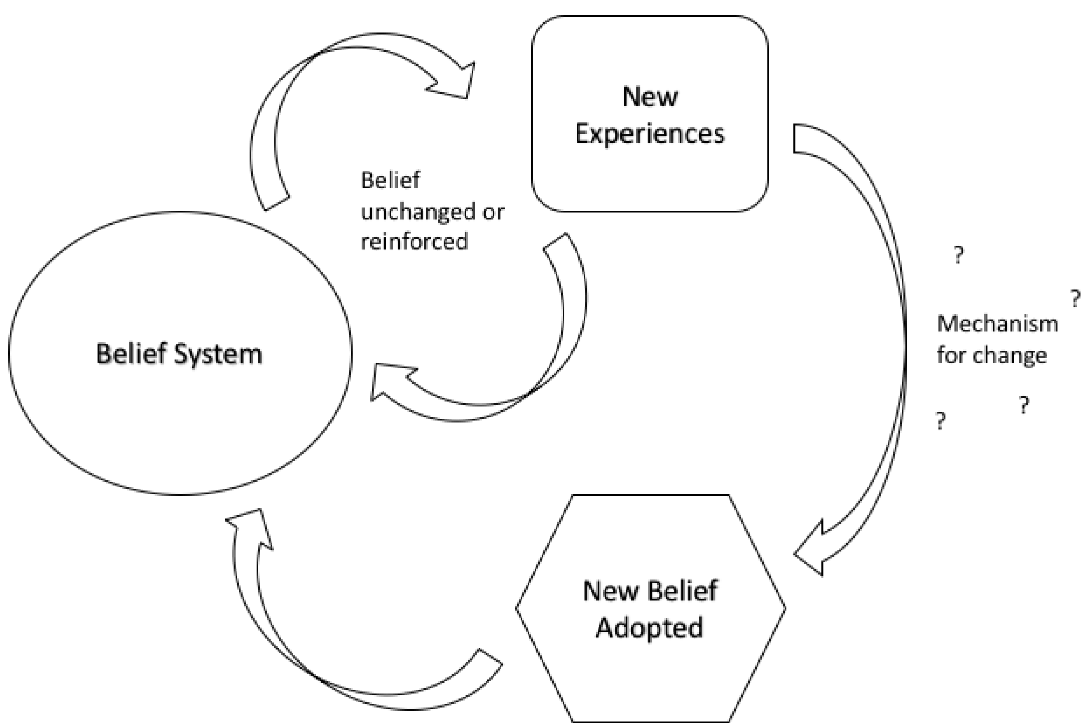


Figure 2. Process of beliefs changing. Adapted from “An Exploration of Pre-service Elementary Teachers’ Mathematical Beliefs” (Doctoral Dissertation) by L. N. Shilling, 2010, Retrieved from ProQuest. (3435525)

Beliefs concerning mathematical mistakes. Pajares (1992) concluded that PTs' beliefs are established though a collection of experiences in learning environments and are resistant to change, citing Lortie's (1975) apprenticeship of observation. However, Pajares (1992) also concluded through his review of beliefs literature that PTs' beliefs can be changed through time and experience, which provides the opportunity for PTs to accept or reject new beliefs and eventually modify their existing beliefs system. Philipp (2007) echoed this idea but added that research demonstrates that PTs can modify beliefs when observing learners. Green's (1971) work suggested that PTs' beliefs would change depending on how central a belief was, how it was connected to other beliefs, and how that cluster of beliefs was isolated from others. Leatham's (2006) work added to Green's (1971) in stating that beliefs would change inasmuch as it made sense to fit into the existing beliefs system.

If the goal is to change PTs' beliefs, this necessitates knowing what those beliefs are. There is a wide array of research concerning PTs' beliefs (e.g., Cooney et al., 1998; Leatham, 2007; Pajares, 1992; Philipp, 2007). Specifically, studies have investigated PTs' beliefs towards mathematics (e.g., Anderson, 2007), beliefs about technology (see Philipp, 2007), beliefs related to gender (see Philipp, 2007), and beliefs about teaching mathematics (e.g., Stuart & Thurlow, 2000).

However, there is little research on beliefs concerning mathematical mistakes. The few studies have focused on ISTs' reactions to mistakes and even fewer have focused on ISTs' beliefs concerning those mistakes. For example, in comparing U.S. teachers' reactions to mathematical mistakes to Italian teachers' reactions to mathematical mistakes, Santagata (2005) extended research on the Third International

Mathematics and Science Study ([TIMSS] Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). She reviewed 30 randomly selected lessons from U.S. teachers and 30 randomly selected lessons from Italian teachers from TIMSS. She found that U.S. teachers demonstrated four major types of reactions to an error: ignoring it, directly solving it, returning the correction back to the student, or redirecting the mistake to another student. These reactions were stated to have occurred in an effort to save students' self-esteem. Additionally, U.S. teachers spent significantly more time having conversations about mistakes with students individually and in private compared to Italian teachers. Tulis (2013) had similar results when she compared the mistakes-handling practices of mathematics classrooms in German elementary and high school classrooms at "Gymnasium" (p. 59) schools, which are school types with the highest level and academic demands. She found in her observations of 16 mathematics classrooms that teachers rarely embarrassed students for making mathematical mistakes. However, they also rarely supported students in taking risks associated with mathematical mistakes or emphasized mathematical mistakes as learning opportunities, often focusing students' effort on the correct answer. Tulis (2013) stated that her results "replicate[d] the empirical findings of Santagata (2005)" (p. 60).

Another example was a study conducted by Bray (2011). This was a multiple-case study of four elementary ISTs from an urban school in a U.S. southeastern state. The school in which the four ISTs were teachers was transitioning to a reform-based curricula as a result of district mandates, and the school hired mathematics educators, including Bray, to provide a year-long PD to assist in the transitioning. By the end of the study, Bray found that three of the four ISTs demonstrated significant shifts in beliefs towards reform-oriented practice. Related to beliefs concerning errors, Bray concluded that there were three dimensions of teacher practices related to how mistakes were treated with their classrooms. The first dimension was the extent to which a teacher made mistakes a focus of whole-class discussions. Bray found that the extent to which this occurred was connected to the teachers' beliefs that the mistake would potentially embarrass the student or that mistakes were opportunities for learning for both the individual and the class. The second dimension was the extent to which the teachers' responses promoted conceptual understanding. This dimension was connected to the belief of whether mathematical concepts were more powerful than remembering procedures. The third and final dimension was the extent to which teachers "mobilize students as a community of learners" (p. 29) when mistakes occurred during discussions. This was connected to the extent to which teachers believed students were limited in how they could support each other's learning and the extent to which they believed the purpose of discussions was to show correct solutions.

In Santagata and Bray's (2016) study, they investigated the changes in four elementary ISTs' practices during a year-long PD. During the PD, the ISTs were exposed to mathematics education literature concerning mistake-handling practices. Additionally, they watched videos of teachers handling mistakes from different countries in their classrooms, presumably from TIMSS, and discussed the implications of teacher moves that occurred in the video. Furthermore, the ISTs worked together with Santagata on strategies they could implement in their classrooms that would promote the use of students' mathematical mistakes as tools for learning. The authors found that overall the teachers designed instruction to allow more mistakes to surface and increase attention given to mistakes. Additionally, the ISTs attempted to respond to mistakes in ways that supported student attention to underlying mathematical concepts and increased the opportunities for students to work with peers to examine mathematical ideas. However, two of the ISTs often limited time spent on mathematical mistakes unless a student's mistake surfaced unintentionally. Additionally, these two ISTs often times focused the students' attention on getting the correct answer and procedural methods of doing so.

The previously described studies provided foundational literature concerning ISTs and their utilization of mathematical mistakes. This included Bray's (2011) study which included the ISTs' beliefs about mathematical mistakes. However, there is a dearth of literature addressing PSTs' beliefs concerning mathematical mistakes.

Significance of the Research

No matter the exact definition used, beliefs serve as guiding forces (Cooney et al., 1998) and lenses through which PTs will interpret (Pajares, 1992) interactions with students, including students' mistakes, and therefore the use of them (Leatham, 2006). As beliefs can be held in varying degrees of psychological strength (i.e., central to peripheral) and inconsistent in different contexts (Leatham, 2006), it is crucial to gain understanding of belief structures in order to better understand how those beliefs influence practices (Jensen, 1993; Pajares, 1992; Philipp, 2007; Thompson, 1992). The previous sections concerning beliefs supported this research in providing the literature in which to situate this study's use of beliefs as well as provided background on PTs' beliefs. This literature was used to help expound on the conceptual framework for the study.

Teacher Education

The following sections will review literature related to teacher education and the influence of teacher educator programs on PTs' classrooms. Specifically, a general overview of teacher preparation programs is provided, followed by a review of how PTs' experiences in their teacher preparation courses impact their beliefs and ultimately their teaching practices.

General Overview of Teacher Preparation

Teacher educators are tasked with supporting PTs in becoming educators in a time in mathematics education where standards and expectations are higher than ever (e.g., AMTE, 2017; CCSSI, 2010; NCTM, 2000; NRC, 2001). CCSSI (2010) stressed that students should understand the mathematics that they are doing at every level, including why their mathematical statements are true and from where a mathematical rule stems. Additionally, teachers should be able to appropriately create, select, and modify tasks in order to support students' learning as well as "understand the mathematical consequences of different choices of numbers, manipulative tools, or problem contexts" (CBSM, 2010, p. 2) when using those tasks. If students are expected to have deep conceptual understanding of topics, then it is evident that teachers must have an even deeper level of understanding of the mathematics that they are teaching (CBMS, 2010) as well as have the instructional tools to convey, support, and challenge their students in mathematical topics. According to Schmidt (2012), teachers must deliver content, using their mathematics and pedagogical knowledge to present content to their students in a way that enables meaningful learning experiences. It is the teacher's responsibility to create an instructional experience in which student learning takes place. Thus, teachers not only need to be trained with mathematical content but also the methods necessary to deliver the content in a manner in which their students can learn, understand, and build mathematical content.

Teacher educators are responsible for educating and teaching PTs in the spirit of these standards so PTs are able to implement them in their future classrooms, where many of these mathematical experiences are in contrast to the experiences that the PTs had in their K-12 classrooms (Bahr et al., 2014; Barlow & Reddish, 2006). CBMS (2010) suggested that teacher preparation courses should be structured so that future teachers of mathematics have time to think about, discuss, and explain mathematical ideas in order to make sense of the mathematics they will eventually teach, which is aligned with how PTs are expected to teach in their classrooms. This is especially true for elementary PTs, as many enter college “with only a superficial knowledge of K-12 mathematics, including the mathematics they intend to teach” (CBSM, 2010, p. 4).

Content Courses’ Impact on PTs’ Teacher Identity

Although methods courses in teacher preparation programs have been found to influence PTs’ beliefs on the teaching of mathematics, studies show that there is a lack of transferability to teaching practices in the classroom (e.g., Goos, 2005; Raymond, 1997; Vacc & Bright, 1999). This is attributed to beliefs about mathematics content being more closely linked to instructional practice than beliefs about mathematics pedagogy (Raymond, 1997). Additionally, productive belief shifts are often diminished by repeated exposure to teaching of mathematics content with teacher-centered practices (Cady, Meier, & Lubinski, 2006; Remillard & Bryans, 2004), which is how PTs develop “what constitutes an appropriate role for the teacher of mathematics” (Wilson & Cooney, 2002,

p. 142). Therefore, one necessary avenue in which PTs' beliefs can be impacted in teacher preparation programs is through content courses (Mizell & Kates, 2004).

For elementary PTs, who often enter teacher preparation with "superficial knowledge" (CBSM, 2010, p. 4) of the mathematics they intend to teach, content courses can be a venue to influence PTs' beliefs about mathematics and to develop their understanding of the mathematics content knowledge needed to teach at the elementary level. With deeper understandings of the mathematics they will teach, elementary PTs will be better able to understand and embrace pedagogical beliefs that are more productive (Swars et al., 2009).

As PTs develop their teaching practice in their teacher preparation courses, they bring with them a set of ideas comprising the characteristics of a good teacher, shaped by prior experiences including those in K-12. Lortie (1975) referred to this as their *apprenticeship of observation,* which consists of beliefs of what it means to be a teacher as well as best practices of a teacher. PTs' identities have been constructed around their apprenticeship of observation and about what they consider the practice of teaching to be. However, their identities are in constant reconstruction with new experiences (Gee, 2001; Ketelaar, Beijaard, Boshuizen, & Den Brok, 2012).

Content courses in teacher preparation programs play a pivotal role in PTs' reconstruction of what the practice of teaching looks like as PTs encounter tasks, class discussions, and deeper examinations of content (Winter, Lemons, Bookman, & Hoese, 2001). Moreover, PTs begin to experience themselves in relation to the practice of mathematics teaching. Whereas PTs begin by viewing themselves as outsiders to the practice of teaching and only see themselves as students, their participation in classes where they are positioned as the teacher and consider pedagogical choices begins to shift their identity to that of a teacher as they develop the skills, knowledge, and dispositions of teachers (Coldron & Smith, 1999; Ketelaar et al., 2012). However, beginning to view themselves as teachers is a “radical departure from [their] long-term identity as a student” (Gormally, 2016, p. 176) and takes significant time and number of experiences focused on this transition. Additionally, their teacher identity is not a consequence of knowledge or experience. A PT’s teacher identity is, in part, “socially given” (Coldron & Smith, 1999, p. 714) by peers, family members, and teachers.

Prospective teachers’ views of what a mathematics teacher is supposed be and do is a culmination of life experiences in mathematics classes. By the time that PTs get to their teacher preparation courses, their views of what a teacher does and how they handle interactions with students, including mathematical mistakes, are well established. However, PTs continuously reform those views, and previous research demonstrates that content courses play a significant role in that process (e.g., Gee, 2001; Ketelaar, Beijaard, Boshuizen, & Den Brok, 2012; Winter, Lemons, Bookman, & Hoese, 2001). In the process of forming and re-forming their views of what a teacher is, PTs also form their own teacher identities. That is, PTs form dispositions as to how they will handle mathematical mistakes, even if they do not see themselves as the teacher at that moment (Coldron & Smith, 1999; Gormally, 2016)

Significance of Research

This section focused on the teacher preparation program’s role in supporting PTs in their transition to becoming practicing teachers. Specifically, two salient features of research presented are significant to this study. First, in order for PTs to be successful in shifting of instructional methods, their beliefs must become the targets of change in content courses. Second, teacher preparation courses, especially content courses, play a significant role in shaping how PTs view themselves as teachers.

Implicit Theories

Educational researchers have stated that beliefs shape the way students and teachers see and interact with the world around them (Dweck et al., 1995). One set of beliefs, implicit theories or beliefs about intelligence and ability (Dweck et al., 1995), have been the focus of educational researchers as a predictor of observable behaviors of students and teachers as well as the classroom culture (Boaler, 2016). The following sections will provide a background on the research concerning implicit theories pertinent to this study, which will be followed by the significance of the research to this study.

Research on Implicit Theories

Dweck's early research on implicit theories (Dweck & Leggett, 1988) was a series of studies that sought to answer why some students responded differently to failure than others. Dweck and Leggett (1988) administered a survey to fifth- and sixth-grade students in an attempt to predict students' persistence when faced with challenges. Students were then categorized as either mastery-oriented or "helpless" (p. 256). Students that were considered mastery-oriented were students that did not give up after initial failure or even "think they were failing" (p. 258) when faced with challenges. The mastery-oriented students viewed "unsolved problems to be mastered through effort" (p. 258). Additionally, the researchers saw increased effort in problem-solving strategies when faced with failure. In contrast, helpless students viewed their difficulties as failures and "insurmountable" (p. 258). Helpless students, or those considered to be performance-oriented, were also observed demonstrating decreased effort in problem solving strategies when faced with difficulty, attributing their struggles to a lack of intelligence, and often gave up when faced with challenges. Since this study, other studies used the term *incremental theorist* to describe those that were master-oriented and *entity theorist* to describe those that were performance-oriented.

Robins and Pals's (2002) longitudinal study of 508 California undergraduate students investigated implicit theories over a four-year period following the undergraduate students throughout their college careers. The researchers had two findings pertinent to this study. First, their findings suggested that implicit theories of intelligence are stable. However, they were not able to conclude that the implicit theories were stable over "transition points" (p. 329) (e.g., elementary grades to secondary grades or college to joining the workforce) as they did not have data from the participants during those years. Second, their findings supported the findings of Dweck and Leggett (1988) related to dealing with challenges. Entity theorists continued to demonstrate a "helpless pattern" (p. 329) and gave up when faced with difficulty blaming their failure on their low ability. In contrast, incremental theorists demonstrated determination when faced with difficulty and continued to pursue success.

Other researchers pursued the influence of one's implicit theory. For example, Blackwell, Trzesniewski, and Dweck (2007) investigated the implicit theory of seventh-grade mathematics students that were considered to be low achieving. In their study, the seventh graders were placed into an experimental and a control group. Both groups were exposed to different study skills and key concepts of how the brain worked when learning. However, the experimental group was presented with the idea that intelligence is like a muscle and “malleable” (p. 254), and that the brain grew through exercise and practice as opposed to lessons provided to the control group on memory and mnemonic devices. The researchers found that students with an incremental theory benefitted the most from the mindset intervention, and “their declining grade trajectory [was] reversed following the intervention, while the grades of students in the control group who endorsed more of an entity theory continued to decline” (p. 258). Other studies’ results supported the findings that a students’ implicit theory is linked to higher mathematics achievement (see Aronson, Fried, & Good, 2002; Claro, Paunesku, & Dweck, 2016; Good, Aronson, & Inzlicht, 2003; Good, Rattan, & Dweck, 2012).

Research on implicit theory was commonly described as a dichotomy of either being an entity theorist or an incremental theorist. However, other researchers have concluded that implicit theory is a continuum. For example, Gelman, Heyman, and Legare (2007) surveyed 195 children and 187 adults in a series of four studies to examine how different essentialist beliefs interrelate, especially over time. Essentialism is a psychological term used to describe beliefs that a trait “may be relatively stable [and] unchanging” (p. 757). Although the authors did not refer to this as implicit theory, the term *essentialism* has been described in psychological literature as “an extension of implicit theory” (Haslam, Bastian, Bain, & Kashima, 2006). In Gelman et al.’s (2007) series of studies that examined children’s and adult’s *essentialism*, they presented their findings on a continuous spectrum from very essentialized to not-at-all essentialized or entity theorist to incremental theorist. Thomas and Samecka (2015) supported the view of implicit theory as a continuum in their two studies that examined the implicit theories of adults. They found the implicit theories of adults concerning intelligence and environmental effects on the brain were a mixture of both entity and incremental theorist characteristics. The authors found no “evidence for two distinct theories in people’s broader beliefs about intelligence and the brain. Instead we found a continuum of beliefs” (para. 52).

Although implicit theory was established through research on individuals’ self-perception of intelligence (Dweck & Leggett, 1988), which did not include mathematical ability, the authors predicted “implicit beliefs about ability predict whether individuals will be oriented toward developing their ability or toward documenting the adequacy of their ability. As such, these theories may be at the root of adaptive and maladaptive patterns [in other qualities]” (p. 263). Through subsequent research, Dweck, Chiu, and Hong (1995) developed an instrument through which to assess an individual’s implicit theory for intelligence, morality, and worldview, which was used in this study. Generalization of implicit theory to other dimensions was supported, including mathematical ability (Willingham, Barlow, Stephens, Lischka, & Hartland, 2016). Willingham et al.’s (2016) study supplemented Dweck et al.’s (1995) instrument measuring implicit theory by adding a mathematical ability dimension, which was supported by Dweck and Leggett (1988) (Willingham et al., 2016). Additionally, Willingham et al. (2016) created an observational protocol to identify observable actions aligned with either an incremental or an entity theorist (see Appendix I), which was used in this study.

Significance of Research

One’s implicit theory, which exists on a continuum (Gelman et al.’s, 2007; Thomas and Samecka, 2015), is a significant predictor of his actions and reactions when faced with challenges and mistakes. Particularly relevant to this study, implicit theory plays a role in persisting when making mathematical mistakes. The provided research suggests that a PT that is an entity theorist would likely give up when faced with mistakes and disengage with activities involving mistakes. Although students' grades and achievement are beyond the scope of this study, the reviewed studies demonstrate the validity of implicit theory and its impact during research.

Conceptual Framework

The error climate framework (Steuer et al., 2013) and the beliefs as sensible systems framework (Leatham, 2006) were the conceptual framework for the study. The error climate of the classroom has been shown to be correlated with other important factors that influence PTs' experiences with mistakes in the classroom. Steuer et al.'s (2013) error climate framework assumed that although learning from mistakes depended on individual characteristics of learners, the classroom context was a precursor to those. Thus, classroom climate facilitates learning from mistakes and the potential to transcend individual attributes that PTs have. Data collection was structured to consider how the error climate of the PTs' classroom impacts their beliefs concerning mathematical mistakes and was used in conjunction with the beliefs as sensible systems framework in providing a way to interpret experiences with mathematical mistakes in the classroom. Additionally, the error climate provided a framework through which the narratives could be described as it accounted for the environment in which each PT experienced mathematical mistakes.

The second framework that guided the study was Leatham's (2006) beliefs as a sensible system framework. All three components of the sensible system framework (i.e., psychological strength, quasi-logical relationship to other beliefs, and extent of isolation in clustering of beliefs) were used to guide data collection to answer both research questions of the study. Furthermore, I used the beliefs as a sensible system framework to organize the PTs' beliefs concerning mathematical mistakes in the context of their classroom with their perceived error climate. This is especially important as beliefs may hold in one particular context but not another, which might otherwise appear contradictory to an outside observer (Leatham, 2006). Together, these theories formed the conceptual framework used to investigate PTs' beliefs concerning mathematical mistakes.

The literature reviewed in this chapter influenced the study in three ways. First, the literature demonstrated that PTs likely held beliefs concerning mathematical mistakes when they entered the classroom in this study, which have been shaped by prior experiences (Pajares, 1992; Philipp, 2007; Thompson, 1992). However, there was no research that specifically addressed what those beliefs were. Research has demonstrated that PTs' beliefs can change or be restructured through interactions with teacher educators and peers (Pajares 1992; Philipp, 2007; Thompson, 1992). This study provided opportunities for PTs to experience mathematical mistakes and reflect on those experiences, thus providing a context to capture PTs' beliefs concerning mathematical mistakes and any change in those beliefs that might occur.

Second, the research described several avenues for research questions that might have been examined. For example, this study could have considered how mathematical ability impacts the beliefs of PTs concerning mathematical mistakes or how mathematical ability impacted the perceived error climate of the classroom. How PTs' affects, or other attributes, influence their beliefs concerning mathematical mistakes could have been observed. Additionally, the impact of PTs' test results on beliefs concerning mathematical mistakes could have been investigated. However, a focus on PTs' beliefs as a sensible system were investigated in conjunction with their perceived error climate which allowed me to examine any of these that were found to be important within each case or across cases.

Finally, the complex nature of belief systems influenced this study's data sources. As PTs were not necessarily able to articulate their beliefs (Leatham, 2006) concerning mathematical mistakes, how they changed, or how the error climate influenced them, multiple data sources were utilized to provide a variety (Leatham, 2006; Pajares, 1992) of ways to capture PTs' beliefs and the structure of those beliefs. Additionally, these data sources provided varied ways of asking PTs about their beliefs concerning mathematical mistakes (i.e., surveys, reflections, interviews, classroom observations) to allow me to draw inferences with "believability" (Leatham, 2006, p. 93) about those beliefs.

Chapter Summary

This chapter contained a review of the theoretical and empirical literature that supported the study. Additionally, the theoretical framework and connections of literature to this study were included. The next chapter will provide details of the research methodology informed by this literature review.

CHAPTER III: METHODOLOGY

Introduction

Mathematics education reform documents (e.g., CCSSI, 2010; NCTM 2000, 2014) suggest ways of teaching that drastically differ from traditional methods of teaching. These documents include teaching strategies in which teachers leverage and explore students' mathematical mistakes, rather than minimize the mistakes' exposure to seek out correct answers. Research shows that this view of teaching contrasts with the teaching in typical U.S. classrooms (Bahr et al., 2014; Barlow & Reddish, 2006; Philipp, 2007). Significant to this study, these typical classrooms played a substantial role in shaping prospective teachers' (PTs) beliefs of mathematics and the role of mathematical mistakes in the classroom (Cooney et al., 1998; Hart, 2002, 2004; Thompson, 1992). Although literature suggests that in-service teachers' (ISTs) beliefs concerning mathematical mistakes are not aligned with views of teaching held by the mathematics education community (e.g., Matteucci et al., 2015; Santagata, 2005; Schleppenbach, Flevares, Sims, & Perry, 2007), there is a dearth of literature on PTs' beliefs concerning mathematical mistakes. Therefore, the purpose of this study was to investigate PTs' beliefs concerning mathematical mistakes while considering the error climate of the classroom, that is, the way that mistakes were perceived in the classroom from the perspective of the PTs.

This chapter contains details of the research methodology utilized in this study. It provides an overview of the study's design and describes the context of the study. This is followed by a description of participant selection, data sources, and the instruments and procedures used to gather the data. Finally, the processes used for analyzing the data in order to address the study's research questions are presented.

Research Overview

This study utilized an exploratory, multiple case design (Yin, 2014) to investigate PTs' beliefs concerning mathematical mistakes and how those beliefs systems change, if at all. Specifically, the study's research questions were:

1. What beliefs do PTs hold about mathematical mistakes?
2. What are the noticeable changes in PTs' beliefs systems concerning mathematical mistakes, if any, given the classroom context?

There were five features of this study that supported this choice in methodology. First, this study focused on understanding beliefs and how those beliefs potentially change while participants experience a particular classroom environment. The research questions required the observation of participants in a real-life learning environment concerning a contemporary phenomenon, thus the nature of the research questions warranted a case study methodology (Yin, 2014). Second, a case study was appropriate for this study, because the purpose was to investigate the cases in depth and in their contexts (Yin, 2014). Third, an exploratory case study was appropriate, because the purpose of the study was to better understand, or pose new insights, in order to generate new ideas or hypotheses when existing theories were insufficient (Yin, 2003). Fourth, the multiple case design had two well-defined units of analysis (i.e., each participant) with the mathematics course as the context for each case. Using two cases, as opposed to one, provided analytical benefits (e.g., the cross-case analysis). Additionally, two cases provided the opportunity for "more powerful analytic conclusions" (Yin, 2014, p. 64) than a study with only one case. Finally, this study was situated in prior research and theoretical constructs, which helped “to guide the design and data collection” (Yin, 2003, p. 1) of the case study while also providing complementary explanations to findings from the study. These five features justified the use of the multiple case design.

Research Context

The elements of the research context are described in this section: the university, the course in which the participants were enrolled, and the instructor of the course. The following sections will describe the relevant details of each of the elements of the study.

University

The university at which the study took place was in a southeastern state. The university had a diverse population consisting of students from within and out of the state as well as international students. It was a public university with a total enrollment of over 22,000 with 19,693 students enrolled at the undergraduate level. The university was 45% male and 66% Caucasian. The university was accredited by the Commission on Colleges of the Southern Association of College and Schools (SACS) to award baccalaureate, masters, and doctorate degrees.

Course

The course in which the participants were enrolled during the study was at the undergraduate level and was a requirement for all PTs pursuing a degree in elementary education. The course explored elementary geometry concepts and was designed to align with NCTM recommendations. It was a 3-credit hour course and required that enrollees complete their college algebra course and a mathematics content course focusing on number concepts and structure in elementary school mathematics before registering for the course. The purpose of the class as described by the course syllabus was to increase PTs' knowledge of geometry and measurement concepts via a problem-solving context. PTs were required to be active participants in group activities and discussions in every class. Their engagement in class activities was worth 10% of their overall grade.

Instructor

Dr. Heart was the instructor for the course and was a tenure-track assistant professor in her second year as a faculty member at the university. She held a Bachelor of Science in mathematics education, a Bachelor of Arts in mathematics and statistics, a Master of Arts in Teaching, and a Doctor of Philosophy in mathematics education. She had published in numerous NCTM journals as well as books endorsed or supported by NCTM. Additionally, she had presented at NCTM conferences, at both the regional and national level, the Research in Undergraduate Mathematics Education Annual Conference, and AMTE conference on numerous occasions. These publications and presentations included topics ranging from promoting classroom discourse to implementing standards-based instruction.

During the class, Dr. Heart positioned mathematical mistakes at the forefront of many classroom tasks, group activities, and evaluation items (e.g., homework, tests). On the class syllabus, which was discussed on the first day of class, the expectations for grading and evaluation were considered (see Figure 3). No mention of correct answers was included. This is not to say that Dr. Heart did not grade for correctness on homework and exams; however, more emphasis was placed on the PTs' process and thinking. In my classroom observations, I observed repeated instances of PTs in the class asking why their responses were incorrect or different from someone else's, to which Dr. Heart replied by asking the class to examine and critique each strategy, both correct and incorrect ones. In fact, Dr. Heart regularly presented the class with her *favorite mistakes* from homework and exams for the class to examine in their groups. These were often conceptual mistakes that a PT from the class made on an assignment. In the cases where she did not have a favorite mistake, Dr. Heart either manufactured mathematical mistakes or presented an elementary student's mathematical mistake for the class to examine. In one class session, I observed the PTs manufacturing different mathematical mistakes to problems for other PTs in the class to examine and critique. Comparatively, there were other instances in which the PTs were asked to attempt to understand a mathematical mistake and what misunderstanding(s) or misconception(s) the person might have on homework and exams. In addition to the exit tickets and in-class reflections that I provided, which are described in later sections, Dr. Heart used significant class time to allow the PTs to examine and critique mathematical mistakes.

GRADING AND EVALUATION

I expect everyone to complete all course requirements with work reflecting the following traits:

- meticulous preparation
- careful consideration of alternatives
- collegial work
- clear expression, with respect for the place and value of precision
- use and application of mathematical knowledge
- genuine curiosity about all ideas
- analysis and reflectiveness
- organization
- timeliness

Figure 3. Grading and evaluation expectations for Dr. Heart's class.

Participant Selection

According to Yin (2014), the case should “capture the circumstances and conditions of an everyday situation—again because of the lessons it might provide about the social processes related to some theoretical interest” (p. 51). With this study being exploratory, I sought to provide a rich description of PTs who would typically take the previously described course. Additionally, with the overwhelming majority of elementary school teachers being female (U.S. Department of Education, National Center for Education Statistics, 2016), female participants were prioritized for this study.

All PTs in the course previously described made up the potential pool of participants for the study. With a variety of criteria available from which to select participants (e.g., ACT score, mathematical content knowledge), I selected the participants based on the characteristics of different implicit theories (see Dweck & Leggett, 1988) they displayed. The decision to select the participants based on implicit theory was made because of the role that implicit theory plays in persisting when making mathematical mistakes (NCTM, 2014). Specifically, PTs that ascribe to an entity theory are more likely to give up when facing mathematical mistakes and not see value in them, whereas PTs that ascribe to an incremental theory are likely to persist when facing mathematical mistakes and see mathematical mistakes as opportunities for learning. Additionally, implicit theory transcends levels of mathematical ability (NCTM, 2014).

None of the participants were on either extreme of the implicit theory continuum. Choosing participants that were not on either extreme of the continuum was done so purposefully. Had a participant been chosen that displayed strong characteristics of an entity theorist, I was concerned that the participant would not be willing to talk about their mathematical mistakes. If the participant displayed strong characteristics of an incremental theorist, I was concerned that they would be too ideal of a candidate given the research questions of this study, which would contaminate the results of the study.

To select the participants for this study, I invited Dr. Heart's entire class of 21 PTs to participate in the study, 20 of which were female. From the remaining 20 PTs, 11 consented to participate further in the study. I evaluated the remaining PTs' implicit theories using a modified version of Dweck, Chiu, and Hong's (1995) implicit theory survey developed by Willingham et al. (2016). Seven PTs displayed characteristics in the middle of the implicit theory continuum, which narrowed the pool of potential participants. Classroom observations during the first two weeks assisted in selecting the final PTs invited to participate in the remainder of the study. During these observations, I looked for PTs that were attentive to the classroom activities and further displayed characteristics of both incremental and entity theorists. Four PTs, two primary and two alternates, met these criteria, all of which were invited and agreed to participate in the duration of the study on September 10, 2017. Four PTs were initially chosen in the event that a primary participant withdrew from the class or the study, which was the case. One of the primary participants withdrew from the study on September 25, 2017. This resulted in me collecting data from three PTs. For the purposes of this study, however, I analyzed and reported on the data drawn from two PTs, hereafter referred to as the participants.

The two participants on which this study focused were Cindy and Harley, both pseudonyms. They were chosen because neither were on the extremes of the implicit theory continuum and both were active participants in class. Cindy and Harley were both female and Caucasian. Cindy's and Harley's majors were in elementary education. Cindy was 22 years old during the study, Harley was 20 years old at the beginning of the study, although Harley was 21 years old at the end of the study. Cindy was a junior, and Harley was a sophomore.

Data Sources and Instruments

The data collected throughout the study focused on Cindy's and Harley's beliefs concerning mathematical mistakes and any noticeable changes in their beliefs systems throughout the course of the study. A PT holding a belief may not be able to articulate what her belief is or be aware that she holds it (Leatham, 2006). Additionally, "beliefs cannot be directly observed or measured but must be inferred from what people say, intend, and do" (Pajares, 1992, p. 207). In order to make those inferences, numerous and varied data sources needed to be collected and examined (Leatham, 2006). By collecting and analyzing numerous data sources, the validity of the findings from the study were strengthened through triangulation (Yin, 2014). Data sources from the study included a mathematical implicit theory survey, an error climate survey, semi-structured interviews, observations of the PTs' class, exit tickets and in-class reflections, and reflective journals kept by the PTs. A description of each of these data sources follows. In addition, as the researcher is an instrument in qualitative inquiry (Patton, 2015), a description of my background is provided.

Implicit Theory Survey

On the first day of class, August 28, 2017, I administered the mathematical implicit theory survey (see Appendix A) to provide a description of the implicit theory of each PT enrolled in the described course. The first nine questions were directly from Dweck et al. (1995). These items addressed implicit theories as they relate to intelligence

(A1-A3), morality (A4-A6), and the world (A7-A9). Each dimension had high internal reliability (Cronbach's α = .85 to .98). Additionally, these items have been used in numerous publications (e.g., Dweck et al., 1995; Pajares, 1996; Wolters, 2004). The final three items (A10-A12) were created by Willingham et al. (2016). These items focused on a person's implicit theory as it relates to mathematical ability, which is distinguished from intelligence with these three items. Although these three additional items are not validated, Willingham et al.'s (2016) factor analysis of these items indicated that implicit theories related to mathematical ability is a separate construct from intelligence, which is aligned with Dweck and Leggett's (1988) view that any meaningful attribute should have a separate implicit theory feature.

For each of the 12 items on the implicit theory survey, PTs indicated their level of agreement with the statement (i.e., strongly agree, agree, somewhat agree, somewhat disagree, disagree, or strongly disagree). For each attribute, scores ranged from 1 to 6, with averages of 3 and below indicating characteristics of an entity theorist for each attribute and scores of 4 and above indicating characteristics of an incremental theorist for each attribute. Averages between 3 and 4 were considered as neither indicating entity nor incremental theorist characteristics.

Error Climate Survey

After Cindy and Harley were invited and agreed to participate further in the study, both completed an error climate survey. The first error climate survey described Cindy's and Harley's perceptions of previous experiences concerning mathematical mistakes (see Appendix B). Furthermore, the survey addressed mathematical errors in relation to the error tolerance by the teacher (B1-B4), the irrelevance of errors for assessment (B5, B6), the teacher support following errors (B7-B10), the absence of negative teacher reactions to errors (B11, B12), the absence of negative classroom reactions to errors (B13, B14), the error-taking risk (B15, B16), the analysis of errors (B17-B19), and the functionality of errors for learning (B20-B23). Sub-scores of each dimension were calculated by taking the average. Some items were negatively worded and thus, re-coded by taking Cindy and Harley's selection and subtracting it from seven. For example, item B3 is negatively worded, so a selection of 5 was re-coded as a 2. Items B3, B5, B6, B8, and B11 through B16 were negatively worded. Furthermore, a superordinate score was computed by taking the average of the average sub-scores. Each dimension had a high reliability (Cronbach's $\alpha = .70$ to .86). For each of the 12 items on the first error climate survey, Cindy and Harley indicated their level of agreement with the statement (i.e., strongly agree, agree, somewhat agree, somewhat disagree, disagree, or strongly disagree) and were asked to talk aloud as they made their choices. The first error climate survey was a modified version of Steuer et al.'s (2013) error climate survey. These changes were supported by Steuer and Dresel's (2015) use of the error climate survey when surveying 1,525 students concerning their perceived classroom error climates. Although Steuer and Dresel (2015) did not adapt the survey in any way, the modifications of the survey for this study were to clearly differentiate participants' previous mathematics classroom error climates with their current one.

The second error climate survey was conducted to describe Cindy's and Harley's perceptions of their current classroom's experiences concerning mathematical mistakes (see Appendix G). Again, this was a modified version of Steuer et al.'s (2013) error climate survey, and the modifications were made to differentiate participants' previous mathematics classroom error climates with their current one. When completing this survey, I asked Cindy and Harley to talk aloud concerning why they were making their decisions.

Semi-structured Interviews

I used semi-structured interviews throughout the study. This allowed me to ask questions that further “explore[d] [and] probe[d]” (Patton, 2015, p. 439) Cindy’s and Harley’s responses. Although an outline was provided for each interview (see Appendices C, E, and H), the sequencing and wording of the questions were flexible in order to gain clarity in responses and to pursue additional contexts and situations provided by Cindy and Harley (Patton, 2015). Eleven interviews were conducted with Cindy, and 12 interviews were conducted with Harley. The inconsistency in the number of interviews was because Cindy was absent on a day that a post classroom observation interview was conducted. The first interview was conducted after Cindy and Harley completed the error climate survey (see Appendix C). This interview served as a means to explore the participants’ responses on the error climate of their past mathematical experiences and to offer an opportunity for the participants to provide more description of their previous experiences concerning mathematical mistakes. The subsequent interviews, excluding the final interview, were conducted after each classroom observation (see Appendix E). These interviews served two purposes. First, they provided an opportunity for the participants to provide further description of their thoughts and insights concerning mathematical mistakes they observed in class as well as the subsequent events. Second, these interviews provided an opportunity for the researcher to establish further rapport with the participants throughout the study. These interviews took place on a bi-weekly basis throughout the term except during the class meetings on area. During these class meetings, interviews occurred after every class meeting. This set of class meetings was chosen because of the abundance of mathematical mistakes that were traditionally prevalent based on the Dr. Heart's experience. I conducted the final interview after Cindy and Harley completed the error climate survey concerning their current mathematics class (see Appendix H). Similar to the first interview (see Appendix C), this interview served as a means to explore the Cindy's and Harley's responses on the error climate of their current mathematical experiences and to offer an opportunity for them to provide more description of their experiences concerning mathematical mistakes. All interviews were audio recorded so that significant exchanges could be transcribed to support analysis.

Classroom Observations

There were two sets of classroom observations. The first set was to assist in selecting participants for the study. An observation protocol (see Appendix I) was used to focus my attention on students' reactions and behaviors during class activities concerning their implicit theory characteristics. These observations will be referred to as *implicit theory observations*. The observation protocol was developed by Willingham et al. (2016). This protocol contained six categories: evaluation of situation, dealing with setbacks, challenges, effort, criticism, and success of others. Examples of behaviors or comments that corresponded to each category and implicit theory were provided in the protocol to assist in clarifying how behaviors or comments were categorized.

The second set of classroom observations provided first-hand accounts of the participants in their classroom environments as mathematical mistakes were made by the participants themselves, their peers, or the instructor. Additionally, these provided a starting point for the interviews. These observations were conducted bi-weekly to coincide with the interviews except during the unit on area as previously described. I developed an observation protocol (see Appendix D) to focus observations on the mathematical mistakes made in the classroom, who made the mistake (e.g., participant or participants' peer), and the actions taken by the instructor and the two participants. The protocol provided general guidelines for the observations with attention to mathematical mistakes. These observations will be referred to as *classroom observations*.

Reflective Journal

Cindy and Harley maintained a reflective journal throughout the course of the study (see Appendix F) on a weekly basis. Cindy completed eight weekly journal entries and Harley completed 10 weekly journal entries. Journal prompts were emailed to Cindy and Harley on days of class meetings. These journal entries were intended to prompt participants' immediate thoughts on their experiences with mathematical mistakes in their class on particular days. These entries were used to support Cindy's and Harley's interview responses and to allow for comparisons across the course of the study.

Exit Tickets and In-class Reflections

Exit tickets (see Appendix J) and in-class reflections (see Appendix K) were administered by the instructor of the course on a weekly basis. Both Cindy and Harley completed nine of these items; however, they were not the same nine as Cindy and Harley were absent on different days. These immediately captured Cindy's and Harley's thoughts and reactions to classroom activities and, specifically, mathematical mistakes that occurred during class time. Dr. Heart administered at least one of these each week after a mathematical mistake occurred, especially when the mathematical mistake was highlighted and explored in the class. The exit tickets and in-class reflections were adapted by the instructor to conform to the day's events, but the focused on participants' reactions to mathematical mistakes that occurred during class and their personal thoughts on the class after those mistakes were made.

The Researcher as an Instrument

A description of my relevant experiences, training, perspective, and purpose are required as I was an instrument in the data collection of this qualitative study (Patton, 2015). Four relevant pieces of information are disclosed. First, three years of doctoral coursework in mathematics and science education provide relevant knowledge regarding the teaching and learning of mathematics and educational research methodologies. Second, throughout this coursework, I was involved in several research projects that used qualitative research methodologies. Third, these research projects were supervised by faculty at the university with expertise in qualitative methods. Finally, I maintained a research journal to add to the credibility to the study. During data collection, I recorded dates in which I observed the class as well as when I sent out and received correspondence from the participants. Additionally, I journaled how my relationship with the participants developed throughout the study. This was especially helpful at the onset of the study in developing rapport with each participant. As the study progressed, I recorded any knowledge that I had of the participants that affected my observations or questioning. For example, one of the participants wrote in a journal reflection that she was not getting along with the other members in her group. This influenced how I questioned her about her learning from mistakes in her groups in the later weeks as well as in her final interview. The research journal provided reflexivity to the study, which included "how [my] presence as an observer or evaluator may have affected what [I] observed" (Patton, 2015, p. 704).

Summary

Data collected during this study was chosen to answer the research questions of the study. Table 3 summarizes how I used the data sources to address each research question of the study.

