A Dual Categorical Equivalence between DiGraph Posets and DiGraph Lattices

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DEDICATION

I dedicate this thesis to my family for their unwavering love, patience, encouragement, and countless sacrifices that made this journey possible. Your support and your belief in me has been my constant motivation and my greatest strength. I would also like to dedicate this thesis to the countless mathematicians who paved the way with their groundbreaking work, inspiring me to explore the elegance of mathematics. I am humbled to contribute, even in a small way, to the math community!

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ABSTRACT

It is well known that there is an incidence poset associated with every directed graph. The problem is that this poset doesn't encode enough information to recreate our directed graph. In his thesis, Crowell solved this problem by considering tripartite posets whose middle elements covers and exactly covered by one element, and which possess a bijection between the maximal elements and the minimal ones. Crowell proved a categorical equivalence between the category DiGraph and the category DiGraph posets [1]. We extend this idea to lattices and we establish a dual categorical equivalence between the categories DiGraph Posets and DiGraph lattices. This will implies a dual categorical equivalence between the categories DiGraph and DiGraph lattices.

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CHAPTER 1

INTRODUCTION

In this thesis, most work will be original, and all original work was completed under the supervision of Dr. James Hart. While all proofs provided in the thesis are my own, some definitions in Chapter 2 and 3 are pulled from source material. Any mathematics that come from a source material will be cited, and any definition, theorem, and lemma not preceded or followed by a citation is original.

In Chapter 2, we will provide the preliminary knowledge in Graph Theory, Order Theory, and Category Theory necessary to understand the thesis.

In Chapter 3, we introduce the categories DiGraph and $\mathcal{DGP}$, and we construct the category $\mathcal{DGL}$.

In Chapter 4, we build our functors between the categories $\mathcal{DGP}$ and $\mathcal{DGL}$, and we show those functors yield a dual categorical equivalence between these two categories.

CHAPTER 2

BACKGROUND

2.1 Graph Theory

In this thesis, only some basic understanding of graph theory definitions is required. We define directed graphs, homomorphisms, and isomorphisms between directed graphs as Crowell [1] defined them.

Definition 2.1.1 A finite directed graph (or DG-object) is a quadruple $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s, t)$ consisting of a non-empty finite set V($\mathcal{G}$) of elements called vertices, a finite set A($\mathcal{G}$) of elements called arrows, as well as a pair of functions $s : A \longrightarrow V$ and $t : A \longrightarrow V$ called source map and target map of $\mathcal{G}$ respectively. Given this, for each $a \in A$, we refer to s(a) as the source of a and t(a) as the target of a. We say the arrow a point from the vertex $s(a)$ to the vertex $t(a)$. We assume V and A are disjoint sets here.

Definition 2.1.2 Suppose that $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s_1, t_1)$ and $\mathcal{H} = (V(\mathcal{H}), A(\mathcal{H}), s_2, t_2)$ are finite directed graphs. A directed graph morphism (DG-morphism) from $\mathcal{G}$ to $\mathcal{H}$ is a pair of maps $\varphi = (\varphi_V, \varphi_A)$ with the following properties.

- $\varphi_V : V(\mathcal{G}) \longrightarrow V(\mathcal{H})$ and $\varphi_A : A(\mathcal{G}) \longrightarrow A(\mathcal{H})$.
- For all $x \in A(\mathcal{G})$, we have $S_2(\varphi_A(x)) = \varphi_V(S_1(x))$ and $t_2(\varphi_A(x)) = \varphi_V(t_1(x))$.

Definition 2.1.3 Two digraphs $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s_1, t_1)$ and $\mathcal{H} = (V(\mathcal{H}), A(\mathcal{H}), s_2, t_2)$ are isomorphic provided there is a DG- isomorphism between them. A DG- isomorphism is a pair (φ, ψ) where

- $\varphi = (\varphi_V, \varphi_A)$ is a DG-morphism from $\mathcal{G}$ to $\mathcal{H}$ and $\psi = (\psi_V, \psi_A)$ is a DG-morphism from $\mathcal{H}$ to $\mathcal{G}$.

- $\psi_V \circ \varphi_V = 1_{V(\mathcal{G})}$ and $\varphi_V \circ \psi_V = 1_{V(\mathcal{H})}$.
- $\psi_A \circ \varphi_A = 1_{A(\mathcal{G})}$ and $\varphi_A \circ \psi_A = 1_{A(\mathcal{H})}$.

2.2 Order Theory

Order theory is a branch of mathematics that investigates the intuitive notion of order using binary relations. In This section, we will introduce some definitions and prove some theorems about partially ordered sets (or posets) that will play a key role in the development of the subsequent future results.

We start by stating some definitions in order to prove the results of this section. All the definitions and theorems in this section are coming from [2] and [4], however, all proofs provided are my own.

Definition 2.2.1 A Poset (or partially ordered set) is a system $\mathcal{P} = (P, \leq)$ consisting of a set P and a binary relation $\leq$ on the set P satisfying the following conditions:

1. For all $x \in P$, we have $x \leq x$ (reflexibility).
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).

If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

The binary relation $\leq$ defined above is called a partial ordering on the set P.

Definition 2.2.2 Let $\mathcal{P} = (P, \leq)$ be any poset. The order dual of $\mathcal{P}$ is defined to be the system $\mathcal{P}^{op} = (P, \leq^{op})$ where $x \leq^{op} y \iff y \leq x$. We usually denote the order dual of a poset $\mathcal{P}$ by simply writing $\mathcal{P}^{op}$.

The duality principle state that:

A statement ϕ is true for all posets if and only if its dual is also true for all posets.

Definition 2.2.3 Let $\mathcal{P} = (P, \leq)$ be any poset and suppose $X \subseteq P$. X is a *lower set* of $\mathcal{P}$ provided $a \in X$ *and* $b \leq a$ together imply that $b \in X$. The *principal lower set* generated by the element x is the set $\downarrow x = \{y \in p : y \leq x\}$.

We say that X is an *upper set* of $\mathcal{P}$ provided X is a lower set of the order dual of $\mathcal{P}$; that is X is an upper set of $\mathcal{P}$ provided $a \in X$ *and* $b \geq a$ together imply that $b \in X$.The *principal upper set* generated by the element x is the set $\uparrow x = \{y \in p : y \geq x\}$.

Definition 2.2.4 Let $\mathcal{P} = (P, \leq)$ be any poset and suppose $X \subseteq P$. An element $u \in P$ is an upper bound for X provided $u \in \uparrow a$ for all $a \in X$. When X has a *smallest upper bound*, we call this element the join (or supremum) of the set X and denoted $\bigvee X$. We will let $j(X)$ be the set of all the upper bounds for X.

An element $l \in P$ is a lower bound for X provided $l \in \downarrow a$ for all $a \in X$. When X has a *greatest lower bound*, we call this element the meet (or infimum) of the set X and denoted $\bigwedge X$. We will let $m(X)$ be the set of all the lower bounds for X.

If $X = \{x_1, ..., x_n\}$, then it is common to practice to set $\bigvee X = x_1 \vee ... \vee x_n$, and $\bigwedge X = x_1 \wedge ...x_n$.

Definition 2.2.5 A poset $\mathcal{P} = (P, \leq)$ is called a lattice provided $\bigvee X$ and $\bigwedge X$ exist in $\mathcal{P}$ for every nonempty finite $X \subseteq P$. It is worth noting that every *finite lattice* must have a largest and smallest element.

For any poset $\mathcal{P} = (P, \leq)$, we will let $\mathcal{L}(\mathcal{P}) = (Low(\mathcal{P}), \subseteq)$ and $\mathcal{U}(\mathcal{P}) = (Up(\mathcal{P}), \subseteq)$ where $Low(\mathcal{P})$ *and* $Up(\mathcal{P})$ denote the families of lower sets and upper sets respectively, for $\mathcal{P}$. It is easy to see that $\mathcal{L}(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})$ are both lattices in which finite joins are set-unions and finite meets are set-intersections. For simplicity, let $\top$ denote the largest element of the lattice, and let $\bot$ denote the smallest element of the lattice.

Definition 2.2.6 Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are lattices. A mapping $f : P \longrightarrow Q$ is a lattice homomorphism provided f preserves all *finite nonempty* meets and joins; that is, we have $f(\bigvee F) = \bigvee f(F)$ and $f(\bigwedge F) = \bigwedge f(F)$ for all finite nonempty $F \subseteq P$. A lattice homomorphism is bounded provided it also preserves the largest and smallest element (when such exist).

Note here that *lattice homomorphisms* are different than *order homomorphisms*. Recall that for any two posets $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$, a function $f : P \longrightarrow Q$ is an order homomorphism provided $x \leq y \implies f(x) \leq f(y)$ for any $x, y \in P$.

Definition 2.2.7 We say a lattice $\mathcal{P} = (P, \leq)$ is *distributive* provided we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in P$.
It is worth noting that a lattice $\mathcal{P} = (P, \leq)$ *is distributive* $\iff$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in P$.

Note here that for any lattice $\mathcal{P} = (P, \leq)$, $y \geq (y \wedge z)$ and $z \geq (y \wedge z)$. Consequently, we also know that $x \vee y \geq x \vee (y \wedge z)$ and $x \vee z \geq x \vee (y \wedge z)$. This allows us to conclude that $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ in any lattice. Therefore, $\mathcal{P}$ is distributive provided the reverse inequality also hold.

It is easy to see here that $\mathcal{L}(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})$ are distributive for any poset $\mathcal{P} = (P, \leq)$.

Definition 2.2.8 An element j in a lattice $\mathcal{P} = (P, \leq)$ is *join-prime* provided for *any finite* $F \subseteq P$, the inequality $j \leq \bigvee F$ implies $j \leq x$ for some $x \in F$.
It is worth noting that the smallest element of a lattice (when it exists) cannot be join-prime.

We will let $JP(\mathcal{P})$ denote the subposet of join-prime elements for the lattice $\mathcal{P}$.

Definition 2.2.9 Let $\mathcal{P} = (P, \leq)$ be any poset and suppose $a \leq b$ in $\mathcal{P}$. We say that *b covers a* provided $\uparrow a \cap \downarrow b = \{a, b\}$, that is $a < b$ and there are no elements "between" a and b. In a finite lattice, every element except the smallest element must cover at least one element.
In a poset with smallest element, an element a is called an *atom* of $\mathcal{P}$ provided a covers the smallest element.

A poset $\mathcal{P} = (P, \leq)$ with smallest element b is <u>atomic</u> provided $\downarrow x$ contains an atom for every $x \in P - \{b\}$. Note that finite lattices are necessarily atomic.

We are now ready to prove some results.

Lemma 2.2.10 *Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are posets and suppose $f : P \longrightarrow Q$. The following statements are logically equivalent.*

1. *The function f is an order homomorphism.*
2. *If X is a lower set of $\mathcal{Q}$, then its preimage under f is a lower set of $\mathcal{P}$.*
3. *If Y is an upper set of $\mathcal{Q}$, then its preimage under f is an upper set of $\mathcal{P}$.*

Proof. We are going to prove $1 \implies 2 \implies 3 \implies 1$.
$(1 \implies 2)$ Suppose that f is an order homomorphism and let X be a lower set of Q. We want to prove that $Pre_f(X)$ is a lower set of $\mathcal{P}$. Let $c, d \in P$ such that $c \in Pre_f(X)$ and $d \leq c$. Since f is an order homomorphism, $d \leq c$ implies that $f(d) \sqsubseteq f(c)$. We know that $f(c) \in X$ since $c \in Pre_f(X)$. Since X is a lower set of $\mathcal{Q}$, we have $f(c) \in X$ and $f(d) \sqsubseteq f(c)$ together implies that $f(d) \in X$. We conclude that $d \in Pre_f(X)$.

$(2 \implies 3)$ Suppose (2), and let Y be an upper set of $\mathcal{Q}$. We want to prove that $Pre_f(Y)$ is an upper set of $\mathcal{P}$. Let $a, b \in P$ such that $a \in Pre_f(Y)$ and $b \geq a$. Consider the lower set $\downarrow f(b)$. By (2), we know that $Pre_f(\downarrow f(b))$ is a lower set of $\mathcal{P}$. Note that $f(b) \in\downarrow f(b)$ implies that $b \in Pre_f(\downarrow f(b)$. Since $Pre_f(\downarrow f(b))$ is a lower set in $\mathcal{P}$, $b \in Pre_f(\downarrow f(b)$ and $a \leq b$ together implies that $a \in Pre_f(\downarrow f(b))$. We may conclude that $f(a) \in\downarrow f(b)$, and therefore $f(a) \sqsubseteq f(b)$. Further, since $a \in Pre_f(Y)$, we know that $f(a) \in Y$. Using the fact that Y is an upper set of $\mathcal{Q}$, $f(a) \sqsubseteq f(b)$ and $f(a) \in Y$ together implies that $f(b) \in Y$. We may therefore conclude that $b \in Pre_f(Y)$.

$(3 \implies 1)$ Suppose (3). we want to prove that f is an order homomorphism.

For that, let $a, b \in P$ such that $a \leq b$. Consider the upper set $\uparrow f(a)$. From our assumption, we know that $Pre_f(\uparrow f(a))$ is an upper set in $\mathcal{P}$.

$$\begin{aligned} f(a) \in \uparrow f(a) &\implies a \in Pre_f(\uparrow f(a)) \\ &\implies b \in Pre_f(\uparrow f(a)) \quad [Pre_f(\uparrow f(a)) \text{ is an upper set and } a \leq b] \\ &\implies f(b) \in \uparrow f(a) \\ &\implies f(b) \sqsupseteq f(a) \end{aligned}$$

■

Lemma 2.2.11 *If $\mathcal{P} = (P, \leq)$ is any poset, then $m(\emptyset) = P$ and $j(\emptyset) = P$, where $j(X)$ is the set of all the upper bounds for X and $m(X)$ is the set of all the lower bounds for X.*

Proof. Let's first prove that $m(\emptyset) = P$. For this purpose, suppose by a way of contradiction that $m(\emptyset) \neq P$. Then there exist $x \in p$ such that $x \notin \downarrow y$ for some $y \in \emptyset$. This contradiction prove that $m(\emptyset) = P$. By similar reasoning, we get $j(\emptyset) = P$.

■

Lemma 2.2.12 *Let $\mathcal{P} = (P, \leq)$ be any poset. The following statements are true.*

1. *The poset $\mathcal{P}$ has a largest element $\iff \bigwedge \emptyset$ exists.*
2. *The poset $\mathcal{P}$ has a smallest element $\iff \bigvee \emptyset$ exists.*

Proof. We will prove the first statement, and the second can be proven using similar reasoning.

($\implies$) Suppose that $\mathcal{P}$ has a largest element. Hence by lemma 2.2.11, $m(\emptyset)$ has a largest element. Therefore, $\bigwedge \emptyset$ exists.

($\impliedby$) Suppose that $\bigwedge \emptyset$ exists. Then, by definition 2.2.4, $m(\emptyset)$ has a largest element. Therefore, by lemma 2.2.11, $\mathcal{P}$ has a largest element.

■

Corollary 2.2.13 *Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are posets. If $f : P \longrightarrow Q$ is an order homomorphism, then the mapping $Low[f] : Low(\mathcal{Q}) \longrightarrow Low(\mathcal{P})$ defined by*

$Low[f](Y) = Pre_f(Y)$ is a bounded lattice homomorphism.

Proof. Recall that $\mathcal{L}(\mathcal{P}) = (Low(\mathcal{P}), \subseteq)$ is a lattice in which finite joins are set-unions and finite meets are set-intersections.

To show that $Low[f]$ is a lattice homomorphism, let $F \subseteq Low(\mathcal{Q})$ finite and non-empty. Let us prove that $Low[f](\bigvee F) = \bigvee Low[f](F)$, and $Low[f](\bigwedge F) = \bigwedge Low[f](F)$ can be proven using similar reasoning.

$$\begin{aligned} Low[f](\bigvee F) &= Pre_f(\bigvee F) \\ &= Pre_f(\bigcup F) \\ &= \bigcup Pre_f(F) \qquad [\ Pre_f(\bigcup F) = \bigcup\{Pre_f(A); A \in F\}\ ] \\ &= \bigvee Pre_f(F) \\ &= \bigvee Low[f](F) \end{aligned}$$

Now want to show that $Low[f]$ is bounded, that is $Low[f]$ preserves the largest and smallest element when such exist.

Let us assume that $Low(\mathcal{Q})$ has a largest element, then this largest element must be equal to Q. We shall prove that $Low[f](Q) = Pre_f(Q) = P$, where P is the largest element of $Low(\mathcal{P})$.

Lemma 2.2.9 tells us that $Pre_f(Q)$ is a lower set of $\mathcal{P}$ since f is an order homomorphism and Q is a lower set of $\mathcal{Q}$. Suppose by a way of contradiction that $Pre_f(Q) \neq P$. This implies that $Pre_f(Q) \subset P$ since $Pre_f(Q)$ is a lower set of $\mathcal{P}$. Thus, $\exists\ a \in P$ such that $a \notin Pre_f(Q)$. We know $f(a) \in Q$ since $a \in P$. Since $f(a) \in Q$, we may conclude that $a \in Pre_f(Q)$. This contradiction proves that $Pre_f(Q) = P$, the largest element of $Low(\mathcal{P})$.

Now let's prove that $Low[f]$ preserves the smallest element. It is obvious that the smallest element of $Low(\mathcal{Q})$ and $Low(\mathcal{P})$ is indeed the empty set since the empty set is a subset of all sets. With that in mind, we shall prove that $Low[f](\emptyset) = Pre_f(\emptyset) = \emptyset$. Suppose by a way of contradiction that $Pre_f(\emptyset) \neq \emptyset$, then $\exists\ a \in P$ such that $a \in Pre_f(\emptyset)$. We know $f(a) \in Q$ since $a \in P$. This contradicts the fact that $f(a) \in \emptyset$. We may therefore conclude that $Low[f](\emptyset) = \emptyset$.

■

Lemma 2.2.14 *Suppose $\mathcal{P} = (P, \leq)$ is a distributive lattice. The following statements are logically equivalent for an element $j \in P$.*

1. *The element j is join-prime.*
2. *For every finite $F \subseteq P$, we have $j = \bigvee F$ implies $j \in F$.*

Proof. ($\Longrightarrow$) Let us assume that the element j is join-prime. Let $F \subseteq P$ be finite and suppose that $j = \bigvee F$. We shall prove $j \in F$.

Notice here that $\bigvee F$ is join prime since we assumed that j is join-prime. Since $\leq$ is reflexive and $\bigvee F \in P$, we get $\bigvee F \leq \bigvee F$.

F is finite, $\bigvee F$ is join prime, and $\bigvee F \leq \bigvee F$ together implies that $\bigvee F \leq x$ for some $x \in F$.(Definition 2.2.8)

Since $x \in F$, we know that $x \leq \bigvee F$. We conclude that $x = \bigvee F$ since $\bigvee F \leq x$ and $x \leq \bigvee F$. Further, $x = \bigvee F$ and $x \in F$ together implies that $j = \bigvee F \in F$.

