Students' Smooth Continuous Covariational Reasoning: A Comparative Case Study

By

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This dissertation is dedicated to my LORD and my SAVIOR Jesus Christ!

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ABSTRACT

Research results from this study reveal students have difficulties understanding and using of the concepts of average rate of change and the derivative function. Students in this study held multiple approach to understand the concepts that made it difficult to develop a strong understanding of the average rate of change and derivative function. In particular, students struggled to visualize or imagine a continuously varying rate of change and had difficulties in making meaning and interpreting concepts of average rate of change and derivative function.

This dissertation presents research on how first-year calculus students develop smooth continuous covariational reasoning abilities in the context of the concepts of rate of change and derivative functions. This study utilizes a comparative case study methodology to explore each research participant's construction of understanding and reasoning pattern development. An initial instructional sequence was designed to support Calculus I students in constructing understandings of average rate of change and derivative function. Students were then supported in reasoning about how two quantities vary and co-vary dynamically. The instruction supported students' reasoning abilities when solving problems related to the concept of average rate of change and derivative function in linear and nonlinear function situations.

The research findings show that the study participants demonstrated different types of reasoning to conceptualize the concept of quantity, variation, and covariation when solving mathematical problems related to the concept of average rate of change and derivative function. Sam, one of the study's participants, demonstrated strong *concrete object-oriented* reasoning to conceptualize the average rate of change and derivative function. Another study participant, Ruby, engaged in *procedure-oriented* reasoning to conceptualize the average rate of change and derivative function. Chris, the third study participant, engaged in *terminology-oriented* reasoning to conceptualize the average rate of change and derivative function. The analysis of the data results of this study shows in detail how these three types of reasoning were a limitation for the participants' mathematical problem-solving ability and conceptualizations of covariation, average rate of change, and the derivative function. This study uncovered the above three types of problematic reasoning orientations as it relates to covariational reasoning and learning average rate of change and derivative, but these types of reasoning orientations are most certainly not the only types of problematic reasoning orientations for Calculus I students —there are likely other problematic reasoning orientations that might be discovered in future studies.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

Introduction

First-year calculus students learning the foundational concepts of calculus, such as rate of change, derivatives, and integrals, depend on their deep mathematical reasoning abilities, especially on their variational, quantitative, and covariational reasoning abilities (Carlson et al., 2002; Castillo-Garsow, 2012; Orhun, 2012, Thompson & Carlson, 2017). In addition to the core mathematical reasoning abilities, current research in the teaching and learning of calculus indicates that a smooth continuous covariational reasoning ability is crucial for fostering students' conceptual understanding in calculus, because the central concepts of calculus are founded on smooth continuous covariational reasoning schema (Ely & Ellis, 2018; Johnson et al., 2017; Oehrtman et al., 2008). Smooth continuous covariational reasoning means having an image of change in one variable value happening simultaneously with changes in another variable value, while both variables progressively change (Castillo-Garsow, 2012; Thompson & Carlson, 2017).

Two reasons form the rationale to study first-year calculus students' smooth continuous covariational processing abilities. The first reason is, in the researcher's two years of teaching experience in first-year university Pre-Calculus and Calculus courses, smooth continuous covariational reasoning ability is a critical reasoning ability that first-year calculus students must develop before they deal with nonlinear mathematical systems (i.e., from a covariational perspective, a nonlinear mathematical system is a system in which the change of the output variable value is not proportional to the change of the input variable) and advanced calculus concepts. The second reason originates from research results in calculus, which indicate that smooth continuous covariational reasoning ability is a fundamental reasoning ability that will support students to develop a conceptual understanding (Castillo-Garsow, 2012; Thompson & Carlson, 2017).

This study will add to the body of research in first year students' calculus learning and teaching. Although many studies have explored first-year calculus students' covariational reasoning abilities, it is rare to find studies that explored smooth continuous covariational reasoning abilities (Carlson et al., 2002, Johnson, 2015; Thompson & Carlson, 2017; Tyne, 2017). Therefore, this dissertation focuses on understanding how first-year calculus students develop smooth continuous covariational reasoning abilities in the context of the concepts of rate of change and derivative.

Background for the Study

Students' covariational reasoning plays a critical role in conceptual understanding of the foundational concepts in calculus (Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013; Ely & Ellis, 2018; Thompson & Carlson, 2017). Covariational reasoning is having an image of two quantities varying together within specified intervals. In calculus, there are several theoretical constructs describing how students develop variational, covariational, and other related reasoning orientations among the major researchers such as Carlson et al. (2002), Confrey and Smith (1994), and Saldanha and Thompson (1998). The researcher describes these constructs below and shows how the constructs are utilized in the study.

Confrey and Smith (1994) describe covariation as coordinating the change of fixed length from y_m to y_{m+1} with the change of fixed length from x_m to x_{m+1} without considering the change within the intervals. For these researchers, variation and covariation are "chunky" in the sense that change in a variable's value is imagined as adding nonzero (typically fixed) amounts repeatedly from the beginning value to the end value of the variable. Students then engage in "chunky" covariational reasoning; they may only coordinate discrete changes Δx with the corresponding discrete changes Δy (Castillo-Garsow, 2012; Confrey& Smith, 1994, 1995; Saldanha &Thompson, 1998; Thompson & Carlson, 2017).

In contrast, Saldanha and Thompson (1998) viewed covariation as "continuous covariation." Saldanha and Thompson describe what they mean by continuous covariation, saying, "In the case of continuous covariation, one understands that if either quantity has different values at different times, it changes from one to another by assuming all intermediate values" (p.2.) The current study interprets and views Saldanha and Thompson's idea of covariation as "smooth continuous covariation" because, for Saldanha and Thompson, variation of a quantity has a meaning of progressive change and at the same time the quantity has a measurable value, while the varying quantities progressively vary at the start and end of the interval (Saldanha & Thompson, 1998). For these researchers, students then engage in "smooth continuous" covariational reasoning when they coordinate change in progress in one variable with the corresponding change in another variable. The current study equally shares the ideas and views of Saldanha and Thompson and defines smooth continuous covariational reasoning, which entails an image of change in one varying quantity or variable's value happening simultaneously with changes in another variable's value, while both variables progressively change (Thompson & Carlson, 2017).

More recently, Castillo-Garsow (2010, 2012) gave a refined view of variation that was developed from his study by extending Saldanha and Thompson's (1998) views of covariation. According to Castillo-Garsow students can think about variation in two ways: “chunky” variation and “smooth” variation. “Chunky” variation means a fixed amount of change of the variable value and the idea is rooted in the sets of integer numbers or sets of rational numbers. Smooth variation means change-in-progress of the value of a variable, and the idea is rooted in sets of real numbers. The current research in calculus supports the idea of Castillo-Garsow and redefines “smooth continuous covariation” as a top-level of covariational reasoning ability among different types of covariational reasoning (Thompson & Carlson, 2017, p.441). The ideas of the two types of reasoning are described below in connection to students’ calculus learning.

Chunky and smooth covariational reasoning play a fundamental role in U.S. mathematics education in different ways, particularly in calculus. Chunky covariational reasoning—not smooth continuous covariational reasoning—is the current dominant form of reasoning in U.S. mathematics education, especially in the study of calculus (Boyer, 1949; Carlson et al., 2002; Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013; Goldstine, 2012; Tall, 1992; Thompson & Carlson, 2017). For instance, a research review in the study of calculus by Thompson and Carlson (2017) reveals that the U.S. curriculum does not have a clear picture of the idea of smooth variation or continuously varying quantity. More importantly, in the U.S. curriculum, it is uncommon to reason covariationally, and this contributes to weakness in students’ conception of the rate of change and derivative (Thompson et al., 2017). The researcher argues that, to benefit calculus students’ conceptual learning, there should be a focus on smooth continuous covariational reasoning instruction in the U.S. curriculum, and to do so, faculty must better understand how calculus students reason covariationally.

Researchers in calculus argue that chunky continuous covariational reasoning may lead students to develop procedural knowledge regarding the concepts of calculus, such as variables, rates of change, and derivatives (Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). However, research has shown that smooth continuous covariational reasoning is a better reasoning ability for promoting a strong conceptual understanding of the core concepts of calculus (Carlson et al., 2002; Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). The researcher contends that in mathematics, particularly in calculus, there should be a greater focus on smooth continuous covariational reasoning practices, because smooth continuous covariational reasoning will facilitate first-year calculus students to a conceptual understanding of the concepts of calculus. In the following section, the researcher discusses the importance of this type of reasoning.

The Importance of the Types of Reasoning

Chunky and smooth reasoning can lead students into different mathematical understandings (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). Castillo-Garsow et al. indicates, “Chunky thinking generates chunky conceptions of variation, whereas smooth thinking generates smooth conceptions of variation, with these conceptions producing different mathematics” (p.36). Consequently, two different forms of reasoning for the same mathematical problem will likely lead students to produce two different mathematical representations and, hence, different mathematical understandings. Therefore, the current study argues that students’ smooth continuous covariational reasoning abilities are critical reasoning abilities when compared to chunky reasoning for students to foster conceptual understanding in calculus. More importantly, the current study shares the idea of Thompson and Carlson (2017) and equally argues that smooth continuous covariational reasoning is epistemologically essential for students to develop the foundational ideas of calculus.

More research on how students develop smooth continuous covariational reasoning needs to occur (Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, 2017). Thus far, research results indicate that students have weak and unproductive covariational and smooth continuous covariational reasoning abilities (Carlson et al., 2002; Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Ely & Ellis, 2018). For instance, a study by Carlson et al. (2002) indicates that many students, even high performing students, show difficulty in covariational reasoning in *dynamic situations* (i.e., a situation that engages students in critical thinking, exploration, and deeper mathematical discussion) and often students' reasoning occurs at the gross variational or covariational level. Gross variational reasoning means a person understands that the value of a variable increases or decreases, but they have no idea that it might take on values in the given interval while changing. One idea that researchers suggest supporting students' development of smooth continuous covariational reasoning is to implement instruction that uses dynamic mathematical tasks with the support of technology (Castillo-Garsow, 2012; Cory & Martin, 2012; Engelke, 2007, 2008; LaForest, 2015; Thompson & Carlson, 2017). Yet, how to implement this is still very much an open problem (Thompson & Carlson, 2017).

This dissertation progresses by setting up a series of lessons that use dynamic mathematics tasks, interactive applets, and an instructional approach that focuses on facilitating students' learning (Clement, 2000; Cobb, 2000; Thompson et al., 2007).

These three aspects support this investigation into students' development of smooth continuous covariational reasoning ability, because research results show that these three aspects of teaching and learning support students' knowledge construction (Clement, 2000; Cobb, 2000; Thompson et al., 2007). While the lessons are not the focus of this study, they are an important part of the methodology because they facilitate opportunities for students to use and develop smooth continuous covariational reasoning. There is a further discussion about the lesson design, dynamic tasks, and technology in Chapter Three.

Statement of the Problem

Research results in calculus indicate that the root cause of first-year calculus students' poor conceptual understanding of the concept of variation, rate of change, and derivatives is connected to their mathematical reasoning abilities, especially their quantity, variational, covariational, and smooth continuous covariational reasoning abilities (Byerley, Hatfield, & Thompson, 2012; Moore et al., 2009). More importantly, research results in calculus indicate that first-year university calculus students hold multiple and unproductive conceptual understandings of the average rate of change and derivative and because the conceptual understanding of the two concepts are significant for students' advanced calculus learning, this should be addressed in early and high school curricula (Dorko& Weber, 2013; Dufour, 2015; Johnson, 2015; Tyne, 2014; Tyne, 2017). In particular, students struggle to visualize or imagine a continuously varying rate of change and have difficulties in constructing meaning and interpreting concepts linked with increasing or decreasing rates for a physical situation (Castillo-Garsow et al., 2013).

The problem that students have while developing an understanding of rates of change and derivatives is understudied. As indicated in the research results in calculus, understanding the concept of rate of change and derivatives requires sophisticated reasoning. Reasoning about the concept of rates of change and derivatives relies on variation and covariation of quantity. To alleviate the students' poor conceptual understanding of the concept of derivative and rate of change, researchers suggest that students first need to develop smooth continuous covariational reasoning in order to overcome their difficulty understanding the basic concepts of calculus (Carlson et al., 2002; Thompson & Carlson, 2017).

Statement of Purpose

The purpose of this study is to examine how first-year calculus students understand and develop smooth continuous covariational reasoning. Researchers emphasize the importance of developing students' smooth continuous covariational reasoning abilities to support a robust understanding of the foundational ideas for calculus, including rate of changes and derivatives (Byerley, Hatfield, & Thompson, 2012; Castillo-Garsow et al., 2013; Moore et al., 2009; Thompson & Carlson, 2017). Additionally, researchers view this type of reasoning as the main agent that allows undergraduate students to succeed in advanced mathematics courses, science, engineering, and STEM education in general (Johnson & McClintock, 2017). Consequently, to understand the development of students' smooth continuous covariational reasoning, this study begins by engaging students in variational and covariational reasoning activities. Next, the study examines how students develop smooth continuous covariational reasoning abilities in first year Calculus I contexts. The purpose of this study is to examine how first-year calculus students develop smooth continuous covariational reasoning in the context of Calculus I. This study utilizes a comparative case study methodology to explore each individual research participant's construction and reasoning pattern development (McKenna et al., 2011; Orlikowski & Baroudi, 199; Moy, 2005; Patton, 2015). This study addresses the following two research questions:

1. What types of reasoning do first-year calculus students engage in to conceptualize the relationship between two progressively covarying quantities?
2. What methods of reasoning do first-year calculus students employ during a rate of change and derivative instructional sequence that supports smooth continuous covariational reasoning?

Significance of the Study

The proposed study is significant in three ways: the research contributes to a larger body of knowledge on students' smooth continuous covariational reasoning in calculus learning; the results of the study inform the calculus curriculum and instruction based on students' foundational reasoning abilities, and the results of this study fill a gap that the research results indicate as to the importance of developing undergraduate students' smooth continuous covariational reasoning abilities (Byerley, Hatfield, & Thompson, 2012; Castillo-Garsow et al., 2013;Moore et al., 2009; Thompson and Carlson, 2017).

Definitions

Throughout this study, the following definitions will be used to clarify the meaning of each key term:

A variable – A symbol that represents a quantity's values within a given interval.

Rate of change - Rate of change between two quantities is a ratio of the corresponding changes in these quantities.

Average rate of change - The average rate of change between the two quantities is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval.

Derivative function - A continuous varying rate of change function whose meaning is rooted in the concept of an average rate of change function that is the derivative of a function is the outputs of the limiting value of the average rate of change of function as the independent variable approaches to zero in a given interval or a refinement of the average rate of change function as the independent quantity continuously vary in the given interval.

Covariational reasoning - The act of holding in mind a continued mental image of two quantities covarying simultaneously; that is, a person's ability to imagine how one of the quantity changes while imagining a change in the other.

Chunky variation - A fixed amount of change of a variable value.

Chunky variational reasoning - Having an image of change of the variable value as the variable values change by intervals of a fixed size.

Chunky covariational reasoning - Having a mental image that entails adding fixed size value between successive values of one variable and coordinating this with another variable value.

Gross variation - The value of a variable increases or decreases but little or no thought is given to the fact that it might have values while changing.

Smooth variation - *Change-in-progress* of the variable value or change by the unnoticeable amount of the variable value.

Smooth continuous variational reasoning – An imagination of a variable value progressively changing.

Smooth continuous covariational reasoning - Image of change in one quantity's or variable's value happening simultaneously with changes in another variable's value, while both variables *progressively* change (Thompson & Carlson, 2017).

CHAPTER 2: LITERATURE REVIEW

Theoretical Perspective and Literature Review

The purpose of this study is to examine how first-year calculus students develop smooth continuous covariational reasoning in the context of the concept of rate of change and derivatives. This study proposes the following two research questions:

1. What types of reasoning do first-year calculus students engage in to conceptualize the relationship between two progressively covarying quantities?
2. What methods of reasoning do first-year calculus students employ during a rate of change and derivative instructional sequence that supports smooth continuous covariational reasoning?

The theories of constructivism (Cobb et al., 1992; Dubinsky & McDonald, 2001) and genetic epistemology (Campbell, 2006; Kitchener, 1986; Piaget, 1970, 1971) guide this study. These two theories are used to assess, measure, and understand the process of the students' knowledge development and the design, implementation, and data analysis of the study. Additionally, this chapter discusses the literature concerning students' conceptual learning of the concepts of rate of change and derivative in connection to covariational and smooth continuous covariational reasoning abilities of which first-year calculus students have limited conceptual understanding.

Theoretical Perspective

This study is built on the idea of a constructivist view of learning; in this view, individual learning is viewed as a process of discovering, constructing, and self-learning by engaging with other learners (Cobb et al., 1992; Dubinsky & McDonald, 2001). This theory views a classroom as a place that a group of students meets to discover, conjecture, and explore by actively participating, asking questions, and justifying their reasoning and the reasoning of other students. In this process of knowledge construction, an individual's knowledge gains will be independent of another individual's knowledge gains; it is a unique experience for individual students. The newly gained knowledge is the result of each individual student's reflection and abstraction activity on the concept they are learning. In this regard, knowledge is viewed as a process of an individual altering his or her prior knowledge due to a change in internal mental structure. More importantly, in this theory, the teacher's role changes from a controlling the classroom to being more of a facilitator. Constructivism does not dismiss the role of teachers in the students' knowledge gain process. In this theory, teachers will be positioned as more a guide of the students' reflection and abstraction process by asking questions, listening, and catalyzing students' work, rather than telling and showing the students what to do. For instance, according to this theory, in the learning of the concept of the derivative, the teacher's role changes from merely presenting the definition of derivative to enabling students to construct or produce their own definition of the derivative.

As Piaget indicated, in the process of knowledge construction, two things happen in the students' minds, that is, students' knowledge assimilation and accommodation (Piaget, 1970, 1971).In this knowledge construction process, students seek a balance between knowledge assimilation and accommodation. For example, for students to construct abstract knowledge, like the concept of derivative function from smooth continuous covariational reasoning, students first need to assimilate the new experience, that is, the progressively covarying nature of two quantities in a dynamic situation.

Students then need to modify their existing knowledge about variations and this knowledge modification is called accommodation.

The glue that creates the balance (the equilibrium or the understanding) between the new experience in the student's mind and the modification of the existing knowledge is what Piaget calls reflective abstraction (Campbell, 2006; Kitchener, 1986; Piaget, 1971). For example, the students' experience and their actions will help them understand that a derivative function is the result of limit value of the average rate of change function as x varies progressively throughout the given interval. This means that the students' abstraction and reflection on their actions will lead students to understand the concept of the derivative function.

Piaget identifies two types of abstraction. The first is an abstraction created from the object, that is, empirical abstraction, and the second is an abstraction originating from the mental activity itself, which is a reflective abstraction (Kitchener, 1986; Piaget, 1970, 1971; Tsou, 2006). This study builds on the idea of both types of abstraction. The empirical abstraction view of knowledge is abstracted from observation or obtained from knowing the property of the object (Kitchener, 1986; Piaget, 1970). For example, knowing that a variable has a fixed value or varying value is a type of empirical abstraction.

Reflective abstraction occurs at a higher-level of mental reorganization. It arises from coordinating between two mental actions. For example, reflective abstraction can arise by coordinating the concepts of ratio and one varying quantity (Kitchener, 1986; Piaget, 1970). Interaction among actions is the basis of reflective abstraction. For instance, the coordination of change, that is, the knowledge of the covarying nature of the two covarying quantity values is the result of reflective abstraction from the action, which is a bit higher knowledge. For example, a rate is a reflectively abstracted conception of a constant ratio between two covarying quantities (i.e., students' image of a rate of change) (Piaget, 1970; Thompson, 1992). The knowledge that draws from the action is logical and mathematical. Therefore, according to the genetic epistemology theory, mathematical knowledge is constructed by the process of repeated reflective abstraction activity inside the students' mind (Kitchener, 1986; Piaget, 1970, 1971). This means that a student constructs his knowledge from his experience through action, not from the transfer of knowledge from another student, person, or teacher.

In this study, the theory of reflection and abstraction is viewed as the means to enable students to construct knowledge. Thus, the goal of teaching is to create opportunities for individuals to participate in abstract and reflective actions. Moreover, teaching must prepare situations in which students face a dynamic change in their cognitive structure so that they can actively engage in constructing knowledge.

Background for the Study

Research in the area of smooth continuous covariational reasoning is limited (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). The lack of research in this area may be due to the complex nature of the topic. The researcher planned to explore smooth, continuous, covariational reasoning because this reasoning type is viewed as a critical reasoning ability that will lead students to a conceptual understanding of the main concepts of calculus. However, the researcher found limited research availability in smooth, continuous, covariational reasoning, and thus research results from covariational reasoning ability, rate of change, and the derivative function comprise a significant amount of the literature review section of this study.

Smooth continuous covariational reasoning entails an image of change in one quantity or variable's value happening simultaneously with changes in another variable's value, while both variables progressively change (Thompson & Carlson, 2017). Research results in first-year calculus suggests that smooth, continuous, covariational reasoning is fundamental for the development of students' conceptual understanding of the rate of change, derivative, and accumulation (Johnson et al., 2017; Thompson & Carlson, 2017.) For instance, Thompson and Carlson (2017), Castillo-Garsow (2012), and Castillo-Garsow et al. (2013) indicate that smooth, continuous, covariational reasoning is vital and critical when students learn advanced ideas of calculus, especially when students learn the ideas of nonlinear functions. Therefore, in this study, the view is that building students' smooth, continuous, covariational reasoning will lead them to the right path of understanding concepts of calculus, such as rate of change and derivatives.

The notion of covariational reasoning means a student has an image of two quantities varying together within specified intervals (Carlson et al., 2002; Castillo-Garsow, 2012; Thompson & Carlson, 2017). The cognitive activity that results from students constructing a covariational relationship by coordinating the values of two covarying quantities provides a foundation for students to reflect and abstract and then develop mathematical understanding. Students' development of conceptual understanding about the concept of rate of change and derivative is connected with their reflection and abstraction ability on the nature of two covarying quantities. For example, according to the Covariation Framework (see Appendix A) the students' covariational reasoning first starts by, “Coordinating the value of one variable with changes in the other,” then students, “Coordinate the direction of change of the two quantities,” and then next students, “Coordinate the values of two quantities.” After that, students “Coordinate average rate of change one quantity with the other quantity,” and finally, students, “Coordinate the instantaneous rate of change of the function with continuous changes in the input variable,” which is the top level of reasoning (MA5) according to the Carlson framework (Carlson et al., 2002). All the above reasoning activities will lead students to construct a new concept and develop mathematical understanding. Further, Carlson classified each student’s mental action into five levels of reasoning: MA1 (L1) is a coordination level, and at this level, the students’ image of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1); Level 2 (L2) coordinating the direction of change of one variable with changes in the other variable; Level 3 (L3) is quantitative coordination; Level 4 (L4) is average rate; and Level 5 (L5) is instantaneous rate (Carlson et al., 2002).Therefore, engaging in this type of reasoning is important for understanding the central concepts of calculus, such as rate of change and derivative (Carlson et al., 2002; Thompson, 1994; Thompson & Carlson, 2017).

Covariational Reasoning in Calculus

Research results in calculus indicate that covariational reasoning is epistemologically necessary for students to develop the foundational ideas of calculus (Ely & Ellis, 2018; Engelke, 2007; Moore & Paoletti, 2015). Covariational reasoning is the act of imagining how one quantity’s value changes while imagining the change in the other quantity's value (Ely & Ellis, 2018; Engelke, 2007; Moore & Paoletti, 2015; Thompson and Carlson, 2017).

Several research results in the study of calculus indicate that students' learning of the foundational concepts of calculus, like the rate of change and derivative function, relies on their clear and explicit covariation reasoning schema (Carlson et al., 2002; Castillo-Garsow, 2012; Engelke, 2004, 2007; Gray et al., 2007; Moore, 2010; Moore & Paoletti, 2013; Moore & Paoletti, 2015; Thompson & Carlson, 2017; Weber & Carlson, 2010). Engelke (2004, 2007) indicates that students' covariational reasoning allows them to create the problem situation in their minds, which helps them to understand how the system works. "For example, consider the following problem:" "Coffee is poured at a uniform rate of 20 cm^3/sec into a cup whose inside is shaped like a truncated cone. If the upper and lower radii of the cup are 4 cm and 2 cm, respectively, and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up the cup." (p.8.) One of the study's participants, Adam, noted that the volume depends on the height, and he said "Just imagine that you already filled it up with coffee to here, um, because who cares about the total volume. All you want to know is how fast the coffee is rising when you're at this volume." For Adam, having the correct reasoning, in this case, the covariational reasoning abilities between how the volume and the height related, enabled him to focus on the problem situation and solve the problem. Similarly, Weber and Carlson (2010, p.8) indicate that, "Students' quantification of the situation in terms of fixed and varying quantities allows them to create a dynamic mental image of the situation." Another study by Moore and Paoletti et al. (2013) also indicates that covariational thinking enabled students to perceive something that can be visualized and quantified by producing the mathematical relationship between two covarying quantities.

A commonality among the above studies is that covariational reasoning is a critical reasoning ability for students to construct concepts like variation, function, rate of change, and derivative in calculus. "For example, consider the following problem:" "A box designer has been charged with the task of determining the volume of various boxes that can be constructed from cutting four equal-sized square corners of a 14-inch by 17-inch sheet of cardboard and turning up the sides. Construct a formula that relates the volume V of the box, to the length of the side of the cutout x" (Weber & Carlson, 2010, p.8). The research participant of the study, Chris, understood that, "The length and width get smaller as you increase *x*, which is the cutout size," (p.10). His covariational reasoning ability helped him to predict patterns of the volume of the box in relation to the length of the cutout based on his understanding of the box. Similarly, a student from Moore and Paoletti et al. (2013) study was tasked to construct the graph of $f(\theta) = 2\theta + 1$ i n the polar coordinate system (PCS) and $y = 2x + 1$ in the Cartesian coordinate system (CCS). This student understood both graphs in terms of a structure of covarying quantities, which enabled him to understand a graph in the PCS, and a graph in the CCS is produced by the same reasoning.

Moreover, all the above research findings asserted that covariational reasoning is fundamental reasoning to develop students' conceptual understanding of the main idea of functions, rate of change, and derivative (Carlson, 1998; Carlson, Larsen, & Jacobs, 2001; Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017).

Therefore, covariational and smooth continuous covariational reasoning is critical for students to construct and develop understandings of the basic concepts of calculus.

Role of Covariational Reasoning

Several studies in calculus indicate that covariational reasoning is a crucial reasoning ability that leads students into a conceptual understanding for many calculus concepts such as rate of change (Castillo-Garsow, 2013; Confrey, 1994; Thompson & Thompson, 1996; Thompson, 1994) and derivative (Ely & Ellis, 2018; Engelke, 2007; Moore &Paoletti, 2015; Moore, 2010). A study by Moore (2010) shows that students, at first, exhibit limited understanding of angle measure, but after, they develop their covariational reasoning by engaging in dynamic mathematical tasks with technology, students develop an understanding of the concept of radius as the unit of measuring arc length. “For example, consider the following problem:” “While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the building. Given that the angle of Bob’s site from the ground was 56 degrees, determine the height of the Empire State Building.” Zac at first constructs a mathematically incorrect equation cos (0.98) =1000; however, after some thought, he imagines determining, in radii, using this measure to determine the radius length and then determining a value that represented a multiplicative relationship between a length and radius. Zac’s correct covariational reasoning enables him to plan the solution by explaining that cos (0.98) represented a fraction of the radius without executing a calculation to determine this value. This correct covariational ability of Zac was due to his uniquely oriented, planned, and executed ability to model the problem situation by constructing a diagram of the situation and by clearly labeling the given variable values.

Similarly, Moore et al. (2013) indicate that covariational reasoning supported students to create a connection between different graphical representational activities. For example, students' covariational reasoning abilities enable them to understand that both graphs $f(\theta) = 2\theta + 1$ in polar and $y = 2x + 1$ in the Cartesian system are representative of the same relationship, a relationship such that there is a constant rate of change between the quantities. Another study by Weber and Carlson (2010) illustrates that students' covariational reasoning enables them to make sense of a complex mathematics problem by creating a diagram of the problem situation. For instance, Zac, one of the research participants, gave the correct explanation to the problem "A box designer has been charged with the task of determining the volume of various boxes that can be constructed from cutting four equal-sized square corners of a 14-inch by 17-inch sheet of cardboard and turning up the sides. Construct a formula that relates the volume V of the box, to the length of the side of the cutout x."

This is due to Zac's developed covariational reasoning abilities about how the length and width of the box change in tandem with the cutout length. He answers, "The length and width get smaller as you increase x, which is the cutout size" (p.10). Collectively, the above studies illustrate that covariational reasoning is foundational for making sense, understanding, constructing, and justifying the mathematics concepts and problems.

Carlson, Larsen, and Jacobs (2001) study 24 students' covariational reasoning abilities. The purpose of the study was to explore the effect of instructional materials on the development of students' covariational reasoning abilities. The study consists of five separate activities designed to promote students' ability to attend to the covariant nature of dynamic functional relationships. Moreover, each activity contains a collection of prompts that encourages students to coordinate an image of the two variables changing and to attend to and represent the way in which the independent and dependent variables change in relationship to one another. That is, as the results of this study, students are able to give correct reasoning when they solve the limit and accumulation problem. In particular, the data analysis results of the pre- and post-test reveal a positive shift in students' covariational reasoning abilities. Additionally, at the end of the semester, out of the 24 students, 23 of them provide the correct responses using the covariational language of concave down, then concave up, when solving the classic bottle problem. Moreover, at the end of the course, the students' level of reasoning had reached a higher point, that is, at the MA5 level of reasoning. Therefore, the study reveals that the development of students' covariational reasoning enables students to understand specific tasks for limit and accumulation.

All three studies reveal that covariational reasoning is a mental activity, because covariational reasoning demands students construct their knowledge from the action and not from the object, and supports students to develop conceptual understanding of the concepts of rate of change. Practically, what the above research results shows is that covariational reasoning is epistemologically necessary for students to develop the foundational ideas of calculus. Moreover, many studies suggest that students' covariational reasoning abilities are developed by engaging students in dynamic mathematics instructional materials with the support of technology.

Students' Covariational Reasoning

Several studies in calculus investigate students' covariational competence using Carlson's covariational framework (2002). The studies' results reveal that undergraduate calculus students, graduate students, and even experts have weak covariational reasoning abilities (Carlson et al., 2002; Hobson & Moore, 2017; Moore & Bowling, 2008; Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017).

Carlson et al. (2002) investigate 20 high-performing calculus students' covariational reasoning abilities. The study's purpose is to explore how students develop covariational reasoning abilities in a mathematical context and also to propose a framework for describing the mental activities involved in applying covariational reasoning. Students in this study engaged in three different mathematical problems: the bottle problem, the temperature problem, and the ladder problem. The results of the study show that many students had difficulty creating images of a continuous rate of change by imagining an increasing and decreasing rate of change function within the given interval. Moreover, the study reveals that all students could not pass MA4 and L4 and had difficulty interpreting the rate of change (see Appendix A for a detailed description of MA4 and L4). Most significantly, the results of the study show that students were unable to interpret the concept of instantaneous rate of change. "For instance, for the bottle problem: Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle," only five (25%) of the students provided the correct answer and other students struggled and were not able to describe inflection points or increasing and decreasing rate of change for dynamic function situations. In particular, students showed difficulty characterizing the nature of change while imagining the independent variable changing continuously and progressively. Similarly, for the temperature problem, only four (20%) students constructed an acceptable temperature graph, but most students were unable to produce a smooth curve with clear indications of concavity changes. In summary, the results of the study show that the majority of the students had difficulties in giving a well-reasoned solution, were unable to coordinate change between two covarying quantities, and had difficulty interpreting why the curve is smooth and what is conveyed by an inflection point on a graph; their reasoning level was below MA4 (see Appendix A).

Moore and Bowling (2008) explore ten college algebra students' covariational reasoning and quantification abilities. The purpose of the study was to classify the students' level of covariational reasoning using the Carlson et al. (2002) covariational framework. The study results reveal that most students exhibited difficulty reasoning at a level higher than L2-L3 covariational reasoning behaviors (see Appendix A.) Most students were able to describe a directional change in volume as the cutout length increased and decreased (i.e., L2-L3) however, students were not able to describe the covariational relationship between the volume of the box and the cutout length. More importantly, they did not have images of varying rate.

Hobson and Moore (2016) explore two students' covariational reasoning. The study focuses on investigating students' reasoning in a dynamic situation. The results of the study indicate that, besides students' similar graph drawing between the two participants, the ways of reasoning between the two students throughout this task unfolded quite differently. Jake did not relate his construction of the tangent line to a coordination of change in one variable with change in the other . Covariational reasoning is to construct equal changes in one variable value by comparing a corresponding change in the second variable values. This dissertation uses a similar idea to this study to determine models of students' mathematics.

All these three studies reveal that covariational reasonings are important to students' conceptual understandings, but students have difficulty engaging with it. Students have difficulty reasoning beyond L3 covariational reasoning and also unpacking their reasoning of the rate of change by saying the rate of change is the quotient of the amount of change in one variable with a change in the other variable values. The currents study engages students in an activity that supports them to construct the idea of changes of one quantity by comparing the corresponding change in the other quantity. More importantly, this study relates the idea of the rate of change with the amount of change of the quotient of two varying quantities.

Students' Smooth Continuous Covariational Reasoning

Studies in calculus suggest that smooth continuous covariational reasoning is critical for students to deal with calculus ideas, particularly ideas related to nonlinear functions (Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). However, there are few research studies that focus on how students develop smooth continuous covariational reasoning in the context of the rate of change and derivative. The current study explores the development of students' smooth, continuous, covariational reasoning and finds that this type of reasoning supports students to a more conceptual understanding of different calculus concepts (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). For instance, smooth continuous covariational thinking can enable students to conceptually understand and distinguish the difference between the concepts of the constant rate of change, the average rate of change, and the derivative function of a function (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017).

The current study, in particular, argues that changing from chunky covariational reasoning learning, which is dominant in school mathematics, to smooth continuous covariational reasoning practices is important because smooth continuous, covariational reasoning abilities leads students to the right path of understanding concepts of calculus (Johnson, McClintock, & Hornbein, 2017; Oehrtman, Carlson, & Thompson 2008). The current study explores students' smooth, continuous, covariational reasoning by engaging students in dynamic mathematics tasks with the support of technology.

Literature on Rate of Change and Derivative

Research available on students' understanding of the rate of change and derivative in calculus provides an insight into students' understanding and application of the two concepts and informs the design of this study. The research results in calculus informs that students hold limited and unproductive understandings of the rate of change and derivative function.

The idea of the rate of change has many meanings in mathematics education and is used to describe the relationship between changing quantities. Many first-year undergraduate calculus students have a poor conceptual understanding of the concept of average rate of change, that is, students often view the rate of change as the arithmetic mean, slope, and rise over run (Bezuidenhout, 1998; Byerley et al., 2012; Dorko& Weber, 2013; Johnson, 2015; Musgrave & Carlson, 2016; Tyne, 2014; Thompson, 1994). The concept of a derivative function in calculus is deeply rooted in the concept of variation and rate of change (Park, 2013; Thompson, 1994). The concept of derivative developed from the idea of an average rate of change to a continuous varying rate of change function (Carlson et al., 2002; Thompson &Ashbrook, 2016; Thompson & Carlson, 2017). Most students don't understand the derivative from this perspective, and this is linked with their procedural knowledge of the average rate of change of a function (Byerley et al., 2012; Dorko & Weber, 2013; Park, 2013; Park, 2015). A detailed literature analysis for the two concepts is provided below.