Table 3

Research Questions in Relation to Data Sources

Research Question	Data Source
1. What beliefs do PTs hold about mathematical mistakes?	Classroom Observations Interviews Error Climate Surveys Reflective Journals Exit Tickets In-class Reflections
2. What are the noticeable changes in PTs' beliefs systems concerning mathematical mistakes, if any, given the classroom context?	Interviews following classroom observations Final Error Climate Survey Final Error Climate Interview Reflective Journals

Note. Information regarding research question 2 may also appear in exit tickets and in-class reflections.

Data Collection Procedures

After approval by the institutional review board was received on August 16, 2017 (see Appendix L), the data were collected in four distinct stages. These stages are described in the following sections.

Participant Selection

The primary data sources for participant selection were the implicit theory survey (see Appendix A) and the initial classroom observations (see Appendix I). I administered the implicit theory survey on August 28, 2017, to all PTs in Dr. Heart's class. Four of the 11 PTs that consented to participate further in the study ascribed to strong incremental theories, selecting either 5 or 6 for every survey item. Priority was given to the remaining seven PTs. Subsequently, I observed the remaining meetings during the first two weeks of class using the implicit theory observation protocol (see Appendix I) focused on the seven participants. Potential participants' actions were recorded and coded using the observation protocol and the implicit theory descriptors (see Appendix I) adapted from Willingham et al. (2016). Once potential participants were identified as not demonstrating either strong incremental or strong entity theorist characteristics and consistently active during class, four PTs, two primary and two alternates, were invited to continue further with the study. After one PT dropped out of the study, I selected Cindy and Harley from the remaining three PTs.

Prior Error Climate Description

After Cindy and Harley agreed to continue in the study, they completed the first error climate survey (see Appendix B). Cindy completed hers on September 12, 2017, and Harley completed hers on September 17, 2017. The error climate survey was purposefully administered within the first two weeks of the start of the class. This was done so that classroom norms and the classroom's use of mathematical mistakes was not well-established with the participants. The error climate survey was used to gain perspective of the error climate of participants' prior mathematics classrooms and to provide context for their choices. Participants were asked to talk aloud as they provided rationale for their choices. Additionally, the participants' choices and rationale provided additional context for the follow-up interview concerning the error climate (see Appendix C). The interview occurred immediately after participants completed the error climate survey. This interview offered an opportunity for the participants to provide more description of their previous experiences concerning mathematical mistakes in an effort to gain an initial understanding of their beliefs concerning mathematical mistakes.

Classroom Meetings and Reflective Journals

After the participants completed the error climate survey and were interviewed, they participated in daily activities and assignments of their mathematics course. During that time, participants were given opportunities to think about mathematical mistakes and were given opportunities to change their beliefs. To record this progression, I conducted 10 classroom observations using the observation protocol (see Appendix D). Cindy was absent on the classroom observation that occurred on September 29, 2017. Thus, there were only nine post classroom observations with Cindy. The classroom observations were conducted bi-weekly with the exception of the unit on area, where I observed every class meeting covering that unit. After I observed the classroom on those dates, follow-up interviews with the participants were conducted immediately after the class meeting to provide further clarification and reactions to classroom events that took place concerning mathematical mistakes (see Appendix E). As a result of both participants having class immediately after the class in which this study took place, I was only able to interview one participate immediately after each class observation. The other participant was either interviewed the following day, or, in some cases where no arrangements could be made, an email prompt was initiated to conduct the interview. To further assist in gathering data focused on participants' beliefs of mathematical mistakes, participants completed reflective journals, throughout the course, on a weekly basis (see Appendix F). Cindy completed nine reflection journal entries, and Harley completed 10. Additionally, each participant completed weekly exit tickets (see Appendix J) and in-class reflections (see Appendix K). These served as additional sources of data to infer participants' beliefs concerning mathematical mistakes.

Current Error Climate Description

Near the end of the semester, Cindy completed the second error climate survey (see Appendix G) on December 4, 2017, and Harley completed the survey on December 7, 2017. This survey was similar to the error climate survey the participants completed at the beginning of the semester, but the vernacular of the survey was changed to focus participants' attention on their current mathematics course. As with the first error climate survey, both participants were asked to talk aloud as they made their choices for each survey item. Additionally, there was a follow-up interview conducted immediately after Cindy and Harley completed the survey (see Appendix H). This interview focused participants' attention on their current mathematics course to provide further description of their beliefs of mathematical mistakes.

Data Analysis

Detailed case descriptions were written for both Cindy and Harley. These case descriptions provided a case study narrative (Patton, 2015) of the participants' beliefs concerning mathematical mistakes and any noticeable changes in those beliefs systems, if any, throughout the course of the study. Additionally, the participants' beliefs were analyzed in a cross-case analysis (Yin, 2014) as members of the classroom and through common themes that emerged within each case. Data collected during this study was stored and analyzed using ATLAS.ti. These analyses are described in the following sections.

Within-case Analysis

A separate account of each participant's beliefs concerning mathematical mistakes was given through a detailed and chronological case narrative, where each case was treated as a separate study (Yin, 2014). Data analysis began during fieldwork in an effort to ensure a strong foundation for qualitative analysis (Patton, 2015) and any observations or initial interpretations were recorded in my researcher journal. Subsequent data analysis continued during transcription of the participants' interviews, classroom observations, post classroom observation interviews, in-class reflections, exit tickets, and participants' journal reflections. Transcriptions were checked several times for accuracy. Additionally, while listening to any audio recordings and reading transcripts for accuracy, any observations or initial interpretations were recorded in my researcher journal.

Data for both Cindy and Harley were organized chronologically and were analyzed separately in a time series fashion (Yin, 2014). Analyzing the data this way provided me the opportunity to notice any changes in the participants' beliefs systems, if any, throughout the study. Open coding for each case was conducted using ATLAS.ti., and codes were compiled into themes consistent with the theoretical framework. That is, this process relied on the “theoretical propositions that led to [this] case study” (Yin, 2014, p. 136). Cindy’s data were analyzed before Harley’s and open coding began by reading Cindy’s Prior Classroom Error Climate Interview and concluded with her Current Classroom Error Climate Interview. It is worth noting that the Implicit Theory Observations were not coded or used to infer any beliefs or changes in the participants’ beliefs systems as those observations focused solely on actions that participants displayed as they related to implicit theory characteristics. There were several codes that I conceived prior to the beginning of the open coding process. These included factors or repeated statements that I noticed either from classroom observations or from the participants’ statements in their interviews, reflection journals, exit tickets, or in-class reflections. I maintained a record of these conceptualized codes with their foci and descriptions in my researcher journal. Those codes are provided in Table 4.

Table 4

Initial Codes List Conceptualized Prior to Open Coding

Conceptualized Code	Focus and Description
Affective influence of mistake	Influence on affective quality – Mistake embarrassed or empowered participant
Discussion worthy	Quality of mistake – Participant found a mistake to be worthy of a classroom discussion
Error climate	Error climate of classroom – Participant statements were related to how errors are handled in a mathematics classroom
Group dynamic	Error climate of the classroom – Participants made statements related to the impact of the group she was in
Isolation mistake	Who made mistake – Participant made a mistake in isolation either on homework or by not telling others in group or class
Learning mistake	Quality of mistake – Participant stated that she learned something from a mistake
Mistake valuable	Quality of mistake – Participant found a mistake valuable for some reason
Major mistake	Type of mistake – Participant classified a mistake as being a conceptual mistake[a]
Minor mistake	Type of mistake – Participant classified a mistake as being a procedural or a computational mistake[a]
Others' mistake	Who made a mistake – Participant assumed others made the same mistake

[a]See Table 1 for definitions of types of mistakes.

Subsequent codes were created based on distinct concepts and categories that originated in the data. A second reading was conducted to ensure that codes were accurate and based on the study's theoretical framework. Codes created along with descriptions of codes were kept in my researcher journal. After open coding of Cindy's Prior Error Climate Interview was completed, I repeated this cycle for the remaining pieces of data chronologically. Throughout this process, new codes were created and, again, kept in my researcher journal with their descriptions. Data items that occurred chronologically before a code was created were again reviewed to incorporate newly generated codes, if necessary. This process was repeated for Cindy's data until all data was reviewed. This resulted in the first set of codes and themes which are provided in Appendix M.

This process was then conducted for Harley's data. As new codes were created through analyzing Harley's data, each code and a description of the code were kept in my researcher journal. Harley's previously reviewed data were then reviewed again to include newly created codes and themes, if necessary. This resulted in a comprehensive list of codes and themes which are provided in Appendix N. Cindy's data were then reviewed again to include codes created from Harley's data. Once all pieces of data were reviewed, the codes themselves were coded for themes that emerged from the data. The codes and themes were used to produce case narratives, and each case narrative was submitted for member checks. The final narrative contained the full results of the study, which are reported in Chapter IV. This process resulted in significant implications of the study, which are presented in Chapter V.

To assist in identifying the central beliefs concerning mathematical mistakes for Cindy and Harley, codes and themes were reviewed to identify beliefs that made up each participant’s beliefs system concerning mathematical mistakes. Each belief was reviewed to examine the extent to which a belief cohered with the rest of the beliefs system and also transcended contexts and foci. To do this, cooccurrence tables were used to initially identify themes that repeatedly surfaced in different contexts and across themes. Subsequently, each data source from each participant was reviewed to examine the coherence of a potential central belief with the other beliefs that were identified.

An auditable trail was provided that described how inferences were made concerning each participant’s beliefs concerning mathematical mistakes along with evidence supporting noticeable changes in those beliefs systems, if any. Additionally, rival interpretations were presented to add to the credibility of the study (Yin, 2014). Ultimately, the previously stated data sources and analyses “coherently [told] the story” (Patton, 2015, p. 551) of each case as well as provided the basis for the cross-case analysis (Yin, 2014).

Cross-case Analysis

The cross-case analysis took place after the within case analysis, although emerging themes noticed during within case analysis were noted in my researcher journal. Yin (2014) recommended cross-case analysis in multiple case designs to provide an expansive generalization of the findings. Additionally, this analysis focused on common and contrasting themes that emerged within each case to extend findings past the individual cases (Yin, 2014). I examined the two participants collectively to establish common trends that occurred during the study (Yin, 2014). Additionally, and in alignment with Yin's (2014) recommendations, I used tables and displays of data to assist in analyzing similarities and differences between the cases of Cindy and Harley. For example, ATLAS.ti allowed me to create networks of codes created during data analysis (see Figure 4). These allowed me to organize codes into groups and subgroups for Cindy and Harley. Additionally, the cooccurrence tables on ATLAS.ti allowed me to the opportunity to see how codes were overlapping (see Figure 5). I kept memos in ATLAS.ti to document commonalities and differences across the cases that I noticed during data collection and analysis, as well as during the writing of the stories for each case. This included supporting inferences across each cases' beliefs concerning mathematical mistakes and any inferred changes in beliefs systems concerning mathematical mistakes between the cases. Again, rival interpretations of inferences are presented to add to the credibility of the study (Yin, 2014).

Figure 4. Network of codes created on ATLAS.ti.

	Big Mistake	Small Mistake
Alt Ways	2 – 0.03	1 – 0.02
Everyone learns fro...	3 – 0.07	1 – 0.02
Learn from Correct		
Learn from mistake	12 – 0.10	13 – 0.11
Learned - Connecti...	9 – 0.18	
Learned - Double C...	1 – 0.02	8 – 0.15
No learn from correct	1 – 0.02	
No Learn Mistake		2 – 0.05
PCK	1 – 0.03	
Persevering mistakes		
Teaching Knowledge	3 – 0.04	3 – 0.05

Figure 5. Cindy's cooccurrence table for codes related to learning from mistakes and the types of mistakes.

Issues of Trustworthiness

When assessing a study, it is important to consider its trustworthiness. This includes it credibility, transferability, dependability, and authenticity (Patton, 2015). Credibility refers to the extent to which the study is actually studying what it intends to study. Patton (2015) suggested several strategies to enhance the credibility of a study. All strategies utilized in this study are outlined in Table 5. This included member checks, where participants had the opportunity to verify their comments and actions as well as my inferences for accuracy. Although neither Cindy nor Harley wanted to read their narratives due to their length, I did offer a brief reminder of what the purpose of the study was and a description of the findings from the analysis. I also included several quotations with my interpretation of each. Neither participant offered any changes. This data source added to the credibility of the study and ensured my interpretations and inferences of the participants' responses and experiences were being described accurately (Patton, 2015). Transferability refers to the degree to which the study's findings can be transferred to other contexts. I provided rich descriptions of the context of the study and its participants to allow readers to make connections between the study and other situations (Merriam, 2009). Finally, a study should be judged on its dependability and authenticity. Dependability is judged on the "systematic process systematically followed" (Patton, 2015, p. 684) by a study. To add to the dependability of this study, I provided an auditable trail in my data collection sources, my data collection procedures, and my data analysis. Authenticity is the "reflexive consciousness about one's own perspective, appreciation for the perspectives of others, and fairness in depicting constructions in the values that undergird them" (Patton, 2015, p. 684). To achieve this, I kept a reflective journal throughout the data collection and data analysis process.

Table 5

Research Quality and Strategies Employed

Research Quality	Strategy Employed
Credibility	Prolonged engagement in the field
	Triangulation of sources
	Provide disconfirming evidence
	Member checks
Transferability	Provide rich, thick descriptions of each case
	Description of participant selection
Dependability	Auditable trail
Authenticity	Reflective journal

Limitations of the Study

Limitations of the study included issues that were largely out of my control. There were three limitations of the study, all of which were physical limitations. The first of which was dealing with observing each participant during class. During the implicit theory classroom observations, I was only able to capture small segments of actions from each participant. This was especially the case during the first week as the pool of 11 participants was only narrowed by those members of the classroom agreeing to participate in the study and participants with implicit theory survey scores that were not near 1 or 6.

The second limitation concerns the participants of the study. Four participants were selected to participate in the duration of the study, which included Cindy and

Harley. Again, this was done in case any participant chose to opt out of the study or withdraw from the class. As the study progressed, I chose two participants, one of which was Cindy, on whom to focus more intently during classroom observations because of their willingness to share their experiences concerning mathematical mistakes. One of those participants withdrew from the study on September 25, 2017, and I replaced her with Harley. This limited the initial four classroom observations with Harley to some degree as my focus was mostly on Cindy and the other participant.

The third and final limitation impacted the post classroom observation interviews. I was not able to interview both of the participants immediately after a classroom observation, as the participants had other classes in geographical opposition to each other. In these cases, both participants had extended periods of time, from 1 to 24 hours, until they were interviewed. During this time, the participants' recollections might have been conflated with other experiences, and they, in some cases, did not remember a particular action that I observed.

Delimitations of the Study

There were four delimitations of this study, choices which were under my control. First, I elected to conduct this study within a content class, as opposed to a methods class or a field-based setting, as a mathematics content class provided an opportunity for the PTs to make mathematical mistakes during the learning process and while still developing their understanding of mathematics content (Lui & Bonner, 2016). Second, the section of the course that I chose to be the context for this study was selected so that the given instructor would explicitly engage PTs with mathematical mistakes and thus provide the opportunity for this study to observe noticeable changes in PTs' beliefs systems, if any, while enrolled in the course. Third, the instructor for the course involved in the study has years of expertise with preparing teachers, specifically related to preparation for teaching with standards-based instructional strategies (e.g., NCTM, 2014). Not all classrooms may have this luxury, but the inclusion of the error climate surveys and interviews that followed the surveys attempted to describe the perceptions of the participants while in the classroom. This should be considered when evaluating the study for transferability. Fourth and finally, this study took place over the entire term of the course, which was a decision that I made for data collection purposes. In that time, the study captured a limited viewed of the participants' beliefs.

Chapter Summary

This chapter contained the research methodology utilized in this study. The details of the exploratory case study with embedded cases was given with support for its use. This chapter also contained the study's context, data collection sources and procedures, and data analysis methods. Finally, the results of this analysis are in the following chapter.

CHAPTER IV: RESULTS

Introduction

Mathematical mistakes play a crucial role in mathematics teaching and learning (Cooney et al., 1998; Hart, 2002, 2004), and mathematics education literature suggests leveraging mathematical mistakes to spur learning (CCSSI, 2010; NCTM, 2000, 2014). Additionally, mathematics research literature (e.g., Bahr, Monroe, & Eggett, 2014; Barlow & Reddish, 2006; Philipp, 2007) provided evidence that teachers' use of mathematical mistakes is not aligned with suggestions from mathematics education literature. Furthermore, other research provided evidence that teachers' beliefs are not aligned with productive beliefs concerning mathematical mistakes (e.g., Matteucci et al., 2015; Santagata, 2005; Schleppenbach et al., 2007). PTs' experiences in the primary, secondary, and post-secondary classrooms helped shaped their own beliefs concerning mathematical mistakes, which will ultimately influence the way they handle mathematical mistakes in their classrooms (Hart, 2004; Pajares, 1992). The role that beliefs play in teaching and learning mathematics is well established, but research is lacking in addressing the beliefs of PTs concerning mathematical mistakes. Therefore, the purpose of this study was to investigate PTs' beliefs concerning mathematical mistakes while considering the error climate of the classroom, that is, the way that mistakes were perceived in the classroom from the perspective of the PTs.

This chapter contains the details of the investigation of two PTs' beliefs concerning mathematical mistakes in three distinct parts. First, the details of the case of Cindy will be presented. Second, the details of the case of Harley will be presented. Within each case's description, a detailed account of the case will be provided, including the results of the prior and current error climate surveys, the case's background, and each case's implicit theory characteristics. This will be followed by the details concerning the case's central beliefs, belief clusters, and noticeable changes in beliefs systems throughout the study. Third and finally, the results of the cross-case analysis will be considered.

The Case of Cindy

The case of Cindy will be described in six parts. The first of which is the details concerning Cindy's implicit theory characteristics. This will inform why Cindy was selected to participate in the study as participants were selected based on their implicit theory. Specifically, participants were chosen that were not on either extreme of the continuum. Second, Cindy's prior mathematics experiences will be provided. This will provide a sense of the experiences that helped to shape Cindy's beliefs concerning mathematical mistakes. Third, the results from Cindy's Prior Error Climate Survey will be presented and connections to Cindy's prior mathematics experiences will be made. The subsequent sections will focus on Cindy's beliefs concerning mathematical mistakes. Specifically, the fourth section will describe Cindy's central beliefs concerning mathematical mistakes followed by the belief clusters concerning mathematical mistakes. Finally, noticeable changes in Cindy's beliefs system concerning mathematical mistakes while considering the perceived error climate of the classroom will be detailed. These will be followed by a summary.

Implicit Theory

Evidence of Cindy's implicit theory characteristics was inferred from two sources: the implicit theory survey (see Appendix A) and from classroom observations using the implicit theory observation protocol (see Appendix I). Cindy completed the implicit theory survey on August 28, 2017. At the time of the survey, Cindy's average results indicated that she ascribed to an incremental theory in all dimensions except worldview: intelligence (M = 6), morality (M = 4.33), worldview (M = 3.67), and mathematical ability (M = 6).

Although Cindy clearly ascribed to incremental theory characteristics in mathematical ability on the survey, I observed mixed characteristics during the classroom observations which focused on enacted implicit theory characteristics from August 28, 2017, through September 8, 2017. During those observations, I observed Cindy enact characteristics of both an incremental and entity theorist. On August 28, 2017, the first day of class, Cindy's group engaged in a task taken from *United We Solve* (Erickson, 1996), where each participant in a group was given a clue about a polygon (e.g., all angles are right angles in this polygon or this polygon is a quadrilateral). The goal for the group was to use the clues to make an accurate drawing of the polygon of which they described. As the group organized to get a sense of how they would draw the polygon, Cindy stated, "This looks hard, but I think that we can do it together" (Implicit Theory Classroom Observation, August 28, 2017). As the group tried different polygons and found that one person's drawing did not work with all four conditions given, the group stalled, and then Cindy offered her drawing to the group stating, "This is what I was thinking" (Implicit Theory Classroom Observation, August 28, 2017). These instances aligned with characteristics of an incremental theorist related to challenges and effort.

However, on September 1, 2017, Cindy's group was responsible for defining angles in words (e.g., obtuse angle) based on examples (e.g., Figure 6). The class was told by Dr. Heart that constructing their own definition of terms would be helpful when doing geometric proofs. At that point, I observed Cindy shake her head and say “Oh, no” (Implicit Theory Classroom Observation, September 1, 2017), which was classified as a descriptor of an entity theorist related to effort. Shortly, thereafter, I observed Cindy not making eye contact with Dr. Heart when she asked the class to operationalize specific constructed definitions (e.g., a member of the class used a group’s definition of acute angle to show why an angle was not acute). Looking away rather than at Dr. Heart was aligned with characteristics of an entity theorist related to challenges. In another observation on September 6, 2017, the class used tracing paper to visualize why vertical angles have equal measure and to develop the parallel postulate. While using the tracing paper, I observed Cindy on two occasions telling her group, “We can do this” (Implicit Theory Classroom Observation, September 6, 2017), as she folded her tracing paper to try to line up her vertical angles to see if they had equal measure. This was an incremental description when evaluating the situation. However, later that class session, Cindy replied to Dr. Heart’s question of what a particular angle measure was and appeared embarrassed by turning slightly red and looking down after realizing her answer was incorrect–an entity descriptor when dealing with setbacks.

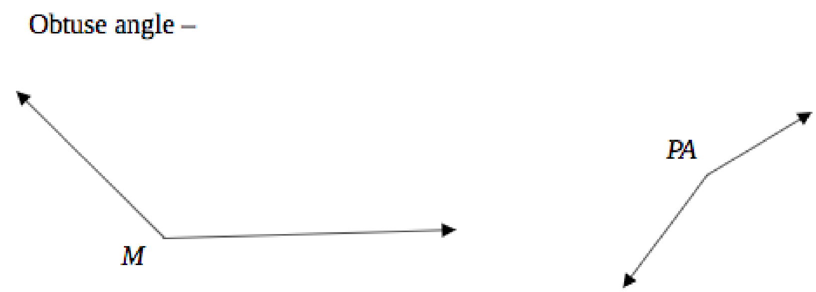


Figure 6. Examples of obtuse angles.

During another observation on September 8, 2017, Cindy's group engaged in a sorting task where they were asked to place given triangles into different categories (e.g., has a right angle, has an obtuse angle, or all angles have the same measure). They were then asked, based on how they sorted their triangles, to identify if certain types of triangles existed (see Figure 7). These two exercises were precursors to using Venn diagrams to organize different types of triangles (e.g., the relationship between acute and equilateral triangles). Cindy displayed characteristics of both an entity and incremental theorist during this part of class. I observed her encouraging a classmate that stated that an obtuse isosceles triangle does not exist. Cindy stated, "That's a good idea. Let's see how you did it" (Implicit Theory Classroom Observation, September 8, 2017). This was an incremental theorist description of supporting others. However, moments later in the same class session, I observed Cindy to be hesitant to share the Venn diagrams that she drew for the relationships between types of triangles, looking down or away when Dr. Heart asked if anyone would like to share what they did–an entity theorist descriptor when evaluating the situation. She later explained during the Prior Error Climate Interview on September 12, 2017, that she knew her Venn diagrams were incorrect. If she knew that her work was incorrect and avoided sharing it when Dr. Heart asked for volunteers, then that was a descriptor of an entity theorist when dealing with setbacks, not when evaluating the situation.

	Equilateral	Isosceles	Scalene
Right			
Acute			
Obtuse			

Figure 7. Which triangles exist table.

Cindy ascribed to strong characteristics of an incremental theorist in both intelligence and mathematical ability based on her responses from the Implicit Theory Survey. However, Cindy enacted characteristics of both entity and incremental theorist during observations. Cindy's actions during my observations implied that Cindy was not as strong of an incremental theorist as she ascribed to be and indicated that Cindy was in the middle of the implicit theory continuum.

Prior Mathematics Experiences

Cindy wanted to teach kindergarten, specifically, when she graduated. She was also a self-described good student in mathematics. In one description of her previous mathematics classroom, she stated that she "was up level-wise, except for geometry, at the top of the class, 'cause I could get it as soon as they would explain it" (Prior Error

Climate Interview, September 12, 2017). Cindy had varied experiences in her mathematical classrooms in the past, and when describing those experiences, Cindy focused her thoughts on three teachers from her secondary classes. When asked to expand on how she learned from assignments that are not done correctly, she explained:

> It happened. It just depends honestly on the teacher. Like, trying to think, my probably, mainly my sophomore and junior year, those teachers really worked with you. “Okay you've got this wrong. Then, you got this right. How did you get that right?” Maybe fifty-fifty or sixty-forty they worked on it, and they explained it. But my freshman and senior year, well I already said it was the same person, she didn't do that. It was more of this the correct thing let me show you how. She didn't explain really or not too often she didn't explain why you got it wrong. (First Post Error Climate Interview, September 12, 2017)

In this quote, Cindy differentiated her experiences with her freshman and senior mathematics teacher from her sophomore and junior teachers, which she did throughout the interview. However, it was important to note that all three teachers described in this quote focused on the correct answer as opposed to considering her incorrect work.

Although Cindy described different experiences within those classrooms (e.g., how nice the teacher was, how she liked one more than another, or what help looked like in each class), she provided common experiences across her secondary mathematics classrooms. When asked to talk more about her classes spending time thinking in detail about students’ incorrect statements, she stated:

> In high school, we never spent time on the incorrect. We've always spent time on the correct and why it was correct. Never, “Alright Tommy got this wrong, why

> did he get it wrong? If he got it right, explain to me why he got it." So, I don't think that it's something that's spent time on. At least in my past experience, it was, it was [*sic*] never the incorrect. It was the correct. (Prior Error Climate Interview, September 12, 2017)

Here, Cindy restated that mathematical mistakes were minimalized and, in her description, nonexistent. This was coupled with a routine that utilized traditional teaching strategies. When asked to provide a description of a typical day in her previous mathematics classrooms, she stated:

> The teacher is either at the desk or at the door, but we'd have some bell work or some getting started assignment that would either be a problem that we'd seen before or had built off the homework. Or to introduce. I never really liked when it introduced, because if I got a bunch of answers wrong, it was kind of hard. Then we'd discuss it. We'd have to work on it and discuss it, and then we'd move straight into the lesson or go over the homework. And then, it would end with new homework. A lesson was either up at the board doing different problems and then sometimes us going up there. Or, it would be us, she'd just put up a bunch [of problems], or he would, and we had to work in a group of four. I mean remembering back it was mainly three or four days of it, it would be lecturing and then we'd work in some groups. (Prior Error Climate Interview, September 12, 2017)

Although she was asked to describe a typical day which did not allow her the opportunity to talk about different classes, I asked if this was common for all of her previous classrooms, and she stated, “They were all pretty similar in how they were setup” (Prior Error Climate Interview, September 12, 2017)

She also elaborated on what her prior mathematics experiences were like as it related to how her mistakes were handled differently from different teachers. First, she differentiated what help looked like:

> I feel like there's two different types of help. There's, “I'm gonna give you the answer” help, and there's actually leading questions like if we're solving for *x*, “Where should you put *x*?” if they're putting *x* in the wrong place. (Prior Post Error Climate Interview, September 12, 2017)

It is important to note that in this quote Cindy perceived that mathematical mistakes were treated differently. In the former, the answer was given, and in the latter, she was asked what the correct answer is. However, neither classroom focused on the mathematical mistake itself, and the focus was still on the correct answer. She went on to describe what those different types of help looked like in other classrooms. Again, she focused on two groups of teachers from her secondary mathematics classrooms.

> She was my freshman algebra teacher and my senior like algebra two or something like that. She was one of those teachers that would just give you the answer. So, I didn't learn much from that class ‘cause if I'd go up and say that I can't find *x* or can't do this, she'd say this is how you do it. So, I'd learn nothing, but I got the homework correct. (Prior Error Climate Interview, September 12, 2017)

Here, Cindy described her freshman and senior teacher as the one who limited her response to giving the correct answer. Cindy elaborated more on her freshman and senior mathematics teacher.

> Even the one that gave me all of the answers. She'd work the process out, she wouldn't let me do it on my own. She just gave me the answers. So, if it was like *x* plus five equals ten and find *x*, she'd be like there you go. There's your answer. And I'm like okay, thanks. (Prior Error Climate Interview, September 12, 2017)

Again, Cindy perceived that this teacher wanted to give her the correct answer or way of doing a problem without considering her work.

Even when describing other teachers that she had, Cindy provided similar characteristics of what help looked like when making mistakes.

> Even when they don't give you the answer. They are still showing you how to get the answer. ‘Cause they, I've had many teachers explain how to get the problem, especially when it comes to math. I'm stronger in some areas than in others and they would take the time to sit down and say that this is how you get the solution to this problem. This is how, and they would work it out. They wouldn't use the exact [problem] from the homework, but they'd bring in another example and say, “Let's work together.” We'd work together, and then I'd have to do it on my own. And then we'd do another one. (Prior Error Climate Interview, September 12, 2017)

Cindy described how her sophomore and junior mathematics teachers worked with her when she made a mistake. Specifically, they showed her how to do a different problem that they could work together before allowing Cindy to try another problem. Cindy differentiated her teachers based on how they responded to her making a mistake. The freshman and senior teacher gave her the answer, whereas the sophomore and junior teachers “work[ed] together” with her. However, the analysis of the data revealed Cindy perceived that none of her prior mathematics classrooms focused on Cindy’s mathematical mistakes and the focus was always on the correct answer or a correct strategy.

Cindy perceived that she had a variety of mathematical learning experiences where teachers had dissimilar personalities and handled mathematical mistakes differently. However, there were two commonalties for Cindy even when she described different mathematics classrooms. First, the data revealed that she perceived most, if not all, of her prior mathematics classroom experiences similarly in that they treated mathematical mistakes as problems to be fixed. Second, it is important to notice that Cindy’s mathematics classroom perceived experiences were descriptive of traditional teaching methods, where there was little or no accounting for student’s thinking. This is especially important to notice when considering the results of Cindy’s Prior Error Climate Survey which follows.

Perceived Error Climate for Previous Mathematics Classes

Cindy was aware of the relevance of the error climate in her previous classrooms. During the first interview, she talked about how mistakes were perceived in her previous mathematics classroom, specifically concerning students not wanting to answer for fear of saying something incorrect:

> My high school experience usually kids don't, I've noticed that a lot of them didn't answer. There's one or two of us that would answer, because we were either really

> confident or we just knew that we got it wrong. So, we would just say that we're not sure if this is correct, but it would only be two. It's kind of like that. You don't answer if you don't fully know. It's like that in high school. I noticed that a lot. You don't answer if you don't fully know. And that's how it was, especially in math. I've had that happen before and sometimes teachers mean to and sometimes they didn't mean to. So, I think that was a big thing. I think that a lot of them didn't answer, because of lot of them didn't want to be belittled, or knocked down a peg. (Prior Error Climate Interview, September 12, 2017)

Cindy noticed and was well aware of how mistakes were treated in her prior mathematics classrooms, and the classroom error climate clearly influenced when she decided to share her mathematical thinking in her previous classes.

During the study, Cindy completed two error climate surveys. The first of which focused on Cindy’s classrooms prior to enrollment in the class in which this study took place. She completed the Previous Error Climate Survey (see Appendix B) on September 12, 2017, which focused on her perception of mathematical mistakes in relation to seven dimensions with a superordinate score. Cindy’s average results are displayed in Table 6. Cindy’s perceived previous mathematics error climate would be described as a positive one, which indicated that Cindy perceived a positive error climate in her past mathematics classrooms. Notably, the error-taking risk sub-score was higher than all other sub-scores, which aligned with Cindy’s description that in high school “You don't answer if you don't fully know” (Prior Error Climate Interview, September 12, 2017).

Table 6

Cindy's Prior Classrooms Error Climate Results

Error Climate Dimension	Average
Error Tolerance by Teacher	*M* = 2
Irrelevance of Errors for Assessment	*M* = 3.5
Teacher Support Following Errors	*M* = 1.85
Absence of Negative Teacher Reactions to Errors	*M* = 3
Absence of Negative Classroom Reactions to Errors	*M* = 2.5
Error-taking Risk	*M* = 5.5
Analysis of Errors	*M* = 2.67
Functionality of Errors for Learning	*M* = 2.25
Superordinate Score	*M* = 2.91

Note. Scores on the survey ranged from 1 (strongly agree) to 6 (strongly disagree), with averages of 3 and below indicating a positive error climate and averages of above 4 indicating a negative error climate. A lower average indicates a more positive error climate.

Connecting to her descriptions of her prior mathematics classrooms, the Analysis of Errors and Functionality of Errors for Learning dimensions appeared misaligned with her descriptions of her prior classrooms. Her survey results indicated that she perceived that mathematical mistakes were analyzed and used to learn something in her prior classrooms. However, her descriptions of her prior mathematics classrooms indicated that mistakes were rarely, if ever, used to learn more than the correct answer. From Cindy's perception, though, learning to get the correct answer was how learning was defined in her prior classrooms.

Cindy's Central Beliefs Concerning Mathematical Mistakes

Throughout Cindy's experiences in Dr. Heart's class, with all the mathematical mistakes that occurred, she was given opportunities to reflect on her experiences with mathematical mistakes through interviews, journal entries, and in-class reflections. From that data, themes emerged concerning mathematical mistakes which revealed Cindy's beliefs concerning mathematical mistakes. Certain beliefs, which will be referred to as *central beliefs concerning mathematical mistakes*, encompassed and influenced all of her other beliefs concerning mathematical mistakes. These central beliefs concerning mathematical mistakes were beliefs around which all of Cindy's other beliefs concerning mathematical mistakes were clustered and were "overarching beliefs about the physical, social, and pedagogical" (Leatham, 2007, p. 192) inclusion of mathematical mistakes in the classroom. Additionally, these were beliefs that cohered with the rest of the system and transcended contexts and foci. The remainder of this section will describe the central beliefs concerning mathematical mistakes for Cindy.

Importance of mathematical mistakes. Cindy believed that mathematical mistakes served a significant role in mathematics teaching and learning. To Cindy, mathematical mistakes were a way to "build on your toolbox or your building of knowledge . . . It just adds," and mathematical mistakes were instances to help things "click" (Prior Error Climate Interview, September 12, 2017). The first time she mentioned mathematical mistakes helping things click was in a story she told during the first interview:

> I remember when I was first learning algebra, we were learning about solving for *x*, ‘cause it clicks now. I've learned how to do this . . . we were learning about negatives and positives. I forgot to bring the negative over, or I didn't switch it to a positive, so I got like three or four answers wrong. And I couldn't figure out why ‘cause I'd work them out and get the same answer and once, I think it was my best friend, she worked it out and she said are you switching this over? Are you doing this? And I said no, and she said that you have to do that. So then, it clicked. I looked back and said that I can't believe I did that. I did that on all of these. I think that it's really important to sometimes, I'm not saying get wrong answers all of the time, but it's important to get those so that you can learn from them and continue doing the right way. (Prior Error Climate Interview, September 12, 2017)

Cindy described this mathematical mistake as an important opportunity to notice that she did not do something correct, and it provided her the opportunity to learn how to “switch it over.” This ultimately helped her to complete the problem the correct way.

Cindy’s belief that mathematical mistakes were important because they helped her learn the correct way was present throughout the study. In a classroom observation on October 18, 2017, the class spent significant time talking about mistakes in how to find the area of triangles. To start class, Dr. Heart displayed the picture in *Figure 8* and asked the class to think about different ways that an elementary student might find the area incorrectly.

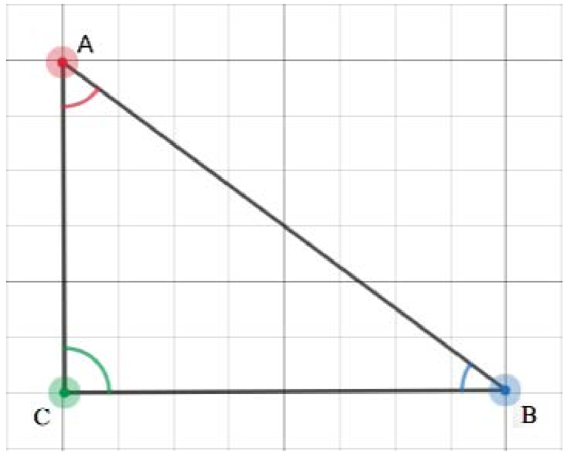


Figure 8. Finding area of triangle on grid paper. Adapted from Mathematics for Elementary Teachers with Activities (5th ed.), by S. Beckmann, 2018, New York, NY: Pearson. Copyright 2018 by Pearson Education, Inc.

Later in that class session, Dr. Heart stopped the class to ask why her work in doubling the shape to find the area, a strategy that she provided to the class, was incorrect (see Figure 9). After the class resolved how she incorrectly doubled the shape, Dr. Heart stated that these types of mistakes can help with connections to other topics like areas of other shapes. After class, I talked briefly with Cindy about why so many mistakes were shown in class and why those might be important to people in the class. She stated:

> It really helped me, and I think seeing those are important for everyone. If I got the answer right but the person next to me got it wrong but couldn't figure out why so then we discussed that as a class, it kind of clicks. Like, okay. Now, I understand. It helps you see other things. There's [*sic*] connections to be made. A lot of people have made those mistakes. (Post Classroom Observation Interview, October 18, 2017)

Here again, mathematical mistakes were important as they served as opportunities for her to understand, which was done by seeing how not to do something. The mathematical mistakes provided by Dr. Heart were even more instances for that to happen.

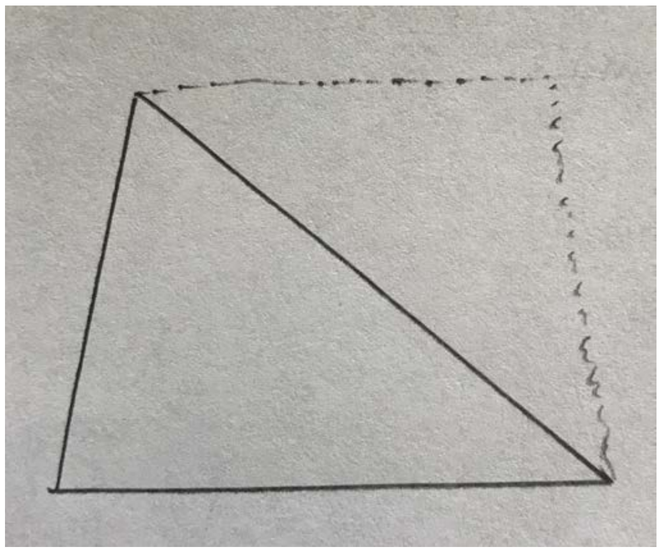


Figure 9. A recreation of Dr. Heart's incorrect strategy of finding the area of the triangle by doubling the shape.

Cindy believed that mathematical mistakes were instances to learn in the classroom and to understand a mathematical concept by learning how not to do a particular problem. Cindy’s belief about the role that mistakes play in mathematics was best captured in her final reflective journal entry when I asked her to write a personal beliefs statement about mathematical mistakes:

> I know no one wants to make mistakes. I will be honest; I know I don't like to make them at all. But even though I don't like to make them, mistakes are a part

> of life. You learn from mistakes, how to fix them or make them better. You can be told how to do it, and sometime pick up on it. But when I make a mistake, I learn why I made the mistake and how to fix it. Then, try again to figure it out. Mistakes will happen, but you need to be able to accept the mistake and learn from it. (Journal Reflection, December 4, 2017)

Mistakes were important for Cindy, because they gave her the opportunity to learn how to fix them and thus be able to avoid that mistake in the future. They were valuable enough that she knew that she had to make them, even though she did not want to, in order to give her the opportunity to "figure it out" (Journal Reflection, December 4, 2017), which correct answers cannot do. Finally, it is important to point out that Cindy believed that mathematical mistakes served a purpose for her as a student of mathematics content. Even in instances where Cindy was positioned in the teacher's role during activities, Cindy focused on the relevancy of the mistake as a service to herself as a student. One example of this was during a class session where Dr. Heart showed a video of fourth-grade students using a broken ruler (i.e., the ends of the ruler were missing) to identify how long a pencil was. I interviewed Cindy after that class meeting and asked her what she thought about the video and the mistakes the students made. She stated, "We really needed to see that. There are so many mistakes that could happen in a math class. Seeing their mistakes was beneficial. *I* [emphasis added] could make that kind of mistake" (Post Classroom Observation, October 2, 2017). Her focus was on what she could learn from seeing others making that mistake.

Mistakes compared to correct answers. Cindy also believed that mathematical mistakes' role in the classroom was differentiated from the role that correct answers played. That is, mathematical mistakes allowed her to do things that correct answers could not. In the final interview with Cindy, I asked her to talk about the difference between getting something correct and getting something incorrect in a mathematics classroom. She explained:

> [A mathematical mistake] opens up a new doorway for you to understand the problem or a different level if you were the one that got it right. You have to understand why you got it right, and how to help others. But if you got it wrong, you kind of dive deeper into why that is wrong, and then, how can I make it right and why is that right. So, I don't make that mistake again. (Current Error Climate Interview, December 4, 2017)

I asked her to expand on exactly what she meant and why you might want to spend more time on the wrong way of doing something. She stated:

> I guess noticing being in college and going through this and being in high school it was more of they focused on the right more than the wrong, but now that we’re in college you need to know why you got it wrong. Whereas in high school and in other classes, you need to get the right answer kind of for the test, but you need to know why it’s right. I feel like those are kind of like two different things. ‘Cause I could get the answer right every single time and be fine, but if I get the answer wrong it could be different every time. It’s one of those things I could look at the right answer and try to recreate that, but it’s hard. But, if I have the wrong answer, I can pinpoint where I got it wrong and make sure to not to do that again. I think that it’s two different [things]. I think that it’s important to see the wrong as much as you see the right. ‘Cause that right is going to be a good example but that

> wrong is going to show you how to fix it. (Current Error Climate Interview, December 4, 2017)

Cindy differentiated the role of correct answers and mathematical mistakes in what she could learn, which necessitated their different roles. Correct answers did not stimulate her to reflect on her work. Mathematical mistakes provided that stimulation for her to examine why she got it wrong and how to fix it.

