THE CATEGORY OF FINITE INCIDENCE POSETS

by

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In particular, I would like to thank Dr. James Hart. Dr. Hart has been setting aside time to teach me mathematics since the day I started my Master's degree at MTSU. He agreed to be my thesis advisor long before he knew how much of a chore that would be, but he helped and encouraged me the whole way. I came within metaphorical millimeters of quitting mathematics, MTSU, and this thesis figurative millions of times, but Dr. Hart's support helped me stick with this whole graduate school thing. His incredible stories and hilarious impressions of mathematicians made me laugh. His vulnerability taught me some of what it means to be a human being and a graduate student at the same time. Dr. Hart is an incredible thesis advisor, and he has been an incredible mentor to me. I appreciate all he has done for me more than I can say.

DEDICATION

I dedicate this thesis to my wonderful board games. Without their constant help and guidance, I never would have had the energy or emotional stability necessary to complete the following thesis. In particular, I would like to thank Mysterium, for many hours of making random, abstract connections between pictures. I would like to thank TIME Stories, for beautiful narratives and intense puzzles that were some of the only excitement I felt in all of graduate school. And, of course, I would like to thank Dungeons and Dragons, for giving me a world of fantasy to escape to when life was too much.

ABSTRACT

In the dissertation *A Study on the Category of Graphs*, Buvaneswari constructs the category of finite, simple graphs and determines the existence of particular, categorical objects within the same. We use Buvaneswari's dissertation as the basis for an investigation into the category of finite incidence posets, which we construct herein. We prove that the category of finite, simple graphs is equivalent to the category of finite incidence posets, and we prove the existence of equalizers, products, and coproducts in the category of finite incidence posets.

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CHAPTER 1

INTRODUCTION

In this thesis, most work will be original, and all original work was completed under the supervision of Dr. James Hart. While all proofs provided in the thesis are my own, many definitions in Chapter 2 come from source material. Where source material is used, a citation and comment will note the use of same. Most importantly, all of the mathematics in this thesis was made possible by *A Study on the Category of Graphs* by S. Buvaneswari [1]. Every categorical proof was inspired by that dissertation, and the current work would not have existed without it. Citing Buvaneswari at every line would be unhelpful, so we honor *A Study on the Category of Graphs* and Buvaneswari here for their importance to this thesis. Any mathematics not preceded or followed by a citation is original, and, as already mentioned, all categorical proofs are original but inspired and informed by [1].

We use concatenation to represent function composition. For example, the composition of the functions $f : A \to B$ and $g : B \to C$ is $gf : A \to C$. All other notation will be introduced as needed.

In Chapter 2, we present the preliminary knowledge necessary to understand the mathematics used in this thesis. We assume a basic comprehension of functions and sets.

In Chapter 3, we prove the category of finite, simple graphs is equivalent to the category of finite incidence posets.

In Chapter 4, we prove the existence of equalizers in the category of finite incidence posets, and we show coequalizers do not exist in the category of finite incidence posets.

In Chapter 5, we prove that both products and coproducts exist in the category of finite incidence posets.

CHAPTER 2

PRELIMINARIES

2.1 Graph Theory

While graph theory is a vast and broad topic within mathematics, we will need only the definition of finite, simple graphs, the classification of some special graphs, and one important theorem.

Definition 2.1.1. *A graph G is a pair of sets $(V(G), E(G))$ such that $E(G)$ consists of two-element subsets of $V(G)$. Such a graph is called a simple graph, though we shall simply call them graphs. We say a graph is finite when the cardinality of $V(G)$ is finite.*

To make discussions about the structure of graphs easier, we say two vertices are adjacent whenever they are connected via an edge, and an edge is incident with a vertex whenever the vertex is an element of the edge. Note that no vertex can be adjacent to itself. If u is adjacent to v, we say $u \sim v$. The degree of a vertex v, denoted $d(v)$, is the number of edges with which v is incident.

Consider the graphs G and H in Figure 1. You may notice that, if we were to remove the edges $\{v, w\}$, $\{w, x\}$, and $\{u, x\}$ from G, the two graphs would appear to be identical except for labeling. Whenever this is the case, we call H a subgraph of G. More formally, H is a subgraph of G whenever $V(H) \subseteq V(G)$, and $e \in E(H) \implies e \in E(G)$.

There are many examples of common subgraphs that are important to the study of graphs in general. While we will not mention it explicitly, much of what we prove in later chapters will fall apart if we do not consider $\emptyset_G$ a graph called the null graph. In fact, $\emptyset_G$ is a graph such that $V(\emptyset_G) = \emptyset$ and $E(\emptyset_G) = \emptyset$. Below are listed several important classes of graphs.

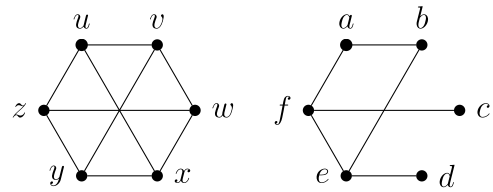

Figure 1: The Graphs G and H, respectively

Definition 2.1.2. *An empty graph E is any graph having no edges. (See Figure 2.)*

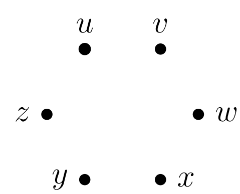

Figure 2: An Empty Graph

Definition 2.1.3. *A path P is a graph with two vertices of degree 1 and every other vertex having degree 2. (See Figure 3.)*

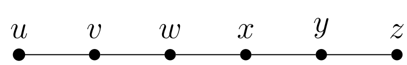

Figure 3: A Path

Definition 2.1.4. *We say a graph is connected when there exists at least one path between any two vertices. Note that every graph shown thus far has been connected.*

Definition 2.1.5. *Similarly, a graph is disconnected exactly when there are at least two vertices with no path between them. (See Figure 4.) The maximal subgraphs such that every pair of vertices in the subgraphs have a path between them are called components.*

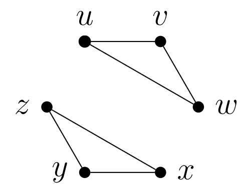

Figure 4: A Disconnected Graph

Definition 2.1.6. *A* <u>*cycle*</u> *is a graph such that every vertex has degree 2. (See Figure 5.) If a graph contains no cycles as subgraphs, we say it is* <u>*acyclic*</u>*.*

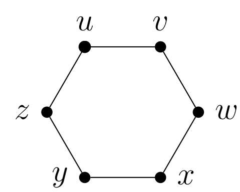

Figure 5: A Cycle

Definition 2.1.7. *A* <u>*tree*</u> *is any acyclic, connected graph. (See Figure 6.) Alternatively, a tree is any graph having a unique path between every pair of vertices.*

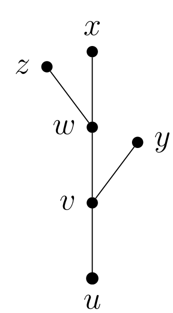

Figure 6: A Tree

Definition 2.1.8. *A* <u>*forest*</u> *is any acyclic graph. In other words, a graph is a forest if each of its connected components is a tree. (See Figure 7.)*

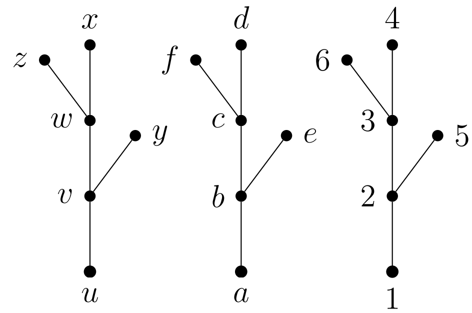

Figure 7: A Forest

Definition 2.1.9. *A* <u>*complete*</u> *graph on* n *vertices, denoted* $\mathbb{K}_n$*, is such that every pair of vertices is adjacent. (See Figure 8.)*

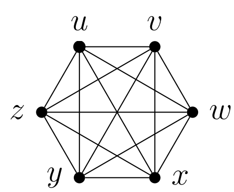

Figure 8: $\mathbb{K}_6$

Definition 2.1.10. *A* <u>*planar*</u> *graph is any graph that can be drawn on an arbitrarily large, two-dimensional surface without any edges crossing. (See Figure 9.)*

One of the most important theorems regarding planar graphs, Kuratowski's Theorem, will be stated below. However, two more types of graphs and a relationship between graphs are needed before Kuratowski's Theorem can be broached.

Definition 2.1.11. *A graph* B *is said to be* <u>*bipartite*</u> *when you can partition* $V(B)$ *into two sets,* V *and* W*, such that no vertex in* V *is adjacent to any vertex in* W*, and neither* V *nor* W *are* $\emptyset$*. (See Figure 10.)*

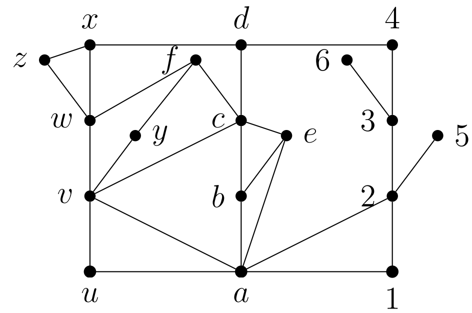

Figure 9: A Planar Graph

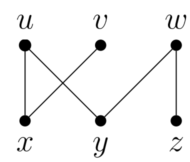

Figure 10: A Bipartite Graph

Definition 2.1.12. *A bipartite graph is called* <u>*complete bipartite*</u> *when every vertex in the partition V is connected to every vertex in the partition W. Such a graph is denoted $\mathbb{K}_{n,m}$, where $n = |V|$ and $m = |W|$. (See Figure 11.)*

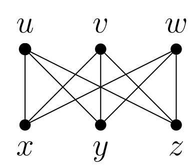

Figure 11: $\mathbb{K}_{3,3}$

Definition 2.1.13. *A graph G has an H* <u>*subdivision*</u> *when a subgraph S of G is such that the edges of H can be put in one-to-one correspondence to paths in S. The subgraph H is said to be an H subdivision. (See Figure 12.)*

Returning to planar graphs, a natural question is whether we can call a graph planar without actually drawing it out. The answer comes in the form of Kuratowski's

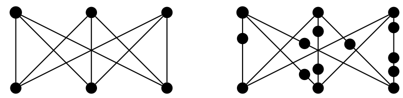

Figure 12: $\mathbb{K}_{3,3}$ and a $\mathbb{K}_{3,3}$ Subdivision

Theorem. Kuratowski's Theorem offers a classification of all planar graphs. Though we will not be proving Kuratowski's Theorem here, Graph Theory by Reinhard Diestel provides a complete proof.

Theorem 2.1.14 (Kuratowski). *A graph G is planar if and only if G has neither a $\mathbb{K}_5$ or $\mathbb{K}_{3,3}$ subdivision.*

Kuratowski's Theorem will play an important roll in the classification of incidence posets corresponding to planar graphs. Why we are using Kuratowski's Theorem as opposed to the more intuitive notion of edge crossings or a function injecting the poset into the plane will be explained in the following section.

We conclude this section with a quick discussion of graph homomorphisms.

Definition 2.1.15. *A graph homomorphism $\varphi : G \to H$ (hereafter simply called a homomorphism) is a function such that*

(a) $V(G) \mapsto V(H)$,

(b) $E(G) \mapsto E(H)$, *and*

(c) For all $u_1, u_2 \in V(G)$, $u_1 \sim u_2 \implies \varphi(u_1) \sim \varphi(u_2)$.

If φ is a bijection, φ is an isomorphism.

A feature of homomorphisms that we will highlight in later chapters is the inability to "delete edges." In Figure 13, φ is a valid homomorphism; however, Figure 14

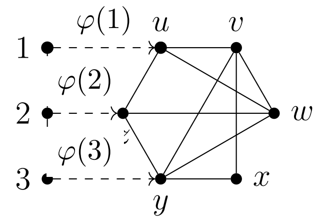

Figure 13: A Homomorphism φ

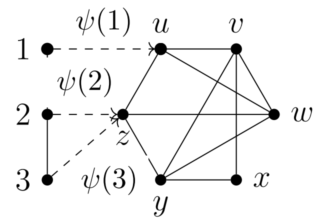

Figure 14: An Invalid Homomorphism ψ

demonstrates a function ψ that is not a valid homomorphism. The function ψ cannot be a homomorphism since $2 \sim 3$ but $\psi(2) \nsim \psi(3)$.

It follows that there is exactly one graph homomorphism from $\emptyset_G$ to any other graph: the function that sends nothing to nothing. While it is uncomfortable to talk about such a function, it is certainly the case that no element of $\emptyset_G$ maps to two elements of the function's domain.

2.2 Order Theory

As with Graph Theory, we will be using an incredibly slim amount of Order Theory. We will define a partial order, a partially ordered set, incidence posets, special elements in incidence posets, special collections of elements in incidence posets, and special classes of incidence posets.

Definition 2.2.1. *A partial order $\leq$ is a relation (much like an equivalence relation) on some set P such that for all $x, y, z \in P$*

(1) If $x \leq y$ and $y \leq z$, $x \leq z$ (transitivity),

(2) If $x \leq y$ and $y \leq x$, $x = y$ (antisymmetry), and

(3) $x \leq x$ (reflexivity).

When an element x is such that $x \leq y$, but $x \neq y$, we say $x < y$, or, x is strictly less than y.

Definition 2.2.2. *A pair $P = (\mathrm{P}, \leq)$ such that P is a set and $\leq$ is a partial order is called a partially ordered set. In this thesis, we shall shorten "partially ordered set" to poset.*

Posets are often represented similarly to graphs. Elements are drawn as nodes, and an "edge" is drawn between two elements x and y if $x \leq y$. However, not every order relation is represented by an edge. The following definitions provide terminology for the relationships between elements in a particular poset $P = (\mathrm{P}, \leq)$.

Definition 2.2.3. *An element y is said to cover another element x when $x < y$, and there exists no element z such that $x < z < y$.*

So, when drawing a poset, lines are drawn between two elements when one covers the other. Such a drawing is called a Hasse Diagram. For an example of a Hasse Diagram, see Figure 15 below. Unlike graphs, however, the way we draw a Hasse Diagram can change what poset the Hasse Diagram represents. If we were to switch any of the horizontal levels, a different poset would result.