Students' Understandings of Rate of Change

The concept of rate of change is central in calculus; an improper conceptual knowledge of rate of change will deter students' abilities to understand other related major concepts in calculus. The concept of rate of change is rests upon conceptualizing the relationship between two dynamically changing quantities. For example, students who develop rich knowledge about concept of rate of change can easily explain that a rate of change is a mathematical relationship between two simultaneously covarying quantities (Musgrave & Carlson, 2016). For example, students who developed a correct conceptual understanding of rate of change can easily imagine that the average rate of change of quantity x with respect to quantity y is the constant rate of change that yields the same change in quantity y as the original relationship over the given interval. However, most research findings in calculus indicate that many calculus students are did not conceptualize the idea of the rate of change function in this way (Bezuidenhout, 1998; Byerley et al., 2012; Carlson, 1998; Musgrave & Carlson, 2016).

Bezuidenhout (1998) investigates 523 first-year calculus students' knowledge of the rate of change. The results of this study reveal that many students (90%) show misconceptions about the concept of the average rate of change. For instance, one of the students, Piet, used the procedural formula $\frac{g'(3)+g'(0)}{2}$, to answer the problem, "Find the average rate of change of $g(x)$ with respect to x over the interval [0, 3]." (p.398) When he was asked to describe his procedure, he said, "The average rate of change over an interval equals the sum of the derivatives at the two endpoints of the interval divided by two." The response of Piet illustrates that he did not understand the fundamental ideas that built the concept of the average rate of change due to his poor conceptual understanding of the rate of change (Bezuidenhout, 1998, p.393). Moreover, the results show that the majority of research participants (90%) hold similar weak reasoning ability about the idea of the rate of change due to their use of the concepts of arithmetic mean to find the average rate of change.

Byerley, Hatfield, and Thompson (2012) investigate seven first-year university calculus students' concepts of division and rate of change. The goal of the study is to understand how calculus students conceptualize the idea of division and rate. The study found that almost all students had a weak understanding of the idea of rate. The study results reveal that the research participant students struggled to interpret quotient as a measure of relative size. Many students showed difficulty abstracting division as a quotient of two covarying quantities, and this difficulty limited students' ability to abstract the main idea behind the concepts of the average rate of change and instantaneous rate of change. For instance, one of the research participants, Arlene, showed difficulty explaining how 29.66 is related to 0.236 for the given equation of $\frac{7}{0.236} = 29.66$; however, she should have said that 29.66 is approximately 7 times as large as $\frac{1}{0.236}$. Although she was a successful high school calculus student, she struggled to relate them. Similarly, like Arlene, many (or more than 75%) of the study participants showed difficulty in conceptualizing the idea of rate and division. In summary, the findings show that many of the study participants exhibited a strong procedural knowledge in calculating and a weak conceptual knowledge to interpret and reason about the concept of rate of change.

Dorko and Weber (2013) study 16 multivariable calculus students' concept images of average rate of change. The purpose of the study was to investigate how the students' meaning of average influenced their conceptions of average rate of change and instantaneous rate of change. The results reveal that most students' mental images of average influenced their concept image of average rate of change. "For instance, for the question: suppose we define a function f, so that $f(x, y) = e^{-cos(xy)}$. Discuss the process you would use to determine the average rate of change of the function with respect to x over the interval [2.0, 2.2]" (p.390), the students' responses to this question demonstrate that many students responded by using the concept images for the average in the concept images for average rate of change. For instance, Jane's concept image for average included the property of an average being 'common' and she thought the average rate of change as the most common value. She said, "The average rate of change tells me the most commonly occurring rate of change of all the rate of change…," and she added, "I find the change in *y* over the change in *x*, and it tells me that, the most common value" (p.390). This response shows that students' images of average were influenced by their images of average rate of change. The researchers conclude that students did not seem to have an image of x, y and z as measure of varying quantities or an image of a quantity value changing as the other quantity value changes at a constant rate.

Thompson (1994) investigates 19 senior and graduate mathematics students' images of rate. The results of the study indicate that senior and graduated mathematics students had poor concepts of rate of change due to their under-developed idea of covariation. "For instance, for the problem: when an object falls from a resting start, the distance it has fallen $t$seconds after being released is given by the function $d(t) = 16t^2$"(p.21), after students develop the average rate of change function $f_h(t) = \frac{d(t+0.1)-d(t)}{0.1}$, when asked to interpret what it means, students interpret it as, "How fast it [the function] is changing," "Divide by 2," without interpreting the details of the expression as an amount of change in one quantity in relation to a change in another. Additionally, for the problem "The volume in cubic meters of a cooling object t hours after removing a heat source is given by the function $v(t)$. Suppose a function $x(t)$ is defined as $x(t) = \frac{v(t+0.1)-v(t)}{0.1}$. State precisely what information $x(t)$ gives about this object" (p.61), students' responses were unproductive and had a poor image towards to the concept of rate of change. For instance, two students said an average, but not an average rate of change, and six students said that $x(t)$ is a derivative. Overall, the results of the study illustrate that students did not have a precise image for the average rate of change function.

Musgrave and Carlson (2016) investigate 21 Ph.D., graduate teaching assistant (GTA) students about the meaning of the average rate of change. The results of this study reveal that GTA students struggled to give accurate interpretations about the meaning of the average rate of change. In the pre-intervention, semi-structured interview, many of the students failed to give a conceptually well-reasoned answer when they were asked, “What does the average rate of change mean to you?” For instance, one of the student’s, Alan's, response was, “A straight line between two points on a graph,” and Edger, another student, responded to the same question as, “Rate of change over an interval.” The pre-intervention participants hold a weak understanding of the meaning of the rate of change, and their ideas were predominantly influenced by their geometric interpretation. Even after the post-intervention, some students struggled to give the correct meaning of the rate of change. One participant indicated, “I will forever think of the average rate of change as the slope of the secant line.”

From the results of these four studies, one can conclude that the concept of rate of change is one of the difficult concepts for students in U.S. mathematics education. Moreover, all four studies indicate that calculus students have difficulty interpreting and reasoning about the rate of change. The difficulty of conceptualizing the idea of the rate of change in this study connected with their reasoning abilities. Moreover, results in the studies of the rate of change show that students had difficulty coordinating the effect of a change in one variable with the effect of the change to the other variable. To overcome students’ problem of understanding of the concept of rate of change, it is important to unpack concept of rate of change by engaging students in a problem that leads them “to coordinate the amount of change of the output variable with a progressive change in the input variables” (Carlson et al., 2002, p.357). Therefore, the results of these four studies inform the current study that developing students’ reasoning about two covarying quantities will likely lead students to develop a conceptual understanding of the concept of rate of change.

Students' Understandings of Derivative

The concept of a derivative function in calculus is deeply rooted in the concepts of the variation and rate of change (Park, 2013; Thompson, 1994.) Typically, students are introduced to the concept of the derivative at post-secondary or after pre-calculus using the concept of limit and the average rate of change function (Hart, 2019; Stewart, 2015.) The concept of derivative develops from the idea of the average rate of change to a continuous varying rate of change function (Carlson et al., 2002; Thompson & Ashbrook, 2016; Thompson & Carlson, 2017.) Many students don't understand the derivative function from this perspective, and this is perhaps linked with their procedural knowledge of the average rate of change of a function (Park, 2013; Tyne, 2014, Tyne,2015). Therefore, the current study is aimed to direct students to understand that derivative function is the result of the quotient of two smoothly and continuously covarying quantities in a dynamic situation or the results of the quotient between two unnoticeable covarying quantities or limiting value of the average rate of change. Moreover, the current study planned for students to have awareness that derivative function resulted from smaller and smaller refinements of the average rate of change of a function.

Park (2013) studied 12 calculus students' conceptual understanding of the concept of the derivatives. The goal of the study is to examine calculus students' utterance about the derivative as a function based on the idea of a function at a point. The study results reveal that students' thinking about the derivative as a function was underdeveloped even after they completed their semester course in the area of the concept of the derivative.

Moreover, the results show that many students had difficulty describing derivative in general, and, in particular derivative as a function. For instance, all 12 students gave correct procedural answers when they were asked to find $f'(2)$ for $f'(x) = x^2 - 7x + 6$, but they showed conceptual difficulty when asked to give the reasoning behind $f'(2)$. One student, when asked to give the reason and meaning behind $f'(2)$, said, "It is a rule … can't remember why."

Tyne (2014) investigates 75 first-year calculus students' conceptual knowledge of slope and derivative. The results of the study indicate that only 7% of students were successful in interpreting the concept of the derivative. The results of the study show that students frequently had more success with slope questions than derivative questions. Students correctly used the slope of a linear relationship to make predictions, but they did not conceptualize derivative as an instantaneous rate of change and an estimate of the marginal change.

Similarly, a study by Tyne (2015) illustrates that, from 100 research participants who were calculus students, only 13% correctly interpreted the derivative for the problem "Let $B(n)$ be the number of bushels of corn produced on a 10-acre tract of farmland that is treated with n pounds of nitrogen, what does the derivative of $B'(20)$ is equals to 2 means?" The results show that many students struggle with knowing what the derivative represents and how to use it appropriately to make predictions. Some of the students incorrectly responded that "$B'(20) = 2$ means that when 20 pounds of nitrogen are applied, the total bushels are equal to 2." Rather, the students should have correctly said, "$B'(20) = 2$ means that when the nitrogen is equal to 20 pounds, the corn yield is increasing at a rate of change value of 2 bushels per pound of nitrogen."

In each of the above studies, the results show that most students lack a conceptual understanding of the concept of the derivative function. Some of the studies link students' difficulty with their procedural knowledge of the average rate of change of a function, and others link students' difficulty with the treatment of such a concept in the curriculum. This means that students, in their school time or learning period, did not have an awareness or had not studied that the derivative function "resulted from smaller and smaller refinements of the average rate of change of a function" or did not discuss a derivative function is the limiting value of the average rate of change of the function (Carlson et al., 2002, p.357). Moreover, students did not demonstrate an understanding of the derivative function as an instantaneous rate of change function. This means that for students to understand the derivative as a function, students must first develop strong covariational reasoning; in particular, students should have a smooth continuous covariational reasoning. For example, the meaning and the conceptual understanding of the concept of rate of change, function, and derivative relied on the students' basic smooth continuous covariational reasoning abilities. More importantly, the conceptual understanding of derivative depends upon the ability to reason that change in one variable value is occurring instantly as the other variable values change, and also having an image that both variables change progressively in the given interval. This dissertation proposes that having in place the smooth continuous covariational reasoning in students' minds at first will enable students to understand that a derivative function is the result of the limit value of the average rate of change function or a derivative function resulting from smaller and smaller refinement of the average rate change of a function.

The results of the above studies clearly show that students have misconceptions about the concept of rate of change and derivative, but none of the studies articulate the cause of the problem for the students' misconception. This dissertation, however, seeks to investigate the link between students' misconceptions and their weak reasoning ability (in particular, shortcomings in their covariational and smooth continuous covariational reasoning) and proposes to develop such reasoning ability. Students' reasoning about the concept of rate of change and the derivative is related to their mental activity of coordinating two or more covarying quantities. A cognitive activity, such as the reasoning about varying quantity, reasoning about the rate of change of two covarying quantities, and reasoning about a continuously varying rate of change of two covarying quantities is needed to understand the key calculus concepts.

Summary of the Chapter

This chapter provides a detailed discussion of the theoretical perspective, which combined the views of constructivism and Piaget's theory of genetic epistemology, builds the basis for the current study. The main idea of the constructivism theory lies in the perspective that each student is unique, and their knowledge construction is independent of other students or persons. Additionally, in this theory, it is viewed that the students' construction of new knowledge or concept lies in their reflection and abstraction abilities.

This chapter also discusses a literature review on the area of rate of change and derivatives. These research results in the area of rate of change and derivative function informed the researcher that first-year calculus students hold a weak and limited conceptual understanding of the concept of rates of change and derivatives. Study results on the rate of change showed that students had difficulty imagining the effect of a change in one variable with the effects of the change to the other variable. Moreover, research results in the studies of derivative indicate that first-year calculus students did not view that the derivative function is a limiting value of the average rate of change function or did not view it resulted from a smaller and smaller refinement of the average rate of change of functions. For the students to develop the conceptual understanding of the concept of rate of change and derivative, students should first construct and reason about the dynamic or unnoticeable changing relationships between two covarying quantities. For students to understand the concept of rate of change and derivative, students must first develop strong, covariational reasoning, in particular, a strong, smooth, continuous, covariational reasoning ability.

This chapter provides a summary of a critical review of covariational reasoning abilities: the role of covariational reasoning in the students' conceptual understanding development. Additionally, the chapter gives some insight into the critical importance of smooth continuous covariational reasoning abilities. Smooth continuous covariational reasoning ability is viewed as a crucial and more important reasoning ability to lead students into a conceptual understanding in calculus. The next chapter provides the methodology of this study, which aims to gain insight into the reasoning abilities and foundational understanding needed to construct a meaningful knowledge of the concept of rate of change and derivative.

CHAPTER 3: METHODOLOGY

The aim of this study is to explore and understand first-year calculus students' reasoning when they have opportunities to develop smooth continuous covariational reasoning and conception of the rate of change and derivative functions. Results of this study might be used to improve students' Calculus I learning. Efforts like this, the purpose of which is to inform calculus instruction, should plan, first, to understand students' ways of reasoning, such as how they try to construct concepts, like rate of change or derivative. Thus, understanding of how students understand and reason about these concepts is critical for informing calculus instruction and research.

The purpose of this chapter is to discuss the methodology of the research that guided the investigation of the research questions of this study:

1. What types of reasoning do first-year calculus students engage in to conceptualize the relationship between two progressively co-varying quantities?
2. What methods of reasoning do first-year calculus students employ during a rate of change and derivative instructional sequence that supports smooth, continuous, covariational reasoning?

In particular, this study utilizes a comparative case study methodology, which is a qualitative methodology used to provide an analysis of similarities and differences to identify patterns found across multiple cases. The purpose is to produce knowledge that assists researchers in generalizing knowledge about the research questions which, in this case, describe students' mathematical reasoning and construction of knowledge related to smooth, continuous, covariation in calculus topics (Bartlett & Vavrus, 2017; Baxter &

Jack, 2008; Dennis & O'hair, 2010; Rashid et al., 2019; Kaarbo & Beasley, 1999). The method of this study includes three phases: a pre-instruction task; instruction session observations; and a post-instruction interview. In this chapter, the researcher describes the participants and setting of the study and then the comparative case study methodology. Finally, the researcher describes the methods of data collection and analysis.

Research Participants and Settings

This study examined Calculus I students' covariational and smooth, covariational reasoning. Three students who enrolled in a Calculus I course at a public university during the Spring semester 2021 participated in this study. The public university is located in the southeastern United States. This university has a total population of 22,000 undergraduate and graduate students with a diverse student body of 34% non-white or underrepresented minority groups and 55% females. Details about the study participants will be provided in Chapter 4.

At this university, the Calculus I course is given for students as an introductory course with an emphasis on analysis of functions, multidisciplinary applications of calculus, and theoretical understanding of differentiation and integration. The topics include the definition of limit and continuity, rate of change, derivative, differentiation techniques, and applications of the derivative function. The course concludes with the fundamental theorem of calculus, the definition of anti-differentiation and the definite integral, basic applications of integration, and introductory techniques of integration. Students who participated in this study learned in a classroom where instruction consisted of lecture, group discussion, and randomly selected individual student presentations.

Passing pre-calculus with a grade of “C” or better is one of the prerequisites for the calculus course. In this study, three students from this class participated voluntarily as research participants. The researcher took observation notes and recorded events that showed the students’ learning development, the interaction between the students and the instructor, and the classroom activities.

The pre-instruction recruitment task was used to document students' prior covariational reasoning abilities. The task was sent to each student through their e-mail. The purpose of the recruitment task (see Appendix B) was to examine and document each student's prior covariational reasoning abilities and helped the researcher to understand each student's prior reasoning ability and conceptions of the rate of change and derivative concepts. From the whole class of students, three students were selected who scored lower, medium, and higher in the recruitment task, and who followed up during the instruction session and post-instruction interview. These students took part voluntarily in the study. The data from the pre-instruction task helped the researcher to compare and contrast and track individual student change or development when they progressed throughout the study period. The purpose of the recruitment task was used to document the entire classroom students’ prior variational, covariational reasoning abilities, and their conception of the rate of change function; because the results of the recruitment task helped the researcher to trace, compare, and evaluate students’ development. The first problem in the task aimed to examine students’ simple variational reasoning abilities. The second and third problems of the task aimed to examine students’ general covariational reasoning abilities.

Comparative Case Study

A comparative case study involves an analysis of two or more cases with the goal of understanding patterns, knowledge development, and why the process of knowledge development in a certain context works or fails to work (Bartlett & Vavrus, 2017; Baxter & Jack, 2008; Brown, 2008; Philips & Schweisfurth, 2014). A comparative case study promotes a model of multi-sited fieldwork that studies through and across contexts (Bartlett & Vavrus, 2017). It enables the researcher to conduct multi-case analysis simultaneously and concurrently by using three axes of comparison: horizontal, which compares how similar or different phenomena unfold in the study context; vertical, which traces phenomena across scales of the study; and transversal, which traces phenomena and cases across the study time (Bartlett & Vavrus, 2017; Philips & Schweisfurth, 2014). That is to say, a comparative case study engages the researcher into two types of logics of comparison: first, compare and contrast between two or more cases; and second, a "tracing across" all cases of the study in the given context to understand, discover, and trace individuals' and groups of students' changes in certain contexts of the study(Bartlett & Vavrus,2017).

Comparative case study methodology supports researchers' investigation of an in-depth analysis of the cases within some specific context by comparing and tracing across all cases of the study reasoning pattern and knowledge development (Bartlett & Vavrus, 2017; Rashid et al., 2019; Kaarbo & Beasley, 1999; Philips & Schweisfurth, 2014). Moreover, a comparative case study as a research methodology helps in the exploration of a phenomenon within some particular context through various data sources, and it undertakes the exploration through a variety of lenses in order to reveal multiple facets of the phenomenon (Bartlett & Vavrus, 2017; Rashid et al., 2019; Kaarbo & Beasley, 1999; Philips & Schweisfurth, 2014). Comparative case study methodology is used in this study to compare, contrast, and trace patterns across all cases of the study to understand how students develop mathematics knowledge. That is, to explore how students construct mathematical knowledge related to developing smooth, continuous variation in calculus courses, and comparative case study methodology facilitates the kind of exploration and investigation necessary for this purpose (Bartlett &Vavrus, 2017; Rashid et al., 2019; Kaarbo & Beasley, 1999; Philips & Schweisfurth, 2014; Greener, 2008; McKenna, Richardson, &Manroop, 2011). According to Bartlett and Vavrus (2017), comparative case study allows the researcher to compare and contrast and trace patterns across all cases of research participants. It focuses on each case's behaviors, actions, and interactions to identify essential factors, processes, and individual knowledge development, and it also makes meaning of relationships among cases. The cases of this study were three calculus students who learned the concept of rate of change and derivative in a class where they had an opportunity to develop smooth continuous covariational reasoning. Their learning was supported with dynamic mathematics tasks, interactive applets, and constructive instructional approaches that focused on facilitating their learning.

Comparative case study methodology is selected for this study because it allowed the researcher to discover, explore, and describe (i.e., to understand and know) the students' reasoning and mathematics conceptual development abilities. In this study, the students' type of reasoning, the method of reasoning that students engaged with, and the ways that students used their developed reasoning to conceptualize mathematics concepts was explored. For instance, in this study, the researcher examined, described, and compared and contrasted each student's mathematics reasoning abilities when they solved problems related to concepts of rate of change and derivative. In particular, the comparative case study allowed the researcher to trace patterns across all cases to identify essential factors, build relationships, discover new patterns among the cases, and understand students' actions as well as the students' explanations, and mistakes that were made in the research as the students come to understand mathematics (Bartlett & Vavrus, 2017; Rashid et al., 2019; Kaarbo & Beasley, 1999; Philips & Schweisfurth, 2014). The selection of the students for the comparative case study employed purposive sampling (Patton, 2015). Specifically, the rationale was to provide an opportunity to compare and contrast and trace patterns across each student with other groups of the students in the study. Thus, three students were selected for this study because they scored lower, average, and higher in their pre-instruction task. The comparative case study students attended their regular remote class and were observed and tracked as they participated in the teaching and learning process. Each activity of the case study was observed and recorded using Zoom video, audio, students' homework and exam written responses, and researcher observation notes. In the following section, the researcher describes the sequence and design process of the three tasks.

The Sequences and Design Process of Mathematical Task

Three mathematics tasks (see Appendix C, D, and E) were used to develop students' smooth, continuous, covariational reasoning ability in phase two of the study. It was anticipated that students who develop smooth, continuous, covariational reasoning were able to understand the concept of rate of change and derivative. In particular, this study was aimed at students to understand that a varying quantity can assume all values in the given interval. Not only this, but it was also proposed that students understand the rate of change as a dynamic relationship between two covarying quantities. Moreover, this study provided opportunities for students to unpack the ideas of the rate of change by solving problems in the task that was asked of them to coordinate the amount of change of the output variable with progressive changes in the input variables. This study proposed students to develop an image of covariation that supports students' ideas related to the instantaneous rate of change of the function with continuous changes in the input variable. More specifically, this study planned to support students to have a mindfulness that the derivative function is the result of the limiting value of the average rate of change or is constructed from smaller and smaller refinements of the average rate of change. Based on the above instructional goal, the mathematics tasks were sequenced to develop students' conceptual understanding of the concept of rate of change and derivative function. Details about the mathematics tasks will be provided in Chapter 4.

Understanding of student knowledge growth and construction of new understandings is rooted in Piaget's genetic epistemology ideas (Campbell, 2006; Kitchener, 1986; Piaget, 1970, 1971; Thompson & Carlson, 2017). Thus, three tasks were designed with the idea that the students' experience would produce knowledge growth and a new understanding of the concept of a rate of change and derivative through reflective abstraction and discussions with other students and the instructor. Three principles were applied to design the task: a task that incorporates students' prior knowledge, a task that encourages the student to anticipate, and a task that used technology.

First, the task is built on students' prior knowledge because this study holds that students are actively building new knowledge from experience or their prior knowledge (Cobb et al., 2003; NCTM, 2000; Steffe & Thompson, 2000; Thompson & Carlson, 2017). Second, students' ideas of imagination are incorporated because imagination is an act of reflecting, and tasks that encourage students to anticipate will help students to reflect on their actions (Castillo-Garsow, 2012; Johnson, 2010; Piaget, 1967; Thompson, 1994; Thompson & Carlson, 2017). Third, the design of the task is supported with technology, because technology has the power to help students visualize mathematics ideas and concepts (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Johnson et al., 2017; Johnson et al., 2013; Thompson & Carlson, 2017).

Research results in the area of varying quantities show that the majority of the students' variable reasoning is at the gross variational reasoning level (Ayalon et al., 2016; Thompson & Carlson, 2017). To conceptualize the variation of quantity, a student first needs to construct a varying quantity value from a situation and attend to the measure of those varying quantities (Ayalon et al., 2016; Philipp, 1992; Carlson et al., 2002; Thompson & Carlson, 2017). This was the purpose of the first task. The first task (see Appendix C) had seven investigation problems and seven homework problems that were intended to develop students' smooth, continuous reasoning. The task was designed to incorporate students' prior knowledge and their imagination through the support of technology that is guided by the analytical framework of this study (Appendix A). The first three problems of the task are focused on engaging students in their intuitive and prior knowledge about continuously varying quantities. In these problems, students are asked to explore the difference between chunky, continuous variation (i.e., assuming the variable has a value at the start and end of the interval) and smooth, continuous variation (i.e., assuming the variable value progressively changes throughout the interval) and the task supported using the GeoGebra applet. The fourth problem focuses on engaging students in their prior knowledge of chunky reasoning that asked students to compare the actual length of a tree and a change that a tree will have after a certain period. This problem was simply intended to engage students in the idea and meaning of a change or a variation. The fifth problem expands students' background knowledge about variation and engages them in more computational ways on how the size of variation will affect the number of lengths of the interval. The sixth problem engages students in a new situation with the support of animation to support their imagination process of variation in a small chunky and smooth variation problem. Finally, the last problem engages students purposely in a mathematics problem that asked them to use their smooth, covariational reasoning abilities to solve the problem. Particularly, the final problem is intended to ask students to use their smooth, continuous, covariational reasoning to solve a mathematical problem. At the end of the phase, students solved homework problems that were intended to build their conceptual understanding and exploration of variational, covariational reasoning, and rate of change concepts.

Research results in the studies of the rate of change show that many students had struggled to conceptualize the rate of change as the quotient of two covarying quantities (Johnson et al., 2017; Thompson & Carlson, 2017). This means that most students exhibited difficulty in higher than L2-L3 covariational reasoning behaviors or were unable to reach MA4 or above covariational reasoning abilities (Carlson et al., 2002). The second task is intended to develop students' conceptual understanding and reasoning ability to interpret the rate of change as as the quotient of the coordinated changes in measures of two covarying quantities that are in a dynamic relationship (see Appendix D). Moreover, in the current study, it was planned to develop students' understanding that the rate of change is a coordinated the quotient of the amount of change of the output variable with progressive change in the input variables. At this stage, the lesson sequence supported students' insight into the relationship between two continuously covarying quantities over smaller intervals. This task helped students to develop an idea that covariational reasoning is having an image of two quantities' values varying simultaneously, and the rate of change is the dynamic relationship between quotient of two covarying quantities within a given interval. Finally, the first five problems of the second task were focused on developing students' smooth continuous covariation reasoning abilities, and the final problem of the task is focused on engaging students in applying their covariational reasoning ability to solve a problem related to the rate of change. Research results show that calculus students often lack conceptual understanding of the concept of the derivative function, and this is linked with their procedural knowledge of the average rate of change function (Park, 2013; Tyne, 2014, Tyne, 2015). This means that students did not have an awareness that the derivative function results from smaller and smaller refinements of the average rate of change of a function or derivative function is the result of the limiting value of the average rate of change function (i.e., L5 or MA1 through MA5 see Appendix A) (Carlson et al., 2002). Moreover, students did not demonstrate an understanding of the derivative as an instantaneous rate of change (i.e., do not reach L5) (Carlson et al., 2002). The final mathematics task is aimed to develop students' understanding of derivatives as a function by first developing their smooth continuous covariational reasoning ability. At this stage of the instructional sequence, it was anticipated that students developed smooth, variational reasoning, covariational reasoning, and smooth, continuous, covariational reasoning. Moreover, at this instructional stage, students developed an understanding concept of variable and average rate of change function. In the final task, the problems (see Appendix E) were designed to engage students in smooth, continuous, covariational reasoning; for instance, in one of the problems, students explored the change of one quantity value with progressive change of the other quantity value. Another problem engaged students to explore the behavior of a given function and its average rate of change of function when the independent variable varies continuously in the given interval. Moreover, the problems in the final task asked students to apply what they constructed in previous problems, and then they were asked to give a reason for what relationship is revealed when the two quantities vary smoothly and continuously on the given interval. In all of the problems, students were asked to explain and understand that the derivative function results from smaller and smaller refinements of the average rate of change of a function. This means that it was expected in this problem that students understood that, when h approaches 0, , the quotients between the dependent variable and the independent variable will be a smaller refinement of the average rate of change of a function or the limiting value of the average rate of change, which is the derivative of the function.

Post-Instruction Interviews

After the instruction session, the researcher conducted a post-instruction interview with the three research participants to get a refined insight into students' mathematics models and reasoning. In this phase, the three students participated in a final one-to-one post-instruction Zoom video interview (task-based interview). A one-to-one, post-instruction Zoom video interview with each student allowed the researcher to get a deep understanding of each student's mathematics reality (see Appendix F for interview questions). In the interview question, students were asked to explain, justify, and show their reasoning, and this helped the researcher to understand and gain insight about each student's reasoning type that was used to solve the mathematics problems. Additionally, the post-instruction interview allowed the researcher to interact and personally know each student's mathematics understanding and reasoning. In general, the main purpose of this interview was to understand each student's conceptual understanding and reasoning ability. In the post-instruction interview, Zoom video, audio data, and students' written work were generated.

Data Collection and Analysis

Throughout the research period, the following data was collected. Table 1 shows the type of data source and the event in which the research data was collected. There are five data sources: pre-instruction task problem and students' written solution, homework and exam problems and students' written solutions, instruction session video, and post-instruction interview and students' written work. The data that comes from all these sources was recorded using Zoom video, audio, students' written responses to the homework and exam problems, and computer screen captures (Zoom). Table 1 describes the type of data that was generated during each event, the total amount of data, and the total amount of time that was taken to record each datum.

Table 1

The Source of Data for the Study

Data sources and purpose	Pre-instruction Data	During Instruction Data	Post-instruction data
Sources	Recruitment task students' response (Three students' written solution or response)	Zoom video and audio of instruction session (approximately 7 hours of class data) Students' written work during the class session (3) Students' solution for the homework problems (3) Students' solution for the exam problems (3) Researcher observation notes (approximately seven notes)	Zoom video and audio of the students' post-instruction online video interview (approximately a three- to six-hour interview data, i.e., one to two hours for each student) Students written work during the post-instruction interview (approximately three to six written works) Researcher field observation notes (approximately three for each student)
Purpose	The data used to answer RQ1	The data used to answer RQ1 & RQ2	The data used to answer RQ1 & RQ2

The preliminary data analysis occurred during and after the instructional sessions. Students' written work, the researcher notes about participant thinking, Zoom audio, and video records, or any student's work during the post-instruction interview were stored in a secured area. The research data from Zoom audio, video, students' written work, and researcher observation notes were transcribed by using ATLAS software and analyzed by using the combination of Thompson and Carlson (2017) and Carlson (2002) frameworks (see Tables 2 & 3 and Appendix A for a more detailed description). Both Thompson and Carlson (2017) and Carlson's frameworks were used to code each student's response by matching with the mental action of the framework (see Appendix A). Additionally, each student's response was interpreted, analyzed, and categorized by creating a corresponding relationship with the mental action descriptions and behaviors that are provided inside the framework. Therefore, the initial stage of the data analysis helped the researcher to understand students' initial or prior mathematics reality and their knowledge construction abilities in the context of the rate of change and derivative function.

Table 2

Mental Actions and Levels of the of the Covariation Framework (Carlson et al., 2002, p. 357-358)

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Labeling the axes with variables indications of coordinating the two variables (e.g., *y* changes with changes in *x*)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of	Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to

	change in the input variable	the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Table 3

Major Levels of Covariational Reasoning of the of the Smooth Covariation Framework

(Thompson & Carlson, 2017, p. 440-441)

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantities or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.

No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

The data that was gathered from the instruction sessions are Zoom video and audio recorded lessons, students' written responses, homework and exam responses, researcher's field observation notes, and daily journal (the researcher recorded the daily journal). All the collected data during this session was analyzed using the framework. The framework was used to code, categorize, level, and interpret each student's response based on the descriptions and behaviors of the mental action of the framework (see Table 2 and Appendix A for a more detailed description). Therefore, this stage of the data analysis supported the researcher in understanding how students develop smooth continuous reasoning during the instruction sessions. The post-instruction data analysis focused on analyzing the post-instruction interview data of the three students. The data that emerged from this session was video and audio recordings of the post-instruction interview data with three students, the research participants' written work, and the researcher's observation notes. All the data collected reflecting students' reasoning during this session initially was mapped to the Thompson and Carlson (2017) and Carlson (2002) framework and then students' mathematics were described. The framework was used to code, interpret, categorize, level, and analyze by creating a corresponding match between the descriptions and behaviors of each mental action of the framework with post-instruction students' data. Additionally, the framework helped the researcher to track the different patterns of students' reasoning in the student response data (see Appendix A.) In general, the data that was collected from the research study helped the researcher to describe, compare and contrast, and understand students' mathematics and reasoning orientations. In particular, the data coded by looking for recurring regularities in the data that were identifying and tracing patterns across each case of behavior by categorizing. The framework was used as a guide for analyzing the students' reasoning. Each coded datum, categorized data, and the classified pattern of students' reasoning was examined by comparing and tracing across with each mental action description and behavior of the framework.

After examining and describing students' reasoning orientations and conceptualizing the mathematics concept, the researcher also looked at the patterns and applied a compare-and- contrast methodology between each of the three students' reasoning orientations. More specifically, the researcher looked at tracing patterns across the students' thinking when the researcher examined their thinking from the prior knowledge by comparing with their instruction and post-instruction interview. Furthermore, the researcher examined each student's ways of reasoning that supports and delimits their construction of new understanding and knowledge about the concept of rate of change and derivative using the framework, and the framework was used to guide the interpretation, analyzing all of the research participants' ways of reasoning by comparing them with each behavior and mental action descriptions of the framework and with each other. Furthermore, the framework guided the researcher as a tool to measure each student's level of reasoning and to identify student mental actions in correspondence with the framework of mental action behavior and descriptions. To remind the reader, the goal of this study was to examine in what ways students engage to reason when they construct mathematics concepts and to understand how students develop smooth continuous covariational reasoning in the context of the concept of rate of change and derivatives throughout the study period.

Data Analysis Procedure

The data analysis procedure focuses on creating themes, finding patterns, reconstructing, and comparing and contrasting the data analysis results from the start to the end of the study about students' mathematical reasoning. This study employed a comparative case study methodology that involves horizontal, vertical, and transversal axes of comparison. In this section, the researcher describes how the analysis was aligned with these three axes of comparison.

First, the primary analysis of the students' reasoning patterns and behavior produced the refined data analysis consisting of ordering, creating themes, and identifying patterns of the student's written work. Particularly, in the data analysis the researcher engaged in comparing how similar or different phenomena unfold between each case across the time of study. During this time, the researcher was taking notes, engaging in reflecting and tracing on the patterns of each student understanding, and making sense of each student's thinking and reasoning. This analysis was a baseline data analysis or a preliminary data analysis.

Next, to refine and detail analysis, the procedure of checking line-by-line a student's reasoning ability or conceptual understanding ability work done by comparing horizontally, vertically, and transversally across each case. For instance, the researcher engaged in analyzing students' particular instances *horizontally* that showed their conceptual understanding of the concept of rate of change and derivative as the phases of the study were completed. In this data analysis stage, the researcher engaged in grouping data, transcribing, interpreting, and analyzing the collected data. During this stage of analysis, the researcher identified each student's understanding and reasoning abilities.

After the primary data analysis was complete, the researcher engaged in *vertical* and *transversal* analysis across the pre-instruction, during instruction, and post-instruction by describing and understanding each student's utterance and written work. In this phase of the analysis process, each student's work was separately examined and described for each student. More importantly, the researcher engaged in a deep examination, comparing and contrasting, and tracing across each case and understanding of each student's reasoning abilities at the different instances that showed each student's action and activity in class or during one-to-one interview sessions. More specifically, the researcher engaged in analyzing any instance that showed students' reasoning ability about the rate of change and derivative during the instruction sequence. In all these analyses, the framework was used as a guiding tool to examine each student's mathematics reasoning with each behavior and description of the mental action of the framework. The framework in the process of examining and describing each student's reasoning was used to level the students' mental action and identify what form of reasoning students engage when constructing their mathematical reality in the context of the rate of change and derivative. Moreover, the framework guided to produce and identify a situation that shows students' conception and reasoning ability when they talk about the concept of rate of change and derivatives. This analysis of students' activity and their behavior led the researcher to describe each student's mathematics reasoning abilities. Additionally, the results of this analysis indirectly informed the role of the instruction sequence, the group set up of the instruction, and the use of technology for the students' conceptual development.

