A Categorical Equivalence between Digraphs and Tripartite Posets

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ABSTRACT

In this thesis we construct a categorical equivalence between the category of quivers and a the category whose objects consist of a subset of tripartite posets. This result takes ideas from Tucker Dowell's thesis: The Category of Finite Incidence Posets[2] and applies them in a new arena.

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CHAPTER 1

INTRODUCTION

We are motivated to construct our categorical equivalence by a desire to apply topologies defined on quivers(as well as digraphs and simple graphs) onto posets and vice versa. In particular, we seek to investigate the results of applying the compatible edge and incompatible edge topologies onto a quiver's equivalent poset, introduced by Khalid Abdulkalek Abdu and Adem Kilicman[1].

In this thesis most of the mathematics will be orignial and all original work was completed under the supervision of Dr.James Hart. All proofs provided are my own. Many of our definitions are standard and are taken from source material. In particular, Introduction to Graph Theory[7], Graph Symmetry: Algebraic Methods and Applications[4], and Quiver Representations and Quiver Varieties[5] will be used as sources for our graph theory definitions. Introduction to Order Theory[3] will be our source for order theory, and Categories for the working Mathematician[6] will be used as our source for category theory. The three previously mentioned resources will provide background for these three fields. We attempt to use the most mainstream vocabulary and definitions in a changing environment. It should be noted that we will not be using any advanced results or topics here. Only an elementary understanding of each topic is necessary for the work provided.

In Chapter 2 we present all background information necessary to understand the thesis. We assume the reader has familiarity with the properties of functions and sets.

In Chapter 3 we build a pair of functors and then show those functors yield a categorical equivalence on our yet-to-be defined categories.

CHAPTER 2

BACKGROUND

2.1 Graph Theory

It is common in elementary Graph Theory to introduce simple graphs before digraphs, so we will follow suit here.

Definition 2.1 *[7] Simple Graph: A simple graph G consists of a non-empty finite set $V(G)$ of elements called vertices and a finite set $E(G)$ of elements called edges. An edge vw is said to join the vertices v and w. Note that vw can also be represented by wv here. For example, Fig X represents the simple graph G whose vertex set $V(G)$ is $\{u, v, w, z\}$, and whose edge set $E(G)$ consists of the edges uv, uw, vw, and wz.*

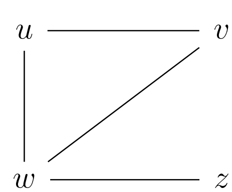

Figure 1: A Simple Graph

We say that two distinct vertices v and w are adjacent when an edge joins v and w, or when $vw \in E(G)$, and write $v \sim w$. We say that that an edge is incident with a vertex whenever that vertex is a member of the pair that defines the given edge. The degree of a vertex is the number of edges incident with that vertex.

We will complete our discussion of simple graphs with a definition of simple graph homomorphisms and, more importantly, simple graph isomorphisms.

Definition 2.2 *[4] Simple Graph Homomorphism: Let G and H be simple graphs. A simple graph homomorphism ϕ from G to H is a function from $V(G)$ to $V(H)$ such that for all $v, w \in V(G), v \sim w \Longrightarrow \phi(v) \sim \phi(w)$.*

If ϕ is a bijection, then we call ϕ a simple graph isomorphism.

As we see in other fields, the homomorphism preserves some of the structure of the input graph, while the isomorphism perfectly preserves the structure of the input graph. Below is an example of a simple graph homomorphism.

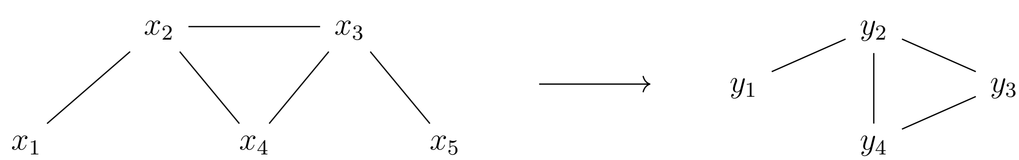

Figure 2: A Simple Graph Homomorphism

Here $\phi(x_1) = y_1, \phi(x_2) = y_2, \phi(x_3) = y_3$ and $\phi(x_4) = \phi(x_5) = y_4$. Effectively, ϕ combines the vertices x_4 and x_5 while retaining each vertex's corresponding edges. It is easy to guess that an isomorphism would produce an exact copy of the input graph, with possibly a relabeling of vertices.

Now that we've completed our discussion on simple graphs, we can generalize to the pertinent case of quivers. We define a quiver as follows.

Definition 2.3 *[5] Quiver: A quiver $G = (V(G), A(G), s, t)$ is an ordered quadruple consisting of a non-empty finite set $V(G)$ of elements called vertices, a finite set $A(G)$ of ordered pairs called arrows, as well as a pair of maps s and t called the source map and target map of G respectively. We call $V(G)$ the vertex set and $A(G)$ the arrow set of G. For an arrow a, we call $s(a)$ the source of a and $t(a)$ the target of a.*

We say that the vertex v "points" to a vertex w if there exists an $a \in A(G)$ such that $s(a) = v$ and $t(a) = w$. Here our arrows take the place of edges in simple graphs. Unlike edges, each arrow has a direction. It may be the case that v points to w and still be true that w does not point to v. Some examples of quivers will be shown below.

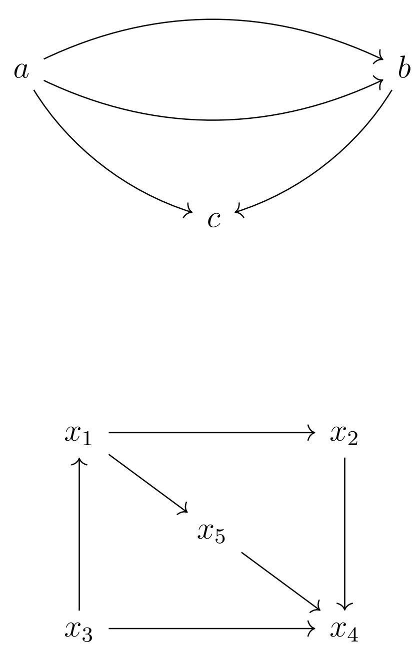

Figure 3: Two Examples of Quivers

It is common to see slightly different definitions for graphs like these from different authors. Our definition allows for cases in which a vertex points to itself. For example it could be the case that there exists $a \in A(G)$ such that $s(a) = t(a) = v$. We will refer to these types of arrows as loops from this point forward. We will also allow distinct arrows to share a source and a target. Arrows of this type are sometimes called parallel arrows.

We previously introduced homomorphisms on simple graphs. Below we define homomorphisms on quivers.