Figure 15 provides several examples of important kinds of elements in posets in general, including covers (see Figure 16).

Definition 2.2.4. *Two elements are said to be incomparable if $x \nleq y$ and $y \nleq x$. (See Figure 17.)*

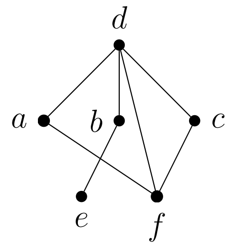

Figure 15: A Hasse Diagram for the poset P

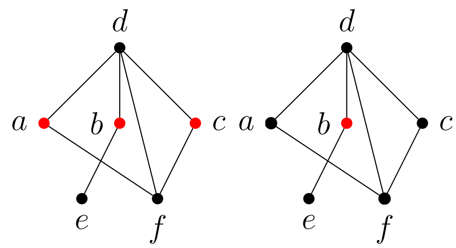

Figure 16: The Covers of f and the Covers of e, respectively

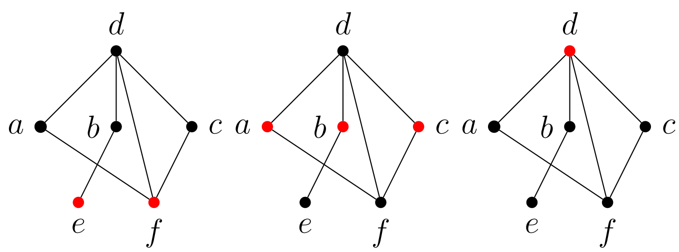

Figure 17: Incomparable Elements in P

Definition 2.2.5. *An element x is said to be* maximal *if there exist no elements y such that $y > x$. (See Figure 18.)*

Definition 2.2.6. *An element y is said to be* minimal *if there exist no elements z such that $y > z$, and there exists some element x such that $x > y$. (See Figure 19.)*

To more easily refer to special elements within a poset, we define the following sets.

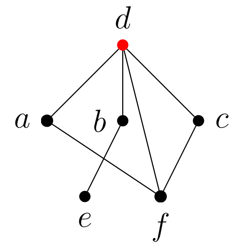

Figure 18: The Maximal Elements of P

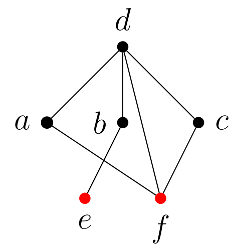

Figure 19: The Minimal Elements of P

Definition 2.2.7. *The set* $\mathbf{C}_P(y) = \{x \mid x \text{ covers } y \text{ in } P\}$*. In Figure 15,* $\mathbf{C}_P(f) = \{a, b, c\}$*, and* $\mathbf{C}_P(e) = \{b\}$*.*

Definition 2.2.8. *The set* $\mathbf{c}_P(x) = \{y \mid y \text{ covers } x \text{ in } P\}$*. In Figure 15,* $\mathbf{c}_P(d) = \{a, b, c\}$*, and* $\mathbf{c}_P(a) = \{f\}$*.*

Definition 2.2.9. *The set* $\mathrm{Max}(P) = \{x \mid x \text{ is maximal in } P\}$*. In Figure 15,* $\mathrm{Max}(P) = \{d\}$*.*

Definition 2.2.10. *The set* $\mathrm{Min}(P) = \{y \mid y \text{ is minimal in } P\}$*. In Figure 15,* $\mathrm{Min}(P) = \{e, f\}$*.*

The particular posets that will be the focus of this study are incidence posets.

Definition 2.2.11. *An incidence poset* IP_A *is a poset with the extra conditions that*

(1) All $x \in IP_A$ *are such that either* $x \in \mathrm{Max}(IP_A)$ *or* $x \in \mathrm{Min}(IP_A)$*,*

(2) $|\mathbf{C}_{IP_A}(y)| = 2, \forall y \in \mathrm{Min}(IP_A)$,

(3) If $y_1 \neq y_2$ *for some* $y_1, y_2 \in \mathrm{Min}(IP_A)$, $\mathbf{C}_{IP_A}(y_1) \neq \mathbf{C}_{IP_A}(y_2)$, *and*

(4) $\mathrm{Max}(IP_A)$ *is finite.*

See Figure 20 for an example of an incidence poset. Incidence posets have the following properties.

- Incidence posets consist of only maximal and minimal elements.

- Every minimal element in an incidence poset is covered by exactly two maximal elements.

- The covers of minimal elements are pairwise distinct.

- Incidence posets have finitely many maximal elements.

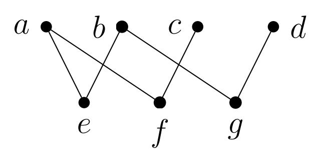

Figure 20: The Incidence Poset IP_A

Definition 2.2.12. *An IP-morphism* $\alpha : IP_A \to IP_B$ *is a function such that*

(1) For all $x \in \mathrm{Max}(IP_A), \alpha(x) \in \mathrm{Max}(IP_B)$,

(2) For all $y \in \mathrm{Min}(IP_A), \alpha(x) \in \mathrm{Min}(IP_B)$*,*

(3) If $a \leq b$ *in* IP_A*, then* $\alpha(a) \leq' \alpha(b)$ *in* IP_B*, and*

(4) $c, d \in \mathrm{Max}(IP_A) \ni c \neq d$ *and* $|\mathbf{c}_{IP_A}(c) \cap \mathbf{c}_{IP_A}(d)| \neq 0 \implies \alpha(c) \neq \alpha(d)$*.*

An isomorphism is a bijective IP-morphism.

One versed in order theory will notice that isomorphisms between incidence posets are standard order isomorphisms between posets. As with graphs, $IP_\emptyset$ is an incidence poset, and there is exactly one function from $IP_\emptyset$ to any other incidence poset. With IP-morphisms defined, we can now develop the concept of one incidence poset being contained within another.

Definition 2.2.13. *We say the incidence poset* IP_B *is a subposet of* IP_A *when there exists an injective IP-morphism* $\beta : IP_B \to IP_A$*. (See Figure 21.)*

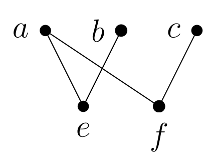

Figure 21: The Subposet IP_B of IP_A Above

With the concept of subposet defined, we can now classify special classes of incidence posets that often show up as subposets.

Definition 2.2.14. *An empty incidence poset is any incidence poset having no minimal elements. (See Figure 22.)*

Definition 2.2.15. *A valley incidence poset (valley)* IP_V *is any incidence poset satisfying the following conditions:*

Figure 22: An Empty Incidence Poset

(1) There are exactly two elements $x_1, x_2 \in \mathrm{Max}(IP_V) \ni |\mathbf{c}_{IP_V}(x_1)| = 1$ *and* $|\mathbf{c}_{IP_V}(x_2)| = 1$*, and*

(2) $|\mathbf{c}_{IP_V}(x)| = 2$ *for all other* $x \in \mathrm{Max}(IP_V)$.

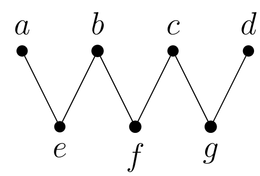

Figure 23: A Valley Incidence Poset

For an example of a valley, see Figure 23. We say a valley <u>connects</u> the maximal elements x_1 and x_2 when x_1 and x_2 are the maximal elements covering only one minimal element each. In Figure 23, a and d are connected by the valley incidence poset. In fact, Figure 23 contains many valleys as subposets.

Definition 2.2.16. *An incidence poset is said to be <u>connected</u> when there exists some valley connecting each pair of vertices. Inversely, an incidence poset is <u>disconnected</u> when there exists some pair of vertices not connected by a valley. (See Figure 24.)*

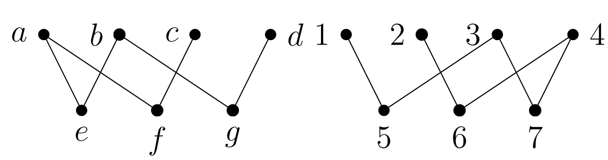

Figure 24: A Disconnected Incidence Poset

Definition 2.2.17. *A cat's cradle is an incidence poset with every maximal element covering exactly two minimal elements. (See Figure 25.) An incidence poset is untied whenever it has no cat's cradle subposet.*

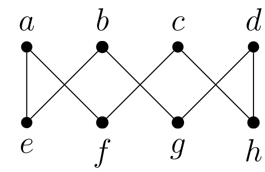

Figure 25: A Cat's Cradle Incidence Poset

Definition 2.2.18. *We call any untied incidence poset a heap. If an incidence poset is untied and connected, we say it is also a string.*

Note that Figures 20, 21, and 23 are all examples of strings, and Figures 22 and 24 are heaps.

Definition 2.2.19. *A saturated incidence poset IP_S is any poset satisfying the condition $|\mathrm{Min}(IP_S)| = \binom{|\mathrm{Max}(IP_S)|}{2}$. Alternatively, IP_S is saturated whenever $|\mathbf{c}_{IP_S}(x)| = |\mathrm{Max}(IP_S)| - 1$ for all $x \in \mathrm{Max}(IP_S)$. Since each saturated incidence poset having n maximal elements is unique up to isomorphism, we will call the saturated incidence poset with n maximal elements IP_n. (See Figure 26.)*

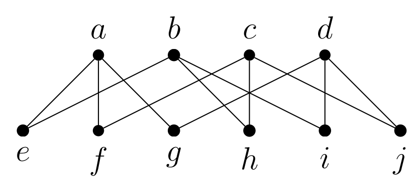

Figure 26: IP_4

To construct all of the incidence posets we need, the following definition will establish one, crucial incidence poset.

Definition 2.2.20. *We define $IP_{3,3}$ as the incidence poset shown in Figure 27 (up to isomorphism).*

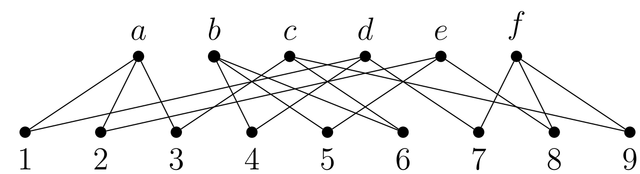

Figure 27: $IP_{3,3}$

Definition 2.2.21. *An incidence poset IP_A is said to* extend *IP_B (or be an IP_B-extension) when each minimal element of IP_B can be put in one-to-one correspondence with a valley between two maximal elements in IP_A. In other words, we can transform IP_A into IP_B by collapsing the minimal elements of each valley into a single minimal element covered by the maximal elements it connects.*

Definition 2.2.22. *An incidence poset IP_P is called* problem-free *if no subposet of IP_P extends IP_5 or $IP_{3,3}$.*

You may have noticed many of the constructions and some of the language concerning incidence posets resembles the same for graphs. This similarity forms the basis for the current thesis, and the intuition linking graphs and incidence posets will be discussed and formalized at greater length in the next chapter.

2.3 Category Theory

We here define the rudimentary elements of Category Theory necessary for the following study; however, the special, categorical objects constituting the bulk of this thesis will be explained in the introductory sections of their respective chapters so the math that follows will be easier to read. We use [5] as the reference for our definition of a category.

Definition 2.3.1. *A category $\mathcal{C}$ is a class $\mathrm{Ob}(\mathcal{C})$ of objects along with a family of mutually disjoint sets $\{\mathrm{Hom}(A, B)\}_{A,B \in \mathcal{C}}$ (the elements of which are called morphisms) such that*

(1) For every morphism $u \in \mathrm{Hom}(A, B)$ and $v \in \mathrm{Hom}(B, C)$, there exists a unique $uv \in \mathrm{Hom}(A, C)$, where uv is called the composition of u and v.

(2) For all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, $f(gh) = (fg)h$, and

(3) For each object $A \in \mathcal{C}$, there exists an identity morphism $\mathbf{1}_A : A \to A$ such that $f\mathbf{1}_A = f$ and $\mathbf{1}_A g = g$ for all $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(C, A)$.

The most intuitive category is $\mathcal{S}$, the category of sets. The objects of $\mathcal{S}$ are sets, and $\mathrm{Hom}(A, B)$ is simply the set of all functions from A to B. Not all categories are so straightforward. A group can be thought of as a category having one object, G, the group, such that the elements of $\mathrm{Hom}(G, G)$ are the elements of G. So, if $a \in G$, $a : G \to G$ would be defined such that $a(x) = ax$ for all $x \in G$. While unintuitive at first, we can see rather easily how associativity, composition of morphisms, and the identity morphism properties of categories would hold in such a category.

When discussing objects and morphisms within a category, it can be helpful to represent them using commutative diagrams.

Definition 2.3.2. *The diagram in Figure 28 commutes when $gf = h$.*

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dr, "h"] & \\ B \arrow[r, "g"] & C \end{tikzcd}$$

Figure 28: A Commutative Diagram

We will generally use diagrams only as physical representations of the objects and morphisms being discussed. This being the case, all diagrams in this thesis will be commutative, unless otherwise stated.

The following definition comes from [2].

Definition 2.3.3. *A category $\mathcal{B}$ is a* <u>*subcategory*</u> *of $\mathcal{C}$ when*

(1) $\mathrm{Ob}(\mathcal{B}) \subseteq \mathrm{Ob}(\mathcal{C})$*,*

(2) $\{\mathrm{Hom}(A, B)\}_{A,B\in\mathcal{B}} \subseteq \{\mathrm{Hom}(A, B)\}_{A,B\in\mathcal{C}}$*,*

(3) Every morphism in $\mathcal{B}$ is the restriction of the corresponding morphism in $\mathcal{C}$, and

(4) Every identity morphism in $\mathcal{B}$ is also an identity morphism in $\mathcal{A}$.