Finally, the researcher compared and contrasted each student's reasoning type and the relationship between each student's mathematics conception of the concept of rate of change and derivative by using horizontal, vertical, and transversal analysis. This stage provided an insight into the critical reasoning ability that students must have to understand the core concept of calculus, in particular, for the concept of rate of change and derivative. Therefore, the results of the data analysis in the pre-instruction and post-instruction interviews helped to understand the students' mathematics reasoning. In particular, the post-instruction data analysis together with instruction data analysis results helped to answer the first and second research questions.

Summary of the Chapter

This chapter discusses the research methodology of the study of Calculus I students' reasoning in the context of the concept of rate of change and derivative. A qualitative, comparative case study methodology was used as a research method and helped the researcher to understand and explore students' mathematics and reasoning abilities. In each instruction stage, a significant amount of data collected about research participants' thinking and understanding about the concept of rate of change and derivative was generated. The collected data analyzed and interpreted by examining, finding patterns, and looking at similarities and differences to examine and understand students' mathematics and reasoning orientations. The next chapter will discuss the results of the data analysis by exploring students' reasoning in the context of the concept of rate of change and derivative.

CHAPTER 4: RESULTS

Introduction

This chapter presents an account of three first-year calculus students' reasoning and problem-solving behaviours revealed during the study, which sought to answer the following research questions:

1. What types of reasoning do first-year calculus students engage in to conceptualize the relationship between two progressively co-varying quantities?
2. What methods of reasoning do first-year calculus students employ during a rate of change and derivative instructional sequence that support smooth continuous covariational reasoning?

The report first provides a description of the three participants' pre-instruction conceptions and reasoning ability, and this result is compared to their shift after their participation in the study within the context of the learning with other students enrolled in the Calculus I course. The report includes a characterization of the three students' initial conceptions of quantity, variation, covariation, and rate of change functions. Following this initial characterization, students' actions during the instruction are reported by tracing patterns in each case that show the productive and unproductive ways students understood content as they tried to solve the mathematics problems in the course. Next, post-instruction interviews of the three students are presented to further understand their reasoning and mathematical Problem-solving abilities. Attending to the three axes of comparative case study analysis in each phase of the study to compare, contrast, and trace across each case of the study horizontally, vertically, and transversally provided a means to conceptualize the types and means of thinking across the cases (Bartlett & Vavrus, 2017). The horizontal axis in the comparative analysis allowed the researcher to compare and contrast each case with the other cases to understand the similarities and differences in reasoning and conceptual understanding abilities that each case applied when they were solving mathematical problems. The vertical axis allowed the researcher to examine a single case development across each phase of the study (i.e., pre-instruction, instruction, post-instruction). Transversal axis in the comparative analysis allowed the researcher to understand how the three cases shifted and changed across time, from the start of the study to the end of the study. Finally, the report utilizes the results from the three-axis analysis to present a conceptualization of the students' reasoning when they were solving mathematical problems related to concepts of rate of change and derivative. For reference, the three students who participated in the study were Sam, Ruby, and Chris (all names are pseudonyms).

The results of the study revealed three core categories in student reasoning. Each category was primarily exemplified in one case, but some cases exhibited multiple instances of these core categories of reasoning. The categories are as follows: (a) concrete object-oriented reasoning — when a student exhibits confusion between object, attribute, unit of quantity; (b) procedure-oriented reasoning — when a student has a strong focus on procedural understanding; and (c) terminology-oriented reasoning — when a student is focused on scientific aspects of the problem situations. In the discussion that follows, the results of the analysis of the data collected in this study are presented to better understand the conceptions of variation, co-variation, average rate of change, and derivative functions exhibited by Sam, Ruby, and Chris. This analysis revealed the core categories stated above. In the following section, the three students' pre-instruction assessment results are presented.

Pre-Instruction Assessment

The pre-instruction assessment (see Appendix B) was developed and adapted from previous research studies that used the task to understand students' variational and co-variational reasoning (Carlson, Oehrtman & Moore, 2010; Schoenfeld & Arcavi, 1988). Similarly in this study, the initial survey (pre-instruction assessment) task was used to understand student pre- or initial conception about variation, co-variation, and rate of change. In the pre-instruction assessment, three questions were asked. The first question is focused on understanding students' variational and covariational reasoning (see Table1). The second question (see Table 2) and third question (see Table3) are focused on understanding students' variational and co-variational reasoning, and rate of change conceptions. In this section, the analysis is presented beginning with an analysis along the horizontal axis, examining how each student reasoned for each of the key ideas that are addressed in the pre-instruction assessment: quantity, variation, covariation, and rate of change.

Result of Sam's Pre-Instruction Assessment

This section presents Sam's reasoning and problem-solving abilities exhibited during the study. This description first provides his pre-instruction assessment results to illustrate his pre -course reasoning type. This is followed by a description of Sam's initial conceptions of quantity, variation, covariation, and rate of change measures as assessed in the pre-course assessment.

Sam is a full-time, traditional student at a large public university in the southern United States. He is a Bachelor of Science major with a focus in Computer Science. Prior to Sam's enrolment at the university, he had high school algebra and pre-calculus courses. Sam was one of the few students who responded to participate voluntarily for this study. Additionally, Sam was selected for this study because he performed below average when compared to all students who responded for the pre-instruction assessment questions.

Sam's Conception of Quantity and Variation

The goal of the pre-instruction assessment was to record students' pre-instruction mathematical reasoning and problem-solving approaches. Sam explained a quantity for the question that asked, "Which is the larger, $2n$ or $2 + n$?" (see Table1) by saying "The one that is larger is $2n$ because $2n$ can be the higher portion if n has been replaced by a number." However, he doesn't specify what number "n" is being replaced with; but his reasoning showed he is viewing "n" as an object, not as a quantity value, whose values are continuously varying. More importantly, Sam's thinking is based on procedural thinking ("$2n$ can be a higher portion than $2 + n$ of the quantity n"), but is not focusing on the quantity's values varying within an interval.

Table 4

Quantity and Variation Problem

Which is the larger, $2n$ or $2 + n$? Explain

Note that in Sam's response, he described a quantity as a replacement for a number or place holder of a number. The answer given by Sam is incorrect because if the value of $n < 2$, $2 + n$ is greater; if $n = 2$, $2n = n + 2$; and if $n > 2$, $2n$ is greater. From Sam's response, it could be deduced that Sam failed to see the continuous or progressive change in the value of variable n to the change in the expression of $2n$ and $n + 2$. This is predominantly due to his thinking about the object of the quantity value (the number that replaces or holds value) that Sam reached the aforementioned conclusion and failed to recognize the increasing, decreasing, or the directional change of "n" on the expressions "$2n$" and "$2 + n$." The findings suggest that he sees the variable itself as an object (e.g., n is replaced by a number or just a symbol rather than thinking of it as a varying quantity value or as continuously changing variable). This showed that Sam was engaged in concrete object-oriented reasoning. This concrete object-oriented reasoning widely influences Sam's thinking, and he lacks knowledge pertaining to variation or a varying quantity. This response shows that Sam's initial conceptions of quantity and variation consisted of a combination of objects, and it is numerical values. As a result, viewing quantity as object or replacement of number impacted and/or impaired Sam's reasoning about quantity, and, therefore, limited his reasoning about varying and fixed quantities. Consequently, Sam was unable to give a correct answer for the question, "Which is the larger, $2n$ or $2 + n$?", because Sam viewed "n" as symbol, place holder, or object, and this impacted his reasoning to provide a solution of the mathematics problem. In the next section, Sam's conception of covariation and rate of change is presented.

Sam's Conception of Covariation and Rate of Change

In the vehicle problem of the pre-instruction (see Table 5) students were asked to, "a. Draw a diagram to illustrate the situation and use variables to represent relevant quantities in their drawing." Sam's response to the problem is illustrated in Figure 1. Sam represented two vehicles (i.e., the objects in the problem) facing each other as points in the coordinate plane, but he did not correctly label all the relevant quantities (the distance that separates the two vehicles in km, the total distance in kilo meter (km), the time since the vehicles began moving in h seconds, and the speed/velocity of the two vehicles in km/h). Sam was able to label the automobile and bus using two points; however, he did not relate them to one another. In addition to this, he drew two vehicles in the plane showing the automobile and the bus travelling at different speeds. His response to the question shows the two cars moving on inclined planes, and the bus moves at a faster rate compared to the automobile, depicted using a steeper incline. Hence, Sam's concrete object-oriented reasoning is clearly evident in this illustration. Likewise, Sam did not draw a diagram that illustrates the relationship between the quantities in the situation. Rather, he drew illustrations of the objects involved, with one illustration depicting the two objects together and a second illustration depicting that one object was moving at a faster rate than the other. Sam's illustration depicts the objects in the situation and not the relationships between the relevant varying and constant quantities. Consequently, Sam did not view the dynamic function relation between the speed of the bus as it varies with time and the speed of the automobile as it varies with time.

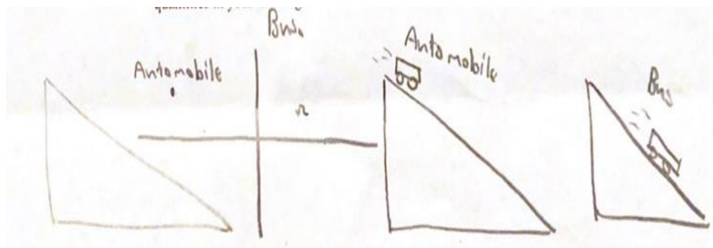


Figure 1. Diagram Constructed by Sam for the Vehicle Problem

Table 5

The Vehicle Problem

Two vehicles, an Automobile, and a Bus are 560 kilometers apart. They start at the same time and drive toward each other. The Automobile travels at a rate of 75 kilometers per hour and the Bus travels 53 kilometers per hour. a. Draw a diagram to illustrate the situation and use variables to represent relevant quantities in your drawing. b. Identify the quantities whose values are continuously changing and those whose values are kept constant in this scenario

In Part 'b' of the vehicle problem question, students were asked to identify the fixed and continuously varying quantities. Similarly, Sam's response for Part b of the vehicle problem, "The one that is kept constant is the Automobile and the continuous is the Bus," reflects his lack of reasoning on quantity variation and quantity covariation. The Automobile and the Bus are not quantities, and they are not representing a process of varying quantities. Sam identified the objects within the scenario and named them as constant and varying quantities without having a proper understanding of what the question seeks to ask. The solution provided by Sam is incorrect, as automobiles and buses are objects, not a measurable quantity. Concerning the solution given by Sam to the proposed problem, it shows that Sam's thinking was inspired more by the object of the quantity.

Table 6

Bacterial Infection Problem

Suppose a town's Board of Health reports that a bacterial infection has been spreading for the last several days. They use a model that relates the number of days since January 1, 2017, to the number of people who are infected. The graph represents the relationship between these two. a. Draw a horizontal line segment on the graph from the point (2, 5) to (10, 5). What does this horizontal line segment represent in the context of this question? How does it vary within this given interval? b. Describe what the graph conveys about how the number of people infected changes over time.

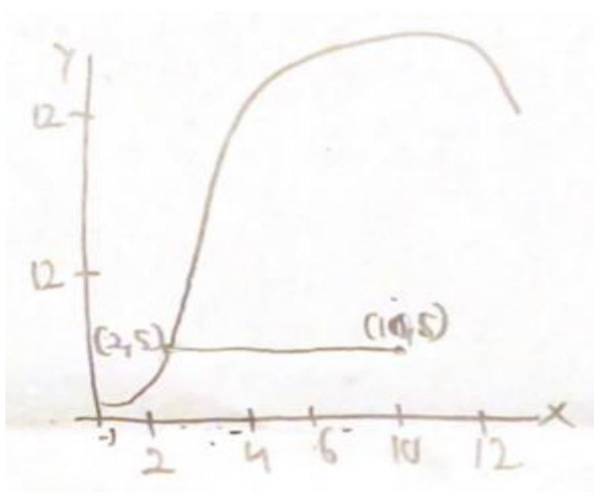


Figure 2. Graph Constructed by Sam for Bacterial Infection Problem

For the Bacterial Infection Problem (see Table 6) Sam responded that, "The line represents the two slopes interacting. It represents the infected people are a kind of constant and it may vary the days that infected." He draws and labels the start and end points of the horizontal line segment (see figure 2). In this response, it is unclear what he means by, "The line represents the two slops interacting. It represents the infected people are a kind of constant." Although Sam was successful in drawing the change using a straight horizontal line, he was unable to identify what exactly was the change in relevant quantities in the situation represented between the two points. Instead, he appears to be treating the two slopes as objects that represent the intersection of infected people. This response indicates that Sam's variational and covariational reasoning is not strong enough to interpret what the line segment "x"represents and how it varies within the given interval.

Following Part a of the Bacteria Problem discussion, Sam responded to Part b of the problem, "It shows that people are getting infected at a slow but increasing rate." Sam did not describe clearly how the number of people at first infected by the bacteria is rapidly growing and then levels off in the rate of infection. Then the rate of infection slows down until the number of infected people begins to decrease, starting on day 10 and the next day after January 10, 2017. Moreover, Sam did not read the graph as representing how two quantities change together between the start and end values. Sam did not show an understanding of how to coordinate changes in individual quantities in a covariational problem to make sense of the overall situation presented in the problem. Specifically, Sam did not clearly reason covariationally as he did not relate the number of people getting infected with the number of days. Sam focused on "people," rather than "number of people," which is a quantity, as he read the graph, and this contributed to his difficulty coordinating the change between the two quantities. Viewing quantity as an object impaired Sam's reasoning about covariation and its relationship to rate of change. Sam employed concrete object-oriented reasoning which addresses the first research question, since Sam's actions during the initial study assessment reveal that his conception of quantity, quantity variation, and covariation consisted of a loose coordination of object (e.g., symbol, automobile, bus, and people) and attributes of this object (e.g., replaced by a number, automobile speed, bus speed, and infected people). Moreover, his responses for the quantity problem, the vehicle problem, and the bacterial problem did not reveal a process view of quantity variation (e.g., the continuous variation of quantity "n" and its mathematical relationship with the expression "$2n$" and "$2 + n$") that consisted of coordinating measure attributes. Concrete object-oriented reasoning is a primary feature of Sam's problem solving. This will also show up in the next student's reasoning, but with different nuances that expose the complexity of the students' approaches. The next section describes Ruby's pre-instruction assessment result.

Results of Ruby Pre-instruction Assessment

This section presents Ruby's reasoning and problem-solving abilities exhibited during the pre-instruction of the study. The narrative provides her pre-instruction assessment results to illustrate her pre-instruction reasoning type, which includes her initial characterization of quantity, variation, covariation, and rate of change.

Ruby was a full-time student at a large, public university in the southern United States. She is a Bachelor of Science major with a focus in Computer Science. She took high school Algebra One, Algebra Two, Geometry, and Pre-calculus courses. After high school, she served in the military for four years as a Marine Corps radio operator. Her future goal is to become a computer programmer. Ruby was one of the few students who responded to the initial survey task and voluntarily participated in this study. In addition to this, Ruby was selected for this study because she performed on average when compared to all students who responded to the survey question.

Ruby's Conception of Quantity and Variation

While assessing the difficulty faced by Ruby in identifying which of the two expression values from $2n$ or $2 + n$ is larger for the quantity problem (see Table 4) Ruby affirmed that "If $n > 2$, then $2n$ will be larger than $2 + n$ because will get doubled instead of just adding two. So, if $n = 3$, then $2(3) = 6$ while $2 + 3 = 5$. If $n < 2$, then $2 + n$ will be larger, for example, if $n = 1$, then $2 + (1) = 3$ while $2(1) = 2$. If $n = 2$, then both answers will be the same." The answer given by Ruby is correct; however, the reasoning given by Ruby to justify her answer was incorrect. She replaced or substituted numbers to justify why $2n$ is larger or smaller when compared to $2 + n$. Ruby failed to recognize the variation in the values of "n" due to her dependencies on the regress computation and her action view. Thus, it can be concluded that Ruby possessed both object and procedural thinking, or her solution had a calculation focus and lacked variational reasoning of the quantity n. Her justification displayed her inclination towards procedural knowledge as she tried to introduce numbers into the equations to support her reasoning.

Ruby's Conception of Covariation and Rate of Change

Concerning the solution drawn by Ruby on the vehicle problem (see Table 5 part "a") Ruby easily drew the problem situation (see Figure3) but without labeling the values of the variables (the distance between the two vehicles or the distance that separates the two vehicles, the total distance in kilo meter, the time t in h second, and the speed/velocity of the two vehicles in km/h). Moreover, Ruby did not present a correct diagram that shows the two vehicles were driving towards each other. This depicts that Ruby did not have a clear understanding of the problem situation. Ruby was successful in coordinating the distance between the bus and the automobile and their speeds, but she incorrectly labeled the quantities' values and the direction of change. The diagram more prominently indicates that the vehicles are separated by some distance, but they do not approach each other.

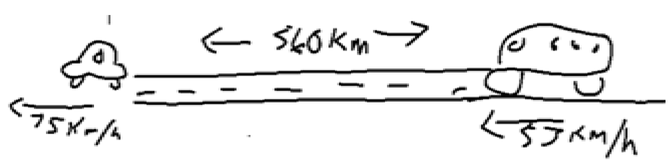


Figure 3. Diagram Constructed by Ruby for the Vehicle Problem

In the context of Part b of the vehicle problem (see Table 5) Ruby stated, "The speed of the automobile (75km/h) and the speed of the bus (53km/h) are constant, neither vehicle changes their speed. The distance (560km) between the two-vehicles, however, changes because of the difference in speed between the two, so it is a variable." From her response, it can be identified that Ruby can easily verify constant quantities, but she incorrectly identified the total distance (560km) as a varying quantity. Moreover, Ruby failed to identify the quantity time and distance between the vehicles as varying quantities, which she confused with the total distance (560km). This may be attributed to the fact that Ruby could not illustrate the problem situation using a graph and could not provide clear variational and covariational reasoning. It can be suggested that Ruby is not well-versed in drawing graphs representing the relationship between quantities that covary. Moreover, the response indicates that Ruby is not clear about the idea of covariation and variation, or reasoning regarding the amount of quantity change, which is shown in that she did not label the variables indicating the coordination of the two variables (e.g., the varying distance between the two vehicles and the time t in hours, the $d_{auto}(t)$, $d_{bus}(t)$and the time t). Furthermore, she did not have awareness of the directional of change of the two varying quantities as they co-vary towards each other.

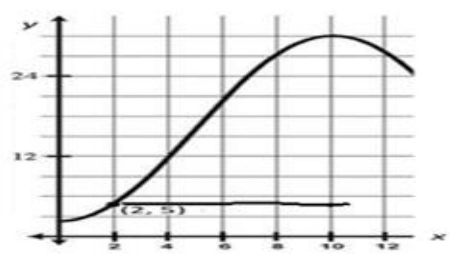


Figure 4. Graph Constructed by Ruby for Bacterial Infection Problem

Next, for the bacterial problem, Ruby asserted that, “This segment is according to the diagram, 8 days”. From the graph plotted (see Figure2,) it can be assumed that, although Ruby could plot the graph correctly and knew the variation is 8 days, she failed to comprehend how x varied, and her variation reasoning is not clearly articulated as she just proclaimed "8 days.” Ruby assumed the number of days between January 2 to January 10 as a single value, not a changing value or the result of the variation. Thus, it is observed that Ruby was unable to produce a meaningful inference about the change of days since the bacterial infection started.

Similarly, for Part b of the bacterial problem (see Table 6) Ruby answered that, “The graph shows a sharp increase in the number of infected patients, growing from 4 patients infected at the start of the graph ($0\ days$) to peaking at 30 infected patients on the tenth day. After the tenth day, the infected cases per day start to decline to 27 infected patients on the $12th$ day.” From the response given by Ruby, it could be confirmed that Ruby read the graph, but she gave loose covariational analysis for describing the problem situation; rather, she should have said, “The graph relates the number of people who were infected by bacteria with the number of days that started infecting people on January 1, 2017. At first the number people who were infected by the bacteria demonstrate rapid or fast growth and then levels off in the rate of infection. Then, the rate of infection slows down until the number of infections begins to decrease, starting on day 10, and the next days after January 10, 2017.” This exhibits that Ruby did not possesses covariational reasoning at the initial point of the study, and, rather, she showed unsettled reasoning type. She showed strong object and procedure-oriented reasoning for the first problem, in which she viewed quantity as replacement of number value due to her action view of the problem situation and loose variational and covariational reasoning ability for the second and third problems. Procedure-oriented reasoning is a primary feature of Ruby’s reasoning, and this will also show up in the next student’s reasoning, along with object reasoning like Sam, but with different nuances that fill out the complexity of the students’ approaches. The following section discussed Chris’s pre-instruction assessment result.

Results of Chris’s Pre-instruction Assessment

This section presents Chris’s reasoning and problem-solving abilities exhibited during the initial study. The narrative provides his pre-instruction assessment results to illustrate his pre-instruction reasoning type, which includes his characterization of quantity, variation, covariation, and rate of change.

Chris was a full-time student at a large public university in the southern United States. He was a Bachelor of Science major with a focus on mechatronics engineering. He graduated from high school where he had high school Algebra One, Algebra Two, Geometry, and Pre-calculus courses. After high school, he graduated with a bachelor's degree in modern German language. After his graduation, he served and worked for the Navy as an electronics technician for more than ten years. His future goal is to become a mechatronics and robotic engineer. Chris was one of the few students who responded to the initial survey task and voluntarily participated for this study. In addition to this, Chris was selected for this study because he scored above average or top score when compared to all students who responded for the survey question. Below, Chris's initial reasoning is described.

Chris's Conception of Quantity and Variation

While assessing which out of the two values $2n$ and $2+n$ is larger (see Table 4 for Quantity and Variation Problem,) Chris replied, "While n is greater than 2, $2n$ will always be larger than $2+n$. While n is equal to 2, $2n$ will be equal to $2+n$. While "n" is less than 2, $2n$ will be less than $2+n$." From the response given by Chris, it can be clearly indicated that Chris easily coordinates the value of one variable with change in the other variable. Moreover, Chris is also able to relate the directional changes in one of the variables with the other variable and reach to a correct conclusion. Chris correctly related that when $n>2$, $2n$ will be greater; when $n<2$, $2+n$ will be greater; and when $n=2$, $2n$ and $2+n$ will be equal, without mentioning or using procedure-oriented reasoning nor was he calculation-focused. From the response, it can be shown that Chris had a strong variational reasoning ability, and he viewed "n" as a varying value of the quantity.

Chris's Conception of Covariation and Rate of Chang

In the diagram constructed by Chris (see Figure 5 below) for the vehicle problem (see Table 5, part a), it is observed that Chris labels all the quantities and all the variables (the distance between them or the distance that separates the two vehicles, the total distance in km, and the speed/velocity of the two vehicles in km/h). Moreover, Chris is successful in illustrating the change in one variable with the change in the other, and he clearly coordinates the amount of change of each variable. Additionally, Chris can coordinate the distance between the two vehicles with their corresponding speeds. Chris correctly labels the direction of change; in the drawing he was also able to show that the two vehicles approached each other (see Figure5).

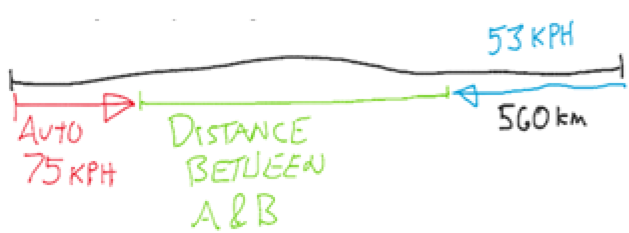


Figure 5. Diagram Constructed by Chris for the Vehicle Problem

From the drawing by Chris, it can be also deduced that he can illustrate the value of change of one of the quantities with the change of the other. Similarly, for Part b of the vehicle problem, Chris replied, "Constant quantity: auto velocity =$75kmph$, bus velocity =$53kmph$ and distance apart =$560km$; variable or varying quantity: auto distance traveled, bus distance traveled." But in his illustration, he missed the time quantity, which is the time since the auto and the bus are traveling towards each other. However, from his response, it can be clearly asserted that Chris describes clearly the constant and varying quantities. Moreover, Chris could easily coordinate the amount of change, the directional change, and the change of one variable with other quantities. It can be proclaimed that Chris exhibited strong quantitative reasoning (e.g., creating and illustrating the quantities relationship using graph). Therefore, in this problem situation it's evident that Chris clearly reasoned as to the amount of quantity change and also engaged in coordinating the amount of change of one variable with change in the other variable.

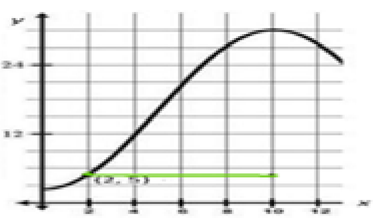


Figure 6. Response of Chris for Bacterial Infection Problem

Similarly, for part a of the bacterial problem (see Table 6,) Chris asserted, "The horizontal line represents the number of people infected on 2 January 2017." From Chris's response, it is observed that Chris could not comprehend the line segment in the graph as a quantity that changed between the start-to-end values of a point. Chris easily and correctly drew the horizontal line segment, but Chris did not comprehend that the value of 'x' varies, or he did not reveal how it varied in between the intervals. Chris describes the horizontal line segment as representing a single day infection value, which is not correct, since the horizontal line segment represents the change in the number of days from January 2 to January 10. Concerning the question in part ‘b’ of the bacteria problem (see Table 6) Chris answered, “The graph shows that infections start off at 2 on the first, increases and peaks at 30 by the 10^{th} and then decreases.” From the response given by Chris, it can be asserted that Chris loosely read the graph and gave quantitative and covariational reasoning using increase and decrease directional change, but he doesn’t use rate of change to describe the number of infected people over the given time or days. Rather, he should have said, “The graph relates the number of people who were infected by bacteria with the number of days that started infecting people on January 1, 2017. At first the number of people who are infected by the bacteria rapidly grows and then levels off in the rate of infection. Then the rate of infection slows down until the number of infections begins to decrease, starting on day 10, and the next days after January 10, 2017.” It can be affirmed that Chris reasoned on the idea of covariation and the amount of quantity change but was limited in reasoning with rate of change with the change of two quantities. Chris reasoned clearly on the idea of amount of quantity change and showed strong quantitative and variational reasoning in Problem 1 and Problem 2 above, but not in Problem 3. Chris showed a loose covariational reasoning type and rate of change conception at the initial point of the study.

Summary of Pre-instruction Assessment

This section summarized the three students reasoning orientations and their level of reasoning during the pre-instruction using the horizontal comparative analysis lens. The horizontal comparative analysis allowed the researcher to compare the three students’ reasoning and conceptual understanding during the pre-instruction phase. First Sam’s responses are summarized, then Ruby’s, and finally Chris’s. Sam’s responses to the pre-instruction assessment showed that his initial reasoning and conception of quantity, variation, covariation, and rate of change consisted of poor coordination of objects (e.g., people, bus) and attributes of these objects (e.g., the bus moves fast, two slopes interacting). Sam's reasoning level was below MA1 on Thompson and Carlson's covariational reasoning framework, and is identified as MA0, which is described as students who are unable to label the axes with variables' values (quantity) and indications of coordinating the two quantities (e.g., unable of coordinating the change in y with the change in x) (Carlson et al., 2002; Thompson &Carlson, 2017). His responses did not reveal a process for measuring quantity and variation of the quantity value that consisted of coordinating measurable attributes. Rather, measurements were pre-defined properties for the quantity objects. For instance, when Sam was solving the vehicle problem, Sam was unable to reason on quantity variation and quantity covariation answering, "The one that is kept constant is the Automobile and the continuous is the Bus." These responses imply that Sam confused object and quantity and did not reason variationally as he was identifying the object and not the quantity. Automobiles and a Bus are not a quantity, and they are not representing a process of a varying quantity.

Ruby's responses to the pre-instruction assessment showed that her initial reasoning and conception of quantity and variation were dominated by her procedure-oriented reasoning (e.g., n is replaced by number, get doubled instead of just adding two). For instance, for the first problem, she justified her reasoning by saying, "$2n$ will be larger than $2 + n$ because will get doubled instead of just adding two." This showed that Ruby, at the initial stages of the study, possessed strong procedural thinking and lacked variational thinking about quantity. Similarly, for the second problem, Ruby correctly depicts the problem situation, but she did not represent relevant quantities in her drawing, coordinate the value of change of one quantity with the change of the other, or coordinate directional change of the quantity. Since Ruby correctly depicts the problem situation but was unable to represent relevant quantities in her drawing, it can be suggested that her mental action can be grouped MA1 level; this implies that Ruby could coordinate the value of change of one quantity with the change of the other, but she did not coordinate directional change of one quantity with changes in the other quantity at the onset of the study (Carlson et al., 2002; Thompson &Carlson, 2017).

Chris's responses during the pre-instruction task reveal that his conception of variation, co-variation, and rate of change consisted of a coordination of quantity and attributes of these. His responses revealed that he had a strong variational and quantitative reasoning ability of process for measuring a variable as a progressively varying quantity (e.g., n as a continuously varying variable) that consisted of coordinating measurable attributes. Chris better possessed strong quantity and variational reasoning ability when compared to Sam and Ruby at the start of the study. This depicted that the variational and covariational reasoning of Chris is strong compared to Sam and Ruby; and it is easy to level his reasoning between MA1 to MA 3, which is he was easily able to coordinate change of one variable with change in the other variable, he understood directional change, and quantitative coordination of two or more co-varying quantities, but he was unable to coordinate the rate of change of the function with uniform increment of change in the input quantity. (Carlson et al., 2002; Thompson &Carlson, 2017). Therefore, at the onset of the study, Sam's reasoning was concrete object-oriented reasoning about quantity, he had no variational and covariational reasoning, and he was unable to coordinate the value of one variable with change in the other variable. Similarly, Ruby, at the initial stage of the study, showed both procedural and concrete object-oriented reasoning about quantity, but she showed loose variational and covariational reasoning when compared to Sam. Chris, on the other hand, at the onset of the study, showed a strong quantity, variational, and covariational reasoning type, but he showed limited reasoning with the coordination of rate of change of one variable with another variable. In the next section, the analysis along the horizontal axis is continuously presented. A trace of each of the cases' reasoning for the instruction phase of the study is presented by analysing their homework assignments and their answers on the exam.

Instruction Assessments

The first six instruction sessions, which included two breakout sessions, consisted of investigations 1-6 (Appendix G.) Only investigation 1, 2, 3, and 6 are reported in this study, because these investigations contained the research problems which were integrated with the instructional materials. This section describes the instructional tasks that were implemented during these teaching sessions. The data analysis from all three students' responses for homework and exams showed the students' preliminary model of thinking and reasoning during the instruction. The instruction sessions provided students with the opportunity to work with conception of variation, quantity, covariation. smooth continuous covariation reasoning, average rate of change function, and derivative function. Homework and exams were given to the students at the end of section 4 and 6, respectively. The homework and exam problems asked students to apply and use what they had learned about quantity, variation, and covariation, rate of change, and derivative concepts. Additionally, the homework and exam response data were collected to understand their learning and conceptual development of rate of change and derivative. The analysis of each student's response is presented below. This is followed by a characterization of each student's response for homework and exam questions. The data analysis result of the homework and the exam showed each student's model of thinking and reasoning. First, Sam's results are presented, next Ruby's results are discussed, and finally, Chris's work is presented.

Sam's Homework Assessment Results

In the water filling problem (see Table 7 below) the question was asked, "Using the grids below, sketch traces that you think would reasonably describe the function $V = f(t)$ in the interval $0 \leq t \leq 10$ and the function $V = g(h)$ in the interval $0 \leq h \leq 12$. You must explain your strategy for full credit." Sam did not draw the graph; instead, he sketched a bottle (the object) without coordinating the way the volume of the water in the bottle (one of the quantities) increased with time, in seconds, within the interval $0 < t < 10$ (another quantity) and the volume of the water's change as the height of the water grew in inches in the interval $0 < h < 12$ (the third quantity.) This suggests that, because Sam was focused on the object, he failed to apply the concepts of covariation and did not graph the relationship between two co-varying quantities. That is, he could not associate by drawing the co-varying quantities that are volume of the water in gallons and time taking to fill the bottle in water in the given time for each second, and he did not formulate the association between the volumes of water in gallons with the height of water in inches in the bottle.

Table 7

Water Filling Problem

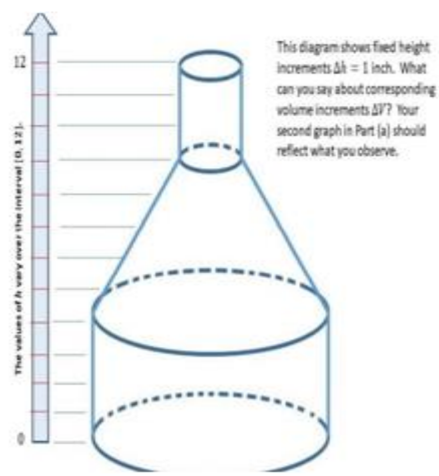
Wilhelmina purchased the beaker shown below at a flea market. The beaker is twelve inches tall and is capable of holding one gallon of liquid. Wilhelmina places the beaker under a faucet and begins to fill it at a constant rate of 0.10 gallons per second.

Problem 1. Imagine the process of water filling the beaker. Let t represent measures of the quantity "time passed since the faucet was turned on, measured in seconds." Let V represent measures of the quantity "the volume of water in the beaker, measured in cubic inches." Let h represent measures of the quantity "the depth of water in the beaker, measured in inches."

Part (a). Using the grids below, sketch traces that you think would reasonably describe the function $V = f(t)$ in the interval $0 \leq t \leq 10$ and the function $V = g(h)$ in the interval $0 \leq h \leq 12$. You must explain your strategy for full credit

Part (b). Use the grid below to sketch a trace that could reasonably describe the function $h = m(t)$ in the interval $0 \leq t \leq 10$. You must explain your strategy for full credit.

Problem 2 (Extension Question): I have included a diagram on the next page to help you think about the graph of volume as a function of depth. You can draw a similar diagram to help you think about the graph of depth as a function of time.



This diagram shows fixed height increments $\Delta h = 1\ inch$. What can you say about corresponding volume increments ΔV? Your second graph in Part (a) should reflect what you observe.

In the justification for his answer, Sam explained, “My reason is that since time has passed and water (object) would be a little full by then in a few seconds. Seeing the width of the beaker, the water (object) should be around there by a few seconds it would vary if the water (object) is still on.” Note that Sam only mentions the water, which is the object, he does not focus on a measurable attribute of the object. He understood that time and volume co-vary together as he suggested, “since time has passed and water (object) would be a little fall by then in a few seconds.” However, he was still thinking on the object “water” not the volume of the water in gallons. Sam’s “concrete object-oriented reasoning” impacted his ability to provide clear explanations about the association between the volume of water in gallons and height of the water that changes in inches as the volume of the water in the bottle varies. Thus, Sam showed a weak understanding about the two co-varying quantities’ relationship, and, rather, he showed a strong concrete object-oriented reasoning type and he reasoned using the physical object (see Figure 7 below). He didn’t apply what he learned, and his reasoning was impaired by his concrete object-oriented reasoning. In the next section Sam’s responses for the exam problem are presented.