Definition 2.4 *[5] Quiver Homomorphism: Let $G = (V(G), A(G), s_1, t_1)$ and $H = (V(H), A(H), s_2, t_2)$ be quivers. A quiver homomorphism $\phi = (\phi_V, \phi_A)$ is a pair of maps such that the following hold.*

1.) $\phi_V : V(G) \to V(H)$
2.) $\phi_A : A(G) \to A(H)$
3.) $\forall a \in A(G), \phi_V(s_1(a)) = s_2(\phi_A(a))$
4.) $\forall a \in A(G), \phi_V(t_1(a)) = t_2(\phi_A(a))$

Conditions 3.) and 4.) simply require that that our vertex and arrow mappings preserve the source and targets of a given arrow. We are primarily concerned with quiver isomorphisms. Just like simple graph isomorphisms, quiver isomorphisms are effectively relabelings of a given quiver's vertices and arrows. However, we will define quiver isomorphisms more precisely.

Definition 2.5 *[5] Quiver Isomorphism: Two quivers $G = (V(G), A(G), s_1, s_2)$ and $H = (V(H), V(A), s_2, t_2)$ are isomorphic as quivers provided there is a quiver isomorphism betweem them. A quiver isomorphism is a pair (ϕ, ψ) where*

1.) $\phi = (\phi_V, \phi_A)$ is a quiver homomorphism from G to H

2.) $\psi = (\psi_V, \psi_A)$ *is a quiver homomorphism from* H *to* G

3.) $\psi_V \circ \phi_V = 1_{V(G)}$ *and* $\phi_V \circ \psi_V = 1_{V(H)}$

4.) $\psi_A \circ \phi_A = 1_{A(G)}$ *and* $\phi_A \circ \psi_A = 1_{A(H)}$

2.2 Order Theory

Repeatedly in mathematics we come across cases in which groups of objects are "ordered" in some fashion. The most popular case of this is when comparing two numbers, specifically how "big" they are in relation to one another. When x is smaller than y we say that x is less than y and write $x < y$. When comparing two sets U and V, we might say that U is a subset of V and write $U \subset V$. Since statements like these are both extremely common and important across many different fields, we seek to generalize this idea.

Definition 2.6 *[3] Partially Ordered Set: A partially ordered set, or "poset" is a pair $(P, \leq)$ satisfying the following:*

1.) For all $x \in P, x \leq x$ (reflexivity)
2.) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)
3.) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

We call the binary relation $\leq$ a partial ordering on P. It is important to remember that $\leq$ is itself a set containing ordered pairs of members of P. Letting $p, q \in P$, we say that $p \leq q \iff (p, q) \in \leq$. This is easy to forget, as in practice we typically think of the relation symbol as sort of function that compares two things and outputs a truth value.

The "partial" in partial ordering refers to the fact that every pair of members of P need not be comparable under $\leq$. For any two real numbers x and y, it is either the case that $x \leq y$ or that $y \leq x$. However this is rarely the case in common orderings.

Let $A = \{1, 2, 3\}$ and $P(A)$ be the set of all subsets of A. Then $(P(A), \subseteq)$ forms a partial ordering on $P(A)$. Certainly, $\{1, 2\}$ and $\{2, 3\}$ are members of $P(A)$. However, we cannot say that $\{1,2\} \subseteq \{2,3\}$ or that $\{2,3\} \subseteq \{1,2\}$. In cases like this, we say that $\{1,2\}$ and $\{2,3\}$ are incomparable under $\subseteq$ and write $\{1,2\}||\{2,3\}$. A poset that contains no pairs of incomparable elements is called a chain. The real numbers form a chain under "less than or equal to", as mentioned previously. Next we assemble all necessary definitions and terms for future results.

Definition 2.7 *[3] Let $(P, \leq)$ be a poset and let $X \subseteq P$. The set*

$$\downarrow X = \{p \in P : p \leq x \textit{ for some } x \in X\}$$

is called the lowerset generated by X. Similarly, the set

$$\uparrow X = \{p \in P : x \leq p \textit{ for some } x \in X\}$$

is called the upperset generated by X.

A lowerset or upperset generated by a singleton $\{x\}$ is called a principal lowerset or upperset respectively. We denote the principal lowerset generated by $\{x\}$ as $\downarrow x$ instead of $\downarrow \{x\}$. We can think of $\downarrow x$ as the set of all members of P that are less than x, and $\uparrow x$ as the set of all members of P that are greater than x. Note that x is a member of $\downarrow x$ and $\uparrow x$. These uppersets and lowersets will be crucial in our later definitions of quiver posets.

Definition 2.8 *[3] Let $(P, \leq)$ be a poset. We say that x is minimal in P provided $\downarrow x = \{x\}$ and we say that x is maximal in P provided $\uparrow x = \{x\}$*

In other words, x is maximal in P if nothing is greater than x in P, and x is minimal in P if nothing is less than x in P.

We have casually discussed chains previously. For completeness we provide a definition for chains, and their opposites, antichains.

Definition 2.9 *[3] A poset $(P, \leq)$ is said to be a chain (or totally ordered) if every element of P is comparable to every other element of P. That is to say, for any $x, y \in P$, $x \leq y$ or $y \leq x$.*

Definition 2.10 *[3] A poset $(P, \leq)$ is said to be an antichain if P contains no pairs of distinct, comparable elements. That is to say, for any $x, y \in P$, $x \leq y \iff x = y$.*

Previously we defined graph homomorphisms, which are mappings that preserve some of the structure of a given graph. We are likewise interested in structure-preserving mappings between posets. We call these mappings order homormorphisms.

Definition 2.11 *[3] Let $(P, \leq)$ and $(Q, \preceq)$ be two posets. A mapping $f : P \to Q$ is called an order homomorphism or an isotone function given that for all $x, y \in P$*

$$x \leq y \Longrightarrow f(x) \preceq f(y)$$

We say that P and Q are isomorphic as posets when there exists a bijective order homomorphism $f : P \to Q$ whose inverse is also an order homomorphism.

It is difficult to work with posets without some tidy diagram that captures the members of the posets and their relationships to each other. We will provide this visual tool soon, but we need one more term before we do so.

Definition 2.12 *[6] Let $(P, \leq)$ be a poset. We say that an element y covers an element x given that $x < y$ and there is no $z \in P$ such that $x < z < y$.*

So, y covers x only when y is "one step above" or "one step greater" than x. We are now ready to define what we call Hasse Diagrams.

Definition 2.13 *[3]Hasse Diagrams are used to represent finite posets. Let $(P, \leq)$ be a finite poset. The members of P will be represented as vertices, while line segments connecting different vertices will represent when one member is less than or greater than another under the given ordering.*

Let $P_1 = P(\{x, y\}$ and $P_2 = P(\{x, y, z\})$ be the two posets containing the power sets of $\{x, y\}$ and $\{x, y, z\}$ under subset inclusion.