For example, the category of finite sets is a subcategory of the category of sets. The category of Abelian groups is a subcategory of the category of groups, and the category of finite, Abelian groups is a subcategory of the category of groups. As with graphs and posets, there are special subcategories within the categories that are the subject of this thesis. To formalize the special subcategories we will be discussing, functors must be defined. The following discussion about functors comes from [4].

Definition 2.3.4. *A* <u>*functor*</u> *is a map $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ consisting of*

(1) An object map assigning $A \in \mathrm{Ob}(\mathcal{C})$ to the object $\mathfrak{F}[A] \in \mathrm{Ob}(\mathcal{D})$, and

(2) A morphism map assigning every morphism $f : A \to B$ to the morphism $\mathfrak{F}[f] : \mathfrak{F}[A] \to \mathfrak{F}[B]$.

The morphism map must satisfy the following conditions:

(a) $\mathfrak{F}[\mathbf{1}_A] = \mathbf{1}_{\mathfrak{F}[A]}$, *and*

(b) $\mathfrak{F}[gf] = \mathfrak{F}[g]\mathfrak{F}[f]$ *whenever* gf *is defined in* $\mathcal{C}$.

Note that the object map and the function map of a functor $\mathfrak{F}$ bears the same name. If $\mathfrak{F}$ is used without any braces, the functor itself is being invoked. If $\mathfrak{F}[A]$ bears a capital letter, we are discussing the object map of $\mathfrak{F}$. If $\mathfrak{F}[f]$ bears a lowercase or Greek letter, the morphism map of $\mathfrak{F}$ is being mentioned.

Definition 2.3.5. *A functor* $\mathfrak{F}$ *is* <u>*faithful*</u> *when, for every pair of objects* $A, B \in \mathrm{Ob}(\mathcal{C})$ *and every pair of morphisms* $f, g \in \mathrm{Hom}(A, B)$, $\mathfrak{F}[f] = \mathfrak{F}[g] \implies f = g$.

Definition 2.3.6. *A functor is* <u>*full*</u> *when, for every pair of objects* $A, B \in \mathrm{Ob}(\mathcal{C})$ *and morphism* $g \in \mathrm{Hom}(\mathfrak{F}[A], \mathfrak{F}[B])$ *in* $\mathcal{D}$, *there exists a morphism* $f \in \mathrm{Hom}(A, B)$ *such that* $g = \mathfrak{F}[f]$.

Though not a perfect analogy, one could think of faithful functors as being approximately injective and full functors as being approximately surjective.

Definition 2.3.7. *A subcategory* $\mathcal{B}$ *of* $\mathcal{C}$ *is* <u>*full*</u> *whenever the inclusion functor from* $\mathcal{B}$ *to* $\mathcal{C}$ *is full.*

Definition 2.3.8. *Two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are* <u>*equivalent*</u> *when there exists a full and faithful functor* $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ *such that each object* $D \in \mathrm{Ob}(\mathcal{D})$ *is isomorphic to* $\mathfrak{F}[C]$ *for some* $C \in \mathrm{Ob}(\mathcal{C})$.

CHAPTER 3

THE CATEGORIES $\mathcal{FSG}$ AND $\mathcal{IP}_{FSG}$

By the end of this chapter, we wish to formalize the previously mentioned connection between finite, simple graphs and finite incidence posets. To do this, we will use the tools of category theory. This will require constructing categories of finite, simple graphs and finite incidence posets, then proving the categories are equivalent.

3.1 $\mathcal{FSG}$

We begin by constructing the category of finite, simple graphs, and then we will discuss special subcategories of the same.

Definition 3.1.1. *Let* $\mathrm{Ob}(\mathcal{FSG})$ *be the class of all finite, simple graphs, and let* $\mathrm{Hom}(G, H)$ *be the set of all graph homomorphisms from* G *to* H*, where* $G, H \in \mathrm{Ob}(\mathcal{FSG})$*.*

Theorem 3.1.2. $\mathcal{FSG}$ *is a category.*

Proof. Let $\varphi \in \mathrm{Hom}(G, H)$ and $\psi \in \mathrm{Hom}(H, J)$, where $G, H, J \in \mathrm{Ob}(\mathcal{FSG})$. We want to show $\psi\varphi \in \mathrm{Hom}(G, J)$. For all $u \in V(G)$, we know $\varphi(u) \in V(H)$ since φ is a graph homomorphism. As ψ is also a graph homomorphism, $\psi[\varphi(u)] \in V(J)$ for all $u \in V(G)$. By similar reasoning, $\psi[\varphi(uv)] \in E(J)$ whenever $uv \in E(G)$. Suppose $u \sim v$ in the graph G. Then, $\varphi(u) \sim \varphi(v)$, and $\psi[\varphi(u)] \sim \psi[\varphi(v)]$. Having satisfied all of the conditions for a graph homomorphism, $\psi\varphi \in \mathrm{Hom}(G, J)$.

Since graph homomorphisms are functions, associativity of graph homomorphism composition follows directly from associativity of function composition.

The homomorphism $\mathbf{1}_G : G \to G$ that maps each vertex and edge of G to itself will serve as the identity for any graph in $\mathrm{Ob}(\mathcal{FSG})$.

Having shown composition of graph homomorphisms yields a graph homomorphism, associativity of graph homomorphisms, and the existence of identity graph homomorphisms, $\mathcal{FSG}$ is a category.

■

Now, we can construct several subcategories of $\mathcal{FSG}$.

Definition 3.1.3. *Let* $\mathrm{Ob}(\mathcal{E}_G)$ *be the class of all finite, empty graphs, and let* $\mathrm{Hom}(E_1, E_2)$ *be the set of all graph homomorphisms between the empty graphs* E_1 *and* E_2*.*

Theorem 3.1.4. $\mathcal{E}_G$ *is a full subcategory of* $\mathcal{FSG}$*.*

Proof. All qualities of a category are inherited from $\mathcal{FSG}$, so $\mathcal{E}_G$ is clearly a category.

Since all finite, empty graphs are finite, simple graphs $\mathrm{Ob}(\mathcal{E}_G) \subseteq \mathrm{Ob}(\mathcal{FSG})$. By similar reasoning, $\{\mathrm{Hom}(E_1, E_2)\}_{E_1, E_2 \in \mathcal{E}_G} \subseteq \{\mathrm{Hom}(G, H)\}_{G, H \in \mathcal{FSG}}$. Every morphism is itself a morphism in $\mathcal{FSG}$, so every morphism is a restriction of the corresponding morphism in $\mathcal{FSG}$, and $\mathbf{1}_E \in \mathrm{Hom}(E, E)$ in both $\mathcal{E}_G$ and $\mathcal{FSG}$. The above being true, $\mathcal{E}_G$ is a subcategory of $\mathcal{FSG}$.

Define the functor $\mathfrak{E} : \mathcal{E}_G \to \mathcal{FSG}$ such that $\mathfrak{E}[E_1] = E_1$ and $\mathfrak{E}[\varphi] = \varphi$, where E_1 is an empty graph and φ is any graph homomorphism between two empty graphs. Consider any two empty graphs $E_1, E_2 \in \mathrm{Ob}(\mathcal{E}_G)$, and consider some morphism $\psi : \mathfrak{E}[E_1] \to \mathfrak{E}[E_2]$. Since $\psi \in \mathrm{Hom}(E_1, E_2)$ such that $\psi = \mathfrak{E}[\psi]$, $\mathfrak{E}$ is a full functor, and $\mathcal{E}_G$ is a full subcategory of $\mathcal{FSG}$.

■

Definition 3.1.5. *Let* $\mathrm{Ob}(\mathcal{P})$ *be the class of all finite paths, and let* $\mathrm{Hom}(P_1, P_2)$ *be the set of all graph homomorphisms between the paths* P_1 *and* P_2*.*

Theorem 3.1.6. $\mathcal{P}$ *is a full subcategory of* $\mathcal{FSG}$*.*

Proof. All qualities of a category are inherited from $\mathcal{FSG}$, so $\mathcal{P}$ is clearly a category.

Since all finite paths are finite, simple graphs $\mathrm{Ob}(\mathcal{P}) \subseteq \mathrm{Ob}(\mathcal{FSG})$. By similar reasoning, $\{\mathrm{Hom}(P_1, P_2)\}_{P_1, P_2 \in \mathcal{P}} \subseteq \{\mathrm{Hom}(G, H)\}_{G, H \in \mathcal{FSG}}$. Every morphism is itself a morphism in $\mathcal{FSG}$, so every morphism is a restriction of the corresponding morphism in $\mathcal{FSG}$. And, $\mathbf{1}_P \in \mathrm{Hom}(P, P)$ in both $\mathcal{P}$ and $\mathcal{FSG}$. The above being true, $\mathcal{P}$ is a subcategory of $\mathcal{FSG}$.

Define the functor $\mathfrak{P} : \mathcal{P} \to \mathcal{FSG}$ such that $\mathfrak{P}[P_1] = P_1$ and $\mathfrak{P}[\varphi] = \varphi$, where P_1 is a path and φ is any graph homomorphism between two paths. Consider any two paths $P_1, P_2 \in \text{Ob}(\mathcal{P})$, and consider some morphism $\psi : \mathfrak{P}[P_1] \to \mathfrak{P}[P_2]$. Since $\psi \in \text{Hom}(P_1, P_2)$ such that $\psi = \mathfrak{P}[\psi]$, $\mathfrak{P}$ is a full functor, and $\mathcal{P}$ is a full subcategory of $\mathcal{FSG}$.

■

As you might have noticed, the two proofs are identical except for names. One can easily confirm that the same proof will work for the class of all connected graphs, the class of all disconnected graphs, the class of all cycles, the class of all trees, the class of all forests, the class of all complete graphs, and the class of all planar graphs. Each special class of subgraphs has a corresponding full subcategory of $\mathcal{FSG}$. The full category of connected graphs will be denoted $\mathcal{C}_G$. The full category of disconnected graphs will be denoted $\mathcal{D}_G$. The full category of cycles will be denoted $\mathcal{O}$. The full category of trees will be denoted $\mathcal{T}$. The full category of forests will be denoted $\mathcal{F}$. The full category of complete graphs will be denoted $\mathcal{K}$. The full category of planar graphs will be denoted $\mathcal{R}$.

3.2 $\mathcal{IP}_{FSG}$

As with $\mathcal{FSG}$, we will construct the category of finite incidence posets, and then we will construct special subcategories of the same.

Definition 3.2.1. *Let* $\text{Ob}(\mathcal{IP}_{FSG})$ *be the class of all incidence posets, and let* $\text{Hom}(IP_A, IP_B)$ *be the set of all IP-morphisms from* IP_A *to* IP_B*, where* $IP_A, IP_B \in \text{Ob}(\mathcal{IP}_{FSG})$*.*

Theorem 3.2.2. $\mathcal{IP}_{FSG}$ *is a category.*

Proof. Let $\alpha \in \text{Hom}(IP_A, IP_B)$ and $\beta \in \text{Hom}(IP_B, IP_C)$, where $IP_A, IP_B, IP_C \in \text{Ob}(\mathcal{IP}_{FSG})$. We want to show $\beta\alpha \in \text{Hom}(IP_A, IP_C)$. As maximal elements are mapped to maximal elements, $(\beta\alpha)(x)$ will be maximal in IP_C for all x maximal in

IP_A. Similarly, $(\beta\alpha)(y)$ will be minimal in IP_C for all y minimal in IP_A. Suppose $a \leq b$ for some $a, b \in IP_A$. Since α and β are IP-morphisms, $\alpha(a) \leq \alpha(b)$ in IP_B, and $(\beta\alpha)(a) \leq (\beta\alpha)(b)$ in IP_C. Suppose $c \neq d$ and $\mathbf{c}_{IP_A}(c) \cap \mathbf{c}_{IP)_A}(d) \neq \emptyset$ for some maximal $c, d \in IP_A$. It follows from the definition of IP-morphisms that $\alpha(c) \neq \alpha(d)$. As α preserves order, $\exists e \in IP_A \ni \alpha(e) \leq \alpha(c)$ and $\alpha(e) \leq \alpha(d)$, meaning $\mathbf{c}_{IP_C}(\alpha(c)) \cap \mathbf{c}_{IP_C}(\alpha(d)) \neq \emptyset$. Therefore, $(\beta\alpha)(c) \neq (\beta\alpha)(d)$, and $\beta\alpha \in \mathrm{Hom}(IP_A, IP_C)$.

Since IP-morphisms are functions, associativity of IP-morphism composition follows directly from associativity of function composition.

The IP-morphism $\mathbf{1}_{IP_A} : IP_A \to IP_A$ that maps each element of IP_A to itself will serve as the identity for any incidence poset $IP_A \in \mathcal{IP}_{FSG}$.

Having shown IP-morphism composition yields an IP-morphism, IP-morphism composition is associative, and the existence of identity IP-morphisms, $\mathcal{IP}_{FSG}$ is a category.

■

Having proved $\mathcal{IP}_{FSG}$ is a category, we can now consider subcategories.

Definition 3.2.3. *Let* $\mathrm{Ob}(\mathcal{L})$ *be the class of all cats' cradles, and let* $\mathrm{Hom}(L_1, L_2)$ *be the set of all IP-morphisms between the cat's cradles* L_1 *and* L_2*.*

Theorem 3.2.4. $\mathcal{L}$ *is a full subcategory of* $\mathcal{IP}_{FSG}$*.*

Proof. The category $\mathcal{L}$ inherits all of the qualities necessary to be a category directly from $\mathcal{IP}_{FSG}$.

Since all cats' cradles are incidence posets, $\mathrm{Ob}(\mathcal{L}) \subseteq \mathrm{Ob}(\mathcal{IP}_{FSG})$. Similarly, $\{\mathrm{Hom}(L_1, L_2)\}_{L_1, L_2 \in \mathcal{L}} \subseteq \{\mathrm{Hom}(IP_A, IP_B)\}_{IP_A, IP_B \in \mathcal{IP}_{FSG}}$. Since every morphism in $\mathcal{L}$ is also a morphism in $\mathcal{IP}_{FSG}$, every morphism in $\mathcal{L}$ is automatically the restriction of the same morphism in $\mathcal{IP}_{FSG}$. It follows that $\mathbf{1}_L \in \mathrm{Hom}(L, L)$ is in both $\mathcal{L}$ and $\mathcal{IP}_{FSG}$, so identity functions in $\mathcal{L}$ are the same as those in $\mathcal{IP}_{FSG}$.