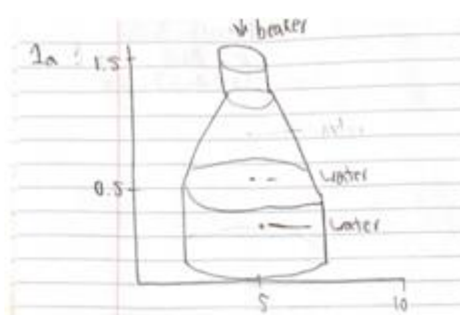


Figure 7. Response of Sam to Part (a) of Water Filling Problem

Sam's Exam Assessment Results

The response given by Sam for linear problem (see Table 8 Problem 1 below) was "$(x + h)(x + h) = x^2 + 2xh + h^2$=>$8 - 6 = 6$ & $-2/3(6) = -4, -4 + 5 = 1$." It can be observed that although Sam did provide the correct answer, his approach towards the problem is unclear, and he didn't apply variational reasoning to solve the problem. Sam failed to apply what he learned about the variation equation in y and x, and he computed the value without any clear descriptions. Sam did not display an understanding of the relationship between the variation in one variable and the corresponding variation in the other variable, instead he produced an unrelated expression "$(x + h)(x + h) = x^2 + 2xh + h^2$," which is not a covariational analysis. The way Sam has calculated the solution is unclear, and it is known that Sam did not use any variational and covariational analysis. Sam approaches the problem by inserting the given number without applying variational or covariational reasoning due to his action view approach or procedure-oriented reasoning to the problem situation, and this hampered the development of his variational and covariational reasoning. Next, for Problem 2, part a, the question asked, "What is the change equation for this linear function? "Sam produced a relationship $\Delta V = -\frac{1}{15}$; he did not write the variation in variable t, but he was able to see variation in the volume (one of the quantities). From the response given by Sam, it can be seen that Sam determined the change in the volume equation, but he did not mention the change in time variable or variation in time 't'. Sam did not write a covariational equation correctly about change in volume 'V' *and* change in time 't,' thus, he displayed poor coordination of reasoning between the two co-varying quantities V and t.

Table 8

Linear Function Problem

Problem 1. Suppose that $y = f(x)$ is a linear function whose change equation is $\Delta y = -\frac{2}{3}\Delta x$. If we happen to know that $f(2) = 5$, what is the value of $f(8)$? You must show your work for full credit.

Problem 2. A bucket of water initially contains 140 ounces of water. The bucket springs a leak, and water seeps out at a constant rate so that two ounces of water drains out every 30 seconds. Let t represent values of the quantity "the time passed since the bucket began leaking, measured in seconds." Let V represent values of the quantity "the volume of water in the bucket, measured in ounces." Let $t = f(V)$ be the function action that gives the values of t in terms of the values of V. Assume that the function f is linear.

Part (a). What is the change equation for this linear function?

Part (b). When $\Delta V = -10$ ounces, what is the corresponding time variation Δt? You must show your work for full credit.

Part (c). We know that $f(140) = 0$ seconds. Use this information and the change equation to construct a formula for the function f.

For part c of Problem 2, Sam produced a formula "$140 - \frac{1}{15}V$" which is not correct, since $f(140)$ should be 0; however, in this formula, f (140) is equal to 130.667. Here, it can be observed that Sam did not use the change function to compute a specific function value also; instead, he substituted number; Sam did not describe his approach and did not use the idea of variation and co-variation to describe the change in the function value with a corresponding variation in time. Sam displayed poor covariational reasoning, and he did not coordinate the change in value of one variable with changes in the other. Therefore, Sam showed a slight improvement in his reasoning of quantity and variation, but he is still struggling and confused with his concrete object-oriented reasoning. His concrete object-oriented reasoning limited him in not developing quantity, variational, and covariational reasoning, and this impaired his conceptualization of the mathematics concept. Sam's concrete object-oriented reasoning dominated his mathematics problem solving and this will also show up in the next student's reasoning, but with different nuances that fill out the complexity of Sam's approaches. Next, Ruby's instructional assessment results are presented.

Ruby's Homework Assessment Results

Ruby was able to draw the relationship between the volume of the water and the time taken to fill the bottle for water problem (See Figure 8 below for the water filling problem of Table 7) for the question that asked, "Using the grids below, sketch traces that you think would reasonably describe the function $V = f(t)$ in the interval $0 \leq t \leq 10$ and the function $V = g(h)$ in the interval $0 \leq h \leq 12$. You must explain your strategy for full credit." The graph sketched by Ruby depicted that there exists a linear relationship between the volume of the water and the time passed in seconds, i.e.,$V = f(t)$. Moreover, Ruby portrayed a nonlinear graph between the volume and height of the water, i.e., $V = g(h)$. In this problem situation, it can be easily justified that Ruby learned and understood variation, covariation, the rate of change function, and function, and she could easily draw the graph for both the situations (linear and nonlinear).

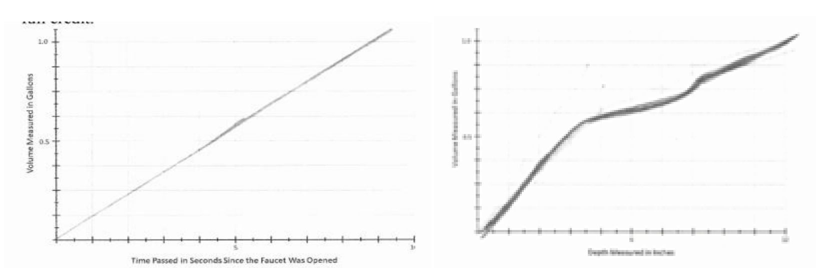


Figure 8. Response of Ruby to Part (a) of Water Filling Problem

Concerning the relationship between the height of the water and the time taken to fill the bottle, Ruby depicted a nonlinear association between the height and time in the graph (see figure 9 below); wherein, the height increases to some point and then decreases after some time. Again, since Ruby learned and understood variation, co-variation, the rate of change functions, and function, she was able to draw correct graph for the two co-varying quantities in this situation. In this context, Ruby replied, “The bottom section can hold more water, so it takes a longer time to fill up as well as the amount of water it takes to fill up is more. The second segment of the beaker takes gradually less time and water to fill up, so the height starts to gradually increase faster. The very top part of the beaker is narrower and holds less liquid, so it fills up fast and increases the height faster.”

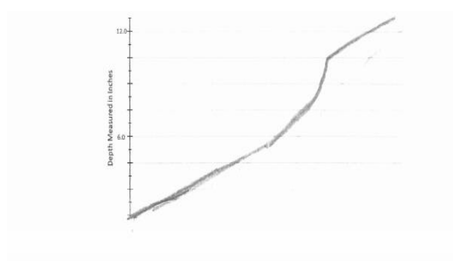


Figure 9. Response of Ruby to Part (b) of Water Filling Problem

Ruby had clarity regarding the co-variational reasoning and, therefore, could easily identify why the lower part of the bottle would take more time to fill, the second portion of the bottle takes less time to fill, and the top part would take little time to fill. Moreover, since Ruby used terms like gradually increasing, increase at a faster rate, fill up fast, and increase fast, it can be identified that Ruby could easily recognize the terms like variation, covariation, the rate of change function, and function. This further indicates that Ruby's reasoning level can be placed in the level of MA1-MA3, since Ruby depicts a clear understanding of coordinating the value of one variable with changes in the other, coordinating the direction of change of one variable with changes in the other variable, coordinating the amount of change of one variable with changes in the other variable, and coordinating the average rate-of-change of the function with uniform increments of change in the input variable. Moreover, the results of this analysis showed that Ruby substantially developed a strong variational and covariational reasoning when compared to her initial reasoning abilities.

Ruby's Exam Assessment Results

Next, for the question that asked, "Suppose that $y = f(x)$ is a linear function whose change equation is $\Delta y = -\frac{2}{3}\Delta x$. If we happen to know that $f(2) = 5$, what is the value of $f(8)$? You must show your work for full credit," Ruby described and showed every step (see Figure 10) to arrive at the correct solution. Ruby computed the variation in y as a $-\frac{2}{3}$ in variation in x, and then she transferred the expression to get the function value.

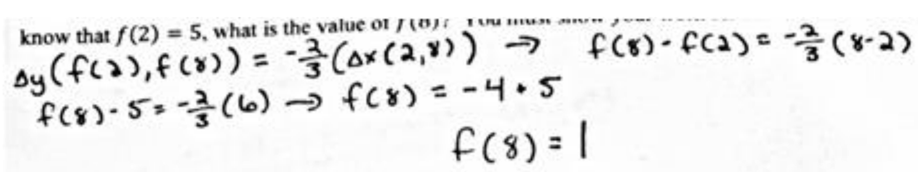


Figure 10. Ruby's Written Solution for Problem 1

Ruby could correctly define the change or variation equation as $(f(8) - f(2) = -\frac{2}{3}(8 - 2))$, and her correct covariational reasoning leads her to justify $f(8) = 1$. Ruby learned and understood how to coordinate the change value of one variable with changes in the other, coordinate the direction of change of one variable with changes in the other variable, and coordinate the amount of change of one variable with changes in the other variable.

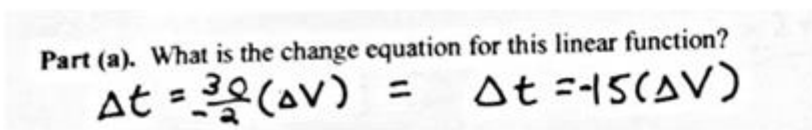


Figure 11. Ruby's Written Solution Problem 2 Part a

Similarly, for the question, "What is the change equation for this linear function?", the response given by Ruby suggests that Ruby could easily identify the change equation (see Figure 11) for the linear function correctly, and she could formulate the relationship between the two varying quantities (e.g., Δt *and* ΔV). Ruby understood the change equation of the given quantity, and she produces a correct solution. The response given by Ruby suggests that Ruby can easily use her covariational analysis to determine the change equation of the linear function of the variation in t and V to conclude $\Delta t = -15\Delta V$.

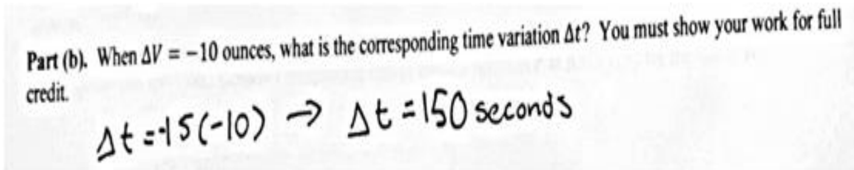


Figure 12. Ruby's Written Solution Problem 2 Part b

In the context, Ruby can correctly compute the corresponding time variation as the volume varies by -10 ounces. She used quantitative and variational reasoning. Ruby understood and analyzed the change equation of the given quantity, and then she produced a correct solution. Since Ruby developed a strong covariational reasoning during the instruction, she was able to comprehend to produce a correct solution.

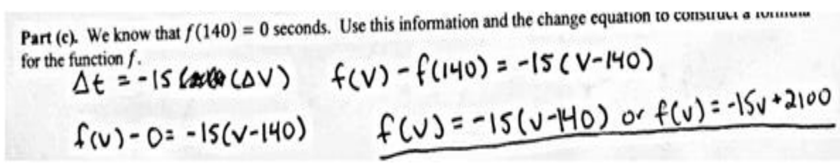


Figure 13. Ruby's Written Solution Problem 2 Part c

For this part of the problem, Ruby developed the variation equation, and she could easily evaluate the variation of time in seconds with the variation in volume of water in gallons, and she produced the function formula in terms of quantity volume. From the response given by Ruby, it can be concluded that Ruby easily comprehended the change equation of the given quantity, and she produced a correct solution. Her covariational reasoning is strong enough and helped her to analyze the problem situation.

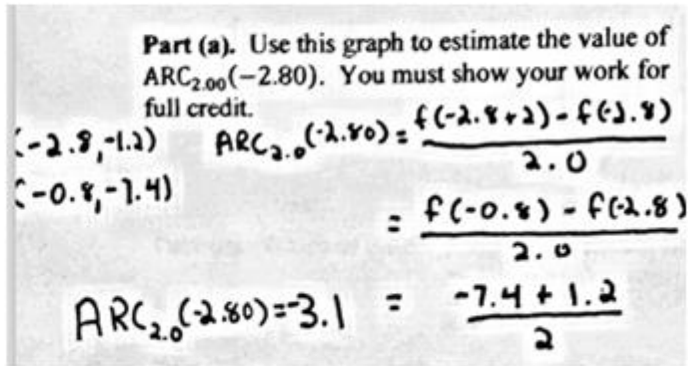


Figure 14. Ruby's written solution Problem 3 Part a

Finally, for the problem, "Use this graph to estimate the value of $ARoC_{2.00}(-2.80)$," Ruby read the graph and she was able to estimate the value of ARoC with clear steps. It was observed that Ruby was able to coordinate the change in the input variable with the change in the output variable to determine the average rate-of-change of the function with uniform increments of change in the input variable, and she was able to reason the covariation between the two covarying quantities. This is because Ruby could easily coordinate the average rate-of-change of the function with uniform increments of change in the input variable.

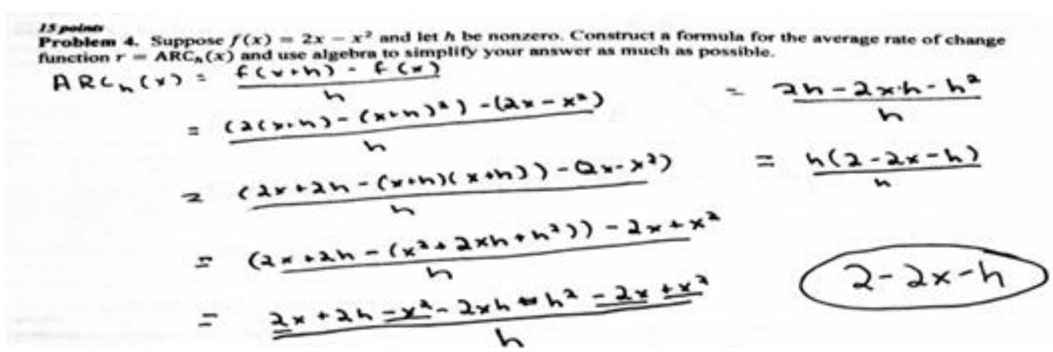


Figure 15. Ruby's written solution for ARoC Problem

Here, it can be observed that Ruby constructed the ARoC function formula and was able to observe her steps or reasoning to conclude $\mathrm{ARoCh}(\mathrm{x}) = 2 - 2\mathrm{x} - \mathrm{h}$. It can be concluded that Ruby can coordinate the average rate-of-change of the function with uniform increments of change in the input variable, and she is able to reason covariationally between two covarying quantities. Ruby's reasoning can be grouped under the MA4 level of the Mental Action Framework. This is because Ruby could easily coordinate the average rate-of-change of the function with uniform increments of change in the input variable. It can be observed that Ruby could answer the question correctly, and she produced a correct $ARoC_{0.2}(0.6) = 8$ (for the problem she was asked to compute $ARoC_{0.2}(06)$) by correctly reading the input and output quantity information from the graph. Ruby used a proper input-output process with the quotient relationship with ARoC at $h = 0.2$. Since Ruby can coordinate the average rate-of-change of the function with uniform increments of change in the input variable, she is able to reason covariation between two covarying quantities. Therefore, during the instruction session, Ruby developed strong quantity, variational, and covariational reasoning abilities when compared to Sam as well as to her initial reasoning. She also used ARoC to create or produce correct covariational relationship between two covarying quantities. Her reasoning was deepening and showed tremendous change when compared to her prior reasoning or pre-instruction reasoning. She developed strong variational, quantity, and covariational reasoning abilities during the instructional session, which helped her to produce mathematically correct or justified solutions for each problem situation. Next, Chris's instructional assessment results are presented.

Chris's Homework Assessment Results

Concerning the homework problem for Part a, it was found that Chris developed a graph of the situation for the relationship between the volume of the water and the time taken to fill the bottle with water (see Figure 16 below).

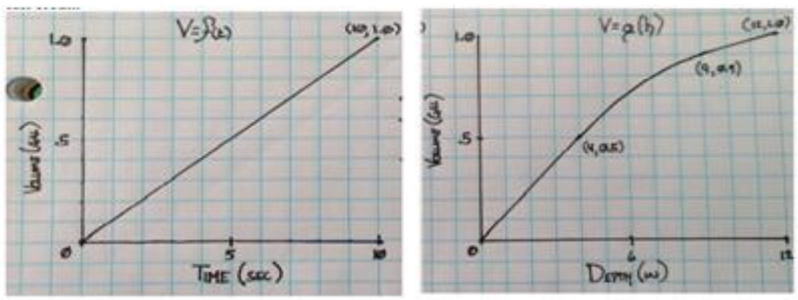


Figure 16. Response of Chris to Part (a) of Water filling Problem

The graph depicted a linear relationship between the volume of water and the time passed to fill the volume of the water. Chris proclaimed, "$V = f(t)$ is a straight line because the faucet fills the beaker 1 gal/sec. The shape of the beaker doesn't matter in this instance, $V = g(h)$ on the other h and does change based on the shape of the beaker. The volume rises steadily as the depth increases during the cylindrically shaped section. In the cone-shaped section, the volume increase will slow down because there are smaller and smaller sections to fill. When the volume is at the small cylinder the volume will increase slowly, but steadily, because each inch of depth only adds a small amount of volume." From the graph plotted and the response given by Chris, it can be deduced that Chris used covariational reasoning to assert that the relationship between the volume of the bottle and the time taken is a straight line irrespective of the shape of the bottle.

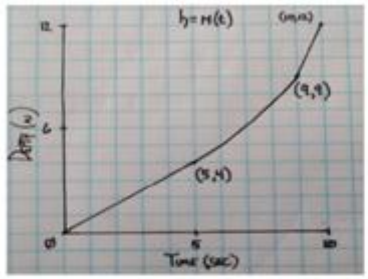


Figure 17. Response of Chris to Part (a) of Water filling Problem

Next, for part b of the water problem, Chris stated that, “As time increases the depth increases steadily over the interval [0,5] because the cylinder is straight up and down the change in h will stay steady. Over the interval [5,9] the change in h will become larger because the fill rate will stay constant but the volume to fill will steadily decrease. Over the interval [9,12] the change in h will remain steady due to another cylindrical section, but the interval for each change in h will be smaller than the change in ‘h’ over the interval [0,5]”. Chris clearly used his co-variational reasoning to analyze the input and output processes between the quantities volume and height. He indicated that the relationship between the volume and height of the bottle is impacted by the shape of the bottle. Moreover, regarding the height of the bottle and the time required to fill the bottle, Chris gave a clear causal relationship between the time taken to fill and the corresponding height increase for each section of the bottle using covariational reasoning. Moreover, Chris correctly used the terms increase, decrease, higher filling rate, and steady rate to describe the covariational relationship between the volume of the water and the height of the water. This exhibits that Chris learned and understood the concepts of variation, covariation, the average rate of change function, and function during the instructional session.

Chris's Exam Assessment Results

The response given by Chris for the exam for the linear problem showed that Chris arrived at his solution by justifying using correct variational and covariational reasoning (see below Figure 18).

$$\Delta y\,(f(8) - f(2)) = -\tfrac{2}{3}(8-2)$$
$$f(8) = -\tfrac{2}{3} * 6 + 5$$
$$f(8) = -4 + 5$$
$$\boldsymbol{f(8) = 1}$$

Figure18. Response of Chris for Problem1

Chris computed the variation in y as a $-\frac{2}{3}$ in variation in x, and then he correctly transferred the expression to get the function value. Chris coordinated the value of the variation in x with a change in the value of y. This implies that Chris learned a coordination change in one variable with the change to the other variable, which helped him to reach a correct conclusion to the problem situation.

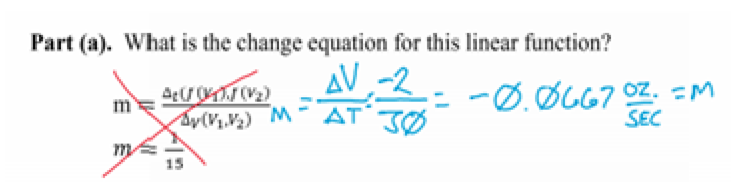


Figure 19. Response of Chris for Part a of Problem 2

Furthermore, for part a of Problem 2 of, it was observed that the change equation was correctly interpreted, and he assigned a new variable 'm,' which may be interpreted as rate of change (see Figure 19). Chris first formulated the relationship between the two varying quantities correctly, and this showed that he can coordinate the value of one variable with changes in the other, coordinate the direction of change of one variable with changes in the other variable, and coordinate the amount of change of one variable with changes in the other variable.

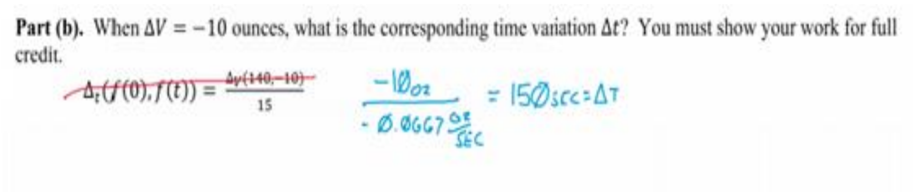
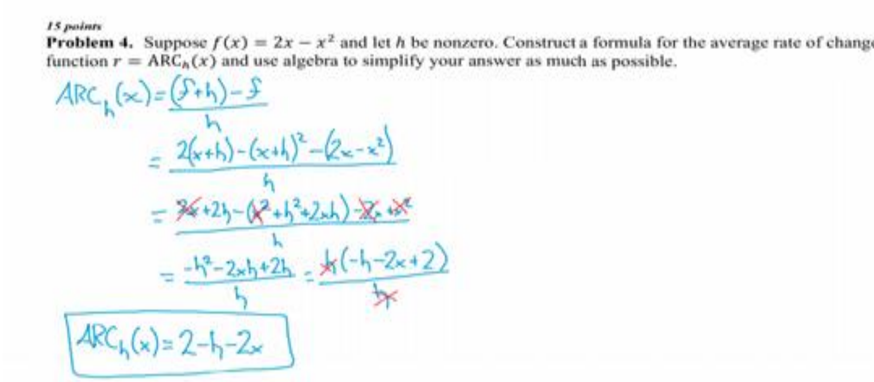


Figure 20. Response of Chris for Problem 2 Part b

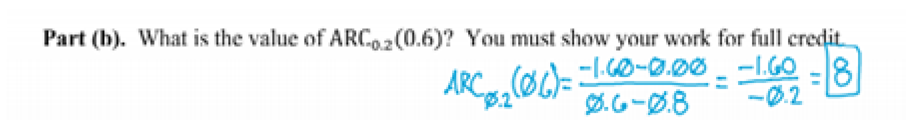
Next, the response given by Chris for part b of Problem 2 of showed that he correctly computed the corresponding time variation in seconds as the volume varies by -10 ounces (see Figure 20). Moreover, it is observed that Chris used quantitative reasoning and variation reasoning for evaluating the relationship between the two quantities. This implies that Chris can coordinate the value of one variable with changes in the other, coordinate the direction of change of one variable with changes in the other variable, and coordinate the amount of change of one variable with changes in the other variable.


15 points
Problem 4. Suppose $f(x) = 2x - x^2$ and let h be nonzero. Construct a formula for the average rate of change function $r = ARC_h(x)$ and use algebra to simplify your answer as much as possible.

$ARC_h(x) = \frac{f(x+h) - f}{h}$
$= \frac{2(x+h) - (x+h)^2 - (2x - x^2)}{h}$
$= \frac{2x + 2h - (x^2 + h^2 + 2xh) - 2x + x^2}{h}$
$= \frac{-h^2 - 2xh + 2h}{h} = \frac{h(-h - 2x + 2)}{h}$
$ARC_h(x) = 2 - h - 2x$

Figure 21. Response of Chris for Problem 2 part c

In addition to this, it can be observed that Chris constructed the ARoC function formula, and he clearly showed his steps and reasoning to produce $\mathrm{ARoC_h(x)} = 2 - 2\mathrm{x} - \mathrm{h}$ for this part of the Problem (see figure 21). From the response given by Chris, it can be asserted that Chris can coordinate the average rate-of-change of the function with uniform increments of change in the input variable, and he can reason covariationally between two covarying quantities.


Part (b). What is the value of $ARC_{0.2}(0.6)$? You must show your work for full credit.

$ARC_{0.2}(0.6) = \frac{-1.60 - 0.00}{0.6 - 0.8} = \frac{-1.60}{-0.2} = 8$

Figure 22. Response of Chris Problem 3 Part b

Finally, for the last problem, Chris produced a correct $ARoC_{0.2}(0.6) = 8$ by correctly reading the input and output quantities information from the graph. He used a proper input-output process with the quotient relationship with ARoC at $h = 0.2$ (see Figure 22). As a result, Chris is able to coordinate the average rate-of-change of the function with uniform increments of change in the input variable, and he is able to display his understanding regarding the concept of co-variation. Both Chris and Ruby demonstrated strong covariational and variational reasoning to analyze the problem situation in the instructional assessment problems. Therefore, from the homework and exam responses, it can be inferred that Chris developed strong quantity, variational, and covariational reasoning when compared to his pre-instruction reasoning abilities.

Summary of the Instructional Assessment

This section summarized the three students' reasoning type and their level of reasoning during the instruction using a horizontal comparative analysis lens. The horizontal comparative analysis allowed comparison of the three students' reasoning and conceptual understanding during the instruction phase. First Sam's responses are summarized, then Ruby's, and finally Chris's. The water filling problem revealed Sam's thought process while constructing a diagram of the object as a graph. He then used his concrete object-oriented reasoning (for instance, he sketched a bottle (the object), he used terms like "the water;" instead he should have said "the volume of the water" which is a measurable quantity) to construct quantities and, subsequently, reason about relationships between these quantities. Sam's reasoning about the rate of change of the volume of water in gallons and the time it takes to fill the bottle was not supported by identifying equal changes of time and comparing corresponding changes of the volume of water, and he did not reach the reasoning level (MA3-MA5). Sam did not envision increases or decreases in one quantity or variable value as it is happening simultaneously with the change in another variable value in the water filling problem.

Similarly, for the exam problems, Sam reasoned procedurally without applying conceptual reasoning due to his poor reasoning ability of the quantity relationship. For instance, in the linear and nonlinear problem, his actions led to Sam to produce an unrelated relationship between the independent quantity, x, and the dependent quantity, y, that reflected his concrete object-oriented reasoning. As it is demonstrated, Sam's responses for the water filling and nonlinear function problems were impaired by his concrete object-oriented reasoning of the two varying quantities, that is, “My reason is that since time has passed and water (object) would be a little full by then in a few seconds. Seeing the width of the beaker, the water (object) should be around there by a few seconds it would vary if the water (object) is still on.” His concrete object-oriented reasoning could not enable him to construct a covariational relationship between the two varying quantities (for instance, Sam thinking about the water but not the volume of the water in a gallon and time taken in seconds). Sam’s concrete object-oriented reasoning also impacted his ability to reason covariationally. His concrete object-oriented reasoning leads him to produce poor mathematics relationships between two covarying quantities (e.g., $\Delta v = -\frac{1}{15}$ and $140 - \frac{1}{15}V$).

On the other hand, Ruby constructed a deep understanding of variation and covariation reasoning that was revealed on her response for the water filling problem. She learned and used her reasoning to coordinate the covariational relationship between the volumes of the water with the time, in seconds, it takes to fill the bottle. Ruby depicted the situation by correctly drawing the graph of the volume of water and the height of the water as the time passed to fill the bottle. She used covariational reasoning to construct the quantities and subsequent relationships. More importantly, Ruby solved the bottle problem by identifying equal changes of time, in seconds, to fill the bottle with water and comparing corresponding changes of the volume of water, and then she reached a correct conclusion. Ruby envisioned increases or decreases in one quantity or variable value as it is happening simultaneously with the change in another variable value in the water filling problem (MA1-MA3). For linear and nonlinear functions, Ruby reasoned and applied covariational reasoning to justify her solution and she used covariational reasoning to analyze and obtain the correct solution. Ruby learned and understood variation, covariation, the average rate of change function, and function compared to her pre-instruction; that is procedure-oriented reasoning and loose variational and covariational reasoning. Her reasoning is grouped under MA1-MA4.

The water filling problem revealed Chris's clear understanding the covariational relationship between the two covarying quantities: the volume of the water and the time taken to fill the volume of water. Chris showed strong covariational reasoning to analyze the input-output quantity relationship; for instance, he said "$V = f(t)$ is a straight line because the faucet fills the beaker 1 gal/sec. The shape of the beaker doesn't matter in this instance." Chris plotted a graph that depicted the covariational relationship between the volume of water and the height of water in the bottle. The graph plotted by Chris depicted an increased, nonlinear relationship between the volume of the water and height, and also the height versus the time throughout the given interval of t. For linear and nonlinear problems, Chris used correct covariational reasoning to analyze the problem situation. Therefore, Chris learned and understood variation, covariation, and the average rate of change; he was also able to draw the graph for two covarying quantities in this situation. His reasoning was placed under MA1-MA4 as he improved his reasoning compared to his pre-instruction reasoning ability. This is because Chris could also coordinate the rate of change of the function with continuous changes in the independent variable for the entire domain of the function.

Both Chris and Ruby developed strong quantity, variation, and covariational reasoning. Then again, Sam still struggled to develop quantity, variational, and co-variational reasoning abilities. Sam used his concrete object-oriented reasoning, and that affected his variational and covariational reasoning development. In the next section, the analysis along the horizontal axis is continuously presented. A trace of each of the cases' reasoning for the post-instruction phase of the study is presented by analyzing interview responses.

Post-Instruction Interview

After the instruction sessions, a post-instruction interview (Appendix F) was conducted with the three students to further understand their reasoning that was revealed in the instructional setting. Four open-ended interview questions were asked to better situate their thinking about quantity, variation, covariation, average rate of change, and derivative concept. The rectangle area problem was used to assess their variational and covariational reasoning as they constructed a relationship between two quantities that included any varying value of one quantity when there is a simultaneously changing value of the other quantity. The distance function and average rate of change function problems were designed to assess students' reasoning ability when solving the average rate of change and the derivative function problems. The results of the post-instruction sessions are also presented to further illustrate each student's mathematics construction during the instruction sessions and to identify the role various reasoning abilities (e.g., variational, quantitative, and covariational reasoning) played in their learning. First, Sam's results are presented, next Ruby's results are presented, and, finally, Chris's results are presented below.

Sam's Post-instruction Interview

After reading the rectangle area problem (see Table 9 below) Sam described quantity as a fixed rate (see excerpt below).

Table 9

Quantity and Variation Reasoning

Area of Rectangle Problem
You have 240 feet of fence to enclose a rectangular lawn. You are free to make the enclosure have any possible length and width, but you must use all the fences. Play the GeoGebra (GG) animation applet of the Covariation and Area function. a. Define the constant variable in this situation. b. Define the varying variables in this situation. State the intervals over which they are varying.

Interviewer: So, the question is, describe the constant quantity in this situation.
Sam: A constant variable is something that just not changes or it's something that does not affect. I try to say the right word, it is a fixed rate.
Interviewer: What do you mean a fixed rate?
Sam: So, is this in a fixed rate being the right word for it to be like a like a fixed rate? Like, I think it cannot change or keep the same value.
Interviewer: So, from these quantities [showing the GeoGebra applet] which one of the quantities keeping the same value.
Sam: Which one keeping the same, I will say the length keep the same,
Interviewer: Which one keeps the same?
Sam: I will say the length keep the same.
Interviewer: The length OK. Why have you said that?
Sam: Ok I see, I am sorry, the length of the rectangle is keep changing [he said that after he saw all the length, width, and area is keep changing in the applet]. Since the width just keeps changing, and so I just put small a

small change, rectangle shape "concrete object-oriented reasoning", just keeps changing constantly,

Interviewer: Nice. It is a nice way of expressing your idea. So now my question for you is, when you say a quantity vary by constant or fixed amount what you mean that

Sam: It just means that the value of the quantity increases by fixed amount or noticeable amount.

Sam was asked to describe the constant variable in the scenario. In this context, Sam revealed that a constant variable is something that just does not change or it's something that is not affected. He further described the constant variable as a "fixed rate." After further investigating the meaning of the word fixed rate, Sam revealed that he meant something that does not change. Here, Sam showed confusion between the concept of "fixed rate" and "constant quantity or fixed quantity." Sam explained that the meanings of fixed rate and constant variable are similar to him. To further assist Sam, the interviewer provided Sam with a GeoGebra applet and asked Sam to make a comparison and observe the change in the applet. After observing the GeoGebra applet for the question asked, "Why you said that?," Sam suggested that, "…so I just put small a small change, rectangle shape, just keeps changing constantly." Here, it was shown that Sam was thinking about the object (e.g., the shape, the rectangular shape,) not the quantity (e.g., length, width, area, & perimeter of the rectangle). The interviewer further assisted Sam and guided him by revealing that the blue part indicated the area of the rectangle, in hopes that Sam could label the length and width of the rectangle. After looking at the applet, he noticed that the length, width, and area keep changing. Sam seems to have adjusted his reasoning here after looking at the applet and suggested that the width is also constantly changing, so the area might be a constant quantity. However, Sam then pointed to the area of the rectangle and asserted that the "rectangle changes continuously." Sam resisted changing his concrete object-oriented reasoning; he used the object (the rectangle) to reason about changing and fixed quantities of the rectangle problem. To gain insight into Sam's reasoning, he was asked further question about varying quantity, and he describes quantity as an object (the rectangular shape).See excerpt below.

> Interviewer: So, the question is, can you identify which one is a varying quantity in this type of situation.
> Sam: You know. So, I'm seeing right now, it just keeps going up, it just keeps changing the shape, the rectangular shape [object] is keeping change.
> Interviewer: Ok. Yeah. Show me. Show me. Exactly.
> Sam: And so right here is the rectangle [object] with variable changes. So, it's not really good drawing. But rectangles right here is changing, [by point to the area of the rectangle]
> Interviewer: What about if the quantity is changing continuously or by an unnoticeable amount or by little amount or by a very small amount. What can you say about this kind of change?
> Sam: I was saying that maybe the quantities were joined together, the area of the rectangle keeps changing continuously.

In this segment, the interviewer asked Sam to define varying quantity. Sam described how the object (rectangle) is changing, but he didn't use a quantity, such as area of the rectangle, length of the rectangle, width of the rectangle, or perimeter of the rectangle as he described which quantity is constant or varying. This response shows that Sam had confusion between the object of the quantity (rectangle) and the quantity (area of rectangle.) Sam struggled to identify the difference between the object of the quantity and the attribute of the quantity. Next, Sam was asked a question to understand the development of his smooth, continuous, covariation reasoning (see below Table 10).

Table 10

Smooth Continuous Covariational Reasoning

Area of Rectangle Problem continued above
You have 240 feet of fence to enclose a rectangular lawn. You are free to make the enclosure have any possible length and width, but you must use all the fences. Play the GeoGebra (GG) animation applet of the Covariation and Area function. c. When the width of the rectangle varies by 10 feet between the interval what relationship do you notice between the rectangle's enclosed area and its width? What can you conclude from the relationship? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not. d. When the width of the rectangle varies smoothly and continuously between the interval (in all points in the interval). What relationship do you notice between the rectangle's enclosed area and its width? How does the area of the rectangle vary as the width of the rectangle varies all intermediate values within the given interval? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not. e. Make graphs for the situation in part (c) and (d). Did you get the same graph for part c and d? If you found different graphs, explain why this is occurred.