($\Longleftarrow$) Assume that [For every finite $F \subseteq P$, we have $j = \bigvee F$ implies $j \in F$]. Let $G \subseteq P$ be finite. Hence, $G = \{x_1, ..., x_n\}$ for some $x_i \in P$, $i \in \{1, ..., n\}$. Let $\bigvee G = x_1 \vee ... \vee x_n$. Suppose that $j \leq \bigvee G$, we shall prove $j \leq x$ for some $x \in G$.

$j \leq \bigvee G$ implies that $j = j \bigwedge (\bigvee G)$. Thus,

$$
\begin{aligned}
j &= j \wedge (x_1 \vee ... \vee x_n) \\
&= (j \wedge x_1) \vee ... \vee (j \wedge x_n) && \text{[Distributive Properties]} \\
&= \bigvee \{j \wedge x_1, ..., j \wedge x_n\} \\
&= \bigvee F \text{ where } F = \{j \wedge x_1, ..., j \wedge x_n\}
\end{aligned}
$$

By our assumption, F is finite since G is finite and $j = \bigvee F$ together implies that $j \in F$. Thus, $j = j \wedge x_i$ for some $x_i \in G$. But we know that $j \wedge x_i \leq x_i$, therefore $j \leq x_i$ for some $x_i \in G$.

■

Lemma 2.2.15 *Suppose $\mathcal{P} = (P, \leq)$ is a finite distributive lattice. An element x is join-prime if and only if x covers exactly one element.*

Proof. ($\Longrightarrow$) Suppose that the element x is join-prime. Suppose by a way of contradiction that x covers two elements a and b. Hence by definition 2.2.9, $x = a \vee b$. If we let $F = \{a, b\}$, then $x = \bigvee F$. By Lemma 2.2.14, it results that $x \in F$. This contradicts the fact that a and b are strictly less that x since x covers a and b. We may therefore conclude that x covers exactly one element.

($\Longleftarrow$) Suppose that x covers exactly one element, let's name it a. Suppose that $F \subseteq P$ is finite, and suppose that $x = \bigvee F$, we want to prove that $x \in F$ (lemma 2.2.14). Suppose by a way of contradiction that $x \notin F$. Observe that $x = \bigvee F$, $x \notin F$, and x covers exactly the element a together imply that $y \leq a$ for all $y \in F$. Hence a is an upper bound for F such that $a < x$. This contradicts the fact that x is the least upper bound of F. We may therefore conclude that $x \in F$, and hence x is a join-prime element.

■

Lemma 2.2.16 *If $\mathcal{P} = (P, \leq)$ is a distributive lattice with smallest element, then every atom of $\mathcal{P}$ is join-prime*

Proof. Let j be an atom of $\mathcal{P}$ and let $\perp$ be the smallest element of $\mathcal{P}$. To prove that j is join prime, let $F \subseteq P$ finite and let $j = \bigvee F$. We want to prove that $j \in F$. Since $j = \bigvee F$ and j covers the smallest element, then $F = \{\perp, j\}$ or $F = \{j\}$. In both cases, $\bigvee F = j \in F$.

■

Theorem 2.2.17 *Suppose $\mathcal{P} = (P, \leq)$ is any finite poset and consider the lattice $\mathcal{L}(\mathcal{P})$ of lower sets of $\mathcal{P}$. The join-prime elements of $\mathcal{L}(\mathcal{P})$ are precisely the principal lower sets of $\mathcal{P}$. Moreover, $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ is order-isomorphic to $\mathcal{P}$ via the maps $\eta_p : P \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{P}))$ and $\vartheta_p : \mathcal{JP}(\mathcal{L}(\mathcal{P})) \longrightarrow P$ defined by the rules $\eta_p(x) = \downarrow x$ and $\vartheta_p : (\downarrow x) = x$.*

Proof. We begin by showing that the join-prime elements of $\mathcal{L}(\mathcal{P})$ are precisely the principal lower sets of $\mathcal{P}$, that is we wish to show that:

$J \in \mathcal{L}(\mathcal{P})$ is join prime $\iff$ $J = \downarrow x$ for some $x \in P$.

($\Longrightarrow$) Suppose that $J \in \mathcal{L}(\mathcal{P})$ is join prime. Suppose by a way of contradiction that J is not a principal lower set of $\mathcal{P}$. Hence, $J = \bigcup \{\downarrow x : \text{ for some } x \in P\} = \bigcup F$, where $F \subseteq Low(\mathcal{P})$ is finite since $\mathcal{P}$ is finite. Moreover, recall that $\mathcal{L}(\mathcal{P})$ is a distributive lattice. Since J is join prime and $J = \bigcup F = \bigvee F$, and satisfying all the conditions of lemma 2.2.14, we may therefore conclude that $J \in F$, which contradicts our assumption that J is not a principal lower set of $\mathcal{P}$. This contradiction proves that J is indeed a principal lower set of $\mathcal{P}$.

($\Longleftarrow$) Now suppose that $J =\downarrow x$ for some $x \in P$. Let $A = J - \{x\} =\downarrow x - \{x\}$. It's clear that J covers A in $\mathcal{L}(\mathcal{P})$ since $A \subset J$ and both J and A are elements of $\mathcal{L}(\mathcal{P})$. Since $\mathcal{L}(\mathcal{P})$ is a distributive lattice, to show that J is join-prime, it's enough to show that J covers exactly one element. Now suppose by a way of contradiction that J covers 2 elements A and $C \in \mathcal{L}(\mathcal{P})$ such that $A \neq C$ and $C \neq J$. Observe here that since J covers both A and C, then $A \cup C = J$. This implies that $x \in C$. Since $C \in \mathcal{L}(\mathcal{P})$, therefore $C = J$. This contradicts our assumption that $C \neq J$. Therefore, J covers exactly one element. This allows us to conclude that J is join prime.

Now we wish now to show that $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ is order-isomorphic to $\mathcal{P}$.

For that, let's first consider the map $\eta_p : P \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{P}))$ defined by the rule $\eta_p(x) =\downarrow x$. It is easy to see that $x \leq y \implies \downarrow x \subseteq \downarrow y \implies \eta_p(x) \subseteq \eta_p(y)$. Next, consider the map $\vartheta_p : \mathcal{JP}(\mathcal{L}(\mathcal{P})) \longrightarrow P$ defined by the rule $\vartheta_p(\downarrow x) = x$. If $\downarrow x \subseteq \downarrow y$, then $x \in\downarrow y$, which implies that $x \leq y$. Moreover, it's clear here that $\eta_p(\vartheta_p(\downarrow x)) =\downarrow x$ and $\vartheta_p(\eta_p(x))) = x$. We may therefore conclude that $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ is order-isomorphic to $\mathcal{P}$.

■

Talking about order isomorphism between these two posets, it is convenient to give an example. We can represent a finite poset $\mathcal{P} = (P, \leq)$ visually using a **Hasse Diagram**. In a Hasse diagram, we indicate $a < b$ for $a,\ b \in P$ by placing a lower than b on the page and connecting them with a line. Hasse diagrams are always drawn with the fewest lines necessary to represent the partial order. This means that if there is already a path of line segments connecting two elements, there is no need to connect these two elements directly by a single line segment. For instance, in the middle diagram of figure 1 below, there is no line segment connecting $\{c\}$ and $\{b, c, d\}$. We get $\{c\} \subseteq \{b, c, d\}$ from transitivity of the partial ordering (here it's the subset inclusion). $\{c\} \subseteq \{c, d\}$ and $\{c, d\} \subseteq \{b, c, d\} \implies \{c\} \subseteq \{b, c, d\}$

Consider the poset $\mathcal{P} = (P, \leq)$ where $P = \{a, b, c, d\}$ and a partial order on P defined by the set $\{(c, a), (d, a), (c, b), (d, b)\} \subseteq P \times P$. It is customary to write $c \leq a$ in $\mathcal{P}$ when (c, a) is an element of the partial order. The below diagrams are the Hasse Diagrams of $\mathcal{P}$ (on the left), $\mathcal{L}(\mathcal{P})$ (on the middle), and $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ (on the right). Notice here that since $\mathcal{L}(\mathcal{P})$ here is a finite distributive lattice, lemma 2.2.15 tells us that the join prime elements of $\mathcal{L}(\mathcal{P})$ are the elements that covers exactly one element.

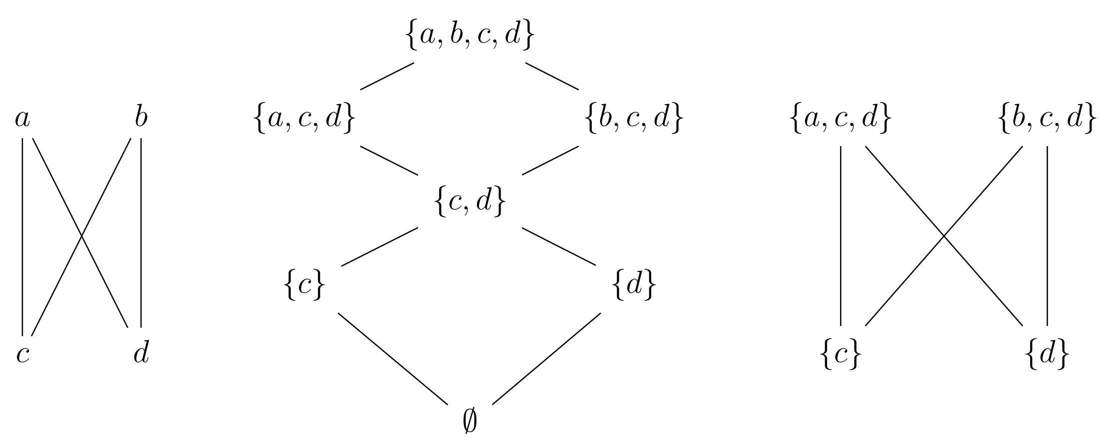

Figure 1: Order Isomorphism between $\mathcal{P}$ and $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$

Lemma 2.2.18 *If every element of a lattice $\mathcal{P} = (P, \leq)$ is the join of a finite set of join prime elements, then $\mathcal{P}$ is distributive.*

Proof. Assume that every element of the lattice $\mathcal{P} = (P, \leq)$ is the join of a finite set of join prime elements. We want to prove that $\mathcal{P}$ is distributive, this means that we want to prove that for all $x \in P$ and $F \subseteq P$ finite, $x \wedge (\bigvee F) \leq \bigvee\{x \wedge y \ : y \in F\}$. Since x *and* $\bigvee F \in P$, and since $\mathcal{P} = (P, \leq)$ is a lattice, therefore $x \wedge (\bigvee F)$ exists and $x \wedge (\bigvee F)$ is an element of P. Thus by our assumption, $x \wedge (\bigvee F)$ is the join of a finite set of join prime elements. Hence, there exists $G \subseteq P$ finite such that $j \in G$ is join-prime for all $j \in G$ and $x \wedge (\bigvee F) = \bigvee G$. Now we know that $\forall j \in G$, $j \leq \bigvee G = x \wedge (\bigvee F) \leq x$. Similarly, we know that $j \leq \bigvee F$. By the definition of join-prime elements, $F \in P$ is finite, j is join-prime and $j \leq \bigvee F$ together imply that $j \leq y$ for some $y \in F$. Therefore, for each $j \in G, j \leq x$ *and* $j \leq y$ for some $y \in F$ together implies that $j \leq x \wedge y$ for some $y \in F$. Thus, for each $j \in G$, $j \leq \bigvee\{x \wedge y : y \in F\}$. This implies that $\bigvee G = x \wedge (\bigvee F) \leq \bigvee\{x \wedge y : y \in F\}$.

■

Lemma 2.2.19 *Suppose $\mathcal{P} = (P, \leq)$ is a finite lattice. Suppose $b \in P$ and let $F \subseteq \downarrow b$. The following statements are logically equivalent.*

1. *We have $b = \bigvee F$.*
2. *If $a \in P$ is such that $b \nleq a$, then there exists $x \in F$ such that $x \nleq a$.*

Proof. ($\Longrightarrow$) Assume that $b = \bigvee F$. Suppose that $a \in P$ is such that $b \nleq a$. Then $b \notin \downarrow a$. We want to prove that there exists $x \in F$ such that $x \nleq a$. For that, suppose by a way of contradiction that $\forall x \in F,\ x \in \downarrow a$. This implies that a is an upper bound of F. Since $b = \bigvee F$, then $b \leq a$. This contradicts the fact that $b \notin \downarrow a$. We may therefore conclude that there exists $x \in F$ such that $x \nleq a$.

($\Longleftarrow$) Suppose that $a \in P$ is such that $b \nleq a$ implies that there exists $x \in F$ such that $x \nleq a$. Suppose by a way of contradiction that $b \neq \bigvee F$. Note here that since

$F \subseteq \downarrow b$, then b is an upper bound of F. Since $b \neq \bigvee F$, we know $\exists\, c \in P$ such that $c = \bigvee F$ and $c \neq b$. Since b is an upper bound of F, $c = \bigvee F$ implies that $c < b$. Hence $b \not\leq c$. By our assumption, we know $\exists\, x \in F$ such that $x \not\leq c$. This contradicts our assumption that $c = \bigvee F$. We may therefore conclude that $b \neq \bigvee F$.

■

Theorem 2.2.20 *A finite lattice $\mathcal{P} = (P, \leq)$ is distributive if and only if every element is the join of a set of join prime elements.*

Proof. ($\implies$) Suppose that $\mathcal{P} = (P, \leq)$ is a finite distributive lattice and let $b \in P$. We want to find $F \subseteq \mathcal{JP}(\mathcal{P})$ such that $b = \bigvee F$. Observe here that if $b \in \mathcal{JP}(\mathcal{P})$, then $b = b \vee b = \bigvee\{b\}$.

Now suppose that $b > \perp$ and let $a \in P$ such that $b \not\leq a$. For each such a, let $X_a = \{y \in P : y \leq b \textit{ and } y \not\leq a\}$. By applying the reflexivity properties, it result that $b \leq b$. Therefore, $b \in X_a$ and $X_a \neq \emptyset$.

Let $\mathcal{M}in(X_a)$ be the set of all minimal elements of X_a. We begin by showing that $\forall j \in \mathcal{M}in(X_a)$, j is join-prime. Note here that since $\mathcal{P} = (P, \leq)$ is a finite distributive lattice, using lemma 2.2.14, we wish to show that for every finite $G \subseteq P$, $j = \bigvee G \implies j \in G$. Suppose by a way of contradiction that $j = \bigvee G$ and $j \notin G$. Then $x < j\ \forall x \in G$. Since $j \in \mathcal{M}in(X_a)$, then $j \leq b$. By applying the transitivity properties, $x < j \textit{ and } j \leq b$ together implies that $x < b \quad \forall x \in G$. Observe that $x \notin X_a$ for all $x \in G$ since $x < j$ and j is a minimal element of X_a. By construction of X_a, $x \notin X_a$ and $x < b$ together imply that $x \leq a\ \forall x \in G$. This implies that $j = \bigvee G \leq a$, which contradicts the fact that $j \in X_a$. We may therefore conclude that $\forall j \in \mathcal{M}in(X_a)$, j is join-prime.

Now let $F = \bigcup \{\mathcal{M}in(X_a) : b \not\leq a\} = \{x \in P : x \leq b \textit{ and } x \in \mathcal{JP}(\mathcal{P})\}$. We need to show that $b = \bigvee F$. It is clear that b is an upper bound for F, hence $\bigvee F \leq b$. We wish to apply Lemma 2.2.19 to conclude that $b = \bigvee F$. Observe that $F \subseteq \downarrow b$. We proved that if $a \in P$ such that $b \not\leq a$, then there exists $x \in \mathcal{JP}(\mathcal{P})$ such that $x \in \mathcal{M}in(X_a)$. Hence, there exists $x \in F$ such that $x \not\leq a$. Therefore, applying lemma 2.2.19, we obtain that $b = \bigvee F$ where F is a set of join prime elements.

($\Longleftarrow$) Follow directly from lemma 2.2.18

■

Definition 2.2.21 Let $\mathcal{P} = (P, \leq)$ be a finite lattice and let $D \subseteq P$. We say that D is Join-Dense in $\mathcal{P}$ provided every element of P is the join of a subcollection from D. Theorem 2.2.20 tells us the join-prime elements constitute a join-dense subset of any finite distributive lattice.

Lemma 2.2.22 *Let $\mathcal{P} = (P, \leq)$ be any finite distributive lattice, and let $I \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$. If $x = \bigvee I$, then $I = \mathcal{JP}(\downarrow x)$*

Proof. Suppose that $I \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and $x = \bigvee I$. We wish to show $I \subseteq \mathcal{JP}(\downarrow x)$ and $\mathcal{JP}(\downarrow x) \subseteq I$.

Let $y \in I$, then $y \leq \bigvee I = x$. Since y is join prime and $y \leq x$, we know $y \in \mathcal{JP}(\downarrow x)$. Therefore, $I \subseteq \mathcal{JP}(\downarrow x)$. Now let $y \in \mathcal{JP}(\downarrow x)$. Then y is join prime and $y \leq x$.

$$
\begin{aligned}
y \leq x \implies & y = y \wedge x \\
\implies & y = y \wedge (\textstyle\bigvee I) \\
\implies & y = \textstyle\bigvee\{y \wedge z : z \in I\} && \text{[Distributivity Properties]} \\
\implies & y \in \{y \wedge z : z \in I\} && \text{[} y \text{ is join prime and } \{y \wedge z : z \in I\} \text{ is finite]} \\
\implies & y = y \wedge Z \text{ for some } z \in I \\
\implies & y \leq z \text{ for some } Z \in I \\
\implies & y \in I && [I \text{ is a lower set}]
\end{aligned}
$$

Therefore, $\mathcal{JP}(\downarrow x) \subseteq I$

■

Corollary 2.2.23 *If $\mathcal{P} = (P, \leq)$ is any finite distributive lattice, then $\mathcal{P}$ is order-isomorphic to $\mathcal{L}(\mathcal{JP}(\mathcal{P}))$ via the maps $\varrho_p : P \longrightarrow \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and $\varsigma_p : \mathcal{L}(\mathcal{JP}(\mathcal{P})) \longrightarrow$*

P defined by $\varrho_p(x) = \mathcal{JP}(\downarrow x)$ and $\varsigma_p(I) = \bigvee I$.

Proof. We first wish to prove that ς_p and ϱ_p are order homomorphisms. Consider the map $\varrho_p : P \longrightarrow \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ defined by $\varrho_p(x) = \mathcal{JP}(\downarrow x)$. Let $x, y \in P$, and assume that $x \leq y$. This implies that $\mathcal{JP}(\downarrow x) \subseteq \mathcal{JP}(\downarrow y)$. Therefore, $\varrho_p(x) \subseteq \varrho_p(y)$. This allow us to conclude that ϱ_p is an order homomorphism.