Define the funtor $\mathfrak{L} : \mathcal{L} \to \mathcal{IP}_{FSG}$ such that $\mathfrak{L}[L_1] = L_1$ and $\mathfrak{L}[\alpha] = \alpha$ for all $L \in \mathrm{Ob}(\mathcal{L})$ and any morphism α. Consider any two cats' cradles $L_1, L_2 \in \mathrm{Ob}(\mathcal{IP})_{FSG})$, and consider some morphism $\beta : \mathfrak{L}[L_1] \to \mathfrak{L}[L_2]$. Since $\beta \in \mathrm{Hom}(L_1, L_2)$ is an IP-morphism such that $\beta = \mathfrak{L}[\beta]$, $\mathfrak{L}$ is a full functor, and $\mathcal{L}$ is a full subcategory of $\mathcal{IP}_{FSG}$.

■

As with graphs, the proof of every full subcategory of $\mathcal{IP}_{FSG}$ will be nigh exactly the same as the previous proof. We now list the names of each subcategory important to the present study: the full subcategory of empty incidence posets will be denoted $\mathcal{E}_{IP}$, the full subcategory of valleys will be denoted $\mathcal{V}$, the full subcategory of connected incidence posets will be denoted $\mathcal{C}_{IP}$, the full subcategory of cat's cradles will be denoted $\mathcal{L}$, the full subcategory of strings will be denoted $\mathcal{G}$, the full subcategory of saturated incidence posets will be denoted $\mathcal{S}$, the full subcategory of problem-free incidence posets will be denoted $\mathcal{PF}$, the full subcategory of disconnected incidence posets will be denoted $\mathcal{D}_{IP}$, and the full subcategory of heaps will be denoted $\mathcal{H}$.

3.3 Categorical Equivalence

Having introduced $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$, the question of their equivalence can now be broached. Intuitively, an incidence poset can be created from a graph. If considering the graph G, its vertex set $V(G)$ constitutes $\mathrm{Max}(IP_G)$ for some incidence poset IP_G, and its edge set $E(G)$ composes $\mathrm{Min}(IP_G)$. An edge is less than a vertex if the vertex is incident with the edge. In Figure 29, a graph H and its corresponding incidence poset are shown.

We formalize this intuitive connection between graphs and their corresponding incidence posets in the following definition.

Definition 3.3.1. *Let $G \in \mathrm{Ob}(\mathcal{FSG})$. Define $\mathfrak{F}[G] = (V(G) \cup E(G), \leq_G)$ such that, $x, y \in \mathfrak{F}[G]$, $x \leq y$ if and only if $x = y$ or $y \in x$ with $x \in E(G)$ and $y \in V(G)$.*

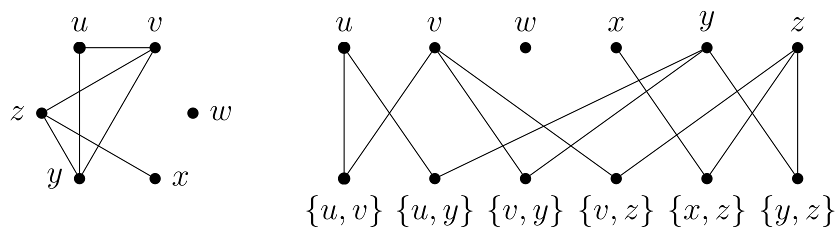

Figure 29: H and the Incidence Poset Corresponding to H

Before we can show $\mathfrak{F}[G]$ is an incidence poset, we must prove $\leq_G$ is a partial order.

Theorem 3.3.2. *If* $G \in \mathrm{Ob}(\mathcal{FSG})$*,* $\leq_G$ *as defined above is a partial order.*

Proof. Reflexivity is immediate since $x \leq_G x$ by construction.

If $x \leq_G y$ and $y \leq_G x$, two possibilities exist: $x = y$, or $x \in y$ and $y \in x$. Recall, $x \in y$ means x is a vertex incident with the edge y, and $y \in x$ means y is a vertex incident with the edge x. However, this means $x \in y$ and $y \in x$ would imply $x, y \in V(G)$ and $x, y \in E(G)$, which is impossible. Thus, $\leq_G$ is antisymmetric.

Suppose $x \leq_G y$ and $y \leq_G z$. There are three options: $y \in x$, $z \in y$, or $x = y = z$. If one of those three options were not the case, an edge in $E(G)$ would contain another edge in $E(G)$, which is impossible. If $y \in x$, y is a vertex incident with the edge x, and z must be a vertex as well. This means $y = z$, which implies $z \in x$, and $x \leq_G z$. If $z \in y$, z is a vertex incident with the edge y, and x must also be an edge. It follows that $x = y$, which also implies $z \in x$, meaning $x \leq_G z$. If $x = y = z$, $x \leq_G z$ by definition of $\leq_G$.

Having shown $\leq_G$ is reflexive, antisymmetric, and transitive, we may conclude $\leq_G$ is a partial order.

∎

Theorem 3.3.3. $\mathfrak{F}[G]$ *is an incidence poset.*

Proof. For purposes of clarity, this proof will be organized with respect to the conditions for incidence posets laid out in Definition 2.2.11.

(1) Suppose $x \in V(G)$. Since x is not a set, there does not exists any element greater than x, so $x \in \mathrm{Max}(\mathfrak{F}[G]), \forall x \in V(G)$. Suppose $y = \{a, b\} \in E(G)$. Then, $y \leq_G a$ and $y \leq_G b$. Since the edges of $E(G)$ comprise only vertices from $V(G)$, there cannot exist another $z \in E(G)$ such that $z \in y$. Therefore, $y \in \mathrm{Min}(\mathfrak{F}[G]), \forall y \in E(G)$. Having exhausted every element of $V(G) \cup E(G)$, we have shown that either $c \in \mathrm{Max}(\mathfrak{F}[G])$ or $c \in \mathrm{Min}(\mathfrak{F}[G])$ for all $c \in \mathfrak{F}[G]$.

(2) Being incident with exactly two vertices in $V(G)$, $\left|\mathbf{C}_{\mathfrak{F}[G]}(y)\right| = 2, \forall y \in \mathrm{Min}(\mathfrak{F}[G])$. In other words, every minimal element of $\mathfrak{F}[G]$ is covered by exactly two maximal elements.

(3) Every edge in $E(G)$ is necessarily distinct. Since every edge in $E(G)$ is a two-element subset of $V(G)$, the set of covers of every minimal element in $\mathfrak{F}[G]$ will also be distinct.

(4) The finite union of finite sets is finite, so $\mathrm{Max}(\mathfrak{F}[G]) = V(G) \cup E(G)$ is finite.

Having satisfied the four conditions for incidence posets, $\mathfrak{F}[G] \in \mathrm{Ob}(\mathcal{IP}_{FSG})$.

■

Naturally, we would also like to construct morphisms in $\{\mathrm{Hom}(IP_A, IP_B)\}_{IP_A, IP_B \in \mathcal{IP}_{FSG}}$ from morphisms in $\{\mathrm{Hom}(G, H)\}_{G,H \in \mathcal{FSG}}$.

Definition 3.3.4. *Let $\mathfrak{F}[\varphi] : \mathfrak{F}[G] \to \mathfrak{F}[H]$ such that $\mathfrak{F}[\varphi](x) = \varphi(x)$, where $\varphi : G \to H$ is some graph homomorphism.*

Theorem 3.3.5. *$\mathfrak{F}[\varphi] : \mathfrak{F}[G] \to \mathfrak{F}[H]$ is an IP-morphism.*

Proof. Recall $\varphi(x) : G \to H$ is a graph homomorphism. Since any $x \in G$ is also in $\mathfrak{F}[G]$, and any $\phi(x) \in H$ is also in $\mathfrak{F}[H]$, we know $\mathfrak{F}[\varphi](x)$ is well-defined. We prove $\mathfrak{F}[\varphi]$ is an IP-morphism as we proved $\mathfrak{F}[G]$ is an incidence poset: by referencing Definition 2.2.12, the definition of IP-morphisms.

(1) We know $\varphi(x) \in V(H), \forall x \in V(G)$, so $\mathfrak{F}[\varphi](x) \in \text{Max}(\mathfrak{F}[H]), \forall x \in \text{Max}(\mathfrak{F}[G])$.

(2) Similarly, $\varphi(y) \in E(H), \forall y \in E(G)$ implies $\mathfrak{F}[\varphi](y) \in \text{Min}(\mathfrak{F}[H]), \forall y \in \text{Min}(\mathfrak{F}[G])$.

(3) Regardless of the nature of any $z \in G$, $\mathfrak{F}[\varphi](z) \leq_H \mathfrak{F}[\varphi](z)$. Let $a \in \text{Min}(\mathfrak{F}[G])$, $b \in \text{Max}(\mathfrak{F}[G])$, and $a <_G b$. It follows that a is an edge incident with the vertex b in the graph G. Therefore, $a = \{b, c\}$ for some $c \in V(G)$. Graph homomorphisms preserve adjacency, so $\varphi(b) \sim \varphi(c)$, and $\varphi(a)$ is incident with $\varphi(b)$ and $\varphi(c)$. So, $\mathfrak{F}[\varphi](a) \in \mathbf{c}_{\mathfrak{F}[H]}(\varphi(b)) \cap \mathbf{c}_{\mathfrak{F}[\mathfrak{H}]}(\varphi(c))$, proving $a \leq_G b \implies \mathfrak{F}[\varphi](a) \leq_H \mathfrak{F}[\varphi](b)$ for all $a, b \in \mathfrak{F}[G]$.

(4) Let $f, g \in \text{Max}(\mathfrak{F}[G]) \ni f \neq g$ and $\left|\mathbf{c}_{\mathfrak{F}[G]}(f) \cap \mathbf{c}_{\mathfrak{F}[G]}(g)\right| \neq 0$. Then, $f, g \in V(G)$, and $f \sim g$. Since φ preserves adjacency, and vertices in a graph are not adjacent with themselves,$\varphi(f) \neq \varphi(g)$, which implies $\mathfrak{F}[\varphi](f) \neq \mathfrak{F}[\varphi](g)$.

Having satisfied the conditions necessary to be an IP-morphism, $\mathfrak{F}[\varphi]$ is also an IP-morphism.

■

Now that we have formalized the connection between graphs and incidence posets and graph homomorphisms and IP-morphisms, we can think of $\mathfrak{F}$ as a functor from $\mathcal{FSG}$ to $\mathcal{IP}_{FSG}$. The two above theorems define the object map and morphism map, respectively, of $\mathfrak{F}$. To prove $\mathfrak{F}$ is a functor, we still need to show $\mathfrak{F}[\mathbf{1}_G] = \mathbf{1}_{\mathfrak{F}[G]}$ and $\mathfrak{F}[\varphi\psi] = \mathfrak{F}[\varphi]\mathfrak{F}[\psi]$.

Theorem 3.3.6. *$\mathfrak{F} : \mathcal{FSG} \to \mathcal{IP}_{FSG}$ is a functor.*

Proof. Consider $\mathfrak{F}[\mathbf{1}_G]$, where $G \in \mathrm{Ob}(\mathcal{FSG})$. We can see that

$$\mathfrak{F}[\mathbf{1}_G](x) = \mathbf{1}_G(x) = x$$

as desired.

Let $\varphi : G \to H$, $\psi : H \to K$, and $G, H, K \in \mathrm{Ob}(\mathcal{FSG})$. Consider $\mathfrak{F}[\varphi\psi]$. Using the properties of φ, ψ, and $\mathfrak{F}$,

$$\begin{aligned}
\mathfrak{F}[\varphi\psi](x) &= (\varphi\psi)(x) \\
&= \varphi(\psi(x)) \\
&= \varphi(\mathfrak{F}[\psi](x)) \\
&= \mathfrak{F}[\varphi](\mathfrak{F}[\psi](x)) \\
&= (\mathfrak{F}[\varphi]\mathfrak{F}[\psi])(x)
\end{aligned}$$

Having shown $\mathfrak{F}$ preserves identities and composition of functions, $\mathfrak{F}$ is a functor. ∎

Categorical equivalence between $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$ being the current goal, we must show $\mathfrak{F}$ is full and faithful, at which point, we will have shown $\mathcal{FSG}$ is equivalent to $\mathcal{IP}_{FSG}$.

Theorem 3.3.7. *The functor* $\mathfrak{F} : \mathcal{FSG} \to \mathcal{IP}_{FSG}$ *is full and faithful.*

Proof. Let $G, H \in \mathrm{Ob}(\mathcal{FSG})$, and let $\alpha \in \mathrm{Hom}(\mathfrak{F}[G], \mathfrak{F}[H])$. Consider the function $\phi : G \to H$ such that $\varphi(x) = \alpha(x)$. We can see that $\mathfrak{F}[\varphi](x) = \varphi(x) = \alpha(x)$, and $\mathfrak{F}$ is, therefore, full.

Let $J, K \in \mathrm{Ob}(\mathcal{FSG})$, and let $\mu, \sigma \in \mathrm{Hom}(J, K)$. Suppose $\mathfrak{F}[\mu] = \mathfrak{F}[\sigma]$. Then,

$$\mathfrak{F}[\mu](x) = \mathfrak{F}[\sigma](x)$$

$$\mu(x) = \mathfrak{F}[\sigma](x)$$

$$\mu(x) = \sigma(x)$$

as desired. Thus, $\mathfrak{F}$ is faithful.

∎

From these theorems, the categorical equivalence of $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$ follows immediately.