After reading parts c, d, and e of the rectangle area problem (see Table 10) Sam explained his solution and he produced the physical object as a graph illustration (see below excerpt.)

> Interviewer: Yeah, what relationships do you notice between the rectangle enclosed area and the width of the rectangle?
> Sam: It keeps changing. Right.
> Interviewer: Ok, how it keeps changing.
> Sam: Keeps changing, both keeps changing. The width it kind of stays the same, but the length and area of the rectangle just keeps changing.
> Interviewer: Ok, how they are changing?
> Sam: I really don't know. I really don't know about this.
> Interviewer: Ok, why part d is tough to draw the graph?
> Sam: It will a different graph and yeah, since its shape of the rectangle keep change on both cases, but for part d since the area is continuously keep change it is hard to draw the graph, since the rectangle is just keeps changing continuously. Part d it is really hard to tell the area but at least in part c we got number like 10 feet.
> Interviewer: Can draw the two graphs, if possible, for you?
> Sam: I will try graph part c but I can tell it is hard to graph part d.
> Interviewer: You can try.
> Sam: Let me use piece of paper.
> Interviewer: Yeah, I can give you time.

Sam: I said, this is my interpretation right here. This is right here. This is for graph for c. [He presented his drawing]
Interviewer: Ok, Show me the graph.
Sam: Just standing right here in the bottom.

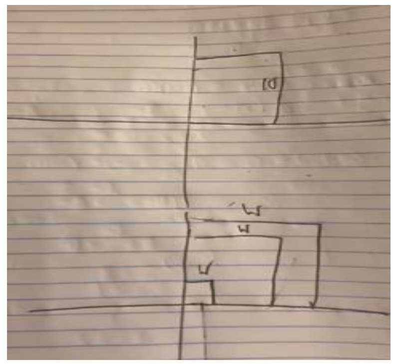


Figure 23. Sam response for the Area Problem

Interviewer: Why you draw this graph for part D?
Sam: Since the width just keeps changing, and so I just put small a small change, rectangle shape, just keeps changing constantly.
Interviewer: Nice. It is a nice way of expressing your idea.
Interviewer: Ok, so that's good. So the other question is when you define two quantities co-varying together, how can you define their relationship? How are they relating to each other?
Sam: I mean, to each other, it is just like one cause the other to change, I just trying to pick the right words for it's like, I think they cause each other.

After reading the question, the interviewer asked Sam to identify the relationship between the rectangle's enclosed area and the width. Sam suggested that the width stays the same, but the length and area of the rectangle just keep changing. However, Sam did not explain how this change exactly occurs. The interviewer next asked Sam regarding the kind of relationship and assisted Sam by displaying the GeoGebra Applet. Sam asserted that, as the width keeps on increasing, the area of the rectangle just keeps changing. Sam further explained that, if the width gets high, then the area of the rectangle gets as close or as high as, or it just disappears. In other words, Sam stated that the area of the rectangle could be smaller and larger as the width just keeps getting larger or even smaller or could be constant at some value. The interviewer further investigated that, if the width is large, what impact would it have on the area? Sam, in this context, affirmed that the area will just be way too high, way too small, or just constant. The interviewer asked Sam to construct a graph for any possible value of the area and width of the rectangle. Sam, in this context, constructed the object of the graph of rectangle, where the width was 10 feet.

The interviewer further asked Sam to establish a relationship between the area of the rectangle and the width of the rectangle when the width of the rectangle smoothly and continuously varies. Sam again asserted that it is all continuous, and, if the width keeps changing, then the area and shape just keep changing. Moreover, Sam affirmed that finding the area is hard, since the rectangle shape just keeps changing (he used here his object thinking "shape") as the width varies. The interviewer asked Sam to draw a graph for parts c & d questions. Sam was ready to draw the graph for part 'c' but found part 'd' difficult. Sam stated that drawing the graph for part 'd' is difficult since the rectangle kept changing continuously (concrete object-oriented reasoning) for part d; however, in part c (see Figure 23,) he was able to draw, since there is a fixed number that is 10 feet. Since Sam's thinking was with symbol or object, it was easy for him to draw and reason about the object's change. Sam then developed a graph of part 'c' of the object of the rectangle (see Figure 23,) wherein, Sam asserted that, since the width just keeps changing, he depicted a small change, rectangular shape (object), which just kept changing constantly. In summary, Sam's action on this problem revealed that Sam's reasoning is about object o not the quantities relationship, and this affects his mathematics problem-solving abilities that demand variational and covariational reasoning abilities. Sam struggled to conceptualize smooth continuous change which changes in a small amount.

Nonlinear Function Problem

After the area of rectangle problem, Sam was asked a nonlinear function problem (see Table 11) that assessed his covariational reasoning ability and his conceptual understanding of rate of change and derivative. Sam used concrete object-oriented reasoning to analysis the problem (see below excerpt).

Table 11

Meaning of Average rate of change and Derivative function

Problem 2. Suppose $d=f(t)=2t^2$ represents the distance (measured in meters) of a car from a stop sign in terms of the number of seconds t since the car started to move away from the stop sign.

a. Determine the average rate of change of the distance of the car from the stop sign on the time intervals from $t = 0$ to $t = 1.5$ seconds.

b. Describe what this average rate of change function is tells you about the change in distance of the car from the stop sign over the time interval from $t = 0$ to $t = 1.5$seconds.

c. Determine $r_h(t)$ when $h = 0.5$ seconds and describe what this tells you about the change in the distance of the car from the stop sign?

d. Determine $r_h(t)$ when $h = 0.1$ seconds and describe what this tells you about the change in the distance of the car from the stop sign.

e. Sketch a graph of $r_h(t)$from the stop sign in terms of number of seconds since the car stared to travel for $h = 0.5$, $h = 0.4$, $h = 0.3$, $h = 0.2$, $h = 0.1$, $h = 0.001$ and $h = 0.00001$ seconds. What can you say about the graph of $r_h(t)$ when $h = 0.000001$ seconds? Explain the graph of $r_h(t)$

Note. The formula $r_h(t) = \frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$ gives the average rate of change for $d = f(t)$ with respect to $t$over any sub-interval from t to $t + h$, where $h \neq 0$ is the length of the interval on which $t$varies.

Interviewer: OK, how did you calculate the average rate of change?
Sam: Yeah, what you replaced with the numbers like 0 and 1.5 one will get 4.5.

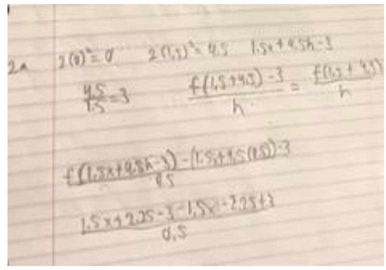


Figure 24. Average rate of change formula when $h = 1.5$

The interviewer asked Sam to determine the average rate of change of the distance of the car from the stop sign on the time intervals from $t = 0$ to $t = 1.5$ seconds. Sam, in this context, suggested replacing $t = 0$ and 1.5 and stated that the answer would be 4.5. Sam depicted the solution on paper (see Figure4.) The interviewer further asked Sam the steps employed for calculating the average rate of change. Sam suggested that he replaced the numbers with 0 and 1.5 and received the solution as 4.5. Furthermore, Sam suggested that the value 4.5 was the average rate of change for the given question. Sam described the variation in quantity t as replacing the variable value of t and as a placeholder here. He didn't show evidence of thinking about variation and covariation between the two co-varying quantities of distance and time to evaluate the average rate of change (ARoC.) Sam confused the variation in t values with a placeholder value, or he wasn't aware that t is a quantity whose value can change or vary between 0 and 1.5. Sam used his action view (he viewed h as placeholder) to evaluate the ARoC function by substituting the initial and end values of the quantity time and variation in distance of the car between these two points; he didn't use the idea of ARoC to determine the value of the ARoC. After Sam was asked to calculate the average rate of change for $h = 1.5$, to gain insight about his reasoning, he was also asked a further question for small change of $h = 0.1$

and for small variation of time t and its meaning as shown below. Sam described ARoC as fast and slow (see below excerpt.)

Interviewer: Yes. Describe what this average rate of change function is for h $=$ 0.1.
Sam: I would say the average rate change function may tell you how far fast or going very slow from the stop sign.
Interviewer: Ok, that's good. Yeah. So, what do you mean it goes fast.
Sam: Asking a question like this is like going fast, growing very slow. That's in the context this I think it might be going a bit slow, but I know most people I just can't close with normal speed.
Interviewer: Yeah. what the Average rate of change is tells you about the distance of the car from the stop sign in this time interval.
Sam: Ultimately, how far back the car right here or just going by really fast or just like really going slow but take his brakes and stuff just right on a stop sign right here. It just slowly going down there.

In this context, Sam described that the average rate of change function suggests how fast or slow one moves from the stop sign. The interviewer further inquired what the average rate of change means in the context of the distance of the car from the stop sign in this time interval ($h = 0.1$). Sam asserted that, "How far back the car right here or just going by really fast or just like really going slow but take his brakes and stuff just right on a stop sign right here. It is just slowly going down there." Moreover, Sam revealed that the stop sign right here tells you it keeps going right here. Sam didn't define ARoC using co-variation between quantities; average rate of change (ARoC) between the two quantities; nor did he explain that ARoC is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval. Instead, he used the term how far, fast, or going very slow for the car to define ARoC, which is not sequitur relative to the definition of ARoC function. The researcher further asked Sam questions to understand his reasoning when the variation in t is a very small or an unnoticeable amount ($h = 0.001$) and aimed to understand his smooth continuous covariational reasoning and concept of rate of change, and Sam related $h = 0.1$, $h = 0.01$ and $h = 0.001$ with slow, fast and very fast (see excerpt below.)

> Interviewer: What can you say about the graph of $\mathrm{r_h(t)}$ when $\mathrm{h} = 0.1$ seconds and when $\mathrm{h} = 0.5$ seconds? Explain.
> Sam: Ok, for $\mathrm{h} = 0.5$ the distance, it's just like I feel like it's definitely going to stop sign, but it's going to really slow. But for $\mathrm{h} = 0.1$ a distance from the stop sign. It'll take a while for it to get their signal speed fast. That's right.
> Interviewer: Ok, yeah, I think so. So finally, I want to ask you if $\mathrm{h} = 0.0001$ what will tell us? You know, you described to me about zero point one that is the case.
> Sam: It's a very small. Will it be a bit faster?

Concerning the graph of $r_h(t)$ when $h = 0.1$ seconds, $h = 0.5$ seconds, for the question the researcher then asked: What can you say about the graph of $r_h(t)$ when $h = 0.1$ seconds and when $h = 0.5$ seconds? Explain. Sam suggested that "for $h = 0.5$, the car is travelling slowly to the stop sign, and the distance covered is extremely large. Moreover, for $h = 0.1$, Sam suggests that it will take a while for it to get to stop sign, and the car has a fast speed. And finally, for h=0.0001, Sam revealed that it is extremely small and was unsure of whether it will be a bit faster." Sam connected the average rate of change function with the idea of fast and slow movement when $h = 0.1$, $h = 0.5$, or $h = 0.001$. The smallest h value related with fast moving and large value of h means the average rate of change of the function tells him how slowly the distance of the car changes (i.e.,$h = 0.0001$ means for him a bit faster compared to $h = 0.1$. That is, when h is small that means fast, and when the h is large, then it is slow) Sam couldn't create a connection or ideate how the ARoC converges to the derivative of the function as h was getting smaller or closer to zero. Sam confused the idea of h approaching zero with the idea of slowness and fast (when $h = 0.1$, fast speed; when $h = 0.001$, a bit fast speed; and when $h = 0.5$, the slow speed.) He did not relate the idea of h approaching zero with the idea of ARoC converging to the derivative function or to some constant rate function or derivative function is the result of the limiting value of the average rate of change. Next, Ruby's post instruction results are presented.

Ruby's Post-instruction Interview Assessment

After reading the rectangle area problem (Table 10) Ruby explained her solution (see below excerpt).

> Interviewer: So, you can you read it out loud the problem.
> Ruby: Ok.
> Interviewer: What is your response for the first problem?
> Ruby: It says we have 240 feet of fence enclosing a lawn and you can have any length and width, but you must use all the fence. So, the constant variable would be 240 feet of fence.
> Interviewer: What about the varying quantity in this situation?
> Ruby: So, the length and width are varying, it has to all equal 240 in the end.
> Interviewer: Yeah, yeah.
> Ruby: So, the length and width is varying. I forgot what the term is. Mm hmm. They're varying together when one is, the other one decreases.
> Interviewer: So, you're right. What about other variables?
> Ruby: The area is also changing.
> Interviewer: Yes.

For the question, "Define the constant quantity in this situation?", Ruby immediately responded that the constant quantity in this context would be 240 feet of fence. Ruby identifies what the constant quantity in this situation is. After this question, the interviewer asked Ruby, "Define the varying quantity in this situation. State the intervals over which they vary." Ruby believed that the length and width are varying; she also said "… the length and width are varying; it has to all equal 240 in the end." Ruby further affirmed that the length and the width vary together, and also the area is changing as she said, "The area is also changing." Moreover, Ruby stated that the length and the width were varying between zero and 120; although, 240 total foot length of fence is a constant. Ruby knew the length, width, and area of the rectangle are varying quantities, and the 240 feet length face is a constant quantity. Interestingly, she identified the constant/fixed quantity and the varying quantity in this situation. Moreover, Ruby described the covariational relationship between the three quantities: length, width, and area of the rectangle. This instance showed that Ruby developed strong variational and covariational reasoning about the problem situation. To further understand Ruby's variational and covariational reasoning, the interviewer asked Ruby a question when the width of the rectangle varies by 10 feet and unnoticeable amount between the intervals $0 \leq w \leq 120$ she analyzed the problem using covariational reasoning (see excerpt below.)

> Interviewer: Ok, yeah, that's right; it is increasing by 10 feet, but how?
> Ruby: It increases by 10, so every time it's increased by 10, until it reached to 120. Mm hmm.
> Interviewer: So that's good observation. So, what will be the area and how the area varies as the length or width of the rectangle change by 10 feet?
> Ruby: So, I'm stuck on because I kind of figure out like this, I'd like to see the formula.
> Interviewer: Ok, let's go back to the formula.
> Ruby: Ok, A(area of the rectangle) equals w (width of the rectangle) times one hundred twenty minus w [$\mathrm{A} = \mathrm{w} * (120 - \mathrm{w})$].
> Interviewer: So, what kinds of relationship exist between the area and the width for part c. What about their relationship?
> Ruby: So, I think between area and width there is a directly varying as width increase that area increase, and the width and length has inverse relationship, as the length increase the width decrease.
> Interviewer: Ok, yeah.
> Ruby: For the length and width I think they're like inverse variation maybe. I get really confused.
> Interviewer: What do you mean they are vary inversely?
> Ruby: They're covering.
> Interviewer: Ok, nice.
> Ruby: Ok, yeah.

For the question that asked, “When the width of the rectangle varies by 10 feet between the intervals, what relationship do you notice between the rectangles enclosed area and its width?”, Ruby was stuck and wanted to see the formula. She said, “So, I’m stuck on because I kind of figure out like this, I'd like to see the formula.” Interestingly, she wanted to refer back to the formula to figure out the relationship between the two covarying quantities (the area and width of the rectangle). Ruby then clarified width of the rectangle increases by 10, so every time it's increased by 10, until it reached to 120. The interviewer again asked Ruby what the area will be, and how the area varies as the length or width of the rectangle changes by 10 feet. Ruby indicated that both the area and the width co-vary with each other by a chunky size of 10 feet and states that, as the width increases, the area increases accordingly, and she knew, after some point, the area starts to decrease. The interviewer further inquired for Ruby to assess her smooth continuous covariational reasoning if it is possible to make a table that shows all possible values of the variables that this relationship produces (see excerpt below.) For the smooth and continuous variations, she did not make a table, and she said, “It is hard or impossible to include every single possible point.” She knew for any continuous change in width of the rectangle (i.e., for all possible values of w in the interval) the area of the rectangle will have all possible real numbers. Ruby said that “It is hard or impossible to include every single possible point …because it is like smoothly or millions of points.” From the responses given by Ruby, it was clear that Ruby affirmed that the area is continuously increasing. While talking about the relationship between area and width, Ruby stated that area and width are directly varying. She suggested that as width increases, area increases; similarly, she said that the width and length had an inverse relationship, that is, as the length increased, the width decreased. Here, Ruby used the ideas of direct and inverse relationships to describe covariational relationships between two covarying quantities. Ruby got confused between covariation and inverse functional relationship. Ruby said, “So, I think between area and width, there is a directly varying as width increase that area increase, and the width and length has inverse relationship, as the length increase the width decrease.” Here, it shows that Ruby was in the state of disequilibrium between her procedure-oriented reasoning (direct relationship and inverse relationship) and the new covariational reasoning (coordination of two or more co-vary quantities.) She reverses her covariational reasoning that was developed during the instructional session, and she used procedure-oriented reasoning to justify the covariational relationship between the area of the rectangle and the width of the rectangle, and she used the formula to describe this relationship. For instance, she used direct relationship and inverse relationship to describe the functional relationship between the area of the rectangle and the width of the rectangle as the width of the rectangle varies smoothly and continuously in the given interval, which is incorrect; rather, she should have used the idea of covariation to describe the functional relationship between the two covarying quantities (see excerpt below).

> Interviewer: Thank you. So, let's go and see part d.
> Ruby: Ok, So I did not make a table for this one.
> Interviewer: Why? You didn’t make a table.
> Ruby: That's because I think it'd be a hard, it is impossible, to include every single possible point that would be on. Because it would be like smoothly, so you'd have, like, millions of them on the table? But I don't think the graph would change. .
> Interviewer: So, now, let's talk about the graph, so what kind of graph can you draw for the graph situation for part d?
> Ruby: I would be any point that would just be a smooth line.
> Interviewer: Can you draw and illustrate using graph what you are thinking?
> Ruby: I said I was confused. Here, the picture [she showed here picture] and point out that the graph is like a parabola.

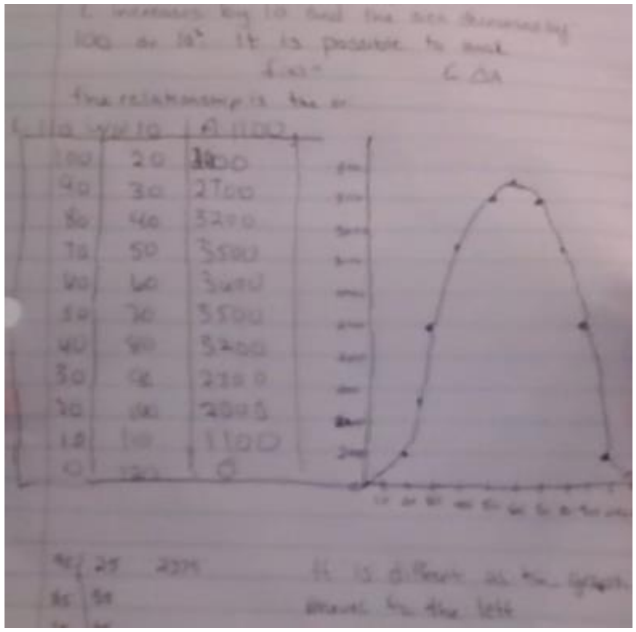


Figure 25. Graph on Situation Chunky and Smooth Continuous Covariation

Nonlinear Function Problem

After the area of rectangle problem, Ruby was asked a nonlinear function Problem (see Table 10) to understand her covariational reasoning ability. The nonlinear distance function problem was used to assess Ruby's reasoning and actions and how she constructed a covariational relationship between two covarying quantities that included the ability to reason about a constant rate of change. Moreover, the distance problem was designed to understand Ruby's conceptual understanding about average rate of change and derivative function. Ruby's response to the problem is presented in the excerpt below.

Ruby had a clear understanding of the problem and suggested that, if d is the distance measured in meters from the starting point to a stop sign, it can be observed that the distance formula is defined as the function of t, which is two times the square of t, and the time is measured in number of seconds passed since the car moved away from the stop sign. Ruby easily calculated the average rate of change of the distance of the car from the stop sign in the given time intervals from $t = 0$ to $t = 1.5$, and she determined that the ARoC is 3 meters. Ruby said “I hate fractions ...hahahaha...so for the first question, I got three. So, the average rate of change of the distance when t varies from 0 to 1.5, I got three. And if you can read my work that I tried to make it readable.” Ruby’s was utilizing covariational reasoning, since she does not use proper units of the average rate of change function; she just said 3 meters, without interpreting what is represented. Further, the interviewer inquired the meaning of average rate of change function. Ruby, in this context, suggested that ARoC would tell her “the car travels three meters,” reasoning was incomplete. The interviewer further assisted Ruby to recognize the actual meaning of ARoC. She interpreted that three is the change in distance when time changes from 0 to 1.5seconds (see excerpt below.) Ruby confused distance and ARoC. She coordinated the change in time with the change in distance that the car traveled, but she interpreted ARoC as distance, which is not covariational reasoning, and she was influenced by rise over run concept and she described ARoC as distance (see below excerpt).

Interviewer: So, go ahead and determine part a, or use the formula in the box to answer part a.
Ruby: What do you square?
Interviewer: One point five.
Ruby: One point five.
Interviewer: Yeah, then one point five means three over two.
Ruby: Ok.
Interviewer: When you square it, it will be nine over four.
Ruby: Really, I hate fractions, hahahahaa ...so I like decimals I will go to decimals. I hate fraction hahahahaha. This was going to go this way.
Interviewer: If you change into fractions, it would be easy.

Ruby: I hate fractions ...hahahaha...So for the first question, I got three. So for the average rate of change from the distance of t zero to t equals one point five, I got three. And if you can read my work that I tried to make it readable.

Interviewer: Yeah. Thank you. So, what is average rate of change function tells you for the relation between the distance of the car in meter and the time t in seconds?

Ruby: So, I thought that would tell me is in a one point five second interval, the car travels three meters.

Interviewer: The car travels three meters OK. So, let us go back; see what 3 meters represent in part a.

Ruby: Three would be the distance from the stop sign.

Interviewer: How did you find three in part a?

Ruby: So, when in one point five seconds, the car moves three meters from the stop sign.

Interviewer: Now Ok. How did you determine three in part a? Look at what it tells you three.

Ruby: So, three is the change in distance when time goes from zero to one point five seconds.

However, she seemed confused regarding how it correlated with the change in distance and the change in time. Ruby associated it with the concept of distance, and she could not relate it with the concepts of average rate of change function due to her strong procedure-oriented reasoning. The interviewer assisted Ruby and explained that the average rate of change is produced when taking the quotient of two covarying quantities, but Ruby resisted to changing her procedural thinking, and said, “Three meters is the distance from the stop sign.” Ruby confused AroC with distance of the car. She demonstrated a strong procedure-oriented reasoning by applying rise over run, thinking, “The change in time is 1.5, and so the change in distance is three,” the reasoning was incomplete. After this question, the researcher further probed Ruby’s deep reasoning about ARoC and derivative function (see excerpt below.)

Interviewer: When h is zero point five, what is the value of the average rate of change function?

Ruby: So, when h is zero point five.

Interviewer: You can calculate for h $= 0.5$.
Ruby: Ok, it is four t plus one [$4t + 1$].
Interviewer: So now we found out ARoC that is $4\mathrm{t} + 1$.
Ruby: Yeah.
Interviewer: What is $4t + 1$? What it tells you about d and t?. Yeah. OK, so when $h = 0.5$, $4t + 1$ is the functions of the average rate of change function.
Ruby: Yes.
Interviewer: So, what it tells you four t plus one [$4t + 1$] about the distance and time relationship. So, describe what this tells you about the change in distance of the car from the stopping sign.
Ruby: OK, so it is a derivative right, it is $4t + 1$ [four t plus one]? No, it's not.
Interviewer: Why you said derivative?
Ruby: Because isn't that what you get when you fill everything out and cancel out h so it would be like when h is like close to zero, just be like….

Ruby correctly determined the formula for ARoC, which was found to be $4t + 1$ and explained that $4t + 1$ was derivative function. The interviewer asked Ruby to describe the meaning of $4t + 1$ in context of distance and time, which Ruby described: "It is a derivative function." The interviewer further enquired the reason behind defining it as a derivative function, in response of which Ruby asserted that "It is what you get when you fill everything out and cancel out h, so it would be like when h is close to zero." Ruby used procedure-oriented reasoning here to justify why she said $4t + 1$ is derivative by memorizing the ARoC function. Ruby was unable to reach and verdict that $4t + 1$ is an average rate of change function when $h = 0.5$. In the next excerpt, Ruby was asked to illustrate her reason about ARoC, and derivative function and she described derivative as when h has no effect on the ARoC.

Interviewer: So, if you see part c and d, can you graph this part when h is $0.5, 0.4, 0.3$ and so on? Can you graph that part?
Ruby: Yeah.
Interviewer: So, let me take that [picture]. OK, let me take that. So, what this graph tells you about ARoC and derivative function, what is their difference. Can you tell me?

Ruby: So, as h is getting closer and closer to zero, it's kind of like moving down there like parallel to each other. But when h approaches zero, like at the smallest value, zero point zero, zero, zero, one [0.0001]. Yeah. It's like as close to zero as possible. So that's close to the derivative.

Interviewer: Why you said it is close to the derivative? Why did you say that is a derivative? How do you know it is a derivative?

Ruby: Because the derivative is when h has almost no effect on the average rate of change.

Interviewer: What do you mean when h has almost no effect on ARoC? How does it relate to the derivative of the function?

Ruby: So, if you take like the formula given like the $\frac{[((f(t+h)-f(t))]}{h}$, like the closer h gets to zero, the less h has no effect on any part of the graph. So, it just is closer to the derivative of $f(t)$.

Interviewer: What does it mean when you said h doesn't have an effect on the ARC?

Ruby: So, it means that like at the end the gap on the graph are getting smaller and smaller and it no longer like the window of error. So h doesn't really affect the graph anymore. There's not really a lot of error in that room I guess [She used her graph to explain…].

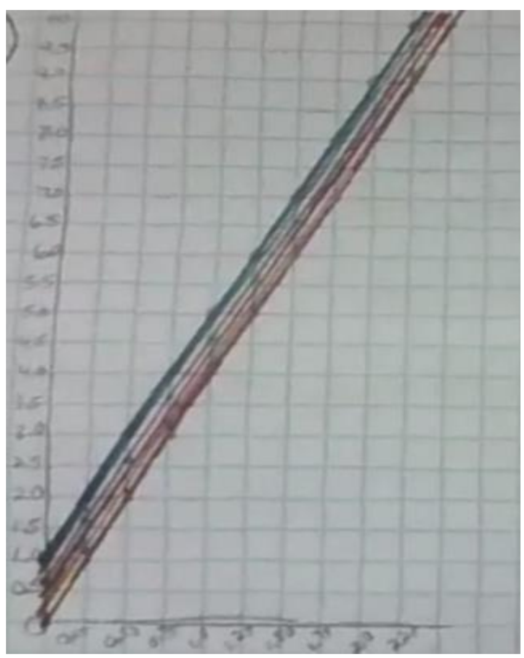


Figure 26. The Average Rate of Change Function when $h = 0.5, 0.4, 0.3, 0.2, 0.1, 0.001$ and 0.00001

For the question, "So what does this graph tell you about ARoC and derivative function," Ruby said, "As h drew closer to zero, the ARoC drew closer to the derivative," but she still did not know the meaning of the ARoC or the derivative function. She still struggled to give the meaning behind ARoC; she does not know ARoC is a constant rate of change that relates the change in distance and change in time on the interval whose size $h = 0.5$, or $h = 0.001$, and as the size of the interval or as the time change in small amount then the ARoC will be the limiting value of the derivative function. She does not know the average rate of change between the two quantities is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval. Ruby said:

> So, if you take like the formula given like the $\frac{(f(t+h)-f(t))}{h}$, like the closer h gets to zero, the less h has no effect on any part of the graph. So, it just is closer to the derivative of $f(t)$.

Interestingly, Ruby confused the idea of derivative with situations in which the value of h has almost no effect of the ARoC. Here, when she said no effect, that means, for her, h is detached from ARoC function. More importantly, h doesn't affect means, for her, no gap between the graph of the derivative and the graph produced when h does not affect ARoC. Ruby did not define ARoC using covariational idea; average rate of change (ARoC) between the two quantities is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval. Ruby did not have any idea that, as h approaches zero, the ARoC converges to the derivative function.

Chris's Post-instruction Interview Assessment

After reading the rectangle problem (see Table 10) Chris explained his solution and Chris described fixed and varying quantity as tiny bit and micro bit (see excerpt below).

> Interviewer: Define the constant variable in this situation.
> Chris: Well, the constant variable in this situation is two hundred and forty feet of the face.
> Interviewer: Ok.
> Chris: Yeah, Ok, so my constant variable is the perimeter of the rectangle.
> Interviewer: Ok so what are the varying variables in this situation?
> Chris: Yeah, Ok, we want to be defining variable or define the varying variables in this situation. We're looking at length, width and area are the ones that are going to vary, and the intervals over which they're going to vary are going to be zero to one hundred and twenty feet, not inclusive just because the smallest area that I can make is going to be like except not quite together and it's going to be almost one hundred and twenty feet long except for that little, tiny bit or little micro bit at the top and the bottom.
> Interviewer: You can use paper maybe to show me or to write the formula or the expression.
> Chris: I'm trying to find the area in terms of width. So, you got to rewrite the length in terms of the width first, that is $2l + 2w = 240$.
> Interviewer: Yeah. And so.
> Chris: Oh, oh. I get it. I need one half of 240 minus. That is Area is equal to width times 120 minus the width [$A = w * (120 - w)$] this should give me the area.

Chris identified the constant quantity and varying quantities in this problem situation. Moreover, Chris suggested the interval over which the length and width of the rectangle are varying. However, when he verbalized how they are going to vary, he did not reason how the area of the rectangle is varying when the length and width are varying. Chris identified the three variables and defined length in terms of width, but he failed to identify the interval over which the area varies. The interval over which the area of the rectangles is varying will be between 0 to 3600 square feet. Chris is not necessarily "incomplete conception" here. He just is not precise in his language. He does not provide an interval, over which area is varying, but he does provide an interval over which length and width is varying. In his reasoning about variation, we see Chris used scientific sounding language, like “a little micro bit,” to define how the quantities vary. This scientific language will show up again later in Chris’s reasoning on later questions. In the next excerpt, Chris is asked further to explain his reasoning when two quantities co-vary in chunky and smooth continuous contexts. Chris used terminology-oriented reasoning to describe chunky and smooth covariation (see excerpt below).

> Interviewer: So, the next question is, part c.
> Chris: Ok.
> Interviewer: The width of the rectangle varies by 10 feet between the intervals, what relationship do you notice between the rectangle’s enclosed area and width of the rectangle and what can you conclude from the relationship? It's possible to make tables.
> Chris: It’s possible to make a table that shows the variables, so I mean, since it's a quadratic equation, we're going to have a curve. But we're only looking for the spot points to points of wherever we're going to have.
> Interviewer: Ok, how the area of the rectangle varies as the width of the rectangle vary by 10 feet.
> Chris: There is at least one spot where the area would stay the same.
> Interviewer: Ok, so what about when the width of the rectangle varies smoothly and continuously between the interval$0 \leq \mathrm{w} \leq 120$, how the area vary? Is that possible to create a table for each value of the width a corresponding value for the area?
> Chris: So, it is impossible to make a table that shows all possible values of the variables that this relationship produces in. Because we have all the little stuff in between one point, it is like a little micro bit, so make the table out of that, because that's too many things to list. We can make a summary of it. What it would be, but again it is impossible to list all the points in a table.
> Interviewer: Yeah, so when you said little stuff or little micro bits, in between one, what do you mean that?
> Chris: So, as we're going. There's always a smaller interval.
> Interviewer: Ok.
> Chris: So if I have. I can make a table that has, let's just say, all the all the averages at an interval of one, and there would still be we can just use the metric system at this point and say, oh, but look, we could do instead of every meter. I know ten feet right now. But let's just I'm thinking in terms at the moment so we could always look at instead of a meter, we can look

at decimeter and that's ten down and oh, we made a table with all the best meters of this, this curve. No, but wait. Now we can move to centimeters, and we can make that table and we can keep going down to very small fraction of meters. And there's still something smaller than that. And we can keep on going until eventually either we find out what the fabric of reality does. And at that point, we would have to start making some definitions or we keep finding smaller and smaller particles or whatever. But and then the other side of it is I could, in fact, have a table that just says, oh, here's my zero point and here's my 120 points. And so average between them is going to be zero.

Interviewer: Yeah, so can you try to graph this idea of what you said for both those ideas for chunky and continuous variation?

Chris: All. Ok, Ok.

Interviewer: Yeah, I like it. Let me take a picture

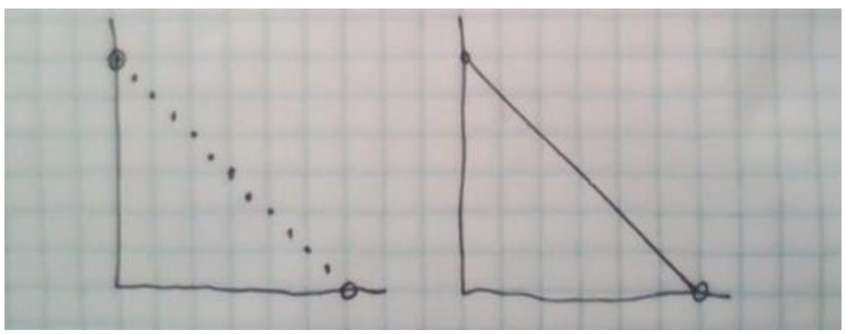


Figure 27. Area of the Rectangle in Chunky and Smooth Continuous Covariation

Interviewer: All right, this is a very nice

Chris: Yeah, so. So, when we have moving in a chunky or in a discreet fixed amount the graphs will look like a discrete. And when we move in and he continues all in small quantities of the graph look like smooth.

Interviewer: Correct.

When Chris was asked to create a table that relates the width and the area of the rectangle as the width of the rectangle varies by 10 feet, he suggested “It’s possible to make a table that shows the variables, so I mean, since it's a quadratic equation, we're going to have a curve. …But we're only looking for the spot points to points of wherever we're going to have.” In this context, he suggested that it is possible to make tables to show variation from point to point. However, Chris did not coordinate the chunky variational analysis to justify how the width and the area of the rectangle co-vary together. Rather, Chris said that "…we're only looking for the spot points to points," he then said, "There is at least one spot where the area would stay the same." Chris did not analyze with reasoning that relates fixed changes in the width of the rectangle to the change in the area of the rectangle. Following this interaction, when the interviewer asked Chris about the possibility of creating a table when the width of the rectangle varies smoothly and continuously throughout the interval, Chris asserted that it is impossible to make a table that shows all possible values of the variables that this relationship produces. However, he used scientific measurement language moving from meters, to decimeters, to centimeters, to, "a very small fraction of meters," in his reasoning. He knew it is impossible to create the table, but the terminology-oriented reasoning he utilized that led him to consider the ultimate "fabric of reality" to explain why it is impossible seems to have impaired his reasoning about the two quantities and the particular aspects of how they vary together. Rather than coordinating variation, Chris used procedure-oriented reasoning to explain that progressive variation was like taking the average of all point in the interval. It appears that Chris's terminology-oriented reasoning impaired his mathematical conceptions of covariation. Even though he knew at the beginning of the excerpt that the graph of the relation between the width and area produces a quadratic function, he produces a linear graph of the situation (see Figure 27 above). The researcher hypothesizes that focusing on the fabric of reality impaired his reasoning focus on the specific covariational relationship being discussed. It seems that Chris did use chunky variational reasoning of some sort here. Maybe it can be called discrete reasoning? In any case, he is noting that, as one variable changes by ten, the other one will change from a point to another point (and not all of the points in between).