We now show that $\varsigma_p : \mathcal{L}(\mathcal{JP}(\mathcal{P})) \longrightarrow P$ defined by $\varsigma_p(I) = \bigvee I$ is an order homomorphism. Let $I, J \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$, and assume that $I \subseteq J$. This implies that $\bigvee I \leq \bigvee J$. Therefore, $\varsigma_p(I) \leq \varsigma_p(J)$. This allow us to conclude that ς_p is indeed an order homomorphism.

Now we wish to show that $\varsigma_p \circ \varrho_p = I_{\mathcal{P}}$ and $\varrho_p \circ \varsigma_p = I_{\mathcal{L}(\mathcal{JP}(\mathcal{P}))}$. Let $I \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and let $x = \bigvee I$. Applying Lemma 2.2.22, it is easy to see that $\varrho_p \circ \varsigma_p(I) = \varrho_p(\bigvee I)$ $= \mathcal{JP}(\downarrow \bigvee \mathcal{I}) = \mathcal{JP}(\downarrow x) = I$. Likewise, $\varsigma_p \circ \varrho_p(x) = \varsigma_p(\mathcal{JP}(\downarrow x)) = \bigvee(\mathcal{JP}(\downarrow x)) = \bigvee I = x$.

■

Figure 1 was an example of an order-isomorphism between a finite poset $\mathcal{P} = (P, \leq)$ and $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$. Figure 2 below is an example of an order-isomorphism between a finite distributive lattice $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ (on the left) and the lattice of lower-sets of join-prime elements $\mathcal{L}(\mathcal{JP}(\mathcal{P}))$ (on the right). The middle diagram correspond to the join-prime elements of $\mathcal{P}$.

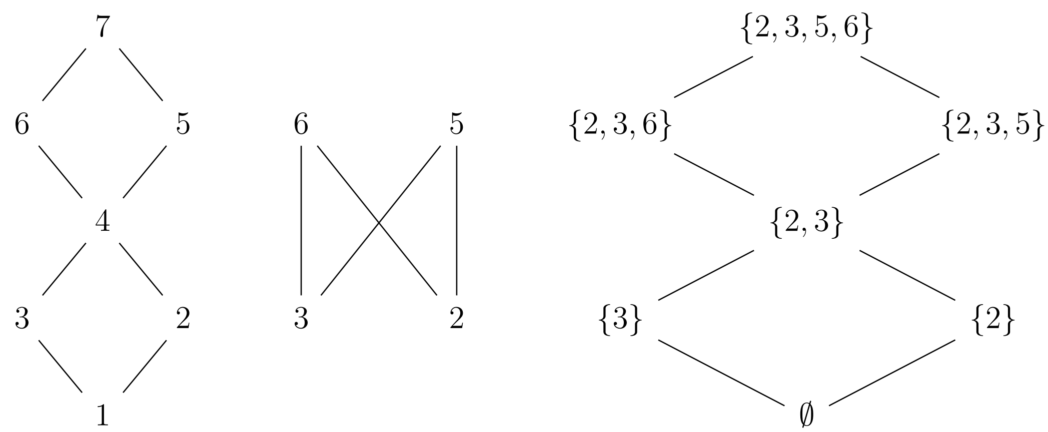

Figure 2: Order Isomorphism between $\mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and $\mathcal{P}$

Now, we introduce some new definitions and we prove some crucial theorems for our future work.

Definition 2.2.24 An element m in a lattice $\mathcal{P} = (P, \leq)$ is meet-prime provided for any finite $F \subseteq P$, the inequality m $\geq \bigwedge F$ implies $m \geq x$ for some $x \in F$. Note that the largest element of a lattice (when it exists) cannot be meet-prime. We will let $\mathcal{MP}(\mathcal{P})$ denote the subposet of meet-prime elements for the lattice $\mathcal{P}$.
Notice that an element of a lattice is meet-prime if and only if it is join-prime in the order-dual of the lattice. (Join-prime elements are sometimes called coprime elements for this reason.) With this in mind, we can deduce a number of useful properties meet-prime elements possess.

• If m is meet prime in a finite lattice, then m is covered by exactly one element.
• If $\mathcal{P}$ is a distributive lattice with greatest element t, then every *coatom* of $\mathcal{P}$ is meet-prime. (An element is a coatom provided it is covered by t.)
• If $\mathcal{P}$ is a distributive lattice, then an element m is meet-prime if and only if $m = \bigwedge F$ implies $m \in F$ for finite $F \subseteq P$.

• If $\mathcal{P}$ is a finite distributive lattice, then every element is the meet of a set of meet-prime elements.

Definition 2.2.25 Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are finite lattices and suppose $f : P \longrightarrow Q$ is a bounded lattice homomorphism. Define a mapping $\tau_f : Q \longrightarrow P$ according to the rule $\tau_f(y) = \bigvee Pre_f(\downarrow y)$. The mapping τ_f is called the upper adjoint *of the function* f.

The importance of the upper adjoint lies with an intimate connection between the posets of meet-prime and join-prime elements in a finite lattice.

It is easy to see that τ_f is an order homomorphism. Indeed, suppose that $a \sqsubseteq b$. It follows that $Pre_f(\downarrow a) \subseteq Pre_f(\downarrow b)$. Hence we know that $\tau_f(a) \leq \tau_f(b)$.

Lemma 2.2.26 *Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are finite lattices and suppose $f : P \longrightarrow Q$ is a bounded lattice homomorphism. For all $a \in P$ and $b \in Q$, we have $a \leq \tau_f(b) \iff f(a) \sqsubseteq b$.*

Proof. ($\Longrightarrow$) Suppose that $a \leq \tau_f(b)$.

$$
\begin{aligned}
a \leq \tau_f(b) &\implies a \vee (\bigvee Pre_f(\downarrow b)) = \bigvee Pre_f(\downarrow b) \\
&\implies f(a \vee (\bigvee Pre_f(\downarrow b))) = f(\bigvee Pre_f(\downarrow b)) \\
&\implies f(a) \vee f(\bigvee Pre_f(\downarrow b)) = f(\bigvee Pre_f(\downarrow b)) && [1] \\
&\implies f(a) \sqsubseteq f(\bigvee Pre_f(\downarrow b)) \\
&\implies f(a) \sqsubseteq \bigvee f(Pre_f(\downarrow b) && [1] \\
&\implies f(a) \sqsubseteq \bigvee(\downarrow b) \\
&\implies f(a) \sqsubseteq b
\end{aligned}
$$

[1] f is a bounded lattice homomorphism.

($\Longleftarrow$) Now suppose that $f(a) \sqsubseteq b$. Fisrt observe that since $f(a) \in \downarrow f(a)$, then $a \in Pre_f(\downarrow f(a))$. Therefore, $a \leq \bigvee Pre_f(\downarrow f(a))$.

$$\begin{aligned} f(a) \sqsubseteq b \quad &\Longrightarrow \tau_f(f(a)) \sqsubseteq \tau_f(b) && [\tau_f \text{ is an order homomorphism}] \\ &\Longrightarrow \bigvee Pre_f(\downarrow f(a)) \leq \tau_f(b) \\ &\Longrightarrow a \leq \bigvee Pre_f(\downarrow f(a)) \leq \tau_f(b) \\ &\Longrightarrow a \leq \tau_f(b) && [\text{transitivity}] \end{aligned}$$

■

Corollary 2.2.27 *Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are finite lattices. If $f : P \longrightarrow Q$ is a bounded lattice homomorphism, then τ_f is a meet homomorphism.*

Proof. Suppose that $f : P \longrightarrow Q$ is a bounded lattice homomorphism and let $F \subseteq Q$ be finite and non-empty. We wish to show that $\tau_f(\bigwedge F) = \bigwedge \tau_f(F)$.

We know that $\tau_f(F) = \{\tau_f(x) : x \in F\}$. It will suffice to show that $\tau_f(\bigwedge F)$ is the greatest lower bound of the set $\tau_f(F)$. Observe that $\bigwedge F \sqsubseteq x \; \forall x \in F$. Since τ_f is an order homomorphism, then $\tau_f(\bigwedge F) \leq \tau_f(x) \; \forall x \in F$. We may therefore conclude that $\tau_f(\bigwedge F)$ is a lower bound of the set $\tau_f(F)$. Suppose now that $y \leq \tau_f(x) \; \forall x \in F$. By Lemma 2.2.26, it result that $f(y) \sqsubseteq x \; \forall x \in F$. This implies that $f(y) \sqsubseteq \bigwedge F$. Hence, Lemma 2.2.26 tells us that $y \leq \tau_f(\bigwedge F)$. We may therefore conclude that $\tau_f(\bigwedge F)$ is the greatest lower bound of the set $\tau_f(F)$.

■

Lemma 2.2.28 *Suppose $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are finite lattices and suppose $f : P \longrightarrow Q$ is a bounded lattice homomorphism. If $m \in \mathcal{MP}(\mathcal{Q})$, then $\tau_f(m) \in \mathcal{MP}(\mathcal{P})$.*

Proof. Assume that m $\in \mathcal{MP}(\mathcal{Q})$. Suppose that $G \subseteq P$ finite such that $\tau_f(m) \geq \bigwedge G$. We want to prove that $\tau_f(m) \geq x$ for some $x \in G$.

$$\begin{aligned}
\bigwedge G \leq \tau_f(m) &\implies f(\bigwedge G) \sqsubseteq m && \text{[Lemma 2.2.26]}\\
&\implies \bigwedge f(G) \sqsubseteq m && \text{[f is a bounded lattice homomorphism]}\\
&\implies f(x) \sqsubseteq m \text{ for some } x \in G && \text{[m is meet prime]}\\
&\implies x \leq \tau_f(m) \text{ for some } x \in G && \text{[Lemma 2.2.26]}
\end{aligned}$$

Now we wish to show that $\tau_f(m) \neq \top_{\mathcal{P}}$ where $\top_{\mathcal{P}}$ is the largest element of $\mathcal{P}$. Since $m \in \mathcal{MP}(\mathcal{Q})$, then $m \neq \top_{\mathcal{Q}}$, where $\top_{\mathcal{Q}}$ is the largest element of Q. Hence, we have $m < \top_{\mathcal{Q}}$. Suppose by a way of contradiction that $\tau_f(m) \sqsupseteq \top_{\mathcal{P}}$. Then by Lemma 2.2.25, we have $m \geq f(\top_{\mathcal{P}})$. Since f is a bounded lattice homomorphism, it results that $m \geq \top_{\mathcal{Q}}$. This contradict the fact that $m < \top_{\mathcal{Q}}$. We may conclude that $\tau_f(m) \sqsubset \top_{\mathcal{P}}$, and therefore it's not the largest element of $\mathcal{P}$.

■

Lemma 2.2.29 *Suppose* $\mathcal{P} = (P, \leq)$, $\mathcal{Q} = (Q, \sqsubseteq)$ *and* $\mathcal{R} = (R, \preceq)$ *are finite lattices. If* $f : P \longrightarrow Q$ *and* $g : Q \longrightarrow R$ *are bounded lattice homomorphisms, then* $\tau_{g\circ f} = \tau_f \circ \tau_g$.

Proof. Let $a \in R$.

$$\begin{aligned}
\tau_f \circ \tau_g(a) &= \tau_f(\tau_g(a))\\
&= \bigvee Pre_f(\downarrow \tau_g(a))\\
&= \bigvee\{x \in P : f(x) \sqsubseteq \tau_g(a)\}\\
&= \bigvee\{x \in P : g(f(x)) \preceq a\} && \text{[lemma 2.2.26]}\\
&= \bigvee\{x \in P : g \circ f(x) \preceq a\}\\
&= Pre_{g\circ f}(\downarrow a)\\
&= \tau_{g\circ f}(a)
\end{aligned}$$

■

Definition 2.2.30 Let $\mathcal{P} = (P, \leq)$ be a finite lattice and suppose a, b $\in P$. We say that the ordered pair <u>(a, b) splits the lattice</u> $\mathcal{P}$ provided $\downarrow a \cap \uparrow b = \emptyset$ *and* $\downarrow a \cup \uparrow b = P$.

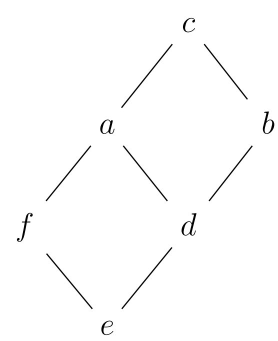

Figure 3: The ordered pair (a, b) splits the lattice $\mathcal{P}$

Consider the lattice $\mathcal{P} = (P, \leq)$ of figure 3 above. Notice here that the ordered pair (a, b) splits the lattice since $\downarrow a \cap \uparrow b = \emptyset$ *and* $\downarrow a \cup \uparrow b = P$.

Lemma 2.2.31 *Let* $\mathcal{P} = (P, \leq)$ *be a finite lattice and suppose* $a, b \in P$. *If* (a, b) *splits* $\mathcal{P}$, *then* a *is meet-prime and* b *is join-prime in* $\mathcal{P}$.

Proof. Assume that $(a\ , b)$ splits $\mathcal{P}$, that is $\downarrow a \cap \uparrow b = \emptyset$ *and* $\downarrow a \cup \uparrow b = P$. We show that b is join-prime, that is for all finite $F \subseteq P,\ b \leq \bigvee F$ implies that $b \leq x$ for some $x \in F$.

Since P is a finite lattice, then without loss of generality, let $F = \{x, y\} \subseteq P$ such that $b \leq x \vee y$. We claim that $b \leq x$ or $b \leq y$. Suppose by a way of contradiction that $b \nleq x$ and $b \nleq y$.

$$\begin{aligned}
b \nleq x \text{ and } b \nleq y &\implies x \notin \uparrow b \text{ and } y \notin \uparrow b \\
&\implies x \in P - (\uparrow b) \text{ and } y \in P - (\uparrow b) \\
&\implies x \in \downarrow a \text{ and } y \in \downarrow a \qquad [\downarrow a \cap \uparrow b = \emptyset \textit{ and } \downarrow a \cup \uparrow b = P] \\
&\implies x \vee y \in \downarrow a \\
&\implies x \vee y \leq a \\
&\implies b \leq a \qquad [\text{ since } b \leq (x \vee y) \leq a]
\end{aligned}$$

This contradicts the fact that $\downarrow a \cap \uparrow b = \emptyset$. We may therefore conclude that $b \leq x$ or $b \leq y$.

Showing that a is meet-prime can be done following similar reasoning.

■

Definition 2.2.32 Let $\mathcal{P} = (P, \leq)$ be a finite lattice and suppose $a \in \mathcal{MP}(\mathcal{P})$ and $b \in \mathcal{JP}(\mathcal{P})$. We let $\underline{jp_{\mathcal{P}}(a) = \bigwedge\{x \in P : x \nleq a\}}$ and $\underline{mp_{\mathcal{P}}(b) = \bigvee\{y \in P : b \nleq y\}}$.

Lemma 2.2.33 *Let $\mathcal{P} = (P, \leq)$ be a finite lattice. If $a \in \mathcal{MP}(\mathcal{P})$ and $b \in \mathcal{JP}(\mathcal{P})$, then $b \nleq mp_{\mathcal{P}}(b)$ and $jp_{\mathcal{P}}(a) \nleq a$.*

Proof. Let $b \in \mathcal{JP}(\mathcal{P})$. We show that $b \nleq mp_{\mathcal{P}}(b)$. Suppose by a way of contradiction that $b \leq mp_{\mathcal{P}}(b)$. Then, $b \leq \bigwedge\{y \in P : b \nleq y\}$. Since b is join-prime, it results that $b \leq y$ for some $y \ngeq b$. This contradiction prove that $b \nleq mp_{\mathcal{P}}(b)$.

Following similar reasoning, we can prove that $jp_{\mathcal{P}}(a) \nleq a$.

■

Lemma 2.2.34 *Let $\mathcal{P} = (P, \leq)$ be a finite lattice. If $a \in \mathcal{MP}(\mathcal{P})$ and $b \in \mathcal{JP}(\mathcal{P})$, then (a, $jp_{\mathcal{P}}(a)$) and $(mp_{\mathcal{P}}(b), b)$ split the lattice $\mathcal{P}$.*

Proof. We show that $(mp_{\mathcal{P}}(b), b)$ splits the lattice $\mathcal{P}$. Since $b \in \mathcal{JP}(\mathcal{P})$, Lemma 2.2.33 tells us that $b \nleq mp_{\mathcal{P}}(b)$. It results that $b \notin \downarrow mp_{\mathcal{P}}(b)$. Therefore, we have $\downarrow mp_{\mathcal{P}}(b) \cap \uparrow b = \emptyset$. In the other hand, $mp_{\mathcal{P}}(b) = \bigvee\{y \in P : b \nleq y\} = \bigvee\{y \in P : y \notin \uparrow b\} = \bigvee\{P - \uparrow b\}$.

Using that $mp_{\mathcal{P}}(b) \notin \uparrow b$ and $mp_{\mathcal{P}}(b) = \bigvee\{P - \uparrow b\}$, we obtain that

$\downarrow mp_{\mathcal{P}}(b) \cup \uparrow b = \mathcal{P}$. Having $\downarrow mp_{\mathcal{P}}(b) \cap \uparrow b = \emptyset$ and $\downarrow mp_{\mathcal{P}}(b) \cup \uparrow b = \mathcal{P}$ together imply that $(mp_{\mathcal{P}}(b), b)$ splits the lattice $\mathcal{P}$.

Proving that (a, $jp_{\mathcal{P}}(a)$) splits the lattice $\mathcal{P}$ can be done using similar reasoning.

■

Theorem 2.2.35 *Let $\mathcal{P} = (P, \leq)$ be a finite lattice. The following statements are true for $a \in \mathcal{MP}(\mathcal{P})$ and $b \in \mathcal{JP}(\mathcal{P})$.*

1− *We have $jp_{\mathcal{P}}(a) \in \mathcal{JP}(\mathcal{P})$*

2− *We have $mp_{\mathcal{P}}(b) \in \mathcal{MP}(\mathcal{P})$*

3− *We have $mp_{\mathcal{P}}(jp_{\mathcal{P}}(a)) = a$ and $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) = b$*

Proof. (Proof for 1) Since $a \in \mathcal{MP}(\mathcal{P})$, then lemma 2.2.34 tells us that (a, $jp_{\mathcal{P}}(a)$) splits the lattice. Hence, lemmma 2.2.31 tells us that $jp_{\mathcal{P}}(a) \in \mathcal{JP}(\mathcal{P})$.

(Proof for 2) Since $b \in \mathcal{JP}(\mathcal{P})$, then lemma 2.2.34 tells us that $(mp_{\mathcal{P}}(b), b)$ splits the lattice. Hence, lemmma 2.2.31 tells us that $mp_{\mathcal{P}}(b) \in \mathcal{MP}(\mathcal{P})$.