Theorem 3.3.8. *$\mathcal{FSG}$ is equivalent to $\mathcal{IP}_{FSG}$.*

3.4 Equivalent Subcategories

It would stand to reason that the subcategories of $\mathcal{FSG}$ are equivalent to subcategories of $\mathcal{IP}_{FSG}$. We have already presented the subcategories of $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$ that will be equivalent to one another, and the proofs will follow. However, we will not go through the steps of proving the restriction of $\mathfrak{F}$ to each subcategory is full and faithful. Rather, we will prove $\mathfrak{F}[G] \in \mathrm{Ob}(\mathcal{Y})$ for all $G \in \mathrm{Ob}(\mathcal{X})$, and we will prove $\exists H \ni \mathfrak{F}[H] \cong IP_H \; \forall IP_H \in \mathrm{Ob}(\mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are subcategories of $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$, respectively. That $\mathfrak{F}$ restricted to the appropriate categories is full and faithful should, at that point, be clear, so we shall consider the proof of the above conditions sufficient to prove $\mathcal{X}$ is equivalent to $\mathcal{Y}$.

Theorem 3.4.1. *The category of empty graphs $\mathcal{E}_G$ is equivalent to the category of empty incidence posets $\mathcal{E}_{IP}$.*

Proof. In this proof and those to follow, we will enumerate the two conditions mentioned above as (1) and (2), with (1) representing $\mathfrak{F}[G] \in \mathrm{Ob}(\mathcal{Y})$ for all $G \in \mathrm{Ob}(\mathcal{X})$, and (2) representing $\exists H \ni \mathfrak{F}[H] \cong IP_H \ \forall IP_H \in \mathrm{Ob}(\mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are the categories under investigation.

(1) Since all $E \in \mathrm{Ob}(\mathcal{E}_G)$ have no edges, $\mathfrak{F}_{|\mathcal{E}_G}[E]$ will have no minimal elements, and $\mathfrak{F}_{|\mathcal{E}_G}[E] \in \mathrm{Ob}(\mathcal{E}_{IP}) \ \forall E \in \mathrm{Ob}(\mathcal{E}_G)$.

(2) Consider some $IP_E \in \mathrm{Ob}(\mathcal{E}_{IP})$. If IP_E has n elements, then any empty graph E_n having n vertices will be such that $\left|\mathrm{Max}(\mathfrak{F}_{|\mathcal{E}_G}[E_n])\right| = n$, and, therefore, $\mathfrak{F}_{|\mathcal{E}_G}[E_n] \cong IP_E$.

■

Theorem 3.4.2. *The category of path graphs $\mathcal{P}$ is equivalent to the category of valley incidence posets $\mathcal{V}$.*

Proof.

(1) Consider some $P \in \mathrm{Ob}(\mathcal{P})$. P contains exactly two vertices of degree 1 that we shall call x_1 and x_2. It follows that $x_1, x_2 \in \mathrm{Max}(\mathfrak{F}_{|\mathcal{P}}[P])$, and $\left|\mathbf{c}_{\mathfrak{F}_{|\mathcal{P}}[P]}(x_1)\right| = \left|\mathbf{c}_{\mathfrak{F}_{|\mathcal{P}}[P]}(x_2)\right| = 1$. Every other $y \in V(P)$ has degree 2, so $\left|\mathbf{c}_{\mathfrak{F}_{|\mathcal{P}}[P]}(y)\right| = 2, \forall y \in V(P)$. As no other vertices exist in P, condition (1) is satisfied.

(2) Let $IP_V \in \mathrm{Ob}(\mathcal{V})$. If $|\mathrm{Max}(IP_V)| = n$, any path $P_n \in \mathrm{Ob}(\mathcal{P})$ having n vertices will be such that $\mathfrak{F}_{|\mathcal{P}}[P_n] \cong IP_V$.

■

Theorem 3.4.3. *The category of connected graphs $\mathcal{C}_G$ is equivalent to the category of connected incidence posets $\mathcal{C}_{IP}$.*

Proof.

(1) Consider some $C \in \mathrm{Ob}(\mathcal{C}_G)$. There exist m paths $P^k_{i,j}$ between any two vertices $x_i, x_j \in V(C)$ with $i \neq j$ and $1 \leq k \leq m$ such that $P^k_{i,j}$ is a subgraph of C. It follows that $\mathfrak{F}_{|\mathcal{C}_G}[P^k_{i,j}]$ is a subposet of $\mathfrak{F}_{|\mathcal{C}_G}[C]$ for all $x_i, x_j \in V(C)$. From the previous theorem, we know $\mathfrak{F}_{|\mathcal{C}_G}[P^k_{i,j}]$ is a valley between x_i and x_j in $\mathfrak{F}_{|\mathcal{C}_G}[C]$. Since every element of $\mathrm{Max}(\mathfrak{F}_{|\mathcal{C}_G}[C])$ is some $x_i \in V(C)$, there exists a valley between every pair of vertices in $\mathfrak{F}_{|\mathcal{C}_G}[C]$. Therefore, $\mathfrak{F}_{|\mathcal{C}_G}[C] \in \mathrm{Ob}(\mathcal{C}_{IP}), \forall C \in \mathrm{Ob}(\mathcal{C}_G)$.

(2) Let $IP_C \in \mathrm{Ob}(\mathcal{C}_{IP})$, and let $|\mathrm{Max}(IP_C)| = n$. Consider the poset $IP_{\widetilde{C}}$ isomorphic to IP_C such that every maximal element of $IP_{\widetilde{C}}$ bears the same name as in IP_C, and every minimal element y has the name $\{a, b\}$, where $\{a, b\} = \mathbf{C}_{IP_C}(y)$. Consider the k valley subposets $IP^\ell_{V_{i,j}}$ of $IP_{\widetilde{C}}$ connecting the maximal elements x_i and x_j of $IP_{\widetilde{C}}$ for $i \neq j$. We know there exists a $\widetilde{P}^\ell_{i,j}$ such that $\mathfrak{F}[\widetilde{P}^\ell_{i,j}] = IP^\ell_{V_{i,j}}$ for every i, j, and ℓ. Let

$$\widetilde{G} = \bigcup_{1 \leq i < j \leq n} \bigcup_{1 \leq \ell \leq k} \widetilde{P}^\ell_{i,j}$$

In other words, $\widetilde{G}$ is the graph whose edge and vertex sets are the union of the edge and vertex sets of every $\widetilde{P}^\ell_{i,j}$. $\widetilde{G}$ is connected automatically, as the vertices come from paths. We can also see that we have constructed the graph whose vertices and edges are exactly the maximal and minimal elements of $IP_{\widetilde{C}}$, meaning $\mathfrak{F}[\widetilde{G}] = IP_{\widetilde{C}} \cong IP_C$, as desired.

■

Theorem 3.4.4. *The category of disconnected graphs $\mathcal{D}_G$ is equivalent to the category of disconnected incidence posets $\mathcal{D}_{IP}$.*

Proof.

(1) Every disconnected graph (and disconnected incidence poset) is the union of connected graphs, i.e., its connected components. So, $\mathfrak{F}_{|\mathcal{D}_G}[D]$ for any disconnected graph D is really $\mathfrak{F}_{|\mathcal{C}_G}[C_1] \cup \mathfrak{F}_{|\mathcal{C}_G}[C_2] \cup \cdots \cup \mathfrak{F}_{|\mathcal{C}_G}[C_n]$, where $D =$

$C_1 \cup \cdots \cup C_n$. From the previous proof, we know $\mathfrak{F}_{|\mathcal{D}_G}[D] = \mathfrak{F}_{|\mathcal{C}_G}[C_1] \cup \mathfrak{F}_{|\mathcal{C}_G}[C_2] \cup \cdots \cup \mathfrak{F}_{|\mathcal{C}_G}[C_n]$, and since every C_i and C_j are mutually exclusive, we know every corresponding $\mathfrak{F}_{|\mathcal{C}_G}[C_i]$ and $\mathfrak{F}_{|\mathcal{C}_G}[C_j]$ are mutually exclusive as well. Since $\mathfrak{F}_{|\mathcal{D}_G}[D]$ is disconnected, we have shown $\mathfrak{F}_{|\mathcal{D}_G}[D] \in \mathrm{Ob}(\mathcal{D}_{IP})$.

(2) Let $IP_D = IP_{C_1} \cup \cdots \cup IP_{C_n}$ be a disconnected incidence poset having the n connected components IP_{C_1} through IP_{C_n}. From the previous proof, we know there exist connected graphs $\widetilde{G}_i$ such that $\mathfrak{F}_{|\mathcal{C}_G}[\widetilde{G}_i] \cong IP_{C_i}$ for every $1 \leq i \leq n$. It follows that

$$\bigcup_{1 \leq i \leq n} \mathfrak{F}_{|\mathcal{C}_G}[\widetilde{G}_i] \cong \bigcup_{1 \leq i \leq n} IP_{C_i}$$

which implies $\bigcup_{1 \leq i \leq n} \widetilde{G} \in \mathrm{Ob}(\mathcal{D}_G)$ such that $\mathfrak{F}_{|\mathcal{D}_G}[\bigcup_{1 \leq i \leq n} \widetilde{G}] \cong IP_D$.

■

Theorem 3.4.5. *The category of cycles $\mathcal{O}$ is equivalent to the category of cats' cradles $\mathcal{L}$.*

Proof.

(1) Let $O \in \mathrm{Ob}(\mathcal{O})$. Since every $o \in V(O)$ has degree two, $\left|\mathbf{c}_{\mathfrak{F}_{|\mathcal{O}}[O]}(x)\right| = 2$ for all $x \in \mathrm{Max}(\mathfrak{F}_{|\mathcal{O}}[O])$, and $\mathfrak{F}_{|\mathcal{O}}[O] \in \mathrm{Ob}(\mathcal{L})$ for all $O \in \mathrm{Ob}(\mathcal{O})$.

(2) Let $IP_L \in \mathrm{Ob}(\mathcal{L})$. Consider the incidence poset $IP_{\widetilde{L}}$ isomorphic to IP_L such that every maximal element bears the same name, and every minimal element $y = \mathbf{C}_{IP_{\widetilde{L}}}(y)$. Let $L = (\mathrm{Max}(IP_{\widetilde{L}}), \mathrm{Min}(IP_{\widetilde{L}}))$. Clearly, $\mathfrak{F}_{|\mathcal{O}}[L] = IP_{\widetilde{L}}$, which implies $\mathfrak{F}_{|\mathcal{O}}[L] \cong IP_L$.

■

Theorem 3.4.6. *The category of forests $\mathcal{F}$ is equivalent to the category of heaps $\mathcal{H}$.*

Proof.

(1) Let $F \in \mathrm{Ob}(\mathcal{F})$. Suppose $\mathfrak{F}_{|\mathcal{F}}[F]$ contains a cat's cradle IP_L as a subposet. Then, $\mathrm{Max}(IP_L) \subseteq V(F)$, and $\mathrm{Min}(IP_L) \subseteq E(F)$. It follows that $O = (\mathrm{Max}(IP_L), \mathrm{Min}(IP_L))$ is a subgraph of F that is a cycle, contradicting F being a forest. Therefore, $\mathfrak{F}_{|\mathcal{F}}[F]$ contains no cat's cradles.

(2) Let $IP_H \in \mathrm{Ob}(\mathcal{H})$. Consider the incidence poset $IP_{\widetilde{H}}$ defined such that $\mathrm{Max}(IP_{\widetilde{H}}) = \mathrm{Max}(IP_H)$ and $y \in \mathrm{Min}(IP_{\widetilde{H}}) \implies y = \mathbf{C}_{IP_{\widetilde{H}}}(y) = \mathbf{C}_{IP_H}(y)$. It is clear that $IP_{\widetilde{H}} \cong IP_H$. As in the previous proofs, the existence of a forest F' such that $\mathfrak{F}_{|\mathcal{F}}[F'] = IP_{\widetilde{H}}$ is immediate. Therefore, $\mathfrak{F}_{|\mathcal{F}}[F'] \cong IP_H$, as desired.

■

Theorem 3.4.7. *The category of trees $\mathcal{T}$ is equivalent to the category of strings $\mathcal{G}$.*

Proof.

(1) Let $T \in \mathrm{Ob}(\mathcal{T})$. Since T is also a forest, $\mathfrak{F}_{|\mathcal{T}}[T]$ will have no cat's cradle subposet. Since T is connected, $\mathfrak{F}_{|\mathcal{T}}[T]$ will be connected. Being connected and having no cat's cradle, $\mathfrak{F}_{|\mathcal{T}}[T] \in \mathrm{Ob}(\mathcal{L})$.

(2) Again, since $\exists F \in \mathrm{Ob}(\mathcal{F}) \ni \mathfrak{F}_{|\mathcal{F}}[F] \cong IP_H$ for any $IP_H \in \mathrm{Ob}(\mathcal{H})$, and $\exists C \in \mathrm{Ob}(\mathcal{C}_G) \ni \mathfrak{F}_{|\mathcal{C}_G}[C] \cong IP_C$ for any $IP_C \in \mathrm{Ob}(\mathcal{C}_{IP})$, it follows that $\exists T \in \mathrm{Ob}(\mathcal{T}) \ni \mathfrak{F}_{|\mathcal{T}}[T] \cong IP_G$ for any $IP_G \in \mathrm{Ob}(\mathcal{G})$, all trees being connected forests, and all strings being connected heaps.

■

Theorem 3.4.8. *The category of complete graphs $\mathcal{K}$ is equivalent to the category of saturated incidence posets $\mathcal{S}$.*

Proof.

(1) Let $\mathbb{K}_n \in \mathrm{Ob}(\mathcal{K})$. $\mathbb{K}_n$ has $\binom{n}{2}$ edges, meaning $\mathfrak{F}_{|\mathcal{K}}[\mathbb{K}_n]$ has $\binom{n}{2} = \binom{|\mathrm{Max}(\mathfrak{F}_{|\mathcal{K}}[\mathbb{K}_n])|}{2}$ minimal elements, and is, therefore, saturated.