Nonlinear Function Problem

After the area of rectangle problem, Chris was asked a nonlinear function problem (see Table 10). The nonlinear distance function problem was used to better understand how Chris constructed a covariational relationship between two covarying quantities, including the ability to reason about a constant rate of change and derivative function. Chris described ARoC as acceleration and his response to the problem is presented below (see excerpt below).

> Interviewer: Yeah, let's talk about it.
>
> Chris: So, at this point, if I'm just doing it the one time, then I would just I just solve it. Yes. So, one point five squared times to whatever that ends up giving me subtract zero squared times two which is zero and that's going to be over one divided by one point five. And that should give me the average rate of change over this particular distance.
>
> Interviewer: Ok, you can describe what this average rate of change function tells you about the change in just over the time interval from zero to one point five. So, what does it tell you?
>
> Chris: We could be going away from the stop sign yet because the number got bigger on level instead of getting smaller. So, I am getting further away from the stop sign. The average rate of change function should tell me the acceleration that I used to get away from the stop sign.
>
> Interviewer: Ok.
>
> Chris: Yeah, that means change in velocity over time.
>
> Interviewer: Yeah.
>
> Chris: The change in y is happening because of the change in t the change in time. The change in distance happens over the change in time. I don't know how else to say that. I just know I'm getting further away this that's.

Chris had showed strong co-variational reasoning in the pre-instruction and instructional assessment; however, in this excerpt, we see him struggling to show a strong understanding of ARoC. Chris states, "The average rate of change function should tell me the acceleration that I used to get away from the stop sign," his reasoning was incomplete. Chris does not define ARoC using co-variation between quantities, for example the average rate of change (ARoC) in this case, is the constant rate of change that produces the same change in the distance over the given interval of time. Interestingly, Chris brings up the notion of acceleration. In this context, acceleration is a related concept, as Chris explains, "Yeah, that means change in velocity over time." Chris brings this prior knowledge from understanding he developed from study of science as an engineering major. Here, it appears this terminology-oriented reasoning impaired his mathematical reasoning. That is, he did not relate the ARoC as a constant rate of change on the given interval over which the two quantities co-vary together, his reasoning was incomplete. Further, Chris was asked questions when the variation in t varies in small and unnoticeable amounts, aimed to understand his reasoning, and assess his smooth continuous covariational reasoning and concept of rate of change and derivative function. Chris described derivative as acceleration (see excerpt below).

> Interviewer: Thank you. We can move to part d.
> Chris: Mm hmm. It says that determine $r_h(t)$ of t when h is equal to 0.1. OK. We care about the smallest unit that will go down to one astronomical unit. Like, I can have a circle that is I draw it. It has one astronomical unit on the side, and it'll still look like a pretty small, good circle if you go far enough away from it.
> Interviewer: When you say the value of h is almost one astronomical unit, then what is happening to ARoC?
> Chris: Yeah, yes, like the smaller, h, or interval or the smaller h is, it gets closer and closer. It's ok if we're off by a thousand miles, because all we got to do is hit the planet or it's ok if we're off by 50 feet because we can correct for that at the end or I don't know. It's I don't necessarily care about what acceleration a car has atone ten thousandth of a second. I care how fast it goes, from zero to 65 in seconds for maybe a tenth of a second.
> Interviewer: Is that possible to construct a graph?
> Chris: Absolutely, it is very possible. Yeah, I don't particularly want to do it, but it's possible.
> Interviewer: Yeah.
> Chris: Ok, so here's a graph, four point five, point one and so on.

Interviewer: Ok.

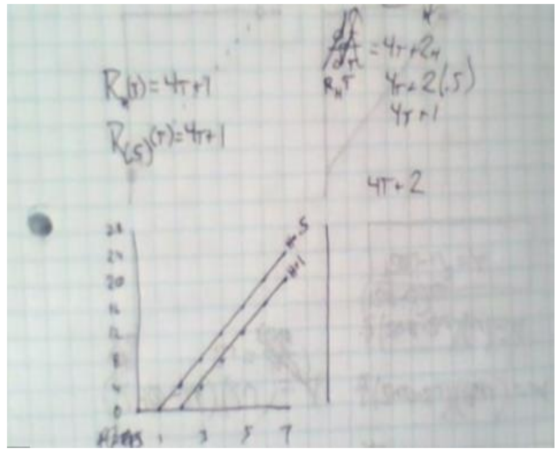


Figure 28. Average Rate of Change Function Graph when $h = 0.5, h = 0.1$ and so on

Interviewer: Oh, yes. Very helpful. Yeah, so now the question is, if you go from 0.5 to 0.1 and then go to 0.00001, something very small. So what can you say about these different graphs. What is the difference between these family ARoC graphs, what do they tell you about the relationship between the distance and the time? Is there anything that can you say about it.

Chris: Well, they're all headed towards they're all headed towards that limit, they're all headed toward that thing. So, yeah, it's all headed towards the acceleration which is the derivative.

Interviewer: Ok. Interesting.

After reading the question, Chris suggested that "We care about the smallest unit that will go down to one astronomical unit," which is the distance from the center of Earth to the center of the sun. He did not see h as variation in time (Δt) rather, he focuses on the smallest unit that we care about, which he says is one astronomical unit. Reasoning with science introduced confusion for Chris as he expressed his understanding of variation and covariation. This limited his ability to produce a meaningful explanation of the ARoC in this context involving distance and time. This limitation is observable when he says, "I don't necessarily care about what acceleration a car has at one ten thousandth of a second, I care how fast it goes." Toward the end of the excerpt, Chris developed a graph to answer part 'c' of the question, depicting on the graph the average rate of change function when $h = 0.5$, $h = 0.1$, and so on. When the interviewer further enquired about the difference between the graphs, Chris asserted, "Well, they're all headed towards that limit, they're all headed toward that thing. So, yeah, it's all headed towards the acceleration which is the derivative." It seems Chris had confused the concept of acceleration with that of derivative. Chris did not use covariational reasoning to relate the change in h with the family of ARoC functions to the derivative function, and he had not built an understanding that as h approach to zero, the families of ARoC converge to the derivative function (not the acceleration of the function). Further, it appears that his focus on other scientific concepts limited his understanding of this relationship. Following this analysis in the next section, the vertical comparative analysis among the three cases is presented.

Summary of Post-Instruction Interview Assessment

This section summarizes the three students' reasoning type and their level of reasoning during the post-instruction using horizontal comparative analysis lens. The horizontal comparative analysis allowed to compare the three students' reasoning and conceptual understanding during the post-instruction phase. First Sam's responses are summarized, then Ruby's, and finally Chris's. The area of rectangle problem revealed Sam's reasoning and conception of quantity variation and covariation. Sam used concrete object-oriented reasoning as means to construct his mathematics. For instance, for the question that asked, "identify continuous co-varying quantity" Sam used concrete object-oriented reasoning to respond, "…so I just put small a small change, rectangle shape, just keeps changing constantly," to justify his construction of the relationship. He then produced a physical object graph to show the covariational relationship between the width of the rectangle and the area of the rectangle. The post-instruction result showed that Sam's persistent concrete object-oriented reasoning limited him from developing variational and covariational reasoning, even after he participated in the instruction. Sam did not develop the concept of covariational relationship he constructed the physical object graph, not a graph that was produced from the covariational relationship between the area of the rectangle and the width of the rectangle. Sam did not reach a reasoning level (MA1-MA3) as described in Carlson framework (Carlson et al., 2002) and his reasoning level is still in gross covariational reasoning as described in Thompson and Carlson's smooth continuous covariational framework. Similarly, for the nonlinear distance problem Sam used his action view (he viewed h as placeholder) to evaluate the ARoC function by substituting the initial and end values of the quantity time and variation in distance of the car between these two points; he didn't use the idea of covariation to determine the value of the ARoC. As result of his concrete object-oriented reasoning Sam did not envision or understand derivative function is the result of the limiting value of the average rate of change or is a refinement of ARoC function as h approaches zero. Sam did not reach a reasoning level (MA1-MA4) as described in Carlson framework (Carlson et al., 2002).

On the other hand, Ruby constructed a deep understanding of variation and covariation reasoning during the instruction, but she reversed during the post-instruction due to her use of procedure-oriented reasoning that showed on her response for area of rectangle and non-liner distance problems. For instance, in the area of rectangle problem she used procedure-oriented reasoning to analyze the covariational relationship between the width and area of the rectangle. Ruby was stuck and wanted to see the formula, and she said, “So, I’m stuck on because I kind of figure out like this, I'd like to see the formula.” Interestingly, she wanted to refer back to the formula to figure out the relationship between the two covarying quantities (the area and width of the rectangle). Not only this, but she also started to use direct relationship and inverse relationship to describe the covariational relationship, which she had not learned during this investigation. She reverses her covariational reasoning that was developed during the instructional session, and she used procedure-oriented reasoning to justify the covariational relationship between the area of the rectangle and the width of the rectangle, and she used the formula to describe this relationship. Similarly, for the nonlinear distance problem, Ruby was engaged in procedure-oriented reasoning, and she was unable to define ARoC and derivative function using covariation ideas. When she was asked to give the meaning of $4t + 1$ she said, “It is what you get when you fill everything out and cancel out h, so it would be like when h is close to zero.” Ruby used a procedure-oriented reasoning to justify why she said $4t + 1$ is derivative by memorizing the ARoC function. Her reasoning level does not reach MA4 or above and she demonstrated strong procedure-oriented reasoning during post-instruction interview.

For the area of rectangle and nonlinear distance problems Chris used terminology-oriented reasoning to engage with the problem situation. For instance, in the area problem in his reasoning about variation, Chris used scientific sounding language, like “a little micro bit” to define how the quantities vary and his uses of terminology-oriented reasoning impaired his reasoning about variation of the two quantities and the particular aspects of how they vary together. Similarly, for nonlinear distance problem his uses of terminology-oriented reasoning impaired his able to engage in covariational reasoning. For example, when he described the ARoC he said, “The average rate of change function should tell me the acceleration that I used to get away from the stop sign,” here Chris used alternative conceptions. Chris does not define ARoC using co-variation between quantities. Chris had a strong co-variational reasoning in the pre-instruction and instructional assessment; however, in post-instruction, he struggled to conceptualize the idea of ARoC and derivative function due to his reliance on terminology-oriented reasoning. As a result, in post-instruction, Chris was not able to reach reasoning level MA4 and his reasoning was impaired by his uses of terminology-oriented reasoning. The post-instruction results show that both Chris and Ruby reversed their strongly developed covariational reasoning due to their use of terminology-oriented reasoning and procedure-oriented reasoning, respectively, to analyze the covariational relationship problem. Because of this, both Chris and Ruby were not able to reach or demonstrate a reasoning level beyond MA1-MA3. Then again, Sam still struggled to develop quantity, variational, and co-variational reasoning abilities due to his use of concrete object-oriented reasoning to analyze the covariational problem. Sam’s concrete object-oriented reasoning affected his development of variational and covariational reasoning and as a result he did not construct meaningful mathematics understanding.

Vertical Axis Comparative Analysis of Students’ Reasoning

This section compares the study participants’ conceptions of variation, co-variation, and average rate of change and derivative functions using vertical comparative analysis. The vertical comparative analysis allows the researcher to look at each student’s development across all three phases, that is, their development from pre-instruction to post-instruction. The students' ways of thinking are compared, while the reasoning abilities that were revealed to be critical for understanding central concepts of the average rate of change and derivative function are highlighted. This section also provides a discussion of the students' problem-solving and reasoning abilities by comparing the primary themes that emerged in an analysis of the students' pre-instruction assessment, during the instruction assessment, and the post-instruction interview assessment. The result of the study shows Sam often displayed what the researcher has termed concrete object-oriented reasoning, which impacted his reasoning while solving problems focused on progressively varying and covarying quantities, as well as his conceptualization of average rate of change and derivative concept. Object-oriented reasoning is reasoning that is focused on the physical object in a problem context in such a way as to hinder focused reasoning about a quantity, the process of measuring a quantity, or an understanding of progressively varying quantities. Similarly, the result of the study shows Ruby had a strong focus on procedure-oriented reasoning and initially some concrete object-oriented reasoning on concept of quantity, and this impacted the types of reasoning she exhibited in solving problems related to covariation, the average rate of change, and derivative function. The researcher uses procedure-oriented reasoning to refer to moments when, prior to understanding and reasoning with the quantities in the problems, the student attempts to use calculations and procedures learned in prior mathematics experiences to obtain an answer. Chris's results show that he displayed what the researcher terms terminology-oriented reasoning, which somehow impaired his covariational reasoning abilities and his conception of the average rate of change and derivative function. Terminology-oriented reasoning is when the student focuses on the scientific sounding terminology of the problem situation when reasoning to a solution, rather than focusing on the quantities of the problem and how the quantities vary together. Although all three of these students' actions revealed some progress in their approaches to problem-solving, the primary themes present in their reasoning in this study speak to their ability to conceptualize and reason about quantities and their relationships.

Below, the vertical comparison analysis between the three students across the three phases is presented. First, Sam's pre-instruction, instruction, and post-instruction vertical comparison analysis is presented. Next, Ruby's pre-instruction, instruction, and post-vertical comparison analysis is presented. And finally, Chris' pre-instruction, instruction, and post-vertical comparison analysis is presented. Recall that the study research questions are:

1. What types of reasoning do first-year calculus students engage in to conceptualize the relationship between two progressively co-varying quantities?
2. What methods of reasoning do first-year calculus students employ during a rate of change and derivative instructional sequence that supports smooth continuous covariational reasoning?

Sam's Pre-Instruction Assessment

Sam's responses during the pre-instructional task reveal that his conception of variation, covariation, and rate of change consisted of loose coordination of objects and attributes of objects. Moreover, his responses did not reveal a process for measuring a variable as progressively varying quantity (e.g., n as a continuously varying variable) that consisted of coordinating measurable attributes. Sam's justifications for his solution had a computational or calculation focus independent of the quantities of the problem. As a result, Sam described a quantity as a replacement of a number, or holder of a number. More importantly, his thinking is based on concrete object-oriented reasoning but not thinking in variational and covariational reasoning. For instance, for vehicle problem, Sam said, "The one that is kept constant is the Automobile and the continuous is the Bus." The Automobile and a Bus are not quantities, and they are not representing a process of a varying quantity. This depicted that the variational and covariational reasoning of Sam at the start of the study is not strong enough, and his covariational reasoning label was below MA1. Therefore, Sam's responses during the pre-instruction revealed that his conception of variation, co-variation, and rate of change consisted of weak coordination of objects and attributes of these objects.

Sam's Instruction Assessment

Sam showed little improvement for his variational and quantitative reasoning during the instruction (e.g., he wrote $\Delta V = -\frac{1}{15}$, instead he should have written $\Delta V = -\frac{1}{15}\Delta t$). He struggled to develop covariational reasoning, and his reasoning can be labeled as gross-coordination covariational reasoning, as he can envision the two quantities varying, but he does not anticipate coordination of the covarying nature of two or more quantities. Sam still struggled with concrete object-oriented reasoning while taking part in the instructional session. For instance, for the water filling problem, Sam constructed the diagram of the object as a graph rather than constructing a covariational relationship graph between the volume of the water and the time taken to fill the bottle with water. He then used his concrete object-oriented reasoning of the situation to construct quantities and, subsequently, reason about relationships between these quantities. Sam did not reason about the rate of change of the volume of water in gallons and the time it takes to fill the bottle. Sam drew the object (bottle of the water,) but he failed to construct the graph out of the two covarying quantity relationships (see Figure 7). Sam did not support his reasoning by identifying equal changes in time and comparing corresponding changes of the volume of water, and he did not reach the reasoning level (MA1-MA3) after participating in the instructional session.

For the linear function Problem, Sam reasoned procedurally, and he used action view without applying conceptual reasoning likely due to his poor reasoning about the quantity value. Sam used his action view for the Problem situation to analyze the covariational relationship between the quantities y and x. For instance, in the linear Function Problem, his actions led him to produce an unrelated relationship between the independent quantity x and the dependent quantity y "$(x+h)(x+h) = x^2 + 2xh + h^2 => 8 - 6 = 6$ & $-\frac{2}{3}(6) = -4, -4 + 5 = 1$." Sam's reasoning is still underdeveloped, and he struggled to develop strong covariational reasoning ability, and his reasoning can be grouped under gross-coordination covariational reasoning. Following this analysis his post-instruction interview result is presented.

Sam's Post-instruction Interview Assessment

Regarding Post-instruction interview assessment for the area and the nonlinear function Problems, Sam's reasoning about varying and constant quantity is still impacted by his concrete object-oriented reasoning (e.g., Rather than stating that the rectangle is continuously varying, he should state that the area of the rectangle is continuously varying), and this impaired his covariational as well as smooth, continuous, covariational reasoning ability. Sam's weak covariational reasoning is exhibited in his drawing for the problem situation (for instance, see Figure 7) where he produced the physical object of the bottle as an illustration of his concrete object-oriented reasoning.

Concerning the nonlinear function problem, Sam reasoned procedurally without applying covariational reasoning between the changes in distance function with the change in time, and he did not relate the average rate of change function with the variation in the input quantity value. Thus, Sam did not construct a smooth, continuous, covariational relationship between the ARoC and h. Even though Sam developed variational reasoning and conception of the quantity within the course of the study, he is still impaired by his object thinking. Therefore, Sam's responses during post-instruction interview revealed that his conception of variation, covariation, and rate of change consisted of loose coordination of objects and attributes of these objects. His response did not reveal a process for measuring a variable as progressively varying quantity (e.g., *n* as continuously varying variable) that consisted of coordinating measurable attributes. Sam's reasoning can be labeled as gross covariational reasoning since his reasoning is loose coordination between two covarying quantity values and his resonating level was gross coordination of value as describe in Thompson and Carlson covariational framework. In the next section Ruby's pre-instruction, instruction, and post-interview result are discussed.

Ruby's Pre-Instruction Assessment

Ruby's responses during the pre-instructional task reveal that her conceptions of variation, covariation, and rate of change consisted of weak coordination of quantity and attributes of these. For instance, for the quantity and variation problem, Ruby said that "If $n > 2$ then $2n$ will be larger than $2 + n$ because will get doubled instead of just adding two. So, if $n = 3$ then $2(3) = 6$ while $2 + 3 = 5$. If $n < 2$ then $2 + n$ will be larger, for example if $n = 1$ then $2 + (1) = 3$ while $2(1) = 2$. If $n = 2$ then both answers will be the same." Ruby used her procedure-oriented reasoning to justify why $2n$ is larger or lesser than 2+n, and she failed to recognize the progressive variation in the values of n due to her dependence on the regress computation and her action view. Her responses did not reveal a process for measuring a variable as progressively varying quantity (e.g., n as continuously varying variable), as similar to Sam's reasoning that consisted of action view. However, Ruby possesses loose variational and covariational reasoning when compared to Sam for the other pre-instructional assessment problem at the start of the study. Therefore, Ruby's mental actions can be placed in the levels of MA1 of Carlson's five levels of reasoning.

Ruby's Instructional Assessment

Ruby developed strong variational and covariational reasoning during an instructional session. For the water problem, Ruby used her developed covariational reasoning to draw a correct graph and justify her solution about the covariational relationship between the volume of the water and the time taken to fill the bottle with water. For instance, Ruby depicted the situation by correctly drawing the graph of the volume of water and the height of the water as the time passed to fill the volume of the bottle with water (see Figure 8). Similarly, for the nonlinear problem, Ruby analyzed and justified her approach to the problem using variational and covariational reasoning to show the correct solution (see Figures 8 to 9). Therefore, Ruby developed strong variation, covariation, average rate of change function, and function ideas during the instruction, and her reasoning could be grouped under MA1-MA4.

Ruby's Post-instruction Interview Assessment

Regarding the post-instruction interview assessment, Ruby used her developed covariational reasoning to validate why the area of the rectangle covaries with the width of the rectangle and she produced a correct illustration for the problem situation (see Figure 25). However, when she engaged with the nonlinear function problem, Ruby's prior procedural knowledge caused confusion between the concept of distance and the average rate of change. Her strong covariational reasoning continued to be hampered by her procedural knowledge, which often caused disequilibrium that was difficult to resolve. Ruby is confused with coordinating the variation in *h* and how that relates to the average rate of change. That is, Ruby's confusion of *h* approaching zero with the concept of subtraction a small amount from the previous value. Ruby had difficulty viewing *h* as the variation or change of the independent variable in the given interval, but she considered *h* as if it is a value that resulted by subtracting by some amount. She confused *h* approaching zero with concept of subtraction that impaired her reasoning about the average rate of change and derivative concepts. Thus, Ruby did not use a smooth continuous covariational reasoning to relate concept of ARoC and *h*. Even though she developed variational reasoning and conception of a quantity within the course of the study, she is impaired by her procedure-oriented reasoning and failed to conceptualize the concept of average rate of change and derivative function and her reasoning level did not reach MA4 as described in Carlson's framework. In the next section Chris's pre-instruction, instruction, and post-interview results are discussed.

Chris's Pre-Instruction Assessment

Chris's responses during the pre-instructional task reveal that his conceptions of variation, co-variation, and rate of change consisted of a strong coordination of quantities and attributes of these. His responses revealed that he had a strong variational and covariational reasoning ability of process view for measuring a quantity as a progressively varying quantity at the start of the study (e.g., *n* as continuously varying variable) that consisted of coordinating measurable attributes of the process. This depicted that at the beginning of the study the variational and covariational reasoning of Chris is between MA1 and MA3, which means he coordinates change of one variable value with change in the other variable value. He understood directional change and quantitative coordination of two or more covarying quantities.

Chris's Instructional Assessment

Chris developed strong variational and covariational reasoning during an instructional session. For the water problem, Chris used his developed covariational reasoning to draw a correct graph and justify his solution about the covariational relationship between the volume of the water and the time taken to fill the bottle of water. For instance, Chris depicted the situation by correctly drawing the graph of the volume of water and the height of the water as the time passed to fill the volume of the bottle with water (see Figure 16.) Chris reached MA1-MA 4 level of covariation reasoning. Chris developed strong covariational and variational reasoning during the instruction, and he used his developed covariational reasoning to analyze the problem situation. For example, Chris used correct covariational reasoning to analyze the problem situation (see Figure 21 to Figure 23).

Chris's Post-Instruction Interview Assessment

Chris had strong covariational reasoning at the start and during the instruction session (see Figure 5) but for this part of the post-instruction assessment, Chris showed weak mathematical reasoning. Regarding the area problem, Chris's reasoning was impacted by his terminology-oriented reasoning; for example, he used "little, tiny bit" or "little micro bit" to describe variation or quantity change. Instead, he should say "varying quantity" or "continuously changing quantity". This may have impaired his variational, as well as covariational, reasoning ability which led him to a incomplete conclusion for the problem situation. Chris reversed his reasoning compared to his pre-instruction and during the instructional assessment, likely due to his use of terminology-oriented reasoning.

Similarly, Chris, for the nonlinear function problem, used his terminology-oriented, and that likely influenced his justification and conclusion about the concept of ARoC and derivative function and led him to a incomplete conclusion. Concerning the average rate of change problem, Chris's reasoning was impaired by his terminology-oriented reasoning of the two covarying quantities, and that impacted his ability to coordinate the average rate of change of the function with uniform increments of changes in the input variable. Thus, Chris did not construct a smooth continuous covariational relationship between the ARoC and h. Even though Chris developed variational reasoning and conception of a quantity within the course of the study, he was impaired by his terminology-oriented reasoning at the end of the course of study, and he was unable to conceptualize the concepts of ARoC and derivative function. Chris's terminology-oriented reasoning caused him to reach a incomplete conclusion about the concept of derivative and acceleration, as he did not view uniform increments of changes in the input variable as coordinated with the quotient of the output quantity, which converges to the derivative of the function or derivative function is the limiting value of the average rate of change function, but not the acceleration of the function, His reasoning level did not reach MA4 as described in the Carlson framework. In the next section, the analysis along the transversal axis is presented. A trace of each of the three cases' reasoning throughout the study time is presented. The table below (Table 12) summarized the vertical analysis of the three students reasoning orientation.

Table 12

Summary results of vertical analysis

Phases	Sam's Response	Ruby's Response	Chris's Response
Pre-instruction	Sam's conception of variation, covariation, and rate of change consisted of loose coordination of objects and attributes of objects. For instance, for vehicle problem, Sam said, "The one that is kept constant is the Automobile and the continuous is the Bus." At the start of the study Sam's	Ruby's conceptions of variation, covariation, and rate of change consisted of weak coordination of quantity and attributes of these. For instance, "If $n > 2$ then $2n$ will be larger than $2 + n$ because will get doubled instead of just adding two. So, if $n = 3$ then	Chris's responses during the pre-instructional task reveal that his conceptions of variation, co-variation, and rate of change consisted of a strong coordination of quantities and attributes of these

	covariational reasoning label was below MA1	$2(3) = 6$ while $2 + 3 = 5$. If $n < 2$ then $2 + n$ will be larger, for example if $n = 1$ then $2 + (1) = 3$ while $2(1) = 2$. If $n = 2$ then both answers will be the same." She used procedure-oriented reasoning to justify why $2n$ is larger or lesser than 2+n	
Instruction	Sam showed little improvement for his variational and quantitative reasoning during the instruction (e.g., he wrote $\Delta V = -\frac{1}{15}$, instead he should write $\Delta V = -\frac{1}{15}\Delta t$). He struggled to develop covariational reasoning, and his reasoning can be labeled as gross-coordination covariational reasoning, as he can envision the two quantities varying, but he does not anticipate coordination of the covarying nature of two or more quantities	Ruby developed strong variation, covariation, average rate of change function, and function ideas during the instruction, and her reasoning could be grouped under MA1-MA4**.** For instance, Ruby depicted the situation by correctly drawing the graph of the volume of water and the height of the water as the time passed to fill the volume of the bottle with water (see Figure 8).	Chris developed strong variational and covariational reasoning during an instructional session. For instance, Chris depicted the situation by correctly drawing the graph of the volume of water and the height of the water as the time passed to fill the volume of the bottle with water (see Figure 16.) Chris reached MA1-MA 4 level of covariation reasoning
Post-instruction	Sam's reasoning about varying and constant quantity still impacted by his object reasoning (e.g., the	Ruby's strong covariational reasoning continued to be hampered by her procedure-	Chris reversed his developed covariational reasoning and he used terminology-

	shape of the rectangle change, the rectangle continuously varying instead he should say the area of the rectangle continuously vary) Sam often displayed concrete object-oriented reasoning through all the three phases	oriented knowledge, which often caused disequilibrium that was difficult to resolve.	oriented reasoning; for example, he used “little, tiny bit” or “little micro bit” to describe variation or quantity change

Transversal Comparative Analysis between Sam, Ruby, and Chris

This section compares the study participants’ conceptions of variation, co-variation, and average rate of change and derivative functions using transversal comparative analysis across the time of the study. At the beginning of the study and during the instruction period, both Ruby and Chris on average constructed functional relationships and rate of change conceptions that were rooted in variational and covariational relationships. After engaging in the instructional activities, they conceptualized processes of measuring varying quantities that stemmed from reasoning about quantities (e.g., area, length and width of the rectangle, the volume of the water, and the time taken to fill the bottle with water and the relationship between the height of the water and the volume of the water) and relationships between these quantities. Throughout the study, when Ruby and Chris engaged with mathematical problems, they first conceived of the relevant quantities, and then they leveraged their understandings of variational and covariational reasoning to solve the tasks. On the contrary, Sam did not appear to construct a process for measuring quantities and their attributes that consisted of quantities and relationships between quantities. Rather, he conceived of varying quantity measures as numerical labels of objects. In general, Sam relied on his concrete object-oriented reasoning and used an action view to solve problems, and this limited his ability to justify with clear reasoning about relationships between two covarying quantities. Therefore, throughout the study when Sam engaged mathematics problems, he used concrete object-oriented reasoning and he did not apply variational and covariational reasoning to solve the problem situations.

However, for the Post-instruction interview assessment, both Ruby and Chris reversed their developed variational and covariational reasoning, and instead engaged with procedural and terminology-oriented, respectively, when they solved problems related to concept of average rate of change and derivative functions. It is not clear why they were trapped by their procedural and terminology-oriented, respectively, but both had difficulty conceptualizing the average rate of change and derivative concept. On the other hand, Sam resisted changing his concrete object-oriented reasoning that limited his ability to develop robust conception of the idea of function, graph, rate of change, and derivative. Therefore, the result of this study shows the study participants rely on this sort of reasoning (procedural and scientific) before they develop robust covariational reasoning. It can impede their progress toward developing deep understanding of the mathematics. For instance, Ruby is a good example of a student who relies too strongly on procedure-oriented reasoning which impaired her covariational reasoning. Similarly, Chris is also an example of someone who has strong terminology-oriented reasoning that is interfering with his learning of the calculus concepts. Likely, after developing covariational reasoning, procedural and terminology-oriented reasoning could potentially be beneficial. Object-oriented reasoning, on the other hand, is always a problem because it confuses objects with quantity. Sam is a good example of how some students really struggle with calculus when they have strong concrete object-oriented reasoning. Therefore, the results of this study can be used to better understand students like Sam, Ruby and Chris in calculus classrooms. The table below (Table 13) summarized the transversal analysis of the three students reasoning orientation.

Table 13

Summary results of transversal analysis

Cases	Pre-instruction	Instruction	Post-instruction
Sam's Response	Sam did not appear to construct a process for measuring quantities and their attributes that consisted of quantities and relationships between quantities. Rather, he conceived of varying quantity measures as numerical labels of objects.	Sam relied on his concrete object-oriented reasoning and used an action view to solve problems, and that limited him to justify with clear reasoning about the functional ratio of multiplicative relationships between quantities.	Sam resisted changing his concrete object orientation reasoning that limited him to develop robust conception of the idea of function, graph, rate of change, and derivative
Ruby's Response	At the beginning of the study Ruby had procedure-oriented conception towards functional relationships and rate of change conceptions that were rooted in variational and covariational relationships	After engaging in the instructional activities, Ruby and Chris conceptualized processes of measuring varying quantities that stemmed from reasoning about quantities (e.g., area, length and width of the rectangle, the volume of the water, and	For the Post-instruction interview assessment, both Ruby and Chris reversed their developed variational and covariational reasoning, and instead engaged

Chris's Response	At the beginning of the study Chris on average constructed functional relationships and rate of change conceptions that were rooted in variational and covariational relationships	the time taken to fill the bottle with water and the relationship between the height of the water and the volume of the water) and relationships between these quantities. Throughout the study, when Ruby and Chris engaged with mathematical problems, they first conceived of the relevant quantities, and then they leveraged their understandings of variational and covariational reasoning to solve the tasks.	with procedure-oriented and terminological oriented reasoning, respectively, both had difficulty conceptualizing the average rate of change and derivative concept

A Conceptualization of Students' Development of Covariational Reasoning

This study explored and analyzed three students' quantity, variation, and covariation reasoning development and their conception of average rate of change and derivative function in light of Carlson et al (2002) and Thompson and Carlson (2017) Covariational and Smooth Continuous covariational framework. The participants of the study exhibited different types of reasoning when they attempted to solve dynamic mathematical problems. To close this results chapter, the results of the horizontal, vertical, and transversal analyses are taken together to theorize the following relationships that may be present in the thinking and reasoning of students similar to the cases in this study.

Understanding a quantity as an object leads students to obscurity when engaging in covariational reasoning.

A strong understanding of quantity is fundamental for students to develop understanding of calculus (for example, variation, covariation, rate of change, and derivative). Clearly describing a quantity is fundamental to understanding variation and covariation. Students who learn the idea of quantity at early school (elementary or high school) by differentiating the object that is being measured, the attribute of the object that is being measured, and the unit used in the measurement will have the opportunity to succeed when learning concepts that build on the idea of quantity. If students did not develop the concept of quantity in their schooling, it will adversely affect their understanding of variation, covariation, rate of change, and derivative, and advanced concept of calculus. In this study, Sam exemplified this result by consistently showing difficulty to differentiate between quantity and object. This impaired his ability to develop understandings of variation, covariation, ARoC, and the derivative function. Sam's concrete object-oriented reasoning was so strong that he was observed graphing the *physical object* to reason with rather than conceptualizing and imagining the quantitative relationship between two or more varying quantities of that object. The concrete object-oriented reasoning that he used prevented him from understanding the variational and covariational relationship when two or more quantities dynamically change together. Sam exemplified students who consistently focus their reasoning on the object, whether it be a symbol, automobile, bus, or people. For students like Sam, it appears that persistent concrete object-oriented reasoning will be a strong barrier, preventing them from developing variational and covariational understandings, thus preventing them from coordinating changes in the dependent variable with the continuous changes in the independent variable. The results of this study show that students who do not develop a clear understanding of quantity beyond an understanding of the object will not be able to develop variational reasoning (fixed, chunky, and continuous variation), and will therefore struggle to understand covariation, ARoC, and derivative function in calculus.

The use of strong procedural understanding has a tendency to impair students' covariational reasoning.

Utilizing procedural understanding will lead students to develop a pseudo-conceptual understanding of the concept of calculus. This is an indication of a tendency towards using formulas or memorized procedures to justify why their approaches to the mathematics solution is correct. For instance, in this study Ruby used her procedure-oriented reasoning to justify why her final verdict is correct when she solved quantitative relationship problem and she reached incomplete conclusion. Students whose understanding is rooted in procedure-oriented reasoning will eventually have confusion or uncertainty when they try to solve mathematics problems that demand their conceptual understanding. In this study it is noted that students' inability to form an image of the dynamically changing events appeared to stem from their procedure-oriented reasoning, as the result the students confused their prior knowledge and newly constructed knowledge of covariation. Ruby was an example of this. In this study, the results suggest that through continuous and regress instructional intervention, students' procedure-oriented reasoning can be developed into more productive covariational reasoning through instruction that focuses on the ideas of quantity, variation, and covariation reasoning.

Reasoning based on science terminology understandings may lead students to incorrect conclusions about covariational relationships.

Student's use of terminology-oriented reasoning to justify their mathematics solution resulted in constructing inappropriate images of the relationship between two covarying quantities. In this study, in one of the problems, Chris was obstructed from using his developed covariational reasoning by introducing related scientific concepts, which limited him to move forward in his use of covariational reasoning for analyzing the mathematics problem. Terminology-oriented reasoning is not problematic, however. In this study, the researcher found that inappropriate use of such type of reasoning impaired Chris's mathematical conceptions of covariation, ARoC, and derivative function. For instance, for the nonlinear problem, when Chris was asked to interpret what the ARoC would tell him about the relationship between two dynamically co-varying quantities he stated that, "The average rate of change function should tell me the acceleration that I used to get away from the stop sign." When students do not develop a strong covariational reasoning ability they may use related scientific understanding to reason, for instance, Chris used incomplete related terminology to justify his solution approach. Students should develop a strong variational and covariational reasoning in their calculus learning first to enable them to easily isolate unrelated concepts when they engage to solve covariational problems in calculus. For instance, Chris was observed communicating information that was not related to situations presented in the tasks. In this study what the researcher found is that there is a variation in how students engage and develop covariational understanding. Moreover, the development of covariational reasoning is not linear or not a one-time encounter, it needs a long-time intervention and emphasis. Therefore, the results of the study showed that procedure-oriented reasoning can serve as a starting point for deeper covariational understanding. Terminology-oriented reasoning can both help and hinder students' development of covariational reasoning; however, concrete object-oriented reasoning is problematic and will not allow students to develop covariational reasoning.