Throughout the study, Cindy often returned to these descriptions of the different roles of correct answers and mathematical mistakes. In the final classroom observation, the class worked on problems provided by Dr. Heart to help review for the final exam. On the review, there were different types of problems, including some that provided an incorrect strategy and asked why the strategy was incorrect. I noticed that Cindy’s group spent a lot more time talking about the problems with mistakes than anything else, so I wondered which one she perceived that they talked about more and why. She responded:

> I would say mistakes. For example, when [another student] made a mistake or got stuck, I had to figure out how and why she got stuck. Then, I have to figure out how to help without giving the answer. I feel I benefit from the whole process. I also get to learn the material on a different level. It is one thing to learn the material and apply it to my own problem but to be able to teach it or help someone through the problem I feel can help you learn the material on a more in-depth level. When you get the correct answer, I feel it's a quicker process. You give the answer, maybe how you did, and then move on. When you get it wrong, you discuss and find where it went wrong. (Post Classroom Observation Interview, November 29, 2017)

Cindy provided a clear distinction about the role that mathematical mistakes played in the classroom from correct answers or correct strategies. Correct answers did not really have a role other than to show the right answer, and the mistake demonstrated that particular answer was correct. However, mathematical mistakes provided opportunities for other experiences to occur, whether this be for Cindy to have to learn the material on a different level or to provide the opportunity to learn another way to work the problem.

Fixing mistakes. Finally, Cindy believed that mathematical mistakes needed to be fixed and made correct so that learning could take place. Although, as previously mentioned, Cindy believed that mathematical mistakes were an indication that something was left to learn, she also believed that fixing the mistake was the way to learning and "seeing the correct answer" (Current Error Climate Interview, December 4, 2017). After the final classroom observation, described earlier where Cindy's group reviewed for the final exam, I also asked her what she gets out of a conversation involving correct and incorrect answers. She stated, "When you get it wrong, you discuss and find where it went wrong. Then, you go over how to fix it and go over the right way or one way to answer the question correctly" (Post Classroom Observation Interview, November 29, 2017). Fixing the mistake was the way to handle mistakes. She saw a mistake, talked about it, fixed it, and then got the question correct. That is how she learned from the mistake. Fixing mistakes to get the correct answer was crucial for Cindy and was the first thing that she thought about when she made a mistake. In one of her reflections, I asked her to think about the mistakes that she made during the week and what she thought about as she recognized that she made them. She stated, "[My group] struggled to reattach part of a shape to make it a regular shape. . . . I wanted to fix [our mistakes] so we could figure out how to get them. To understand them" (Reflection Journal Entry, October 20, 2017). Again, Cindy's focus was on how to fix the mistake, so she could understand how to get those types of problems correct.

Cindy talked about fixing mathematical mistakes in terms of the structure of class. That is, class is the time where mathematical mistakes should be shared so that the teacher can draw attention to them and correct them. As she stated:

> That's what class is. Your, that's your time to shout out your ideas, your thoughts. And for the professor, the teacher, to correct them or your classmates if you're working in groups. So, if I said like two plus two equals five, you could say no. This is why it doesn't. (Prior Error Climate Interview, September 12, 2017)

This quote was in response to her being asked if something incorrect in class could affect her grade in class. In this quote, her focus was on how class time was used to elicit the mistakes in an effort to have them fixed. She described this structure of class similarly near the end of the study as well:

> Everyone has said something wrong in that class. But it doesn't affect your grade unless you don't fix it. Like if you don't fix, if no one tells you that it's wrong and you just keep going with it, that could affect your grade. That's why we share them and fix them. (Current Error Climate Interview, December 4, 2017)

Although Cindy focused on fixing the mistake so it did not affect her grade, class still served as the mechanism for sharing mistakes so that they can be fixed, which allowed her to get the answer right on the test or some other assessment.

Cindy believed that a mathematical mistake had to be fixed in order to learn from it. However, she was not satisfied with just seeing the correct answer but needed to know why the wrong answer was the wrong one. She explained that she would not “learn much” (Prior Error Climate Interview, September 12, 2017) if she was given the answer or if she did not figure out why it was wrong. Cindy believed that it also had to include “why [she] made the mistake and how to fix it” (Reflection Journal Entry, December 4, 2017) in order to learn. Cindy’s need to know why a mathematical mistake occurred was linked to the importance of mathematical mistakes in the classroom. That is, the mistakes provided her an opportunity that correct solutions did not. However, the process of fixing the mistake was the path to learning, or as Cindy stated, “But when I make a mistake, I learn why I made the mistake and how to fix it” (Journal Reflection, December 4, 2017).

Belief Clusters

Cindy’s *central beliefs concerning mathematical mistakes* can be viewed as Cindy’s stance of how mathematical mistakes should be used in teaching and learning mathematics. Cindy believed that mathematical mistakes served a significant role, and they were opportunities for learning if the mistake was fixed or corrected. Finally, Cindy’s beliefs concerning mathematical mistakes were in contrast to how correct responses should be used in the mathematics classroom. Her *central beliefs concerning mathematical mistakes* influenced her other beliefs concerning mathematical mistakes. Particularly, the clusters of beliefs that emerged from this study were when mathematical mistakes were important, the knowledge that can be gained from mathematical mistakes, and the influence of mathematical mistakes on affective qualities. Each of these will be discussed in detail.

When mistakes are important. Although Cindy did not like to make mistakes, she believed that “mistakes [were] a part of life. You learn from mistakes, how to fix them or make them better" (Journal Reflection, December 4, 2017). Mistakes were where most of mathematical learning took place, especially in comparison to correct answers. However, there were times that Cindy believed that mathematical mistakes were more important. Particularly, there were two factors that influenced if Cindy believed a mathematical mistake to be important: her perception of who made the mistake and if the mistake offered the opportunity for others to participant in fixing it.

Perception of who made the mistake. Who Cindy perceived to have made, or could have made, a mathematical mistake played a significant role in if Cindy believed a mathematical mistake was valuable to the class. At first, I started to infer that Cindy believed mathematical mistakes were important to the class just if there were several people in the class that made the mistake. However, Cindy's data suggested that, for her, it was about whether or not she perceived that others made or could have made the mistake. This is not to say that the more people in the class that claimed to have made the mistake was not important to Cindy. A mistake with collegial ownership was the most influential in Cindy deciding if a mathematical mistake was important, because then she knew the others made the mistake.

The importance of a mathematical mistake as it related to her personal ownership of the mistake was present from the first interview with Cindy. For example, when talking about if mathematical mistakes were something bad for her teachers, Cindy talked about an instance during the first week in Dr. Heart's class in which she made a mistake. She stated, "I've made mistakes in math classes before, but I'm sure that others had similar thought processes in getting the wrong answer because it's been a while since we've had geometry. But, we learn from them" (Prior Error Climate Interview, September 12, 2017). It is important to notice that Cindy took a mistake that she made and projected that mistake onto others in the class. Another instance, in the same interview, Cindy responded to Survey Item B16–In our mathematics classroom, a lot of students hope they will not be called on, because they are afraid that they will say something wrong. In her response, she told me about her fear of being called on in class, to which I wondered why that was the case if she, or others, could learn from the mistake. She stated:

> I feel like it goes back to feeling dumb or *out of place* [emphasis added]. Like you should've gotten this right, but you can't get it. But as soon as you, as soon as they call on you, why did you call on me? So that always scared me. That constant, it sounds silly. That constant fear of being called on, especially if it's a large group, because I don’t know how they’re thinking. If it's a small group, I'm okay with it, ‘cause it's less people, and we could all relate to messing up. But when it's a larger class, or like twenty or more, it's that fear of saying something wrong and then having like, nineteen to however, everyone just like, that moment of judgement I guess. There might be a lot of people that didn’t do it the way that I did. (Prior Error Climate Interview, September 12, 2017)

Again, Cindy took this opportunity to think about others owning her mistake. If she saw that the other students in the class would not be able to assume some ownership of the mistake that she made, then there would be no need to share it. Another instance in the Prior Error Climate Interview (September 12, 2017) where her personal ownership of a mathematical mistake came up was in her response to me asking her what a mathematics teacher was like. I gave her the options of conductor, entertainer, a coach, gardener, news caster, or a doctor. Cindy chose a coach and stated:

> I'd [*sic*] have more math teachers or professors kind of cheer you on. I don't know. When I think of a coach, I think of someone that pushes you to do better. Gives you advice and helps you get to where you need to be. And like game help. If we're using football, you watch game film. So, game film for us would be the homework and assignments. So, show me how you got it wrong. Let's look at what you did wrong. I think that's kind of what a coach does. They push you to do more. They explain how you got it wrong. Then, you review it and then you knock it out, the test, which would be the game.

I went on to ask Cindy how a teacher like a coach would handle mathematical mistakes made in class. She stated:

> If I made the mistake, then they'd ask you to come back or go to study hall or office hours in college. And review what you are doing wrong. But, if there were others that maybe did the way that I did it, in a wrong way, then the coach would need to go over it if we're thinking the same way.

Cindy believed her mathematical mistake was not valuable to the class unless others thought about the problem in the same incorrect way.

This notion that Cindy needed others to make the mistake in similar ways in order for a mistake to be important enough for the class to see was present throughout the study as well. In a journal reflection, Cindy referred to a class session where her group made several mistakes in determining if certain quadrilaterals satisfied certain properties. They were asked to check all properties that were satisfied and be able justify their choices and show counterexamples of properties that a quadrilateral did not satisfy (see Figure 10). I asked Cindy to write about the mistakes that she observed and which ones that she thought were important, and she referenced this exercise in class. She stated:

> I was confused, because my thought was, if they are congruent, would they not be bisect [*sic*]? Because bisect means to cut in half. I am still struggling with why that is not technically correct. Just because it bisect [*sic*] does not mean it's congruent. I think there were *other groups* [emphasis added] confused on this too. I think it was important because others were struggling too. I wish we would have gone in more depth with it. We could have really talked about that more. (Journal Reflection, October 6, 2017)

In this quote, Cindy again took a mistake that she made with her group and stated that it was important, because she thought others would make the same mistake.

	Quadrilateral	Parallelogram	Trapezoid #1	Trapezoid #2	Kite	Rectangle	Rhombus	Square
At least one pair of parallel sides								
Exactly one pair of parallel sides								
Two pairs of parallel sides								
Opposite angles are congruent								
Opposite sides are congruent								
Equiangular								
Equilateral								
Diagonals bisect each other								
Diagonals are congruent								
Diagonals are perpendicular								

Figure 10. Quadrilateral properties checklist.

In another reflection, I asked Cindy to put herself as a teacher in a mathematics class. I asked her to imagine that she observed a student making a mistake and what she would do thereafter. She wrote:

> Well, I would take that problem and work it out the wrong way, like we did in class one day. Then, as a class we work through why it is wrong. Because if they made the mistake, then there are probably more that have or may not understand why they got the answer. For example, we are in groups in class, if I had my students in groups, sometimes you have one or two that lead the group or understand. So, it is easy for the others to just copy the answer or get nervous to

> ask questions. So, this is where we address it as a class then we can cover the other students who were missed. (Journal Reflection, November 20, 2017)

In this quote, there were two salient features. First, Cindy took the ownership of the mistake away from the student that made the mistake by making it the teachers' when she was "working it out the wrong way." Second, she again assumed that others had to make the same mistake, and thus, the mistake was important enough to share. Both features place ownership of the mistake with the group.

In other instances, Cindy provided similar responses. In a classroom observation, the class was given an activity, where Cindy was given three ways that different students found the perimeter of a shape (see Figure 11). I asked Cindy after that class meeting why she thought that Dr. Heart provided those incorrect ways of finding the perimeter. She stated, "I could have made one of those mistakes. I'm pretty sure that others did. I'm pretty sure that I heard them say that" (Post Classroom Observation Interview, October 18, 2017). Again, there was an assumption of ownership by others in the class on that particular mistake and the reason why Cindy perceived the mistake to be important enough to share with the class. In her final interview, she provided several instances of affording importance of mistakes to others' ownership of mistakes. In one instance, when expanding on her choice to the first question related to if she perceived if Dr. Heart was okay with assignments not being done correctly, she stated:

> I think I haven't really met anyone in the class that gets frustrated by people getting the wrong answers. Mainly because I know talking to some of them, they have either made that mistake before in their past or they've made it in the class or they just weren't either called on or they just didn't speak up. So, it's just one of

> those things that we just all understand that, especially with geometry that, or just math in general, that sometimes it's harder for some and everyone's been there. (Current Error Climate Interview, December 4, 2017)

Here again Cindy perceived mathematical mistakes as okay with Dr. Heart because everyone has made mistakes on their assignments. Later in the same interview, she expanded on her choice for Survey Item G19–In our mathematics classroom, assignments that are not done correctly are discussed in the class. She stated:

> I'd agree with that. We've done that before. I think that it's mainly when it's an overwhelming amount of people thinking that way that's missed it where if it's just me, we don't really touch on it. We've discussed the perimeter one, there was another one, we took time at the beginning of the next class to do it. But we've done it before where a lot of people were thinking the same thing, we've taken time in class to go over it. (Current Error Climate Interview, December 4, 2017)

In all of these instances, Cindy believed that the mistake was important enough for the class to spend significant time on if she believed that it was not just her mistake. As long as she thought others can stake a claim in the mistake, then it was important enough to bring to the attention of the class.

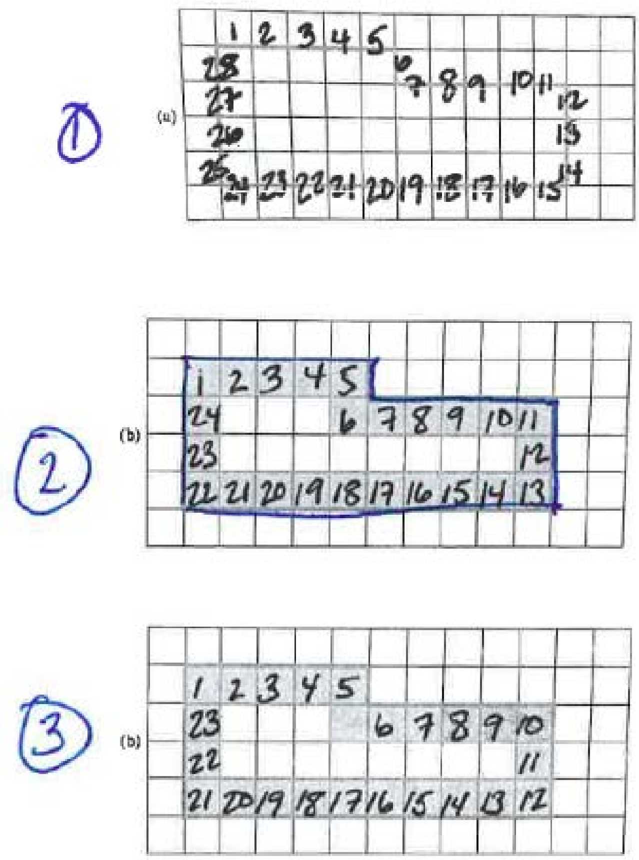


Figure 11. In-class task with two mistakes in finding the perimeter. The first picture contains a correct way of finding the perimeter while the second and third have mathematical mistakes. Adapted from *Mathematics for Elementary Teachers with Activities* (5th ed.), by S. Beckmann, 2018, New York, NY: Pearson. Copyright 2018 by Pearson Education, Inc. Reprinted with permission.

However, others in the class making the mistake or Cindy believing that others in the class made or could have made the same mistake was not the only criteria for when a mathematical mistake was important. Cindy also believed that mathematical mistakes were important when she initially had no ownership in them. As she stated, “If someone else made a mistake, let's talk about it. Let's put it on the board" (Current Error Climate, December 4, 2017). Cindy found activities where someone else's mistake was on display as being important for the class. For example, during a class meeting Dr. Heart showed a video where fourth-grade students used a broken ruler (i.e., the ends of the ruler were missing) to identify how long a pencil was. I interviewed Cindy after that class meeting and asked her what she thought about the video and the mistakes the students made. She stated:

> We really needed to see that. There are so many mistakes that could happen in a math class. Seeing their mistakes was beneficial. I could make that kind of mistake if I'm not careful, and we needed to see how kids might use a ruler incorrectly. (Post Classroom Observation, October 2, 2017)

Here, Cindy believed that the mistake was useful for the class, because she could make that mistake and students might make the mistake in similar ways.

Most of the instances where Cindy was provided with someone else's work were important to her. In another classroom activity, Dr. Heart provided a solution that was not taken from another PT in the class (see Figure 12). After being given the mathematical mistake, Cindy was asked if this mistake was useful or important for the class. She wrote:

> This is extremely valuable! The mistake that this student made is something that I can relate to. *We* [emphasis added] can all relate to. The 2 can be confusing in the picture and can be confused as part of the base, and we would all need to know that. (In-class Reflection, October 23, 2017)

Note that Cindy saw the mistake as being valuable because she perceived that others might relate to the same mistake.

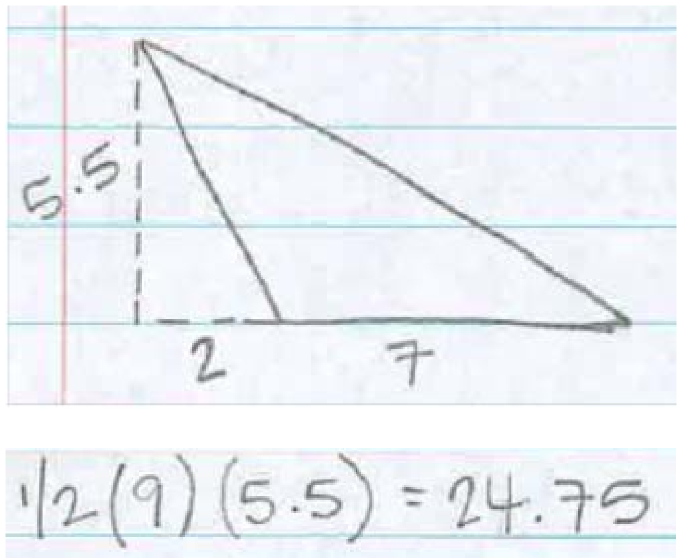
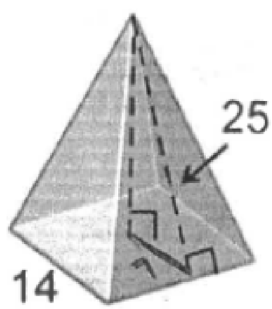


Figure 12. Cindy's replication of a student's mathematical mistake presented by Dr. Heart.

The case could be made at this point that Cindy saw these mathematical mistakes as important, because they showed common misconceptions held by learners of mathematics and were thus rightly seen as important by Cindy. However, other data suggested that, for Cindy, mathematical mistakes were important because others, and more importantly not her, made the mistake. For example, the class was given three examples of mathematical mistakes provided by Dr. Heart (see Figure 13). During the in-class reflection, Cindy was asked how she might use these mistakes if they occurred with her students. For Student 1, she wrote, "Just to go over making sure having double checked your answers" (In-class Reflection, December 1, 2017). For Student 2, she wrote, "I would use this to ask the class why is it 7 not 14 then go over why we take half" (In-class Reflection, December 1, 2017). For Student 3, she wrote, "I would use this for going back over the volume process because they missed a big step in this problem" (In-class Reflection, December 1, 2017). After that class meeting, I asked Cindy why she would do that for each student's mistake. She stated, "Because if one student made the mistake, then I'm pretty sure others did as well. So, I would need to take the time to talk to them about that" (Post Classroom Observation Interview, December 1, 2017). Cindy would choose to use class time exploring a mathematical mistake, because of her assumption that others made the mistake too, seemingly no matter what the mistake was.



Note: The base is a square and the units are feet.

Student 1	Student 2	Student 3
The volume of a prism is always the area of the base of the prism times the height. I would first need to find the height by using the Pythagorean theorem: $7^2 + b^2 = 25^2$ $49 + b^2 = 625$ $b^2 = 576$ $b = 24$ So the volume is (14*14)*(24) 1/3 9,936 ft^3 4,704 ft^3	The volume of a prism is always the area of the base of the prism times the height. I would first need to find the height by using the Pythagorean theorem: $14^2 + b^2 = 25^2$ $196 + b^2 = 625$ $b^2 = 429$ $b = 20.7$ So the volume is (14*14)(20.7) 4,057.2 ft^3	The volume of a prism is always the area of the base of the prism times the height. I know that the base of the triangle is 7 because half of 14 is 7. I would first need to find the height by using the Pythagorean theorem: $7^2 + b^2 = 25^2$ $49 + b^2 = 625$ $b^2 = 576$ $b = 24$ So the volume is 7*24 168 ft^3

Figure 13. Cindy's in-class reflection work. (December 1, 2017)

The importance of a mistake was not about the mistake itself or why the mistake was made. Cindy believed that mathematical mistakes were important if she perceived that others made or could have made the mistake. When Dr. Heart provided the class with a mistake, she perceived these mistakes to be important as she assumed that they were common mistakes made by students. This limited the value in her own mistakes as she was not always able to perceive that others might have made the same mistake.

Opportunity for others to engage. Cindy also believed that mathematical mistakes were of value if the mistake engaged others. Albeit, the teacher had to provide the space for others to engage with the mistake, but Cindy perceived the mistake as important if others engaged with the mistake. This belief was a consequence of the previous belief that mistakes were important if she could see other classmates making the same mistake. However, if that condition was satisfied, then a mistake was even more important, for Cindy, if others would engage in some activity with it. This engagement ranged from being motivated to work on correcting a mistake to participating in a whole-class discussion related to the mistake.

This belief was first evident in the Prior Error Climate Interview (September 12, 2017) when Cindy and I talked about the impact of mathematical mistakes on grades in a class. She mentioned that mistakes in class would not have an immediate impact on her grade, but mistakes on a test or a quiz would. This made me wonder what Cindy thought about those different mistakes and if one was more or less beneficial to her. She responded:

> I mean you can learn something from any mistake. That’s why they’re important. You know in class, that’s your time to shout out your ideas, your thoughts, and

> your professor can correct them or your classmates if you're in groups. But mistakes where we can all go back and work on our mistakes, like when we are given a test back and we can make corrections, we can all learn from those. (Prior Error Climate Interview, September 12, 2017)

In this context, all students had some sort of engagement with mathematical mistakes because they were able and motivated to rework those problems that contained mistakes. Shortly thereafter in the same interview, Cindy continued by describing how working with mistakes during the class worked. She stated:

> I liked it in high school when they'd write it up on the board and say how did you get it? You could go up to the board and write it and if it was wrong someone could speak out and say this is why I thought it was wrong or this is why I thought it was right. It's just talking trying to figure it out as a class or you as a person trying to figure it out to get the right answer or get that problem correct. (Prior Error Climate Interview, September 12, 2017)

In this quote, she described liking a mathematical mistake where others were able to engage with the mistake and work on it together to get the right answer. Again, later in the same interview, she described the value in mathematical mistakes through their ability to engage others in the class. She expanded on her choice to Survey Item B17–In our mathematics classroom, there is a detailed discussion when something is done incorrectly. She stated:

> Yeah, I'd agree with that. That's *the thing* that opens up the floor at least in the past, in high school, whether it's college or high school, it opens up that discussion why is this wrong. "Can someone explain it?" 'Cause if some people

> get it wrong but you get it right, you could probably explain it to us. And, also, you get to see other people's views. I think, I would agree with that.

Although she was responding to a prompt that asked about discussing a mathematical mistake, she perceived the mistake as the catalyst for opening the discussion where others could participate.

Cindy's belief that mathematical mistakes were useful when they engaged others was noticeable throughout the study as well. In the journal reflection focused on the quadrilateral activity where Cindy's group struggled with a quadrilateral with bisecting diagonals (see Figure 10), she stated, "I think it was important because others were struggling too. I wish we would have gone in more depth with it. We could have really talked about that more" (Journal Reflection, October 6, 2017). This mistake was important because others made the same mistake with her, the class as a whole could discuss that mistake. Looking back at Cindy's statements in her reflection journal where she imagined herself as the teacher encountering mistakes in her own classroom, she stated:

> For example, we are in groups in class, if I had my students in groups, sometimes you have one or two that lead the group or understand. So, it is easy for the others to just copy the answer or get nervous to ask questions. So, if we address it as a class then we can cover the other students who were missed. (Journal Reflection, November 20, 2017)

Here, Cindy imagined using the mistakes to get other students involved in thinking about the mistake. In another situation that focused on why Cindy thought Dr. Heart chose the mathematical mistakes on finding the perimeter (see Figure 11), Cindy stated, "I could have made one of those mistakes. I'm pretty sure that others did. I'm pretty sure that I heard them say that" (Post Classroom Observation Interview, November 18, 2017). I followed up her response by asking why she thought that other types of mistakes were not provided. Cindy responded, "Because those are ones that [the class] could really talk about. Like [Dr. Heart] mentioned not counting the corners twice, but we really wouldn't have been able to say much past that. She pointed that mistake out, but there wasn't anything else to say." Here, Cindy provided a case where a mathematical mistake was not as important, because the rest of the class could not really engage past what Dr. Heart had already stated. However, the other mistakes provided the opportunity for the rest of the class to engage further.

A mathematical mistake was considered valuable for Cindy as long as the mistake engaged others to participate further. It was not about the type of mistake but if Cindy perceived that others would have a need to talk about or engage with the mistake in some way. This need could be that they themselves made the mistake of that it was common mistake that their future students might make. Both of which were enough cause to necessitate further engagement.

Knowledge gained from mathematical mistakes. Cindy believed that mathematical mistakes provided her, and others, the opportunities to gain knowledge from either making the mistakes or discussing the mistakes with others. This was connected to her belief in the role that mathematical mistakes served as compared to correct answers or solution paths. As Cindy stated, "You gotta learn from your mistakes. You gotta use them to build on what you know" (Prior Error Climate Interview, September 12, 2017). There were a variety of aspects of mathematics that Cindy stated could be learned from a mathematical mistake. For example, in her Current Error Climate Interview, she talked about the difference in a correct answer being stated in class compared to an incorrect one, and she stated that everyone could learn from the incorrect one. I asked her to explain more. She stated:

> So, it's one of those things that you might learn. I might learn a better way is [*sic*] quicker or easier than the long process that I took so I think that it's one of those things. It can only build onto your knowledge. So, if I got the answer wrong and you got it correct, or vice versa, that just builds onto you. One, having to listen to it, I think again, but two, you're also figuring out you're just following through and you might be able to learn something from it. (Current Error Climate Interview, December 4, 2017)

Here, Cindy described learning as building onto existing knowledge even if she was not the one that made the mistake as it might open up opportunities to learn quick methods for solving that type of problem. Cindy was not always clear about what, specifically, could be learned from a mathematical mistake, but there were two distinct trends in what Cindy believed could be learned when interacting with a mathematical mistake.

Mathematical connections. Although Cindy was not always clear on what mathematical connections could be made when interacting with a mathematical mistake, she believed that mathematical mistakes provided the opportunity for connections among topics. For example, in the classroom observation where she was given different strategies for finding the perimeter of a shape (see Figure 11), I asked Cindy about why she would be shown strategies like those. She stated, “It helps you see other things, other things kids might be seeing. There’s connections to be made that you might be able to use later even" (Post Classroom Observation Interview, November 18, 2017). Here, Cindy saw mathematical mistakes as a way to learn even more than just learning how to fix a particular mistake, particularly how her future students might think about future topics. Similarly, in another instance, Cindy spent time during class on the day of one of my classroom observations working with a Venn diagram of squares and rectangles (see *Figure 14*). During that classroom observation, Cindy was given an exit slip asking which mistakes she took notice of the most and why. She wrote, "Trying to make other shapes square. I think it got confusing because squares fits [*sic*] in with other shapes, but those shapes don't fit in a square. It made me think about how other shapes are related" (Exit Slip, September 20, 2017). Again, Cindy saw mathematical mistakes provided her the opportunity to think about more than the correct answer and how other geometric figures are related to each other.

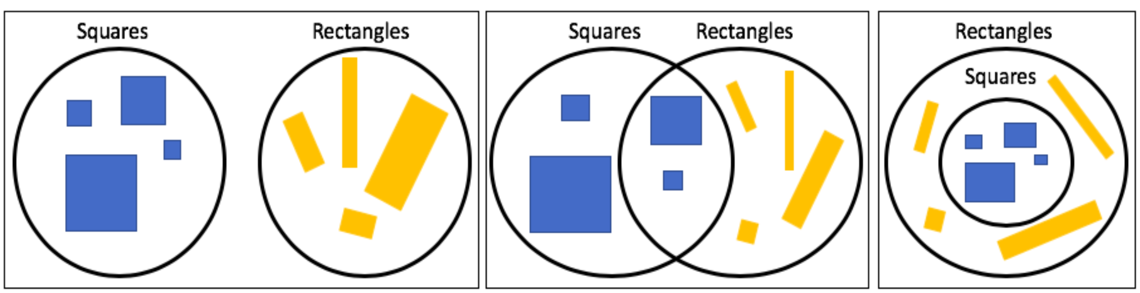


Figure 14. Venn diagrams of squares and rectangles. Presented during the classroom observation on September 20, 2017. Adapted from Mathematics for Elementary Teachers with Activities (5th ed.), by S. Beckmann, 2018, New York, NY: Pearson. Copyright 2018 by Pearson Education, Inc.

This notion came up in other instances as well. For example, in another instance in working with definitions of polygons, I asked Cindy to reflect on her experiences for the week and think about her favorite mistake as well as why it was her favorite. In the following quote, Cindy referred to an activity where she used Geogebra, an online interactive geometry applet, to identify quadrilaterals based on how a shape that started as a square could be manipulated but stay within the definition of a *hidden* shape (see Figure 15). She stated:

> I didn't understand what diagonals were and a few of the other properties. This was my favorite, because I have never gone over quadrilateral or polygons like this. Yes, I got some or half of them wrong, but, I, when we went over it, I could see why. That really helped me learn more about quadrilaterals overall and their definitions. (Journal Reflection, September 25, 2017)

Here, again, Cindy viewed mathematical mistakes as a way to gain understanding in something more than just the definitions of different quadrilaterals but something about quadrilaterals in general. I followed up with Cindy after the following class session on September 27, 2017, to ask if it was the activity that helped her or if it was the mistakes as it was unclear in her reflection entry. She stated:

> Both. The applet was a really neat way to play with the shapes and try to figure out the shape that it was supposed to be. When I realized that I was wrong on a lot of them, I went back and figured out why, which was frustrating but good, because I realized what I did wrong and why. Now I feel that I really know each quadrilateral better. At a deeper level.

In this situation, Cindy again mentioned how mathematical mistakes were used to learn more and at a deeper level than just with the activity itself.

Manipulate each shape maker on your screen and identify the hidden shape maker. Justify your responses.

Shape A is the ________________ Maker

Shape B is the ________________ Maker

Shape C is the ________________ Maker

Shape D is the ________________ Maker

Shape E is the ________________ Maker

Shape F is the ________________ Maker

Shape G is the ________________ Maker

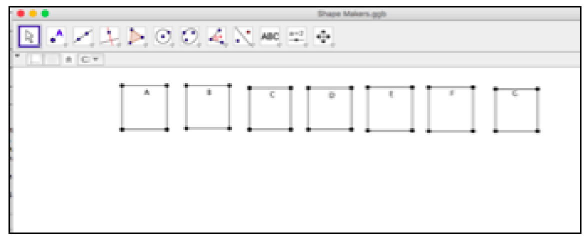


Figure 15. Hidden shape maker activity. Adapted from "Shape Makers: A Computer Environment that Engenders Students' Construction of Geometric Ideas and Reasoning," by M. T. Battista, *Computers in the Schools*, *17*, pp. 105-120. Copyright 2001 by The Haworth Press.

In another classroom observation that took place on November 29, 2017, I observed Cindy conversing with her group and the rest of the class while reviewing for the final exam. On that day, the students were given a review sheet concerning surface area and volume of three-dimensional figures. On the sheet, there were some problems worked out incorrectly where the students were expected to identify the error and talk about its significance. Additionally, the review sheet contained problems that the students were expected to work as a group, and they made mistakes while doing so. I asked Cindy again which mistakes she thought were the most fruitful. She stated:

> I feel I benefit from the whole process. I also get to learn the material on a different level. It is one thing to learn the material and apply it to my own problem, but to be able to teach it or help someone through the problem when they mess up, I feel can help you learn the material on a more in-depth level. (Post Classroom Observation, November 29, 2017)

In this case, Cindy saw that mathematical mistakes afforded her the opportunity to learn mathematics in a way to be able to teach someone else the concepts and at some deeper level than before.

Cindy talked about learning at different levels in her Current Error Climate Interview (December 4, 2017) several times. One quote representative of how mathematical mistakes help a person learn at different levels was in our exchange focused on how someone that got the answer correct on a problem can use mathematical mistakes to learn more. I again asked Cindy to explain more on how that was the case. She stated:

> It's worth it, because you are having to kind of teach in like, not recycle, but you're having to teach what you know. So, you have that like that base like that first level knowledge of how to get the correct answer and then you have that next, relating it to a building, you get that next level that next story of having to teach it. But then a mistake lets you go further. So, you have to figure out, okay, one, where did they go wrong and two how am I going to do this or how am I going to explain this where it's easier and then the next level is just kind of how to

> teach it to them without telling them the answer. That's been hard. Working with some people in my group, it's hard to, you can't give the answer, or they won't learn more than that. (Current Error Climate Interview, December 4, 2017).

In this case, mathematical mistakes were at the epicenter of learning for her and the other classmate. As Cindy stated, she used the mathematical mistake to get to the next level of understanding and to help a fellow classmate. Giving the correct answer would have stopped any learning in progress.

Cindy believed that mathematical mistakes were opportunities to learn more. Specifically, mathematical mistakes were opportunities to learn more about that mathematical concept and how that concept was connected to other mathematical concepts. In each case, this meant how to teach that concept, how to learn more about that particular concept, or how to think about other concepts similarly.

Slowing down and double checking work. Cindy not only believed that she could learn mathematical connections from mistakes, but she also believed that mathematical mistakes could teach her to "slow down" and to "double check" (Current Error Climate Interview, December 4, 2017) her work. Throughout the study, Cindy talked about the importance of mathematical mistakes for learning, so I continuously asked Cindy why they were important and what she was learning from making them or seeing them in class.

The first instance that Cindy mentioned learning something different from mathematical mistakes not related to content was in her Prior Error Climate Interview (September 12, 2017). She was finishing up talking about how mathematical mistakes were used to make sure that she really understood something (Survey Item B20). She had agreed with that statement, but I was not sure what she thought you could understand from making a mistake, so I asked her to elaborate. She responded:

> Because, that's what you learn from. The homework we'd get back, we'd get it wrong. Some classrooms we'd go over why we got it wrong. Some were kind of just like you got it wrong. So, I think that it helps you see or know where you made that mistake. Like if you didn't carry the one or you did didn't subtract the right side, you see where you went wrong, and you could redo it. (Prior Error Climate Interview, September 12, 2017)

In this quote, the mistake got her attention, and she was aware of where that mistake was made. Cindy stated that a mathematical mistake allowed her to understand that she did not perform some arithmetic operation correctly. Immediately after this quote, Cindy responded to Survey Item B21. She stated:

> It's important, like the correct way of doing it or the correct, the correct answer, but it's also [important] to see why it's incorrect because you can easily make those mistakes. Especially if you are on a test and you are timed, you're trying to zoom through and get all the answers done. Going too fast. So, if you work on that, I don't think you should spend amounts and amounts of time on it, but, if you do take out time to work on why they're incorrect or why they're wrong, I think it only builds up your I don't know. It kind of builds on your toolbox or your building of knowledge. (Prior Climate Interview, September 12, 2017)

Here again, Cindy attributed the mathematical mistake to her rushing though problems and the mathematical mistake would “build on her knowledge” by working on not going too fast. After this interview, I wondered if Cindy would continue to use mathematical mistakes to learn to slow down, so I continued to ask Cindy what she was learning when mathematical mistakes were presented to her.

Cindy shared numerous instances throughout the study on how she learned to slow down and to take her time from a mathematical mistake. In the journal reflection after the working with the Geogebra applet (see Figure 15), not only did Cindy share that she learned more about quadrilaterals, but she also stated:

> My mistake that I made this week was on homework #5. I thought I did so well on it, but I did not do well at all. I missed 7 of the angles out of 13. I learned that I need to take it slower and double check my work. I got overwhelmed with all the missing angles. (Journal Reflection, September 25, 2017)

Although Cindy had previously stated that she learned to make connections among the quadrilaterals, she also attributed the mistakes to her rushing through the problems. Seeing that she made mistakes after feeling confident in getting them correct allowed her to learn to slow down in the future. In another reflection, Cindy provided similar sentiments. In one of the reflection prompts for this journal entry, I asked Cindy to think about what her mathematical mistakes looked like for that week before asking her to imagine that she was a teacher in a class where several mistakes were made by her students. She wrote, "I struggled with homework. I need to start taking my time and looking over my work and double checking my work. I looking [*sic*] over my missed problems, and I realized that I had made little mistakes" (November 20, 2017). Again, Cindy used mathematical mistakes as a way to learn to double check her work. Additionally, Cindy also brought up that in her reflecting of her mistakes she noticed they were "little mistakes" which she usually called "minor mistakes" (Current Error Climate

Interview, December 4, 2017). I started to think at this point that Cindy believed that she learned to slow down or to double check her work only when the mistake that she was referring to was a procedural error.

However, there were two other instances where this was not the case, or at least, it was not clear that the mistake was procedural. The first instance occurred on October 20, 2017, during a classroom observation. Dr. Heart displayed a mistake on the board from a homework problem (see Figure 16). After showing the mistake, Dr. Heart asked the class why this strategy did not work. Immediately, I observed Cindy stating, “I did it that way” (Classroom Observation, October 20, 2017). The class talked in groups and had a group discussion coming to the conclusion that the 8 and the 10 did not match up. After that classroom observation, I talked with Cindy about, what we called, the *why conversation*.

Cindy: Yeah. I did it that way. I thought it was right until she said to say why it was wrong. I did exactly what she said not to do or what was wrong to do. I didn’t think about the 8 or the 10.

Me: So why show this?

Cindy: To make sure that we slow down and take our time. That way we can do the steps and double check it.

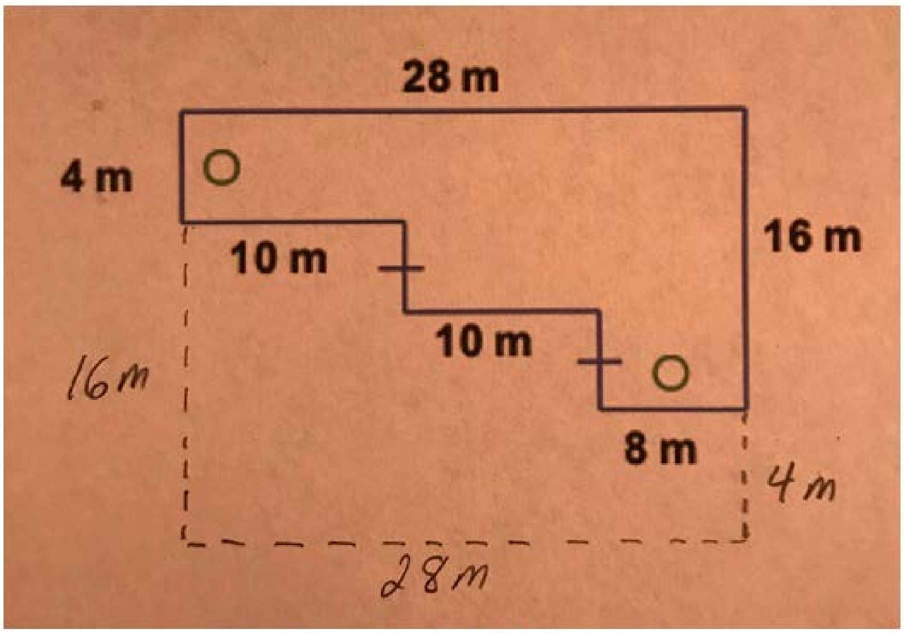
Me: Okay. Was this what you would have called a minor mistake.

Cindy: I don’t think so.

Me: So, why show this one to you and talk about it as a class?

Cindy: I think we can get into a rush and this helps me think about slowing down and making sure I or we do the steps correctly. (Post Classroom Observation, October 20, 2017)

In this case, even though Dr. Heart showed an error that Cindy did not classify as a procedural or what she sometimes called a “minor mistake” (Current Error Climate Interview, December 4, 2017), it was still an error that she believed taught her to slow down and to double check her work.



Area of original figure is

$$\frac{1}{2}(20)(28) = 280m^2$$

Figure 16. Recreated non-procedural mistake provided by Dr. Heart.

The other instance, where the mistake was not necessarily what Cindy would classify as a minor mistake but still taught her to slow down, occurred during the in-class reflection where Cindy examined a student’s work concerning the volume of a pyramid (see Figure 13). Cindy was asked to identify the error in each of these students’ work.

When responding to Student 1's work, she stated, "The error here was made at the end of the problem. They did not multiply the equation correctly. Other than that great job. [They] forgot to 1/3 [it], and it's not a prism" (In-class Reflection, December 1, 2017). She correctly identified the error in the calculation and pointed out that the students identified the shape as a prism, which utilized an area of the base times height formula and excluded the one-third, which I would classify as a conceptual error. When responding to how she would use these mistakes, if at all, to help with students' understanding of volume, Cindy stated that she would use it "just to go over making sure having [*sic*] double checked your answers" for Student 1's work. In Student 1's work, the volume was incorrectly found by not taking one-third of the area of the base of the prism multiplied by the height. Although this could be interpreted that the student simply forgot the one-third in the formula, Cindy recognizing the shape as not being a prism suggested that it was more than a procedural error.

Cindy's statements in the interview that followed the Current Error Climate Survey (December 4, 2017) supported my thought that she learned to slow down and double check her answers only on minor mistakes. She made two statements specifically pertaining to what is learned from a mathematical mistake in this interview. The first instance was in her response to how much can be learned from assignments that are not done correctly (Survey Item G21). She stated:

> I've learned a lot from our homework assignments, the ones that I've gotten wrong. Whether it was major or minor. If it's minor, I've learned to take my time with the assignment and not rush through. If it's major, I kind of learn that, I learn

> that missing, I'm missing a key part of the process, and it kind of helps me. (Current Error Climate Interview, December 4, 2017)

Here, Cindy stated that she learns to slow down only on minor mistakes, and major mistakes was where she learned more about the content itself. The second instance was her response to me asking what a mathematical mistake told her about a person. She responded:

> I think that it kind of tells me where they lie, how to figure out the problem. If it's just little tiny mistakes, I know they understand the concept, they just need to take the time to slow down and write it out and two, double check their work. It's kind of like making sure they focus on that too. If it's something that's a major problem where they are not understanding the concept. You have to take that time to kind of pinpoint what they are not understanding and try to build off that. Try to figure out how to word it a different way or bring up different like manipulatives or something like that to help them. (Current Error Climate Interview, December 4, 2017)

This quote again confirmed that Cindy believed that learning to slow down and to double check her work comes from experiences with minor mistakes. In both of these instances, Cindy stated that a minor mistake offered the opportunity to learn to slow down and to double check the work, and a major mistake offered the opportunity to learn more about the content itself or connections to other mathematics, hence, the need to use manipulatives.