(2) Let $IP_{\mathcal{S}} \in \mathrm{Ob}(\mathcal{S})$. Consider $IP_{\widetilde{\mathcal{S}}} \in \mathrm{Ob}(\mathcal{S})$ such that $IP_{\widetilde{\mathcal{S}}} \cong IP_{\mathcal{S}}$, $\mathrm{Max}(IP_{\widetilde{\mathcal{S}}}) = \mathrm{Max}(IP_{\mathcal{S}})$, and $y = \mathbf{C}_{IP_{\mathcal{S}}}(y)$ for all $y \in \mathrm{Min}(IP_{\widetilde{\mathcal{S}}})$. Since $|\mathrm{Min}(IP_{\widetilde{\mathcal{S}}})| = \binom{|\mathrm{Max}(IP_{\widetilde{\mathcal{S}}})|}{2}$, $G \ni \mathfrak{F}[G] = IP_{\widetilde{\mathcal{S}}}$ must be complete. The theorem follows.

■

Theorem 3.4.9. *The category of planar graphs $\mathcal{R}$ is equivalent to the category of problem-free incidence posets $\mathcal{PF}$.*

Proof.

(1) From the previous theorem, it follows that $\mathfrak{F}[\mathbb{K}_5] \cong IP_5$. It is also easy to verify that $\mathfrak{F}[\mathbb{K}_{3,3}] \cong IP_{3,3}$. This being so, $R \in \mathrm{Ob}(\mathcal{R})$ will be such that $\mathfrak{F}_{|\mathcal{R}}[R]$ has neither an IP_5 nor an $IP_{3,3}$ subdivision, which implies $\mathfrak{F}_{|\mathcal{R}}[R] \in \mathrm{Ob}(\mathcal{PF})$.

(2) Using the same method of construction and same logic as in all of the previous theorems showing one subcategory to be equivalent to another, it is easy to construct a R' such that $\mathfrak{F}_{|\mathcal{R}}[R'] \cong IP_R$ for any $IP_R \in \mathrm{Ob}(\mathcal{PF})$.

■

CHAPTER 4

EQUALIZERS AND COEQUALIZERS

4.1 Equalizers

Having shown the categories $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$ are equivalent, we can now leave $\mathcal{FSG}$ behind and construct in $\mathcal{IP}_{FSG}$ some of the categorical objects Buvaneswari constructed in $\mathcal{FSG}$. The first objects we will deal with, equalizers, have an intuitive role within categories: they restrict the domain of two functions so that, under a specific composition, the functions are equal.

Definition 4.1.1. *Consider two morphisms $f : A \to B$ and $g : A \to B$ in the category $\mathcal{C}$. An* equalizer *of f and g in $\mathcal{C}$ is an object E along with a function e such that*

(1) $e : E \to A \in \mathrm{Hom}(E, A)$,

(2) $fe = ge$, *and*

(3) For any morphism $e' : E' \to A$ satisfying conditions (1) and (2), there exists a unique morphism $\overline{e} : E' \to E$ such that $e\overline{e} = e'$.

We say equalizers exist in the category *$\mathcal{C}$ when every pair of objects A and B in $\mathrm{Ob}(\mathcal{C})$ and every two morphisms in $\mathrm{Hom}(A, B)$ have a corresponding equalizer (E, e). We call E the* equalizer object *and e the* equalizer function.

See Figure 30 for the diagrammatic definition of an equalizer. Though not mentioned in the proof, the existence of equalizers in $\mathcal{IP}_{FSG}$ depends on $IP_\emptyset$, the incidence poset having no elements. We will define the equalizer object of two IP-morphisms

$$\begin{tikzcd} E \arrow[d, dashed, "\exists! e'"'] \arrow[dr, "e"] & & \\ E' \arrow[r, "\bar{e}"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Figure 30: An Equalizer Diagram

α and β as the incidence poset containing all elements for which α and β are equal. You may ask, however, what happens when α and β are equal for no elements in the domain? Then, $IP_\emptyset$ will be the equalizer object of α and β, and the function taking $IP_\emptyset$ to the domain of α and β will be the equalizer function. This edge case will make more sense after we have proven the following theorem.

Theorem 4.1.2. *Equalizers exist in the category $\mathcal{IP}_{FSG}$.*

Proof. If $IP_A = (A, \leq)$ and $IP_B = (B, \leq')$ are finite incidence posets, let $\pi : IP_A \to IP_B$ and $\tau : IP_A \to IP_B$ be IP-morphisms. Consider the object $IP_E = \{x \mid \pi(x) = \tau(x)\}$ endowed with the partial order $\leq$ from IP_A restricted to the elements of IP_E. Finitely many maximal and minimal elements comprise IP_E, so, to show $IP_E \in \mathrm{OB}(\mathcal{IP}_{FSG})$, we need only show each minimal element is covered by exactly two maximal elements. (Note that no two minimal elements of IP_E can have the same upperset since the same is true for IP_A.) If IP_E consists of only maximal elements, we are done, so consider some $y \in IP_E$ such that $\exists x > y \in IP_E$. Since $x, y \in IP_A$, there exists $x' \ni x' \neq x$ and $x' > y$. Since π is an IP-morphism, $\pi(x) \neq \pi(x')$, $\pi(x) > \pi(y)$, and $\pi(x') > \pi(y)$. The same is true for τ, meaning $\pi(x') = \tau(x')$, and $x' \in IP_E$. As no other element in IP_A is greater than y, there cannot exist another element in IP_E greater than y. We have shown an arbitrary minimal element y is covered by exactly two elements in IP_E, so $IP_E \in \mathrm{Ob}(\mathcal{IP}_{FSG})$.

Define $\varphi : IP_E \to IP_A$ such that $\varphi(x) = x$. As φ is simply an inclusion map, and the partial order on IP_E is inherited from IP_A, φ is an IP-morphism. It follows that $\pi\varphi = \tau\varphi$.

Suppose $IP_{E'}$ and $\psi : IP_{E'} \to IP_A$ satisfy the same properties as IP_E and φ, respectively. Let $\theta : IP_{E'} \to IP_E$ such that $\theta(z) = \psi(z)$. It follows that

$$(\varphi\theta)(k) = \varphi(\theta(k))$$

$$= \varphi(\psi(k))$$

$$= \psi(k)$$

as desired. Let $\theta' : IP_{E'} \to IP_E$ be some other morphism such that $\varphi\theta' = \psi$. Then,

$$\psi(k) = \psi(k)$$

$$(\varphi\theta)(k) = (\varphi\theta')(k)$$

$$\varphi(\theta(k)) = \varphi(\theta'(k))$$

$$\theta(k) = \theta'(k)$$

The final step holds since θ and θ' are injective. Having shown the existence and uniqueness of θ, the theorem is proven.

■

It is easy to verify (and rather intuitive) that, if (IP_E, η) is an equalizer for some two morphisms between IP_A and IP_B, then IP_E is a subposet of IP_A. This fact will be important in the remaining theorems of this section. See Figure 31 for an example of an equalizer. Note that $IP_E \subseteq IP_A$, but IP_E is not a cat's cradle. Even though equalizers exist in $\mathcal{IP}_{FSG}$, equalizers do not necessarily exist in every subcategory of $\mathcal{IP}_{FSG}$. As a matter of fact, very few subcategories have equalizers.

Theorem 4.1.3. *Equalizers do not exist in $\mathcal{L}$, the category of cats' cradles.*

Proof. See Figure 31 for a counterexample to the existence of equalizers in $\mathcal{L}$.

■

Theorem 4.1.4. *Equalizers do not exist in $\mathcal{C}_{IP}$, the category of connected incidence posets.*

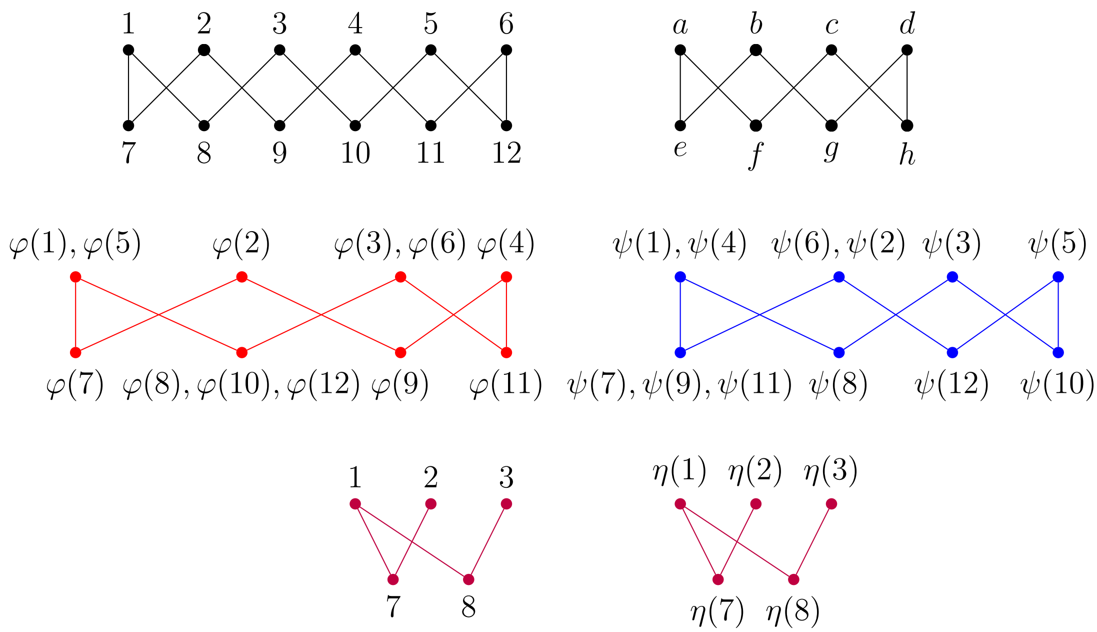

Figure 31: Top Row: IP_A and IP_B; Middle Row: $\varphi, \psi \in \text{Hom}(IP_A, IP_B)$; Bottom Row: The Equalizer (IP_E, η) of φ and ψ

Proof. Since cat's cradles are connected, Figure 31 will be a sufficient counterexample to the existence of equalizers in $\mathcal{C}_{IP}$ as well.

■

Theorem 4.1.5. *Equalizers do not exist in $\mathcal{V}$, the category of valleys.*

Proof. Figure 32 demonstrates a counterexample to equalizers existing in $\mathcal{V}$. Since IP_E is not a valley, α and β do not have an equalizer.

■

Theorem 4.1.6. *Equalizers do not exist in $\mathcal{G}$, the category of strings.*

Proof. Since all valleys are strings, and equalizers do not exist in $\mathcal{V}$, $\mathcal{G}$ cannot have equalizers.

■

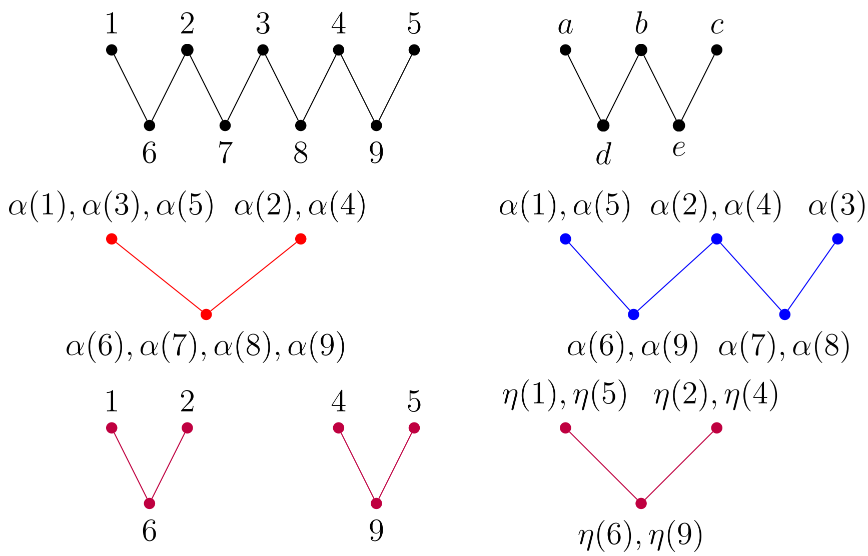

Figure 32: Top Row: IP_{V_1} and IP_{V_2}; Middle Row: $\alpha, \beta \in \mathrm{Hom}(IP_{V_1}, IP_{V_2})$; Bottom Row: The Equalizer (IP_E, η) of α and β

Theorem 4.1.7. *Equalizers exist in $\mathcal{H}$, the category of heaps.*

Proof. Figure 32 demonstrates the main problem with equalizers in categories containing only connected incidence posets: the equalizer can easily be disconnected. However, the equalizer cannot create new structure, i.e., since equalizers are subsets of those incidence posets they map into, they cannot have elements their codomain do not have. More formally, if IP_{H_1} and IP_{H_2} are heaps, and $\alpha, \beta \in \mathrm{Hom}(IP_{H_1}, IP_{H_2})$, then any equalizer (IP_E, η) will be such that $IP_E \subseteq IP_{H_1}$. Theorem 4.1.2 shows (IP_E, η) exists, and IP_E cannot have any cat's cradles since IP_{H_1} does not have any cat's cradles; therefore, the equalizer of any two morphisms between any two objects in $\mathcal{H}$ will also be in $\mathcal{H}$.

■

Theorem 4.1.8. *Equalizers exist in $\mathcal{E}_{IP}$, the category of empty incidence posets.*

Proof. Using the same logic as in the previous theorem, the equalizer of any two morphisms between any two empty incidence posets will be empty.

■

Theorem 4.1.9. *Equalizers do not exist in $\mathcal{D}_{IP}$, the category of disconnected incidence posets.*

Proof. Figure 33 shows two morphisms between disconnected incidence posets whose equalizer is connected, and, therefore, not in $\mathrm{Ob}(\mathcal{D}_{IP})$.