CHAPTER 5: CONCLUSION

Conclusions

This chapter concludes the study's investigation of three student's types of reasoning and conceptual understanding of the average rate of change and derivative functions. The three students' ways of thinking are reported in this chapter, while highlighting the reasoning abilities that were revealed to be critical for understanding central concepts of the average rate of change and derivative functions. This conclusion is discussed in light of related research results in mathematics education. The fundamental reasoning abilities, such as quantity, variation, and covariation, were found to be critical to learning calculus concepts, in particular the concept of rate of change and derivative functions. The students' actions in this study revealed that their approaches to problem-solving were related to their ability to conceptualize and reason about quantities and their relationships. Considering this, the chapter provides a discussion of the relationship between terminology-oriented reasoning and covariational reasoning, procedure-oriented reasoning and development of covariational reasoning, and concrete object-oriented reasoning and its influence on students' variational and covariational reasoning development. Suggestions for curriculum and instruction are also provided in the context of this dissertation's findings. Finally, this dissertation is concluded by addressing the limitations of this study, as well as this study's directions for future mathematics education research.

Quantity, Variation, and Covariational Reasoning

The three students' responses during the instructional tasks offered insights into this study's research questions. Specifically, data analysis revealed various conceptual understanding that the students demonstrated and the reasoning abilities the students used when learning and using the ideas of the rate of change and derivative function. The concept map below (Figure 29) illustrates the type of reasoning that students engaged with while solving covariational reasoning related problems. The concept map shows students may utilize procedural, concrete object, and terminology-oriented reasoning to solve covariational relationship problems. The peach oval shape shows the type of reasoning that students utilized when they solved covariational reasoning problems. The arrow from each type of reasoning shows the means of reasoning. For instance, in procedure-oriented reasoning students utilize the idea of formula, equation, and memorizing procedures to engage and solve the covariational problems. Similarly, for concrete object-oriented reasoning, students use the idea of physical objects, symbols, and numbers to justify their solution procedures. In terminology-oriented reasoning students use the ideas of micro bit, average, velocity, and acceleration while solving the covariational problem. The results of the study showed that procedure-oriented reasoning may be developed into covariational reasoning; however, terminology-oriented reasoning may likely hinder the development of students' covariational reasoning, and concrete object-oriented reasoning is problematic to develop into covariational reasoning. This section outlines these findings by comparing the ways of thinking exhibited by the three students in this study. First, the students' understanding of quantity and variation are described, and then the students' conceptions of covariation are characterized. Next, the students' conceptions of the average rate of change and derivative function are presented, and the role of variational and covariational reasoning is also addressed throughout the characterization of the students' thinking for their conceptualization of the idea of the rate of change and derivative concept.

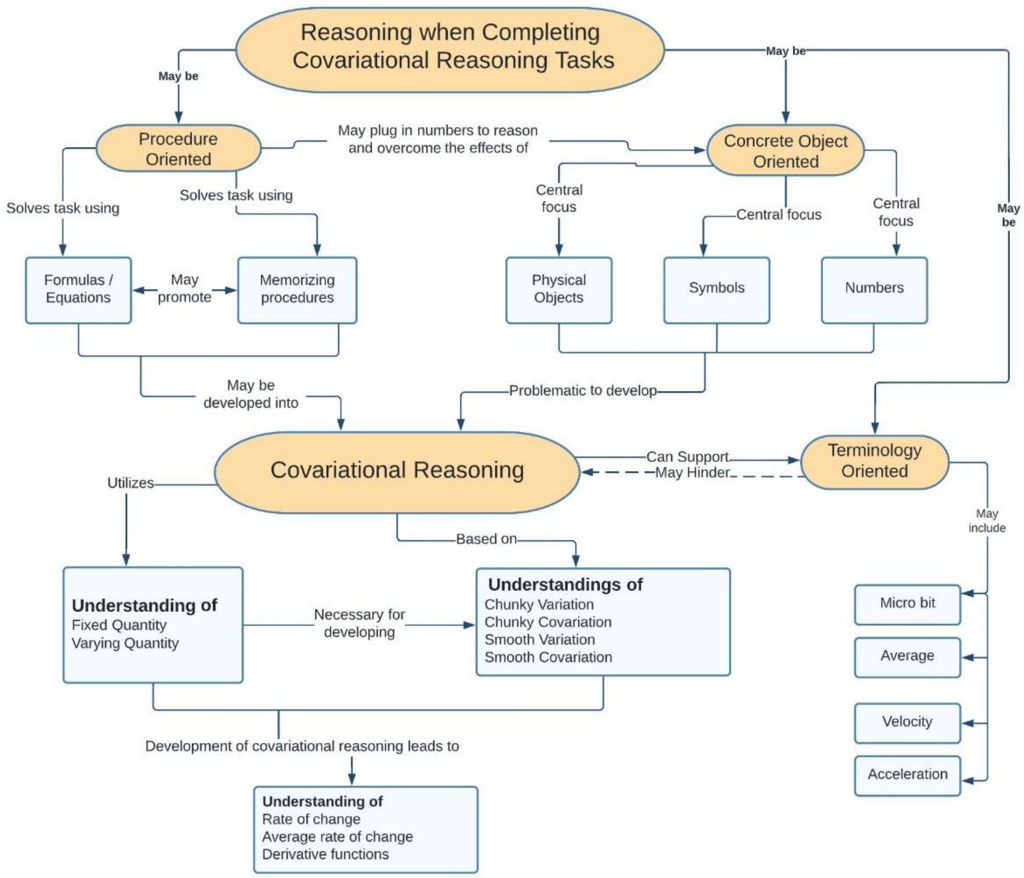


Figure 29. Concept Map of Students' Reasoning Orientations

Students' Conceptions of Quantity and Variation

This section describes procedure-oriented and concrete object-oriented reasoning and how that influences the ways that students came to understand the concept of quantity and variation. The participants in this study showed different levels of understanding upon entry to this study related to their conception of quantity and variable. Each student showed a different type of reasoning when they tried to define and describe the meaning of a quantity and variation. For instance, Chris showed strong quantitative reasoning at the beginning of the study and he defined quantity as fixed and varying, but Sam utilized concrete object-oriented reasoning when he was asked about properties of a quantity (e.g., he said that the automobile is kept constant, and the bus is continuous—these statements focus on the physical object or concrete object, as described in the concept map of Figure 29, and not quantities) rather than utilizing quantitative reasoning focused on a measurement process involving measurable attributes. Due to his inability to reason about the process of measuring the varying quantity n in terms of coordinating quantities, he did not justify why $2n$ is greater than $2 + n$ when given sufficient details to accomplish this task. Similarly, Ruby failed to recognize the variable n due to her focus on procedure-oriented reasoning, which depended on computation. Sam and Ruby were also unable to give meaningful explanations of the calculations they performed when trying to solve the pre-interview task, revealing that they engaged with concrete object and procedure-oriented reasoning. This result adds to the body of research that shows calculus students have difficulty using variables as measures of varying quantity value. Trigueros and Ursini (2003) found that there was strong evidence among the 164 first-year undergraduates in their study that the students did not understand the concept of a variable when it is was used as a varying quantity in variational situations. Gray, Loud, and Sokolowski (2007) indicated in their study that most of their research participants had difficulty using variables as generalized numbers and varying quantities. However, this study compared to other studies specifically identified three types of resonating that hinder students' development of covariational reasoning abilities: concrete object-oriented, procedure-oriented, and terminology-oriented reasoning.

As this study progressed, there were moments when Ruby correctly conceptualized quantity as both fixed and progressively varying, and she used her developed quantity and variational reasoning ability to analyze problems. When she completed the area problem in the post-instruction task, Ruby identified fixed quantities and varying quantities as measurable attributes of a rectangle (see Figure 29). Then, as she reflected on the relationship between the area and width of the rectangle of various sizes, she determined that the area of the rectangle varied continuously as the width of the rectangle varied continuously throughout the interval. This showed that procedure-oriented reasoning can developed into covariational reasoning abilities. In contrast to Ruby's thinking, Sam did not show evidence that he developed the ability to reason about quantity and variation as a process of measuring in each situation. Rather, Sam conceptualized quantity and variation as numerical labels of concrete objects (see Figure 29).

In summary, in this study, the results showed that concrete object-oriented reasoning centrally focuses on reasoning using physical objects, symbols, and plugging of numbers (see Figure 29). This type of reasoning limited students' (for example, Sam) understanding and conceptualizing the concept of quantity and variation. For instance, Sam did not appear to construct a process for measuring quantity, quantity variation, and relationships between covarying quantities. Rather, he conceived of quantity and quantity variation as numerical labels of objects, as opposed to the result of a process of change measuring the number of length or width in feet. As he encountered variational or covariational problems that demand his quantity and variational reasoning, he was unable to apply covariational reasoning to analyze the problem situation due to his strong concrete object-oriented reasoning.

The result of this study showed that procedure-oriented reasoning is focused on solving task using formulas, equations, and memorized procedures (see Figure 29). This type of reasoning led students to develop pseudo or a weak type of reasoning towards quantity and variation. Ruby, at the beginning of the study, used procedure-oriented reasoning to analyze quantitative relationship. However, in this study the result showed that procedure-oriented reasoning can be developed into covariational reasoning when compared to concrete object-oriented reasoning. For instance, in this study Ruby, after she engaged in the instruction session, she developed a strong conception of quantity and quantity variation that was rooted in quantitative relationships that helped her to overcome her initial conception of quantity and variation. She used her developed quantitative and variational reasoning to analyze homework and exam problems during the instruction period.

Students' Conceptions of Covariation

This section describes concrete object-oriented reasoning, procedure-oriented reasoning, and terminology-oriented reasoning and how that influence the ways that students came to understanding covariation. Object-oriented reasoning is centrally focused on reasoning with the physical object or concrete object, symbols and numbers and is problematic to develop into covariational reasoning (see Figure 29). For instance, Sam did not engage with covariational reasoning (compared to Ruby and Chris), nor did he appear to use this reasoning to solve the problem throughout the study period due to his dependence on concrete object-oriented reasoning. His use of concrete object-oriented reasoning to analyze two smoothly and continuously co-varying quantities' relationship led him to produce a concrete object without applying and using covariational reasoning to analyze the problem. Sam struggled and confused with his concrete object-oriented reasoning, and that limited him to not develop quantity, variational, and covariational reasoning. This also impaired his conceptualization of the mathematics concept. Reasoning about concrete objects instead of quantities that covary impaired Sam's development of covariational reasoning. For example, in the vehicle problem Sam did not draw a diagram that illustrates the interaction between the quantities in the situation. Rather, he drew illustrations of the concrete object involved, with one illustration depicting the two concrete objects together and a second illustration depicting that one object was moving at a faster rate than the other. Sam's use of concrete object-oriented reasoning to engage in a different mathematics problem (linear, nonlinear, and rate of change mathematics problems) limited him to develop covariational reasoning.

Ruby and Chris reversed their variational and covariational reasoning during post-instruction that was strongly developed during the instructional period. Ruby had used procedure-oriented reasoning to interpret the quantitative relationship between two covarying quantities and this limited her ability to conceptualize and give correct meaning for the average rate of change. Procedure-oriented reasoning utilized formulas, equations, and memorizing procedures to solve mathematics problems. Students who depend on the use of procedure-oriented reasoning as a means to analyze covariational relationship problems may not likely develop strong conceptual understanding. For instance, when Ruby engaged with the nonlinear function problem, Ruby's prior procedural knowledge caused confusion between the concept of distance and the average rate of change. Ruby is confused with coordinating the variation in *h* and how that relates to the average rate of change. Procedure-oriented reasoning may be developed into covariational reasoning through support of dynamic mathematics instruction. For instance, during the instruction session of this study, Ruby developed a strong covariational reasoning even though she reversed her developed covariation reasoning during post-instruction. For example, when presented with a water filling problem, Ruby conceived the graph of the two dynamically varying quantities as a nonlinear graph (e.g., for the water filling problem, the graph represents continuously co-varying quantity between the height of the water and the time taken to fill the bottle of the water) and then reasoned about an increasing or decreasing rate of change of the highest of the water with the time taken to fill the bottle with water (MA5 of Carlson et al.'s Covariation Framework). She supported her reasoning during instruction problem response by comparing changes of the height of the water in the bottle with continuously varying time *t* in seconds (MA3), which resulted in the graph and the formula of the nonlinear function emerging from this reasoning. This showed that procedure-oriented reasoning can be developed into covariational reasoning with support of dynamic mathematics tasks.

Similarly, Chris used terminology-oriented reasoning while he engaged in the post-instructional assessment. For instance, Chris states, "The average rate of change function should tell me the acceleration that I used to get away from the stop sign." Chris does not define ARoC using co-variation between quantities; he did not comprehend the quotient of a corresponding change in one quantity value by comparing a corresponding change in the second quantity value to define what the ARoC means. Terminology-oriented reasoning may include a reasoning with micro bit, average, velocity, and acceleration which are related terms to variation, average rate of change, and derivative. Terminology reasoning can hinder development of students covariational reasoning. Here, it appears that the students' concrete object-oriented, procedure-oriented, and terminology-oriented reasoning impaired their mathematical reasoning, especially their use of covariational reasoning to analyze mathematical concepts like ARoC and derivate function. Similar studies reveal that students have difficulty engaging with covariational reasoning, and many students have difficulty reasoning beyond L3 covariational reasoning and also unpacking their reasoning of the rate of change by saying rate of change between two quantities is a ratio of the corresponding changes in these quantities (Carlson et al.,2002; Moore & Bowling,2008; Hobson & Moore, 2016).

Students' Conceptions of Rate of Change and Derivative Function

This section summarizes the influence of concrete object-oriented, procedure-oriented, and terminology-oriented reasoning on the students' conception of average rate of change and derivative function (see Figure 29). Consistent with a research study on students' conception of the rate of change and derivative functions (Carlson et al., 2002; Moore & Bowling,2008; Thompson,1994), the findings from this study showed that the students' prior reasoning (for instance, in this study concrete object-oriented, procedure-oriented, and terminology-oriented reasoning) influenced their conception of the average rate of change and derivative function. Each student was trying to conceptualize the concept of the average rate of change function and derivative function using of properties of physical objects, procedure-oriented, and terminology-oriented reasoning rather than using idea of a measurement of the covariational relationship between two or more quantities. The study findings by Carlson (2002) reveal that many students had difficulty creating images of a continuous rate of change by imagining an increasing and decreasing rate of change function within the given interval. Similarly, Moore and Bowling (2008) indicate that most students did not have images of varying rate. Additionally, other research findings indicate that students did not have a precise image of the average rate of change function (Thompson, 1994). However, this study uniquely identified what type of reasoning students engaged in when solving covariational related problems: concrete object-oriented, procedure-oriented and terminology-oriented reasoning, that hinder their construction of covariational reasoning and conceptualization of ARoC and derivative function. More importantly, this study results reveals that all students could not pass MA4 and L4 and had difficulty interpreting the rate of change due to the types of reasoning they were engaged (i.e., object-oriented, procedure-oriented and terminology oriented reasoning) (see Appendix A for a detailed description of MA4 and L4). For instance, Sam used concrete object-oriented reasoning to define ARoC, but he did not define ARoC using idea of co-variation; he used the term how far, fast, or going very slow to define ARoC, which are not relevant to the definition of ARoC function. Ruby used procedure-oriented reasoning, "It is what you get when you fill everything out and cancel out h so it would be like when h is close to zero," to the connection between ARoC and derivative. Similarly, Chris states, "The average rate of change function should tell me the acceleration that I used to get away from the stop sign," using his terminology-oriented reasoning. Most significantly, the results of this study show that students were unable to interpret the concept of derivative function as related to the concept of the average rate of change function. That is, students do not have the awareness that the average rate of change between the two quantities is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval and derivative function is the limiting value of the average rate of change function or results from smaller and smaller refinements of the average rate of change of a function.

Ruby and Chris developed strong covariational reasoning during the instructional session; however, both students reversed their developed covariational reasoning and had difficulty conceptualizing the idea of the average rate of change and derivative function during the post-instruction interview. For instance, for the nonlinear problem, Ruby confused distance and ARoC. She coordinates the change in time with the distance that the car travels, but she interpreted ARoC as distance by defining procedurally and she did not define and use the idea of covariation in her understanding of ARoC. More importantly, for the question that asked, "Why you said derivative instead of ARoC," she indicated that, "It is what you get when you fill everything out and cancel out h so it would be like when h is close to zero." She used procedure-oriented reasoning to justify why she said $4t + 1$ is derivative when rather she should have engaged in covariational reasoning and said this is ARoC when $h = 0.5$. Ruby tends to depend on procedure-oriented reasoning (canceling out h) when making sense of mathematical situations. This procedure-oriented reasoning is impeding her understanding of ARoC.

Like Ruby, Chris reversed his developed covariational reasoning and he used strong terminology-oriented reasoning to engage in post-instruction interview tasks, which impaired his conceptualized idea of the rate of change and derivative function. For instance, when Chris was asked to give the meaning of the ARoC function he said, “The average rate of change function should tell me the acceleration that I used to get away from the stop sign.” Rather he should define using the idea of covariation by saying the average rate of change between the two quantities is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval. When the interviewer further asked Chris about what will happen as h approaches to zero or for h=0.0001for nonlinear problem, Chris asserted, “Well, they're all headed towards that limit, they're all headed toward that thing. So, yeah, it's all headed towards the acceleration which is the derivative.” It seems Chris had confused the concept of acceleration with that of derivative. This study finding shows that Chris's terminology-oriented reasoning impaired his developed covariational reasoning and, as result, he showed a weak conception of the rate of change and derivative function. More importantly, in this study, it appears that familiarity with science terminology concepts is impeding Chris’s covariational reasoning. He seems to be skipping over the coordination of two quantities in thinking about ARoC and derivative function and going straight to the science terminology ideas. This is impeding his understanding of the covariational relationships in the problem. In this study Chris terminology-oriented reasoning impaired his covariational reasoning development.

Sam’s reasoning during the study was dominated by concrete object-oriented reasoning and concrete object-oriented reasoning influenced his conceptual development to mathematics concepts. Throughout the study period, he resisted changing his concrete object-oriented reasoning, and that impaired his development of covariation reasoning. As a result, he did not show evidence of thinking about variation and covariation between the two co-varying quantities of distance and time to evaluate the average rate of change (ARoC). Sam connected the average rate of change function with the idea of the fast and slow move when $h = 0.1$, $h = 0.5$, or $h = 0.001$. Sam showed confused the ideas of h approaching 0 with the idea of slow and fast (when $h = 0.1$ fast speed, when $h = 0.001$ a bit fast speed, and when $h = 0.5$ the slow speed). Reasoning about concrete or physical objects instead of quantities that covary led Sam to have difficulty conceptualizing the idea of the average rate of change and derivative function. Moreover, his concrete object-oriented reasoning limited his development of variational and covariational reasoning also contributed to his inability to reason about covariation relationships and conceptualize concepts like average rate of change and derivative function.

In general, research findings in calculus and science indicated that well-developed covariational reasoning ability will help students to understand functional relationships, concepts like ARoC and derivative, and science concepts like gravity and acceleration (Weber & Carlson, 2010; Panorkou & Germia, 2021). However, many studies reported that students had limited understandings of covariational relationships as and a result they exhibited limited conceptions for the idea of ARoC and derivative (Moore, 2010; Park, 2013; Tyne, 2014; Tyne, 2015; Weber, 2005). But most often, studies did not clearly identify the types of reasoning that hinder the students' development of covariational reasoning and conception of ARoC and derivative function. In this study, results identified three types of such problematic reason: concrete object, procedure, and terminology-oriented reasoning that hinder students' covariational development and their conceptualization of ARoC and derivative function. The findings of this study shows that these types of reasoning (i.e., concrete object, procedural, and terminology-oriented reasoning) impaired students' development of quantity, variation, covariation reasoning as well as their conception of rate of change, function, ARoC, and derivative function.

Implications for Curriculum and Instruction

This section presents an implication to inform curriculum and instruction on students' reasoning type such as object, procedural, and terminology-oriented reasoning and its influence on the students' calculus learning. This study implemented a sequence of instruction that was designed to support students to develop variational and covariational reasoning abilities by engaging in dynamic mathematics tasks. However, the result of the study showed that the study participant did not always engage in reasoning on the level that was set as a goal in the instructional sequence. The results of the study showed that students did not reach a level of reasoning that promotes conceptual understanding of mathematics. For example, the results of this study showed that some students demonstrated a strong concrete object-oriented reasoning, procedural, and terminology-oriented. For instance, in this study, Sam demonstrated strong concrete object-oriented reasoning and his approach to mathematics problems did not help him to construct quantity, variational, covariational reasoning. Similarly, with Ruby and Chris, their use of procedural and terminology-oriented reasoning respectively did not help them to engage in covariational reasoning ability. To support students' development of productive covariational understanding curriculum needs to help students progress in this regard. Thus, the current curriculum and instruction must promote a problem-solving approach that is founded on quantity, variational, and covariational reasoning. When students participate in such kind of instruction which promotes quantity, variational, and covariational reasoning in a dynamic instructional environment they will likely construct and develop mathematical understanding. For instance, in the case of Sam, if he had exposure to the concepts of quantity, variation, and covariation in his high school mathematics, he might not be confused between the quantity and the physical object. A study by Thompson and Carlson (2017) revealed that the U.S. curriculum does not have a clear picture of the idea of smooth variation or continuously varying quantity. A student's approach to mathematics problem solving is influenced by their previous experiences in mathematics courses. Ruby is an example of this, at the beginning of the study she demonstrated strong procedure-oriented reasoning and as she participated in the instruction session she quickly learned and developed covariational reasoning. However, her developed covariational reasoning would not last long as she reversed and used her strong procedure-oriented reasoning again during post instruction. This shows that developing students' covariational reasoning is not a one-time or a linear process, it needs a long period intervention and emphasis. Therefore, in the school of mathematics curriculum and instruction, the need is to focus on teaching students the fundamental reasoning abilities such as quantity, variation, and covariation reasoning to promote strong mathematics learning and to help students to develop conceptual understanding in mathematics class, especially for their future advanced calculus learning by designing instruction that promotes covariational reasoning in elementary and secondary schools. The findings of this study suggest developing students' quantity, variational and covariational reasoning requires a long period of intervention by designing dynamic instructional tasks, and it is understood that implementing and practicing such kind of instruction is beneficial for the students' mathematics problem-solving ability and for their future advanced mathematics learning.

Findings of this study show that student conceptions of quantity, variational, and covariational reasoning were critical for their understanding of the average rate of change function and derivative function. The concept of the average rate of change and derivative function is rooted in understanding quantity, variation, and covariation. Also, conceptualizing quantitative and covariational relationship and its process at the end will lead students to conceptualize the idea of a function as an input-output relationship between two co-varying quantities. The average rate of change is the constant rate of change that produces the same change in the dependent quantity as the original relationship over the given interval as the independent quantity progressively vary, and derivative function is the limiting value of the average rate of change of function as the independent variable approaches to zero in the given interval or a refinement of the average rate of change function as the independent quantity continuously vary in the given interval. Developing such types of reasoning at an early stage will support students to have the imagine of the input-output process without needing any procedure or calculation of the average rate of change and derivative function. Findings from previous studies show that students' learning the foundational concepts of calculus such as rate of change, derivative, and integral depend on their deep mathematical reasoning abilities, especially on their variational, quantitative, and covariational reasoning abilities (Carlson et al., 2002; Castillo-Garsow, 2012; Orhun, 2012, Thompson & Carlson, 2017). The result of this study avail three types of reasoning: concrete object, procedure, terminology-oriented reasoning that instructor and curriculum designer to be aware of them when they are teaching and design instruction materials to avoid in engaging in these types of reasoning. Instructors could use examples of each type of reasoning when they teach by comparing with correct reasoning types such as, variational and covariational reasoning. Curriculum developers could develop content that reflect concrete object-oriented, procedure-oriented reasoning and terminology-oriented reasoning by comparing with variational and covariational reasoning type. This study also reveals the importance of students engaging in learning quantities and constructing relationships between quantities. Therefore, in schools of mathematics instruction, it's important to focus on quantity, variation, and covariational reasoning teaching and learning rather than procedural and computational-based mathematics teaching and learning.

Limitations of the Study

This study examines three calculus students' quantity, variation, and covariation reasoning abilities and also their average rate of change and derivative conceptual understanding abilities. Each student engages in different and unique conceptual and reasoning abilities. The results of this qualitative study may not be generalized to all levels of first-year calculus students. This is one of the limitations of this study. The main goal of the study is to understand students' variational and covariational reasoning abilities and their conceptual development of the concept average rate of change and derivative. To explore such kind of development in this study, students were engaged in dynamic instructional activity with the support of technology, which is a much different instructional approach to traditional calculus instruction. For this, this study result might not reflect and generalize to all calculus classroom situations.

A second limitation of this study was the zoom video group setting in which the students engaged in the instructional activities. This setting made it difficult to capture the entire progress of each student on the instructional activities, as each student's level of participation varied during these activities, and it is impossible to follow and record the entire class activity. An interview setting was used to pose additional tasks to the students to test the researcher's models of their mathematics, but their engagement during the initial instructional tasks was not captured at this individualized level.

The third limitation was that the study only tracked what the students did during the videotaped instruction. Additionally, the study only occurs over a three-week period of time and that cannot represent the entire semester of calculus study and mathematics experience.

Lastly, this study collected the students' written work data during the pre-course and instruction period due to the remote instruction setup. Thus, the results presented should be read as the researcher's interpretation of the students' understandings and reasoning, where this interpretation was grounded in building and testing models of the students' mathematics, which were inferred from the students' observable behaviors from collected data.

Directions for Future Research

This study investigated the role of quantity, variational, and covariational reasoning in first-year calculus students' learning of the average rate of change and derivative function. Specifically, this dissertation focused on calculus students constructing and reasoning about the average rate of change and derivative functions in the contexts of the linear and nonlinear function mathematics problems. To explore the role of a quantity, variational, and covariational reasoning of students' development of conception in the area of average rate of change and derivative function, future studies could explore the average rate of change function by comparing with derivative function learning. Future study could especially explore students' smooth continuous covariational reasoning and average rate of change as *h* varies progressively throughout the given interval.

Second, future studies could explore the cause of students' concrete object-oriented reasoning and implications of how to support students to move from such type of reasoning to variational and covariational reasoning abilities. The variation and covariation conceptions held by the students of this study appeared to play a role in their thinking and reasoning abilities. Given students' varying conceptions of measurement labels as an object, the role of students' variation and covariation conceptions should be further explored in the context of the average rate of change and derivative function.

Third, this dissertation revealed insights into the relationship between variational and covariational reasoning with terminology-oriented reasoning on students' problem-solving behaviors. The findings of this study show that students' use of terminology-oriented reasoning limited students' conception of the average rate of change and derivative function. Future studies may see the relationship between terminology-oriented reasoning and covariational reasoning and its influence on understanding mathematics as well as science concepts.

In summary, this study uniquely identified three types of reasoning (i.e., concrete object-oriented, procedure-oriented, and terminology-oriented reasoning) that hinder the students' conceptualization of concepts of ARoC and derivative function, but these are most certainly not the only types of problematic reasoning orientations—there are likely others. It would serve the field well future researcher to identify additional problematic reasoning orientations for students when they are using covariational reasoning to understand ARoC and the derivative function.

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APPENDICES

Appendix A: Analytical Framework

Table 1

Mental Actions and Levels of the of the Covariation Framework (Carlson et al., 2002, p. 357-358)

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Labeling the axes with variables indications of coordinating the two variables (e.g., *y* changes with changes in *x*)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable	Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Table 2

Covariational reasoning levels

Covariational Reasoning Levels
The covariation framework describes five levels of development of images of covariation. These images of covariation are presented in terms of the mental actions supported by each image.
Level (L1). *Coordination* At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).
Level (L2). *Direction* At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are both supported by L2 images.
Level (L3). *Quantitative Coordination* At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.
Level (L4). *Average Rate* At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.
Level (L5). *Instantaneous Rate* At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

Table 3

Major Levels of Covariational Reasoning of the of the Smooth Covariation Framework (Thompson & Carlson, 2017, p. 440-441)

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantities or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y)
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Precoordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values

Appendix B: Recruitment Task

The purpose of the task

- The task is designed to examine students general covariational reasoning abilities.

Anticipation for the students' response

- The students' response to this task will vary. Some students may engage in covariational reasoning and others may not engage in covariational reasoning to give answers to the questions.
- All students' response will be examine using the analytical framework of this study.
- The goal of the task
- To recruit students for this study who scored low, medium, and high.
- To further study their smooth continuous covariational reasoning abilities.

Possible solution for this task

1. Which is the larger, 2n or 2+n? Explain

   Answer

   - The correct answer for this question is "It depends. If n<2, 2+n is greater, if n=2, 2n =n+2, if n>2, 2n is greater." (Schoenfeld&Arcavi, 1988)
   - The students' answer will be scored using Carlson framework of this study (see Appendix A)

2. Two vehicles, an Automobile and a Bus, are 560 kilometers apart. They start at the same time and drive toward each other. The Automobile travels at a rate of 75 kilometers per hour and the Bus travels 53 kilometers per hour.

a. Illustrate the situation by drawing and use variable to represent each situation in your drawing.

Answer

- Drawing of two vehicles facing each other (i.e., drawing of Automobile and Bus) separated by varying distance measured in kilometers (with a clear label a variable “d” or “D” that represent the number of kilometers between the two vehicles), with labels of total distance measured in kilometers (a constant variable), and a variable t that represent the number of hours since the vehicles started moving. A formula that relates the distance covered by each vehicle while they travel within the given rate of change (d=560-128t) or a formula that show the covariational relationship between d in the number of km between the two vehicles in terms of the amount of time in hours since they started driving towards each other.
- Below is one possible drawing of the situation

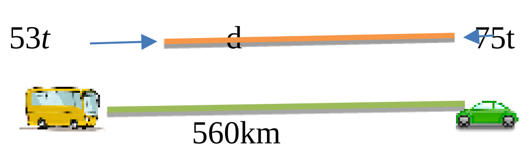


Figure 1. Two vehicle problem

b. Identify the quantities whose values are continuously changing those whose values are kept constant in this scenario.

Answer

Continuously varying quantities

- The distance d in km between the two vehicles.

- The time t in hours since the vehicles started moving.
- Each vehicle distance that vary with time (53t and 75t).

  Continuously Constant quantities

- The total distance in kms (560 kms)
- The rate of change of the two vehicles (53km/h and 75km/h)
- The students' answer will be scored using the analytical framework of this study (see table 3).
- The students' answer will be scored using the variational reasoning analytical framework of this study (see table 3)

3. Suppose a town's Board of Health reports that a bacterial infection has been spreading for the last several days. They use a model that relates the number of days since January 1, 2017, *x*, to the number of people who are infected with the infection, *y*. The graph represents the relationship between these two variables in this model.

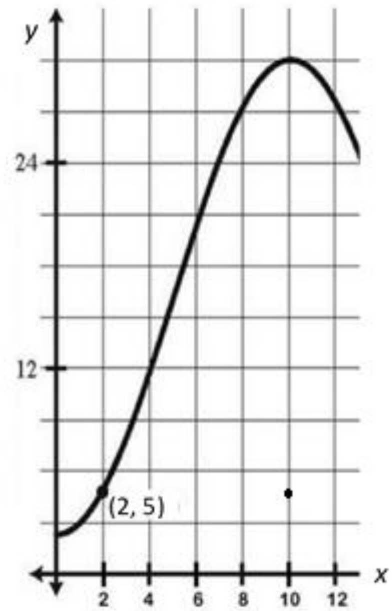


Figure 2. Bacterial infection problem

a. Draw a horizontal line segment on the graph from the point (2, 5) to (10, 5). What does this horizontal line segment represent in the context of this question? How is x vary within this given interval?

Answer

- A correct drawing of a horizontal line segment that joined point (2, 5) to (10, 5).
- The horizontal line segment represents the change in the number of days from January 2 to January 10.
- The horizontal line segment represents a change of 8 days.

  - x vary smoothly and continuously within the given interval from 2 to 10 or x varies progressively in all intermediate values within the given interval from 2 to 10.

b. Describe what the graph conveys about how the number of people infected changes over time.

   Answer

  - The graph relates the number of people who infected by bacteria with the number of days that started infecting people on January 1, 2017. At first the number people who infected by the bacteria are rapidly or fast grow and then levels off in the rate of infection. Then the rate of infection slows down until the number of infections begins to decrease starting day 10 and the next days after January 10, 2017.
  - The students' answer will be scored using the variational, covariational, and smooth continuous covariational reasoning analytical framework of this study (see table 3).

Appendix C: Mathematics Task 1

This investigation is focused to develop students' image of variation.

Some important terminology:

- If the value of a quantity does not change, the quantity is called a fixed quantity and its value is a constant.
- If the value of the quantity does change, the quantity is called a varying quantity and the quantity can assume more than one value or multiple values.
- Key concept: imagining of variation happen within a unnoticeable amount of change or imagining the varying quantity change in a progress.

This investigation has two goals

- To create an image that variation in a quantity's value can happen over smaller intervals.
- To give insight into at variation over smaller intervals often reveals more of a relationship's overall behavior.

Problem 1. Open the GeoGebra applet 1 and describe how you anticipate variation of v and u. How u is varying compared to v? What types of variation do you notice between v and u? Explain the different variations that you notice between the two varying quantities? By how much do v vary? By how much do u vary? Is that possible to notice by how much amount does u vary?

(show the screen of the applet)

Answer

- The purpose of this question is to develop students' knowledge of smooth continuous variation of a variable value. I anticipate that students will answer correctly for chunky continuous variation, and they will say *v* is varying by 1 unit each time (i.e., by 1 cm) but student may not give exact answer about the variation of u. To help student to understand the variation of the variable *u* the instructor must ask students what they notice, or they imagine about the variation in the value of *u* until students understand u vary smoothly and continuously in the given interval or until they know u vary in all intermediate values of the given intervals GG applet will help for this part of discussion.

Problem 2. Let $x =$ the tree's height measured in feet. What is the difference between saying $x = 1.5$ feet and $\Delta x = 1.5$ feet?

Answer

- $x = 1.5$ feet represent the actual height of the tree as it is measured.
- $\Delta x = 1.5$ feet represent the change of the height of the tree when it is measured in between certain time interval.

Problem 3. The variable x varies through the interval [2,7] by varying through intervals of length $\Delta x = 0.0001$. Through how many subintervals of length Δx will the value of x vary as it varies from x=2 to x=7?

Answer

- The length of the interval [2,7] is 5 where *x* can vary and the number of subintervals of length $\Delta x = 0.0001$ is $\frac{5}{0.0001} = 50{,}000$(or 50,000 number of subintervals of length equals to 0.0001 of the bigger interval of length 5) . The purpose of this problem is to show students a large variation can be made from small variation or large variation can be partition into a very small size variation and within that size interval *x* can also vary or there is variation.