Whether Cindy correlated learning to slow down and double-checking work with minor mistakes or not, Cindy believed that mathematical mistakes offered the opportunity for her to learn to do that. Additionally, this is where Cindy's central belief that mistakes and correct answers or strategies play a role. Correct answers cannot offer the opportunities for learning like mathematical mistakes can and do. Additionally, it is worth noting again that Cindy's beliefs concerning what she could learn from a mathematical mistake focused on herself learning mathematics as a student. In this case, the mistakes served a role for her to re-examine her work on a problem because she was going too fast.

Affective component of mathematical mistakes. Finally, Cindy believed that mathematical mistakes had a strong affective component. Although Cindy believed mathematical mistakes were where "you learned" (Prior Error Climate Interview, September 12, 2017) and it was "okay when you mess up" (Post Classroom Observation, November 15, 2017), Cindy also believed that mathematical mistakes were strong enough to embarrass her to the point of shutting down. This was especially true when Cindy made a mistake in isolation, and she was not able to imagine other students in the class taking ownership in the mistake. As Cindy stated, "When I get told I am wrong, most times I feel, this may sound silly, bad. Most times, I feel embarrassed about making the mistake. In a way, I shut down" (Post Classroom Observation Interview, November 20, 2017). A mathematical mistake was powerful enough that Cindy would quit even though Cindy saw the value in mathematical mistakes for learning.

The affective component of mathematical mistakes was always at the forefront of Cindy's thinking, and it permeated throughout the study. Near the beginning of the study, Cindy shared experiences from her previous classroom where she was embarrassed or perceived a classmate to be embarrassed because of a mistake. In one instance, she explained why she agreed that a student could be mocked for making a mistake. She stated:

> I could see, not mocking. It's never happened to me. I've answered a question and it be way off based and some people laugh or kind of chuckle ‘cause they got the right answer. I could see that, especially like, not honestly in a standard class. ‘Cause in a standard class in high school, it's one of those where you all knew that you were all on the same level or a little under. But I know like when I was some honor classes, but not for math, other students kind of put you on a higher pedestal so you should know this. So, when you get it wrong they're kind of like “Psssh. Like how could you not get that right?” So, I'd say, I'd somewhat disagree ‘cause I don't think that happens often or at least that didn't happen often to me. It like one or two times. So, I'd say somewhat disagree. (Prior Error Climate Interview, September 12, 2017)

There was an expectation for her to get the correct answer, because she was in an honors class. In either case, Cindy felt embarrassment because she made a mathematical mistake. Her feeling embarrassed because of an expectation to get the correct answer resurfaced later in the interview. She elaborated on why students in a mathematics class might not want to provide an answer to a question for fear of being wrong, and she shared more of her thoughts related to being embarrassed because of her expectation of getting the answer correct:

> The teacher would be like, okay more people answer, and no one would ‘cause it's like that thought process [of] I don't want to say this wrong. They do think they are going to get teased, [but] it never happened. But, it's one of those things, I also

> don't want to look dumb. Like I don't. That's my thought. I don't want to appear like I didn't give it all or that I'm dumb. That's what stopped me sometimes from asking questions ‘cause I don't want that, I don't want that professor or teacher to think, “Wow, she is dumb.” Which I know that they don't think that, ‘cause I mean that’s all in the process. But it is still hard to get that out of your head that they're gonna think that “Why did you not get this? How could you have not gotten this?” (Prior Error Climate Interview, September 12, 2017)

Here again, Cindy described a situation where a potential mathematical mistake made her feel embarrassed because of personal expectation to respond correctly. Throughout the Prior Error Climate Interview, Cindy described how mathematical mistakes made her feel “sad” and “disappointing” (Prior Error Climate Interview, September 12, 2017), because she felt that she was supposed to be getting those questions correct in class.

However, Dr. Heart’s class was designed differently, and Cindy noticed the difference. As she stated, Dr. Heart “welcomes mistakes. She wants to see them” (Current Error Climate Interview, December 4, 2017). I asked Cindy about how mathematical mistakes made her feel in one of her reflections during Dr. Heart’s class. I asked her what was the first thing that came to mind when she made mistakes in a mathematics class. She responded:

> My first thought is always, “Oh my word how could I make that mistake? How can I fix it?” Nine times out of ten I am embarrasses [*sic*] that I have made the mistake. I will shut down and not want to really talk to anyone. I just don't like to make a mistake in front of everyone. I feel dumb too. Honest. I just don't like to mess up in front of others. I need to remind myself that it is okay, because that is

> a part of life. It is hard to take your own advice. (Journal Reflection, October 26, 2017)

Her embarrassment and notions of feeling dumb had not changed, but she still saw the importance of mathematical mistakes. She made similar comments after a classroom observation where the class was given two-dimensional views of three-dimensional shapes and was asked to recreate the three-dimensional shape on isometric paper. During that class session, I observed Cindy making several mistakes in recreating the three-dimensional shapes. After class, I talked with her about the times that she found out that she was incorrect by either being told by Dr. Heart or another person in the group. I asked her how being told that she was incorrect differed from when she was told that she was correct. Specifically, what did she do differently in response? She stated:

> When I get told I am wrong, most times I feel, this may sound silly, bad. Most times, I feel embarrassed about making the mistake. In a way, I shut down for a few seconds. . . . I have no smile. (Post Classroom Observation, November 20, 2017)

Even though this was early in the process of working with three-dimensional figures and the isometric paper, Cindy still felt embarrassed from making a mistake and did not want to interact with anyone for a short period.

In most situations, Cindy expressed that she felt the embarrassment from a mistake, but there were times that she did not. In the following exchange that occurred during the Current Error Climate Interview, she offered an instance of when that happened. In this exchange, Phillip, a pseudonym, was a supplemental instructor for the class.

Me: So, you seem very active, that might not be the right word. Willing? Maybe that's the right word. Willing to share a mistake now?

Cindy: Oh yes!

Me: Previously you were saying that you didn't want to be called on because you could be wrong.

Cindy: Definitely from when we started the course to now. Now, I mean, I might get a little embarrassed, but at the same time, I'm just kind of like alright I made this dumb mistake, or I've made this. I shouldn't call it dumb. I made this mistake or this major mistake. Someone help me. Whereas the first few weeks of class, I was just like, okay, don't tell anyone that you made a mistake, just talk to Phillip in tutoring and move on with your business. Or, you talk to Dr. Heart, but I definitely, I'm definitely, I've grown more. Okay, it's a mistake. Someone has made this mistake, and I've kind of more comfortable with it. Still a little embarrassed, but I'm better about saying alright I did this wrong. Someone help me. Where, I was just like, until no one is around, and then I'll talk about it. But, it's definitely, I feel more comfortable with it. I'm pretty sure that I could go into another math class and take a week or so, and get to know those people, and then I'd be like I made a mistake. Someone help me. (Current Error Climate Interview, December 4, 2017)

Although Cindy still recognized mathematical mistakes' influence on affective qualities, here, she expressed sentiments that she was starting to overcome those in light of the value of mathematical mistakes and how she can learn from them.

Cindy believed that mathematical mistakes were a powerful force that influenced how she participated in class. The way that making a mathematical mistake made her feel forced her in some cases to not participate further and shut down altogether. Cindy strongly believed that mathematical mistakes influenced affective qualities, especially when she made the mistake in isolation, or said another way, when she believed that no one else would claim ownership of a mistake she made.

Noticeable Changes in Beliefs System

First, this section necessitates a comparison of Cindy's Prior Error Climate Survey and Current Error Climate Survey (see Table 7). The most noticeable difference in Cindy's survey results is how she perceived mistakes to be received by both Dr. Heart and her classmates (i.e., absence of negative teacher and classroom reactions to errors). These differences were also accounted for in other data collected from Cindy. Although Cindy described experiences in Dr. Heart's class as embarrassing at times (see Affective component of mathematical mistakes), she described Dr. Heart as:

> [Willing to] open up the class if mistakes are made. She opens up like group discussions on how we could fix it or what went wrong or whatever with the problem, so I'd say she welcomes it. She's not mad about it or anything like that. (Current Error Climate Interview, December 4, 2017)

She also described her classmates in a similar fashion. She described the classroom as "comfortable . . . [because] . . . we've all made major and minor mistakes in the class" (Current Error Climate Interview, December 4, 2017), and "We're all still learning. So, I think that everyone is okay when you mess up" (Post Classroom Observation, November 15, 2017).

Table 7

Cindy's Prior and Current Error Climate Results

Error Climate Dimension	Prior Error Climate Average	Current Error Climate Average
Error Tolerance by Teacher	*M* = 2	*M* = 2.25
Irrelevance of Errors for Assessment	*M* = 3.5	*M* = 2.5
Teacher Support Following Errors	*M* = 1.85	*M* = 1
Absence of Negative Teacher Reactions to Errors	*M* = 3	*M* = 1
Absence of Negative Classroom Reactions to Errors	*M* = 2.5	*M* = 1
Error-taking Risk	*M* = 5.5	*M* = 3.5
Analysis of Errors	*M* = 2.67	*M = 1.33*
Functionality of Errors for Learning	*M* = 2.25	*M* = 1
Superordinate Score	*M* = 2.91	*M* = 1.70

Note. Scores on the survey ranged from 1 (strongly agree) to 6 (strongly disagree), with averages of 3 and below indicating a positive error climate and averages of above 3 indicating a negative error climate. A lower average indicates a more positive error climate.

Additionally, two other dimensions were notably different in her surveys: analysis of errors and functionality of errors for learning. These two dimensions demonstrated that Cindy perceived mathematical mistakes as being treated differently in Dr. Heart's class in terms of how she examined mistakes and what she was able to learn from inspecting a mistake.

The differences between Cindy's perceived error climate in past mathematics classrooms compared to Dr. Heart's classroom were important to notice in light of the changes in Cindy's beliefs system throughout the study. The next two sub-sections will describe the changes that the data revealed in Cindy's beliefs system concerning mathematical mistake. A summary for the case of Cindy will follow.

What is learned from a mistake. Throughout the study, Cindy believed that mathematical mistakes were used to learn the correct way of doing a problem. As she stated, mathematical mistakes were a way to "seeing the correct answer" (Current Error Climate Interview, December 4, 2017). This belief endured throughout the study, but Cindy also provided comments during the study that revealed small changes in the connections to this belief. Specifically, Cindy believed that if there was a mistake present, then it had to be fixed and that was the end of the interaction with the mistake. From that, she could learn mathematical content or that she should slow down. However, throughout the study, Cindy made small changes in her belief of what could be learned from a mathematical mistake.

In Cindy's Prior Error Climate Interview, Cindy acknowledged the benefit of seeing a mistake even if she had worked a particular problem correctly. She told a short story of a time when she helped someone that had made a mathematical mistake.

Cindy: Yes. In high school, I had a really good friend, we, I don't know how we were always in the same math class together. It was one of those things that was a better subject for me than it was for him. So, once I got it, I would explain it to him and start working with him. He'd usually ask. He would rather ask the person beside him than the teacher and so, we'd work together on homework or I would write out all of his steps and show him how I did it. It was, it was constantly. We were always sitting next to each other. We worked mainly together.

Me: So, when you are showing him your work. How do you think that helped him?

Cindy: It gave him a different perspective than what the teacher had. ‘Cause sometimes, nine times out of ten, I did it exactly how the teacher taught me, but sometimes I'd do it a little differently or try to find a shorter way or it may be longer way but it was a different way for him to look at it, and it came from a peer. So sometimes I think that it clicks more when it comes from someone at the level as you. But, that, helped him because he could see it two different ways or two different people and it came like a wave of her or him discussing it and a way of me talking to him about it. (Prior Error Climate Interview, September 12, 2017)

In this quote, Cindy recognized the importance of seeing different strategies to working a problem. However, in this quote, Cindy was referring to different correct strategies, not strategies that contained a mistake. She told a similar story in the same interview when explaining how wrong answers were used to learn something.

> I think so we'd go over the homework or if we did group work, we'd explain our answers. So, if we got something wrong or another group got something wrong or another person did, you kind of *saw* [emphasis added]. Like if they had to write it

> on the board. You could see, you could kind of pinpoint. Like if you got it right, you could pinpoint how they got it wrong. Or, if I was up there and got it wrong, someone else could tell me. I would say that would happen often. Like we would go up to the board and write our answers. Because we were showing the class like this is how we got it and then it was either correct. Junior year, she'd say is this correct or incorrect like as a class and so if it was incorrect the whole class would say incorrect and if they got the answer correct. (Prior Error Climate Interview, September 12, 2017).

It is important to take note that Cindy “saw” something when others were explaining their answers that were incorrect. It appeared that she was seeing how to get the correct answer, but there is a possibility that she was also seeing other strategies that she did not think of initially, even if they resulted in an incorrect answer. Although this might be an unsupported inference, I point this instance out as Cindy was already noticing the usefulness of alternative strategies in her past mathematics classrooms without the teacher, or anyone else, pointing this out, and thus, she was primed to make changes in her beliefs of what could be learned from a mathematical mistake.

As I analyzed more of Cindy’s data, I noticed that Cindy was taking more away from a mathematical mistake than how to do it correctly. For example, during an early Journal Reflection, I asked Cindy what she learned from her favorite mistake from the week. She wrote:

> I learned that I need to take it slower and double check my work. I got overwhelmed with all the missing angles. I learned that I could have taken the problem from a different approach. Dr. Heart showed us in class another way to

> look at how to solve the angles, in a wrong way. I am still confused on how I went wrong, but I am going to figure it out. (Journal Reflection, September 26, 2017)

Although Cindy also stated that she learned to double check her answers, she stated that she was learning different approaches to finding missing angles. In another instance that occurred during class, the class was given the following exit slip: “If a student stated, ‘as the perimeter of a closed polygon increases, the area also increases,’ what does it tell you about the student’s thinking?” She responded:

> They are probably only thinking of one type of polygon. I made the same mistake I only had a thought of one shape. I think they understand if they increase one side it will. But the student didn’t play around with other shapes. One group here did triangles. They messed up. I wouldn’t have thought of that to use that shape. (Exit Slip, October 4, 2017)

In this case, Cindy again stated that she was able to gain something more from a mathematical mistake than the answer. There was another instance, excluding the final interview, where Cindy provided evidence that she believed that mathematical mistakes provided more than insight into mathematical content. In a Journal Reflection, I asked Cindy to think about the mathematical mistakes from class that week, and I asked her what her general thoughts were on activities where mistakes were the focus. In the journal entry, she referred to an activity from class where they were asked to imagine that one of their students told them that she had figured out a new theory that as the perimeter of a closed polygon increased, the area also increased. The student also provided two examples, a four by four and a four by eight rectangle with the perimeter and area calculated for each, to support this new theory. Cindy wrote:

> I like them. I think this gave us a chance to see another perspective, but also how to see if that is correct. The goal, I feel, was to see what could happen if a child brought this idea to us. Does it work? Does it work for every shape or etc. [*sic*]? I think this was worth going over. I did not occur to me that students would do this or even to try something like this. (Journal Reflection, October 11, 2017)

Again, Cindy shared that even though this activity contained a mathematical mistake it provided her with insight into how to employ different strategies in mathematics.

In the Current Error Climate Interview, I wanted to get a better sense of what Cindy believed that she could learn from the examination of a mathematical mistake. All of the following quotes and exchanges occurred during that interview on December 4, 2017. In the first instance, Cindy talked about why mistakes were not viewed as a bad thing for Dr. Heart (Survey Item G4), and Cindy explained it was partially because of the benefits of mathematical mistakes. She then explained how mistakes were beneficial:

> I mean for me I look back on what I did, because I've learned, in math in college with math, I could get the answer one way and you could get it another way and they could be the same, both correct. So, it's one of those things that you might learn, I might learn that your way is quicker or easier than the long process that I took, so I think that it's one of those things. (Current Error Climate Interview, December 4, 2017)

Again, Cindy stated that mathematical mistakes provided her insight into other strategies that she would not have seen if no mistake made. Similarly, in another instance, she talked about how the class spent time on students' incorrect statements:

> ‘Cause we'd have to watch those videos. Like you've have the one kid that got it right, we'd discuss that and how they got it right but then you'd get three or four that got it wrong and so but they'd each get it wrong differently. But they'd each get it wrong differently, so we each *got* [emphasis added] to see four different ways that a you could get one problem wrong, so we'd *get* [emphasis added] to talk about those. (Current Error Climate Interview, December 4, 2017)

Here, Cindy expressed that mathematical mistakes allowed the opportunity to examine different strategies and the importance of doing that. In other instances during the interview, she described mathematical mistakes as ways to “invite criticism on different ideas” and as ways to “give [others] your ideas.” However, there is one other quote that was representative and summative of her beliefs concerning what else can be learned from a mathematical mistake in addition to content. Cindy provided the quote below after explaining how she thought a mathematics teacher was now like a conductor and how a mathematics teacher like a conductor would treat mistakes. Then, I asked how this was different compared to other content areas (e.g., history or literature). She replied:

> In math, there are so many different possibilities and so many ways to get answers. I feel, you have to take more time to discuss the problems and the answers, the correct and the wrong answers. ‘Cause there's so many different ways to get them. Whereas in other classes, it's “this is one way. Learn this one way.” Then, you kind of move on. In math, it kind of opens the door to multiple different things. Whether you make a mistake three or four different ways, or we can get two different answers. *We get to see that, those ways* [emphasis added] and how they went about it. (Current Error Climate Interview, December 4, 2017)

Here, the different ways of working a problem, both correct and incorrect, were worth noticing to Cindy. It was no longer limited to seeing the right way or even to be able to point out the mistake. Cindy now believed that the strategies that led to those mistakes were beneficial in their own right for her learning as a student of mathematics.

Embarrassment from a mistake. As described in the affective component of mathematical mistakes section, Cindy believed that mathematical mistakes were powerful influences on affective qualities. Particularly, Cindy believed that mathematical mistakes could embarrass her to the point of quitting. However, as Cindy's experiences with mathematical mistakes accumulated, I noticed that Cindy started to believe in her own advice that mathematical mistakes were "part of the learning process" (Prior Error Climate Interview, September 12, 2017) and nothing of which to be embarrassed. I first started to notice this change on Cindy's second Reflection Journal entry. She was writing about her experience with a task involving the Geogebra applet (see Figure 7), and she realized she made a lot of mistakes. In her reflection, she stated:

> This was my favorite, because I have never gone over quadrilaterals or polygons like this. Yes, I got some or half of them wrong, but when we went over it I could see why. It helped me learn more about quadrilaterals. (September 17, 2017)

Cindy expressed the slightest insight that she was able to overcome the embarrassment of a mathematical mistake because of its usefulness. Later in the study, I observed a class session where the class explored the Pythagorean theorem and found Pythagorean triplets in the groups using trial and error. Although I observed Cindy make some calculation mistakes, neither Cindy nor the class made mistakes that were highlighted in class. I talked with Cindy after that class session, and I asked her if she thought she got more out of a class meeting where mathematical mistakes were the focus or class meetings like the one that I observed on that day. She stated:

> I feel when there are many mistakes made it makes things harder. I know that may sound crazy but when the whole class is struggling, that could cause more confusion if we are struggling with different parts of it. At the same time, it has been beneficial. Because we get a chance to see how others may come to the correct answer. And try different things to get the correct answer. When there is [*sic*] less mistakes made, I honestly feel a class is more upbeat. Everyone seems more confident and less upset and angry, but I'm getting better at that with what we're doing. I would have to say, there is a happy medium. I hope that is an okay answer. Because when there are mistakes it gives us a chance to help others or learn from them but at the same time when there are not many mistakes it makes class more upbeat. (Post Classroom Observation Interview, November 15, 2017)

Although Cindy still demonstrated that mathematical mistakes can embarrass or upset her, she stated that the benefits could outweigh the impact on her affective qualities.

Near the end of the study, I became curious if Cindy's beliefs concerning mathematical mistakes and their influence on affective qualities had changed at all, so I spent several instances in the final interview asking Cindy questions related to those beliefs. The first instance was when she shared if she thought students would rather say nothing than say something incorrect (Survey Item G15). She mentioned that the class made her comfortable enough saying things that were wrong, so I asked in what way. She responded:

> I guess it's more of like as the class has gone on, I've made those mistakes in front of the class, and I just know, we're kind of like a closer group and I've gotten more comfortable making mistakes in front of them. Like the very first time I made a mistake, I just kind of completely shut down and quit talking for the rest of the class 'cause I was embarrassed. I was like that was dumb. You're in college kind of thing. That's just how I felt. But now, it's kind of like okay, it's okay. Like, we've done that. I feel it's like more, it's a more comfortable setting than it was at the beginning of class. Even if it's major or minor [mistake]. (Current Error Climate Interview, December 4, 2017)

Cindy started to believe that mathematical mistakes can be made in an open setting without having the fear of embarrassment if it is the right audience. A few minutes later in the interview, I asked her why she would not want to be called on in light of how she had described mathematical mistakes as being beneficial.

> But I still somewhat agree with that. I still think people don't want to be called on. I guess I don't want to be called on, because I know that I have the wrong answer or I'm on the fence. This is probably incorrect, so I'm just like call on Lacey, call on Andy (both pseudonyms) or somebody else. (Current Error Climate Interview, December 4, 2017)

I responded by telling her that it was interesting that she said that because of how she had talked about mistakes thus far. She responded:

> I know. It's more of, I'm totally fine if someone else makes a mistake. I'm like it's fine, let's talk about it. But if, I don't know if it's just, I just get so embarrassed. Sometimes for me, I just draw back in. I know that it's okay to make a mistake. I

> tell myself all day long, but as soon as it gets put up in front of the class, I kind of just like, oh, don't look at me kind of thing. Except for this last time. We were doing block, the visual thing and I added an extra block. That was the first time in the class that where I was just like eh. . . . I was just like. Yep. Now where can I go from here? (Current Error Climate Interview, December 4, 2017)

Cindy referred to a time in class when she incorrectly drew a three-dimensional shape on isometric paper from the two-dimensional faces of the shape. She experienced a time where her mathematical mistake was not embarrassing, and it actually spurred a whole-class discussion about why it would not be what Cindy drew for the class (Classroom Observation, November 20, 2017). As she later said, “It open[ed] a new doorway to learn something else.”

Although Cindy’s belief that mathematical mistakes could still be embarrassing, there was evidence to suggest that the classroom climate influenced that belief to change. She still believed that mathematical mistakes could be embarrassing, but given the right circumstances, she believed that there should not be embarrassment. There was one more time in the Current Error Climate Interview that Cindy provided evidence that this belief was changing. It was near the end of the interview, and she was talking about how she can learn different things depending on what a mistake was. I challenged her on her statement because she mentioned earlier that students do not usually like to share their mistakes. She replied:

> Definitely from when we started the course to now. Now, I mean, I might get a little embarrassed, but at the same time, I'm just kind of like alright I made this dumb mistake or I've made this, I shouldn't call it dumb, I made this mistake or

> this major mistake. Someone help me. Whereas the first few weeks of class, I was just like, okay don't tell anyone that you made a mistake, just talk to Phillip in tutoring and move on with your business. Or you talk to Dr. Heart, but I definitely, I'm definitely, I've grown more. Okay, it's a mistake. Someone has made this mistake and I've kind of more comfortable with it, still a little embarrassed, but I'm better about saying alright I did this wrong. Someone help me. Where, I was just like, until no one is around and then I'll talk about it. But, it's definitely, I feel more comfortable with it. I'm pretty sure that I could go into another math class and take a week or so, and get to know those people, and then I'd be like I made a mistake. Someone help me. (Current Error Climate Interview, December 4, 2017)

Cindy chronicled the change in her belief in this quote. She even showed promise of the belief enduring to the next class.

Although it was difficult to state with certainty that Cindy made any lasting changes to her beliefs system, the evidence presented in this section illustrated that Cindy reconciled experiences in Dr. Heart’s class and this study that were dissonant with her current beliefs system. Cindy showed that she took the experiences from the class and made them “make sense” (Leatham, 2007, p. 187) into her beliefs system with changes in how mathematical mistakes are used in the classroom and what can be learned from a mathematical mistake.

Summary

The previous sections focused on the case of Cindy. Specifically, this section described Cindy’s implicit theory as being neither an extreme incremental or entity theorist and provided a description of Cindy's perception of her prior mathematical experiences. This provided insight into Cindy's experiences that helped shape her beliefs concerning mathematical mistakes.

Cindy's central beliefs concerning mathematical mistakes were also described. Figure 17 provides a visualization of the different roles Cindy believed that mathematical mistakes and correct strategies had in the classroom. Furthermore, the figure shows how Cindy believed that through mathematical mistakes being fixed she learned mathematical connections and learned to slow down on certain parts of problems. It is important to note that all mathematical mistakes, both procedural and conceptual, are together in this figure as Cindy believed that mathematical mistakes were where learning took place.

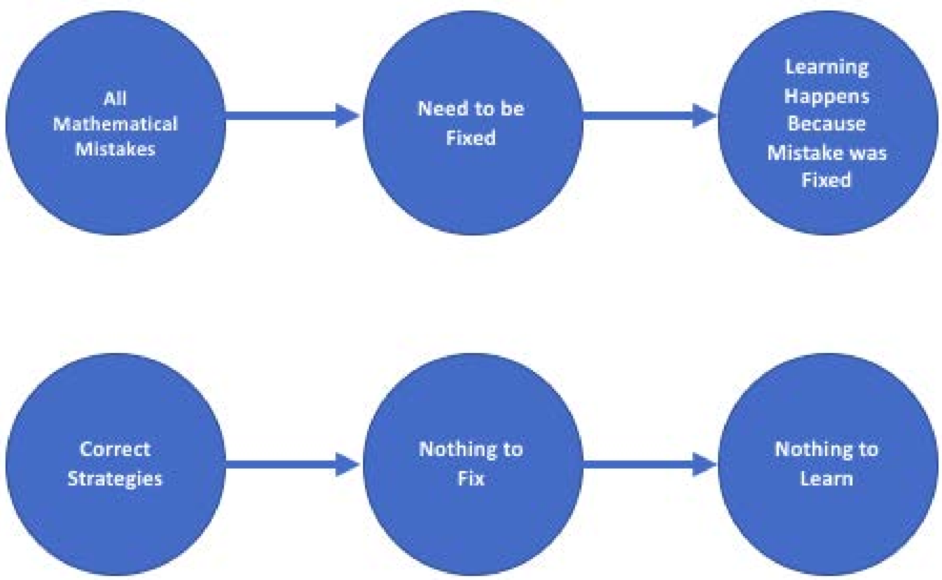


Figure 17. Cindy's central beliefs concerning mathematical mistakes.

Cindy’s belief clusters influenced by those central beliefs concerning mathematical mistakes were also detailed. Specifically, Cindy believed that there were times when mathematical mistakes were more important for the class to see (i.e., when she perceived others made or could have made the same mistake and when the mistake provided the opportunity for others to engage with the mistake), that she could learn mathematical connections and to slow down when a mathematical mistake is fixed, and that a mathematical mistake was influence by its impact on affective qualities. Figure 18 displays Cindy’s beliefs clusters concerning mathematical mistakes.

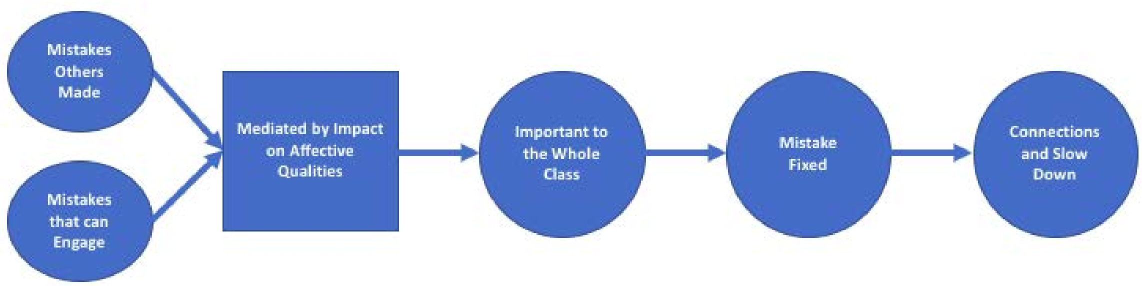


Figure 18. Cindy's beliefs clusters concerning mathematical mistakes.

Finally, noticeable changes in Cindy’s beliefs system concerning mathematical mistakes were presented. Specifically, the data revealed that Cindy believed that the impact of a mathematical mistake on affective qualities was diminished given a classroom environment where the class was accustomed to making mistakes and that she found “comfortable” (Current Error Climate Interview, December 4, 2017). Additionally, Cindy believed by the end of the study that there was more that could be learned from a mathematical mistake than connections to other mathematical topics and for her to slow down when working problems. Figure 19 shows the changes in Cindy's beliefs system at the end of the study.

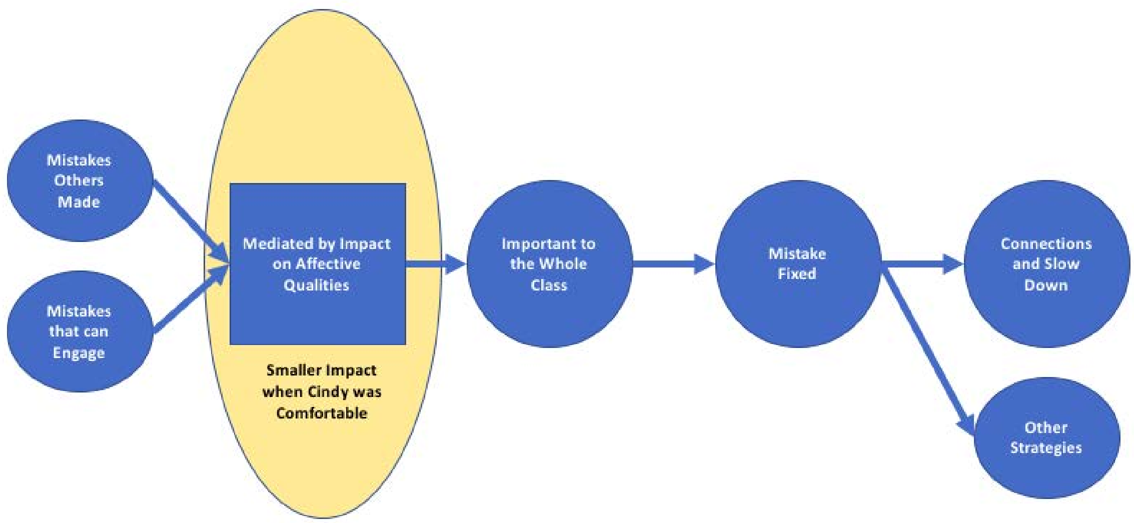


Figure 19. Cindy's beliefs system at the end of the study.

The previous sections focused on the case of Cindy. The following sections will focus on the case of Harley. Harley's case is organized in a similar fashion compared to the case of Cindy.

The Case of Harley

The case of Harley will be described, similarly to the case of Cindy, in six parts. The first of which is the details concerning Harley's implicit theory characteristics. This will inform why Harley was selected to participate in the study as participants were selected based on their implicit theory. Specifically, participants were chosen that were not on either extreme of the continuum. Second, Harley's prior mathematics experiences will be provided. This will provide a sense of the experiences that helped to shape

Harley's beliefs concerning mathematical mistakes. Third, the results from Harley's Prior Error Climate Survey will be presented and connections to Harley's prior mathematics experiences will be made. The subsequent sections will focus on Harley's beliefs concerning mathematical mistakes. Specifically, the fourth section will describe Harley's central beliefs concerning mathematical mistakes followed by the belief clusters concerning mathematical mistakes. Finally, changes that the data revealed in Harley's beliefs system concerning mathematical mistakes while considering the perceived error climate of the classroom will be detailed. These will be followed by a summary.

Implicit Theory

Evidence of Harley's implicit theory characteristics was inferred from two sources: the implicit theory survey (see Appendix A) and from classroom observations using the implicit theory observation protocol (see Appendix I). Harley completed the implicit theory survey on August 28, 2017. At the time of the survey, Harley's average results were in the middle of the implicit theory continuum: intelligence (M = 3.33), morality (M = 3.67), worldview (M = 2.67), and mathematical ability (M = 4.33). Notably, her mathematical ability average was higher than other dimensions. This average was lowered by somewhat agreeing with the statement: A person's mathematical ability is something about them that they can't change very much (A11). I point this out as this item was the only item that addressed the degree to which mathematical ability can increase and the other items concerning mathematical ability ask only if it can change.

Harley ascribed to a mix of characteristics of both an entity and incremental theorist. Her mathematical ability dimension was more aligned with that of an incremental theorist, but during my implicit theory observations (i.e., August 28, 2017 through September 8, 2017), I observed Harley enact characteristics indicative of an entity theorist (see Willingham et al., 2016). On September 6, 2017, the class explored linear pairs, vertical angles, and the parallel postulate using tracing paper (i.e., folding the paper to examine each concept). During which, Harley exhibited descriptors of an entity theorist by hiding her work when the instructor came by and repeatedly asking the other three partners in her group what they were doing. Per Willingham et al. (2016), these were entity theorist descriptors when dealing with setbacks and evaluation of situation, respectively. Similarly, during my observation on September 6, 2017, Harley displayed entity theorist characteristics while practicing using a protractor to measure different angles with the same vertex- a task designed to promote thinking about the different ways that elementary students might incorrectly use a protractor. In one instance, Dr. Heart walked by Harley's group and asked Harley what she was thinking. Harley stated, "I don't know. What were we thinking" (Implicit Theory Classroom Observation, September 6, 2017). Here, Harley looked to the other members of her group to explain what the group was doing when Dr. Heart asked Harley specifically, which was an entity descriptor in the evaluation of situation. I also observed Harley as giving up at one point during the same class meeting. Harley asked Dr. Heart if she could ask questions yet, and Dr. Heart asked her to continue working with her group. Harley appeared frustrated and appeared to give up on the task by reaching for her phone and not paying attention to what the group was doing, which was an entity descriptor in dealing with effort.

Other entity theorist descriptors were observed on September 8, 2017, during a sorting task of polygons and triangles. This was a task where the class was given several triangles and asked to sort them based on characteristics they saw (e.g., triangles with equal sides, at least one acute angle, or an obtuse angle). I observed Harley disengaging from the task by looking down at her paper and not talking to the rest of the group after a group member suggested sorting the shapes in a different way from how Harley suggested, which was an entity descriptor in dealing with challenges. She eventually worked with the group again after they moved to the next section of identifying which types of triangles exist (see Figure 7), but I observed her continually deferring to other members of the groups saying, “What did you write down?” and “What are we doing” (Implicit Theory Classroom Observation, September 8, 2017). Both of these were entity theorist descriptors when evaluating the situation.

This was not to say that Harley’s actions were void of incremental theorist descriptors. During my first implicit theory observation on August 28, 2017, I observed Harley repeatedly saying, “I can do this,” and, “I am going to learn this” (Implicit Theory Classroom Observation, August 28, 2017), when her group first received the instructions in a task taken from *United We Solve* (Erickson, 1996). These statements were classified as incremental theorist characteristics during the evaluation of situations and challenges. I also observed her encourage her group members saying, “Great job!” and, “Good for you!” (Implicit Theory Classroom Observation, September 6, 2017) when they got the correct answer while practicing with the protractor, which were classified as incremental theorist descriptions with the success of others.

The results from the implicit theory survey (Appendix A) and my implicit theory observations (Appendix I) yielded mixed results. Although Harley espoused to more of an incremental theorist in her survey, particularly in mathematical ability, Harley regularly enacted characteristics more aligned with an entity theorist during my observations. Thus, I determined that Harley was on neither extreme of the implicit theory continuum.

Prior Mathematics Experiences

Harley was a PT that wanted to teach at the elementary level when she graduated and described herself as never being good at mathematics. In my first interview with her, she described mathematics as her “weak point” and stated that she’s “always been bad at math, so it's not surprising to be wrong at things” (Prior Error Climate Interview, September 17, 2017). Her “frustration” with mathematics grew, in many cases, to the point that she “didn’t even want to go to class” (Prior Error Climate Interview, September 17, 2017). Harley’s previous mathematical experiences varied to some degree; however, Harley’s previous mathematical experiences were very similar in her middle and high school grades. She made a clear distinction between her mathematics classroom experiences at the middle and high school level compared to her post-secondary experiences. When expanding on her perception of how teachers perceive mathematical mistakes, she explained:

> [Mistakes are] highly looked up to in college. Teachers like that when students can plan out, “Oh well that's wrong. I would do it this way.” . . . In [middle and high school], the teachers you know, students aren't gonna stand up and say something to the teacher and teachers [*sic*]. I just remember that my high school teachers were never like, “Oh I want to hear what you were thinking.” “No, like this is the way that you do it. Like this!” But in college it's not like that. You can

> say, "Oh I don't think that's right" and the teacher's okay with it if it makes sense. (Prior Error Climate Interview, September 17, 2017)

Harley perceived her secondary classroom teachers as not wanting to hear an incorrect way of working a problem. There was one way. However, in college, Harley perceived that students' thoughts and, particularly, mathematical mistakes were welcomed. I further inquired why she thought that was the case. She stated that "it was higher levels of thinking. Maybe. And because we're older" (Prior Error Climate Interview, September 17, 2017). Later in that interview, this distinction between mathematics classrooms came up again when she explained how mathematical mistakes can impact her grade, so I again asked her to talk about why that was the case. She explained:

> Because of standardized testing, and so, [the students] have to learn things that way. But in college, because you're older, I feel like it's, they want you to think outside the box. They know that it's not going to escalate into anything bad. (Prior Error Climate Interview, September 17, 2017)

Harley's perceptions of her prior classroom experiences were shaped by the influence of standardized testing and a comparison to her more recent mathematical experiences in college. However, it was unclear when she mentioned "college" (Prior Error Climate Interview, September 17, 2017) if that was in reference to her classes before the one that the involved in this study or if she was referencing all of her college experiences together.

Furthermore, Harley's perception of her previous mathematics classrooms was procedural and rarely took into consideration what students were thinking. When asked to provide a description of a typical day in her previous mathematics classrooms, she stated:

Harley: Well, pretty much just thinking about my favorite teachers. Mainly in high school. ‘Cause that had the most impact. I mean usually we'd walk in, sit at a desk facing the board, and, it's been so long since high school. I really don't remember. I just know that we, we did a lot on the whiteboard. Like she would write down you know show us how to do equations and take notes. I remember that. I feel like we'd do activities a lot with shapes and stuff like that. We'd usually get a worksheet or two per class. So that's not fun.

Me: What's a worksheet? What does that entail?

Harley: Sometimes we would have to do it by ourselves and sometimes we'd be able to do it in a group. Usually has ten-ish problems with whatever we're learning about that day. And then we'd go over it and find out if we got it right or wrong and then we'd get homework. We could start that in class. Lot of textbook homework.

Me: Doing problems out of the textbook?

Harley: Yes. (Prior Error Climate Interview, September 17, 2017)

She described a routine of lecture followed by worksheets of problems from the book. I was not sure if this description was just one instance that she remembered, but this routine was mentioned at other times. This was best captured in a quote that followed her response to me asking if it was okay to give up on a problem. She, again, compared to how a teacher’s response might differ from high school compared to college:

> Well in high school, they would just pretty much write it out for me. But I feel like now, they would just ask me what is it you don't understand. . . . The high

> school teachers want you to know it so bad. They're not going to sit there and question you. Like, “Well, oh, what is it? Blah blah blah. Oh, here. This is the right equation. Take it. Memorize it. There you go.” (Prior Error Climate Interview, September 17, 2017)

Again, Harley described her experiences prior to college as very procedural and not accounting for what she was thinking, particularly when she made a mistake.

Harley’s perceptions of her mathematics classrooms were similar across her experiences. This was particularly the case when comparing Harley’s classrooms in middle and high school. However, she noticed a significant change in her college mathematics classrooms. Particularly, she expressed a difference in how her thinking was accounted for and the freedom to which she was allowed to think about mathematics in different ways. This was especially important to notice when considering the results of Cindy’s Previous Error Climate Survey which follows.

Perceived Error Climate for Previous Mathematics Classes

Harley was also aware of the relevance of the error climate in her previous classrooms. When Harley spoke about her past mathematical experiences in the Prior Error Climate Interview (see Appendix C), she stated that mathematical mistakes were useful for her teachers, because the mistakes “help them to be able to help you more” (September 17, 2017). However, she also acknowledged that teachers’ perceptions of mathematical mistakes could change if students were “not getting it. I'm sure that the teacher could get frustrated. Like, why don't you understand this? What can I do differently to make you understand it?” (Prior Error Climate Interview, September 17, 2017). Harley shared instances where the rest of the class’ perception of mathematical mistakes played a significant role for her. This was best captured when expanding on her response to Survey Item B13–If someone in our mathematics classroom gets a wrong answer, she can be mocked by his classmates. She stated:

> I'm like. [I] really don't like speaking up in class now ‘cause I'm scared that they're gonna laugh ‘cause I'm a blonde or was. It's like brown now, but [I] was a blonde. And people just love using it when you say something wrong in class. Yes. I definitely got mocked in class sometimes. Not a lot, but I can remember, especially in math class, that was my weak point. (September 17, 2017)

For Harley, the classroom error climate was influenced by her classmates, which did not appear to be mediated by the teacher, as much as it was influenced by the teacher. She elaborated on how the classroom error climate was different from talking with the teacher privately when expanding on Survey Item B15–In our mathematics classroom, a lot of students would rather say nothing than say something wrong. In the following exchange, Phillip, a pseudonym, was an undergraduate assistant that was a supplemental instructor for the class:

> Harley: Yes! Strongly agree with that one! ‘Cause that is me. All the way. Because of getting teased like that.
>
> Me: So, you wouldn't want to say something wrong because of getting teased in public. What about privately?
>
> Harley: If I was with my teacher, is that what you mean?
>
> Me: Yeah.
>
> Harley: Just with my teacher, no. I feel like that's a safe space. I'd be able to. Like the other night with Phillip, I was talking the whole time asking

> stupid questions, and I didn't feel bad at all. So, I guess it's just when you are around other people. You feel scared to mess up. (Prior Error Climate Interview, September 17, 2017)

As Harley's comments demonstrated, the error climate of the classroom clearly had an impact on her participation in mathematics classes. To further investigate Harley's perception of her previous mathematics classrooms' error climate, the results of the error climate survey are explained in the following paragraph.