■

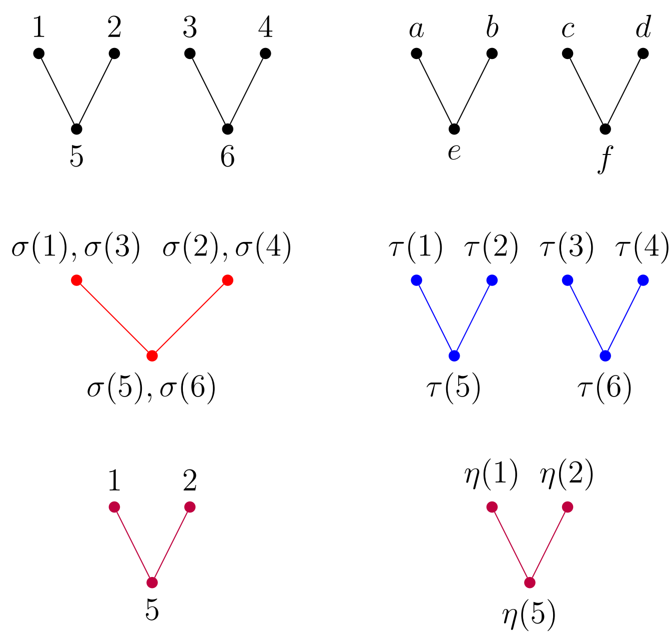

Figure 33: Top Row: IP_{D_1} and IP_{D_2}; Middle Row: $\sigma, \tau \in \mathrm{Hom}(IP_{D_1}, IP_{D_2})$; Bottom Row: The Equalizer (IP_E, η) of σ and τ

Theorem 4.1.10. *Equalizers exist in $\mathcal{PF}$, the category of problem-free incidence posets.*

Proof. As stated in Theorem 4.1.7, if we couple two incidence posets without cat's cradles with any two IP-morphisms between them, then the corresponding equalizer will also lack a cat's cradle. Likewise, we can couple two incidence posets IP_{PF_1} and

IP_{PF_2} without any IP_5 or $IP_{3,3}$ subdivisions with any two IP-morphisms between IP_{PF_1} and IP_{PF_2}, and the resulting equalizer must also be without any IP_5 or $IP_{3,3}$ subdivisions.

■

Theorem 4.1.11. *Equalizers exist in $\mathcal{S}$, the category of saturated incidence posets.*

Proof. By definition, IP-morphisms cannot map two, distinct, maximal elements covering the same minimal element in the domain to the same maximal element in the codomain. This condition ensures that any equalizer of two morphisms between saturated incidence posets will also be saturated.

■

4.2 Coequalizers

Coequalizers are the duals of equalizers in a category. In other words, coequalizers share a structure similar to equalizers, except the domains and codomains of every function are swapped. One can observe this relationship by comparing Figures 30 and 34.

Definition 4.2.1. *Consider two morphisms $h : D \to C$ and $k : D \to C$ in the category $\mathcal{C}$. A coequalizer of h and k in $\mathcal{C}$ is an object Q along with a function q such that*

(1) $q : C \to Q \in \mathrm{Hom}(Q, C)$,

(2) $qh = qk$, and

(3) For any morphism $q' : C \to Q'$ satisfying conditions (1) and (2), there exists a unique morphism $\overline{q} : Q \to Q'$ such that $\overline{q}q = q'$.

We say coequalizers exist in the category $\mathcal{C}$ *when every pair of objects* D *and* C *in* $\mathrm{Ob}(\mathcal{C})$ *and every two morphisms in* $\mathrm{Hom}(D, C)$ *have a corresponding coequalizer* (Q, q).

$$\begin{tikzcd} Q' & & \\ Q \arrow[u, dashed, "\exists!\overline{q}"] & C \arrow[l, "q"] \arrow[ul, "q'"'] & D \arrow[l, shift right, "h"'] \arrow[l, shift left, "k"] \end{tikzcd}$$

Figure 34: A Coequalizer Diagram

Theorem 4.2.2. *Coequalizers do not exist in* $\mathcal{IP}_{FSG}$.

Proof. Consider $IP_C = (C, \leq)$, where $C = \{c\}$, and $\leq = \{(c, c)\}$. Consider also $IP_D = (D, \leq')$, where $D = \{d, d', e\}$, and $\leq' = \{(d, d), (d', d'), (e, e), (e, d), (e, d')\}$. Define $f : IP_C \to IP_D$ such that $f(c) = d$, and define $g : IP_C \to IP_D$ such that $g(c) = d'$.

For some object IP_F and morphism h to be a coequalizer of f and g, it must be the case that $hf = hg$. However, this would imply $h(d) = h(d')$, which is forbidden under the definition of IP-morphisms.

The theorem is proven.

■

Figure 35 below shows the incidence posets and IP-morphisms discussed in the previous proof. We can find almost exactly the same problem in almost all of the subcategories we have studied. At least, IP_C is a subset of or equal to some valley, cat's cradle, connected incidence poset, string, heap, problem-free incidence poset, or saturated incidence poset, and IP_D is an object of the category corresponding to each of those classes of incidence posets as well. It follows that none of those categories can have coequalizers (as stated in the theorem below).

Theorem 4.2.3. *Coequalizers do not exist in* $\mathcal{V}$, $\mathcal{L}$, $\mathcal{C}_{IP}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{PF}$, *or* $\mathcal{S}$.

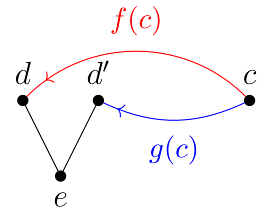

Figure 35: IP_C, IP_D, f, and g

By making a simple change, we can find an incredibly similar problem in the category of disconnected incidence posets (see Figure 36). As before, there does not exist a function h from IP_D to any incidence poset that will be able to satisfy $(hf)(c) = (hg)(c)$. It follows that coequalizers do not exist in $\mathcal{D}_{IP}$.

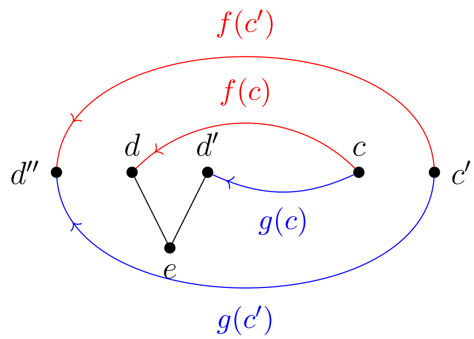

Figure 36: IP_C, IP_D, f, and g

Theorem 4.2.4. *Coequalizers do not exist in $\mathcal{D}_{IP}$.*

Having demonstrated when equalizers and coequalizers exist or do not exist within $\mathcal{IP}_{FSG}$, we can move on to the more complicated problem of products and coproducts. Though more complicated, products are fundamental within many, disparate areas of mathematics. One of the first topologies a student learns is the Product Topology, and even middle or high school students are exposed to products when they start studying the Cartesian coordinate system. This being the case, we are often interested in the structure of products within categories, as the structure is often rich enough to be interesting, yet accessible enough to be one of the first categorical objects studied when investigating a new category.

4.3 Products

The most natural notion of product is the Cartesian product. Usually, the product of two or more partially ordered sets, $\Pi_{i=1}^{n} P_i = P_1 \times \cdots \times P_n$, is the set Cartesian product of the vertices along with the partial order $(p_1, \ldots, p_n) \leq_\times (q_1, \ldots, q_n) \iff p_i \leq_i q_i$ for all i, where $\leq_i$ is the partial order of P_i. We can think of the Cartesian product inheriting its order from the posets composing it. Consider Figure 37 below. On the left are two incidence posets, IP_A and IP_B, and their Cartesian product is on the right.

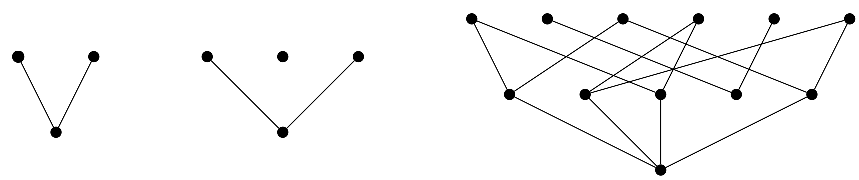

Figure 37: IP_A, IP_B, and $IP_A \times IP_B$

We are concerned with the categorical notion of products, and we consult [3] for the following definition. We define the categorical product for finitely many objects of the category $\mathcal{C}$ since all of the objects in $\mathcal{FSG}$ and $\mathcal{IP}_{FSG}$ are finite.

Definition 4.3.1. *A* <u>*product*</u> *in* $\mathcal{C}$ *for the family of objects* $\{A_i \mid A_i \in \mathrm{Ob}(\mathcal{C}), 1 \leq i \leq n\}$ *is an object* $D \in \mathrm{Ob}(\mathcal{C})$ *together with a family of morphisms* $\{\pi_i \mid \pi_i \in \mathrm{Hom}(D, A_i)\}$ *such that for every object* C *and family of morphisms* $\{\varphi_i \mid \varphi_i \in \mathrm{Hom}(C, A_i)\}$*, there exists a unique morphism* $\varphi \in \mathrm{Hom}(C, D)$ *such that* $\pi_i \varphi = \varphi_i$ *for all* $1 \leq i \leq n$.

That the Cartesian product will not be the categorical object in $\mathcal{IP}_{FSG}$ is plain. Most clearly, there are elements of $IP_A \times IP_B$ that are neither maximal nor minimal, and the minimal element is covered by more than two elements. To fix these issues, we construct an incidence poset whose minimal elements are generated by a rule applied to the Cartesian product of the maximal elements, rather than pulling the minimal elements from the Cartesian product of the minimal elements.

Define $IP_{A_1} \mathrm{X}\, IP_{A_2} \mathrm{X} \cdots \mathrm{X}\, IP_{A_n} = \mathrm{X}_{i=1}^{n} IP_{A_i}$ such that

(i) $\mathrm{Max}(\mathrm{X}_{i=1}^{n} IP_{A_i}) = \mathrm{Max}(IP_{A_1}) \times \mathrm{Max}(IP_{A_2}) \times \cdots \times \mathrm{Max}(IP_{A_n})$, and

(ii) Whenever two maximal elements, $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with $a_i, b_i \in \mathrm{Max}(IP_{A_i})$, are such that each $\left|\mathbf{c}_{IP_{A_i}}(a_i) \cap \mathbf{c}_{IP_{A_i}}(b_i)\right| = 1$ and no $\mathbf{c}_{IP_{A_i}}(a_i)$ or $\mathbf{c}_{IP_{A_i}}(b_i)$ is $\emptyset$, create a minimal element called $\{a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n\}$ that is strictly less than $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$.

In other words, $\mathrm{Max}(\mathrm{X}_{i=1}^{n} IP_{A_i})$ comprises the elements of the set Cartesian product of maximal elements of IP_{A_1} through IP_{A_n}, and there exists a minimal element of $\mathrm{X}_{i=1}^{n} IP_{A_i}$ covered by two maximal elements from $\mathrm{X}_{i=1}^{n} IP_{A_i}$ only when every coordinate of the two maximal elements cover one element in common in their respective incidence posets, IP_{A_i}. For an example of $\mathrm{X}_{i=1}^{n} IP_{A_i}$, see Figure 38.

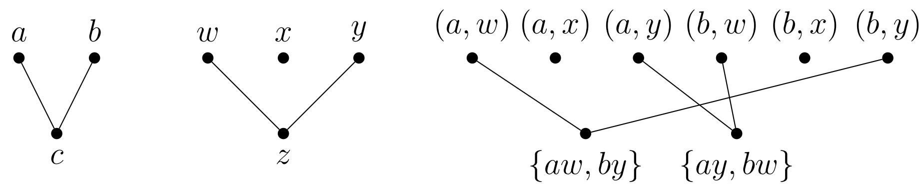

Figure 38: IP_A, IP_B, and $IP_A \mathrm{X}\, IP_B$

Theorem 4.3.2. *The construction* $\mathrm{X}_{i=1}^{n} IP_{A_i}$ *is an object of* $\mathcal{IP}_{FSG}$.

Proof. Every element of $\mathrm{X}_{i=1}^{n} IP_{A_i}$ is maximal or minimal. Each minimal element $\{a_1 \ldots a_n, b_1 \ldots b_2\}$ is in the lowerset of exactly two maximal elements, $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, and is therefore covered by exactly two maximal elements. The uniqueness of the minimal elements guarantees distinct minimal elements will have distinct uppersets. The set Cartesian product of finitely many finite sets is finite, so there are finite maximal elements in $\text{X}_{i=1}^{n} IP_{A_i}$, which implies there are finite minimal elements as well.

Therefore, $\text{X}_{i=1}^{n} IP_{A_i} \in \text{Ob}(\mathcal{IP}_{FSG})$.

■

The above theorem leads us to the main theorem of this section.

Theorem 4.3.3. *The category $\mathcal{IP}_{FSG}$ has finite products.*

Proof. Consider the incidence poset $IP_D = \text{X}_{i=1}^{n} IP_{A_i}$. Define the family of maps $\{\alpha_i : IP_D \to IP_{A_i}\}_{i=1}^{n}$ as follows:

(1) $\alpha_i(x_1, \ldots, x_n) = x_i$

(2) If $z = \{x_1 x_2 \ldots x_n, y_1 y_2 \ldots y_n\}$, $\alpha_i(z) = z'$, where $z' \in \downarrow x_i \cap \downarrow y_i$.

We want to show $\{\alpha_i\}_{i=1}^{n}$ is a family of IP-morphisms. Every input of each α_i yields exactly one output, meaning every α_i is a function. The first condition guarantees maximal elements will be mapped to maximal elements while the second does the same for minimal elements. Consider some $a, b \in IP_D \ni a \leq b$. If $a = b$, $\alpha_i(a) \leq \alpha_i(b)$ automatically, so suppose $a < b$. This means $a = \downarrow b \cap \downarrow c$ for some $c \neq b$ in $\text{Max}(IP_D)$. So, $\alpha_i(a) = a'$, where $a' \in \downarrow b_i \cap \downarrow c_i$, confirming $\alpha_i(a) < \alpha_i(b)$. Now, consider some $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ such that $|\downarrow (e_1, \ldots, e_n) \cap \downarrow (f_1, \ldots, f_n)| = 1$. It follows that $|\downarrow e_i \cap \downarrow f_i| = 1$. Therefore, each α_i is an IP-morphism. Note, also, that each α_i is surjective.