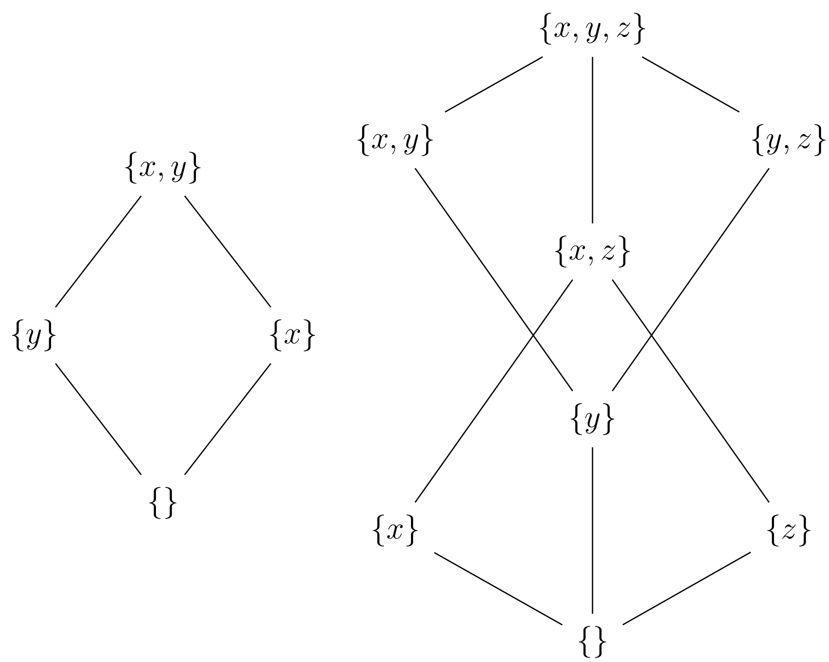

Figure 4: Hasse Diagrams of P_1 and P_2

The above diagrams are the Hasse diagrams for P_1(on the left) and P_2(on the right). We are meant to read the diagram "bottom up". That is to say, when one vertex is connected to another, the lower vertex is less than the upper. In this context, that means that $\{x\} \subset \{x, y\}$ from the diagram for P_1.

Notice there is no line from $\{\}$ to $\{x, y\}$ in P_1. In Hasse diagrams, we only draw a minimal number of line segments. We do not need one directly connecting the empty set and $\{x, y\}$ since there is already a path of line segments connecting them. We sometimes call this a transitive reduction in the context of graph theory. We do not think of these objects as graphs in any formal sense, although the connection between them is clear. It will be important later on to distinguish between a diagram being a graph or a Hasse Diagram of a poset.

2.3 Category Theory

This section will be the longest and most pertinent to the thesis. As mentioned previously, we seek to show a categorical equivalence between the category of quivers and the category of directed quiver posets. Of course, this goal is meaningless until we provide the relevant definitions. We will first provide some much needed intuition.

It is common to see that distinct mathematical structures act very similarly, or even identitically, whilst having radically different origins and representations. For example, D_3, the dihedral group of order 6, is group isomorphic to S_3, the permutation group of order 6. This is surprising given that the members and compositions of each group are very different, yet they are "essentially" the same thing from a group theory perspective. Category theory generalizes this concept further, allowing us to say that entire classes of objects are "essentially" the same as opposed to just two objects.

Definition 2.14 *[6] Category: A category C consists of a class of objects, $Obj(C)$, and a class of morphisms, $Hom(C)$, between objects such that:*

1.) For every pair of morphisms $f : A \to B$ and $g : B \to C$, there exists a unique morphism $g \circ f : A \to C$ called the composition of f and g.

2.) For every triplet of morphisms $f : A \to B, g : B \to C$, and $h : C \to D$, we have that $(h \circ g) \circ f = h \circ (g \circ f)$.

3.) For every object A, there exists an identity morphism 1_A such that $f \circ 1_A = f$ and $1_A \circ g = g$ for any morphisms $f : A \to X$ and $g : Y \to A$.

We sometimes call these morphisms between objects mappings instead. We've seen examples of homomorphisms and isomorphisms already; both being examples of structure preserving mappings. A morphism is a generalization of this idea that will change depending on the category we are working in. For example, a category whose objects are topologies would most likely have continuous functions as it's morphisms. If the category contains vector spaces as its objects, the morphisms might be linear transformations.

Definition 2.15 *<u>Notation:</u> Given a category C and $X, Y \in Obj(C)$, we denote $Hom(X, Y)$ as the class containing all C-morphisms from X to Y.*

Once again, we able to represent another concept with a diagram consisting of nodes and line segments between them. Let C be a category with X, Y, and Z as objects in C. Let $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$. We can represent these objects and morphisms like so.

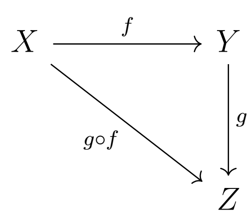

Figure 5: A Categorical Diagram

Here our objects take the form of nodes and morphisms the form of arrows. Essentially, we can represent any finite collection of objects and morphisms as a quiver. It is important to note we may not represent every morphism in a given diagram. By our definition there is an identity morphism from each of X, Y, and Z back to itself. These would be represented as loops around each node, as mentioned previously.

Our goal is to show that two specific categories are effectively identical. We will do this in a similar manner as before. Previously we used isomorphisms to state that two objects are identitical, with isomorphisms being invertible structure preserving functions. We will instead construct the categorical equivalent of an isomorphism, but first we will define the categorical equivalent of a function.

Definition 2.16 *[6] <u>Functor:</u> Let C and D be categories. A functor F from C to D is a mapping that:*

1.) Associates each object X in C to an object $F(X)$ in D

2.) Associates each morhpism in C to a morphism in D such that

a.) If $f \in Hom(X,Y)$, then $F(f) \in Hom(F(X), F(Y))$

b.) $F(I_X) = I_{F(X)}$ for each $X \in Obj(C)$

c.) $F(g \circ f) = F(g) \circ F(f)$ for all $f \in Hom(X,Y)$ and $g \in Hom(Y,Z)$

Functors are distinct from functions in that they map members of classes to members of classes, whereas functions map members of sets to members of sets. We use functors here to avoid any spooky paradoxes, but they are the same idea in spirit. It is helpful to think of morphisms in a given category C to be mappings entirely inside of C, while functors are mappings that pull objects in C outside of C. Morphisms are "internal" mappings and functors are "external" mappings.

It should be noted that the definition above describes what is called a <u>covariant functor</u>. A covariant functor preserves the direction of morphisms in C. This can be seen in part 2a, where f is a morphism **from** X **to** Y in C and $F(f)$ is a morphism **from** $F(X)$ **to** $F(Y)$ in D. Likewise, <u>contravariant functor</u> reverses the order of these morphisms. Suppose F was instead a contravariant functor, then if f was a morphism

from X **to** Y, then $F(f)$ would be a morphism **from** $F(Y)$ **to** $F(X)$. The functors we will define later will all be covariant functors.