During the study, Harley completed two error climate surveys. The first of which focused on Harley's classrooms prior to enrollment in the class in which this study took place. She completed the Previous Error Climate Survey (see Appendix B) on September 17, 2017, which focused on her perception of mathematical mistakes in relation to seven dimensions with a superordinate score. The average results from Harley at the time of the survey are presented in Table 8. Harley's prior error climate would be described as a negative one with a superordinate score over 3. Additionally, the most notable error climate dimension was the error-taking risk (i.e., a measurement of her perception of students' willingness to share mistakes) of her previous classrooms. This was aligned with Harley's explanation of not wanting to share her comments with the class, especially if they might be incorrect, but willing to share with the teacher or teacher helper (e.g., Phillip) privately.

Table 8

Harley's Prior Classrooms Error Climate Results

Error Climate Dimension	Average
Error Tolerance by Teacher	*M* = 3.5
Irrelevance of Errors for Assessment	*M* = 2.5
Teacher Support Following Errors	*M* = 2
Absence of Negative Teacher Reactions to Errors	*M* = 3.5
Absence of Negative Classroom Reactions to Errors	*M* = 4
Error-taking Risk	*M* = 6
Analysis of Errors	*M* = 2.67
Functionality of Errors for Learning	*M* = 3.75
Superordinate Score	*M* = 3.49

Note. Scores on the survey ranged from 1 (strongly agree) to 6 (strongly disagree), with averages of 3 and below indicating a positive error climate and averages of above 3 indicating a negative error climate. A lower average indicates a more positive error climate.

Harley's Central Beliefs Concerning Mathematical Mistakes

With Harley's experiences in prior classrooms and the class in which this study took place, she was given opportunities to reflect on her experiences with mathematical mistakes through interviews, journal entries, and in-class reflections. From that data, themes emerged related to beliefs concerning mathematical mistakes in which I was able to infer Harley's beliefs concerning mathematical mistakes. Certain beliefs, which will be referred to as *central beliefs concerning mathematical mistakes*, encompassed and influenced her beliefs. These central beliefs concerning mathematical mistakes were beliefs around which Harley's other beliefs concerning mathematical mistakes were clustered and were "overarching beliefs about the physical, social, and pedagogical" (Leatham, 2007, p. 192) inclusion of mathematical mistakes in the classroom. Additionally, these were beliefs that cohered with the rest of the system and transcended contexts and foci. The remainder of this section will describe the central beliefs concerning mathematical mistakes for Harley.

Role of mistakes for the teacher. For Harley, mathematical mistakes served a significant role for the teacher. Specifically, Harley believed that mathematical mistakes were signals for the teacher. Harley stated in her Prior Error Climate Interview that mistakes are a way for the "teacher to understand where you are going wrong . . . [and to] . . . see what you're struggling with and what you're not good at" (September 17, 2017). Harley saw mathematical mistakes as a way to help teachers. In her response to being asked if teachers liked it when students do something incorrectly, she stated:

> Harley: Even though [teachers] don't like it, it helps them to be able to help you more. To see where you're struggling and they can help you further with seeing what you do correctly.
>
> Me: You were saying that you didn't think they would prefer it though?
>
> Harley: Well, I don't they'd be happy that their students are not doing well. But, in a way it also helps them to be able to teach the students better and find out what they don't know. (Prior Error Climate Interview, September 17, 2017)

Here, Harley saw mathematical mistakes as an instance for the teacher to learn something about the students in order to teach them better. Harley's belief that mistakes were helpful for the teacher was relevant throughout the study. When responding to the same question during the Current Error Climate Interview, she talked about the usefulness of mistakes for the teacher again:

> 'Cause if you're doing something incorrectly, you've made a mistake, and she can teach from that. Even people that in the class that might have made that mistake and didn't realize that they made the mistake, so she can use those. (Current Error Climate Interview, December 7, 2017)

For Harley, a mistake was useful because it provided the teacher insight into what the students know and do not know. However, Harley's belief in the utility of mathematical mistakes for the teacher expanded to giving the teacher the opportunity to correct something that the student had said.

Specifically, Harley described these instances as a "reason to correct their definition of whatever they're talking about" (Current Error Climate Interview, December 7, 2017). The teacher was able to use the mistake as an opportunity to take corrective action. One instance where Harley talked about using mathematical mistakes to correct what a student had said was after a class meeting where the class had a sort of quiz at the beginning of class on perimeter where they were asked to identify which ways of finding the perimeter were incorrect and why (see Figure 20). I asked Harley why she thought certain incorrect answers were on a quiz concerning how to find the perimeter of geometric figures. She stated:

> I believe she showed us the examples of the assignment [*sic*], because it is important for us as future teachers to know the incorrect answers and correct answers of problems. I think this is important for her to teach us incorrect answers, because children might do problems in a way that are [*sic*] easier for them to understand. I think it will also help us to teach it better and teach our future students how to not do a problem and correct what they did, so they don't get it wrong on tests. (Post Classroom Observation Interview, October 18, 2017)

The examples of mathematical mistakes were instances for Harley to learn what students could do, and thus, she would be prepared as a teacher and know how to correct them.

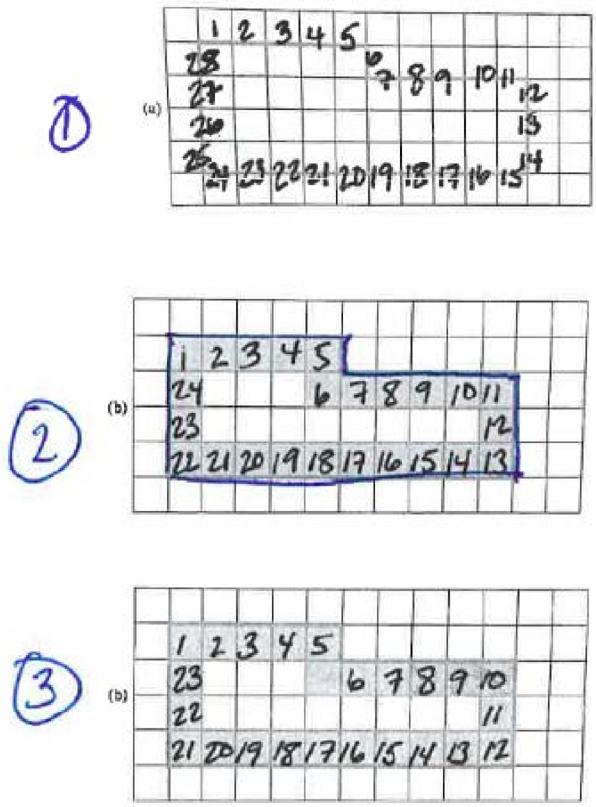


Figure 20. In-class activity with two mistakes in finding the perimeter. The first picture contains a correct way of finding the perimeter while the second and third have mathematical mistakes. Reproduced from *Mathematics for Elementary Teachers with Activities* (5th ed.), by S. Beckmann, 2018, New York, NY : Pearson. Copyright 2018 by Pearson Education, Inc. Reprinted with permission.

For Harley, mathematical mistakes were instances where teachers could learn what students were doing wrong and use that as an opportunity to correct the mistake. In the reflection that occurred after taking the previously mentioned quiz, Harley chose to elaborate on why those mistakes might be valuable by talking about herself becoming a teacher. Harley described the mistakes on the quiz as being “good to know [so you can] show your students. Because, it teaches them the correct way to find perimeter and keep [*sic*] them from doing it wrong” (In-class Reflection, October 18, 2017). For Harley, mathematical mistakes were not just signals that students were doing something incorrect, but also as a signal for corrective action or error avoidance. It is important to note mathematical mistakes served the teacher, specifically, for Harley. No matter where Harley was positioned, as the learner or teacher, mathematical mistakes were important, because they were signals for the teacher and not usually the student.

Fixing mistakes for the test. As previously described, Harley believed that mathematical mistakes were a signal to correct a student’s definition or process and help teachers to teach better, but Harley also believed that this was done so for a specific purpose. Harley believed that those mathematical mistakes should be fixed because of standardized testing in high school. As she stated:

> High school teachers want you to know it so bad. They're not going to sit there and question you. Like, “Well, oh what is it? Blah, blah, blah. Oh here, this is the right equation. Take it. Memorize it. There you go.” Because of the testing. (Prior Error Climate Interview, September 17, 2017)

Additionally, in that interview, she described mistakes as a “signal that this teacher needs to try better to explain the material, because on the standardized test thing. It's gotta [*sic*] be that way” (September 17, 2017). With standardized testing, Harley believed that teachers must fix mathematical mistakes in very specific ways to support the students in doing well on the test. Again, in the same interview, she talked about what support looked like in response to someone making a mathematical error in previous classrooms:

> Well in high school, it was a little easier to explain things ‘cause it was more equations, and in college it's more like deeper thinking. Like putting things in words and critically thinking about things. So, in high school, I was able to help people sometimes. Just because I can write out numbers easily. Like, *this* [emphasis added] is how you do it. (September 17, 2017)

For Harley, especially in high school, mathematical mistakes were not only signals for the teacher to fix something, but they were signals to fix the mistake in a specific way. As she explained it, they have to do that because of the looming standardized exam. However, Harley’s belief concerning how teachers should handle mistakes did not change when talking about her post-secondary classes.

Although Harley believed that mistakes should be fixed in a specific way in high school because of standardized testing, her belief that mistakes should be fixed in a specific way persisted throughout her college experiences. When she compared her prior mathematics experiences with her more recent ones, she stated:

> Even though high school didn't have discussions like that, mistakes were not treated differently. If someone said something wrong in high school or college, the teacher would still try to fix the answer and like get to the bottom of it. Figure out where they went wrong. So, I don't think it would be any different. Treated different. (Current Error Climate Interview, December 7, 2017)

It is pertinent to note that Harley described the treatment of mistakes by her teachers in the same way, or better stated, for a specific reason. Although, Harley’s teachers did not have to deal with standardized testing in college, Harley still believed that mathematical mistakes should be fixed for a test (i.e., the class test). Harley mentioned this on several occasions. The first instance that she specifically referenced tests in her class was in a journal reflection, where she stated:

> Most of the time I do understand it after they explain the correct way to do the problem. Mistakes are good to make, because I would have made the same mistake on the test if the teacher would not have caught my mistake in class. (Journal Reflection, October 24, 2017)

Using mistakes as opportunities to fix something in preparation for the exam was salient in the Current Error Climate Interview (December 7, 2017) as well. When talking about what she would expect to happen when the teacher notices a mistake, she stated:

> In a class period where there's a lot of mistakes going on and a lot of people are making them, she would talk about that, because the whole class as a whole is gonna fail if she doesn't fix that. (Current Error Climate Interview, December 7, 2017)

Later in that interview, she continued to provide descriptions of how teachers use mistakes to correct students to prepare for a test. When describing what a mathematics teacher is like, she responded that they were like a coach. I asked her how a mathematics teacher like a coach handles mathematical mistakes, and she stated:

> They immediately try to fix it. So, you will win the test or make that A on the test. Just like in football. Like that's why they have those timeouts, I'm assuming. They bring them in and we're gonna do this ‘cause y’all messed up on this. So, this is how we're gonna do. They send them back out and get an A. They win! They're definitely like a coach. They teach, too. They don't boss them around. They have to teach them. I guess like senior football players . . . they don't have to teach

> them ‘cause they already know what they're doing, but all the new people that aren't good at football, like no one is good at math. So, we have to be taught. We don’t know what we're doing. (Current Error Climate Interview, December 7, 2017)

Harley believed that the test, standardized or not, was the reason that a teacher would need to see a student’s mistake, and although Harley did not always bring up the test specifically, I inferred that this was at the forefront of her thoughts in other instances. For example, I walked with Harley after a classroom observation where the class made posters that included mathematical mistakes for finding the area of different geometric figures. I asked Harley why she thought the class was asked to do that. She stated that it was “to make sure that they fully [understood] it for later . . . [and to] . . . remember it for next time” (Post Classroom Observation Interview, October 25, 2017). The way that she used later and next time indicated a specific moment, like the test. She used a similar tone and wording in the Current Error Climate Interview (December 7, 2017) when I asked her to describe why she thought a mathematical mistake would be valuable. She stated:

> It helps you learn and make all that information [understandable]. ‘Cause when you go over a mistake and figure out how to correct it, you remember it and you are going to remember, oh I remember I did that wrong last time. So, you'll do it right from now on. You won't get it wrong on homework or tests if you fix it.

Harley believed that mathematical mistakes were useful for the teacher in preparing students for the exam, even using them as a tool for preparation for the exam.

The necessity of mistakes. Throughout this study, Harley had a positive view of mathematical mistakes and believed that mathematical mistakes were an aspect of a mathematics class that would and should happen. In one of her earlier reflections, I noticed that she shared some of her mistakes with her group but not others during two classroom observations (September 27, 2017; September 29, 2017). During these sessions, the class was working on dimensional analysis which included working with fractions. I asked her to share why she might want to share a mistake with a classmate or the class as a whole. She responded:

> Mistakes in math are a beneficial thing, because it shows you are learning. Even though students get nervous or discouraged when they make a mistake, it is still a good thing if they realize it or not. So, we have to make those mistakes and talk about them to learn from them. (Journal Reflection, September 29, 2017)

For Harley, doing something incorrectly in a mathematics class was a necessity in order to learn. In a Journal Reflection near the end of the study, Harley emphasized the need for mistakes in mathematics. It was the last Journal Reflection, and I asked Harley to write a personal statement concerning mistakes in mathematics classes. She stated:

> If students can easily solve math problems, there will be little growth. There has to be a challenge to keep them learning and growing. I also think this makes learning more interesting. If we went to class every day and we knew everything, what is the point of going to class? There would be no growth. *Mistakes are key to learning* [emphasis added]. (Journal Reflection, December 3, 2017)

As Harley stated, mathematical mistakes was the mechanism for learning. Without mathematical mistakes, she would not be able to learn. Granted, this statement was at the end of the study after Harley was repeatedly asked about her mathematical mistakes.

However, Harley's belief that mathematical mistakes were a necessity for the class was present throughout the study and in all contexts.

In my first interview with Harley, she talked about why the teacher would be okay with a submitted assignment that contained mistakes (Survey Item B4). She stated, "I can somewhat agree with that because, I don't want to say arguments, but [the mistakes] are what make you think" (Prior Error Climate Interview, September 17, 2017). Although she later attributed using mistakes to promote thinking and disagreement in college to them being at a "higher level of thinking" (Prior Error Climate Interview, September 17, 2017), she still stated that "mistakes helped the teacher to know where to help us" (Prior Error Climate Interview, September 17, 2017). Without the mistakes in the classroom, there would be nothing with which her colleagues to disagree, or the teacher would not know how to help the students.

When she considered herself learning to become a teacher, the notion that mathematical mistakes are important was unchanged. This was best captured during an in-class reflection (October 18, 2017) where Harley was asked why seeing different mistakes in finding the perimeter of an irregular shape might be valuable for her. She stated that mistakes are "good to know to show my students because it teaches them the correct way to find perimeter and keep them from doing it wrong." Although Harley's focus was still on using the mistake to know the right way, or at least to know how not to find the perimeter, she would not know this if she was not there to experience the mistake. It was apparent that, for Harley, learning was about learning how to do it correctly and ultimately for the test. Nonetheless, experiencing mistakes was the way for her to do that.

For Harley, mathematical mistakes were an aspect that the class could not do without and essentially where learning happened. She talked on several occasions about learning from mathematical mistakes during the final interview. Two quotes from this interview are provided that are summative of Harley's belief that mathematical mistakes were a necessity for the class. First, when elaborating on why the teacher would be okay with an assignment submitted that contained mistakes (Survey Item G4), she stated:

> For me, I make a lot of incorrect ones, so I would say that I learn, cause when I already know something, I'm not learning anything about it, but when I'm wrong, my way of thinking about it completely changes and I have to learn a new thing or way to do 'cause I was doing it wrong. (Current Error Climate Interview, December 7, 2017)

Again, learning for Harley came from making mistakes in mathematics. If she did something correct, then there would be nothing more for her to do. This was similarly stated in her response to learning from wrong answers (Survey Item G22). She stated, "Wrong answers, like wrong assignments, are beneficial to the class. 'Cause you're learning when you're making these mistakes, 'cause you're retaining it while also learning to do it correctly" (Current Error Climate Interview, December 7, 2017). Harley believed that learning happened when mistakes were made, by retaining it and doing it correctly. If it was not for mistakes, then she would not need to attend class.

Belief Clusters

With belief system's central location, Harley's *central beliefs concerning mathematical mistakes* can be viewed as Harley's position of how mathematical mistakes should be used in teaching and learning mathematics. Harley believed that mathematical mistakes were signals for the teacher and a way to help the teacher. Harley also believed that mistakes were signals for the teacher to fix something for the student in a specific way. Although she differentiated how this was done in middle and high school compared to college (i.e., being told how to fix it compared to discussing the mistake), fixing mistakes for a test was how Harley positioned the role of mathematical mistakes in the classroom. Harley believed that role to be a necessity in the mathematics classroom and an essential reason to come to class. These *central beliefs concerning mathematical mistakes* influenced and encompassed other belief clusters concerning mathematical mistakes. Particularly, the clusters of beliefs that emerged from this study were when mathematical mistakes were important, the opportunities to gain knowledge from mistakes, and the influence of mathematical mistake on affective qualities. Each of these will be subsequently discussed in detail.

When mistakes are important. Harley held strong beliefs about when mathematical mistakes were important or of value. Although Harley believed that "all mistakes are valuable" (Post Classroom Observation Interview, September 22, 2017), she also believed that there were other factors involved in how mathematical mistakes should be treated in the classroom. Particularly, Harley believed that mathematical mistakes would be more important depending on who made the mistake and if the mistake was worth discussing as a class.

Who made the mistake. For Harley, she believed mathematical mistakes that were, or could be, common to the class were the ones that were of the most value. As she put it, a mistake "would be relevant to the class if it is a common mistake, especially when first learning [a] topic" (Post Classroom Observation Interview, December 12, 2017). Although Harley mentioned in this quote that common mistakes were useful when *first* learning a topic, the utility of a mistake was not limited by the familiarity with content when accounting for ubiquity of the mistake in the classroom. In a Journal Reflection (October 28, 2017), I asked Harley to think about and to describe a mathematical mistake about which it was worth spending time thinking and talking, and she stated:

> A mistake that I make often in math is doing the wrong formula for a problem, because they all confuse me so often. I would spend more time on this mistake, because I feel as if it is a common problem in math. (Journal Reflection, October 28, 2017)

Furthermore, she described talking about mistakes in other contexts. I asked Harley to elaborate on her response to Survey Item B19–In our mathematics classroom, assignments that are not done correctly are discussed in class. This was our exchange:

> Harley: I would agree with that one, definitely not all of the assignments. Not every day we're gonna talk about the homework from last night or the assignment from last night, but in high school, college, middle school, we definitely went over problems that we did–that a lot of students got wrong. The teacher would go over that and you know ask if anyone had any questions and show them how to do it correctly so they could do it better on tests.
>
> Me: At the beginning of that, you said not every time. Were there specific times that you're thinking that you would?

Harley: Yeah, well I'm just not every assignment we did. The teacher wouldn't go over every single assignment. So, I don't know how they would pick those. Maybe some of those assignments, maybe the majority of the class got a lot wrong. So, that's why they went over it. Some of the assignments the majority did really good, so there was no reason to go over it.

Me: But only if a lot of people missed it?

Harley: Yeah, I mean, I guess I'm not the teacher, so I don't know what they were thinking, but she didn't tell us. “Oh, we're doing this because”, yeah, I actually guess they would tell. “Hey, a lot of ya'll got this messed up. A lot of ya'll mess this up, so we're gonna go over it.” So, it is if the majority of the class messed up, we'd go over that. To clarify. (Prior Error Climate Interview, September 17, 2017)

Throughout her explanation, mathematical mistakes were in need of explanation or further discussion if a majority of the class made the mistake. It is also pertinent to notice that Harley believed mathematical mistakes were important in similar circumstances to when she perceived mathematical mistakes were important for her previous teachers. That is, mathematical mistakes were important enough to bring to the attention of the whole class when a lot of the class “messed up” (Prior Error Climate Interview, September 17, 2017).

For Harley, the usefulness of a mathematical mistake when a majority of students made the mistake was not limited to homework or tests or specific mathematical content. Harley shared more concerning this belief in an interview following one of my classroom observations. During this class meeting, Harley was in a group sorting quadrilaterals into different sets based on their characteristics (e.g., parallel sides, number of congruent angles) and used the sets to talk about polygon definitions. I asked her about the mistakes that she made and if she thought any of them were worth sharing with the class. She stated:

> I didn't make many mistakes but it was fairly easy for me and my partner. I'm trying hard to remember everything we did or what I did wrong. I know I was confused on how to tell if the shape has diagonals that are perpendicular. I had to draw it out on the shape, and if I didn't draw it out, I wouldn't be able to tell if the diagonals are perpendicular or not. That is the only thing I struggled with. I believe it would be worth discussing, so we can figure out how to tell if something has diagonals that are perpendicular without drawing it out. (Post Classroom Observation Interview, September 25, 2017)

I was not clear how she was using we in that quote, meaning the group or the class, so I asked her to whom she was referring. She stated, “We as in my table! Unless, the rest of the class had an issue with that same problem too, then the whole class should discuss it so we can clarify how to identify a shape with perpendicular diagonals.” It is pertinent to note that Harley only wanted to discuss her mistake with her personal group and not the class unless more of the class made the same mistake. Harley’s belief that mathematical mistakes were important depending on how many students in the class was best captured in one of her journal reflections when I asked her to describe when a mathematical mistake was worth exploring. She responded:

> I think a mistake that is worth exploring is based on the class. If the majority of the class made the same kinds of mistakes on the same problem, then I believe it is beneficial to go over it. If a teacher doesn’t go over the mistake and correct it, then how will the students ever be able to correct their mistake in the future? So, I think a mistake is worth exploring or sharing if the majority of the class does not understand it. (Journal Reflection, October 28, 2017)

The extent to which Harley would want to spend time on mistakes that others, and more importantly the majority of others, made was almost altruistic. During an observation of class, I perceived that Harley was struggling with converting units (i.e., dimensional analysis), so I asked her to tell me what she was thinking about during that moment. She stated, “When we went from smaller to bigger, I didn’t want to slow everyone down. I wasn’t sure that if everyone else was having the same problem I was having, so I didn’t want to ask” (Post Classroom Observation Interview, September 29, 2017). Harley believed that her mistakes were not as important if more people made a different mistake, even if her mistake had an opportunity for learning more. This belief was best captured during the Current Error Climate Interview (December 7, 2017) when the following exchanged occurred during the discussion of Harley’s selection on H9–If someone in our mathematics class says something incorrect, the teacher will explain the problem:

> Harley: I guess I'm gonna say somewhat agree, because we only have 55 minutes in this period so she really has to pick and choose what mistakes are probably going to be the most beneficial for us to know. Like maybe if I'm the only one making a mistake on something, she probably

wouldn't put it on the board. But if everybody was making, you know this mistake or this one, I feel like she'd talk about it. But, also if nobody was making mistakes that day and I just made one mistake, she'd have time to talk about my one mistake, but sometimes we don't have enough time to talk about this one little mistake that one person made when the whole class is making another really, really big mistake that she wants to fix.

Me: Two things I’m thinking about. One, when you say little mistake are you referring to little as in a silly mistake or I didn't know if you meant silly as you meant earlier.

Harley: Yeah, not like an error in numbers or something.

Me: So, you're talking about a big mistake?

Harley: I guess when I say little now, I'm meaning, I'm the only person making it. So, it's not like the whole class isn't making it, so I feel like it's not big or detrimental mistake that needs to be put on the board if I'm the only one making it. If all these other people are making the same mistake. Even though I feel like she might come back to me when she's done talking about that, you know kind of talk about mine privately to see where I went wrong. We definitely only have a certain amount of time, so she has to decipher which mistake she's gonna have time to talk about or which one is going to be the most important. (Current Error Climate Interview, December 7, 2017)

Harley described a situation where she had a mistake that might have been fruitful for the class to discuss or review. However, she saw the majority of others were making a different mistake which made that mistake more useful for utilizing class time.

For Harley, sharing mathematical mistakes was important when a lot of students made that mistake or if she was "sure [that she wasn't] the only one that forgot what that was" (Post Classroom Observation Interview, November 6, 2017). This belief was not mediated by what could be learned from the mistake, when the mistake happened, or if she thought the mistake was important in itself, but rather by the quantity of classmates that made the same mistake. These beliefs, like all non-central beliefs, are influenced by Harley's *central beliefs concerning mathematical mistakes*. This was best captured in the Prior Error Climate Interview (September 17, 2017) when Harley expanded on why a mathematical mistake might be used to make sure that an individual really understands a concept:

> We definitely went over problems that we did that a lot of students got wrong. The teacher would go over that and you know ask if anyone had any questions and show them how to do it correctly so they could do it better on tests.

As Harley stated, the teacher has to figure out "what the whole, the majority of the class is struggling with first before you start individually finding mistakes" (Current Error Climate Interview, December 7, 2017).

Discussing mathematical mistakes. Whether or not a mathematical mistake should be discussed was another dimension that emerged during the study. I differentiated this dimension from the previous one because of the explicit nature in which reform documents state the usefulness of mathematical mistakes for discussions.

Additionally, the data collection processes provided opportunities for Harley to explicitly talk about discussing mathematical mistakes. For Harley, discussing mathematical mistakes was valuable as a learner. Mathematical mistakes were “super valuable” (Current Error Climate Interview, December 7, 2017), and they allowed her to “see how others were getting it wrong” (Journal Reflection, September 24, 2017) and to “go deeper into the problem” (Post Classroom Observation Interview, November 29, 2017). However, there were certain instances that a mathematical mistake was worth discussing and some mathematical mistakes that were “weighted heavier” (Current Error Climate Interview, December 7, 2017) or “more [discussion] worthy” (Journal Reflection, September 29, 2017) than others. Harley always talked about this dimension from the viewpoint of the teacher. That is, whether a mathematical mistake was discussed in the class was based on the teacher deciding if it was.

Harley believed that mathematical mistakes were worth discussing based on three factors. First, it depended on how much time there was in the class for discussing mathematical mistakes. This first emerged in the first interview when Harley explained her choice on Survey Item B17–In our mathematics classroom, there was a detailed discussion when something is done incorrectly. She explained:

> There were discussions [in high school] when something was done incorrectly but not all the time. I guess that only happens when the teacher has time during the class to actually have a discussion about a problem, but most classes in high school, we didn't really have discussions on this stuff. It was more like worksheets, teach something on the board, worksheet, homework, go home. In

> college, we definitely have a lot more discussions. (Prior Error Climate Interview, September 17, 2017)

The time aspect not only played a factor in her prior mathematics classrooms but also in the class in which this study took place. During the final interview, I asked Harley to expand on the same survey item that focused on Dr. Heart's classroom (i.e., Survey Item G17). She stated, "We definitely only have a certain amount of time, so she has to decipher which mistake she's gonna have time to talk about or which one is going to be the most important" (Current Error Climate Interview, December 7, 2017). She later continued, "So, somewhat agree. I think that it depends on the situation. If we had a two-hour math class, I feel like that would also be different. She'd get to everybody's question or mistake" (Current Error Climate Interview, December 7, 2017). Harley saw the time constraint of the classroom as a limiting factor in which mathematical mistakes were examined in her past classrooms and Dr. Heart's.

The second factor related to whether or not a mathematical mistake should be discussed was the type of mistake that was made. Specifically, Harley differentiated between small mistakes, which she also referred to as tiny or minor (e.g., procedural mistakes), and big mistakes, which she also referred to as major mistakes (e.g., conceptual mistakes). In a journal entry, she stated:

> I believe that ALL of our mistakes are valued in this particular class. This class makes you think critically about problems and finding new ways to solve problems rather than just the certain way you learned it. So, when someone in class made a mistake, Dr. Heart would talk to them one on one, or if it spiked a great interest in her, then she would talk about it to the whole class. I feel like all

> mistakes are valued but some are more worthy to explain to the class than others because of their difficulty and the kind of mistake it is. Was it a tiny mistake or something big? (Journal Reflection, September 29, 2017)

Here, Harley started to differentiate the importance of a mistake based on what type of mistake it was. Different types of mistakes was something that Harley and I discussed for some time in the Current Error Climate Interview (December 7, 2017). We were talking about whether or not she perceived mistakes as something bad for the teacher and the conversation moved to whether or not she got anything out of correct answers. I asked if she thought Dr. Heart would view a student calling out a correct answer differently than the student calling out an incorrect answer, and the following exchange occurred:

> Harley: Oh. Actually, I think that, I feel like she might. I don't think that she'd talk to the whole class about that little mistake. Like on my homework, I've messed up a lot of calculations, like tiny stupid stuff, and I look back at my homework and I'm like why did I do that. That was such a tiny mistake and I got a wrong answer because of it, but she'll write, "Hey this was supposed to be this number." And it was just a wrong calculation, but she doesn't talk to the class about my wrong calculation or my wrong number because I feel like it's more of an accident than, the small mistakes are more like accidents, than like understanding the problem. Yeah. Do you get that?
>
> Me: But bigger ones?
>
> Harley: But the big ones, where you actually understand the problem, where you're getting the equation wrong or the formula wrong or you don't

even know how to do it. She would share that with the class. Because an error or calculations could be fixed easily. Just be more careful with these next time. That's why on tests, I'm super easy with my calculator and I make sure that I do the numbers right. But, yeah, I feel like that there is a difference. I mean they both give me the wrong answer, they're both a mistake, but one if weighted more heavily.

Me: And it sounds like one is worth more, worth sharing?

Harley: Yeah, yeah.

Me: Even if you were the only one that did it?

Harley: Like the big mistake?

Me: Either one.

Harley: If I was the only one that did it, and she's been a teacher for a while, so if she's seen multiple people before make the same mistake I did, I feel like she probably would have taken it and put it up on the board and say why did this person go wrong. Yeah, but I feel like if I just made it, I feel like I wouldn’t be the only one. Because, I don’t know. It's a hard question. (Current Error Climate Interview, December 7, 2017)

As previously stated and evident here, Harley believed that mathematical mistakes are signals for the teacher to do something differently, and in this case, that would mean bringing it to the attention of the class and discussing the mistake. However, Harley believed that the type of mistake (i.e., major or minor) would influence that decision by the teacher.

Finally, the last dimension of when a mathematical mistake was worth discussing was related to how many people made the mistake. I described Harley's belief that mathematical mistakes are more important "to the class if it is a common mistake" (Post Classroom Observation Interview, December 12, 2017) in the previous section, but this belief was very much related to and was a precursor for mathematical mistakes to be discussed in the class as well. One instance where I perceived Harley to be passionate about discussing mathematical mistakes, based on her excited tone, was during a class session where they watched a video. The video showed fourth grade students using a broken ruler (i.e., the ends of the ruler were missing) to identify how long a pencil was. Students in the video incorrectly identified how long the pencil was because of where the end of the pencil was placed on the ruler. In this case, the student stated that the pencil was 8 inches long because that was where the end of the pencil was. In the interview that took place immediately after this class session, I asked Harley why she thought Dr. Heart facilitated a discussion concerning a video they watched in class, and she stated, "Because that's where most students would mess up, and I needed to know that for my students. That's a common error" (Post Classroom Observation Interview, October 2, 2017). I additionally asked her what she remembered thinking about while she watched the video and talked with her classmates. She stated:

> This makes me scared to even become a teacher, because what if I don't teach them every single detail they need to know so they don't make these tiny mistakes. And there's so many ways that they can all mess up, so we needed to talk about what were common ways that our students might mess up measuring something. The [discussion] helped me think about ways that I could help [the

> students] avoid making those common errors. (Post Classroom Observation Interview, October 2, 2017)

The notion that a mathematical error was common or likely to be made was influential for Harley in deciding if it should be discussed.

This was not only the case when Harley positioned herself as a prospective teacher but also as a mathematics learner. In another classroom observation that took place on November 29, 2017, I observed Harley conversing with her group and the rest of the class while reviewing for the final exam. On that day, the students were given a review sheet concerning surface area and volume of three-dimensional figures. On the sheet, there were problems worked incorrectly where the students were expected to identify the error and talk about its significance. Additionally, the review sheet had problems that the students were expected to work as a group, where they inadvertently made mistakes themselves. I questioned Harley concerning which mathematical mistakes she thought were significant and about which the whole class should talk. She responded:

> We talked about a lot, and we made some mistakes ourselves. I think that the ones that the other students made are the ones that we should talk about next time. That's where everyone could really mess up, not remembering a one-half here or not getting the right base or formula right. (Post Classroom Observation Interview, November 29, 2017)

I then asked Harley about the mistakes that they made and why she did not think those were important enough to discuss. She stated, "We made some big mistakes, but I don't think a lot of people would have done it that way. Wrong that way" (Post Classroom

Observation Interview, November 29, 2017). Again, her focus was on how many people made the mistake, which overruled the type of mistake made.

For Harley, discussion concerning mathematical mistakes takes place based on certain conditions: time remaining in class, the type of mistake made, and how many people made it. The following quote from Harley in the Current Error Climate Interview (December 7, 2017) was summative of her beliefs concerning when mathematical mistakes are important for discussion. This was near the end of the interview, and I asked Harley general questions about some things that she said throughout the interview. She was responding to whether or not she thought mathematical mistakes were valuable. She responded:

> Yes! Completely. Super valuable. Like, some, like I said, more important to talk about when you only have a certain time of class period and everyone's making a mistake you really need to focus on that one, but all mistakes are gonna be very valuable, even those silly calculator mistakes. But the bigger mistakes give us more to talk about and get out. That was valuable ‘cause I got things wrong ‘cause I was making the wrong click on a calculator. The silly, calculator mistakes can really affect my grade too, so now I know to fix my calculator and click it right. It helps you fix. It helps you get things right. It helps you learn and retain and be able to teach it better. (Current Error Climate Interview, December 7, 2017)

As Harley described, mistakes are valuable for learning, but when those mistakes are important for the class depends on certain conditions where some take priority over others.

Mathematical knowledge from mistakes. Harley also believed that certain mathematical knowledge could be learned from mathematical mistakes. This knowledge could be gained from anyone that was around when the mistake was made, even if they were not the one who made it. As Harley stated:

> Every mistake is different. . . . It doesn't matter where in the process you made a mistake. You still made one. So, going on that, you're still gonna see where you went wrong in it [*sic*]. Like some people could have made a calculation error and got the wrong answer and did all the steps right, and some people could have got the whole formula wrong or used the wrong formula. So, it helps everybody if they made different mistakes. It benefits everyone. (Current Error Climate Interview, December 7, 2017)

Not only were mistakes important for everyone to experience, but there was certain knowledge that Harley believed could be gained from that experience. This applied to every type of mistake, even procedural ones. As Harley stated:

> All mistakes are gonna be very valuable, even those silly calculator mistakes. That was valuable ‘cause I got things wrong ‘cause I was making the wrong click on a calculator. It was so silly but it really will affect my grade. So now I know to fix my calculator and click it right. It helps you fix. It helps you get things right. It helps you learn and retain and be able to teach it better. (Current Error Climate Interview, December 7, 2017)

However, there were some mistakes that were more important than others as described in the previous section (e.g., which mistakes to discuss). For Harley, there were two dimensions of mathematical knowledge that could be gained from mistakes that emerged from the data: knowledge for learning mathematics and mathematical knowledge for teaching. In the Current Error Climate Interview (December 7, 2017), I asked Harley specifically about what she could learn from a mistake. She stated:

> I mean, incorrect answers, mistakes, helping us [*sic*] learn as teachers and helping us learn as students. And when Dr. Heart talks about others' mistakes or yours, like throughout that whole process of all of these mistakes happening. You're constantly learning more, and you're retaining more and more of that concept every time you go over someone else's mistake even if it's not one that you've made. (Current Error Climate Interview, December 7, 2017)

Making mistakes in class provided the opportunity for others to learn even if they answered the question correctly or knew how to work a problem correctly. Once a mistake was made, it set the foundation for knowledge to be gained by everyone.

Knowledge for learning mathematics. Before detailing Harley's beliefs concerning knowledge for learning mathematics, it is worth noting that this section's title does not use the word *content knowledge*. Harley never referenced specific content knowledge gained from a mistake, as most of our conversations were focused on mistakes in general. Nevertheless, Harley believed that there was knowledge that could be learned only from making a mistake as opposed to getting the right answer, and she described "mistakes are a way of learning" (Current Error Climate Interview, December 7, 2017). As Harley stated, "If you get the answer correct, you just move on. Like, here it is. Done. But with mistakes, you can go deeper into what happened and learn from that" (Post Classroom Observation Interview, October 23, 2017). She talked about learning from mistakes and not learning from correct responses on several occasions during the study (e.g., Current Error Climate Interview, December 7, 2017; Journal Reflection, November 7, 2017; October 24, 2017), always saying something similar to the above quote. The “deeper” was where Harley believed certain knowledge could be gained for learning mathematics.

For Harley, examining mistakes and discussing them with her peers was a way to “see why it works the way it does” (Journal Reflection, October 9, 2017). On this journal reflection, I asked Harley to think about why Dr. Heart chose particular problems for the classroom activities, especially the ones that contained mathematical mistakes. She stated, “I think she chose those mistakes because it will help us learn deeper thinking into the subject of area and perimeter.” For Harley, mathematical mistakes were opportunities for learning more mathematics. She talked about these opportunities for learning on several occasions. On the last Journal Reflection (December 3, 2017), I asked Harley to write a personal statement concerning mathematical mistakes, and Harley chose to write about how mathematical mistakes provided these opportunities for learning. She wrote:

> Mistakes help make information “sticky”. It sticks to your brain when you go over something again and again to figure out what went wrong. IF [*sic*] students can easily solve math problems, there will be little growth. There has to be a challenge to keep them learning and growing. (Journal Reflection, December 3, 2017)

Harley also acknowledged that she had to reflect on those mistakes and purposefully correct them in order to learn from them. In the same reflection, she wrote, “But, I think when you realized you made a mistake that you should figure it out yourself to make it ‘sticky’” (Journal Reflection, December 3, 2017).

Another instance where Harley shared her beliefs concerning knowledge for learning mathematics occurred after a class meeting where groups in the class were asked to make a mathematical mistake on purpose when finding the area of a shape that they were given. Harley's group had an intense conversation to which I was not privy, so I asked Harley if she would share what they were talking about. The following was the exchange that took place after that class meeting:

Me: You and your group talked about how useful this activity was and how you could use this. Can you tell me more about what y'all were saying? I didn't catch all of it.

Harley: Oh yeah. It was really helpful to see different mistakes and how one thing, one mistake could affect the answer. Just that one thing.

Me: So it helped you?

Harley: Yeah for sure.

Me: How?

Harley: Well like with the trapezoid, a group wrote it as ½ height (b1*b2) but it should be b1+b2. All the other formulas are multiplication so that would be a mistake you could easily make. A kid could make.

Me: I see.

Harley: And like making the mistake helps me in the future.

Me: How is that?

Harley: Well, when I saw that a group forgot to half the area for a circle.

Me: The semicircle?

Harley: Yeah. Now, I can see myself making that mistake but now I won't. Because that burned it into my brain. I can see that circle, semicircle, and I see that it's half now.

Me: So, could this not have happened if you saw a correct way of computing that area?

Harley: Maybe, but I don't think it would have worked as well as seeing it this way. (Post Classroom Observation Interview, October 25, 2017)

In this context, Harley used a particular mathematical mistake as an opportunity to learn a formula and what the different parts of the formula represented.

Learning not to forget. Two types of mathematical knowledge that could be learned from mistakes emerged from Harley's data. First, Harley believed that mathematical mistakes could help her learn how to not do something. This type of knowledge was usually connected to, but not limited to, mathematical mistakes that she classified as "small mistakes" (Current Error Climate Interview, December 7, 2017). For example, during a classroom observation, the class was working on developing the surface area formulas for different prisms. While Harley's group was working with a pyramid, I noticed that Harley did not understand where one of the side's length was coming from, and a classmate suggested using the Pythagorean Theorem. In the Post Classroom Observation Interview (November 6, 2017), I asked her about that moment in class. She stated that she "had forgot that formula [which was] an error on [her] part," and her making this mistake in class would "help [her] remember that." In a Journal Reflection (October 24, 2017), which occurred shortly after an exam, Harley wrote about preparing for the exam and the mistakes that she made. She described them as "stupid calculation mistakes . . . [that would have made her] bomb this exam." She also stated that she used those mistakes as opportunities to correct herself, so she would not "forget it when [she is] doing that certain equation" (Journal Reflection, October 24, 2017). In another class meeting, the class was asked to complete an exit ticket in an effort to reflect on the mistakes that happened in class. That day, the class explored the area of shapes that did not fit the area formulas they developed (e.g., semicircle, irregular shapes). In response to being asked what the most meaningful mistake was for Harley that day, Harley chose to write about "seeing people forgetting to divide by 2 because many problems had a half circle" (Exit Ticket, October 25, 2017). She also stated that "it was good to [see] that mistake so [she] could remember it for next time" (Exit Ticket, October 25, 2017). Again, the mistake was useful for remembering how to not do something.

For Harley, some mathematical mistakes were opportunities for her to remember something or, more specifically, remember how to not do something. In her Prior Error Climate Interview when talking about learning from mathematical mistakes, she stated:

> I don't think that there's anything else you could learn but the correct way to do it. You could just learn how to not do it and how to do it correctly. I feel like there's not much else you could learn from it. (September 17, 2017)

A few minutes later in that interview, she elaborated more:

> Well, I guess now you'll know how to not do it. For sure. But, you'll, I mean, I hope that the teacher would help how to fix it, too. Yeah, I guess you won't keep making the same mistake again when you see that the answer is wrong. (September 17, 2017)

Harley's central belief of using mathematical mistakes to fix something in a specific way was highlighted here. She believed that fixing a mathematical mistake helped her to learn how to avoid that mistake or she "would have made the same mistake on the test" (Journal Reflection, October 24, 2017).

Learning connections. Harley also believed that mathematical mistakes provided opportunities to learn other mathematics and to "make connections" (Post Classroom Observation Interview, October 23, 2017) or to learn mathematics at "a deeper" level (Post Classroom Observation Interview, November 29, 2017). For Harley, these opportunities for learning mathematics at a deeper level always coincided with a mathematical mistake that she would describe as a "bigger mistake" (Current Error Climate Interview, December 7, 2017). In one instance, I talked with Harley after a class session that was dedicated entirely to examining work, provided from Dr. Heart from students not in the class, containing conceptual mistakes. An example is provided in Figure 21. I asked Harley why Dr. Heart would select these types of problems for the class to examine and discuss in one of her journal reflections. She stated:

> I think she chose those mistakes because it will help us learn deeper thinking into the subject of area and perimeter. If we get into the complicated parts of area and perimeter and actually SEE [*sic*] why it works the way it does . . . This worksheet teaches us to see why and how perimeter and area can work together. (Journal Reflection, October 4, 2017)

Harley perceived that the mistakes chosen by Dr. Heart provided her the opportunity to learn the mathematical content at a deeper level as well as how different mathematical concepts were connected.