Let IP_C be some incidence poset in $\text{Ob}(\mathcal{IP}_{FSG})$ with a family of IP-morphisms $\{\beta_i : IP_C \to IP_{A_i}\}_{i=1}^{n}$ satisfying the same properties as $\{\alpha_i\}_{i=1}^{n}$. For each

$$c = (c_1, \ldots, c_n) \in IP_C$$

we know $\beta_i(c) \in IP_{A_i}$, so

$$(\beta_1(c), \beta_2(c), \ldots, \beta_n(c)) \in \text{Max}(IP_D).$$

Let $\varphi : IP_C \to IP_D$ such that $\varphi(c) = (\beta_1(c), \beta_2(c), \dots, \beta_n(c))$. For any maximal element of IP_C, we see that $\alpha_i\varphi = \beta_i$. Suppose $w = \{u_1 \cdots u_n, v_1 \cdots v_n\} \in \text{Min}(IP_D)$. We know

$$|\mathbf{c}_{IP_C}((u_1, \dots, u_n)) \cap \mathbf{c}_{IP_C}((v_1, \dots, v_n))| = 1$$

for all i, so the same will be true for $(\beta_1(u_1), \dots, \beta_n(u_n))$ and $(\beta_1(v_1), \dots, \beta_n(v_n))$. If we let

$$\varphi(w) = \{\beta_1(u_1) \cdots \beta_n(u_n), \beta_1(v_1) \cdots \beta_n(v_n)\}$$

we see that $\alpha_i\varphi = \beta_i$ for all elements in IP_C.

We wish to show φ is unique. Suppose there exists some $\psi : IP_C \to IP_D$ such that $\alpha_i\psi = \beta_i$ for all $1 \leq i \leq n$. Note $\alpha_i\varphi = \alpha_i\psi = \beta_i$. Let $d \in \text{Max}(IP_C)$ with $\varphi(d) = (g_1, \dots, g_n)$ and $\psi(d) = (h_1, \dots, h_n)$. It follows that $g_i = \beta_i(d) = h_i$ for all i. Since $\varphi(d) = \psi(d)$ for all maximal d in IP_C, it follows immediately that $\varphi(k) = \psi(k)$ for all minimal k in IP_C.

Having shown any ψ satisfying the same conditions as φ must equal φ, φ is unique, and IP_D is a categorical product for the incidence posets IP_{A_1} through IP_{A_n}. The theorem is proven.

■

As with Equalizers and Coequalizers, we are interested in which subcategories of $\mathcal{IP}_{FSG}$ contain their products. We can knock three of the categories out with one example (see Figure 39). Since IP_{V_1} and IP_{V_2} are both valleys, connected, and strings, but $IP_{V_1} \text{X}\, IP_{V_2}$ is neither a valley, connected, nor a string, so products do not exist in those categories.

Theorem 4.3.4. *Finite products do not exist in $\mathcal{V}$, $\mathcal{C}_{IP}$, or $\mathcal{G}$.*

Similarly, finite products do not exist in the category of cats' cradles, but a proof will do better than a figure.

Theorem 4.3.5. *Finite products do not exist in $\mathcal{L}$.*

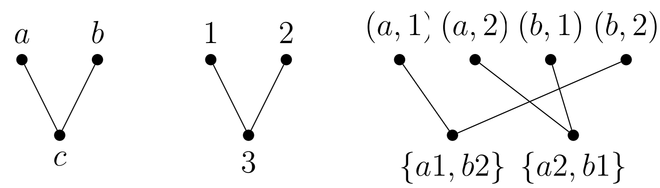

Figure 39: IP_{V_1}, IP_{V_2}, and $IP_{V_1}\mathrm{X}\,IP_{V_2}$

Proof. Consider the cat's cradle in Figure 40 having 3 maximal elements, IP_3. In $IP_3\mathrm{X}\,IP_3$, $|\mathbf{c}_{IP_3\mathrm{X}\,IP_3}((a,a)) \cap \mathbf{c}_{IP_3\mathrm{X}\,IP_3}((b,b))| = 1$, $|\mathbf{c}_{IP_3\mathrm{X}\,IP_3}((a,a)) \cap \mathbf{c}_{IP_3\mathrm{X}\,IP_3}((b,c))| = 1$, and $|\mathbf{c}_{IP_3\mathrm{X}\,IP_3}((a,a)) \cap \mathbf{c}_{IP_3\mathrm{X}\,IP_3}((c,c))| = 1$. However, this means $|\mathbf{c}_{IP_3\mathrm{X}\,IP_3}((a,a))| \geq 3$, which means $IP_3\mathrm{X}\,IP_3$ is not a cat's cradle, and the theorem is proven. ■

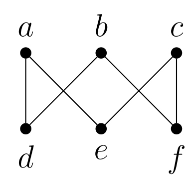

Figure 40: IP_3

Notice that, in $IP_A\mathrm{X}\,IP_B$, (x, y) cannot cover a common minimal element with (x, z), where $x \in IP_A$ and $y, z \in IP_B$. From this, it follows that the finite product of nontrivial saturated incidence posets will not be saturated.

Theorem 4.3.6. *Finite products do not exist in $\mathcal{S}$, the category of saturated incidence posets.*

Theorem 4.3.7. *Finite products exist in $\mathcal{D}_{IP}$, the category of disconnected incidence posets.*

Proof. Let $\mathrm{X}_{i=1}^{n} IP_{D_i}$ be the categorical product of n disconnected incidence posets. Consider $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathrm{Max}(\mathrm{X}_{i=1}^{n} IP_{D_i})$ such that no valley connects x_i and y_i in any IP_{D_i}.

Suppose there exists a valley between $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$. Then, there would exist a sequence of elements $\{z_j^i\}_{j=1}^{k}$ such that $\left|\mathbf{c}_{IP_{D_i}}(x_i)\cap\mathbf{c}_{IP_{D_i}}(z_1^i)\right| = 1$, $\left|\mathbf{c}_{IP_{D_i}}(z_j^i)\cap\mathbf{c}_{IP_{D_i}}(z_{j+1}^i)\right| = 1$, and $\left|\mathbf{c}_{IP_{D_i}}(z_k^i)\cap\mathbf{c}_{IP_{D_i}}(y_i)\right| = 1$ for all i. However, this sequence would also constitute a valley between x_i and y_i in IP_{D_i}, which is a contradiction.

Therefore, there is no valley between $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$, and $\mathrm{X}_{i=1}^{n} IP_{D_i}$ is disconnected, as desired.

■

4.4 Coproducts

As with equalizers and coequalizers, there exists a dual notion of products called 'coproducts.'

Definition 4.4.1. *A coproduct for the family of objects $\{A_i \mid i\in I\}$ in the category $\mathcal{C}$ is an object $\coprod_{i\in I} A_i = E$ of $\mathcal{C}$ and a family of morphisms $\{\iota_i : A_i \to E \mid i\in I\}$ such that, for any object B and family of morphisms $\{\kappa_i : A_i \to B \mid i\in I\}$, there exists a unique morphism $\lambda : E\to B \ni \lambda\iota_i = \kappa_i,\ \forall i\in I$.*

Define $IP_{A_1}\coprod\cdots\coprod IP_{A_n} = \coprod_{i=1}^{n} IP_{A_i}$ such that

(i) $\mathrm{Max}\left(\coprod_{i=1}^{n} IP_{A_i}\right) = [\mathrm{Max}(IP_{A_1})\times\{1\}]\cup\cdots\cup[\mathrm{Max}(IP_{A_n})\times\{n\}]$, and

(ii) Whenever two maximal elements, $\left(x_u^{(i)}, i\right)$ and $\left(x_v^{(j)}, j\right)$, with $x_u^{(i)}\in\mathrm{Max}(IP_{A_i})$, $x_v^{(j)}\in\mathrm{Max}(IP_{A_j})$, and $i,j\in\mathbb{N}$, are such that $i=j$, $x_u^{(i)}\neq x_v^{(j)}$, and $\left|\downarrow x_u^{(i)}\cap\downarrow x_v^{(j)}\right|$, create a minimal element $\left\{x_u^{(i)}, x_v^{(j)}, i\right\}$ that is strictly less than $\left(x_u^{(i)}, i\right)$ and $\left(x_v^{(j)}, j\right)$.

Basically, $\coprod_{i=1}^{n} IP_{A_i}$ represents the disjoint union of IP_{A_1} through IP_{A_n}. The coproduct of IP_{A_1} through IP_{A_n} could be thought of as drawing each IP_{A_i} on a different sheet of paper, then taping the n pieces of paper together and declaring it a new poset.

Theorem 4.4.2. *The construction $\coprod_{i=1}^{n} IP_{A_i}$ is an object in the category $\mathcal{IP}_{FSG}$.*

Proof. Every element of $\coprod_{i=1}^{n} IP_{A_i}$ is maximal or minimal. Each minimal element, $\left\{x_u^{(i)}, x_v^{(j)}, i\right\}$, is in the lowerset of exactly two maximal elements, $\left(x_u^{(i)}, i\right)$ and $\left(x_v^{(j)}, j\right)$, and must be covered by exactly two maximal elements. The uniqueness of the minimal elements guarantees distinct, minimal elements will have distinct uppersets. Since the union of finite sets is finite, there are finite maximal elements in $\coprod_{i=1}^{n} IP_{A_i}$, which implies there are finite minimal elements as well.

Therefore, $\coprod_{i=1}^{n} IP_{A_i} \in \mathrm{Ob}(\mathcal{IP}_{FSG})$.

■

This construction leads to the main theorem of this section.

Theorem 4.4.3. *Finite coproducts exist and are unique (up to isomorphism) in the category $\mathcal{IP}_{FSG}$.*

Proof. Consider the incidence poset $IP_E = \coprod_{i=1}^{n} IP_{A_i}$. Define the family of IP-morphisms $\{\iota_i : IP_{A_i} \to IP_E \mid \quad 1 \leq i \leq n\}$ such that

(a) For $x \in \mathrm{Max}(IP_{A_i})$, $\iota_i(x) = (x, i)$, and

(b) $\iota_i(y) = \{a, b, i\}$, where $y \in \mathrm{Min}(IP_{A_i})$, $a \neq b$, and $a, b \in \uparrow y$.

We want to show $\{\iota_i\}_{i=1}^{n}$ is a family of IP-morphisms. Every input of each ι_i yields exactly one output, meaning every ι_i is a function. The first condition guarantees maximal elements will be mapped to maximal elements while the second does the same for minimal elements. Consider some $c, d \in IP_{A_i} \ni c \leq d$. If $c = d$, $\iota_i(c) \leq \iota_i(d)$ automatically, so suppose $c < d$. This means $c \in \downarrow d \cap \downarrow b$ for some $b \neq d$ in IP_{A_i}. So, $\iota_i(c) = \{b, d, i\}$, where $\{b, d, i\} \in \downarrow (b, i) \cap \downarrow (d, i)$, confirming $\iota_i(c) < \iota_i(d)$. Now, consider some arbitrary (u, i) and (v, j) such that $|\downarrow (u,i) \cap \downarrow (v,j)| = 1$. It follows that $i = j$, and $|\downarrow u \cap \downarrow v| = 1$. Therefore, each ι_i is an IP-morphism. Note, also, that each ι_i is injective, as $(g,h,i) = (l,k,j) \implies \downarrow g \cap \downarrow h = \downarrow l \cap \downarrow k$.

Let IP_F satisfy the same conditions as IP_E, and let the family of IP-morphisms $\{\kappa_i : IP_{A_i} \to IP_F \mid 1 \leq i \leq n\}$ satisfy the same conditions as $\{\iota_i\}_{i=1}^n$. Define a function $\lambda : IP_E \to IP_F$ such that $\lambda(z,i) = (z,i)$, and $\lambda(x,y,i) = (x,y,i)$. It follows immediately that $\lambda\iota_i = \kappa_i, \forall i \ni 1 \leq i \leq n$. Suppose another function, ρ, also satisfies the condition $\rho\iota_i = \kappa_i$. Since each ι_i is injective,

$$\rho\iota_i = \kappa_i$$

$$\rho\iota_i = \lambda\iota_i$$

$$\rho = \lambda$$

which proves λ is unique up to isomorphism.

■

While not complex, the result that the coproduct of n incidence posets is their disjoint union immediately yields which subcategories of $\mathcal{IP}_{FSG}$ will also have finite coproducts. Obviously, no category of connected incidence posets will remain connected when gathered in a coproduct. The disjoint union of problem-free incidence posets will be problem-free, the disjoint union of disconnected incidence posets will be disconnected, and the disjoint union of heaps will still be a heap. Therefore, the corresponding categories will have finite coproducts.

Theorem 4.4.4. *Finite coproducts do not exist in $\mathcal{V}$, $\mathcal{C}_{IP}$, $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{S}$, the categories of valleys, connected incidence posets, cats' cradles, strings, and saturated incidence posets, respectively.*

Theorem 4.4.5. *$\mathcal{PF}$, $\mathcal{D}_{IP}$, and $\mathcal{H}$ have finite coproducts.*

CHAPTER 5

CONCLUSION

We have established the equivalence of the category of finite, simple graphs, $\mathcal{FSG}$, and the category of finite incidence posets, $\mathcal{IP}_{FSG}$, and we constructed common, categorical objects in $\mathcal{IP}_{FSG}$. In later research, it would be interesting to examine some categorical objects we decided not to pursue, such as intersections. The existence of other universal and couniversal objects within $\mathcal{IP}_{FSG}$ would also be a fruitful area of study.

One could also expand $\mathcal{FSG}$ to include graphs with loops and multiple edges (Pseudo-graphs), which would loosen the conditions necessary on graph homomorphisms, leading to a loosening of the conditions for IP-morphisms in the corresponding equivalent, order-theoretic category. While most of the proofs for the special objects of the new category of incidence posets would be almost identical, the concept of minors from Graph Theory could be extended to incidence posets. The study of minors within Graph Theory is rich and has yielded many interesting, fruitful results, so the corresponding theory within incidence posets could also be of note.

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