(Proof for 3) Let $b \in \mathcal{JP}(\mathcal{P})$. We show that $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) = b$. First observe that $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) = \bigwedge\{x \in P : x \nleq mp_{\mathcal{P}}(b)\}$.

$$
\begin{aligned}
b \in \mathcal{JP}(\mathcal{P}) &\implies b \nleq mp_{\mathcal{P}}(b) && \text{[lemma 2.2.33]}\\
&\implies b \in \{x \in P : x \nleq mp_{\mathcal{P}}(b)\}\\
&\implies b \geq \bigwedge\{x \in P : x \nleq mp_{\mathcal{P}}(b)\}\\
&\implies b \geq jp_{\mathcal{P}}(mp_{\mathcal{P}}(b))
\end{aligned}
$$

Lemma 2.2.34 tells us that since $b \in \mathcal{JP}(\mathcal{P})$, then $(mp_{\mathcal{P}}(b), b)$ splits the lattice $\mathcal{P}$, which implies that either $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) \leq mp_{\mathcal{P}}(b)$ or $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) \geq b$. Suppose by a way of contradiction that $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) \leq mp_{\mathcal{P}}(b)$. This implies $mp_{\mathcal{P}}(b) \geq \bigwedge\{x \in P : x \nleq mp_{\mathcal{P}}(b)\}$. Since $mp_{\mathcal{P}}(b)\}$ is meet prime, then $mp_{\mathcal{P}}(b)\} \geq x$ for some $x \nleq mp_{\mathcal{P}}(b)\}$. This contradiction prove that our assumption is false. We may conclude that $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) \geq b$.

Since we proved that $b \geq jp_{\mathcal{P}}(mp_{\mathcal{P}}(b))$ and $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) \geq b$, we may therefore conclude that $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) = b$.

Following similar strategy, we can prove that $mp_{\mathcal{P}}(jp_{\mathcal{P}}(a)) = a$.

■

Theorem 2.2.35 tells us that we may define mutually inverse functions $jp_{\mathcal{P}} : \mathcal{MP}(\mathcal{P}) \longrightarrow \mathcal{JP}(\mathcal{P})$ and $mp_{\mathcal{P}} : \mathcal{JP}(\mathcal{P}) \longrightarrow \mathcal{MP}(\mathcal{P})$. It is easy to see that these functions are order homomorphisms.
Consider the lattice $\mathcal{P}$ below. In this lattice, it is the case that $a \in \mathcal{MP}(\mathcal{P})$ and $b \in \mathcal{JP}(\mathcal{P})$. It is easy to see how (a, $jp_{\mathcal{P}}(a)$) and $(mp_{\mathcal{P}}(b), b)$ split the lattice $\mathcal{P}$. Observe here that $jp_{\mathcal{P}}(a) \in \mathcal{JP}(\mathcal{P})$ and $mp_{\mathcal{P}}(b) \in \mathcal{MP}(\mathcal{P})$. Moreover, you can easily get that $mp_{\mathcal{P}}(jp_{\mathcal{P}}(a)) = a$ and $jp_{\mathcal{P}}(mp_{\mathcal{P}}(b)) = b$.

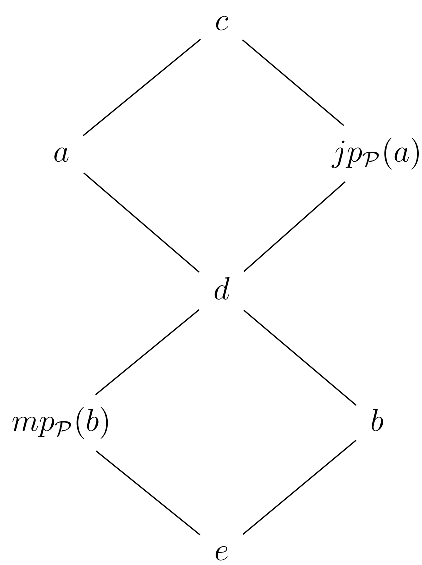

Figure 4: The inverse functions $jp_{\mathcal{P}}$ and $mp_{\mathcal{P}}$

At this point, we have led the order theory groundwork necessary to prove our subsequent results.

2.3 Category Theory

Category theory is a branch of abstract mathematics that focuses on studying the relationships and structures that exist within various mathematical objects and systems. A category consists of objects that represent various mathematical entities, and morphisms that represent the relationships or mappings between these objects. Categorical equivalence is a concept that asserts that two categories are essentially the same, even if their objects and morphisms may look different. Functors play a central role when it comes to proving categorical equivalence between two categories. They are mappings between categories that preserve the structure and relationships between objects and morphisms. In other words, they establish a correspondence between the objects and morphisms in such a way that key properties and relationships are maintained. Since our goal is to prove a categorical equivalence, it is relevant here to provide some category theory definitions. These definitions are coming from [5] and [6].

Definition 2.3.1 A Category $\mathcal{C}$ is a class of objects $Obj(\mathcal{C})$ along with a class of morphisms $Hom(\mathcal{C})$ between objects such that

1. For every pair of morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, there exists a unique morphism $g \circ f : A \longrightarrow C$ called composition of f and g.
2. Composition of morphisms is associative, that is, for every triplet of morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$, we have that $(h \circ g) \circ f = h \circ (g \circ f)$.
3. For every object $A \in \mathcal{C}$, there exists an identity morphism $1_A : A \longrightarrow A$ such that for any morphisms $f : A \longrightarrow B$ and $g : D \longrightarrow A$, $f \circ 1_A = f$ and $1_A \circ g = g$.

For notational simplicity, we denote $Hom(X, Y)$ be the class of all the morphisms from the object X to the object Y in the category $\mathcal{C}$.

As with directed graphs and posets, we can represent categorical objects and morphisms with a diagram. Those diagrams are physical representations of the objects and morphisms. Consider the diagram below. We let $\mathcal{C}$ be a category with A, B, C, and D as objects. Let $f \in Hom(A, C)$, $g \in Hom(C, D)$, $h \in Hom(A, B)$, and $K \in Hom(B, D)$. As you can see, we represent morphisms as arrows between the objects.

We say that the below diagram commute if $g \circ f = k \circ h$. Since functors are used to prove categorical equivalence between two categories, functors must be defined.

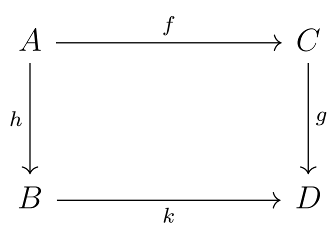

Figure 5: A Commutative Diagram

Definition 2.3.2 A functor $\mathcal{F} = (\mathcal{F}_O, \mathcal{F}_M)$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of maps $\mathcal{F}_O$ and $\mathcal{F}_M$ called object and morphism maps respectively such that :

1. $\mathcal{F}_O : Obj(\mathcal{C}) \longrightarrow Obj(\mathcal{D})$ associate each object A in $\mathcal{C}$ to the object $\mathcal{F}_O[A]$ in $\mathcal{D}$.

2. $\mathcal{F}_M$ associate each morphism $f \in Hom(\mathcal{C})$ to a morphism $g \in Hom(\mathcal{D})$ such that the following two conditions hold:

- $\mathcal{F}_M[1_A] = 1_{\mathcal{F}_M[A]}$

- $\mathcal{F}_M[g \circ f] = \mathcal{F}_M[g] \circ \mathcal{F}_M[f]$ for all $f, g \in Hom(\mathcal{C})$

This definition describe what we call a covariant functor. A covariant functor preserves the direction of morphisms in $\mathcal{C}$, that is if $f \in Hom(A, B)$ in $\mathcal{C}$, then $\mathcal{F}_M[f] \in Hom(\mathcal{F}_O[A], \mathcal{F}_O[B])$ in $\mathcal{D}$. In the other hand, a contravariant functor reverse the order of composition, that is, if $\mathcal{F}$ is a contravariant functor and if $f \in Hom(A, B)$ in $\mathcal{C}$, then $\mathcal{F}_M[f] \in Hom(\mathcal{F}_O[B], \mathcal{F}_O[A])$ in $\mathcal{D}$.

Definition 2.3.3 A covariant functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ yield a categorical equivalence if it satisfies the following three criteria:

1. The functor $\mathcal{F}$ is full, that is, for any $X, Y \in Obj(\mathcal{C})$ and any $g \in Hom(\mathcal{F}_O[X], \mathcal{F}_O[Y])$, there exists an $f\ in\ Hom(X, Y)$ such that $\mathcal{F}_M[f] = g$

2. The functor $\mathcal{F}$ is faithful, that is, if $f, g\ in\ Hom(X, Y)$ and $\mathcal{F}_M[f] = \mathcal{F}_M[g]$, then f = g.

3. The Functor $\mathcal{F}$ is essentially surjective, that is, for every $B \in Obj(\mathcal{D})$, there exists an $A \in Obj(\mathcal{C})$ such that $\mathcal{F}_O[A]$ is isomorphic to B.

Notice here the a functor being full is equivalent to being surjective along maps, while a functor being faithful is equivalent to being injective along maps. A functor being essentially surjective is equivalent to being surjective along objects.

In this thesis, we will define contravariant functors between the categories $\mathcal{DGP}$ and $\mathcal{DGL}$, then we will prove that our functors are full, faithful, and essentially surjective, which yield a dual categorical equivalence between the categories $\mathcal{DGP}$ and $\mathcal{DGL}$.

CHAPTER 3

THE CATEGORIES DIGRAPH, $\mathcal{DGP}$ & $\mathcal{DGL}$

In his thesis, Jordan Crowell [1] proved a categorical equivalence between the category of directed graphs and the category of directed graph posets. We're building on that to prove a dual categorical equivalence between the category of directed graph posets and a category of new objects we will call directed graph lattices.

3.1 The Categories DiGraph and $\mathcal{DGP}$

In this section, we are going to introduce the categories of Digraph and the category of DiGraph Posets ($\mathcal{DGP}$) and define the objects lying inside of these categories. We will then establish the connection between these two categories. All definitions and theorems in this subsection are coming from Crowell's work [1].

Suppose $\mathcal{P} = (P, \leq)$ is a nonempty, finite poset that can be written as the union of maximal chains of length one or three. Let S($\mathcal{P}$) denote the set of suprema of length-three chains and let T($\mathcal{P}$) denote the infima of length-three chains. Let A($\mathcal{P}$) denote the set of elements covered by a member of S($\mathcal{P}$)(or, equivalently, covering an element of T($\mathcal{P}$)). The maximal singleton chains represent elements that are both maximal and minimal in $\mathcal{P}$; it does not not matter how we choose to classify these elements. Divide this collection into disjoint sets $I_1(\mathcal{P})$ and $I_2(\mathcal{P})$. Let $Max(\mathcal{P}) = S(\mathcal{P}) \cup I_1(\mathcal{P})$ and let $Min(\mathcal{P}) = T(\mathcal{P}) \cup I_2(\mathcal{P})$. It is the case that $Max(\mathcal{P}),\ A(\mathcal{P}),\ and\ Min(\mathcal{P})$ are necessarily antichains.

Definition 3.1.1 We say that $\mathcal{P}$ is a directed graph poset provided the following conditions are met.

1. There exists a bijection $\upsilon : Max(\mathcal{P}) \longrightarrow Min(\mathcal{P})$.
2. For all $a \in A(\mathcal{P})$, the sets $\uparrow a \cap S(\mathcal{P})$ and $\downarrow a \cap T(\mathcal{P})$ are singletons.

Note here that throughout this thesis, the bijections υ will be called vertex maps. The elements of $Max(\mathcal{P})$ and $Min(\mathcal{P})$ will be called pseudo-vertices of $\mathcal{P}$, and the elements of $A(\mathcal{P})$ will be called arrows of $\mathcal{P}$.

Definition 3.1.2 A $\underline{DGP - object}$ (DGP) is an ordered pair $(\mathcal{P}, \upsilon)$ where $\mathcal{P}$ is a directed graph poset and $\upsilon : Max(\mathcal{P}) \longrightarrow Min(\mathcal{P})$ is any vertex map.

Definition 3.1.3 Suppose $(\mathcal{P}, \upsilon)$ is a DGP, and suppose $Max(\mathcal{P}) = \{1, ..., n\}$. The vertex map υ induces a natural indexing for the set $Min(\mathcal{P}) = \{x_1, ..., x_n\}$ in a natural way: For $1 \leq i \leq n$, and $x_i \in Min(\mathcal{P})$, we will assume $x_i = \upsilon(i)$. While this convention is not necessary, it will prove very convenient.

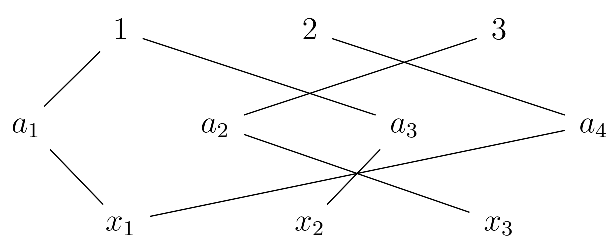

Figure 6: A directed graph poset object $(\mathcal{P},\ \upsilon)$ under the natural indexing.

Above is an example of a directed graph poset object $(\mathcal{P},\ \upsilon)$ under the natural indexing. Note here that $Max(\mathcal{P}) = \{1, 2, 3\}$, $A(\mathcal{P}) = \{a_1, a_2, a_3, a_4\}$, and $Min(\mathcal{P}) = \{x_1, x_2, x_3\}$. Observe here that the sets $I_1(\mathcal{P})$ and $I_2(\mathcal{P})$ of maximal singleton chains are empty since there isn't any element of $\mathcal{P}$ that is incomparable with all the other elements of $\mathcal{P}$. It is convenient to draw these diagrams in such a way that the bijection mapping is "vertical". Consider the directed graph poset object $(\mathcal{P},\ \upsilon)$ represented in the above Hass Diagram, it is easy to see that the vertex map $\upsilon : Max(\mathcal{P}) \longrightarrow Min(\mathcal{P})$ is defined by : $\upsilon(1) = x_1,\ \upsilon(2) = x_2,\ \upsilon(3) = x_3$.

Theorem 3.1.3 *Every DGP-object* $(\mathcal{P},\ \upsilon)$ *induces a directed graph* $\mathcal{G}[(\mathcal{P},\upsilon)]$*, where* $V_{\mathcal{P}}(\mathcal{G}) = \{(x,\upsilon(x)) : x \in Max(\mathcal{P})\}$ *and* $A_{\mathcal{P}}(\mathcal{G}) = A(\mathcal{P})$*, and the source and target maps are defined by* $s_{\mathcal{P}}(a) = (x,\ \upsilon(x))$ *where we have* $\{x\} =\uparrow a \cap S(\mathcal{P})$ *and* $t_{\mathcal{P}}(a) = (y,\upsilon(y))$ *where we have* $\{\upsilon(y)\} =\downarrow a \cap T(\mathcal{P})$*.*

For instance, consider the directed graph poset object $(\mathcal{P},\ \upsilon)$ of figure 6 under the natural indexing. Then $\mathcal{G}[(\mathcal{P},\upsilon)]$ of figure 7 bellow would be the directed gaph induced by $(\mathcal{P},\ \upsilon)$.[1]

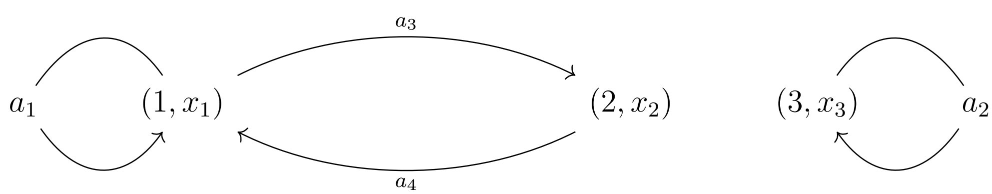

Figure 7: The directed graph $\mathcal{G}[(\mathcal{P},\upsilon)]$ induced by $(\mathcal{P},\ \upsilon)$

Lemma 3.1.4 *Every finite DG-object* $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s, t)$ *induces a DGP-object* $(\mathcal{P}(\mathcal{G}),\upsilon_{\mathcal{G}})$ *where* $\mathcal{P}(\mathcal{G})= (P_{\mathcal{G}},\leq)$ *defined below is a directed graph poset and the assignment* $\upsilon_{\mathcal{G}} : Max(P_{\mathcal{G}}) \longrightarrow Min(P_{\mathcal{G}})$ *defined by* $\upsilon_{\mathcal{G}}(i) = \upsilon_i$ *is a vertex map.*
Suppose that $V(\mathcal{G})=\{\upsilon_1,\upsilon_2,...,\upsilon_n\}$*. Let* $Max\ (P_{\mathcal{G}}) = \{1,...,n\}$*,* $A\ (P_{\mathcal{G}}) = A(\mathcal{G})$*, and* $Min(P_{\mathcal{G}}) = V(\mathcal{G})$*.*
The partial order $\leq$ *is defined by the following rule:*
1- We have $x = y$ *provided* x *and* y *are the same element.*
2- We have $x < y$ *if and only if one of the following conditions is met:*
a. We have $y \in Max(P_{\mathcal{G}})$ *and* $x \in A(P_{\mathcal{G}})$ *and* $s(x) = \upsilon_y$*.*

[1]Note that the arrows a_1 and a_2 are loops, but they're represented this way due to some technical limitations

b. We have $y \in A(P_{\mathcal{G}})$ and $x \in Min(P_{\mathcal{G}})$ and $t(y) = x$.

c. We have $y \in Max(P_{\mathcal{G}})$ and $x \in Min(P_{\mathcal{G}})$ and there exist $a \in A\ (\mathcal{G})$ such that $s(a) = \upsilon_y$ and $t(a) = x$.

Below is an example of a DG-object $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s, t)$ (from the left) and the DGP-object ($\mathcal{P}(\mathcal{G})$,$\upsilon_{\mathcal{G}}$) (from the right) induced by $\mathcal{G}$.

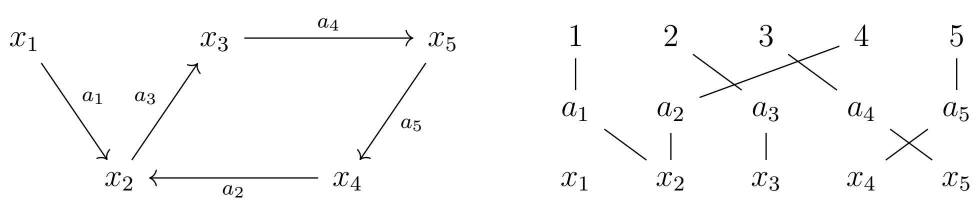

Figure 8: DGP-object ($\mathcal{P}(\mathcal{G})$,$\upsilon_{\mathcal{G}}$) induced by DG-object $\mathcal{G}$

Definition 3.1.5 *Suppose that ($\mathcal{P}$, υ) and ($\mathcal{Q}$, u) are DGP's. A DGP-morphism is an order homomorphism map $F : P \longrightarrow Q$ with the following properties.*

•$F(Max(\mathcal{P})) \subseteq Max(\mathcal{Q})$

•$F(Min(\mathcal{P})) \subseteq Min(\mathcal{Q})$

•$F(A(\mathcal{P})) \subseteq A(\mathcal{Q})$

• For all $x \in Max(\mathcal{P})$, we have $F(\upsilon(x)) = u(F(x))$.