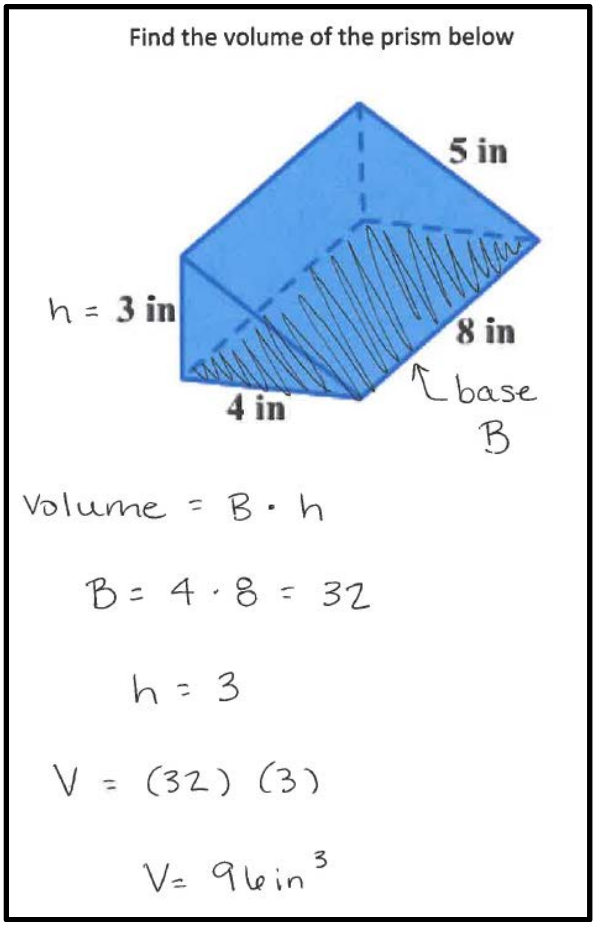


Figure 21. Example of a student's conceptual mistake provided by Dr. Heart.

Making connections in mathematics because of mistakes also came up after observing Harley on November 29, 2017, while she was working with her group on the final exam review sheet. During the interview after that class meeting, I asked Harley what she learned while working on the review sheet with her group. She stated:

> I learned a lot. I learned how to fix the mistakes that these kids made and how to avoid making that same mistake. . . . These also made me think how these were related to other things, especially the formulas. Why there's a two here or a one-half there. There's so many places you can mess up and only one right answer.

> Seeing different mistakes made me think about all of that. (Post Classroom Observation Interview, November 29, 2017)

For Harley, mathematical mistakes not only provided her opportunities to learn how to not complete a problem, but the mistakes also provided her the opportunity to consider her other aspects of the mathematics she was doing (e.g., meaning of values in formulas). As Harley described this, “mistakes are used to make sure that you really understand something” (Current Error Climate Interview, December 7, 2017).

Mathematical knowledge for teaching. Not only did Harley believe that mathematical mistakes could be used for her to learn something about mathematics, she also believed that mathematical mistakes provided her opportunities to gain knowledge for teaching mathematics. Throughout the study, Harley repeatedly put herself in the role of the teacher when thinking about mathematical mistakes. There were cases where Dr. Heart asked the class to “think like a teacher” (Post Classroom Observation, October 2, 2017), but there were other cases that Harley did this herself stating, “I’m going to be a school teacher, so I’m thinking like a school teacher here” (Prior Error Climate Interview, September 17, 2017). In these instances, Harley described how mathematical mistakes provided opportunities for her to learn as a teacher.

Specifically, Harley saw mathematical mistakes as opportunities to see how her future students “can mess up and come up with different answers” (Journal Reflection, October 9, 2017). The mathematical mistakes that she saw in class were helpful in seeing the “importance of explaining a subject very deeply and answering as many questions as possible during class” (Journal Reflection, October 9, 2017). This was especially true when Harley was exposed to multiple ways of getting the wrong answer. For example, during the class session where the class watched the fourth graders use the broken ruler, I asked Harley what she remembered thinking as she watched those videos. She stated:

> I'm so glad we have to take this class, because it really teaches you some things that we need to make a high level of importance to teach our future students. Like the different mistakes. We can so easily misguide them by missing such a small piece of the puzzle. (Post Classroom Observation Interview, October 2, 2017)

Harley saw the different mistakes as evidence of what to stress in her classrooms. The different mistakes were strategies of which she needed to be aware as a teacher.

Another instance was on an activity that occurred at the beginning of a class meeting, where Harley was given three ways that different students found the perimeter of a shape (see Figure 20). On the activity, she was asked how these might help her as a teacher. She stated, “These are good to know to show your students, because it teaches them the correct way to find perimeter and kept them from doing it wrong” (In-class Reflection, October 18, 2017). The mistakes in this case were tools that she could use as a teacher. After that class meeting, I asked Harley what she thought about the two incorrect ways of finding the perimeter and if she thought the task was helpful in any way. She stated:

> I believe these examples were important for us as future teachers to know the incorrect answers and correct answers of problems. I think this is important for her to teach us incorrect answers, because we have to be able to adapt to all the different ways that children might do problems in a way that are easier for them to understand. I think it will also help us to teach it better and teach our future

> students how to not do a problem, so they don’t get it wrong on tests. (Post Classroom Observation Interview, October 18, 2017)

Here, the mistakes shown in class added to Harley’s repertoire of strategies that her students might use in her classroom. I continued to talk to Harley about the different mistakes on this exercise, and I wondered what she thought about the two mistakes in particular that she was shown. She again chose to speak on how those mistakes applied to her as a prospective teacher:

> If there are different ways, I think she should have showed us those so we can know more ways to teach our future students in case someone does not understand the certain way we might be teaching it. (Post Classroom Observation Interview, October 18, 2017).

Again, Harley looked at the mathematical mistakes as strategies that her future students might use, and thus, she needed to be aware of them. Harley believed that seeing mathematical mistakes in class was an opportunity for her to become a better teacher. Seeing these would help her “as a teacher [in] recognizing that there's people are [*sic*] gonna do things differently and make different mistakes” (Post Classroom Observation Interview, October 18, 2017).

Near the end of the study, I was curious how Harley might handle mistakes in her future classrooms, so I asked her to imagine that she was the teacher in an elementary mathematics classroom. As she saw herself walking around listening to what students were doing and saying, she noticed several mistakes in the class. Then, I asked how she imagined herself handling the mistakes that she saw. She stated:

> I definitely would not call out the students that made the mistakes. Instead, when I walk by I will talk to them and try to understand what they wrote down. Since it was incorrect, I will take one of the correct problems and the incorrect problems and see if the class can decide which one is correct. This helps the students that made the mistakes not feel embarrassed by being called out. I would then explain why the correct answer is correct. I would even break down the incorrect answer and see what mistakes were made so the class knows not to make it again. (Journal Reflection, November 27, 2017)

This quote is summative of Harley's beliefs in how mathematical mistakes should be used. Harley was appropriating the actions that she perceived happened in Dr. Heart's class in asking the students to analyze mistakes and helping them get the correct answer. Here, Harley demonstrated the impact of her beliefs on how mathematical mistakes should be used in the classroom, especially for the teacher: highlighting them so that they can be fixed and therefore avoided in the future, most likely on the test.

Affective component of mathematical mistakes. In addition to believing that mathematical mistakes provided Harley with opportunities for gaining knowledge for mathematics and knowledge for teaching mathematics, Harley also believed that mathematical mistakes had an affective component for students. Although Harley believed that mistakes were the "key to learning" (Journal Reflection, December 3, 3017), Harley also believed that mathematical mistakes could hinder learning because of how mathematical mistakes made her feel.

This belief concerning mathematical mistakes was first displayed in the Prior Error Climate Interview. I asked Harley about what a mathematical mistake meant to her when she made one and how that made her feel. She responded:

> I get really stressed out when I mess up ‘cause like a lot of people around me are getting it, and I just can't for some reason. I don't know why it's so hard for me, but I get really down and frustrated and sometimes I don't even want to go to class. (September 17, 2017).

This was related to her past experiences in mathematics classrooms as she shared stories of getting “mocked a little bit” (Current Error Climate Interview, September 17, 2017) and being told that she was wrong, which she described as “Oh Man! [A] punch to your ego a little [that would] make [her] want to give up” (Current Error Climate Interview, September 17, 2017).

However, she described similar circumstances in Dr. Heart’s class. During a week of class meetings, I noticed Harley acting differently when Dr. Heart was near her group and appeared to be listening to the group’s conversation. In a Journal Reflection (October 24, 2017) prompt following that week’s meetings, I asked Harley to think about how she thought the conversations at her table changed, if at all, when Dr. Heart was listening. She replied:

> When Dr. Heart listens to our conversations, I get quiet. I am scared to tell my group my opinion with fear of being wrong and called out by Dr. Heart. The rest of my table group keeps talking, because they don’t seem as scared as I am to be wrong. They will ask her for help if she is around listening, but I won’t. (Journal Reflection, October 24, 2017)

Harley was aware when Dr. Heart was listening to the group's conversation and did not want to share her thoughts for fear of being wrong.

Harley mentioned quitting during a task in Dr. Heart's class at other times as well. In a journal entry prompt, I asked Harley to think about what she thinks about when she makes a mistake in class. She stated, "My face turns bright red and I get so embarrassed. Sometimes I even get frustrated and angry, but I try not to show it. I shutdown" (Journal Reflection, October 26, 2017). She mentioned shutting down when she made a mistake in another journal entry. I asked her to think about specific examples of mistakes that happened in class that week and what she did after she made them. She wrote, "I made several mistakes every day in math class [this week]. I usually feel embarrassed to make mistakes, and I often don't want to let anyone know that I've made one. So, I didn't tell anyone about them" (Journal Reflection, November 7, 2017). Again, Harley did not want to share her mistakes because of the level of embarrassment that she would feel. In other instances, she similarly described "being intimidated" (Prior Error Climate Interview, September 17, 2017) and "freaking out" (Current Error Climate Interview, December 7, 2017) when thinking about making a mathematical mistake.

Although most of Harley's statements concerning the affective influence of mathematical mistakes was focused on her own mistakes, Harley's sentiments did not change when thinking about others' mistakes. During the Prior Error Climate Interview (September 17, 2017), Harley described how a teacher that was like an entertainer would treat a mistake. She stated, "A teacher like an entertainer would make you feel better when you made a mistake. They would make you more comfortable after you made one. Like 'don't feel bad. We've all done it.' Something like that." Here, the teacher like an entertainer would make her feel better because of how the mistake might make her feel. Following her response, I asked if she thought that is what a teacher should do after a student makes a mistake. She stated:

> Maybe, I know that some of my teachers would be like, shucks, that's a good idea but let's try it, what if we tried it this way. I think this is a better way to do it instead of just saying no, you're wrong. Kind of like redirecting you to make you feel better. (Prior Error Climate Interview, September 17, 2017)

Harley expressed a focus on how the mathematical mistake made her feel and how the teacher could assist in alleviating that embarrassment.

For Harley, mathematical mistakes were a force to be reckoned with in terms of how they made the person making the mistake feel and something that the teacher needed to mitigate when seeing students make mistakes. Harley believed that mathematical mistakes influenced her affectively, and, in some cases, caused her to quit and to stop any learning that was taking place. Although she recognized that mathematical mistakes were necessary for learning and “a good thing if [you] realize it or not” (Journal Reflection, October 26, 2017), Harley also believed that mathematical mistakes were a powerful aspect of the classroom of which the teacher needed to be aware.

Noticeable Changes in Beliefs System

Before describing the noticeable differences in Harley’s beliefs system, this section necessitates a comparison of Harley’s Prior Error Climate Survey and Current Error Climate Survey (see Table 9). The two averages that are noticeably different for Harley are the error-taking risk and functionality of error for learning dimensions. The error-taking risk dimension focused on Harley’s perception of how likely students are to take risks with mistakes in her mathematics classrooms. When she elaborated on her choices on the survey for this dimension in the Prior Error Climate Interview (September 17, 2017), she described herself as being “scared of being called on . . . for fear of being wrong” and “[hating] being called on even if [she] thought [she] knew the answer.” For comparison, in her Current Error Climate Interview (December 7, 2017), she described her willingness to make errors as situational. In her words, “It depends on the day . . . [and] . . . how the teacher is and how the class is and if you feel comfortable in that classroom.” Although Harley’s perception of Dr. Heart’s class still showed an unwillingness to be wrong in certain instances, there was a noticeable difference in Harley’s perception of students’ willingness to make mistakes in her prior classrooms compared to Dr. Heart’s. She stated that Dr. Heart “*liked* [emphasis added] seeing our mistakes and talking about them” (Current Error Climate Interview, December 7, 2107). Whereas when talking about her prior classrooms, she made statements similar to, “Why would [the teacher] want to hear a wrong answer?” (Prior Error Climate Interview, September 17, 2017). These two quotations are in opposition to one another. Harley described her previous classrooms as, essentially, not ever talking about mistakes, and she described Dr. Heart as a teacher that wanted to hear students’ mistakes.

Table 9

Harley's Prior and Current Error Climate Results

Error Climate Dimension	Prior Error Climate Average	Current Error Climate Average
Error Tolerance by Teacher	*M* = 3.5	*M* = 2.25
Irrelevance of Errors for Assessment	*M* = 2.5	*M* = 2
Teacher Support Following Errors	*M* = 2	*M* = 2.25
Absence of Negative Teacher Reactions to Errors	*M* = 3.5	*M* = 2
Absence of Negative Classroom Reactions to Errors	*M* = 4	*M* = 3
Error-taking Risk	*M* = 6	*M* = 3
Analysis of Errors	*M* = 2.67	*M* = 2.67
Functionality of Errors for Learning	*M* = 3.75	*M* = 1.75
Superordinate Score	*M* = 3.49	*M* = 2.37

Note. Scores on the survey ranged from 1 (strongly agree) to 6 (strongly disagree), with averages of 3 and below indicating a positive error climate and averages of above 3 indicating a negative error climate. A lower average indicates a more positive error climate.

In terms of the functionality of errors for learning dimension, a dimension focused on how mathematical mistakes are used to learn in the classroom, Harley perceived her prior classrooms and Dr. Heart's similar in terms of using mathematical mistakes for learning. That is, she described her prior classrooms and Dr. Heart's as using "mistakes to learn something” (Current Error Climate Interview, December 7, 2017; Prior Error Climate Interview, September 17, 2017). However, she perceived her prior mathematics classrooms as using mathematical mistakes as more of an opportunity for the teacher to “fix the mistake” and give everyone more opportunities to “practice getting the right answer” (Prior Error Climate Interview, September 17, 2017). In her description of Dr. Heart’s class, mathematical mistakes were seen more as opportunities for “everyone to learn more [than they already did]” (Current Error Climate Interview, December 7, 2017) and not just “not how to do it” (Prior Error Climate Interview, September 17, 2017).

This leads into the changes that Harley made in her beliefs system throughout the study. There were two areas of change in Harley’s beliefs system concerning mathematical mistakes. Each of which will be described and detailed in the following sections.

The purpose of mathematical mistakes in the classroom. At the beginning of the study, Harley saw mistakes in the classroom as obstacles that learners of mathematics had to overcome. In her Prior Error Climate Interview (September 17, 2017), Harley stated, “Mistakes happen [and you have to] fix them.” However, they did serve some useful purpose. She conceded that mathematical mistakes would happen and that they would be helpful for the teacher to know where students were lacking in knowledge. Harley talked about mistakes like they were a way of knowing if teaching was going well or not. Harley described these as “signals for the teacher to [teach better]” (Prior Error Climate Interview, September 17, 2017). Mathematical mistakes were signals to not continue the action in progress. Harley described what this interaction would look like for the teacher. When expanding on her choice to B8–If someone in a mathematics class does something incorrect, he will get very little support from the teacher–Harley stated:

> Well, I somewhat agree. ‘Cause if, if he does something wrong, the teacher is obviously not going to support his answer but at least support his idea for trying. “Good job, you did great! But, you should do it like this.” Support the problem. (Prior Error Climate Interview, September 17, 2017)

In this quote, Harley explained how when a mathematical mistake was present corrective action was needed on the part of the teacher.

For the student, mathematical mistakes served a similar purpose. She never described mathematical mistakes as signals, exactly, for the student, but she did describe them as “ways to not do something” (Post Classroom Observation Interview, October 2, 2017). When describing how a teacher in her prior classrooms used mathematical mistakes to learn something, she stated:

> I would agree with that. Because even though the teacher doesn’t want you to do assignments incorrectly, I guess a lot can be learned if you mess something up. Because then they can go over it and you know people who already know how to do it are getting more practice and the people who don't know how to do it are learning how to do it. So, I think that I would agree with that. (Prior Error Climate Interview, September 17, 2017)

Here, mathematical mistakes served as an opportunity to practice the problem more, whether a student initially got the correct answer or not. She later described in that interview that going “over it” meant to show the class “how to do it correctly” (Prior Error Climate Interview, September 17, 2017). In another instance during class, Harley was given the image, similar to the one in Figure 22, along with a statement from a child that angle 1 was smaller than angle 2. Harley was then asked if she thought the child was incorrect to describe what she thought the child understood and misunderstood about the concept of angle. Harley replied by saying, “The child is not correct. Angle 1 and Angle 2 are the same size angle” (Exit Slip, August 30, 2017). That was Harley’s entire response. In this quote, Harley simply pointed out that the student was not correct, and that was the end of it. In both instances, mathematical mistakes were moments to learn how to not do a specific problem correctly, and thus, corrective action on the part of the student was needed.

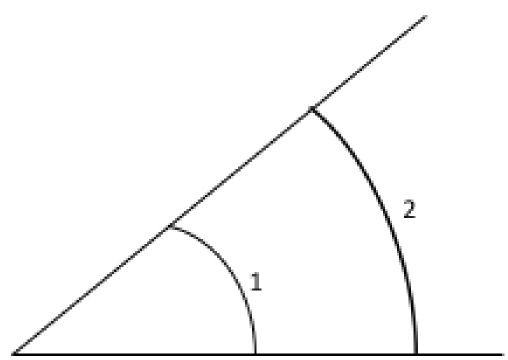


Figure 22. Angles provided in an exit slip given to Harley on August 30, 2017.

However, as Harley spent more time in Dr. Heart’s class, Harley was confronted with numerous occasions where mathematical mistakes were not treated in the same fashion as they were in her past. The notion that mathematical mistakes were signals for corrective action permeated for some time, but experiences in the class that were in direct opposition to this notion forced Harley to make sense of these experiences with her established beliefs system concerning mathematical mistakes. For example, in her first Journal Reflection, Harley was asked to think about the mathematical mistakes that she remembered in class that week and to reflect on the interactions that took place thereafter. She wrote about not knowing the definition of perpendicular and the interaction among herself, the group, and Dr. Heart after no one in her group could remember what it meant. She first stated, “[Dr. Heart] tried to talk it out with us rather than giving us the answer right up front. She asked us what we thought perpendicular meant or looked like? I wished that she had just told us” (Journal Reflection, September 19, 2017). In this quote, Harley was in contention with what she expected to happen and what actually happened. She found it somewhat odd that she would be asked about what she thought instead of the teacher using this moment to correct the students, as this was in contrast to her description of her experiences prior to this class (i.e., Prior Error Climate Interview, September 17, 2017). Near the beginning of the study she had already started inquiring about activities that were occurring in Dr. Heart’s class. For example, she stated, “I’ve never seen a favorite mistake before. Showing a mistake that she liked” (Prior Error Climate Interview, September 17, 2017). Similarly, in her second Journal Reflection (September 24, 2017), Harley was asked to write about her favorite mathematical mistake from class that week. I used the term *favorite mistake* because Dr. Heart used that terminology on an activity during class. In Harley’s reflection, she described an interaction with another classmate in which they disagreed on whether a square was a rectangle or rectangle is a square. They asked Dr. Heart, and Dr. Heart posed this question back to the group for inspection attending to what the definitions of each are.

Harley wondered why “[the group] wasn’t just given the answer. I ended up being right anyways” (Journal Reflection, September 24, 2017). Here again, Harley’s group was not just given the answer, and she was asked to think more about the problem even though she was correct. In this quote, Harley started to question how experiences in Dr. Heart’s class fit into her belief structure. As the class was prevalent with experiences where mathematical mistakes were presented and discussed and mathematical mistakes were not used just for corrective action, noticeable changes in Harley’s beliefs system were present in terms of how mathematical mistakes were used in the classroom.

First, Harley thought the teacher used mistakes differently by the end of the study. As opposed to using mathematical mistakes as just an opportunity to show the correct answer, Harley used mathematical mistakes as opportunity to understand what a mathematical mistake meant in terms of understanding or, more so, not understanding. Take, for example, her response to the Exit Slip provided earlier in Figure 22, where Harley simply stated that Angle 1 and Angle 2 were not different angles. Harley’s response was simply to say that the child was not correct and to provide the correct answer. However, in a future Exit Slip (October 4, 2017), Harley’s response was different. The exit slip offered the following: If a student stated, ‘as the perimeter of a closed polygon increases, the area also increases,’ what does it tell you about the student’s thinking? She responded, “The student may only be thinking of one specific polygon that this works with. However, it does not work with all polygons. So, the student needs to be more specific about the shape they are talking about” (Exit Slip, October 4, 2017). Although Harley was primed to think about the student’s thinking in the prompt, Harley started to change the way in which she responded to a student making a mathematical mistake. Here, she still focused on stating that the response is incorrect, but she also wanted to know more about what the student is thinking. It was no longer just about right and wrong but also the rationale that the student is bringing with the response.

There were other instances that provided similar evidence. In one classroom observation that occurred on November 15, 2017, I did not observe many errors in class by anyone. I was curious how Harley thought about class sessions where mistakes were the focus as opposed to class sessions without this focus. She stated, “I think that class sessions where mistakes are there are more beneficial . . . [because] . . . I’m more engaged in the class when I have to brainstorm and talk to my group about where we might have went [*sic*] wrong” (Post Classroom Observation Interview, November 15, 2017). In this case, Harley saw mistakes as catalyst for learning, which was in stark contrast to her view of mistakes prior to being in this study. In another classroom observation that occurred on October 18, 2017, I asked Harley about the activity where she was given three ways that different students found the perimeter of a shape (see Figure 20). On the activity, she was asked how these might help her as a teacher. She responded:

> I think this is important for her to teach us incorrect answers, because we have to be able to *adapt* [emphasis added] to all the different ways that children might do problems in a way that are easier for them to understand. I think it will also help us to teach it better and teach our future students how to not do a problem, so they don’t get it wrong on tests. (Post Classroom Observation Interview, October 18, 2017)

Although Harley was still focused on providing the students with how to not do a problem, she chose to talk about adapting as a teacher here. This suggested that mathematical mistakes would be used differently than simply replying with the correct answer or showing the student(s) how to do the problem correctly, and Harley would need to be able to help students in different ways.

Harley also saw that different types of mistakes served different roles in the classroom. In the Current Error Climate Interview (December 7, 2017), Harley talked about what teacher support looked like, and she differentiated between support with a “big mistake” and a “small mistake.” I stopped her to ask about the difference. She stated, “That’s another thing. I’ve never really thought about the difference between a big mistake and a small mistake, but there is a difference” (Current Error Climate Interview, December 7, 2017). Here, different mistakes might mean different things. This was in stark contrast to her quotes in the Prior Error Climate Interview (September 17, 2017) where she wondered “why would [the teacher] want to hear a wrong answer?” She went on to say:

> With a big mistake, that’s something that we need to talk about, sometimes as a whole class. But with a small mistake, that’s something that can be corrected easy [*sic*]. [Dr. Heart] wouldn’t need to talk about that. She could just correct it real quick, like she does on my tests. “Hey, you didn’t multiply correctly here.” But with a big mistake, that’s different. Those we might spend a lot longer talking about. (Current Error Climate Interview, December 7, 2017)

Before Dr. Heart’s class and Harley’s participation in this study, a mistake was a mistake, and the person that made the mistake was wrong. At the end of the study, Harley attended to what type of mistake it was and a different way of handling it with the student or the class as a whole.

What is learned from a mistake. During the course of the study, there were also noticeable changes in how Harley considered learning from a mathematical mistake. As stated previously, Harley never talked about mathematical content specifically, unless given the opportunity during data collection (e.g., exit slips, in-class reflections). However, she did make statements such as:

> I don't think that there's anything else you could learn but the correct way to do it. You could just learn how to not do it and how to do it correctly. I feel like there's not much else you could learn from [the mistake]. (Prior Error Climate Interview, September 17, 2017)

In this quote, Harley stated that an individual could only learn the right way from a mistake. There is nothing else available to learn. At the beginning of the study, Harley believed that mathematical mistakes were used to learn how to do the problem correctly, and that was the end of it. This was aligned with how she believed teachers should use mathematical mistakes in the classroom. However, similar to making changes in how teachers use mathematical mistakes, Harley also made changes in her beliefs system in terms of what an individual could learn from a mathematical mistake.

In Harley’s Journal Reflection (September 29, 2017), I prompted her to write her thoughts concerning if there were mistakes that were valued or not valued in class that week. Additionally, I asked her to reflect on what actions indicated value being placed on a mistake or not. She responded, “I believe that ALL of our mistakes are valued in this particular class. This class makes you think critically about problems and finding new ways to solve problems rather than just the certain way you learned it" (Journal Reflection, September 29, 2017). Here, Harley started to think about how mathematical mistakes allowed her insight into multiple ways of working the problem and not just *the way* to solve it. Harley expressed similar sentiments in another journal reflection where I prompted her to consider how she felt and what she thought about when she made a mistake. She stated:

> I make several mistakes every day in math class. I usually feel embarrassed to make mistakes and I often don't want to let anyone know that I've made one. But, with this new group I'm in, we've *used* [emphasis added] my mistakes and [the mistakes] pushed me to learn *new things* [emphasis added] with them. (Journal Reflection, November 7, 2017)

Here, the mathematical mistake was an actionable object, something that could be used to learn something else. In another journal reflection that focused on why Harley thought certain mathematical mistakes were highlighted in class, she described Dr. Heart's selection of mathematical mistakes as an access point to "deeper thinking . . . [and to] . . . see why it works the way that it does" (Journal Reflection, October 9, 2017). In this instance, mathematical mistakes were not just a way to the correct answer but also an opportunity to gain understanding of something else, whether that be an understanding of the underlying concept that the problem is using or insight into other ways to solve a problem.

In another classroom observation, I observed Harley talking to one of her partners in her group on several occasions about the mistakes that they made on the worksheet to help prepare for the final exam. I asked Harley what she got out of those conversations. Harley stated:

> When a mistake occurs, you get to go deeper into the question/problem and figure out where you went wrong. I am actually doing that tonight while studying and it really has helped me to make these mistakes and backtrack to see where I messed up. That is when I recognize my lack in knowledge, and I can fix it. Also, by redoing a question/problem that you made a mistake on, you are retaining it and learning it even more by repeating it and talking about it. There is little conversation about correct answers. Most of the time there is no conversation at all, because there is little to be explained or talked about. Mistakes bring up tons of conversation among the class groups, and we can all learn something when we do that. (Post Classroom Observation Interview, November 29, 2017)

This quote was representative of Harley taking the experiences from the class and fitting them into her beliefs system concerning mathematical mistakes. She saw the opportunities that mathematical mistakes afford, but her central beliefs also influenced that this was all in an effort to get the right answer and maybe she can learn something else in the process.

Harley saw mathematical mistakes as more than how to not do a problem at this point. However, in light of her belief clusters described earlier, Harley was still working on how this all makes sense in her beliefs system. Even when Harley talked about learning the correct way to do a particular problem, mathematical mistakes were viewed as a better way of learning it. In the interview after the classroom observation with few mistakes, I asked Harley if she thought class sessions with or without mistakes were more beneficial to her. She stated, “Sometimes mistakes are difficult, because I don’t understand everything. But when I do, I get so excited and I usually remember things better when I had to struggle to understand it” (Post Classroom Observation Interview, November 15, 2017). Here, Harley saw a benefit of mathematical mistakes as a way to understand the mathematical concept, but she also struggled with not understanding. After all, she had been accustomed to being given the right answer for so long.

Although it was difficult to state with certainty that Harley made any lasting changes to her beliefs system, the evidence presented in this section illustrated that Harley reconciled experiences in Dr. Heart’s class and this study that were dissonant with her current beliefs system. Harley showed that she took the experiences from the class and made them “make sense” (Leatham, 2007, p. 187) into her beliefs system with changes in how mathematical mistakes are used in the classroom and what can be learned from a mathematical mistake.

Summary

The previous sections focused on the case of Harley. Specifically, this section described why Harley’s implicit theory was determined to be neither an extreme incremental nor entity theorist. Additionally, a description of Harley’s perception of her prior mathematical experiences was discussed. This provided insight into Harley’s experiences that helped shape her beliefs concerning mathematical mistakes.

Harley’s central beliefs concerning mathematical mistakes were also described. Specifically, Harley believed that mistakes needed to be fixed in specific ways to better prepare her for the test. Additionally, students’ mathematical mistakes are necessary in the classroom for the teacher to be able to learn how to fix them. Figure 23 provides a visual representation of Harley's central beliefs concerning mathematical mistakes.

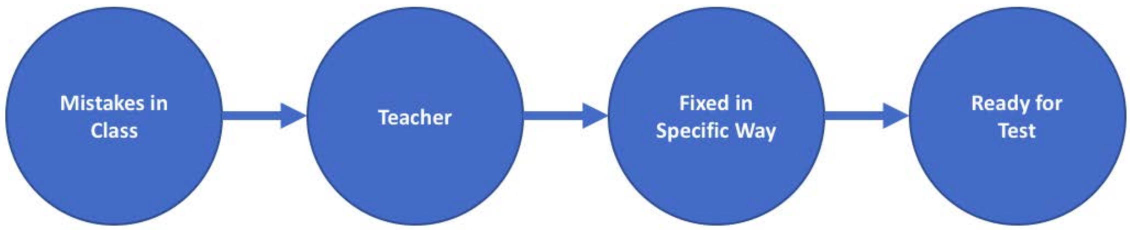


Figure 23. Harley's central beliefs concerning mathematical mistakes.

Harley's beliefs clusters were also detailed. This included when Harley believed that mathematical mistakes were important enough to share with the class. However, the sharing of those beliefs was mediated by the level of embarrassment that the mathematical mistake being made public could bring. Additionally, Harley believed that there were times that mathematical mistakes could be used to learn about mathematics and teaching mathematics in addition to preparing for the test from fixing a mistake in a specific way. Figure 24 provides a visualization of Harley's beliefs system concerning mathematical mistakes.

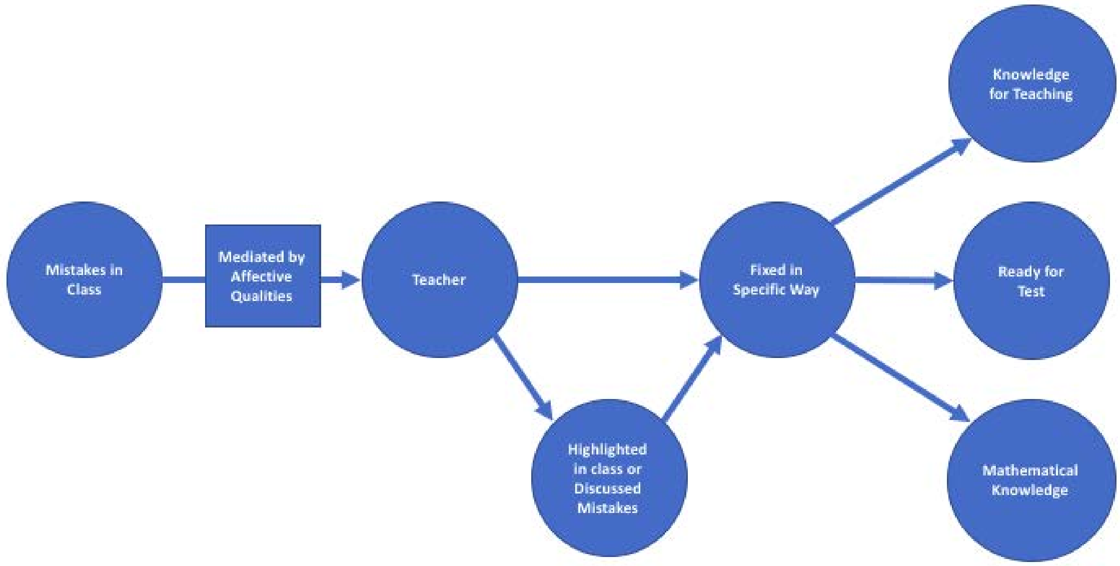


Figure 24. Harley's belief clusters.

Finally, the noticeable changes in Harley’s beliefs system were described. Harley made small changes in how mathematical mistakes were used for in the classroom. She believed that mathematical mistakes did not have to necessarily be fixed in a specific way in order to learn from them. For Harley, engaging with the mistake in itself offered opportunities to learn. Additionally, and related to the previous change in Harley’s beliefs system, knowledge could be gained from directly engaging with mathematical mistakes rather than seeing how the mistake was fixed in a specific way for the test. Those changes are illustrated in Figure 25 and represented with dashed lines.

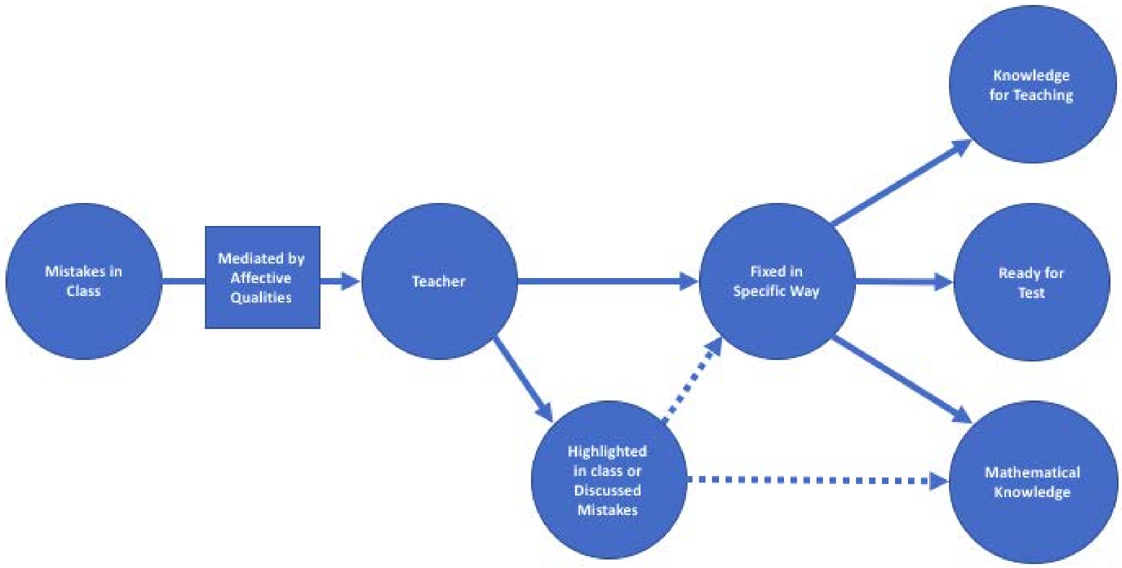


Figure 25. Harley's beliefs systems at the end of the study.

The previous sections focused on the cases of Cindy and Harley separately. The next section will detail the cross-case analysis of Cindy and Harley. Specifically, it will detail the similarities and differences between each case's background, perception of error climates, beliefs systems, and the changes in their beliefs system.

Cross-Case Analysis of Cindy and Harley

This study investigated the beliefs concerning mathematical mistakes of Cindy and Harley as well as how those beliefs changed during enrollment in Dr. Heart's class, which focused on using mathematical mistakes in ways aligned with current reform documents. Using a cross-case analysis based on the overarching themes in each case identified during analysis of each case, I will discuss the similarities and differences between Cindy and Harley, including their backgrounds and beliefs systems concerning mathematical mistakes. Furthermore, I will compare the changes in their beliefs systems concerning mathematical mistakes in the following sections.

Implicit Theory

Cindy and Harley were similar in their ascription to an implicit theory (see Table 10). The two notable differences were on the Intelligence and Mathematical Ability dimensions. Cindy strongly ascribed to an incremental theory in both dimensions while Harley ascribed to neither an entity nor incremental theory for the Intelligence dimension. Related to the Mathematical Ability dimension, Harley ascribed to an incremental theory even with her average being lowered by somewhat agreeing that a person's mathematical ability was something about them that they cannot change very much (Survey Item A11)– the only item that addressed the degree to which mathematical ability can increase compared to if it could change.

Table 10

Cindy and Harley's Implicit Theory Averages

Implicit Theory Dimension	Cindy	Harley
Intelligence	*M* = 6	*M* = 3.33
Morality	*M* = 4.33	*M* = 3.67
Worldview	*M* = 3.67	*M* = 2.67
Mathematical Ability	*M* = 6	*M* = 4.33

Note. Scores ranged from 1 to 6, with averages of 3 and below indicating characteristics of an entity theorist for each attribute and scores of 4 and above indicating characteristics of an incremental theorist for each attribute.

During the implicit theory observations, I observed Cindy and Harley enacting similar implicit theory descriptors. When working in their groups on the *United We Solve* task (Erickson, 1996), both participants were observed making similar comments, "This looks hard, but I think that we can do it together" (Cindy, Implicit Theory Observation, August 28, 2017) and "I can do this. I am going to learn this" (Harley, Implicit Theory Observation, August 28, 2017), both of which are incremental descriptors. However, both participants enacted similar entity theorist descriptors when dealing with setbacks and challenges by avoiding responding to Dr. Heart's prompts or deferring to their group when asked for their thoughts.

Collectively, both participants were similar in their implicit theory. Specifically, they were neither on the extremes of the implicit theory continuum, which was a reason both of them were selected to participate in the study. Although, Cindy held a stronger incremental theory, the implicit theory observations yielded similarities in their enacted descriptors of their implicit theory.

Mathematics Background

Cindy's and Harley's prior mathematical experiences were also very similar. Although Cindy described herself as a "level-up" (Prior Error Climate Interview, September 12, 2017) student, always in honor classes, and Harley described herself as "always [being] bad at math" (Prior Error Climate Interview, September 17, 2017), their mathematical classrooms, especially in high school, were very similar in terms of how they perceived mathematical mistakes were treated. During the first interview, Cindy stated:

> In high school, we never spent time on the incorrect. We've always spent time on the correct and why it was correct. Never, "Alright Tommy got this wrong, why did he get it wrong? If he got it right, explain to me why he got it." So, I don't think that it's something that's spent time on. At least in my past experience, it was, it was [*sic*] never the incorrect. It was the correct. (First Post Error Climate Interview, September 12, 2017)

Here, Cindy explained that in her high school mathematics classrooms, there was never a time that students' mathematical mistakes were talked about or shared with the class. The focus was always on the correct answer. Harley stated something similar in my first interview with her:

> I just remember that my high school teachers were never like, "Oh I want to hear what you were thinking." "No, like this is the way that you do it. Like this!" But

> in college it's not like that. You can say, “Oh I don't think that's right,” and the teacher's okay with it if it makes sense. (Prior Error Climate Interview, September 17, 2017)

Harley, like Cindy, perceived her mathematics classrooms as being focused on the correct answer and choosing not to highlight the mistakes students made.

Cindy and Harley also similarly described a typical day in their prior classrooms. Again in the first interview, Cindy stated:

> The teacher is either at the desk or at the door, but we'd have some bell work or some getting started assignment that would either be a problem that we'd seen before or had built off the homework. Or to introduce. I never really liked when it introduced, because if I got a bunch of answers wrong, it was kind of hard. Then we'd discuss it. We'd have to work on it and discuss it, and then we'd move straight into the lesson or go over the homework. And then, it would end with new homework. A lesson was either up at the board doing different problems and then sometimes us going up there. Or, it would be us, she'd just put up a bunch [of problems], or he would, and we had to work in a group of four. I mean remembering back it was mainly three or four days of it, it would be lecturing and then we'd work in some groups. (Prior Error Climate Interview, September 12, 2017)

Cindy described a teacher-centered classroom with occasional instances of her working with groups on worksheets that she described as “homework problems” (Prior Error Climate Interview, September 12, 2017). Harley’s description of a typical day in her prior mathematics classrooms was similar:

> Well, pretty much just thinking about my favorite teachers. Mainly in high school. ‘Cause that had the most impact. I mean usually we'd walk in, sit at a desk facing the board, and, it's been so long since high school. I really don't remember. I just know that we, we did a lot on the whiteboard. Like she would write down you know show us how to do equations and take notes. I remember that. I feel like we'd do activities a lot with shapes and stuff like that. We'd usually get a worksheet or two per class.

Harley’s description of her prior classrooms was teacher-centered classroom and lecture-heavy.

Cindy and Harley had similar prior mathematics experiences, especially when dealing with mathematical mistakes. Their descriptions are important to note as they helped to shape their beliefs system concerning mathematical mistakes, which are compared in the next two sections. As a reminder, both participants were selected prior to any knowledge of their mathematical backgrounds, and this similarity was coincidental.

Central Beliefs Concerning Mathematical Mistakes

Cindy and Harley held similar central beliefs concerning mathematical mistakes. That is, they saw the role of mathematical mistakes in the classroom similarly. Cindy believed that mathematical mistakes were essential to the classroom. She described them as important for the classroom, because mistakes were where “you build on your knowledge” (Prior Error Climate Interview, September 12, 2017) and “make things click” (Post Classroom Observation Interview, October 18, 2017). She also stated that “even though I don't like to make them, mistakes are a part of life. You learn from mistakes, how to fix them or make them better” (Journal Reflection, December 4, 2017).

Comparatively, Harley believed mathematical mistakes were important to the class as well. Even though she described them as a “key to learning” (Journal Reflection, December 3, 2017), she believed that they were more of a necessity for the classroom. When Harley described mathematical mistakes, they were signals that she did not know something. As she stated:

> When I already know something, I'm not learning anything about it, but when I'm wrong, my way of thinking about it completely changes and I have to learn a new thing or way to do ‘cause I was doing it wrong. (Current Error Climate Interview, December 7, 2017)

If it were not for mistakes, then she would not know if she needed to change what she was doing or as she described in one instance, “What would be the point [of going to class]?” (Journal Reflection, December 3, 2017). Mathematical mistakes were important for the classroom for both Cindy and Harley, although in slightly different ways. Cindy believed mathematical mistakes were opportunities for learning, and Harley believed they were signals to change what she was doing.