Problem 4. You have 240 feet of fence to enclose a rectangular lawn. You are free to make the enclosure have any possible length and width, but you must use all the fence. Play the GeoGebra (GG) animation applet2. Drag point D to the left and to the right to see how the enclosed rectangular area vary as the width and the length of the rectangle varying.

a. Define the constant variable in this situation.

Answer

- The constant variable in this situation is the total fence length that used to enclose a rectangular lawn (240 feet).

b. Define the varying variables in this situation. State the intervals over which they vary.

Answer

- The varying variable in this situation are
- The area of the enclose rectangular lawn and the area of the rectangle can vary in the interval between 0 to $3600ft^2$.
- The length of the enclose rectangular lawn and the length of the rectangle can vary between the interval 0 to $120ft$.
- The width of the enclose rectangular lawn and the width of the rectangle can vary between the interval 0 to $120ft$.
-

c. Express the relationship between the rectangle's width and length using a formula.

Answer

- Let w represents the width of the enclose rectangular lawn in ft and l represents the length of the enclose rectangular lawn in ft
- $2w + 2l = 240ft \Rightarrow w + l = 120ft \Rightarrow w = (120 - l)ft$ or $l = (120 - w)ft$

d. Express the relationship between the rectangle's enclosed area and either its width or its length using a formula.

Answer

- Let $A(w)$ represents the area of the enclose rectangular lawn in ft^2 and w represents the length of the enclose rectangular lawn in ft
- $A(w) = w(120 - w)$

e. When the width of the rectangle varies by 10 feet between the interval $0 \leq w \leq 120$. What relationship do you notice between the rectangle's enclosed area and its width? What can you conclude from the relationship? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not.

Answer

- The answer for this part is depends on the classroom discussion with the instructor but students expected to notice the rectangle's enclosed area and its width coordinately covary by 10 ft length of intervals. Students should produce a table for all possible values of the two variables. The big message of this part is a chunky variation in the width of the rectangle will produced corresponding chunky variation in the area of the enclosed rectangle and students should understand there is a chunky covariation between the area of the rectangle and the width of the rectangle.

f. When the width of the rectangle varies smoothly and continuously between the interval $0 \leq w \leq 120$ (open GG applet 2 to explore this question). What relationship do you notice between the rectangle's enclosed area and its width? How does the area of the rectangle vary as the width of the rectangle vary all intermediate values within the given interval? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not.

Answer

- The answer for this part depends on the classroom discussion with the instructor but students expected to notice the rectangle's enclosed area and its width coordinately covary smoothly and continuously in given interval the GG applet will help for this part the discussion. If they can, students can produce a table for all possible values of the two variables (i.e., the instructor can use this question to discuss a question why or why not produce a table. It is true that students cannot produce a table for smoothly and continuously covary quantities, since each variation is in a bit or cannot be noticed). The big message of this part is a smooth variation in the width of the rectangle will produce a corresponding smooth variation in the area of the enclosed rectangle or students should understand there is a smooth continuous covariation between the area of the rectangle and the width of the rectangle or there is a smooth continuous covariation relationship between the width and area of the rectangle.

g. Make graphs for the situation in part (e) and (f). Did you get the same graph for part e and f? If you found different graphs, explain why this is occurred.

Answer

- Possible graph for part (e)

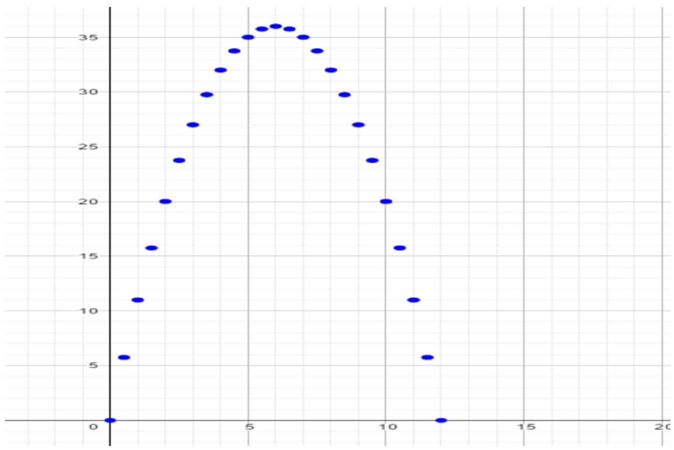


Figure 3. Chunky covariational graph

- Possible graph for part (f)

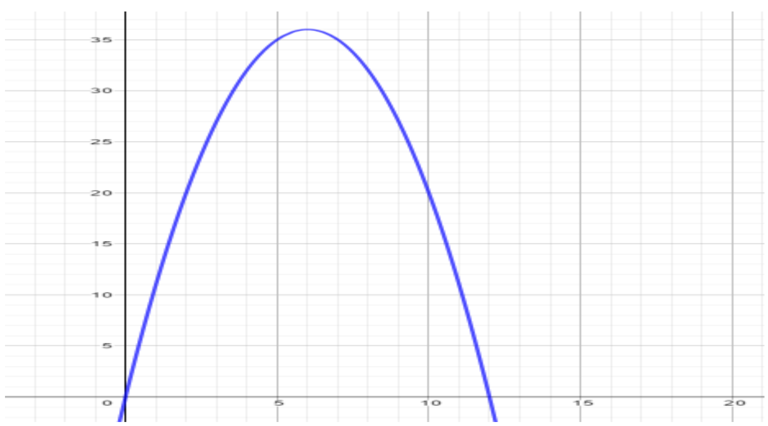


Figure 4. Smooth covariational graph

Appendix D: Mathematical Task 2

This investigation is focused to develop students' image of covariation reasoning.

This investigation has two goals

- To create an image that variation in a quantity's value can happen over smaller intervals and the change can also be unnoticeable.
- To give insight of the relationship between two continuously covarying variables over smaller intervals.

Note:

- We will use the letter "d" preceding a variable to mean that the variable's value "varies a little bit". The symbols "dx" and "$\varDelta x$" have different meanings. "dx" represents a change of variable that we cannot notice in our eyes but we have awareness or consciousness which has a value that is imaginable or very little bit. Or "dx" represents a small variation in the variable x. "$\varDelta x$" represents a change that we can notice or have fixed values that we can measured.

Problem 1. Open GG applet 3 and give your explanation how du and dv vary.

a) By how much do you think du varies?

Answer

- Δu varies by 0.2 amount

b) By how much do you think Δv varies?

Answer

- Δv varies by 0.4 amount

c) What meaning do you have for Δu and Δv?

Answer

- Δu represents 0.2 variation in the variable u
- Δv represents 0.4 variation in the variable v

d) What relationship can you conclude from these two covarying quantities?

Answer

- One possible answer for this question is when every time du vary by 0.2 amount dv vary by 0.4 amount or $dv = 2du$ or $\frac{1}{2}dv = du$ (for further discussion the instructor possibly can ask the meaning of 2 or ½ in this relationship).
- In this part of the task I want students to know that two quantities can covary in a chuck continuous covariation relationship.

Problem 2. Open GG applet 4 and first explore how dv and du vary independently and then explore how the two small variations co-vary together.

a) What do you notice when the two quantities co-vary simultaneously?

Answer

- One possible answer for this question is du and dv covary simultaneously by unnoticeable amount or they covary in a bit.
- Another possible answer for this part could be both dv and du vary smoothly and continuously in the given interval.

b) What is $\frac{dv}{du}$? What is $\frac{du}{dv}$?

Answer

- One possible answer for this question is students may say it is a derivative or students may say something else like rate of change or ratio (further discussion can be facilitated by the instructor based on students' response).
- Another possible answer for this part could be students see $\frac{du}{dv}$ or $\frac{dv}{du}$ as the quotient of two smoothly and continuously covarying quantities (further discussion can be facilitated by the instructor based on students' response).

Problem 3. Suppose the mass *m* of a bacterial culture, measured in grams, varied at a constant rate of 5 $grams/hr$ with respect to time, measured in hours, for *t* hours. This means that within this period, any variation in the culture's number of grams will be 5 times as large as the variation in the number of hours since measurements began that produced it.

Let dm represent a varying change in the culture's mass during this period and let dt represent a varying change in the number of hours during which dm happens. Then $dm = 5dt$ as the value of dt varies.

a. When Δt = 0.1 hours compute Δd and explain how the mass varied.

Answer

- When Δt=0.1hours, Δd=5grams/ hours *0.1hours=0.5grams, that means the mass varied by 0.5 grams for0.1 hours
- b. When $\Delta t = -3.2$ hours compute Δd and explain how the mass varied.
- Answer
- When Δt=-3.2hours, Δd=5grams/ hours *-3.2hours=-16grams, that means the mass varied by -16 grams for -3.2 hours

c. When $\Delta t = 0.000\ 001$hours compute Δd and explain how the mass varied.

Answer

- When Δt=0.000001hours, Δd=5grams/ hours *0.000001hrs=0.00005grams, that means the mass varied by 0.00005 grams for 0.000001 hours

d. When $\Delta d = -0.3288732$ compute $\Delta t$and explain how the time varied.

Answer

- When Δd=-0.3288732 grams, $-0.3288732 grams$ =5grams/ hours *Δt, so Δt=-0.0657464hours, that means the time varied by -0.0657464hrs when the mass is -0.3288732 grams

Problem 4. Open GG applet 5 and explore the two covarying quantities.

a) By how much does r vary?

Answer

- Possible answer for this question is r vary by a little bit or by small dr amount or by unnoticeable amount

b) By how much does k vary?

Answer

- Possible answer for this question is k vary by a little bit or by small dk amount or by unnoticeable amount
- c) What relationship can you predict from these continuously covarying quantities?

Answer

- Possible answer for this question is when the two quantities covary smoothly and continuously in the given interval the value of k vary in a little bit (dk) and the value of r is also varying simultaneously by small dr amount or by unnoticeable amount.

- Another possible answer is r and k have a smooth continuous covariation relationship.

d) Plot the graph of r as a function of k.

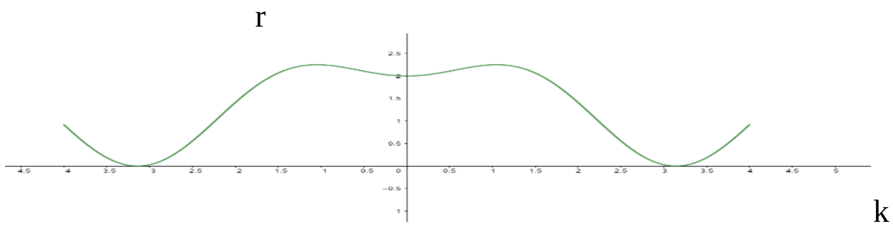


Figure 5. Covariational graph

Problem 5. Answer the question below by using the graph of a function f that represents the amount of money in a banking account (measured in thousands of dollars) as a function of the number of years since 1995.

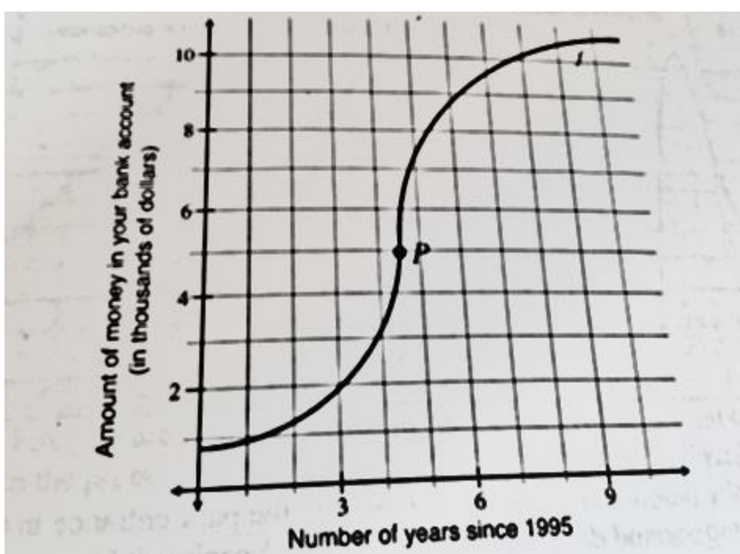


Figure 6. Graph of the amount of money in the bank

a. How did the amount of money in the bank account change from 0 to 5 years after 1995?

Answer

- As the number of years since 1995 increases from 0 to 5 years the total amount of money in the account increases from approximately 1,000 to7, 000 dollars, or the amount of money in the account changes from approximately \$1,000 to \$7,000. This is a change of \$7000-\$1000≅\$6000.

b. What does the point P on the graph convey about the amount of money in the account is changing with time?

Answer

- The point P conveys two things about the amount of money in the account:
- Below the point P (4.5 years, \$5,000) conveys for any smooth continuous increase of time since the begging until 4.5 years the change in the amount of money in the account increase and yield \$5,000.
- After the time passes through 4.5 years the point P (4.5 years, \$5,000) conveys the change in the amount of money in the account decreases as the time increase smoothly and continuously from 4.5 years and for the continuous years.

c. The graph is concave down on the interval $4.5<t<10$. How is the amount of money in the account changing on this time interval? How is the rate of change of the amount of money in the account changing on this time interval?

Answer

- The change in the amount of money in the account continuously decreases as the time increase smoothly and continuously from 4.5 years to 10 years.
- The rate of change of the amount of money continuously decreases as the time increase smoothly and continuously from 4.5 years to 10 years.

Appendix E: Mathematical Task 3

This investigation is focused to help students to apply smooth continuous covariation reasoning ability to solve mathematics problem.

The goal of investigation 3

- is to apply students' smooth continuous covariation reasoning.
- is to develop students' conceptual understanding of derivative function

Problem 1. Suppose Fatima deposits \$100 into an account at Incredible Bank on January 1. The table below shows the amount of money in her account, rounded to the nearest dollar, at the end of each month passed since January 1.

Table 4

Trend of the amount of money in the account in each month

Month	1	2	3	4	5	6	7	8	9	10	11	12
Amount in the Account	135	182	246	332	448	605	817	1,103	1,489	2,011	2,714	3,664

Let A represent the amount of money in the account, measured in dollars; and let t represent the time passed since January 1, measured in months. If we assume t and the amount of money in the account continuously increases over the interval from $t = 1$ to $t = 12$.

a) What amount of money will Fatima have in the account at the second day of the month of January? By how much interest rate is the amount of money in the account growing? Explain your reasons.

Answer

- There are many ways to solve this problem. One may approach this problem by looking each month trend and he/she may relate the accumulated amount of money per each month $A(t)$ and the monthly interest rate of the bank r_m, that is, the time *t* measured in months, and r_m is the interest rate in month. Or one may use the regular compound interest formula $A(t) = 100(1 + r_m)^n$, where n is the number of periods that the amount of money compounded (i.e, n=m*t) and r_m is the monthly interest rate. However, this approach may not be helpful for students, since every time students need to change the interest rate when they asked to calculate the amount of money in the account in each period or time, that is, for each day, minute, second, or even microsecond.
- This approach for me has two problems for the student to solve this particular question, first, since the amount days in each month is different (i.e., January has 31 days, but February has 28 days, 28 days in common years and 29 days in leap years), because of this, each month will have different daily interest rate and solving the interest rate using this approach will be problematic for the students in the context of this problem.
- Second, when we converted 365 days by dividing into 12 equal months, and this also will have a fraction of days (~30.41666666…days per each month) in each month and this will also be problematic for the students to solve this particular problem.
- Another approach to solve this particular problem is using a smooth continuous covariational thinking, i.e., if students think all variables are

covary smoothly and continuously in a given interval it may help them to solve this particular problem easily.

- For instance, for this particular problem if students think at first the amount of money in the account can smoothly and continuously covary within the given interval of time, that is, thinking the amount of money in the account can vary or change continuously within the year of each day, month, hour, second, microsecond and so on. Like laying a ruler and seeing the total length, seeing the small part of the ruler within the total length, seeing within each small part of the ruler another very little or bit small part. For example, within 1cm there is a 10^{-6} cm … so on. or assuming in this case the time t is varying progressively or gradually by a very bit amount or unnoticeable amount.

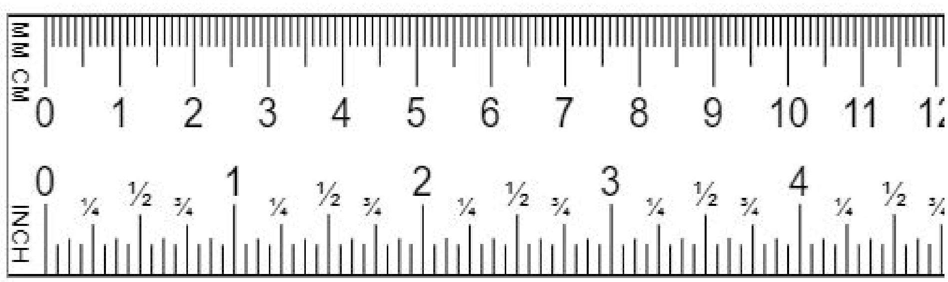


Figure 7. Marked ruler

- Additionally, students need also reason the interest rate can also by continuously co-vary with time, that is, student who think at each variation in time the amount of the interest rate will also change, that is thinking of for each day they will have a corresponding daily interest rate, for each month there is monthly interest rate, for each second there is interest rate within the second… so on. The mathematics relationship that will develop using this way of thinking may need students at first to reason in smooth continuous covariational and if students have this reasoning at first it is easy to produce a meaningful mathematics relationship between the smoothly and continuously co-varying quantities of $A(t), r_t$, and t .

- One possible mathematics relationship that can student produce for this particular problem using smooth continuous covariation reasoning is $A(t) = 100(1 + r_t)^t$. Where A(t) is the function that represents the amount of money in the account covary smoothly and continuously with time t, t represents the time measured in month, day, second, or even microsecond or any bit time in the given interval and r_t represent the interest rate that covary smoothly and continuously in a given interval of time t.
- Let us solve this particular problem using smooth continuous covariational reasoning, to solve this particular problem students first think how the amount of money vary within 365 days by partitioning the given 12 months (365 days are 12 months). This will lead them easily to develop a formula that focus to solve the daily interest rate at first and then solve the daily amount of money in the account.
- For instance, at the end of the year or 12 months the amount of money in the account is 3664 dollars and the account started with 100 dollar bills, this means that, $3664 = 100(1 + r_d)^t$, $r_d$represents the daily interest rate, t represent number of days, so after finding r_d it is easy to calculate the daily amount of money in the account. For instance, for this particular problem

$r_d = 0.009915$ and t=2, then it is easy to find A (2) = $100(1 + 0.009915)^2$=$101.9928. where, $r_d = 0.009915$ is the daily interest rate that the amount of money will grow in the account.

- To solve this particular problem using the regular compound interest formula it will be difficult for students, since each month will have a different number of days that means each days of the 12 months will have a different interest rate for each day in the month and it is impossible for the students solve the problem using this ways of reasoning for this particular problem unless they partitioned the 12 months into 365days.

b) What amount of money will Fatima have in the account at the 15th day of the month of February? Explain your reason.

Answer

- Students can answer this problem after they develop the smooth continuous covariational reasoning by using the idea in part a.
- They can use the pervious formula that developed by the idea of SCCR that is $A(t) = 100(1 + r_t)^t = A(46) = 100(1 + 0.00915)^{46}$, where t is in number of days, r_t the interest rate per day, and A(t) is the amount of money in the account per given day.

c) What amount of money will Fatima have in the account after 1 second, she deposited $100 into the account? Is reasonable to ask the bank to calculate the amount of money in the account after microseconds Fatima deposited her $100?

Answer

- This problem will be difficult if students used the regular compounding formula to solve the problem, but it will be easy when they first approach using SCCR (That is everything change smoothly and continuously with time).
- Students can answer this problem after they develop the smooth continuous covariational reasoning
- They can use the previous formula that develop by the idea of SCCR that is $A(t) = 100(1 + r_s)^t = A(1) = 100(1 + r_s)^1$, where t is in number of seconds, r_s = 0.000000114191423941 interest rate per seconds, and A(t) is the amount of money in the account in each second. A(1)=100*(1.000000114191423941)=100.0000114191423941, this is the amount of money in the account after one second the money deposited.
- Yes, it is possible to ask the bank to calculate the amount of money in the account after microsecond of deposited.

d) Is that possible to think the deposited amount of money in nanosecond (i.e., one billionth of a second)? Can you calculate the amount of money in the bank after one billionth of a second deposited? What change can you notice on the $100 bill after a nanosecond you deposited?

Answer

- This problem will be difficult if students used the regular compounding formula to solve the problem, but it will be easy when they first approach using SCCR (That is everything change smoothly and continuously with time).

- Students can answer this problem instantly after they develop the smooth continuous covariational reasoning
- They can use the previous formula that develop by the idea of SCCR that is $A(t) = 100(1 + r_s)^t = A(0.000000001) = 100(1 + r_s)^{10^{-9}}$, where t is in number of seconds, r_s the interest rate per seconds, and A(t) is the amount of money in the account in each second., where r_s = 0.000000114191423941 and $A(0.000000001) = 100(1 + 0.000000114191423941\ )^{10^{-9}}$=
- Yes, it is possible to think the deposited amount of money in nanosecond.
- The change for the $100 bill is unnoticeable or is a small bit but there is a change, or a very small change
- $A(0.000000001) - A(0) = 100.0000000000000114191417421159 -$ 100.000000000000000000=0.0000000000000114191417421159 $\cong$ $1.14 * 10^{-14}$, for unit of money this change is unnoticeable or meaningless, however, if the deposited amount of money is big enough or in trillion dollars this change is noticeable.

e) Produce a formula of the function $A = f(t)$ as t varies over the interval from $t = 1$ to $t = 12$?

Answer

- $A(t) = 100(1 + r_t)^t$, where t is in number of periods, r_t the interest rate per that period of time, and A(t) is the amount of money in the account in that period of time or the given time.
- $A(t), r_t, and\ t$ covary continuously and smoothly simultaneously

f) Describe how the rate of change of the function is change as t varies in interval from $t = 1$ to $t = 12$.

Answer

- The rate of change of the amount of money in the account function at first grow slowly and then grow or change exponentially over the given interval of time t.

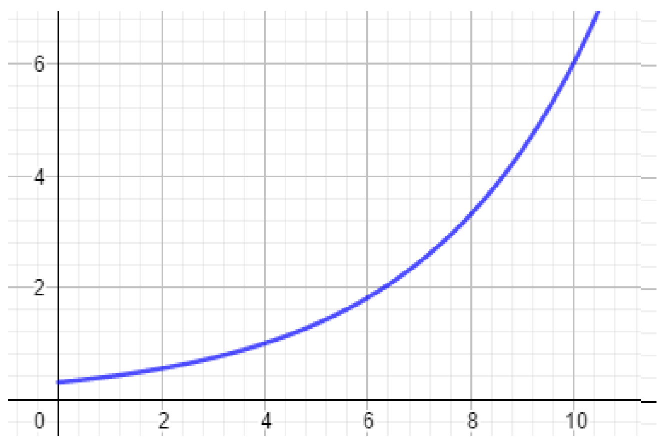


Figure 8. Rate of change graph

g) Produce a graph that could represent the function $A = f(t)$ on this interval.

Answer

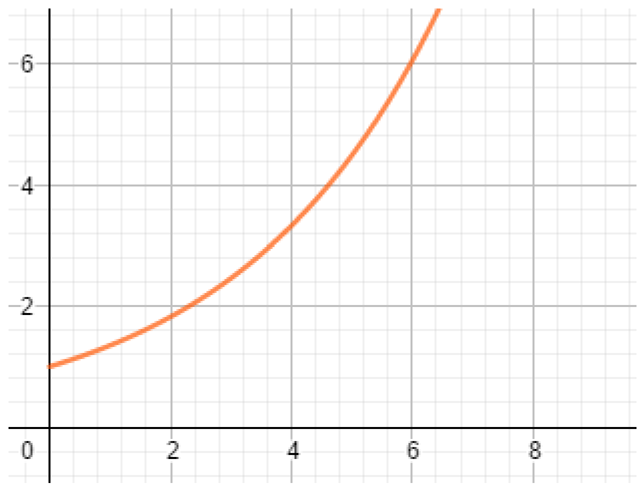
○ 

Figure 9. Rate of change function at A=f(t)

Problem2. Suppose $d = f(t) = 2t^2$ represents the distance d (measured in meters) of a car from a stop sign in terms of the number of seconds t since the car started to move away from the stop sign.

a. Determine the average rate of change of the distance of the car from the stop sign on the time intervals from $t = 0$ to $t = 1.5$ seconds.

Note:

- The formula $r_h(t) = \frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$ gives the average rate of change for $d = f(t)$ with respect to t over any subinterval from t to $t + h$, where $h \neq 0$ is the length of the interval on which t varies.

Answer

- Let $r_h(t) = \frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$ represents the average rate of change of the distance over the time interval $h = t + h - t = 1.5 - 0$=1.5.
- $r_h(t) = \frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$ => $r_h(t) = \frac{2(t+h)^2-2t^2}{h}$=$\frac{2t^2+4th+h^2-2t^2}{h}$, h≠ 0
- =>$r_h(t) = \frac{2(t+h)^2-2t^2}{h}$=$\frac{2t^2+4th+h^2-2t^2}{h}$=$4t + h, h \neq 0$
- Within the interval $h = 1.5$
- $r_{1.5}(t) = \frac{f(t+1.5)-f(t)}{t+1.5-t} = \frac{f(t+1.5)-f(t)}{1.5}$=>$r_{1.5}(t) =$ $\frac{2(t+1.5)\text{^}2-2t^2}{1.5}$=$\frac{2t^2+4th+h^2-2t^2}{h}$=$4t + 1.5$

a. Describe what this average rate of change function is tells you about the change in distance of the car from the stop sign over the time interval from $t = 0$ to $t = 1.5$ seconds.

Answer

- $r_{1.5}(t) = 4t + 1.5$ is the average rate of change of the distance of the car or in the given interval h=1.5 or in the given interval h=1.5 as t continually vary the car distance measured in meters change with average rate of change function of $r_{1.5}(t) = 4t + 1.5$

b. Determine $r_h(t)$ when h=0.5 seconds and describe what this tells you about the change in the distance of the car from the stop sign?

Answer

- $r_{0.5}(t) = 4t + 0.5$ is the average rate of change of the distance of the car in the interval h=0.5 or as t vary the car distance in meters change by the average rate of change function of $r_{1.5}(t) = 4t + 0.5$ as t continuously vary in the given interval h=0.5

c. Determine $r_h(t)$ when h=0.1 seconds and describe what this tells you about the change in the distance of the car from the stop sign.

Answer

- $r_{0.1}(t) = 4t + 0.1$ is the average rate of change of the distance of the car in the given interval h=0.1 or in the given interval h=0.1 as t continually vary the car distance in meters change with average rate of change function of $r_{1.5}(t) = 4t + 0.1$

- Sketch a graph of $\mathrm{r_h(t)}$from the stop sign in terms of number of seconds since the car stared to travel for h=0.5, h=0.4, h=0.3, h=0.2, h=0.1, h=0.001 and h=0.00001 seconds. What can you say about the graph of $r_h(t)$ when h=0.000001 seconds?

- Answer
- The graph of $r_h(t)$ when h=0.000001 seconds is almost behaved similarly like the graph of 4t and their difference is almost unnoticeable, which at this instance change in t average rate of change function and derivative function of f(t) are equal or the same.

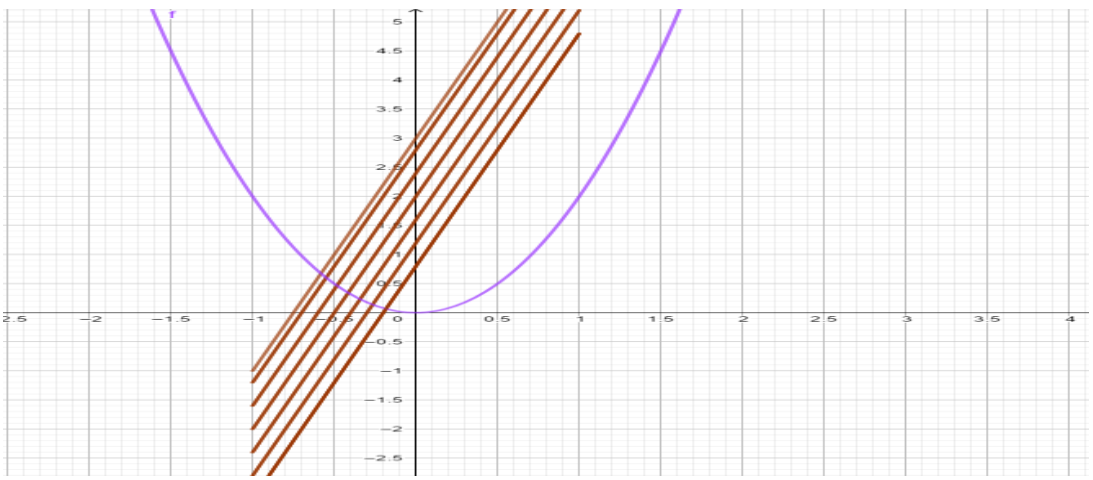


Figure 10. Family of average rate of change functions of $f(t) = 2t^2$

d. What is the graph of $r_h(t)$ tell you about the change in the distance of the car from the stop sign when h=0.5, h=0.4, h=0.3, h=0.2, h=0.1, and h=0.000001seconds?

Problem 3. Open GG1 and explore the graph of the function $f(x) = 2^x$ and the average rate of change function$G(x)$.

a. As you vary h by sliding from right to left through the interval 0 to 1 in small amount, what mathematical relationship do you notice about the average rate of change function $G(x)$ and the input variable$x$?

Answer

- When you slide h from left to right the average rate of change of the function almost behave like the derivative of the function f(x) for a very small h=0.0000001, that is, for a very small change in x or in a bit change in x or for a very unnoticeable amount of h the average rate of change of the function f(x) will be the derivative of the function.
- Students may relegalize and understand that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change $g_h(x)$. If students realize this fact it may show that students have a smooth continuous covariational reasoning abilities or they have an awareness of both x and $g_h(x)$varying smoothly and continuously as size of h is vary in a bit or in small amount or in unnoticeable amount.
- The following graph illustrates this fact

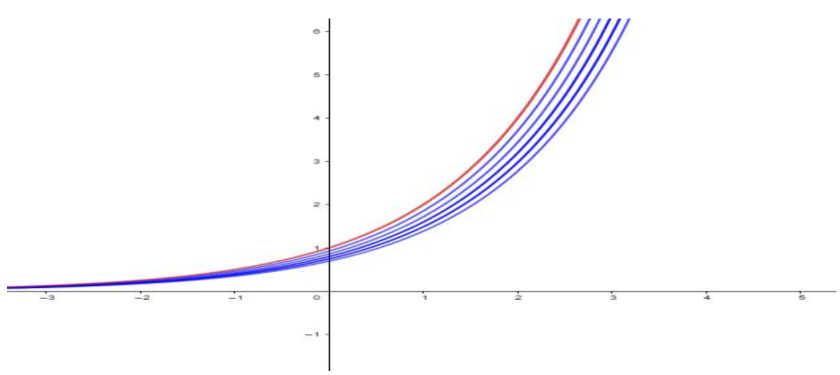


Figure 11. Family of average rate of change functions of $f(t) = 2^x$

Problem 4. Open GG2 and explore the graph of any function by inserting any function in the open dialog box.

a. After inserting any function in the open dialog box drag h from right to left and record what do you notice on the varying value of $g_h(x)$.

Answer

a. When students drag h from left to right, they can record any value of$g_h(x)$.

b. As you vary h by a very small amount what mathematical relationship revealed between the function that you inserted and the average rate of change function $g_h(x)$.

Answer

- When you slide or vary value of h from left to right the average rate of change of the function for a very small h=0.0000001 will almost behave like the derivative of the function f(x), that is, for a very small change in x or in a bit change in x or for a very unnoticeable value of h the average rate of change of the function $f(x)$ is the derivative of the function.
- When you slide h from left to right the average rate of change of the function will almost behave like the derivative of the given function f(x) or it is a

derivative function, for a very small h=0.0000001. That is, for a very small change in x or for a bit change in x or for a very unnoticeable value of h the average rate of change of the function f(x) is same as the derivative of the function.

- Students may relegalize and understand that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change $g_h(x)$. This may be due to the fact that students develop a smooth continuous covariational reasoning or their awareness of both x and $g_h(x)$varying smoothly and continuously as size of h is a bit or small amount.

Appendix F: Post-Instruction Interview Questions and Protocol

The following sample or hypothetical interview questions guided the post-instruction interview questions. The design of the interview questions is guided by the theoretical framework of this study (see Appendix A).

1. Sample questions that will examine students' variational reasoning for problem 1 below:

- By how much does this quantity varies? What do you notice when this quantity varies by unnoticed amount? By how much does it vary? Small amount or unnoticed amount? Can you justify your reasoning by drawing graph or anything you can?
- How does the two quantities vary? What do you notice about their variations?
- By how much does the variable value vary in this interval? Explain your reason.

2. Sample questions that will examine students' covariational and smooth continuous covariational reasoning for problem 2 below:

- By how much does the change in one variable changes the other variable?
- What happen when two variables change by unnoticed amount simultaneously? What are the results of the quotient of the two unnoticeable changing quantities?
- What kind of variation do you noticed between these two co-varying quantities? Can you justify or reason out about this?

Problem 1. You have 240 feet of fence to enclose a rectangular lawn. You are free to make the enclosure have any possible length and width, but you must use all the fence.

Play the GeoGebra (GG) animation applet2. Drag point D to the left and to the right to see how the enclosed rectangular area vary as the width and the length of the rectangle varying.

a. Define the constant variable in this situation.
b. Define the varying variables in this situation. State the intervals over which they vary.
c. When the width of the rectangle varies by 10 feet between the interval What relationship do you notice between the rectangle's enclosed area and its width? What can you conclude from the relationship? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not.
d. When the width of the rectangle varies smoothly and continuously between the interval (in all points in the interval). What relationship do you notice between the rectangle's enclosed area and its width? How does the area of the rectangle vary as the width of the rectangle varies all intermediate values within the given interval? Is it possible to make a table that shows all possible values of the variables that this relationship produces? If so, make the table, if not, explain why not.

Make graphs for the situation in part (c) and (d). Did you get the same graph for part c and d? If you found different graphs, explain why this is occurred

Problem 2. Suppose $d=f(t)=2t^2$ represents the distance (measured in meters) of a car from a stop sign in terms of the number of seconds *t* since the car started to move away from the stop sign.

a. Determine the average rate of change of the distance of the car from the stop sign on the time intervals from $t = 0$ to $t = 1.5$ seconds.

b. Describe what this average rate of change function is tells you about the change in distance of the car from the stop sign over the time interval from $t = 0$ to $t = 1.5$seconds.

c. Determine $r_h(t)$ when $h = 0.5$ seconds and describe what this tells you about the change in the distance of the car from the stop sign?

d. Determine $r_h(t)$ when $h = 0.1$ seconds and describe what this tells you about the change in the distance of the car from the stop sign.

e. Sketch a graph of $r_h(t)$from the stop sign in terms of number of seconds since the car stared to travel for $h = 0.5$, $h = 0.4$, $h = 0.3$, $h = 0.2$, $h = 0.1$, $h = 0.001$ and $h = 0.00001$ seconds. What can you say about the graph of $r_h(t)$ when $h = 0.000001$ seconds? Explain the graph of $r_h(t)$

Note. The formula $r_h(t) = \frac{f(t+h)-f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$ gives the average rate of change for $d = f(t)$ with respect to t over any subinterval from t to $t + h$, where $h \neq 0$ is the length of the interval on which $t$varies.