In other words, F is a strict order homomorphism that respect the bijection.

Given any nonempty set X, we will let 1_X denote the identity map from X to X. That is, $1_X : X \longrightarrow X$ is defined by $1_X(a) = a$ for all $a \in X$.

Theorem 3.1.6 *The class $\mathcal{DGP}$ consisting of all DGP-objects coupled with DGP-morphisms constitutes a category in which morphism composition is function composition.*

Theorem 3.1.7 *The class DiGraph consisting of all finite DG-objects coupled with DG-morphisms constitutes a category in which morphism composition is component-wise function composition. The identity morphism for any DG-object* $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), s, t)$ *is the pair* $1_{\mathcal{G}} = (1_V, 1_A)$.

Definition 3.1.8 Two DGP-objects $(\mathcal{P}, \upsilon)$ and $(\mathcal{Q}, u)$ are isomorphic provided there is a DGP-isomorphism between them. A DGP-isomorphism is a pair (F, G) where

- $F\ is\ a\ DGP - morphism\ from$ $(\mathcal{P}, \upsilon)$ to $(\mathcal{Q}, u)$ and G is a DGP-morphism from $(\mathcal{Q}, u)$ to $(\mathcal{P}, \upsilon)$.
- $G \circ F = 1_P$ and $F \circ G = 1_Q$.

In other words, (F, G) constitutes an order-isomorphism between $\mathcal{P}$ and $\mathcal{Q}$ whose component functions respect the bijections υ and u.

3.2 The Categories $\mathcal{DGP}$ and $\mathcal{DGL}$

As with the categories DiGraph and $\mathcal{DGP}$, we will construct the category of directed graph lattices $\mathcal{DGL}$. We will establish the connections between the category of directed graph posets $\mathcal{DGP}$ and the category of directed graph lattices $\mathcal{DGL}$. It's worth noting that this section marks the begining of our original work.

Definition 3.2.1 Let $\mathcal{P} = (P, \leq)$ be a finite distributive lattice. We say $\mathcal{P}$ is a digraph lattice provided $\mathcal{JP}(\mathcal{P})$ is a digraph poset.

Notice here that in Theorem 2.2.17, we proved that $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ is order-isomorphic to $\mathcal{P}$ via the maps $\eta_p : P \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{P}))$ and $\vartheta_p : \mathcal{JP}(\mathcal{L}(\mathcal{P})) \longrightarrow P$ defined by the rules $\eta_p(x) = \downarrow x$ and $\vartheta_p : (\downarrow x) = x$. Hence, Theorem 2.2.17 tells us that every digraph poset induces a digraph lattice, namely its poset of lower sets.

Definition 3.2.2 Suppose $\mathcal{P} = (P, \leq)$ is a digraph lattice. A mapping $\mu : P \longrightarrow P$ is called a meta vertex map provided the following criteria are met.

1. The mapping μ is a join homomorphism.
2. The set $\downarrow \mu(x)$ contains only minimal members of $\mathcal{JP}(\mathcal{P})$.
3. The set $\downarrow x \cap Max(\mathcal{JP}(\mathcal{P}))$ contains the same number of elements as $\downarrow \mu(x) \cap Min(\mathcal{JP}(\mathcal{P}))$.
4. The restriction $\hat{\mu}$ of the mapping μ to $Max(\mathcal{JP}(\mathcal{P}))$ is a vertex map.

Let $\mathcal{P} = (P, \leq)$ be a digraph lattice with smallest element b and suppose $\mu : P \longrightarrow P$ is a vertex map. Observe here that Criterion 3 of Definition 3.3.2 tells us that we have $\mu(x) = b$ if and only if $\downarrow x$ contains no maximal join-prime elements. Note also that Criteria 2 and 3 together tell us that $\mu(x) \in Min(\mathcal{JP}(\mathcal{P}))$ if and only if $x \in Max(\mathcal{JP}(\mathcal{P}))$.

Definition 3.2.3 A <u>DGL object</u> is a pair $(\mathcal{P}, \mu)$ where $\mathcal{P} = (P, \leq)$ is a digraph lattice and $\mu : P \longrightarrow P$ is a meta vertex map.

Lemma 3.2.4 *Let μ_v be the map $\mu_v : \mathcal{L}(\mathcal{Q}) \longrightarrow \mathcal{L}(\mathcal{Q})$ defined by $\mu_v(I) = \{v(x) : x \in Max(\mathcal{Q}) \cap I\}$. If $(\mathcal{Q}, v)$ is a DGP-object, then $(\mathcal{L}(\mathcal{Q}), \mu_v)$ is a DGL-object. If $(\mathcal{P}, \mu)$ is a DGL-object, then $(\mathcal{JP}(\mathcal{P}), \hat{\mu})$ is a DGP-object.*

Proof. We first show that if $(\mathcal{Q}, v)$ is a DGP-object, then $(\mathcal{L}(\mathcal{Q}), \mu_v)$ is a DGL-object. By theorem 2.2.17, every digraph poset induces a digraph lattice, namely its poset of lower sets. Hence, since $\mathcal{Q}$ is a digraph poset, then $\mathcal{L}(\mathcal{Q})$ is a digraph lattice. Thus, to prove that $(\mathcal{L}(\mathcal{Q}), \mu_v)$ is a DGL-object, we just show that μ_v is a meta vertex map by showing each criterion required for meta vertex maps.

Note here that $\forall\ I \in \mathcal{L}(\mathcal{Q})$, $\mu_v(I)$ is a possibly empty subset of $Min(Q)$ and therefore is indeed a lower set of $\mathcal{Q}$.

We show now that μ_v is a join homomorphism. Let $I_1, I_2 \in \mathcal{L}(\mathcal{Q})$. Then,

$$\begin{aligned}
\mu_v(I_1 \vee I_2) &= \mu_v(I_1 \cup I_2) \\
&= \{v(x) : x \in Max(\mathcal{Q}) \cap (I_1 \cup I_2)\} \\
&= \{v(x) : x \in Max(\mathcal{Q}) \cap I_1)\} \bigcup \{v(x) : x \in Max(\mathcal{Q}) \cap I_2)\} \\
&= \mu_v(I_1) \bigcup \mu_v(I_2) \\
&= \mu_v(I_1) \bigvee \mu_v(I_2)
\end{aligned}$$

We know that v is a bijection from $Max(\mathcal{Q})$ to $Min(\mathcal{Q})$, we also know from Theorem 2.2.17 that $\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ is order-isomorphic to $\mathcal{Q}$. Therefore, we may conclude that the set $\downarrow I \cap Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q})))$ contains the same number of elements as $\downarrow \mu_v(I) \cap Min(\mathcal{JP}(\mathcal{L}(\mathcal{Q})))$. Moreover, by definition of μ_v, $\mu_v(I) \subseteq Min(\mathcal{Q})$, which implies that $\downarrow \mu_v(I) \subseteq Min(\mathcal{JP}(\mathcal{L}(\mathcal{Q})))$. We may therefore conclude $\downarrow \mu_v(I)$ contains only minimal members of $\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$.

Now since $\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ is order-isomorphic to $\mathcal{Q}$, then $Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q}))) = \{\downarrow x : x \in Max(Q)\}$. Hence, $\forall I \in Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q})))$, $\widehat{\mu_\upsilon(I)} = \{\upsilon(x)\} \in Min(\mathcal{JP}(\mathcal{L}(\mathcal{Q})))$. It follows that $\widehat{\mu_\upsilon}$ is a vertex map since υ is a vertex map.

We now wish to show that if $(\mathcal{P}, \mu)$ is a DGL-object, then $(\mathcal{JP}(\mathcal{P}), \hat{\mu})$ is a DGP-object. This is trivial since by the definition of digraph lattice, $\mathcal{P}$ is a digraph lattice provided $\mathcal{JP}(\mathcal{P})$ is a digraph poset. Also, by the definition of meta vertex maps, the restriction $\hat{\mu}$ of the mapping μ to $Max(\mathcal{JP}(\mathcal{P}))$ is a vertex map.

■

Corollary 3.2.5 *If $(\mathcal{Q}, \upsilon)$ is a DGP-object, then $(\mathcal{Q}, \upsilon)$ is isomorphic to $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_\upsilon})$ in the category $\mathcal{DGP}$.*

$$\begin{array}{ccc} Max(\mathcal{Q}) & \xrightarrow[\eta_{\mathcal{Q}}(x)=\downarrow x]{} & Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q})) \\ \Big\downarrow{\scriptstyle \upsilon} & & {\scriptstyle \widehat{\mu_\upsilon}}\Big\downarrow \\ Min(\mathcal{Q}) & \xleftarrow{\vartheta_{\mathcal{Q}}:(\downarrow x)=x} & Min(\mathcal{JP}(\mathcal{L}(\mathcal{Q})) \end{array}$$

Figure 9: Diagram for corollary 3.2.5

Proof. We already proved in Theorem 2.2.17 that any digraph poset $\mathcal{Q} = (Q, \leq)$ is order-isomorphic to the poset $\mathcal{JP}(\mathcal{L}(\mathcal{Q})$ via the pair (η_Q, ϑ_Q) where $\eta_Q : Q \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ and $\vartheta_Q : \mathcal{JP}(\mathcal{L}(\mathcal{Q})) \longrightarrow Q$ are defined by the rules $\eta_Q(x) =\downarrow x$ and $\vartheta_Q : (\downarrow x) = x$.

We wish to show first that $\eta_Q(x)$ is a DGP-morphism from $(\mathcal{Q}, \upsilon)$ to $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_\upsilon})$. It is clear that the first three conditions of Definition 3.1.5 are satisfied. It is left to show that $\forall x \in Max(\mathcal{Q})$, $\eta_Q \circ \upsilon(x) = \widehat{\mu_\upsilon} \circ \eta_Q(x)$.

$$\begin{aligned}
\eta_Q(\upsilon(x)) &= \downarrow \upsilon(x) \\
&= \{\ \upsilon(x)\} && [1] \\
&= \mu_\upsilon(\downarrow x) && [2] \\
&= \mu_\upsilon(\eta_Q(x)) \\
&= \widehat{\mu_\upsilon}(\eta_Q(x)) && [3]
\end{aligned}$$

[1] Because $\upsilon(x) \in Min(\mathcal{Q})$.

[2] Since $x \in Max(\mathcal{Q})$, $Max(\mathcal{Q}) \cap \downarrow x = x$, which by definition of μ_υ implies that $\mu_\upsilon(\downarrow x) = \{\upsilon(x)\}$.

[3] $x \in Max(\mathcal{Q})$ implies $\eta_Q(x) \in Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$.

We now wish to show that ϑ_Q is a DGP-morphism from $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_\upsilon})$ to $(\mathcal{Q}, \upsilon)$. It is clear that the first three conditions of Definition 3.1.5 are satisfied. It is left to show that $\forall \downarrow x \in Max(\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$, we have $\vartheta_\mathcal{Q} \circ \widehat{\mu_\upsilon}(\downarrow x) = \upsilon \circ \vartheta_\mathcal{Q}(\downarrow x)$.

$$\begin{aligned}
\vartheta_\mathcal{Q}(\widehat{\mu_\upsilon})(\downarrow x) &= \vartheta_\mathcal{Q}(\mu_\upsilon)(\downarrow x) \\
&= \vartheta_\mathcal{Q}(\{\upsilon(x)\}) \\
&= \vartheta_\mathcal{Q}(\downarrow \upsilon(x)) \\
&= \upsilon(x) \\
&= \upsilon(\vartheta_\mathcal{Q}(\downarrow x))
\end{aligned}$$

Finally, since $\eta_Q(\vartheta_Q(\downarrow x)) = \eta_Q(x) = \downarrow x$ and $\vartheta_Q(\eta_Q(x)) = \vartheta_Q(\downarrow x) = x$, it follows that $\eta_Q \circ \vartheta_Q = 1_{\mathcal{JP}(\mathcal{L}(\mathcal{Q})}$ and $\vartheta_Q \circ \eta_Q = 1_\mathcal{Q}$.

We may therefore conclude that $(\mathcal{Q}, \upsilon)$ is isomorphic to $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_\upsilon})$ in the category $\mathcal{DGP}$.

■

Lemma 3.2.6 *Suppose that* $(\mathcal{P}, \mu)$ *is a DGL-object and suppose* $\mu : P \longrightarrow P$ *is a meta vertex map. For all* $y \in P$*, it is the case that* $\mu(y) = \varsigma_p(\mu_{\widehat{\mu}}(\varrho_p(y)))$*. In particular,* $\varrho_p \circ \mu = \mu_{\widehat{\mu}} \circ \varrho_p$ *and* $\varsigma_p \circ \mu_{\widehat{\mu}} = \mu \circ \varsigma_p$*.*

Proof. Recall that in Corollary 2.2.23, we proved that if $\mathcal{P} = (P, \leq)$ is any finite distributive lattice, then $\mathcal{P}$ is order-isomorphic to $\mathcal{L}(\mathcal{JP}(\mathcal{P}))$ via the maps
$\varrho_p : P \longrightarrow \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and $\varsigma_p : \mathcal{L}(\mathcal{JP}(\mathcal{P})) \longrightarrow P$ defined by $\varrho_p(x) = \mathcal{JP}(\downarrow x)$ and $\varsigma_p(I) = \bigvee I$.

Moreover, Lemma 3.2.4 tells us that if $(\mathcal{P}, \mu)$ is a DGL-object, then $(\mathcal{JP}(\mathcal{P}), \hat{\mu})$ is a DGP-object, which in turn implies that $(\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\widehat{\mu}})$ is a DGL-object such that $\mu_{\widehat{\mu}}(I)= \{ \ \widehat{\mu}(x) : x \in Max(\mathcal{JP}(\mathcal{P})) \cap I\} = \{\mu(x) : x \in Max(\mathcal{JP}(\mathcal{P})) \cap I\}$.

Let us now prove that for all $y \in P$, it is the case that $\mu(y) = \varsigma_p(\mu_{\widehat{\mu}}(\varrho_p(y)))$. Let $y \in P$, then there exists exactly one $I_y \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ such that $y = \bigvee I_y$, namely $I_y =\downarrow y \cap \mathcal{JP}(\mathcal{P}) = \varrho_p(y)$.

$$
\begin{aligned}
\mu(y) &= \mu(\bigvee I_y) \\
&= \mu(\bigvee(Max(\mathcal{JP}(\mathcal{P}) \ ) \cap I_y)) \\
&= \bigvee\{ \ \mu(x) : x \in Max(\mathcal{JP}(\mathcal{P})) \cap I_y\} \qquad [\ \mu \text{ is join homomorphism }] \\
&= \bigvee \mu_{\widehat{\mu}}(I_y) \\
&= \varsigma_p(\mu_{\widehat{\mu}}(I_y)) \\
&= \varsigma_p(\mu_{\widehat{\mu}}(\varrho_p(y)))
\end{aligned}
$$

■

Corollary 3.2.7 *If* $\mathcal{P} = (P, \leq)$ *is any finite lattice, then the subposets* $\mathcal{MP}(\mathcal{P})$ *and* $\mathcal{JP}(\mathcal{P})$ *are order isomorphic. In particular, if* $\mathcal{P} = (P, \leq)$ *is any digraph lattice, then* $\mathcal{MP}(\mathcal{P})$ *is a digraph poset.*

Proof. Recall that Theorem 2.2.35 tells us that we may define mutually inverse functions $jp_{\mathcal{P}} : \mathcal{MP}(\mathcal{P}) \longrightarrow \mathcal{JP}(\mathcal{P})$ and $mp_{\mathcal{P}} : \mathcal{JP}(\mathcal{P}) \longrightarrow \mathcal{MP}(\mathcal{P})$ defined by the rules $jp_{\mathcal{P}}(a) = \bigwedge\{x \in P : x \nleq a\}$ and $mp_{\mathcal{P}}(b) = \bigvee\{y \in P : b \nleq y\}$. We show that these functions are order homomorphisms.

Observe that for $a_1 \in \mathcal{MP}(\mathcal{P})$, $jp_{\mathcal{P}}(a_1) = \bigwedge\{x \in P : x \nleq a_1\} = \bigwedge\{x \in P : x \notin \downarrow a_1\}$.
Now let $a_1, a_2 \in \mathcal{MP}(\mathcal{P})$ such that $a_1 \leq a_2$.

$$\begin{aligned} a_1 \leq a_2 &\implies \downarrow a_1 \subseteq \downarrow a_2 \\ &\implies P - \downarrow a_2 \subseteq P - \downarrow a_1 \\ &\implies \bigwedge(P - \downarrow a_1) \leq \bigwedge(P - \downarrow a_2) \\ &\implies jp_{\mathcal{P}}(a_1) \leq jp_{\mathcal{P}}(a_2) \end{aligned}$$

Therefore, $jp_{\mathcal{P}}$ is an order homomorphism. Proving that $mp_{\mathcal{P}}$ is an order homomorphism can be done using similar reasoning.
Since we proved in Theorem 2.2.35 that $mp_{\mathcal{P}} \circ jp_{\mathcal{P}} = 1_{\mathcal{MP}(\mathcal{P})}$ and $jp_{\mathcal{P}} \circ mp_{\mathcal{P}} = 1_{\mathcal{JP}(\mathcal{P})}$, we may therefore conclude that $\mathcal{MP}(\mathcal{P})$ and $\mathcal{JP}(\mathcal{P})$ are order isomorphic.

■

Suppose that $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$ are finite lattices and suppose $f : P \longrightarrow Q$ is a bounded lattice homomorphism. Let $\widehat{\tau_f}$ denote the restriction of τ_f to $\mathcal{MP}(\mathcal{P})$. We define the map $jp[f] : \mathcal{JP}(\mathcal{Q}) \longrightarrow \mathcal{JP}(\mathcal{P})$ by the rule $jp[f](a) = jp_{\mathcal{P}} \circ \widehat{\tau_f} \circ mp_{\mathcal{Q}}(a)$.

Definition 3.2.8 Suppose that $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ are DGL-objects. A DGL-morphism is a bounded lattice homomorphism $f : P \longrightarrow Q$ which satisfies the following additional criteria.

1. The function f "respects the meta vertex maps"; that is, for all $x \in P$, we have $f(\mu_1(x)) = \mu_2(f(x))$.
2. The function $jp[f]$ is a DGP-morphism.