The role that mathematical mistakes served in the classroom was an area where Cindy’s and Harley’s beliefs differed. Cindy believed mathematical mistakes were an aspect of the classroom that was distinctly different than how correct answers were used. An instance that was representative of Cindy’s belief of the role of mistakes and correct answers in the classroom was after the class meeting where she worked in her group to review for the final exam. She stated:

> When [another student] made a mistake or got stuck, I had to figure out how and why she got stuck. Then, I have to figure out how to help without giving the

> answer. I feel I benefit from the whole process. I also get to learn the material on a different level. It is one thing to learn the material and apply it to my own problem but to be able to teach it or help someone through the problem I feel can help you learn the material on a more in-depth level. When you get the correct answer, I feel it's a quicker process. You give the answer, maybe how you did, and then move on. When you get it wrong, you discuss and find where it went wrong. (Post Classroom Observation Interview, November 29, 2017)

Mathematical mistakes were ways for Cindy to learn something, whereas correct answers were only an opportunity to share how you got it right only to move on to the next problem. In contrast, Harley believed mathematical mistakes were more of an opportunity for the teacher. Specifically, they were opportunities for teacher to take corrective action and “teach better” (Prior Error Climate Interview, September 17, 2017). She described how she would use mathematical mistakes in her future classroom after a class session that focused on incorrect ways of finding the perimeter of an irregular polygon (see *Figure 8*):

> I think this is important for her to teach us incorrect answers, because children might do problems in a way that are [*sic*] easier for them to understand. I think it will also help us to teach it better and teach our future students how to not do a problem and correct what they did, so they don’t get it wrong on tests. (October 18, 2017)

Cindy and Harley believed that mathematical mistakes served different roles in the classroom. Cindy believed that they were primarily opportunities for the student, and Harley believed that they were primarily opportunities for the teacher.

Finally, Cindy and Harley believed that mathematical mistakes needed to be fixed. Both of them believed that the process of fixing the mistake was where learning occurred. Cindy believed that finding where the mistake was and how to fix it was where the learning took place. This was best captured after the class meeting where she reviewed for the final exam with her group, and I asked her what she gained from conversations focused on mistakes. She stated, “When you get it wrong, you discuss and find where it went wrong. Then, you go over how to fix it and go over the right way or one way to answer the question correctly” (Post Classroom Observation Interview, November 29, 2017). Mathematical mistakes served as opportunities to fix the mistake and to figure out how to answer the question correctly for Cindy. Similarly, Harley believed that mathematical mistakes needed to be fixed in order to learn from them as well. However, Harley believed that mathematical mistakes needed to be fixed in specific ways because of class tests. She stated that her high school experiences with mathematical mistakes were treated as, “‘This is the right equation. Take it. Memorize it. There you go.’ Because of the testing” (Prior Error Climate Interview, September 17, 2017). Although she described this type of interaction in high school because of standardized testing, she viewed mathematical mistakes in college similarly:

> Even though high school didn't have discussions like that, mistakes were not treated differently. If someone said something wrong in high school or college, the teacher would still try to fix the answer and like get to the bottom of it. Figure out where they went wrong. So, I don't think it would be any different. Treated different. (Current Error Climate Interview, December 7, 2017)

They both believed that mathematical mistakes needed to be fixed for learning to happen. Cindy believed learning was in the process of getting the correct answer, but Harley believed learning was in fixing the mistake in a specific way because of class assessments.

Cindy's and Harley's beliefs were similar in many respects, especially in the belief that mathematical mistakes needed to be fixed in order to learn from them. However, there were small differences in why they should be fixed as well as why mathematical mistakes were important for the class and who the role of mathematical mistakes served in the classroom. One feature that is important to point out is the difference in how Cindy and Harley viewed the role of mistakes and who they served. Cindy believed that mathematical mistakes were instances for her to learn mathematical content. In contrast, Harley believed that their main role was to alert the teacher of a mistake. Table 11 provides a comparison of Cindy's and Harley's central beliefs concerning mathematical mistakes.

Table 11

Central Beliefs Comparison of Cindy and Harley

Dimension of Mathematical Mistake	Cindy’s Central Beliefs	Harley’s Central Beliefs
The Importance of Mathematical Mistakes	Important as they are opportunities to learn	Important as they identify where the error took place
The Role of Mistakes	Served the student as places to learn content	Served the teacher as places to correct
Necessity of Fixing Mistakes	Needed to fixed in order to learn how to do the problem correctly	Needed to be fixed in a specific way to prepare for the test

Belief Clusters Concerning Mathematical Mistakes

The previous section focused on the central beliefs of Cindy and Harley concerning mathematical mistakes, which were beliefs that were overarching in terms of the role that mathematical mistakes play in a mathematics classroom. This section will focus on the belief clusters for Cindy and Harley. The clustering of beliefs is useful for explaining contextual variables in which a belief may be held in one instance or context but not in another (Leatham, 2006). Cindy and Harley shared similarities and differences in contexts for when mathematical mistakes were important, knowledge gained from engaging with mathematical mistakes, and the impact of mathematical mistakes on affective qualities.

When mistakes are important. Cindy and Harley believed that mathematical mistakes were more important for the class in certain circumstances. First, both of them saw mathematical mistakes as being important for the class if many students in the class made the mistake. Cindy did not have to actually know or see the students making the mistake, but she did need to perceive that students made the mistake or could have made the mistake. For example, she stated in her first interview, “I’ve made mistakes in math classes before, but I’m sure that others had similar thought processes in getting the wrong answer because it’s been a while since we’ve had geometry. But, we learn from them” (Prior Error Climate Interview, September 12, 2017). Here, Cindy assumed that others made a mistake in a similar way and thus the class could learn from those. In another instance on a journal reflection where she was asked to think of herself as the teacher, she stated:

> Well, I would take that problem and work it out the wrong way, like we did in class one day. Then, as a class we work through why it is wrong. Because *if they made the mistake, then there are probably more that have or may not understand why they got the answer* [emphasis added]. For example, we are in groups in class, if I had my students in groups, sometimes you have one or two that lead the group or understand. So, it is easy for the others to just copy the answer or get nervous to ask questions. So, this is where we address it as a class then we can *cover the other students who were missed* [emphasis added]. (Journal Reflection, November 20, 2017)

Here again, Cindy perceived that there were others in the class that made the mistake in a similar way, and thus, the class needed to spend time thinking about that mistake.

Similarly, Harley believed that mathematical mistakes were important when the majority of the class made the mistake. As she stated in one of her journal reflections:

> I think a mistake that is worth exploring is based on the class. *If the majority of the class made the same kinds of mistakes on the same problem* [emphasis added], then I believe it is beneficial to go over it. If a teacher doesn't go over the mistake and correct it, then how will the students ever be able to correct their mistake in the future? So, I think a mistake is worth exploring or sharing if the majority of the class does not understand it. (Journal Reflection, October 28, 2017)

She did not have to perceive that others made the mistake, but rather, the majority of the class making the mistake was more of a condition for the mistake to be important.

Cindy and Harley also believed that mistakes were more important for the class if the mistake provided the opportunity for the class to engage beyond correcting the mistake. Cindy and Harley made very similar statements when referring to using mathematical mistakes to engage the class. Cindy believed, "It was important because others were struggling too. I wish we would have gone in more depth with it. We could have really talked about that more" (Journal Reflection, October 6, 2017). This mistake was important for Cindy because others made the same mistake with her, the class as a whole could discuss that mistake. Similarly, Harley believed a mathematical mistake was important for the class if they could discuss it. However, Harley also believed that the mistakes that were discussed in class needed to be "common error[s]" (Post Classroom Observation Interview, October 2, 2017) and there needed to be enough time in class to do this, as discussing mistakes takes significant time.

What is gained from working with mistakes. Cindy and Harley believed that working with mathematical mistakes allowed for them to gain knowledge. The knowledge that Cindy and Harley believed to be gained were similar overall. Both of them believed that mathematical mistakes allowed them to “dive deeper” (Cindy, Current Error Climate Interview, December 4, 2017) or to “go deeper into the problem” (Harley, Post Classroom Observation Interview, November 29, 2017). They also both believed that certain mathematical mistakes allowed for them to “make connections” (Harley, Post Classroom Observation Interview, October 23, 2017) among mathematical topics. Cindy stated it as, “[Mathematical mistakes allow] connections to be made that you might be able to use later even” (Post Classroom Observation Interview, November 18, 2017).

Additionally, Cindy and Harley both believed that minor mathematical mistakes provided opportunities to gain knowledge. Cindy believed these types of mathematical mistakes provided her the opportunity learn that she “needed to take it slower and double check my work” (Journal Reflection, September 25, 2017). In contrast, Harley described this process as “help[ing] remember . . . a formula” (Post Classroom Observation Interview, November 6, 2017) that she had forgotten. Furthermore, Harley believed that she gained knowledge for teaching from the mathematical mistakes that occurred in Dr. Heart’s class. A summative comment from Harley was after a classroom observation, where she stated:

> I believe these [mistakes] were important for us as future teachers to know the incorrect answers and correct answers of problems. I think this is important for her to teach us incorrect answers, because we have to be able to adapt to all the different ways that children might do problems in a way that are easier for them to understand. (Post Classroom Observation Interview, October 18, 2017)

The mistakes in the classroom allowed Harley to gain knowledge for teaching mathematics in her future classroom. This is not to say that Cindy did not ever express that she could use a moment in class to learn as a teacher. However, Cindy focused on learning more as a student, where Harley often focused her thoughts on learning as a teacher. The saliency of these beliefs were influenced by their central beliefs, as Cindy's central beliefs focused on learning mathematics from a student's perspective and Harley's on how the teacher could "be a better teacher" (Prior Error Climate Interview, September 17, 2017).

Influence of affective qualities on mistakes. Both participants believed that mathematical mistakes were influenced by affective qualities. Both Cindy and Harley believed that mathematical mistakes and the engagement that takes place after a mathematical mistake were mediated by affective qualities. Specifically, Cindy and Harley believed that mathematical mistakes could push them to the point of shutting down their engagement in a mathematical task or, for Harley, even the class. Both stated that mathematical mistakes could make them feel "dumb" (Cindy, Journal Reflection, October 26, 2017) and "embarrassed" (Harley, Journal Reflection, October 26, 2017). Although they both acknowledged that making mistakes was good for them, especially in a mathematics class, they believed that mathematical mistakes could cause them to not share more of their mathematical mistakes in the future.

Cindy's and Harley's clustering of beliefs shared similarities related to what is gained from working with mathematical mistakes as well as the role that affective qualities play in sharing mistakes with the class. However, there were differences in when mathematical mistakes were important and what each believed that they gained from mathematical mistakes. Table 12 provides an overview of each's belief clusters concerning mathematical mistakes.

Table 12

Comparison of Cindy's and Harley's Clusters of Beliefs

Cindy's Beliefs	Harley's Beliefs
When Mathematical Mistakes are Important	
Perceived that other students made or could have made the same mistake	Other students in the class made the same mistake
Opportunity was provided to engage further with the mistake: • Group collaboration • Whole-class discussions	If the class discussed the mistake: • Mistake needed to be a common mistake • Needed to be time in class for the discussion
What is Gained from a Mathematical Mistake	
Make connections to other mathematical content with major mistakes	Make connections to other mathematical content with major and minor mistakes
Slow down and double check work with minor mistakes	Remember something forgotten with minor mistakes
	Knowledge for teaching mathematics
Mathematical Mistakes Influence on Affective Qualities	
Affective qualities mediate if mathematical mistakes are shared with the class	Affective qualities mediate if mathematical mistakes are shared with the class

Changes in Beliefs Systems Concerning Mathematical Mistakes

Before comparing Cindy's and Harley's changes in beliefs systems concerning mathematical mistakes, the perceived error climates for both Cindy and Harley will be reviewed. Cindy and Harley's perceived error climate averages are provided in Table 13. Cindy and Harley's perceived error climates of their prior classrooms were similar based on the results from their surveys. However, Harley perceived her prior classrooms as negative in the absence of negative reactions to errors dimension, whereas Cindy perceived her previous classrooms as positive in that dimension. Both also perceived Dr. Heart's class as a more positive error climate in all dimensions. The only exceptions were in the error tolerance by teacher for Cindy and the teacher support following errors dimension for Harley. Most notably for Cindy and Harley were the differences in the prior error climate average and the current error climate average for the error-taking risk, functionality for errors dimensions, and absence of negative reactions to errors dimensions. Cindy and Harley both perceived Dr. Heart's class similarly in those dimensions.

Table 13

Cindy's and Harley's Perceived Error Climate Averages

	Cindy		Harley	
Error Climate Dimension	Prior Error Climate Average	Current Error Climate Average	Prior Error Climate Average	Current Error Climate Average
Error Tolerance by Teacher	$M = 2$	$M = 2.25$	$M = 3.5$	$M = 2.25$
Irrelevance of Errors for Assessment	$M = 3.5$	$M = 2.5$	$M = 2.5$	$M = 2$
Teacher Support Following Errors	$M = 1.85$	$M = 1$	$M = 2$	$M = 2.25$
Absence of Negative Teacher Reactions to Errors	$M = 3$	$M = 1$	$M = 3.5$	$M = 2$
Absence of Negative Classroom Reactions to Errors	$M = 2.5$	$M = 1$	$M = 4$	$M = 3$
Error-taking Risk	$M = 5.5$	$M = 3.5$	$M = 6$	$M = 3$
Analysis of Errors	$M = 2.67$	$M = 1.33$	$M = 2.67$	$M = 2.67$
Functionality of Errors for Learning	$M = 2.25$	$M = 1$	$M = 3.75$	$M = 1.75$
Superordinate Scores	$M = 2.91$	$M = 1.70$	$M = 3.49$	$M = 2.37$

Note. Scores on the survey ranged from 1 (strongly agree) to 6 (strongly disagree), with averages of 3 and below indicating a positive error climate and averages of above 4 indicating a negative error climate. A lower average indicates a more positive error climate.

Harley and Cindy made similar changes in their beliefs systems related to what is learned from a mathematical mistake. Both of them made small changes in that they could learn more from a mathematical mistake other than how to get the correct answer. Although getting the correct answer was still part of their central beliefs, they both experienced similar changes in that they expanded what could be learned from mathematical mistakes. Cindy believed that mathematical mistakes offered the opportunity to learn different strategies and opportunities to critique others' work. In contrast, Harley believed that mathematical mistakes afforded her the opportunity to engage with others in a conversation about mathematics as well as acknowledge the different roles that major and minor mistakes could serve. A mathematical mistake was no longer just a mistake that needed to be fixed to only move on to the next problem. Examining mistakes provided Cindy and Harley to learn more than just mathematical content.

The other changes that occurred in their beliefs system concerning mathematical mistakes were different for Cindy and Harley. Cindy believed that mathematical mistakes could have a smaller impact on affective qualities given the right circumstances and even stated that she would take this belief into her other classes (Current Error Climate Interview, December 4, 2017). However, Harley did not make changes in the impact of mathematical mistakes on affective qualities, but she changed her belief of the role that mathematical mistakes served in the classroom for the teacher. They were no longer signals for a teacher to correct the student necessarily. Mathematical mistakes could be used to learn mathematics without them being fixed.

Summary

Cindy's and Harley's changes in their beliefs systems concerning mathematical mistakes were related to the differences in their perceived error climates from their prior classrooms and Dr. Heart's. Cindy's changes in her beliefs system were related to the impact of mistakes on affective qualities and the knowledge that could be gained from mathematical mistakes (see Figure 19), which were related to the absence of negative reactions, error-taking risk, and functionality of errors for learning dimensions (see Table 13). Comparatively, the changes in Harley's beliefs systems were related to role of mathematical mistakes in the classroom and the knowledge that could be gained from a mathematical mistake (see Figure 25), which were related to the error-taking risk and functionality of errors for learning dimensions (see Table 13).

Chapter Summary

This chapter detailed the cases of Cindy and Harley, including the beliefs systems of each participant as well as the changes in those beliefs systems. Additionally, a cross-case analysis of the two cases was included. The next chapter summarizes the study as well as provides a discussion of the results and recommendations for future research.

CHAPTER V: SUMMARY AND DISCUSSION

Introduction

Mathematics education reform documents (e.g., CCSSI, 2010; NCTM 2000, 2014) suggest ways of teaching that drastically differ from traditional methods of teaching. These documents include teaching strategies in which teachers leverage and explore students' mathematical mistakes, rather than minimize the mistakes' exposure to seek out correct answers. Research shows that this view of teaching contrasts with the teaching in typical U.S. classrooms (Bahr et al., 2014; Barlow & Reddish, 2006; Philipp, 2007). Significant to this study, these typical classrooms played a substantial role in shaping prospective teachers' (PTs) beliefs of mathematics and the role of mathematical mistakes in the classroom (Cooney et al., 1998; Hart, 2002, 2004; Thompson, 1992). Although literature suggests that in-service teachers' beliefs concerning mathematical mistakes are not aligned with standards and research (e.g., Matteucci et al., 2015; Santagata, 2005; Schleppenbach, Flevares, Sims, & Perry, 2007), there is limited available literature on PTs' beliefs concerning mathematical mistakes.

Therefore, the purpose of this study was to investigate PTs' beliefs concerning mathematical mistakes while considering the error climate of the classroom, that is, the way that mistakes were perceived in the classroom from the perspective of the PTs. As an aid to the reader the final chapter will contain a restatement of the research problem, a review of the methodology utilized in the study, and a summary of the results of the study. This review will be followed by a discussion of the results of the study, which will include its connections to prior research, theoretical and practical implications, and recommendations for future research.

The Research Problem

Mathematical mistakes play a crucial role in learning within the environment described by reform documents (see CCSSI, 2010; NCTM 2000, 2014). Additionally, these documents call for teachers to teach in drastically different ways compared to the classroom experiences they themselves had in their primary and secondary years. These experiences shape PTs' beliefs and eventually the way that they will teach in the classroom (Cooney et al., 1998; Hart, 2002, 2004; Thompson, 1992). Understanding those beliefs and how they change is a crucial step in changing how a teacher will teach and use mistakes in the classroom (Jensen, 1993; Thompson, 1992). As Thompson (1992) stated, "It is not until we have a clearer picture of how teachers modify and reorganize their beliefs in the presence of classroom demands and problems . . . that we can claim to understand the relationship between beliefs and practice" (p. 135). Empirical research is needed to examine the beliefs that PTs hold concerning mathematical mistakes and how those beliefs change, if at all, during teacher preparation. Additionally, calls from researchers have recommended the examination of teachers' beliefs and practice through integrating situational, biographical, and historical information (Britzman, 1991; Hargreaves, 1994). A case study design, reviewed in the next section, was developed to address these concerns.

Review of Methodology

An exploratory multiple case study (Yin, 2014) was used to capture the beliefs of PTs concerning mathematical mistakes and how those beliefs change, if at all, while enrolled in a teacher preparation course focused on elementary geometric content–the second course in which the elementary PTs considered the mathematics they will be teaching at deeper levels than in their previous experiences. Cindy and Harley, were the PTs on which the study focused. A multitude of data sources, including an implicit theory survey, classroom observations focused on enacted implicit theory descriptors, error climate surveys, classroom observations focused on the PTs' engagement with mathematical mistakes, interviews following those classroom observations, journal reflections, in-class reflections, and exit tickets, were utilized to collect data from the two participants. The data was used to develop rich descriptions of Cindy's and Harley's prior and current perceived mathematical experiences from which a conceptualization of their beliefs systems concerning mathematical mistakes and how those beliefs systems changed, if at all, were developed. The data collected were organized and analyzed in a chronological fashion. Themes were generated through this analysis that represented Cindy's and Harley's beliefs systems concerning mathematical mistakes. A description of these themes produced the full results of the study, which are summarized in the next section.

Summary of Results

The results in Chapter IV presented the case of Cindy and Harley. Each case was presented separately and in a cross-case analysis. The overall findings for each of area of results will be presented briefly in the next sections.

Implicit Theory

Cindy and Harley were similar in their implicit theories. Their Implicit Theory Survey results indicated that Cindy ascribed to a strong incremental theory while Harley was in the middle of ascribing to an entity and incremental theory (see Table 10). However, during classroom observations that took place between August 28, 2017, and

September 8, 2017, that focused on the participants' enacted descriptors of their implicit theory, I observed both participants enacting entity and incremental theorists' descriptors, particularly when evaluating the situations in class, dealing with setbacks, and challenges (see Appendix I). Thus, it was determined that neither participant was on either extreme of the implicit theory continuum and ultimately selected to participate further in the study.

Prior Mathematical Experiences

Cindy and Harley perceived their mathematical ability differently. Cindy saw herself as a good mathematics student in her past classrooms. She described herself as being one of the students on which teachers would depend for answers and was always in honor classes. Harley perceived herself as always being bad at mathematics. She described herself as always having to rely on tutors for her to get through her mathematics classes and commonly answered questions incorrectly.

However, Cindy and Harley perceived their prior mathematics classrooms in similar ways. Both participants described classrooms in their past as not taking into account what they were thinking, especially related to mathematical mistakes. Both participants described classrooms that were lecture-heavy and rarely, if ever, took class time to talk about their mathematical mistakes. Cindy and Harley both described mathematics classrooms that focused on the correct answer and showed mathematical mistakes to the class to only fix them. This was especially important for Harley as she perceived mathematical mistakes being fixed in very specific ways because of standardized testing. However, Cindy described some differences in her mathematics classrooms from the past. Where Harley's perception of her classrooms were teacher-centered and lecture heavy, Cindy described instances in which she worked with classmates in groups and sometimes was at the front of the class discussing work with other classmates. However, she also described these instances as focusing on correct strategies and the correct answer. Although there were nuanced differences in Cindy's and Harley's perception of their prior mathematics experiences, they perceived the treatment of mathematical mistakes similarly.

Central Beliefs Concerning Mathematical Mistakes

The central beliefs concerning mathematical mistakes of each participant were presented as well. These central beliefs encompassed and influenced all of their other beliefs concerning mathematical mistakes. These central beliefs concerning mathematical mistakes were beliefs around which all of Cindy's and Harley's other beliefs concerning mathematical mistakes were clustered and were "overarching beliefs about the physical, social, and pedagogical" (Leatham, 2007, p. 192) inclusion of mathematical mistakes in the classroom. Additionally, these were the beliefs that were the most coherent with the rest of Cindy's and Harley's beliefs system concerning mathematical mistakes. Cindy's central beliefs concerning mathematical mistakes focused on the role of mistakes for herself as a student. Her central beliefs included the importance of mistakes in the mathematics classroom, the different role that mathematical mistakes had compared to correct answers or strategies, and that mathematical mistakes needed to be fixed in order to learn from them (see Figure 17). These three central beliefs can be viewed as Cindy's stance of how mathematical mistakes should be used in teaching and learning mathematics, and these central beliefs influenced her other beliefs concerning mathematical mistakes. Cindy believed that mathematical mistakes served a significant purpose in the classroom, because “that’s where you learn” (Prior Error Climate Interview, September 12, 2017). However, to learn from those mistakes, you have to “fix them and make them better” (Prior Error Climate Interview, September 12, 2017). This cannot be done with correct answers, although getting the answers correct was always the goal. Her prior experiences helped shape these central beliefs (see Pajares, 1992; Philipp, 2007).

Harley’s central beliefs concerning mathematical mistakes focused mainly on the role that mistakes served for the teacher. Her central beliefs included that mathematical mistakes were necessary for a mathematics class, those mistakes had to be fixed to prepare for the exam, and that the teacher used mathematical mistakes to teach better (see Figure 23). Additionally, when the teacher observed a student’s mistake, Harley believed that this was an opportunity for the teacher to teach better and to correct the mistake. All of these efforts were in service of preparing for exams. In her secondary classrooms, the preparation was for the standardized exam, but in college, it was for the upcoming class exam.

Belief Clusters Concerning Mathematical Mistakes

The belief clusters concerning mathematical mistakes were useful for explaining contextual variables in which a belief may be held in one instance or context but not in another (Leatham, 2006). The clusters of beliefs that emerged from this study for Cindy were when mathematical mistakes were important, when opportunities to gain knowledge from mistakes were present, and the influence of mistakes on affective qualities. Cindy believed mathematical mistakes were important for the class when she perceived that there were others that made a mathematical mistake and when others would have a need to talk about or engage with the mistake in some way. Additionally, Cindy believed that by engaging with a mathematical mistake she could learn to either slow down and double check her work or learn mathematical connections to related topics. However, Cindy also believed that sharing these important mistakes or mistakes with which others could engage was mediated by the impact of the mistakes on affective qualities. Cindy believed that if she felt that sharing the mistake would cause her too much embarrassment, then she would not be willing to share it even in spite of the learning that could take place from it.

In contrast, Harley believed that mathematical mistakes were important when others made the same mistake and when mathematical mistakes spurred discussions in the class. She also believed that she could learn mathematical knowledge and knowledge for teaching mathematics from fixing the mistake. Additionally, there were certain mistakes that she believed should be shared with the class. However, there were three factors that she believed determined if the mistake should be discussed publicly: if there was enough time, if it was a conceptual mistake, and if there were enough people that made the mistake. Finally, Harley believed that the impact of mathematical mistakes on affective qualities mediated if she would share a mathematical mistake with the teacher or class.

Noticeable Changes in Beliefs System

Cindy and Harley both perceived Dr. Heart's class as a more positive error climate than their prior mathematics classrooms (see Table 13), and the changes in their beliefs systems were reflective of the differences in certain error climate dimensions. By the end of the study, Cindy believed that mathematical mistakes offered the opportunity to learn different strategies and opportunities to examine other students' work in addition to learning to slow down to double check her work and make connections to other mathematical topics. Additionally, she believed that the level that affective qualities mediated her sharing mistakes with the teacher or class could be diminished if she was "comfortable" (Current Error Climate Interview, December 7, 2017). She explained her being comfortable as a classroom environment where mathematical mistakes were made regularly and the teacher was welcoming of students' mathematical mistakes (see Figure 19). By the end of the study, Harley believed that the teacher used mathematical mistakes for more than just fixing the mistake. Although fixing a mistake was a pervasive belief, Harley also believed that mathematical mistakes could be used to learn knowledge for mathematics by engaging in an activity with a mistake and not by just fixing it. Connected to using mathematical mistakes for more than fixing the mistake, Harley believed that she learned more avoiding incorrect solution strategies (see Figure 25).

Discussion of Results

The results of the current study are significant in four ways. First, the results connect to prior research by providing evidence that further supports beliefs as a sensible system (Leatham, 2006) and the influence of the perceived error climate (Steuer et al., 2013) from which the conceptual framework of this study was derived. Additionally, the results of this study add to literature concerning PTs' identity as students and as teachers. Second, the results offer theoretical implications as they extend the perceived error climate as part of the mechanism through which PTs' beliefs can change. Third, this study offers suggestions for practice for teacher educators and designers of teacher education courses regarding the use of mathematical mistakes. Finally, the results of the study recommend questions and considerations for future research to extend the findings of this study. The remaining sections of this chapter present a discussion for each of these.

Connections to Prior Research

The results of this study connect to prior research in five ways. First, they provide further evidence to support the use beliefs as a sensible system (Leatham, 2006). Second, the results extend the use of the perceived error climate (Steuer et al., 2013) into a qualitative study which assists in understanding the influences of the perceived error climate on other facets, particularly beliefs in this study. Additionally, they offer a rich description of change in PTs' beliefs during a teacher preparation course. Fourth, the results support findings of previous research concerning the impact of a positive classroom error climate. Fifth and finally, they offer insight into how PTs' identities influence their beliefs concerning mathematical mistakes. This section details the connections of each of these.

Support for beliefs as a sensible system. The descriptions of the cases of Cindy and Harley presented in Chapter IV provided two PTs starting the practice of teaching with well-established beliefs systems in place and in a class that focused on the mathematics that they were going to teach. Additionally, these two PTs were studied during a time in which they experienced drastically different teaching methods compared to their prior mathematics classrooms. Throughout the study, Cindy and Harley were forced to reconcile previously conceived notions of what teaching mathematics, and geometry specifically, looked like and what they were being presented with in Dr. Heart's class. The reconciliation process was not instantaneous and required an extensive network to conceptualize established beliefs and how those beliefs changed. The beliefs as a sensible system framework (Leatham, 2006) provided a construct with which to organize Cindy's and Harley's beliefs systems concerning mathematical mistakes.

The models of Cindy's and Harley's beliefs systems concerning mathematical mistakes portrayed how mathematical mistakes were filtered through each participant's beliefs systems. Cindy's and Harley's central beliefs, presented in Figure 17 and Figure 23, respectively, portrayed the beliefs that have the most psychological strength and are "overarching beliefs about the physical, social, and pedagogical" (Leatham, 2007, p. 192) use of mathematical mistakes. These were beliefs that are the hardest to change and also the beliefs by which all mathematical mistakes were mediated. For example, both Cindy and Harley believed that mathematical mistakes needed to be fixed in order for learning to occur. All other beliefs were secondary to mathematical mistakes needing to be fixed for Cindy and Harley.

The beliefs as a sensible system framework also provided structure to how other beliefs were organized, particularly in belief clusters. The beliefs as a sensible system framework provided belief clusters as a useful structure for explaining contextual variables in which a belief may be held in one instance or context but not in another. Additionally, the belief clusters were useful for explaining why an observer may find contradictions in beliefs, especially in teachers' beliefs related to practice. For example, Cindy and Harley both believed that mathematical mistakes were important for the mathematics classroom, and they both believed that there were times that their mathematical mistakes should be shared with the class and potentially discussed. However, their belief that mathematical mistakes impacted affective qualities mediated when they shared their mistakes. That is, if the mistake provided too much potential impact on their affective quality, then they would deem others' mathematical mistakes were more important.

Although Cindy's and Harley's beliefs were well supported by the use of the beliefs as a sensible system framework, some results from the study were difficult to describe using the framework. For example, describing changes that occurred in Cindy's and Harley's beliefs systems concerning mathematical mistakes was difficult considering the inclusion of the clustering of beliefs. Specifically, preliminary data analysis indicated, initially, that there might be a changes in their beliefs system (e.g., conceptual mistakes should be the only mistakes shared with the class). However, considering that the clustering of beliefs allows for contradictions with contextual variables, it was impossible to claim additional changes in their beliefs systems concerning mathematical mistakes due to the possibility that I did not have access to data revealing other belief clusters.

Support for the perceived error climate. The framework with which to capture Cindy's and Harley's perception of Dr. Heart's classroom and the treatment of mathematical mistakes within her classroom was a difficult choice. However, considering that PTs' beliefs are influenced by past and current experiences, this study needed a way to account for Cindy's and Harley's past mathematics classrooms' treatment of mathematical mistakes in addition to Dr. Heart's. Steuer et al.'s (2013) perceived error climate offered a framework that did that. Additionally, the perceived error climate framework accounted for other metacognitive activities that follow mathematical mistakes, such as motivation, implicit theory, attitude, the conception of the mistake, affect, and the teacher's attitude and beliefs concerning mistakes (Grassinger & Dresel, 2015).

Furthermore, a comparison of the differences in Cindy's and Harley's prior and current error climate survey results with changes in their beliefs systems concerning mathematical mistakes aligned quite well. For example, Cindy and Harley both perceived Dr. Heart's classroom to have a more positive error climate in terms of the functionality for errors dimension than in their previous classrooms. As their data were analyzed, a change in the belief of how mathematical mistakes were used in the classroom was revealed.

However, the perceived error climate framework had its limitations. For example, without having Cindy and Harley to talk aloud as they made their choices on the survey, nuances in why choices were made would be overlooked and assumptions would be made concerning their perception of the error climates. Admittedly, Steuer et al. (2013) designed the framework to predict learning outcomes. However, the cumulative nature of the average scores for each dimension should be used with caution when using the averages alone to interpret lived experiences.

Impact of positive error climate. Research using the classroom error climate as a framework to understand students' reactions to mathematical mistakes demonstrated that a classroom with a perceived positive error climate tended to show *affective adaptivity* and fostered affective reactions to mathematical mistakes (e.g., Grassinger & Dresel, 2017; Steuer & Dresel, 2015; Steuer et al., 2013; Tulis et al., 2016). Additionally, the results of this study supported the findings of previous research. Both Cindy and Harley perceived Dr. Heart's class as having a positive error climate in all but one dimension. The only dimension not perceived as a positive error climate was the Error-taking Risk dimension for Cindy (*M* = 3.5). The changes in Cindy's and Harley's beliefs systems concerning mathematical mistakes aligned with the notable differences between their prior classroom's perceived error climate and their perceived error climate in Dr. Heart's class in two important ways.

First, Cindy's beliefs concerning the impact of mathematical mistakes on affective qualities was diminished by the end of the study as long as the classroom met certain conditions, which Dr. Heart's class did based on Cindy's results on the of the classroom error climate survey. This finding was supported by Steuer et al. (2013) and Tulis et al. (2016) in that Cindy demonstrated affective adaptivity in a positive classroom error climate. Although there was not enough data to support similar findings for Harley, data suggested that Harley was aware of the positive error climate and its impact on her affective qualities.

Second, the results of this study supported the findings from Tulis et al.'s (2016) process model for reactions and learning from mistakes. In Tulis et al.'s (2016) model, the classroom error climate was conceptualized as the mediator for learners' direct reactions to mistakes, the learning processes that occurred as a result of the error, and ultimately the learning from mistakes. As Cindy and Harley both perceived Dr. Heart's class as having a positive error climate, they were both dispositioned to learn from mathematical mistakes (Tulis et al., 2016). Specifically, as detailed in the changes in the PTs' beliefs systems, Cindy and Harley experienced changes in what they learned from mathematical mistakes. Without the participants' perception of the classroom error climate as positive, these changes in their beliefs systems may not have occurred.

Teacher education and identity. Research on identity states that PTs' views of what it means to be a teacher is a culmination of previous experiences as a student and classroom interactions (Lortie, 1975). These *apprenticeships of practice*, as Lortie (1975) described them, is what PTs bring with them to their teacher preparation courses. Additionally, PTs' views of what teaching is and consists of are in constant adjustment for coherence during new experiences, especially during experiences dissonant with current views (Gee, 2001; Ketelaar, Beijaard, Boshuizen, & Den Brok, 2012). Dr. Heart's class was a classroom in contrast with Cindy's and Harley's past mathematics experiences (Bahr et al., 2014). Dr. Heart's class modeled the "consequences of different choices" (CBSM, 2010, p. 2), including mathematical mistakes, and the spirit of reform documents.

This study took place in a content course for elementary teachers and offered several opportunities for Cindy and Harley to position themselves as teachers as compared to students, a difficult change to make (Gormally, 2016). Although the focus of this study was not on PTs and their development of their teacher identity, the ways in which Cindy and Harley positioned the importance and role of mathematical mistakes in the classroom as a student and teacher, respectively, emerged as an important factor. Cindy positioned herself as the student in most situations, even when prompted to consider an elementary student's work. This suggested that Cindy still saw herself as a student of mathematics and had not started to develop a sense of her teacher identity. In contrast, Harley believed that mathematical mistakes mostly served the teacher, even when positioned as a learner of mathematics content. This suggested that Harley had already started to develop her sense of teacher identity. In both cases, Cindy's and Harley's identities clearly impacted their beliefs concerning mathematical mistakes.

Cindy's not yet having a developed sense of her teacher identity impacted her beliefs concerning mathematical mistakes as they focused on the utility of a mistake for herself as a student. For example, she believed sharing a mathematical mistake with the class offered her the opportunity to either learn to slow down or make connections with other mathematical content. In contrast, Harley had already developed a sense of her teacher identity, and her beliefs concerning mathematical mistakes reflected this. She believed that mistakes being shared with the class provided her the opportunity to learn mathematical knowledge for teaching in addition to supplementing her own mathematical knowledge.

In both cases, however, the content course in which this study took place provided impactful experiences for both Cindy and Harley. Additionally, the class potentially impacted their conception of what the practice of teaching looks like and how mathematical mistakes are handled (Wilson & Cooney, 2002; Winter et al., 2001), even though Cindy had not yet developed a sense of her teacher identity. Furthermore, for both Cindy and Harley, Dr. Heart's class contributed to their *apprenticeship of observation* (Lortie, 1975), which will potentially impact how Cindy and Harley view the role of mathematical mistakes in their future classroom.

Theoretical Implications

The results of this study present at least one important theoretical implication. Particularly, this study's results extend the use of the perceived error climate into the mechanisms for which PTs' beliefs can change. This section discusses that theoretical implication.

As previously stated, the beliefs as a sensible system and perceived error climate frameworks proved to be useful in investigating the beliefs concerning mathematical mistakes of Cindy and Harley. One of the most important results from the study was how the participants' beliefs changed while considering the participants' perception of the error climate for Dr. Heart's class. The alignment of the changes that occurred in Cindy's and Harley's beliefs systems concerning mathematical mistakes with the differences in their perceived prior error climate and their current perceived error climate are important to notice when considering change in beliefs concerning mathematical mistakes. Figure 2 provides a visualization of the process of beliefs changing. In Figure 2, question marks appear around the mechanism of belief change as there is a wide spectrum of factors that research shows impact change. This study adds the perceived error climate as part of that mechanism for belief change and situates the new experiences and beliefs of others and the new beliefs, if adopted, in the context in which those experiences occurred and the new beliefs were formed.

Suggestions for Practice

The results of this study offer practical suggestions to inform teacher educators and designers of teacher preparation courses. These suggestions revolve around creating a classroom environment that challenges PTs' beliefs in a way that requires them to restructure their beliefs system in a more productive way and align with reform documents. This section will discuss the application of this study's results.

Considerations for teacher educators. The results of this study offer insight into PTs' beliefs concerning mathematical mistakes, and more globally, this study's results offer insights into the beliefs that PTs bring to teacher preparation classrooms. For teacher educators, understanding the beliefs that their prospective teachers bring with them to teacher preparation classrooms is pivotal if teacher preparation courses are to "play [the] crucial role" (AMTE, 2017, p. 54) in developing PTs' beliefs. Furthermore, understanding PTs' beliefs is the first step in knowing what experiences teacher educators need to provide in an effort to nurture desired beliefs.

Cindy's and Harley's beliefs concerning mathematical mistakes aligned with the way reform documents encourage mistakes to be used. For example, reform documents and research suggests that there should be discussions in class that focus on students' mistakes (AMTE, 2017; CCSSI, 2010; NCTM, 2014). Cindy believed that mistakes were important for the class if the mistakes engaged others in the class. Class discussions would fit this criteria. Harley also believed that mistakes that were discussed in class were important. However, both Cindy's and Harley's belief of sharing these types of mistakes were mediated by the potential impact that the mistakes had on their affective qualities. Additionally, reform documents suggest that mathematical mistakes are a natural part of learning and serve as opportunities to learn (AMTE, 2017; NCTM, 2014; Kazemi & Stipek, 2001). Again, Cindy and Harley believed that mathematical mistakes were crucial for the classroom and essentially where mathematical learning took place. Again, these beliefs were mediated by the impact that the mistakes could have on their affective qualities. Another recommendation from reform documents concerning mathematical mistakes is that "well-prepared" (AMTE, 2017, p. 82) teachers should be aware of common mistakes that students will make. Cindy’s and Harley’s beliefs concerning mathematical mistakes were well-aligned in this area, especially by the end of the study.

However, other beliefs were not aligned with reform documents’ recommendations. For example, both Cindy and Harley believed that mathematical mistakes needed to be fixed in order for learning to occur. This was especially important for Harley as she believed that mathematical mistakes needed to be fixed in specific ways to prepare for exams. These beliefs lend themselves to correcting mathematical mistakes and demonstrating the correct approaches and overlooking students’ thinking, similarly to Cindy’s and Harley’s secondary classroom experiences.

Considerations for designing teacher preparation courses. For designers of teacher preparation courses, this study’s results offer considerations for the treatment of mathematical mistakes. Previous research stated that reoccurring experiences are required to nurture desired belief structures (Aston & Hyle, 1997; Hart, 2004) as well as extensive reflection on those experiences (Maher et al., 2013; Hart, 2004). The course in which this study took place repeatedly utilized mathematical mistakes in ways aligned with reform documents. Specifically, the PTs were regularly engaged with mathematical tasks and homework focused on incorrect solution strategies. Additionally, the PTs were held accountable for continuing to think about mathematical mistakes as the PTs were aware that novel mathematical mistakes were on the class exams. Furthermore, the two participants in this study were engaged in reflecting on mistakes from the class by participating in the study.

For teacher educators, two aspects of the study are useful in designing teacher preparation courses. The first of which is understanding the impact that mathematical mistakes can have on PTs' affective qualities (see Grassinger & Dresel, 2015; Tulis et al., 2016). In this study, the impact of mathematical mistakes on affective qualities mediated the PTs' beliefs concerning the utility of mistakes in the classroom. Additionally, this study's results demonstrated that a positive error climate diminished mathematical mistakes impact on affective qualities for Cindy. To assist in the design of teacher preparation courses that reduce the impact of mathematical mistakes on affective qualities, teacher educators should consider the importance of the positive error climate when designing tasks and structuring the class. However, creating a positive error climate is not sufficient for aligning PTs' beliefs with reform documents' recommendations as PTs will be left to reconcile positive error climate experiences with their beliefs systems.

This leads to the second aspect from the study that is useful for teacher educators in designing teacher preparation courses. Prior research stated that opportunities for changing PTs' beliefs (Hart, 2004; Maher et al., 2013) and developing PTs' teacher identity (Coldron & Smith, 1999; Gormally, 2016; Ketelaar et al., 2012) occur with repeated exposure to experiences and reflection. The PTs in this study had additional opportunities to reflect on the experiences in Dr. Heart's class by participating in this study. Explicit attention and opportunities for reflection on experiences, especially those believed to be in dissonance with PTs' prior experiences, should be given to focus PTs' reflections on those experiences in an effort to make small changes to align their beliefs with productive beliefs concerning mathematical mistakes (see NCTM, 2014).

The previous section discussed the considerations for practice. This included considerations for teacher educators and designers of teacher preparation courses. The next section will discuss recommendations for future research.

Recommendations for Future Research

This multiple case study investigated the beliefs concerning mathematical mistakes of two PTs and how those beliefs changed during the course of the study. Though the results revealed similarities in Cindy’s and Harley’s beliefs systems, including changes in those beliefs systems while considering the perceived error climate of the class in which they were enrolled, more research is needed in this area. There is a limited research base examining the beliefs of PTs concerning mathematical mistakes, and this study only provided the results from two PTs from one class. Thus, there are five directions for future research that can be considered based on the results of this study.