The way we define a categorical equivalence will mirror our previous definitions of order and quiver isomorphisms. Recall that these isomorphisms are bijective mappings that preserve structure. To show a categorical equivalence, we will provide a functor between categories that preserves structure in a similar fashion.

Definition 2.17 *[6] Equivalent Categories: A functor $F : C \to D$ yields an equivalence of categories if the following hold*

1.) Let $X, Y \in Obj(C)$. Then for any $g \in Hom(F(X), F(Y))$, there exists an $f \in Hom(X, Y)$ such that $F(f) = g$.

2.) If $f_1, f_2 \in Hom(X, Y)$ and $F(f_1) = F(f_2)$, then $f_1 = f_2$.

3.) For every $Y \in Obj(D)$, there exists an $X \in Obj(C)$ such that $F(X)$ is isomorphic to Y.

We say that a functor F is full if F satisfies requirement 1. and that F is faithful if F satisfies requirement 2. Fullness is the functor equivalent of being surjective along morphisms, while faithfulness is the functor equivalent of being injective along morphisms. Requirement 3 is sometimes referred to as Essential Surjectivity and is the functor equivalent of being surjective along objects.

This idea is the centerpiece of the thesis, as we will be using it to show that the category of quivers is equivalent to the category of Quiver Posets.

CHAPTER 3

RESULTS

Before we begin building our equivalence, we must first define the objects lying inside of our new category.

Definition 3.18 *Quiver Poset: Suppose $P = (P, \leq)$ is a nonempty, finite poset that can be written as the union of maximal chains of length one or three. Let $S(P)$ denote the set of suprema of length-three chains and let $T(P)$ denote the infima of length-three chains. Let $A(P)$ denote the set of elements covered by a member of $S(P)$ (or, equivalently, covering an element of $T(P)$). The maximal singleton chains represent elements that are both maximal and minimal in P. Partition this collection into sets $I_1(P)$ and $I_2(P)$. Let $Max(P) = S(P) \cup I_1(P)$ and $Min(P) = T(P) \cup I_2(P)$. We say P is a quiver poset provided the following conditions are met.*

1.) There exists a bijection $v : Max(P) \to Min(P)$.

2.) For all $a \in A(P)$, the sets $\uparrow a \cap S(P)$ and $\downarrow a \cap T(P)$ are singletons.

Definition 3.19 *Natural Indexing: Suppose (P, v) is a quiver poset and that $|Max(P)| = n = |Min(P)|$. We can relabel the sets $Max(P)$ and $Min(P)$ such that $Max(P) = \{1, ..., n\}, Min(P) = \{x_1, ..., x_n\}$ and $v(i) = x_i$ for each i.*

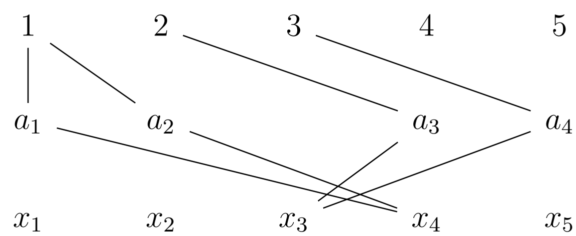

Figure 6: A Quiver Poset (P, v)

Above is an example of a quiver poset, P, under our natural indexing. Here $Max(P) = \{1, 2, 3, 4, 5\}, A(P) = \{a_1, a_2, a_3, a_4\}$ and $Min(P) = \{x_1, x_2, x_3, x_4, x_5\}$. In particular, $S(P) = \{1, 2, 3\}, T(P) = \{x_3, x_4\}, I_1(P) = \{4, 5\}$ and $I_2(P) = \{x_1, x_2, x_5\}$. Our maximal singleton chains are simply the members of P that are incomparable with all other members of P. Every other element will be a member of chain of length three. For example $\{1, a_2, x_4\}$ is such a chain.

It should be noted that we can relabel this quiver poset in such a way that the bijection is less obvious. However, any poset can be relabelled in this more convenient manner. In general, we will display these diagrams in such a way that the bijection mapping is "vertical". For example the following quiver (Q, v) is implied to have the bijection $v : Max(Q) \to Min(Q)$ defined by:

$$v(e) = w, v(f) = x, v(g) = y, v(h) = z$$

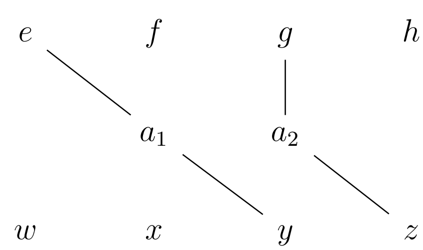

Figure 7: A Quiver (Q, u)

Notation: We are often concerned with the singleton sets $\uparrow a \cap S(P)$ and $\downarrow a \cap T(P)$ for a given quiver poset P and an $a \in A(P)$, as they represent the elements directly above and below our arrow a in P. We will denote the member of $\uparrow a \cap Max(P)$ as $C_P(a)$ and the member of $\downarrow a \cap Min(P)$ as $c_P(a)$.

Definition 3.20 *<u>QP-morphism:</u> Suppose (P, v) and (Q, u) are Quiver Posets. A QP-morphism is an order homomorphism $F : P \to Q$ with the following properties*

1.) $F(Max(P)) \subseteq Max(Q)$
2.) $F(Min(P)) \subseteq Min(Q)$
3.) $F(A(P)) \subseteq A(Q)$
4.) For all $x \in Max(P)$, we have that $F(v(x)) = u(F(x))$
5.) For all $a \in A(P)$ we have that
  a.) $F(C_P(a)) = C_Q(F(a))$ *or* $F(\uparrow a \cap Max(P)) = (\uparrow F(a) \cap Max(Q))$
  b.) $F(c_P(a)) = c_Q(F(a))$ *or* $F(\downarrow a \cap Min(P)) = (\downarrow F(a) \cap Min(Q))$

These $QP-$morphisms are essentially order homomorphisms that respect the bijections equipped to P and Q.

As previously mentioned, we seek to show a categorical equivalence between quivers and our new objects that we have referred to as quiver objects. It is reasonable to first show that these two classes of objects and their morphisms are indeed categories.

Theorem 3.21 *The class **QP** consisting of all quiver posets coupled with QP-morphisms constitutes a category in which morphism composition is function composition.*

Proof: The associativity and uniqueness of morphism composition is guaranteed by the properties of function composition. We define 1_P to be the usual identity mapping on P. This works as the identity morphism on any object $P \in QP$.