Theorem 3.2.9 *The class $\mathcal{DGL}$ consisting of all DGL-objects coupled with DGL-morphisms constitutes a category in which morphism composition is function composition. For each DGL-object $(\mathcal{P}, \mu)$, the identity morphism is the identity map $1_{\mathcal{P}}$.*

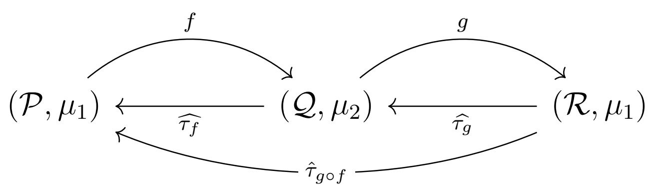

Figure 10: Diagram for Theorem 3.2.9

Proof. Let $f : P \longrightarrow Q$ and $g : Q \longrightarrow R$ be DGL-morphisms. By Lemma 2.2.29, $jp[g \circ f] = jp_{\mathcal{P}} \circ \hat{\tau}_{g\circ f} \circ mp_{\mathcal{R}} = jp_{\mathcal{P}} \circ \widehat{\tau_f} \circ \widehat{\tau_g} \circ mp_{\mathcal{R}}$. Hence, $jp[g \circ f]$ is a DGP-morphism, and $g \circ f$ is indeed a DGL-morphism. The associativity and uniqueness of morphism composition is guaranteed by the properties of function composition.

If $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ are DGL-objects, then the identity morphisms $1_{\mathcal{P}}$ and $1_{\mathcal{Q}}$ are such that $1_{\mathcal{Q}} \circ f = f$ and $f \circ 1_{\mathcal{P}} = f$ for all DGL-morphisms $f : P \longrightarrow Q$.

■

CHAPTER 4

FUNCTORS AND CATEGORICAL EQUIVALENCE

In this section, we seek to show a dual categorical equivalence between the categories $\mathcal{DGP}$ and $\mathcal{DGL}$ defined in the previous sections. We do that by defining the functors that yield the dual equivalence of these categories.

4.1 The contravariant functors DiGP and DiGL

We seek to define our functors DiGP and DiGL by defining their objects and morphisms maps, and then prove that our functors are well defined up to isomorphism classes on each categories.

Definition 4.1.1 Suppose $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ are DGL-objects. We say these objects are isomorphic provided there exist mutually inverse DGL-morphisms $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$.

Lemma 4.1.2 *If $(\mathcal{P}, \mu)$ is a DGL-object, then $jp[\varrho_p]$ and $jp[\varsigma_p]$ are DGP-morphisms.*
Proof. We prove that $jp[\varrho_p]$ is a DGP-morphism, and $jp[\varsigma_p]$ is a DGP-morphism can be proven using almost identical argument.
Theorem 2.2.17 tells us that the elements of $\mathcal{JP}(\mathcal{L}(\mathcal{JP}(\mathcal{P})))$ are precisely the principle lower sets of $\mathcal{JP}(\mathcal{P})$. Thus, for all $J \in JP(L(JP(P)))$, there exists a unique $x \in \mathcal{JP}(\mathcal{P})$ such that $J = \downarrow x$.
Corollary 2.2.23 tells us that $\mathcal{P}$ is order-isomorphic to $\mathcal{L}(\mathcal{JP}(\mathcal{P}))$ via the maps $\varrho_p : P \longrightarrow \mathcal{L}(\mathcal{JP}(\mathcal{P}))$ and $\varsigma_p : \mathcal{L}(\mathcal{JP}(\mathcal{P})) \longrightarrow P$ defined by the rules $\varrho_p(x) = \mathcal{JP}(\downarrow x)$ and $\varsigma_p(I) = \bigvee I$. Hence, it is the case that $Pre_{\varrho_p}(I) = \varsigma_p(I) = \bigvee I$.

Recall that a DGP-morphism is an order homomorphism map $F : P \longrightarrow Q$ with the properties that, F(Max($\mathcal{P}$)) $\subseteq Max(\mathcal{Q})$, $F(Min(\mathcal{P})) \subseteq Min(\mathcal{Q})$, $F(A(\mathcal{P})) \subseteq A(\mathcal{Q})$, and for all $x \in Max(\mathcal{P})$, we have $F(\upsilon(x)) = u(F(x))$.

For notational simplicity, we will let $\mathcal{Q} = \mathcal{L}(\mathcal{JP}(\mathcal{P}))$. We begin by showing that $jp[\varrho_p]$ is an order homomorphism. We know that $jp[\varrho_p] = jp_{\mathcal{P}} \circ \widehat{\tau_{\varrho_p}} \circ mp_{\mathcal{Q}}$. Since $jp_{\mathcal{P}}, \widehat{\tau_{\varrho_p}}$, and $mp_{\mathcal{Q}}$ are all order homomorphisms, it follows that $jp[\varrho_p]$ is a strict order homomorphisms.

We now show that $jp[\varrho_p](Max(\mathcal{JP}(\mathcal{Q}))) \subseteq Max(\mathcal{JP}(\mathcal{P}))$. Proving that $jp[\varrho_p](Min(\mathcal{JP}(\mathcal{Q}))) \subseteq Min(\mathcal{JP}(\mathcal{P}))$ and $jp[\varrho_p](A(\mathcal{JP}(\mathcal{Q}))) \subseteq A(\mathcal{JP}(\mathcal{P}))$ can be done similalirly.

Since we know that for all $x \in \mathcal{JP}(\mathcal{P}), \downarrow x \in \mathcal{JP}(\mathcal{Q})$, thus we have

$$\begin{aligned}
\widehat{\tau_{\varrho_p}} \circ mp_{\mathcal{Q}}(\downarrow x) &= \bigvee Pre_{\varrho_p}(\downarrow mp_{\mathcal{Q}}(\downarrow x)) \\
&= \bigvee Pre_{\varrho_p}(mp_{\mathcal{Q}}(\downarrow x)) && [\widehat{\tau_{\varrho_p}} \text{ is an order homomorphism}] \\
&= \bigvee \varsigma_P(mp_{\mathcal{Q}}(\downarrow x)) \\
&= Pre_{\varrho_p}(mp_{\mathcal{Q}}(\downarrow x)). && [\varrho_p \text{ is a bijection with inverse } \varsigma_p]
\end{aligned}$$

Let $\downarrow x \in Max(\mathcal{JP}(\mathcal{Q}))$ for some $x \in \mathcal{JP}(\mathcal{P})$. We have $jp[\varrho_p](\downarrow x) = jp_{\mathcal{P}} \circ \widehat{\tau_{\varrho_p}} \circ mp_{\mathcal{Q}}(\downarrow x) = jp_{\mathcal{P}} \circ \varsigma_p \circ mp_{\mathcal{Q}}(\downarrow x)$.

Hence,

$$\begin{aligned}
\downarrow x \in Max(\mathcal{JP}(\mathcal{Q})) &\iff mp_{\mathcal{Q}}(\downarrow x) \in Max(\mathcal{MP}(\mathcal{Q})) && [1] \\
&\iff \varsigma_p \circ mp_{\mathcal{Q}}(\downarrow x) \in Max(\mathcal{MP}(\mathcal{P})) && [2] \\
&\iff jp_{\mathcal{P}} \circ \varsigma_p \circ mp_{\mathcal{Q}}(\downarrow x) \in Max(\mathcal{JP}(\mathcal{P})). && [3]
\end{aligned}$$

[1] $mp_{\mathcal{Q}}$ is an order homomorphism from $\mathcal{JP}(\mathcal{Q})$ to $\mathcal{MP}(\mathcal{Q})$.

[2] ς_p is an order homomorphism from $\mathcal{Q}$ to $\mathcal{P}$.

[3] $jp_{\mathcal{P}}$ is an order homomorphism from $\mathcal{MP}(\mathcal{P})$ to $\mathcal{JP}(\mathcal{P})$.

We may therefore conclude that $jp[\varrho_p](Max(\mathcal{JP}(\mathcal{Q}))) \subseteq Max(\mathcal{JP}(\mathcal{P}))$.

Now since $(\mathcal{P}, \mu)$ is a DGL-object, then $\mu_{\hat{\mu}} : \mathcal{Q} \longrightarrow \mathcal{Q}$ defined by $\mu_{\hat{\mu}}(I) = \{\hat{\mu}(x) : x \in Max(\mathcal{JP}(\mathcal{P})) \cap I\}$ is a meta vertex map. The restriction $\widehat{\mu}_{\hat{\mu}}$ of the mapping $\mu_{\hat{\mu}}$ to $Max(\mathcal{JP}(\mathcal{Q}))$ is a vertex map. Thus, $\widehat{\mu}_{\hat{\mu}} : Max(\mathcal{JP}(\mathcal{Q})) \longrightarrow Min(\mathcal{JP}(\mathcal{Q}))$ is defined by $\widehat{\mu}_{\hat{\mu}}(\downarrow x) = \{\hat{\mu}(x) : x \in Max(\mathcal{JP}(\mathcal{P}))\}$.

We now wish to show that $jp[\varrho_p]$ respects the bijections, that is $\forall \downarrow x \in Max(\mathcal{JP}(\mathcal{Q}))$, $jp[\varrho_p](\widehat{\mu_{\hat{\mu}}}(\downarrow x)) = \hat{\mu}(jp[\varrho_p])(\downarrow x)$.

Let's first prove that $\bigvee(mp_{\mathcal{Q}}(\downarrow \hat{\mu}(x))) = mp_{\mathcal{P}}(\hat{\mu}(x))$. Recall that we proved in Corollary 2.2.23 that $\mathcal{P}$ is order-isomorphic to $\mathcal{Q}$ via the maps $\varrho_p : P \longrightarrow \mathcal{Q}$ and $\varsigma_p : \mathcal{Q} \longrightarrow P$. Hence, it is the case that $\downarrow \hat{\mu}(x) \in Min(\mathcal{JP}(\mathcal{Q})) \iff \hat{\mu}(x) \in Min(\mathcal{JP}(\mathcal{P})$.

Observe that
$$\begin{aligned}\bigvee(mp_{\mathcal{Q}}(\downarrow \hat{\mu}(x))) &= \bigvee[\cup\{I \in \mathcal{Q} : \downarrow \hat{\mu}(x) \nsubseteq I\}] \\ &= \bigvee\{y \in P : \hat{\mu}(x) \nleq y\} \\ &= mp_{\mathcal{P}}(\hat{\mu}(x))\end{aligned}$$

Observe also that $\downarrow x \in Max(\mathcal{JP}(\mathcal{Q})) \implies x \in Max(\mathcal{JP}(\mathcal{P})) \implies \widehat{\mu_{\hat{\mu}}}(\downarrow x) = \{\hat{\mu}(x)\} = \downarrow \hat{\mu}(x)$.

Now let $\downarrow x \in Max(\mathcal{JP}(\mathcal{Q}))$.

$$\begin{aligned}jp[\varrho_p](\widehat{\mu_{\hat{\mu}}}(\downarrow x)) &= jp[\varrho_p](\downarrow \hat{\mu}(x)) && \text{[Proved above]} \\ &= jp_{\mathcal{P}} \circ \widehat{\tau_{\varrho_p}} \circ mp_{\mathcal{Q}}(\downarrow \hat{\mu}(x)) \\ &= jp_{\mathcal{P}} \circ \varsigma_p(mp_{\mathcal{Q}}(\downarrow \hat{\mu}(x))) \\ &= jp_{\mathcal{P}}(\bigvee(mp_{\mathcal{Q}}(\downarrow \hat{\mu}(x)))) \\ &= jp_{\mathcal{P}}(mp_{\mathcal{P}}(\hat{\mu}(x))) && \text{[Proved above]} \\ &= jp_{\mathcal{P}} \circ mp_{\mathcal{P}}(\hat{\mu}(x))) \\ &= \hat{\mu}(x)\end{aligned}$$

In the other hand observe that $\downarrow x \in Max(\mathcal{JP}(\mathcal{Q})) \implies jp[\varrho_p](\downarrow x) = x \implies \hat{\mu}(jp[\varrho_p](\downarrow x)) = \hat{\mu}(x)$.

Since we have $jp[\varrho_p](\widehat{\mu_{\hat{\mu}}}(\downarrow x)) = \hat{\mu}(x)$ and $\hat{\mu}(jp[\varrho_p](\downarrow x)) = \hat{\mu}(x)$, we may conclude $jp[\varrho_p](\widehat{\mu_{\hat{\mu}}}(\downarrow x)) = \hat{\mu}(jp[\varrho_p])(\downarrow x)$.

■

Lemma 4.1.3 *If $(\mathcal{P}, \mu)$ is a DGL-object, then* $\bigvee(I \cup J) = (\bigvee I) \vee (\bigvee J)$ *and* $\bigvee(I \cap J) = (\bigvee I) \wedge (\bigvee J)$ *for all* $I,\ J \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$.

Proof. We prove first that $\bigvee(I \cup J) = (\bigvee I) \vee (\bigvee J)$. Let $I, J \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$. Observe that $(\bigvee I) \vee (\bigvee J) \geq \bigvee I \geq x \ \ \forall x \in I$. Similarly, $(\bigvee I) \vee (\bigvee J) \geq \bigvee J \geq y \ \ \forall y \in J$. It follows that $(\bigvee I) \vee (\bigvee J) \geq z \ \ \forall z \in I \cup J$. Thus, $(\bigvee I) \vee (\bigvee J)$ is an upper bound of $I \cup J$. We now show that $(\bigvee I) \vee (\bigvee J)$ is the least upper bound of $I \cup J$. Suppose that there exists $a \in Q$ such that $a \geq z \ \forall z \in I \cup J$. This implies that $a \geq x \ \forall x \in I$ and $a \geq y \ \forall y \in J$. It follows that $a \geq \bigvee I$ and $a \geq \bigvee J$. Therefore, a is an upper bound of $\{(\bigvee I), (\bigvee J)\}$. We may therefore conclude that $a \geq (\bigvee I) \vee (\bigvee J)$.

Now we show that $\bigvee(I \cap J) = (\bigvee I) \wedge (\bigvee J)$. Let $I, J \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, it results that $\bigvee(I \cap J) \leq \bigvee I$ and $\bigvee(I \cap J) \leq \bigvee J$. It follows that $\bigvee(I \cap J)$ is a lower bound for $\{(\bigvee I), (\bigvee J)\}$. We now show that $\bigvee(I \cap J)$ is the greatest lower bound of $\{(\bigvee I), (\bigvee J)\}$. Suppose that a is a lower bound of $\{(\bigvee I), (\bigvee J)\}$, we wish to show that $a \leq \bigvee(I \cap J)$. Since $a \leq \bigvee I$ and $a \leq \bigvee J$, then $a = a \wedge (\bigvee I)$ and $a = a \wedge (\bigvee J)$. It follows that $a = \bigvee\{a \wedge x : x \in I\}$ and $a = \bigvee\{a \wedge y : y \in J\}$. This implies that $a = \bigvee\{a \wedge z : z \in I \cap J\}$. Hence, $a = a \wedge (\bigvee(I \cap J))$. We may therefore conclude that $a \leq \bigvee(I \cap J)$.

■

Theorem 4.1.4 *If $(\mathcal{P}, \mu)$ is a DGL-object, then $(\mathcal{P}, \mu)$ is isomorphic to $(\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\hat{\mu}})$ in the category $\mathcal{DGL}$.*

Proof. We begin by showing that ϱ_p and ς_p are DGL-morphisms. Lemma 3.2.6 tells us that ϱ_p and ς_p respect the meta vertex maps. Lemma 4.1.2 tells us that $jp[\varrho_p]$ and $jp[\varsigma_p]$ are DGP-morphisms. Hence, we just show that ϱ_p and ς_p are bounded lattice homomorphisms.

Let us begin by proving that ς_p is a bounded lattice homomorphism, i.e we prove that for all $I, J \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$, $\varsigma_p(I \bigvee J) = \varsigma_p(I) \bigvee \varsigma_p(J)$, $\varsigma_p(I \bigwedge J) = \varsigma_p(I) \bigwedge \varsigma_p(J)$, and ς_p preserves the largest and smallest element.

Observe that $\varsigma_p(I \bigvee J) = \varsigma_p(I \cup J)$

$$= \bigvee(I \cup J)$$

$$= (\bigvee I) \vee (\bigvee J) \qquad \text{[Lemma 4.1.3]}$$

$$= \varsigma_p(I) \vee \varsigma_p(J)$$

Observe that $\varsigma_p(I \bigwedge J) = \varsigma_p(I \cap J)$

$$= \bigvee(I \cap J)$$

$$= (\bigvee I) \wedge (\bigvee J) \qquad \text{[Lemma 4.1.3]}$$

$$= \varsigma_p(I) \wedge \varsigma_p(J)$$

It's left to show that ς_p preserves the largest and smallest element. **For notational simplicity, we will let $\mathcal{Q} = \mathcal{L}(\mathcal{JP}(\mathcal{P}))$.**

Let $\top_{\mathcal{Q}}$ be the largest element of $\mathcal{Q}$. Then $\top_{\mathcal{Q}} = \mathcal{JP}(\mathcal{P}) = \mathcal{JP}(\downarrow \top_{\mathcal{P}})$. We proved in Corollary 2.2.23 that $\mathcal{P}$ is order-isomorphic to $\mathcal{Q}$ via the maps $\varrho_p : P \longrightarrow \mathcal{Q}$ and $\varsigma_p : \mathcal{Q} \longrightarrow P$ defined by the rules $\varrho_p(x) = \mathcal{JP}(\downarrow x)$ and $\varsigma_p(I) = \bigvee I$. It result that $\varsigma_p(\top_{\mathcal{Q}}) =$

$\varsigma_p(JP(\downarrow \top_{\mathcal{P}})) = \varsigma_p(\varrho_p(\top_{\mathcal{P}})) = \top_{\mathcal{P}}$.

Now, let $\bot_{\mathcal{Q}}$ be the smallest element of $\mathcal{Q}$. Then $\bot_{\mathcal{Q}} = \emptyset$. It follows that $\varsigma_p(\bot_{\mathcal{Q}}) = \varsigma_p(\emptyset) = \varsigma_p(\mathcal{JP}(\downarrow \bot_{\mathcal{P}})) = \varsigma_p(\varrho_p(\bot_{\mathcal{P}})) = \bot_{\mathcal{P}}$.

We may therefore conclude that ς_p is a bounded lattice homomorphism.

Now, let us prove that ϱ_p is a bounded lattice homomorphism, i.e we prove that for all $x, y \in P$, $\varrho_p(x \vee y) = \varrho_p(x) \vee \varrho_p(y)$, $\varrho_p(x \wedge y) = \varrho_p(x) \wedge \varrho_p(y)$, and ϱ_p preserves the largest and smallest elements.