First, this study collected data on only two PTs in one content course for prospective teachers. With the limited research available on PTs’ beliefs concerning mathematical mistakes, more research was needed to gain a holistic understanding of the beliefs that PTs have as they enter into their teacher preparation programs. Additionally, more research examining the change in PTs’ beliefs concerning mathematical mistakes is needed to gain a deeper understanding of the factors that influence the changes in beliefs systems.

Second, the results of this study demonstrated the effects of a perceived positive error climate on PTs’ beliefs concerning mathematical mistakes. Future studies should investigate the instructional activities and practices that foster a positive error climate. This would not only add to the findings of this study but also studies investigating classroom error climate (e.g., Grassinger & Dresel, 2017; Steuer et al., 2013; Tulis et al., 2016).

Third, this study took place over the course of one semester and for approximately three months. This limited time span for which to collect data from the PTs limited understanding the changes in the beliefs systems that will endure. As beliefs are difficult to change (Hart, 2002; Pajares, 1992; Philipp, 2007), it is problematic to state with certainty that the changes that occurred in Cindy's and Harley's beliefs systems will endure the rest of their teacher preparation program and, ultimately, impact their instructional choices as practicing teachers. Longitudinal studies investigating the change in PTs' beliefs over the course of their teacher preparation program would expand this study's limited scope.

Fourth, the results of this study found that the PTs' sense of their teacher identities, or lack thereof, influenced their beliefs concerning mathematical mistakes and the changes in those beliefs. Future studies should account for the PTs' sense of their teacher identities when investigating beliefs concerning mathematical mistakes. The importance of the PTs' teacher identities arose during data analysis in this study. Future studies accounting for the participants' sense of teacher identity at the onset of the study would be able to investigate the extent to which the PTs' teacher identify influences the changes in beliefs systems concerning mathematical mistakes.

Fifth and finally, the two participants from this study were selected purposefully because their implicit theories were on neither extreme of the implicit theory continuum. This was done so, in part, for data collection purposes and resulted in two participants with similar implicit theories. Subsequently, to some degree, the results from the study indicated that the participants experienced some similarities in their beliefs and changes in beliefs systems concerning mathematical mistakes. For example, both participants believed that mathematical mistakes were mediated by their impact on affective qualities, and both participants believed that by fixing a mathematical mistake they learned connections in mathematical content. However, this might not be the case with PTs that are an extreme entity or incremental theorist. Future studies designed similar to this one, but selecting PTs on the extremes of the continuum, would add to the literature base of PTs' beliefs concerning mathematical mistakes and thus teacher preparation programs.

Chapter Summary

Prospective teachers' beliefs concerning mathematical mistakes are comprised from a breadth of experiences from prior classrooms. These systems of beliefs comprise a complex relationship of what type of mistakes are important, when they decide to share their mistakes, what can be learned from mistakes, and the teacher's role in how they engage with mathematical mistakes. This study examined the beliefs of two PTs and how those beliefs changed while enrolled in a teacher preparation content course that utilized mathematical mistakes in ways aligned with mathematics reform documents recommendations. Additionally, this course strived to produce a positive error climate, which both participants perceived as such. A rich description of the beliefs concerning mathematical mistakes of each participants were provided as well as how those beliefs changed during the course of the study.

This chapter discussed the background of the study, its methodology, and its results. Additionally, connections to prior research were provided, and the results supported the theoretical framework. Furthermore, theoretical and practical implications were discussed, including considerations for teacher educators in preparing future teacher education courses. Recommendations for future studies were provided which included investigating further implications of the classroom error climate.

REFERENCES

Anderson, R. (2007). Being a mathematics learner: Four faces of identity. *The Mathematics Educator*, *17*, 7-14.

Anderson, A., Hamilton, R., & Hattie, J. (2004). Classroom climate and motivated behavior in secondary schools. *Learning Environments Research*, *7*, 211–225.

Aronson, J., Fried, C. B., & Good, C. (2002). Reducing the effects of stereotype threat on African American college students by shaping theories of intelligence. *Journal of Experimental Social Psychology, 38*, 113-125.

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Retrieved from www.amte.net/standards

Aston, M., & Hyle, A. (1997, March). *Social networks, teacher beliefs and educational change*. Paper presented at the annual meeting of the University Council of Educational Administration, Orlando, FL.

Ayers, M. S., & Reder, L. M. (1998). A theoretical review of the misinformation effect. Predictions from an activation-based memory model. *Psychonomic Bulletin and Review, 5*, 1–21.

Bahr, D. L., Monroe, E. E., & Eggett, D. (2014). Structural and conceptual interweaving of mathematics methods coursework and field practica. *Journal of Mathematics Teacher Education, 17*, 271-297.

Barlow, A. T., & Reddish, J. M. (2006). Mathematical myths: Teacher candidates' beliefs and the implications for teacher educators. *The Teacher Educator*, *41*, 145–157.

Beckmann, S. (2018). *Mathematics for elementary teachers with activities* (5th ed.). New York, NY: Pearson Education, Inc.

Blackwell, L. S., Trzesniewski, K. H., & Dweck, C. S. (2007). Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development*, *78*, 246-263.

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning (*revised, expanded edition). Mahwah, NJ: Erlbaum.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey Bass.

Borasi, R. (1994). Capitalizing on errors as 'springboards for inquiry': A teaching experiment. *Journal for Research in Mathematics Education*, *25*, 166–208.

Bray, W. S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education*, *42*, 2–38.

Bray, W. S., & Santagata, R. (2014). Making mathematical errors springboards for learning. In K. S. Karp & A. Roth McDuffie (Eds.), *Using research to improve instruction* (pp. 239-248). Reston, VA: National Council of Teachers of Mathematics.

Britzman, D. P. (1991). *Practice makes practice: A critical study of learning to teach*. Albany, New York: State University of New York Press.

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics*, *85*, 221-239.

Cady, J., Meier, S. L., & Lubinski, C. A. (2006). Developing mathematics teachers: The transition from preservice to experienced teacher. *The Journal of Educational Research, 99*, 295-306.

Cannon, M. D., & Edmondson, A. C. (2001). Confronting failure: Antecedents and consequences of shared beliefs about failure in organizational work groups. *Journal of Organizational Behavior, 22*, 161-177.

Cheng, Y. C. (1994). Classroom environment and student affective performance: An effective profile. *Journal of Experimental Education, 62*, 221–239.

Children’s Mathematical Learning. (n.d.). Measurement broken ruler. Retrieved from https://vimeo.com/user7358057/review/105290013/522e63750f

Church, M. A., Elliot, A. J., & Gable, S. L. (2001). Perceptions of classroom environment, achievement goals, and achievement outcomes. *Journal of Educational Psychology, 93*, 43–54.

Claro, S., Paunesku, D., & Dweck, C. S. (2016). Growth mindset tempers the effects of poverty on academic achievement. *Proceedings of the National Academy of Sciences, 113*, 8664-8668.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Mayer, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 125–146). Reston, VA: National Council of Teachers of Mathematics.

Coldron, J., & Smith, R. (1999). Active location in teachers’ construction of their professional identities. *Journal of Curriculum Studies, 31*, 711-726.

College Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29*, 306-333.

Demirdag, S. (2015). Management of errors in classrooms: Students mistakes and teachers. *International Journal of Humanities and Social Science, 5*, 77-83.

Dewey, J. (1933). *How we think*. Boston, MA: D. C. Health.

Dresel, M., & Ziegler, A. (2002, September). *Failure as an element of adaptive learning*. Paper presented at the 8th Biennial Conference of the European Association for Research on Adolescence, Oxford, England.

Dweck, C. S., Chiu, C. Y., & Hong, Y. Y. (1995). Implicit theories and their role in judgments and reactions: A word from two perspectives. *Psychological Inquiry, 6*, 267-285.

Dweck, C. S., & Leggett, E. L. (1988). A social-cognitive approach to motivation and personality. *Psychological Review*, *95*, 256-273.

Erickson, Tim. (1996). *United we solve: 116 math problems for groups*. Oakland, CA: Eeps Media.

Franke, M., Fennema, E., & Carpenter, T. P. (1997). Changing teachers: Interactions between beliefs and classroom practice. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 255–282). Mahwah, NJ: Erlbaum.

Fraser, B. J. (1989). Twenty years of classroom climate work: Progress and prospect. *Journal of Curriculum Studies, 21,* 307–327.

Gattegno, C. (1954). Mathematics and the child. III: The use of mistakes in the teaching of mathematics. *The Mathematical Gazette, 38*(323), 11-14.

Gee, J. P. (2001). Identity as an analytic lens for research in education. *Review of Research in Education, 25,* 99-125.

Gelman S. A., Heyman G. D., & Legare C. H. (2007). Developmental changes in the coherence of essentialist beliefs about psychological characteristics. *Child* Development, 78, 757–774.

Good, C., Aronson, J., & Inzlicht, M. (2003). Improving adolescents' standardized test performance: An intervention to reduce the effects of stereotype threat. *Journal of Applied Developmental Psychology, 24*, 645-662.

Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology, 102,* 700-717.

Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8,* 35-59.

Gormally, C. (2016). Developing a teacher identity: TAs’ Perspectives about learning to teach inquiry-based biology labs. *International Journal of Teaching and Learning in Higher Education, 28*, 176-192.

Green, T. F. (1971). *The activities of teaching*. New York, NY: McGraw-Hill.

Grassinger, R., & Dresel, M. (2017). Who learns from errors on a class test? Antecedents and profiles of adaptive reactions to errors in a failure situation. *Learning and Individual Differences, 53*, 61-68.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching, 15*, 273-289.

Hargreaves, A. (1994) *Changing teachers, changing times: Teachers’ work and culture in the postmodern age*. New York, NY: Teachers College Press.

Hart, L. C. (2002). Preservice teachers' beliefs and practice after participating in an integrated content/methods course. *School Science and Mathematics, 102*, 4-14.

Hart, L. C. (2004). Beliefs and perspectives of the first-year, alternative preparation, elementary teachers in urban classrooms. *School Science and Mathematics, 104*, 79-88.

Haslam, N., Bastian, B., Brain, P., & Kashima, Y. (2006). Psychological essentialism, implicit theories, and intergroup relations. *Group Processes & Intergroup Relations, 9*(1), 63-76

Joram, E., & Gabriele, A. J. (1998). Preservice teachers’ prior beliefs: Transforming obstacles into opportunities. *Teaching and Teacher Education, 14*, 175-191.

Jensen, R. J. (1993). *Research ideas for the classroom: Early childhood mathematics*. New York, NY: Macmillan Publishing Company.

Kagan, D. M. (1992). Professional growth among preservice and beginning teachers. *Review of Educational Research, 62*, 129-169.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*, 59–80.

Keith, N., & Frese, M. (2008). Effectiveness of error management training: A meta-analysis. *Journal of Applied Psychology, 93*, 59-69.

Ketelaar, E., Beijaard, D., Boshuizen, H. P. A., & Den Brok, P. J. (2012). Teachers' positioning towards an educational innovation in the light of ownership, sense-making and agency. *Teaching and Teacher Education, 28*, 273-282.

Lambdin, D., & Walcott, C. (2007). Changes through the years: Connections between psychological learning theories and the school mathematics curriculum. In W. G. Martin, M. E. Struchens, & P. C. Elliott (Eds.), *The learning of mathematics: Sixty-ninth yearbook* (pp. 3–25). Reston, VA: National Council of Teachers of Mathematics

Lampert, M. (1992). Practices and problems in teaching authentic mathematics in school. In F. Oser, A. Dick, & J. L. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295-314). New York, NY: Jossey-Bass.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*, 91-102.

Leatham, K. R. (2007). Pre-service secondary mathematics teachers' beliefs about the nature of technology in the classroom. *Canadian Journal of Science, Mathematics and Technology Education, 7*, 183-207.

Lortie, D. (1975). *Schoolteacher: A sociological study*. Chicago, IL: University of Chicago Press.

Lui, A. M., & Bonner, S. M. (2016). Preservice and inservice teachers' knowledge, beliefs, and instructional planning in primary school mathematics. *Teaching and Teacher Education, 56*, 1-13.

Maher, P. A., Bailey, J. M., Etheridge, D. A., & Warby, D. B. (2013). Preservice teachers' beliefs and confidence after working with STEM faculty mentors: An exploratory study. *Teacher Education and Practice, 26*, 266-284.

Matteucci, M. C., Corazza, M., & Santagata, R. (2015). Learning from errors, or not. An analysis of teachers' beliefs about errors and error-handling strategies through questionnaire and video. *Progress in Education, 37*, 33-54

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York, NY: Macmillan.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: John Wiley & Sons, Inc.

Mizell, J. A., & Cates, J. (2004). The impact of additional content courses on teachers candidates' beliefs regarding mathematics content and pedagogy. *Issues in the Undergraduate Mathematics Preparation of School Teachers, 4*. Retrieved from https://files.eric.ed.gov/fulltext/EJ835512.pdf

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.), Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19*, 317-328.

Ng, W., Nicholas, H., & Williams, A. (2010). School experience influences on pre-service teachers' evolving beliefs about effective teaching. *Teaching and Teacher Education, 26*, 278-289.

Nisbett, R., & Ross, L. (1980). *Human inference: Strategies and shortcomings of social judgment*. Englewood Cliffs, NJ: Prentice.

Oser, F., & Spychiger, M. (2005). *Learning is painful. On the theory of negative knowledge and the practice of error culture*. Weinheim, Germany: Beltz.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307-332.

Pajares, M. F. (1996). Self-efficacy beliefs in academic settings. *Review of Educational Research, 66*, 543-578.

Patton, M. Q. (2015). *Qualitative research and evaluation methods* (4th ed.). Thousand Oaks, CA: Sage Publications.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal of Research and Development Education, 28*, 550-576.

Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientation toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education, 35*, 352-388.

Robins, R. W., & Pals, J. L. (2002). Implicit self-theories in the academic domain: Implications for goal orientation, attributions, affect, and self-esteem change. *Self and Identity, 1*, 313-336.

Rokeach, M. (1968). *Beliefs, attitudes, and values: A theory of organization and change*. San Francisco, CA: Jossey-Bass.

Rybowiak, V., Garst, H., Frese, M., & Batinic, B. (1999). Error Orientation Questionnaire (EOQ): Reliability, validity, and different language equivalence. *Journal of Organizational Behavior, 20*, 527-547.

Santagata, R. (2004). “Are you joking or are you sleeping?” Cultural beliefs and practices in Italian and U.S. teachers' mistake-handling strategies. *Linguistics and Education, 15*, 141–164.

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and U.S. mathematics lessons. *Teaching and Teacher Education, 21*, 491–508.

Santagata, R., & Bray, W. (2016). Professional development processes that promote teacher change: The case of a video-based program focused on leveraging students' mathematical errors. *Professional Development in Education, 42*, 547-568.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and U. S. mathematics classrooms. *The Elementary School Journal, 108*, 131-147.

Schmidt, William, H. (2012) At the Precipice: The story of mathematics education in the United States. *Peabody Journal of Education, 87,* 133-156.

Steuer, G., & Dresel, M. (2015). A constructive error climate as an element of effective learning environments. *Psychological Test and Assessment Modeling, 57*, 262-275.

Steuer, G., Rosentritt-Brunn G., & Dresel, M. (2013). Dealing with errors in mathematics classrooms: Structure and relevance of perceived error climate. *Contemporary Educational Psychology, 38*, 196-210.

Stevenson, H. W., & Stigler, J.W. (1992). *The learning gap: Why our schools are failing and what we can learn from Japanese and Chinese education*. New York, NY: Touchstone.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States*. Washington, DC: National Center for Educational Statistics.

Stigler, W. J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Stuart, C., & Thurlow, D. (2000). Making it their own: Preservice teachers' experiences, beliefs, and classroom practices. *Journal of Teacher Education, 51*, 113-121.

Swars, S., Smith, S., Smith, M., & Hart, L. (2009). A longitudinal study of effects of a developmental teacher preparation program on elementary prospective teachers' mathematics beliefs. *Journal of Mathematics Teacher Education, 21*, 47-66.

Teo, T., & Noyes, J. (2011). An assessment of the influence of perceived enjoyment and attitude on the intention to use technology among pre-service teachers: A structural equation modeling approach. *Computers and Education*, *57*, 1645-1653.

Thomas, A. J., & Sarnecka, B. W. (2015). Exploring the relation between people's theories of intelligence and beliefs about brain development. *Frontiers in Psychology, 6*. Retrieved from https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4490215/

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York, NY: MacMillan.

Tulis, M. (2013). Error management behavior in classrooms: Teachers' responses to students' mistakes. *Teaching und Teacher Education, 33,* 56-68.

Tulis, M., Grassinger, R., & Dresel, M. (2011). Adaptive handling of errors as an aspect of learning motivation and self-regulated learning of overachievers. In M. Dresel & L. Lämmle (Eds.), *Motivation, Selbstregulation und Leistungsexzellenz* (pp. 29-51). Münster: LIT.

Tulis, M., Steuer, G., & Dresel, M. (2016). Learning from errors: A model of individual processes. *Frontline Learning Research, 4*(2), 12-26.

U.S. Department of Education, National Center for Education Statistics. (2016). *Digest of Education Statistics, 2015* (NCES 2016-014). Retrieved from https://nces.ed.gov/pubs2016/2016014.pdf

Van Dyck, C., Frese, M., Baer, M., & Sonnentag, S. (2005). Organizational error management culture and its impact on performance: A two-study replication. *Journal of Applied Psychology, 90*, 1228-1240.

Wang, T., & Murphy, J. (2004). An examination of coherence in a Chinese mathematics classroom. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 107–123). Hackensack, NJ: World Scientific.

Willingham, J. C., Barlow, A. T., Stephens, D. C., Lischka, A. E., & Hartland, K. (2016). *Mindset regarding mathematical ability in K-12 teachers*. Manuscript submitted for publication..

Wilson, S., & Cooney, T. (2002). Mathematics teacher change and development. In G. C. Leder, E.Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education*? (pp. 127–147). Dordrecht, Netherlands: Kluwer Academic Publishers.

Winter, D., Lemons, P., Bookman, J., & Hoese, W. (2001). Novice instructors and student-centered instruction: Identifying and addressing obstacles to learning in the college science laboratory. *The Journal of Scholarship of Teaching and Learning, 2*(1), 15-42.

Wolters, C. A. (2004). Advancing achievement goal theory: Using goal structures and goal orientations to predict students' motivation, cognition, and achievement. *Journal of Educational Psychology*, *96*, 236-250.

Wrightstone, J. W. (1933). Analyzing and measuring democracy in the classroom. *Nation's Schools, 11*, 31-35.

Wrightstone, J. W. (1951). Measuring the social climate of a classroom. *The Journal of Educational Research*, *44*, 341-351.

Yin, R. K. (2003). *Applications of case study research* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). Thousand Oaks, CA: Sage Publications.

APPENDICES

APPENDIX A: IMPLICIT THEORY SURVEY

For each of the following statements, rate how strongly you agree or disagree with the statement.	Strongly Agree	Agree	Somewhat Agree	Somewhat Disagree	Disagree	Strongly Disagree
A1. You have a certain amount of intelligence and you really can't do much to change it.	**1**	**2**	**3**	**4**	**5**	**6**
A2. Your intelligence is something about you that you can't change very much.	**1**	**2**	**3**	**4**	**5**	**6**
A3. You can learn new things, but you can't really change your basic intelligence.	**1**	**2**	**3**	**4**	**5**	**6**
A4. A person's moral character is something very basic about them and it can't be changed much.	**1**	**2**	**3**	**4**	**5**	**6**
A5. Whether a person is responsible and sincere or not is deeply ingrained in their personality. It cannot be changed very much.	**1**	**2**	**3**	**4**	**5**	**6**
A6. There is not much that can be done to change a person's moral traits (e.g. conscientiousness, uprightness, and honesty).	**1**	**2**	**3**	**4**	**5**	**6**
A7. Though we can change some phenomena, it is unlikely that we can alter the core dispositions of our world.	**1**	**2**	**3**	**4**	**5**	**6**
A8. Our world has its basic and ingrained dispositions, and you really can't do much to change them.	**1**	**2**	**3**	**4**	**5**	**6**
A9. Some societal trends may dominate for a while, but the fundamental nature of our world is something that cannot be changed much.	**1**	**2**	**3**	**4**	**5**	**6**
A10. A person has a certain amount of mathematical ability and they really can't do much to change it.	**1**	**2**	**3**	**4**	**5**	**6**
A11. A person's mathematical ability is something about them that they can't change very much.	**1**	**2**	**3**	**4**	**5**	**6**
A12. A person can learn new things about mathematics, but they can't really change their basic mathematical ability.	**1**	**2**	**3**	**4**	**5**	**6**

APPENDIX B: PREVIOUS MATHEMATICS CLASSROOM ERROR CLIMATE

For each of the following statements, rate how strongly you agree or disagree with the statement.	**Strongly Agree**	**Agree**	**Somewhat Agree**	**Somewhat Disagree**	**Disagree**	**Strongly Disagree**
B1. In a mathematics class, it is okay with the teacher if assignments are not done correctly.	**1**	**2**	**3**	**4**	**5**	**6**
B2. In a mathematics class, mistakes are nothing bad for our teacher.	**1**	**2**	**3**	**4**	**5**	**6**
B3. In a mathematics class, the teacher doesn't like if something is done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
B4. In a mathematics class, it is not at all bad for our teacher if someone says something incorrect.	**1**	**2**	**3**	**4**	**5**	**6**
B5. If someone in a mathematics class makes a mistake, he will get a bad grade.	**1**	**2**	**3**	**4**	**5**	**6**
B6. If someone in a mathematics class says something wrong, it has an immediate effect on his grade.	**1**	**2**	**3**	**4**	**5**	**6**
B7. If someone in a mathematics class cannot solve an exercise correctly, the teacher will help him.	**1**	**2**	**3**	**4**	**5**	**6**
B8. If someone in a mathematics class does something incorrect, he will get very little support from the teacher.	**1**	**2**	**3**	**4**	**5**	**6**
B9. If someone in a mathematics class says something incorrect, the teacher will explain the problem.	**1**	**2**	**3**	**4**	**5**	**6**
B10. If someone in a mathematics class says something incorrect, the teacher will explain how to get the solution to the problem.	**1**	**2**	**3**	**4**	**5**	**6**
B11. If someone in a mathematics class makes a mistake, the teacher often looks annoyed.	**1**	**2**	**3**	**4**	**5**	**6**
B12. If someone in a mathematics class solves a problem incorrectly, the teacher can become impatient.	**1**	**2**	**3**	**4**	**5**	**6**

B13. If someone in a mathematics class gets a wrong answer, he can be mocked by his classmates.	**1**	**2**	**3**	**4**	**5**	**6**
B14. If someone in a mathematics class makes a mistake, his classmates will tease him.	**1**	**2**	**3**	**4**	**5**	**6**
B15. In a mathematics classroom, a lot of students would rather say nothing at all than say something wrong.	**1**	**2**	**3**	**4**	**5**	**6**
B16. In a mathematics classroom, a lot of students hope they will not be called on, because they are afraid they will say something wrong.	**1**	**2**	**3**	**4**	**5**	**6**
B17. In a mathematics classroom, there is a detailed discussion when something is done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
B18. In a mathematics classroom, the class spends time thinking in detail about students' incorrect statements.	**1**	**2**	**3**	**4**	**5**	**6**
B19. In a mathematics classroom, assignments that are done incorrectly are discussed in the class.	**1**	**2**	**3**	**4**	**5**	**6**
B20. In a mathematics classroom, mistakes are often used to make sure that you really understand a concept.	**1**	**2**	**3**	**4**	**5**	**6**
B21. In a mathematics classroom, a lot can be learned from assignments that are done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
B22. In a mathematics classroom, wrong answers are used to learn something.	**1**	**2**	**3**	**4**	**5**	**6**
B23. In a mathematics classroom, incorrect answers are often used as an opportunity to understand something.	**1**	**2**	**3**	**4**	**5**	**6**

APPENDIX C: FIRST POST ERROR CLIMATE INTERVIEW PROTOCOL

C1. How would you describe your past mathematics experiences?

- Bring up context, activities, tests, and attitude

C2. Can you describe what a typical day in your previous mathematics courses would look like?

- What would the teacher be doing?
- What would you be doing?
- What questions would you ask in your past mathematics courses?
- When you asked a question in your past mathematics course, how would you expect the teacher to respond?

C3. Complete this sentence: A mathematics teacher is like a:

News broadcaster	Entertainer	Doctor
Orchestrator/Conductor	Gardener	Coach

- Why do you say that?

C4. How do you think that a mathematics teacher that is like a (answer choice from C3) handles students' mistakes in the classroom?

C5. Can you describe what a mathematical mistake is?

- What it looks like?
- How you know when you made one?
- How a mathematical mistake different than a mistake in another course?

C6. When someone makes a mathematical mistake in class, what do you think the teacher should do?

C7. When you did something wrong in previous mathematics courses, what did that indicate to *you*, specifically?

- How did it make you feel?
- Did you ever reflect, if at all, on mistakes that you made, be it on an activity that you were doing in class, an assignment, or a test? How so?

C8. What would you expect the teacher to do if you made a mathematical mistake?

- Why do you think they would react that way? What was their goal in doing so?
- What were you hoping for them to do?

C9. I want you to recall a time that you helped a fellow classmate in a mathematics class. Can you describe how you helped them?

C10. Do you feel that it is okay to say that you don't know how to do something in mathematics?

C11. In your past mathematics classes, do you think that it was okay to give up on a problem?

- When would you give up?
- What did you expect to happen if you did?

C12. Do you think that there is any value in making a mistake in a mathematics classroom?

C13. In mathematics, can you can learn from a mistake?

- Does it matter when you make it (e.g., homework, test, in class)?
- Who makes it (e.g., you, peer, or teacher)?

C14. Are there any thoughts or questions that you had that you didn’t get a chance to say during this interview?

APPENDIX D: CLASSROOM OBSERVATION PROTOCOL

Time	PT(s) Who Made the Mistake	Description of Mistake and Actions Taken by Professor and PTs

APPENDIX E: POST CLASSROOM OBSERVATION INTERVIEW PROTOCOL

E1. What mathematical mistakes did you observe in class today? If you remember what each mistake was and who made it, please provide that as well.

E2. Can you describe what you were thinking right after the mistake(s) was made?

E3. Do you think that the mistake was worth talking about? Why or why not?

E4. Was the mistake along the same line of thinking as yours or different? How so?

- If the mistake was different than what you were thinking, what did that mean to you?
- If the mistake was along the same thought process as yours, what did that mean to you?

E5. How did you expect the teacher or other students to react after the mistake was made?

E6. Do you think anything was learned from the mistake either by you, your classmates, or the professor? Why or why not?

APPENDIX F: REFLECTION JOURNAL PROMPTS

F1. What mistakes did you or someone else make in your mathematics class today? If multiple mistakes were made, please include as many as you can remember.

F2. How do you think that the person making the mistake felt?

F3. What do you think the teacher was thinking when that mistake was made?

F4. Why do you think the mistake was made?

F5. Why do you think that the mistake was handled the way that it was?

APPENDIX G: CURRENT CLASSROOM ERROR CLIMATE

For each of the following statements, rate how strongly you agree or disagree with the statement.	**Strongly Agree**	**Agree**	**Somewhat Agree**	**Somewhat Disagree**	**Disagree**	**Strongly Disagree**
G1. In our mathematics class, it is okay with the teacher if assignments are not done correctly.	**1**	**2**	**3**	**4**	**5**	**6**
G2. In our mathematics class, mistakes are nothing bad for our teacher.	**1**	**2**	**3**	**4**	**5**	**6**
G3. In our mathematics class, the teacher doesn't like if something is done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
G4. In our mathematics class, it is not at all bad for our teacher if someone says something incorrect.	**1**	**2**	**3**	**4**	**5**	**6**
G5. If someone in our mathematics class makes a mistake, he will get a bad grade.	**1**	**2**	**3**	**4**	**5**	**6**
G6. If someone in our mathematics class says something wrong, it has an immediate effect on his grade.	**1**	**2**	**3**	**4**	**5**	**6**
G7. If someone in our mathematics class cannot solve an exercise correctly, the teacher will help him.	**1**	**2**	**3**	**4**	**5**	**6**
G8. If someone in our mathematics class does something incorrect, he will get very little support from the teacher.	**1**	**2**	**3**	**4**	**5**	**6**
G9. If someone in our mathematics class says something incorrect, the teacher will explain the problem.	**1**	**2**	**3**	**4**	**5**	**6**
G10. If someone in our mathematics class says something incorrect, the teacher will explain how to get the solution to the problem.	**1**	**2**	**3**	**4**	**5**	**6**
G11. If someone in our mathematics class makes a mistake, the teacher often looks annoyed.	**1**	**2**	**3**	**4**	**5**	**6**
G12. If someone in our mathematics class solves a problem incorrectly, the teacher can become impatient.	**1**	**2**	**3**	**4**	**5**	**6**

G13. If someone in our mathematics class gets a wrong answer, he can be mocked by his classmates.	**1**	**2**	**3**	**4**	**5**	**6**
G14. If someone in our mathematics class makes a mistake, his classmates will tease him.	**1**	**2**	**3**	**4**	**5**	**6**
G15. In our mathematics classroom, a lot of students would rather say nothing at all than say something wrong.	**1**	**2**	**3**	**4**	**5**	**6**
G16. In our mathematics classroom, a lot of students hope they will not be called on, because they are afraid they will say something wrong.	**1**	**2**	**3**	**4**	**5**	**6**
G17. In our mathematics classroom, there is a detailed discussion when something is done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
G18. In our mathematics classroom, the class spends time thinking in detail about students' incorrect statements.	**1**	**2**	**3**	**4**	**5**	**6**
G19. In our mathematics classroom, assignments that are done incorrectly are discussed in the class.	**1**	**2**	**3**	**4**	**5**	**6**
G20. In our mathematics classroom, mistakes are often used to make sure that you really understand a concept.	**1**	**2**	**3**	**4**	**5**	**6**
G21. In our mathematics classroom, a lot can be learned from assignments that are done incorrectly.	**1**	**2**	**3**	**4**	**5**	**6**
G22. In our mathematics classroom, wrong answers are used to learn something.	**1**	**2**	**3**	**4**	**5**	**6**
G23. In our mathematics classroom, incorrect answers are often used as an opportunity to understand something.	**1**	**2**	**3**	**4**	**5**	**6**

APPENDIX H: FINAL ERROR CLIMATE INTERVIEW PROTOCOL

H1. How would you describe the mathematics experiences you had in this class?

- Bring up context, activities, tests, and attitude

H2. Can you describe what a typical day in this mathematics courses would look like?

- What would the teacher be doing?
- What would you be doing?
- What questions would you ask in this mathematics course?
- When you asked a question in this mathematics course, how would you expect the teacher to respond?

H3. Complete this sentence: A mathematics teacher is like a:

News broadcaster	Entertainer	Doctor
Orchestrator/Conductor	Gardener	Coach

- Why do you say that?

H4. How do you think that a mathematics teacher that is like a <u>(answer choice from H3)</u> handles students' mistakes in the classroom?

H5. Can you describe what a mathematical mistake is?

- What it looks like?
- How you know when you made one?
- How a mathematical mistake different than a mistake in another course?

H6. When someone makes a mathematical mistake in class, what do you think the teacher should do?

H7. When you did something wrong in this mathematics course, what did that indicate to *you*, specifically?

- How did it make you feel?
- Did you ever reflect, if at all, on mistakes that you made, be it on an activity that you were doing in class, an assignment, or a test? How so?

H8. What would you expect the teacher to do if you made a mathematical mistake?

- What were you hoping for them to do?
- Why do you think they would react that way? What was their goal in doing so?

H9. I want you to recall a time that you helped a fellow classmate in this mathematics class. Can you describe how you helped them?

H10. Do you feel that it is okay to say that you don’t know how to do something in mathematics?

H11. In this mathematics classes, do you think that it was okay to give up on a problem?

- When would you give up?
- What did you expect to happen if you did?

H12. Do you think that there is any value in making a mistake in a mathematics classroom?

H13. In mathematics, can you can learn from a mistake?

- Does it matter when you make it (e.g., homework, test, in class)?
- Who makes it (e.g., you, peer, or teacher)?

H14. Are there any thoughts or questions that you had that you didn’t get a chance to say during this interview?

APPENDIX I: IMPLICIT THEORY OBSERVATION PROTOCOL

Potential Participants' Actions	Category	Implicit Theory (i.e., incremental or entity)

Implicit Theory Categories and Descriptions

	Implicit Theory Descriptors	
Category	**Entity**	**Incremental**
Evaluation of Situation	• May avoid situation involving mistakes • Looks for others to respond • “This is boring” • “Why are we doing this?”	• Asks questions if unknown • “I’ll try and see if we can do this”
Dealing with Setbacks	• Hides mistakes • Gives up after seeing a mistake • “I’m so stupid” • “I should have known that”	• External setbacks do not discourage • Not worried about how others perceive their mistakes • “I’ll try harder”
Challenges	• Avoids challenges • Gives up easily or gets defensive • Avoids eye contact	• Embraces challenges and doesn’t hold back when listening or explaining • Shows signs of encouragement towards self and others • “We can do this”
Effort	• Tries to look smart by giving answers with no relevance • “I can’t do this. This is too hard?”	• Works hard, always ready • Grabs for paper to continue working • Shows examples to others
Criticism	• Ignores constructive criticism • Gives up after criticism	• Learns from criticism • “That is a great idea, I didn’t see that” • Uses criticism to better one’s own ability
Success of Other	• Feels threatened by success of others	• Supports and encourages other classmates

Source: Willingham, J. C., Barlow, A. T., Stephens, D. C., Lischka, A. E., & Hartland, K. (2016). Mindset regarding mathematical ability in K-12 teachers. Manuscript submitted to *School Science and Mathematics.*

APPENDIX J: EXAMPLE EXIT TICKETS

J1. Today in class, there were several mathematical mistakes that were made. Which one meant the most to you? Why?

J2. Today in class, we discussed making this mathematical mistake. What are your reactions to our discussion of that mistake?

J3. What were the mathematical mistakes that you noticed today in class? What did those mistakes mean to you?

APPENDIX K: EXAMPLE IN-CLASS REFLECTIONS

K1. Just now, there was a mathematical mistake made in our class. What are your immediate thoughts on that?

K2. If you made a mistake on the assignment last night, describe what that mistake means to you.

K3. What are at least potential mathematical mistakes that someone could make on this assignment? What would each mistake indicate?

APPENDIX L: INSTITUTIONAL REVIEW BOARD APPROVAL

IRB
INSTITUTIONAL REVIEW BOARD
Office of Research Compliance,
010A Sam Ingram Building,
2269 Middle Tennessee Blvd
Murfreesboro, TN 37129

IRBN001 - EXPEDITED PROTOCOL APPROVAL NOTICE

Wednesday, August 16, 2017

Principal Investigator	**Matthew Duncan** (Student)
Faculty Advisor	Angela Barlow and Alyson Lischka
Co-Investigators	NONE
Investigator Email(s)	*matthew.duncan@mtsu.edu; angela.barlow@mtsu.edu; alyson.lischka@mtsu.edu*
Department	Mathematics and Science Education
Protocol Title	***Prospective teachers' mistakes and their beliefs about them: an exploratory, multiple case study***
Protocol ID	**18-2004**

Dear Investigator(s),

The above identified research proposal has been reviewed by the MTSU Institutional Review Board (IRB) through the **EXPEDITED** mechanism under 45 CFR 46.110 and 21 CFR 56.110 within the category *(7) Research on individual or group characteristics or behavior* A summary of the IRB action and other particulars in regard to this protocol application is tabulated as shown below:

IRB Action	APPROVED for one year from the date of this notification
Date of expiration	8/31/2018
Participant Size	25 (TWENTY FIVE)
Participant Pool	General adult MTSU students (18 years or older) enrolled in the teacher preparation courses.
Exceptions	1. Permitted to collect full name, email ID and telephone numbers for project administration or follow up. 2. Approved to withdraw participants who do not continue the required education or attend the required classes.
Restrictions	**1. Mandatory informed consent; The PI must provide a signed copy of the informed consent document to each participant. 2. The protocol must satisfy Education Research requirements. 3. Identifiable information must be destroyed once data are analyzed.**
Comments	NONE

This protocol can be continued for up to THREE years (8/31/2020) by obtaining a continuation approval prior to 8/31/2018. Refer to the following schedule to plan your annual project reports and be aware that you may not receive a separate reminder to complete your continuing reviews. Failure in obtaining an approval for continuation will automatically result in cancellation of this protocol. Moreover, the completion of this study MUST be notified to the Office of Compliance by filing a final report in order to close-out the protocol.

Continuing Review Schedule:

Reporting Period	Requisition Deadline	IRB Comments
First year report	7/31/2018	TO BE COMPLETED
Second year report	7/31/2019	TO BE COMPLETED
Final report	7/31/2020	TO BE COMPLETED

Post-approval Protocol Amendments:

Date	Amendment(s)	IRB Comments
NONE	NONE	NONE

The investigator(s) indicated in this notification should read and abide by all of the post-approval conditions imposed with this approval. Refer to the post-approval guidelines posted in the MTSU IRB's website. Any unanticipated harms to participants or adverse events must be reported to the Office of Compliance at (615) 494-8918 within 48 hours of the incident. Amendments to this protocol must be approved by the IRB. Inclusion of new researchers must also be approved by the Office of Compliance before they begin to work on the project.

All of the research-related records, which include signed consent forms, investigator information and other documents related to the study, must be retained by the PI or the faculty advisor (if the PI is a student) at the secure location mentioned in the protocol application. The data storage must be maintained for at least three (3) years after study completion. Subsequently, the researcher may destroy the data in a manner that maintains confidentiality and anonymity. IRB reserves the right to modify, change or cancel the terms of this letter without prior notice. Be advised that IRB also reserves the right to inspect or audit your records if needed.

Sincerely,

Institutional Review Board
Middle Tennessee State University

Quick Links:

Click here for a detailed list of the post-approval responsibilities.
More information on expedited procedures can be found here.

APPENDIX M: FIRST SET OF CODES AND THEMES

Open Codes		Emergent Themes
Current Error Climate Prior Error Climate Prior Error Climate-K12 Negative Error Climate Event Positive Error Climate Event		Context of Mistake a) Prior Mathematics Classroom b) Current Mathematics Classroom
Isolation Others' Group Dynamic Group Examination Fake Student		Ownership of Mistakes a) Their Own b) Someone's in the Class c) Elementary Student
Affective Impact Correct Encouragement Mistake Embarrassment Mistake Encouragement Math Ability Mistake Not Valued Mistake Avoided	No Embarrassment Response Not Valued	Influence of Mistake on Affective Qualities
Learn from Mistake Double Check Connections Everyone Learns Opportunity for Learning	Learn from Correct Mistake Fixed	Learning from Mistakes
Learn from Mistake Double Check Discuss Mistake Positioned as Teacher Content Common Mistake		Function of Mistake
Error Climate Opportunity for Learning Teacher Frustration Math Ability Mistake Not Discussed Error Analysis-Teacher Error Analysis-PTs Error Analysis-Other	Response Not Valued Mistake Not Valued Mistake Fixed Teacher Welcomes Mistake	Error Climate of Classroom a) Positive Perceived b) Negatively Perceived
Common Discuss Correct Discuss Mistakes Homework Isolation Major Minor	Others' Repetitive Test	Type of Mistake

APPENDIX N: FINAL CODES LIST

Open Codes		Emergent Themes
Current Error Climate Prior Error Climate Prior Error Climate-K12 Prior Error Climate-College	Negative Error Climate Event Positive Error Climate Event Non-reform Strategy[a] Reform Strategy[a]	Context of Mistake a) Prior Mathematics Classroom b) Current Mathematics Classroom
Isolation Others' Group Dynamic	Group Examination Fake Student Future Elementary Student[a]	Ownership of Mistakes a) Their Own b) Someone's in the Class c) Elementary Student
Affective Impact Correct Encouragement Mistake Embarrassment Mistake Encouragement Math Ability Mistake Not Valued	Mistake Avoided No Embarrassment Response Not Valued Apprehension to Mistake[a] Mistakes Hard[a]	Influence of Mistake on Affective Qualities
Learn from Mistake Double Check Connections Everyone Learns Opportunity for Learning Persevering Past Mistake	Learn from Correct Mistake Fixed Teacher Knowledge PCK[a] Alternative Strategies[a]	Learning from Mistakes
Learn from Mistake Double Check Discuss Mistake Positioned as Teacher Content Common Mistake Teacher Knowledge	Discussion Worthy Mistakes Happen[a] Mistakes Good[a] Mistakes Examples[a] Necessity of Mistakes[a] Mistakes = Bad Grade	Function of Mistake
Error Climate Opportunity for Learning Teacher Frustration Math Ability Mistake Not Discussed Error Analysis-Teacher Correct Focus	Error Analysis-PTs Error Analysis-Other Response Not Valued Mistake Not Valued Mistake Fixed Teacher Welcomes Mistake[a] Mistake Fixed This Way[a]	Error Climate of Classroom c) Positive Perceived d) Negatively Perceived
Common Discuss Correct Discuss Mistakes Homework Isolation Major One on One	Minor Others' Repetitive Test Get it Right[a] If There's Time[a]	Type of Mistake
Student[a]	Teacher[a]	Position of PT[a]

[a]Indicates codes created from analyzing Harley's data.

APPENDIX O: COPYRIGHT PERMISSION FOR FIGURE 11 AND FIGURE 20

Pearson

Permissions
4th Floor, Auto Atlantic
Corner, Hertzog Boulevard &
Heerengracht
Cape town, 8001
South Africa
USAPermissions@pearson.com

Jun 11, 2018 PE Ref # 205228

Matthew Duncan
MIDDLE TENESSEE STATE UNIVERSITY

616 DORSHIRE LN
Nashville, TN 37132

Dear Matthew Duncan,

You have our permission to include content from our text, ***MATHEMATICS FOR ELEMENTARY TEACHERS WITH ACTIVITIES, 5th Ed. by BECKMANN, SYBILLA,*** in your dissertation or masters thesis at MIDDLE TENNESSEE STATE UNIVERSITY.

Content to be included is:
CA-257,

Please credit our material as follows:
BECKMANN, SYBILLA, MATHEMATICS FOR ELEMENTARY TEACHERS WITH ACTIVITIES, 5th, ©2018. **Reprinted by permission of Pearson Education, Inc., New York, New York.**

Sincerely,
Julia Alexander
Global Rights/Permissions Analyst