Theorem 3.22 *The class* **Q** *consisting of all quivers coupled with quiver morphisms constitutes a category in which morphism composition is component-wise function composition.*

Proof:

Associativity: Let $\alpha, \beta, \gamma \in Hom(Q)$ with

$$\alpha : G_1 \to G_2,\ \beta : G_2 \to G_3, \text{ and } \gamma : G_3 \to G_4$$

$$\text{Then } \gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$

$$\Updownarrow$$

$$\gamma_V \circ (\beta_V \circ \alpha_V) = (\gamma_V \circ \beta_V) \circ \alpha_V \text{ and } \gamma_A \circ (\beta_A \circ \alpha_A) = (\gamma_A \circ \beta_A) \circ \alpha_A$$

Which is guaranteed by the associativity of function composition.

Identity Morphism: Let $G \in Q$ and $\phi_1, \phi_2 \in Hom(Q)$ with

$$\phi_1 : H_1 \to G \text{ and } \phi_2 : G \to H_2.$$

Then

$$1_G \circ \phi_1 = (1_V \circ \phi_{1V}, 1_A \circ \phi_{1A}) = (\phi_{1V}, \phi_{1A}) = \phi_1$$
$$\text{and}$$
$$\phi_2 \circ 1_G = (\phi_{2V} \circ 1_V, \phi_{2A} \circ 1_A) = (\phi_{2V}, \phi_{2A}) = \phi_2$$

Thus, for any $G \in Q$, 1_G acts as an identity morphism on G.

Theorem 3.23 *Every quiver poset* (P, v) *induces a quiver* $G[(P, v)]$, *where*

$$V(G) = \{(i, x_i) : i \in Max(P)\} \text{ and } A(G) = A(P)$$

and the source and target maps are defined by $s(a) = (i, x_i)$ *and* $t(a) = (j, x_j)$ *where*

$$i = C(a) \text{ and } j = c(a).$$

Proof: Let (P, v) be a QP. Let G be the quiver with $V(G) = \{(i, x_i) : i \in Max(P)\}$, $A(G) = A(P)$. Since $Max(P)$ and $A(P)$ are both finite, $V(G)$ and $A(G)$ must also both be finite. These are also clearly disjoint, as $V(G)$ contains only ordered pairs while $A(G)$ contains none. $V(G)$ is nonempty exactly when $Max(P)$ is nonempty.

To show that s and t are functions, we only need to show their outputs are unique, however this follows from the fact that $\uparrow a \cap S(P)$ and $\downarrow a \cap T(P)$ are singletons, or from part 2 of our definition for quiver poset.

For example, let (P, v) be the poset shown below.

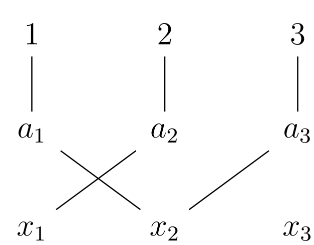

Figure 8: Another Quiver Poset (P, v)

Then $G[(P, v)]$ would be the quiver

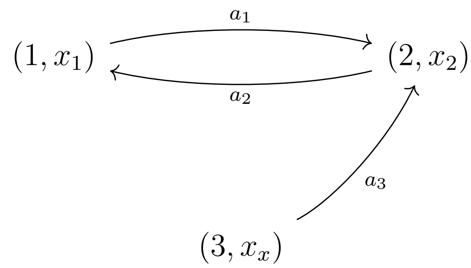

Figure 9: The Quiver $G[(P, v)]$ induced by (P, v)

Theorem 3.24 *Every quiver $G = (V(G), A(G), s, t)$ induces a quiver poset $(P(G), v_G))$ where*

$$Max(P(G)) = \{1, ..., |V(G)|\}$$
$$A(P(G)) = A(G)$$
$$Min(P(G)) = V(G)$$
$$v_G(i) = v_i \text{ for all } i \in Max(P(G))$$

and $\leq$, the partial ordering on $(P(G), v_G)$, is defined by the following rule:

We have that $x = y$ if x and y are the same element. We also have that $x < y$ if and only if one of the following is met.

a.) $y \in Max(P(G))$ and $x \in A(P(G))$ and $s(x) = v_y$.

b.) $y \in A(P(G))$ and $x \in Min(P(G))$ and $t(y) = x$.

c.) $y \in Max(P(G))$ and $x \in Min(P(G))$ and there exists an $a \in A(G)$ such that $s(a) = v_y$ and $t(a) = x$.

Proof: First we show that $P(G)$ is in fact a poset. Since reflexivity is clear, suppose $x, y \in P$ with $x \leq y$ and $y \leq x$. Suppose further that $x \leq y$ and $y \leq x$. By our relation definition, it must be the case that both x and y lie in two of the three disjoint sets. This contradicts $x < y$ and $y < x$. Thus, it must be the case that $x = y$. Hence, our relation is anti-symmetric.

Now suppose that $x, y, z \in P$ with $x < y$ and $y < z$. By our relation definition, it must be the case that $x \in Min(P), y \in A(P)$, and $z \in Max(P)$. By (a), $t(y) = x$ and by (b), $s(y) = v_z$. Then by (c) $x < z$. Thus, our relation is transitive and hence, a partial ordering.

It remains to be shown that $P(G)$ is a quiver poset. Since $A(G)$ and $V(G)$ are both finite sets, $P(G)$ is also finite. To show that $P(G)$ can be written as the union of maximal chains of length 1 or 3, recall that our ordering does not allow chains of length 4 or higher. To show that chains of length 2 cannot exist in $P(G)$, suppose that $\{x, y\}$ is a maximal chain in $P(G)$. Then $x < y$ under one of our three conditions (a), (b), and (c). Notice that (c) implies a contradiction immediately as we could write $\{x, a, y\}$ as a chain in $P(G)$. Now suppose $x < y$ under either (a) or (b). Then either x or y is in $A(P)$. However, all members of $A(P)$ both cover and are covered by elements in $Min(P)$ and $Max(P)$ respectively. These would imply the existence of chains of length three in either case. Thus, $P(G)$ contains no chains of length 2. It follows that $P(G)$ can be written as the given union.

Alternatively, define $E_M = \{x \in Max(P) : c(x) = \emptyset\}$ and $E_m = \{x \in Min(P) : C(x) = \emptyset.\}$ Then we can write

$$P(G) = \bigcup_{i=1}^{n} [\bigcup_{a_k \in s^{-1}(v_i)} \{i, a_k, t(a_k)\}] \bigcup E_M \bigcup E_m = A.$$

For any $i \in \{1, ..., n\} = Max(P)$, and for any $a_k \in s^{-1}(v_i)$, $\{1, a_k, t(a_k) \subset P(G)$. We've also defined E_M and E_m such that they are subsets of $P(G)$ as well. So,

$\cup_{i=1}^{n}[\cup_{a_k \in s^{-1}(v_i)}\{i, a_k, t(a_k)\}] \cup E_M \cup E_m \subseteq P(G)$. Now suppose that $x \in P(G)$. If $x \in A(P)$, then $s(x) = v_i$ for some i. Thus, $x \in A$. Now suppose $x \in Max(P)$. Then $x \in E_M$ or $c(x) = a_i$ for some i. So, $s(a_i) = x \Longrightarrow a_i \in s^{-1}(x) \Longrightarrow \{a_i, x, t(a_i)\} \subset A \Longrightarrow x \in A$. Similarly, suppose that $x \in Min(P)$. Then $x \in E_m$ or $C(x) = a_i$ for some i. So, $t(a_i) = x$. Then $a_i \in s^{-1}(v_j)$ for some j. So, $\{a_i, i, x\} \subset A \Longrightarrow x \in A$. Thus, $P(G) = \bigcup_{i=1}^{n}[\bigcup_{a_k \in s^{-1}(v_i)}\{i, a_k, t(a_k)\}] \bigcup E_M \cup E_m$, as desired.