Let $x, y \in P$. Observe that $\varsigma_p(\varrho_p(x \vee y)) = x \vee y$

$$= \varsigma_p(\varrho_p(x)) \vee \varsigma_p(\varrho_p(y))$$

$$= \varsigma_p(\varrho_p(x) \vee \varrho_p(y)) \qquad [\varsigma_p \text{ preserves joins}]$$

Since $\mathcal{P}$ is order-isomorphic to $\mathcal{Q}$ via the maps ϱ_p and ς_p, we may therefore conclude that $\varrho_p(x \vee y) = \varrho_p(x) \vee \varrho_p(y)$. We can prove that $\varrho_p(x \wedge y) = \varrho_p(x) \wedge \varrho_p(y)$.
It's left to show that ϱ_p preserves the largest and smallest elements. For that, let $\top_{\mathcal{P}}$ be the largest element of $\mathcal{P}$. It is clear that $\varrho_p(\top_{\mathcal{P}}) = \mathcal{JP}(\downarrow \top_{\mathcal{P}}) = \mathcal{JP}(\mathcal{P}) = \top_{\mathcal{Q}}$ and $\varrho_p(\bot_{\mathcal{P}}) = \mathcal{JP}(\downarrow \bot_{\mathcal{P}}) = \emptyset = \bot_{\mathcal{Q}}$. We may therefore conclude that ϱ_p is a bounded lattice homomorphism.
Since we proved in Corollary 2.2.23 that $\varrho_p \circ \varsigma_p = I_{\mathcal{Q}}$ and $\varsigma_p \circ \varrho_p = I_{\mathcal{P}}$, then there exists mutually inverse DGL-morphisms $\varrho_p : P \longrightarrow \mathcal{Q}$ and $\varsigma_p : \mathcal{Q} \longrightarrow P$. Therefore, $(\mathcal{P}, \mu)$ is isomorphic to $(\mathcal{Q}, \mu_{\hat{\mu}})$ in the category $\mathcal{DGL}$.

■

Lemma 4.1.5 *Suppose $(\mathcal{P}, \upsilon_1)$ is a DGP-object. $M \in \mathcal{L}(\mathcal{P})$ is meet prime if and only if there exists $x \in \mathcal{P}$ such that $M = mp_{\mathcal{L}(\mathcal{P})}(\downarrow x) = \mathcal{P} - \uparrow x$.*

Proof. Theorem 2.2.35 tells us that $M \in \mathcal{L}(\mathcal{P})$ is meet prime if and only if $\exists\ J \in \mathcal{JP}(\mathcal{L}(\mathcal{P}))$ such that $M = mp_{\mathcal{L}(\mathcal{P})}(J)$. But we know that the join prime elements of $\mathcal{L}(\mathcal{P})$ are the principal lower sets of $\mathcal{P}$. It follows that $J = \downarrow x$ for some $x \in \mathcal{P}$. It results that

$$\begin{aligned} M &= mp_{\mathcal{L}(\mathcal{P})}(J) \\ &= mp_{\mathcal{L}(\mathcal{P})}(\downarrow x) \\ &= \cup\{I \in \mathcal{L}(\mathcal{P}) : \downarrow x \nsubseteq I\} \\ &= \cup\{I \in \mathcal{L}(\mathcal{P}) :\ x \notin I\} \\ &= P - \uparrow x \end{aligned}$$

■

Lemma 4.1.6 *Suppose $(\mathcal{P}, \upsilon_1)$ and $(\mathcal{Q}, \upsilon_2)$ are DGP-objects. If $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is an order homomorphism, then $\hat{\tau}_{Low[f]}(P - \uparrow x) = Q - \uparrow f(x)$.*

Proof. Recall that in Corollary 2.2.13 we proved that for the posets $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \sqsubseteq)$, if $f : P \longrightarrow Q$ is an order homomorphism, then the mapping $Low[f] : Low(\mathcal{Q}) \longrightarrow Low(\mathcal{P})$ defined by $Low[f](Y) = Pre_f(Y)$ is a bounded lattice homomorphism. Observe that since we proved in Lemma 4.1.5 that $(P- \uparrow x) \in \mathcal{MP}(\mathcal{L}(\mathcal{P}))$, then Lemma 2.2.28 tells us that $\hat{\tau}_{Low[f]}(P- \uparrow x) \in \mathcal{MP}(\mathcal{L}(\mathcal{Q}))$. It follows that $\hat{\tau}_{Low[f]}(P- \uparrow x) = Q- \uparrow y$ for some $y \in Q$. We wish to show that $y = f(x)$. Let's first show that $f(x) \sqsubseteq y$.

$$\begin{aligned}
\downarrow y \nsubseteq Q- \uparrow y &\implies \downarrow y \nsubseteq \hat{\tau}_{Low[f]}(P- \uparrow x) \\
&\implies Low[f](\downarrow y) \nsubseteq P- \uparrow x && \text{[Lemma 2.2.26]} \\
&\implies Pre_f(\downarrow y) \nsubseteq P- \uparrow x \\
&\implies \exists w \in Pre_f(\downarrow y) \text{ such that } w \notin P- \uparrow x \\
&\implies \exists w \in Pre_f(\downarrow y) \text{ such that } w \in \uparrow x \\
&\implies x \leq w \\
&\implies f(x) \sqsubseteq f(w) && \text{[f is an order homomorphism]} \\
&\implies f(x) \sqsubseteq y && [w \in Pre_f(\downarrow y) \implies f(w) \in \downarrow y \implies f(w) \sqsubseteq y]
\end{aligned}$$

Now suppose by a way of contradiction that $f(x) \sqsubset y$.

$$\begin{aligned}
f(x) \sqsubset y &\implies f(x) \in Q- \uparrow y \\
&\implies f(x) \in \hat{\tau}_{Low[f]}(P- \uparrow x) \\
&\implies f(x) \in \cup\{I \in \mathcal{L}(\mathcal{Q}) : Pre_f(I) \subseteq P- \uparrow x\} \\
&\implies \exists\, I \in \mathcal{L}(\mathcal{Q}) \text{ such that } f(x) \in I \text{ and } Pre_f(I) \subseteq P- \uparrow x \\
&\implies Pre_f(f(x)) \in P- \uparrow x \\
&\implies x \in P- \uparrow x && \text{[A contradiction]}
\end{aligned}$$

This contradiction forces us to conclude that $y = f(x)$ as desired.

■

Lemma 4.1.7 *Suppose $(\mathcal{P}, \upsilon_1)$ and $(\mathcal{Q}, \upsilon_2)$ are DGP-objects. If $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is an order homomorphism, then we have $jp[Low[f]] = \eta_Q \circ f \circ \vartheta_P$. In particular, $jp[Low[f]]$ is a DGP-morphism when f is a DGP-morphism.*

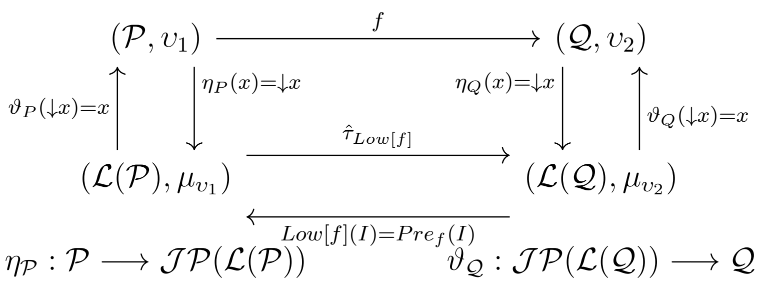

Figure 11: Diagram for Lemma 4.1.7

Proof. We already proved in Theorem 2.2.17 that any digraph poset $\mathcal{Q} = (Q, \leq)$ is order-isomorphic to the poset $\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ via the pair (η_Q, ϑ_Q) where $\eta_Q : Q \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ and $\vartheta_Q : \mathcal{JP}(\mathcal{L}(\mathcal{Q})) \longrightarrow Q$ are defined by the rules $\eta_Q(x) = \downarrow x$ and $\vartheta_Q : (\downarrow x) = x$. We also proved in Corollary 3.2.5 that if $(\mathcal{Q}, \upsilon)$ is a *DGP*-object, then $(\mathcal{Q}, \upsilon)$ is isomorphic to $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_\upsilon})$ in the category $\mathcal{DGP}$. Define $jp[Low[f]] : \mathcal{JP}(\mathcal{L}(\mathcal{P})) \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{P}))$ by $jp[Low[f]](\downarrow x) = jp_{\mathcal{L}(\mathcal{Q})} \circ \hat{\tau}_{Low[f]} \circ mp_{\mathcal{L}(\mathcal{P})}(\downarrow x)$. We wish to show that $jp[Low[f]](\downarrow x) = \eta_Q \circ f \circ \vartheta_P(\downarrow x)$. Let $\downarrow x \in \mathcal{JP}(\mathcal{L}(\mathcal{P}))$.

$$
\begin{aligned}
jp[Low[f]](\downarrow x) &= jp_{\mathcal{L}(\mathcal{Q})} \circ \hat{\tau}_{Low[f]} \circ mp_{\mathcal{L}(\mathcal{P})}(\downarrow x) \\
&= jp_{\mathcal{L}(\mathcal{Q})} \circ \hat{\tau}_{Low[f]}(P- \uparrow x) && \text{[Lemma 4.1.5]} \\
&= jp_{\mathcal{L}(\mathcal{Q})}(Q- \uparrow f(x)) && \text{[Lemma 4.1.6]} \\
&= jp_{\mathcal{L}(\mathcal{Q})}(mp_{\mathcal{L}(\mathcal{Q})}(\downarrow f(x))) && \text{[Lemma 4.1.5]} \\
&= \downarrow f(x) && \text{[Theorem 2.2.35]} \\
&= \eta_Q(f(x)) && \text{[Theorem 2.2.17]} \\
&= \eta_Q(f(\vartheta_P(\downarrow x))) && \text{[Theorem 2.2.17]} \\
&= \eta_Q \circ f \circ \vartheta_P(\downarrow x)
\end{aligned}
$$

We proved in Theorem 2.2.17 that the join prime elements of $\mathcal{L}(\mathcal{P})$ are precisely the principal lower sets of P. We also proved in theorem 2.2.17 that $\mathcal{Q} = (Q, \leq)$ is order-isomorphic to the poset $\mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ via the pair (η_Q, ϑ_Q) and $\mathcal{P} = (P, \leq)$ is order-isomorphic to the poset $\mathcal{JP}(\mathcal{L}(\mathcal{P}))$ via the pair (η_P, ϑ_P). Since we proved that $jp[Low[f]] = \eta_Q \circ f \circ \vartheta_P$, it follows that $jp[Low[f]]$ is a DGP-morphism when f is a DGP-morphism.

■

Theorem 4.1.8 *Suppose $(\mathcal{P}, \upsilon_1)$ and $(\mathcal{Q}, \upsilon_2)$ are DGP-objects. If $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a DGP-morphism, then $Low[f] : Low(\mathcal{Q}) \longrightarrow Low(\mathcal{P})$ defined in Corollary 2.2.13 by $Low[f](Y) = Pre_f(Y)$ is a DGL-morphism from $(\mathcal{L}(\mathcal{Q}), \mu_{\upsilon_2})$ to $(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})$.*

Proof. We already proved in Corollary 2.2.13 that $Low[f]$ is a bounded lattice homomorphism. We also proved in Lemma 4.1.7 that $jp[Low[f]]$ is a DGP-morphism. It left to prove that $Low[f]$ respects the meta vertex maps, that is $\forall\ I \in \mathcal{L}(\mathcal{Q}), Low[f](\mu_{\upsilon_2}(I)) = \mu_{\upsilon_1}(Low[f](I))$. Observe that since $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a DGP-morphism, then $f(\upsilon_1(y)) = \upsilon_2(f(y))$. It follows that $\upsilon_1(y) = Pre_f(\upsilon_2(f(y)))$. Now let $I \in \mathcal{L}(\mathcal{Q})$.

$$
\begin{aligned}
\mu_{\upsilon_1}(Low[f](I)) &= \mu_{\upsilon_1}(Pre_f(I)) \\
&= \{\mu_{\upsilon_1}(y) : y \in Max(P) \cap Pre_f(I)\} \\
&= \{Pre_f(\upsilon_2(f(y))) : f(y) \in Max(Q) \cap I\} \\
&= Pre_f\{\upsilon_2(f(y)) : f(y) \in Max(Q) \cap I\} \\
&= Low[f](\mu_{\upsilon_2}(I))
\end{aligned}
$$

We may therefore conclude that $Low[f]$ is a DGL-morphism from $(\mathcal{L}(\mathcal{Q}), \mu_{\upsilon_2})$ to $(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})$.

■

Theorem 4.1.8 tells us that DGL-morphisms are quite abundant since they arise essentially as the "preimage maps" of DGP-morphisms. Our next task will be to show that all DGL-morphisms arise in this manner.

Theorem 4.1.9 *Suppose $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ are DGL-objects. If $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a bounded lattice homomorphism, then it is the case that $\varrho_Q \circ f \circ \varsigma_p = Low[jp[f]]$.*

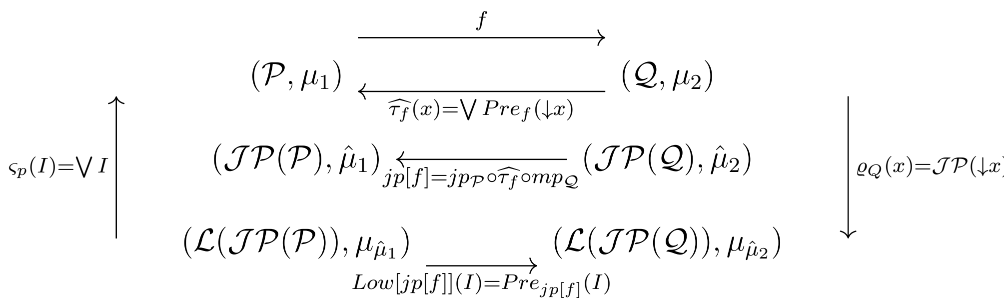

Figure 12: Diagram for Theorem 4.1.9

Proof. Let $I \in \mathcal{L}(\mathcal{JP}(\mathcal{P}))$. We wish to show that $Low[jp[f]](I) = \varrho_Q \circ f \circ \varsigma_p(I)$. For this purpose, let's first prove that if $jp_{\mathcal{P}} \circ \widehat{\tau_f} \circ mp_{\mathcal{Q}}(x) \in I$, then $x \leq \bigvee f(I)$.

$$
\begin{aligned}
jp_{\mathcal{P}} \circ \widehat{\tau_f} \circ mp_{\mathcal{Q}}(x) \in I &\implies jp_{\mathcal{P}}(\widehat{\tau_f}(mp_{\mathcal{Q}}(x))) \in I \\
&\implies I \nsubseteq \downarrow (\widehat{\tau_f}(mp_{\mathcal{Q}}(x))) \qquad [1] \\
&\implies f(I) \nsubseteq f(\downarrow (\widehat{\tau_f}(mp_{\mathcal{Q}}(x)))) \\
&\implies f(I) \nsubseteq \downarrow f((\widehat{\tau_f}(mp_{\mathcal{Q}}(x)))) \qquad [2] \\
&\implies f(I) \nsubseteq \downarrow f(\bigvee Pre_f(\downarrow mp_{\mathcal{Q}}(x))) \\
&\implies f(I) \nsubseteq \downarrow (\bigvee f(Pre_f(\downarrow mp_{\mathcal{Q}}(x)))) \\
&\implies f(I) \nsubseteq \downarrow (\bigvee \downarrow mp_{\mathcal{Q}}(x)) \\
&\implies f(I) \nsubseteq \downarrow mp_{\mathcal{Q}}(x) \\
&\implies \exists\, y \in f(I) \text{ such that } y \geq x \qquad [3] \\
&\implies \bigvee f(I) \geq x
\end{aligned}
$$

[1] Lemma 2.2.33 $[jp_{\mathcal{P}} \circ \widehat{\tau_f} \circ mp_{\mathcal{Q}}(x) \nleq \widehat{\tau_f} \circ mp_{\mathcal{Q}}(x)]$.

[2] $f(\downarrow x) = \downarrow f(x)$.

[3] Lemma 2.2.33 $mp_{\mathcal{Q}}(x) \ngeq x$.

Now we can prove our theorem.

$$
\begin{aligned}
Low[jp[f]](I) &= Pre_{jp[f]}(I) \\
&= \{x \in \mathcal{JP}(\mathcal{Q}) : jp[f](x) \in I\} \\
&= \{x \in \mathcal{JP}(\mathcal{Q}) : jp_{\mathcal{P}} \circ \widehat{\tau}_f \circ mp_{\mathcal{Q}}(x) \in I\ \} \\
&= \{x \in \mathcal{JP}(\mathcal{Q}) : x \leq \bigvee f(I)\} && \text{[Proved above]} \\
&= \mathcal{JP}(\downarrow (\bigvee f(I))) \\
&= \varrho_Q(\bigvee f(I)) \\
&= \varrho_Q(f(\bigvee I)) && \text{[f is a bounded lattice homomorphism]} \\
&= \varrho_Q \circ f \circ \varsigma_p(I)
\end{aligned}
$$

■

Suppose $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ are DGL-objects and suppose $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a DGL-morphism. Note that Theorem 4.1.9 tells us f can be realized as the "preimage map" of some DGP-morphism. In particular, we have $f = \varsigma_Q \circ Low[jp[f]] \circ \varrho_p$.

At this point, we have laid the groundwork necessary to establish that the category $\mathcal{DGP}$ is *dually equivalent* to the category $\mathcal{DGL}$. Now, we are able to define the functors that will yield this dual categorical equivalence.

Definition 4.1.10. <u>The Functor DiGL</u>

1. Define a mapping $DiGL_O : Obj(DGP) \longrightarrow Obj(DGL)$ as follows: For each DGP-object $(\mathcal{Q}, \upsilon)$, let $DiGL_O[(\mathcal{Q}, \upsilon)] = (\mathcal{L}(\mathcal{Q}), \mu_\upsilon)$.

2. Define a mapping $DiGL_M : Hom(DGP) \longrightarrow Hom(DGL)$ as follows: For each DGP-morphism f from $(\mathcal{P}, \upsilon_1)$ to $(\mathcal{Q}, \upsilon_2)$, let $DiGL_M[f] = Low[f]$.

Note that the morphism component of DiGL "reverses arrows" in the sense $f : P \longrightarrow Q$ implies $DiGL_M[f] : Low[\mathcal{Q}] \longrightarrow Low[\mathcal{P}]$. This is the key feature of a *contravariant* functor, as is the fact the morphism assignment "reverses" the order of composition.