Assuming that $V(G) = \{v_1, ..., v_n\}$, gives us that $Max(P(G)) = \{1, ..., n\}$. As previously defined, the natural bijection $v_G(i) = v_i$ for all $i \in Max(P(G))$.

Finally, we show that for all $y \in A(P)$, the sets $\uparrow y \cap S(P)$ and $\downarrow y \cap T(P)$ are singletons. From our order definition, we can deduce that $\uparrow y = \{y, s(y)\}$ and $\downarrow y = \{y, t(y)\}$. Clearly $y \notin S(P)$ and $y \notin T(P)$. Since s and t are functions mapping into $S(P)$ and $T(P)$, we have that $\uparrow a \cap S(P) = \{s(y)\}$ and $\downarrow a \cap T(P) = \{t(y)\}$, proving the result.

Again we provide an example. Let G be the quiver shown below.

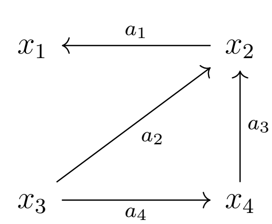

Figure 10: Another Quiver G

Then we would represent $(P(G), v_G)$ by

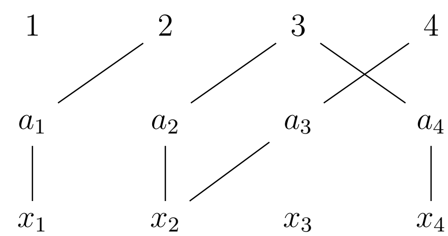

Figure 11: The Quiver Poset $(P(G), v_G)$ induced by G

Next, we are able to define the functors that will yield the equivalence of categories.

Definition 3.25 *The mapping P_O:*

Define a mapping $P_O : Obj(Quiv) \to Obj(QP)$ as follows. For each quiver $G = (V(G), A(G), s_1, t_1)$, let $P_O[G] = (P(G), v_G)$, as defined in Theorem 3.23.

Definition 3.26 *The Mapping Q_O:*

Define a mapping $Q_O : Obj(QP) \to Obj(Quiv)$ as follows: For each quiver poset (P, v), let $Q_O[(P, v)] = G[(P, v)]$

These will be the object maps on our functors defined later on. Now we show that each of these maps are well-defined up to isomorphism classes. We will need a few lemmas to do this.

Lemma 3.27 *Suppose (P, v) is a quiver poset with induced quiver $G[(P, v)]$. The induced quiver poset $(P(G[(P, v)]), v_G)$ is isomorphic to (P, v).*

Proof: Suppose that $P = Max(P) \cup A(P) \cup Min(P)$ with $Max(P) = \{1, ..., n\}, A(P) = \{a_1, ..., a_m\}$, and $Min(P) = \{v_1, ..., v_n\}$. Then $G[(P, v)] = G$ has $V(G) = \{(1, v_1), ..., (n, v_n)\}$ and $A(G) = \{a_1, ..., a_m\}$. For clarity, relabel (i, v_i) as w_i for all i. Then let $(P(G[(P, v)]), v_G) = (Q, u) = Max(Q) \cup A(Q) \cup Min(Q)$, with $Max(Q) = \{1', ..., n'\}, A(Q) = A(G) = A(P)$, and $Min(Q) = \{w_1, ..., w_n\}$. Let $\lambda, \xi \in Hom(QP)$ be defined by

$$\lambda : P \to Q \text{ and } \xi : Q \to P$$
with
$$\lambda(i) = i', \lambda(a_i) = a_i, \lambda(v_i) = w_i \text{ and}$$
$$\xi(i') = i, \xi(a_i) = a_i, \xi(w_i) = v_i.$$

First we must verify that λ and ξ are indeed in $Hom(QP)$. Our first three requirements are clear. To show (4), Suppose $i \in Max(P)$. Then $\lambda(v(i)) = \lambda(v_i) = w_i = u(i') = u(\lambda(i))$. Now suppose that $i' \in Max(Q)$. Then $\xi(u(i')) = \xi(w_i) = v_i = v(i) = v(\xi(i'))$. To show (5), note that

$$C_P(a) = i \Longleftrightarrow s(a) = (i, v_i) = w_i \Longleftrightarrow C_Q(a) = i'$$
and that
$$c_P(a) = i \Longleftrightarrow s(a) = (i, v_i) = w_i \Longleftrightarrow c_Q(a) = i'$$

So, let $a \in A(P)$ and suppose $C_P(a) = i$. Then $\lambda(C_P(a)) = \lambda(i) = i' = C_Q(a) = C_Q(\lambda(a))$. Similarly, suppose $c_P(a) = j$. Then $\lambda(c_P(a)) = \lambda(j) = j' = c_Q(a) = c_Q(\lambda(a))$. Since $\xi(a) = \lambda(a)$ for all a, we can replace λ with ξ here. Thus, λ and ξ are QP-morphisms.

Lemma 3.28 *Suppose $G = (V(G), A(G), s_1, t_1)$ is a quiver with induced quiver poset $(P(G), v)$. The induced quiver $G[(P(G), v)]$ is isomorphic to G.*

Let $G[(P(G), v)] = H = (V(H), A(H), s_2, t_2)$ and suppose that $V(G) = \{v_1, ..., v_n\}$ and $A(G) = \{a_1, ..., a_m\}$. Then $(P(G), v) = Max(P(G)) \cup A(P(G)) \cup Min(P(G))$, where $Max(P(G)) = \{1, ..., n\}, A(P(G)) = A(G)$, and $Min(P(G)) = \{v_1, ..., v_n\}$. Likewise, $V(H) = \{(1, v_1), ..., (n, v_n)\}$ and $A(H) = \{a_1, ..., a_n\}$.. Let ρ, ϑ be mappings defined by

$$\rho : G \to H \text{ and } \vartheta : H \to G$$
with
$$\rho_V(v_i) = (i, v_i), \rho_A(a_i) = a_i, \vartheta_v(i, v_i) = v_i, \text{ and } \vartheta_A(a_i) = a_i$$