Lemma 4.1.11 *The pair $DiGL = (DiGL_O, DiGL_M)$ is a contravariant functor from DGP to DGL.*

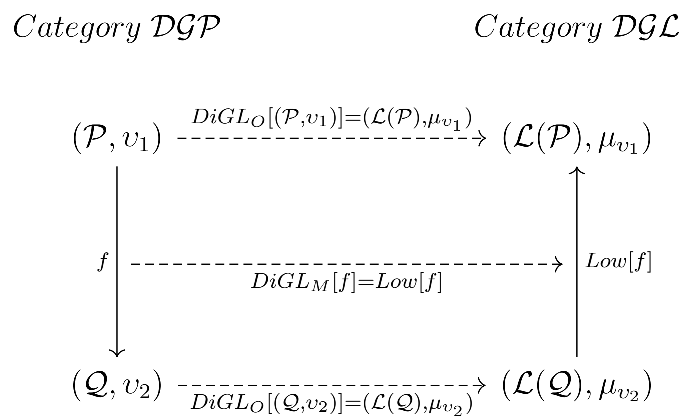

Figure 13: Diagram for Lemma 4.1.11

Proof. The map $DiGL_O : Obj(DGP) \longrightarrow Obj(DGL)$ associate each DGP-object $(\mathcal{P}, \upsilon_1)$ in the category $\mathcal{DGP}$ to the DGL-object $(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})$ in the category $\mathcal{DGL}$. Lemma 3.2.4 tells us that this map is well defined. The map $DiGL_M : Hom(DGP) \longrightarrow Hom(DGL)$ associate each DGP-morphism f from $(\mathcal{P}, \upsilon_1)$ to $(\mathcal{Q}, \upsilon_2)$ in the category $\mathcal{DGP}$ to the DGL-morphism $Low[f] : Low[\mathcal{Q}] \longrightarrow Low[\mathcal{P}]$ in the category $\mathcal{DGL}$. Theorem 4.1.8 tells us that this map is well defined.

We show now that $DiGL_M$ dually preservers composition of morphisms, that is, $DiGL_M[g \circ f] = DiGL_M[f] \circ DiGL_M[g]$. Let f be a DGP-morphism from $(\mathcal{P}, \upsilon_1)$ to $(\mathcal{Q}, \upsilon_2)$ and g be a DGP-morphism from $(\mathcal{Q}, \upsilon_2)$ to $(\mathcal{S}, \upsilon_3)$. Let $I \in \mathcal{L}(\mathcal{S})$.

$$\begin{aligned} DiGL_M[g \circ f](I) &= Low[g \circ f](I) \\ &= Pre_{g\circ f}(I) \\ &= \{x \in P : g(f(x)) \in I\} \\ &= \{x \in P : f(x) \in Pre_f(I)\} \\ &= Pre_f(Pre_g(I)) \end{aligned}$$

$$= Low[f] \circ Low[g](I)$$
$$= DiGL_M[f] \circ DiGL_M[g]$$

Let us now show that the functor $DiGL$ preserves identity morphisms, that is, $DiGL_M[1_P] = 1_{DiCL_O[(P,\upsilon_1)]}$. Let $(\mathcal{P}, \upsilon_1)$ be a DGP-object in the category $\mathcal{DGP}$. Let $DiGL_O[(\mathcal{P}, \upsilon_1)] = (\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})$, and let $I \in \mathcal{L}(\mathcal{P})$. Then,

$$DiGL_M[1_P] = Low[1_P](I)$$
$$= Pre_{1_P}(I)$$
$$= I$$
$$= 1_{\mathcal{L}(\mathcal{P})}(I)$$
$$= 1_{DiCL_O[(P,\upsilon_1)]}$$

We may therefore conclude that the pair $DiCL = (DiGL_O, DiGL_M)$ is a contravariant functor from DGP to DGL.

■

Definition 4.1.12. **The Functor DiGP**

1. Define a mapping $DiGP_O : Obj(DGL) \longrightarrow Obj(DGP)$ as follows: For each DGL-object $(\mathcal{P}, \mu)$, let $DiGP_O[(\mathcal{P}, \mu)] = (\mathcal{JP}(\mathcal{P}), \hat{\mu})$.

2. Define a mapping $DiGP_M : Hom(DGL) \longrightarrow Hom(DGP)$ as follows: For each DGL- morphism f from $(\mathcal{P}, \mu_1)$ to $(\mathcal{Q}, \mu_2)$, let $DiGP_M[f] = jp[f]$.

Lemma 4.1.13 *The pair* $DiGP = (DiGP_O, DiGP_M)$ *is a contravariant functor from DGL to DGP.*

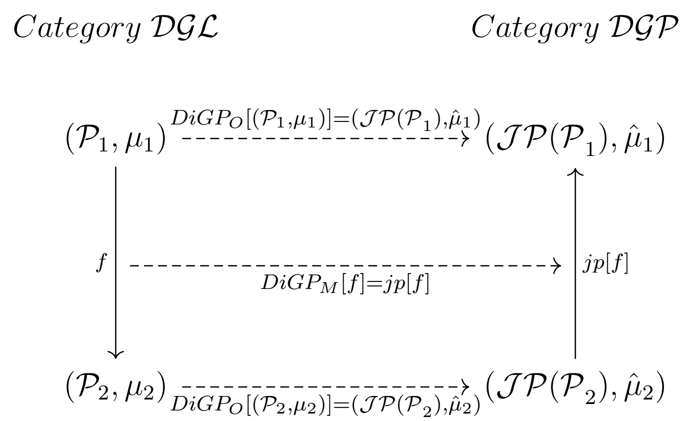

Figure 14: Diagram for Lemma 4.1.13

Proof. Definition 3.2.8 tells us that if f is a DGL-morphism, then $jp[f]$ is a DGP-morphism. Hence, the map $DiGP_M$ is well defined. By Definition 3.2.1, $\mathcal{P} = (P, \leq)$ is a digraph lattice provided $\mathcal{JP}(\mathcal{P})$ is a digraph poset. Thus, the map $DiGP_O$ is well defined. We wish to show that $DiGP_M$ preserves the identity morphism, and dually preserve the composition of morphisms.

Let f be a DGL-morphism from $(\mathcal{P}_1, \mu_1)$ to $(\mathcal{P}_2, \mu_2)$ and g be a DGL-morphism from $(\mathcal{P}_2, \mu_2)$ to $(\mathcal{P}_3, \mu_3)$. We wish to show that $DiGP_M[g \circ f] = DiGP_M[f] \circ DiGP_M[g]$. Let $x \in \mathcal{JP}(\mathcal{P}_1)$.

$$
\begin{aligned}
DiGP_M[g \circ f](x) &= jp[g \circ f](x) \\
&= jp_{\mathcal{P}_1} \circ \hat{\tau}_{g\circ f} \circ mp_{\mathcal{P}_3}(x) \\
&= jp_{\mathcal{P}_1} \circ \hat{\tau}_f \circ \hat{\tau}_g \circ mp_{\mathcal{P}_3}(x) && \text{[Lemma 2.2.29]} \\
&= jp_{\mathcal{P}_1} \circ \hat{\tau}_f \circ mp_{\mathcal{P}_2} \circ jp_{\mathcal{P}_2} \circ \hat{\tau}_g \circ mp_{\mathcal{P}_3}(x) && \text{[Theorem 2.2.35]} \\
&= jp[f] \circ jp[g](x) \\
&= DiGP_M[f] \circ DiGP_M[g](x)
\end{aligned}
$$

Now, let $x \in \mathcal{JP}(\mathcal{P}_1)$. We show that $DiGP_M[1_{\mathcal{P}_1}] = 1_{DiGP_O[(\mathcal{P}_1,\mu_1)]}$.

$$\begin{aligned} DiGP_M[1_{\mathcal{P}_1}](x) &= jp[1_{\mathcal{P}_1}](x) \\ &= jp_{\mathcal{P}_1} \circ \hat{\tau}_{1_{\mathcal{P}_1}} \circ mp_{\mathcal{P}_1}(x) \\ &= jp_{\mathcal{P}_1} \circ mp_{\mathcal{P}_1}(x) \\ &= x \\ &= 1_{jp_{\mathcal{P}_1}}(x) \\ &= 1_{DiGP_O[(\mathcal{P}_1,\mu_1)]}(x) \end{aligned}$$

■

We are now ready to prove our final results.

4.2 Dual Categorical Equivalence

Since we defined the functors DiGP and DiGL in the previous section, we are now ready to prove the dual categorical equivalence between the categories $\mathcal{DGP}$ and $\mathcal{DGL}$. We do that by proving that each of the functors DiGP and DiGL is full, faithful, and essentially surjective. We then conclude a dual categorical equivalence between $\mathcal{DGP}$ and $\mathcal{DGL}$. It follows that the categories DiGraph and $\mathcal{DGL}$ are dually equivalent.

Theorem 4.2.1 *The Categories DiGraph and $\mathcal{DGP}$ are equivalent via the covariant functors DGP : DiGraph $\longrightarrow \mathcal{DGP}$ and $DIG : \mathcal{DGP} \longrightarrow$ DiGraph [1].*

Theorem 4.2.2 *The functor DiGP is full, faithful, and essentially surjective.*

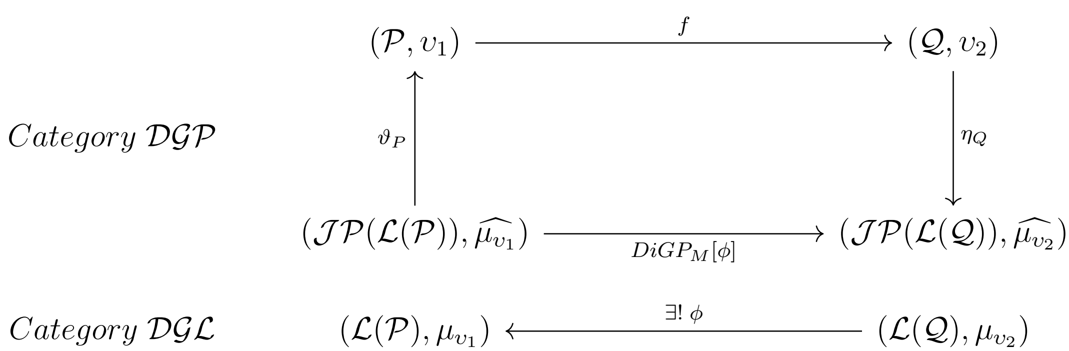

Figure 15: Diagram for Theorem 4.2.2

Proof. Let us first prove that the functor DiGP is full. Consider the diagram above. We are given DGP-objects $(\mathcal{P}, \upsilon_1)$ and $(\mathcal{Q}, \upsilon_2)$ and DGP-morphism $f : P \longrightarrow Q$. We proved in Corollary 3.2.5 that if $(\mathcal{Q}, \upsilon_2)$ is a DGP-object, then $(\mathcal{Q}, \upsilon_2)$ is isomorphic to $(\mathcal{JP}(\mathcal{L}(\mathcal{Q})), \widehat{\mu_{\upsilon_2}})$ in the category $\mathcal{DGP}$ via the pair (η_Q, ϑ_Q) where $\eta_Q : Q \longrightarrow \mathcal{JP}(\mathcal{L}(\mathcal{Q}))$ and $\vartheta_Q : \mathcal{JP}(\mathcal{L}(\mathcal{Q})) \longrightarrow Q$. Similalirly, $(\mathcal{P}, \upsilon_1)$ is isomorphic to

$(\mathcal{JP}(\mathcal{L}(\mathcal{P})), \widehat{\mu_{\upsilon_1}})$ in the category $\mathcal{DGP}$ via the pairs (η_P, ϑ_P). To prove that DiGP is full, our task is to identify a unique DGL-morphism $\phi : Low(\mathcal{Q}) \longrightarrow Low(\mathcal{P})$ such that the diagram above commute. This tells us we want $\eta_Q \circ f = DiGP_M[\phi] \circ \eta_P$. Since $\eta_P \circ \vartheta_P = 1_{\mathcal{JP}(\mathcal{L}(\mathcal{P}))}$, this equation tells us that we want $\eta_Q \circ f \circ \vartheta_P = DiGP_M[\phi]$. Lemma 4.1.7 tells us this will occur if we let $\phi = Low[f]$ since $jp[Low[f]] = \eta_Q \circ f \circ \vartheta_P$. Since such ϕ exists, we may conclude that DiGP is full.

Let us now prove that the functor DiGP is faithful, that is if $\phi, \psi : Low(\mathcal{Q}) \longrightarrow Low(\mathcal{P})$ are DGL-morphisms such that $DiGP_M[\phi] = DiGP_M[\psi]$, then $\phi = \psi$. Assume $DiGP_M[\phi] = DiGP_M[\psi]$. Since $DiGP_M[\phi] = DiGP_M[\psi] = \eta_Q \circ f \circ \vartheta_P$, then $jp[\phi] = jp[\psi] = \eta_Q \circ f \circ \vartheta_P$. Lemma 4.1.7 tells us that this implies that $\phi = \psi = Low[f]$. We may therefore conclude that DiGP is faithful.

It is left to prove that the functor DiGP is essentially surjective. We proved in Corollary 3.2.5 that if $(\mathcal{P}, \upsilon_1) \in Obj(DGP)$, then $(\mathcal{P}, \upsilon_1)$ is isomorphic to $(\mathcal{JP}(\mathcal{L}(\mathcal{P})), \widehat{\mu_{\upsilon_1}})$ in the category $\mathcal{DGP}$. Since $DiGP_O[(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})] = (\mathcal{JP}(\mathcal{L}(\mathcal{P})), \widehat{\mu_{\upsilon_1}})$, then $(\mathcal{P}, \upsilon_1)$ is isomorphic to $DiGP_O[(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1})]$ for $(\mathcal{L}(\mathcal{P}), \mu_{\upsilon_1}) \in Obj(DGL)$.

■

Theorem 4.2.3 *The functor DiGL is full, faithful, and essentially surjective.*

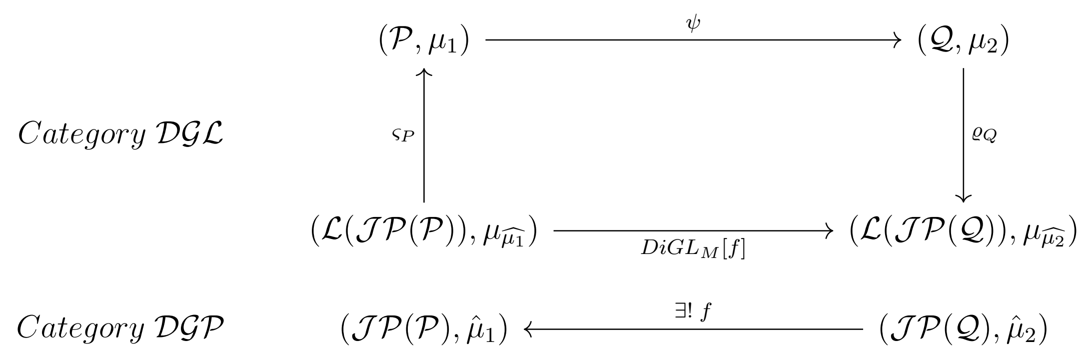

Figure 16: Diagram for Theorem 4.2.3

Proof. Let us first prove that the functor DiGL is full. Consider the diagram above. We are given DGL-objects $(\mathcal{P}, \mu_1)$ and $(\mathcal{Q}, \mu_2)$ and DGL-morphism $\psi : P \longrightarrow Q$. We proved in Theorem 4.1.4 that if $(\mathcal{P}, \mu_1)$ is a DGL-object, then $(\mathcal{P}, \mu_1)$ is isomorphic to $(\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\hat{\mu}_1})$ in the category $\mathcal{DGL}$ via the maps ς_p and ϱ_p. Similarly, $(\mathcal{Q}, \mu_2)$ is isomorphic to $(\mathcal{L}(\mathcal{JP}(\mathcal{Q})), \mu_{\hat{\mu}_2})$ in the category $\mathcal{DGL}$. To prove that DiGL is full, our task is to identify a unique DGP-morphism $f : \mathcal{JP}(\mathcal{Q}) \longrightarrow \mathcal{JP}(\mathcal{P})$ such that the diagram above commute. This tells us we want $\varrho_Q \circ \psi = DiGL_M[f] \circ \varrho_p$. Since $\varrho_Q \circ \varsigma_P = 1_{\mathcal{L}(\mathcal{JP}(\mathcal{P}))}$, this equation tells us we want $\varrho_Q \circ \psi \circ \varsigma_P = DiGL_M[f]$. Since Theorem 4.1.9 tells us that $\varrho_Q \circ \psi \circ \varsigma_p = Low[jp[\psi]]$, $\varrho_Q \circ \psi \circ \varsigma_P = DiGL_M[f]$ will occur if we let $f = jp[\psi]$. Since such f exists, we may conclude that DiGL is full.
Let us now prove that the functor $DiGL$ is faithful, that is if $f, g : \mathcal{JP}(\mathcal{Q}) \longrightarrow \mathcal{JP}(\mathcal{P})$ are DGP-morphisms such that $DiGL_M[f] = DiGL_M[g]$, then $f = g$. Assume $DiGL_M[f] = DiGL_M[g]$. Then, $Low[f] = Low[g] = \varrho_Q \circ \psi \circ \varsigma_P$. By Theorem 4.1.9, we may conclude that $f = g = jp[\psi]$. It is left to prove that the functor DiGL is essentially surjective. We proved in Theorem 4.1.4 that if

$(\mathcal{P}, \mu_1) \in Obj(DGL)$, then $(\mathcal{P}, \mu_1)$ is isomorphic to $(\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\hat{\mu}_1})$ in the category $\mathcal{DGL}$. Since $DiGL_O[(\mathcal{JP}(\mathcal{P}), \hat{\mu}_1)] = (\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\hat{\mu}_1})$, then $(\mathcal{P}, \mu_1)$ is isomorphic to $DiGL_O[(\mathcal{JP}(\mathcal{P}), \hat{\mu}_1)] = (\mathcal{L}(\mathcal{JP}(\mathcal{P})), \mu_{\hat{\mu}_1})$ where $(\mathcal{JP}(\mathcal{P}), \hat{\mu}_1) \in Obj(DGP)$. Therefore, $DiGL$ is essentially surjective.

■

The following theorem follows directly from Theorems 4.2.2 and 4.2.3.

Theorem 4.2.4 *The categories $\mathcal{DGP}$ and $\mathcal{DGL}$ are dually equivalent via the functors $DiGP$ and $DiGL$.*

The following theorem follows from the facts that the categories DiGraph and $\mathcal{DGP}$ are equivalent via the covariant functors $DGP : DiGraph \longrightarrow \mathcal{DGP}$ and $DIG : \mathcal{DGP} \longrightarrow DiGraph$, and the categories $\mathcal{DGP}$ and $\mathcal{DGL}$ are dually equivalent via the functors $DiGP : \mathcal{DGL} \longrightarrow \mathcal{DGP}$ and $DiGL : \mathcal{DGP} \longrightarrow \mathcal{DGL}$.

Theorem 4.2.5 *The categories DiGraph and $\mathcal{DGL}$ are dually equivalent via the functors $(DIG \circ DiGP) : \mathcal{DGL} \longrightarrow DiGraph$ and $(DiGL \circ DGP) : DiGraph \longrightarrow \mathcal{DGL}$.*

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