First we must verify that ρ and ϑ are indeed in $Hom(Quiv)$. Let $a_i \in A(G)$. From our construction, we know that

$$s_1(a_i) = v_j \Longleftrightarrow j = C_{P(G)}(a_i) \Longleftrightarrow s_2(a_i) = (j, v_j)$$

and

$$t_1(a_i) = v_k \Longleftrightarrow k = c_{P(G)}(a_i) \Longleftrightarrow t_2(a_i) = (k, v_k)$$

Let $a_i \in A(G)$ and suppose that $s_1(a_i) = v_j$ Then $\rho_V(s_1(a_i)) = \rho_V(v_j) = (j, v_j) = s_2(a_i) = s_2(\rho(a_i))$. Similarly, suppose that $t_1(a_i) = v_k$. Then $\rho_V(t_1(a_i)) = \rho_V(v_k) = (k, v_k) = t_2(a_i) = t_2(\rho_A(a_i))$. Thus, ρ is a quiver morphism.

Let $a_i \in A(H)$ and suppose that $s_2(a_i) = (j, v_j)$. Then $\vartheta_V(s_1(a_i)) = \vartheta_V((j, v_j)) = v_j = s_1(a_i) = s_1(\vartheta_A(a_i))$. Similarly, suppose that $t_2(a_i) = (k, v_k)$. Then $\vartheta_V(t_2(a_i)) = \vartheta_V(k, v_k) = v_k = t_1(a_i) = t_1(\vartheta_A(a_i))$. Thus, ϑ is also a quiver morphism.

It is clear that (ϑ_v, ρ_v) and (ϑ_A, ρ_A) form inverse pairs. Hence, G is isomorphic to $H = G[P(G), v)]$, as desired.

Lemma 3.29 *Suppose* (P, v) *and* (Q, u) *are quiver posets. If the induced graphs* $G[(P, v)]$ *and* $G[(Q, u)]$ *are isomorphic, then* (P, v) *and* (Q, u) *are also isomorphic.*

Proof: Suppose that $G(P)$ and $G(Q)$ are isomorphic, where $V(G(P)) = \{(i, v_i) : i \in Max(P)\}$, $A(G(P)) = A(P)$, $V(G(Q)) = \{(j, u_j) : j \in Max(Q)\}$, and $A(G(Q)) = A(Q)$. Then there exists an invertible quiver-morphism $\phi : G(P) \to G(Q)$. Define $F : P \to Q$ by

$$F(i) = j, \text{ when } \phi_v(i, v_i) = (j, u_j)$$
$$F(v_i) = u_j, \text{ when } \phi_v(i, v_i) = (j, u_j) \text{ and}$$
$$F(a_k) = b_l, \text{ when } \phi_a(a_k) = b_l.$$

Suppose $i \in Max(P)$ and that $\phi_V(i, v_i) = (j, u_j)$. Then $F(v(i)) = F(v_i) = u_j = u(j) = u(F(i))$. Now suppose that $a_k \in A(P)$. Notice that $F(C_P(a_k)) = j \iff \phi_v(s_1(a_k)) = (j, u_j)$ and that $C_Q(F(a_k)) = j \iff s_2(\phi_A(a_k)) = (j, u_j)$. Then $F(C_P(a_k)) = j \iff \phi_v(s_1(a_k)) = (j, u_j) = s_2(\phi_A(a_k)) \iff C_Q(F(a_k)) = j$. So, for all $a_k \in A(P)$, $C_Q(F(a_k)) = F_P(C(a_k))$. Thus, F is a QP-morphism.

Because ϕ_A and ϕ_V are bijections, so is F. Define $H : Q \to P$ to be the inverse of F. Specifically,

$$H(j) = i, \text{ when } \phi_V(i, v_i) = (ju_j)$$
$$H(u_j) = v_i, \text{ when } \phi_V(i, v_i) = (j, u_j) \text{ and}$$
$$H(b_l) = a_k, \text{ when } \phi_A(a_k) = b_l$$

Suppose $j \in Max(Q)$ and that $\phi_V(i, v_i) = (j, u_j)$. Then $H(u(j)) = H(u_j) = v_i = v(i) = v(H(j))$. Now suppose that $a_k \in A(Q)$. Notice that $H(C_Q(a_k)) = i \iff \phi_V^{-1}(s_2(a_k)) = (i, v_i)$ and that $C_P(H(a_k)) = i \iff s_1(\phi_A^{-1}(a_k)) = (i, v_i)$. Then $H(C_Q(a_k)) = i \iff \phi_V^{-1}(s_2(a_k)) = (i, v_i) = s_1(\phi_A^{-1}(a_k)) \iff C_P(H(a_k)) = i$. So, for all $a_k \in A(Q)$, $C_P(H(a_k)) = H(C_Q(a_k))$. Thus, H is also a QP-morphism. Since F and H are inverse morphisms, they form an isomorphism on P and Q, as desired.

Lemma 3.30 *Suppose that* $G = (VG), A(G), s_1, t_1)$ *and* $H = (V(H), A(H), s_2, t_2)$ *are quivers and that* $(P(G), v)$ *and* $(P(H), u)$ *are their quiver posets. If* $(P(G), v)$ *and* $(P(H), u)$ *are isomorphic, then* G *and* H *are also isomorphic.*

Proof: Suppose that $V(G) = \{v_1, ..., v_n\}, V(H) = \{u_1, ..., u_n\}, A(G) = \{a_1, ..., a_m\}$, and that $A(H) = \{b_1, ..., b_m\}$. Then there exists an invertible QP-morphism $F : P(G) \to P(Q)$ such that $C_Q(F(a)) = F(C_P(a))$ and $c_Q(F(a)) = F(c_P(a))$ for all $a \in A(P(G))$. Define $\phi = (\phi_V, \phi_A)$ as follows

$$\phi_V(v_i) = u_j, \text{ when } F(i) = j \iff F(v_i) = u_j \text{ and}$$
$$\phi_A(a_k) = b_l, \text{ when } F(a_k) = b_l.$$

It remains to be shown that ϕ is a quiver morphism. Suppose that $a_k \in A(G)$. Then $s_2(\phi_A(a_k)) = u_j \iff C_Q(F(a_k) = j = F(C_P(a_k)) \iff \phi_V(s_1(a_k)) = u_j$. Thus, $s_2(\phi_A(a)) = \phi_V(s_1(a))$ for all $a \in A(G)$.

We also have that $t_2(\phi_A(a_k)) = u_j \iff c_Q(F(a_k)) = j = F(c_P(a_k)) \iff \phi_V(s_1(a_k)) = u_j$. Thus, $t_2(\phi_A(a_K)) = \phi_V(t_1(a_k))$ for all $a_k \in A(G)$.

Hence, ϕ is a quiver morphism. We define γ to be the inverse of ϕ. Specifically, $\gamma = (\gamma_V, \gamma_A)$ defined by,

$$\gamma_V(u_j) = v_i, \text{ when } F(i) = j \iff F(v_i) = u_j \text{ and}$$
$$\gamma_A(b_l) = a_k, \text{ when } F(a_k) = b_l.$$

To show γ is a quiver morphism, suppose that $a_k \in A(H)$. Then $s_1(\gamma_A(a)) = v_i \iff C_P(F^{-1}(a_k)) = i = F^{-1}(C_Q(a_k) \iff \gamma_V(s_2(a_k))$. Thus, $s_1(\gamma_A(a_k)) = \gamma_V(s_2(a_k))$ for all $a_k \in A(H)$.

We also have that $t_1(\gamma_A(a_k)) = v_i \iff c_P(F^{-1}(a_k)) = i = F^{-1}(c_Q(a_k)) \iff \gamma_V(t_2(a_k)) = v_i$.Thus, $t_1(\gamma_A(a_k)) = \gamma_V(t_2(a_k))$ for all $a_k \in A(H)$. So, γ is also a quiver morphism. Since ϕ and γ are inverse morphisms, they form an isomorphism on G and H, as desired.

We can now begin the process of showing that P_O and Q_O, defined in **Definition 3.25** and **Definition 3.26** respectively, are well-defined. This is shown in the following corollary.

Corollary 3.31 *The following statements are true.*

1.) If (P, v) and (Q, u) are isomorphic QPs, then their induced quiver's $G(P)$ and $G(Q)$ are also isomorphic.

2.) If $G = (V(G), A(G), s_1, t_1)$ and $H = (V(H), A(H), s_2, t_2)$ are isomorphic quivers, then their induced QP's $P(G)$ and $P(H)$ are isomorphic.

Proof of 1: Suppose that P and Q are isomorphic QP's. Let $G(P)$ and $G(Q)$ be the quivers induced by P and Q. Then let P' and Q' be the QP's induced by $G(P)$ and $G(Q)$. By Theorem 3.26, P is isomorphic to P' and Q is isomorphic to Q'. So, P' is also isomorphic to Q'. By Theorem 3.29, $G(P)$ must be isomorphic to $G(Q)$.

Proof of 2: Suppose that G and H are isomorphic quivers. Let $P(G)$ and $P(H)$ be the QP's induced by G and H. Then let G' and H' be the quivers induced by $P(G)$ and $P(H)$. By Theorem 3.27 G' is isomorphic to G, and H' is isomorphic to H. So, G' is also isomorphic to H'. By Theorem 3.28, it must be that $P(G)$ is isomorphic to $P(H)$.

This shows that each of our functors' object mappings are well-defined up to isomorphism classes on each of their appropriate categories. Next we will begin defining our morphism maps.

Definition 3.32 *Induced QP-morphism: Suppose (P, v) and (Q, u) are QP's and suppose ϕ is a Q-morphism from $G[(P, v)]$ to $G[(Q, u)]$. Note that $x \in V(G[(P, v)])$ if and only if $x = (i, v(i))$ for exactly one $i \in Max(P)$. With this in mind, define a mapping $F^{(\phi)} : P \to Q$ as follows. Assume the "natural indexing on the minimal elements of P and Q.*

1.) For all $a \in A(P)$, *let* $F^{(\phi)}(a) = \phi_A(a)$

2.) For all $i \in Max(P)$, *let* $F^{(\phi)}(i) = j \iff \phi_V((i, x_i)) = (j, y_j)$

3.) For all $x_i \in Min(P)$, *let* $F^{(\phi)}(x_i) = y_j \iff \phi_V((i, x_i)) = (j, y_j)$

Theorem 3.33 *The mapping* $F^{(\phi)}$ *is a QP-morphism.*

Proof: As before, the first three requirements are clear. To show (4), suppose $i \in Max(P)$ and that $\phi_V(i, x_i) = (j, y_j)$. Then $F^{(\phi)}(v(i)) = F^{(\phi)}(x_i) = y_j = u(j) = u(F^{(\phi)}(i))$.

To show (5), suppose that $a \in A(P)$. Let $G[(P, v)]$ have source and target maps (s_1, t_1) and let $G[(Q, u)]$ have source and target maps (s_2, t_2). Then $F^{(\phi)}(C_P(a)) = j \iff \phi_V(s_1(a)) = (j, y_j) = s_2(\phi_A(a)) \iff C_Q(F^{(\phi)}(a)) = j$. Thus, $F^{(\phi)}(C_P(a)) = C_Q(F^{(\phi)}(a)$ for all $a \in A(P)$. Similarly, $F^{(\phi)}(c_P(a)) = y_j \iff \phi_V(t_1(a)) = (j, y_j) = t_2(\phi_A(a)) \iff c_Q(F^{(\phi)}(a)) = y_j$. Thus, $F^{(\phi)}$ is a QP-morphism.

Definition 3.34 *Suppose* $G = (V(G), A(G), s, t)$ *is a quiver and that* $G' = G[(P(G), v)]$ *Then we may define a pair of mutually inverse Q-morphisms* ρ_G *and* ϑ_G *as follows:*

1.) $\rho_G = (\rho_V, \rho_A) : G \to G'$ *with*

a.) For all $a \in A(G)$ *let* $\rho_A(a) = a$

b.) For all $x_i \in V(G)$ *let* $\rho_V(x_i) = (i, x_i)$

2.) $\vartheta_G = (\vartheta_V, \vartheta_A) : G' \to G$ *with*

a.) For all $a \in A(G')$ *let* $\vartheta_A(a) = a$

b.) For all $(i, x_i) \in V(G')$ *let* $\vartheta_V(i, x_i) = x_i$

These are actually the same mappings from Lemma 3.28 and form a quiver isomorphism on G and G'. We can now define our morphism mapping for the functor Q.

Definition 3.35 *The Morphism Mapping P_M: Define a mapping $P_M : Hom(Quiv) \to Hom(QP)$ as follows. Suppose that $\phi \in Hom(Quiv)$ is a morphism from a quiver G to a quiver H. Then we define $P_M[\phi] = F^{(\rho_H \circ \phi \circ \vartheta_G)}$*

We may now build up our corresponding morphism mapping Q_M in a similar fashion.

Definition 3.36 *Induced Quiver Morphism: Suppose $G = (V(G), A(G), s_1, t_1)$ and $H = (V(H), A(H), s_2, t_2)$ are quivers and suppose F is a QP-morphism from $(P(G), v_G)$ to $(P(H), v_H)$. Construct a pair $\phi^{(F)} = (\phi_V^{(F)}, \phi_A^{(F)})$ as follows.*

1.) For all $a \in A(G)$ let $\phi_A^{(F)}(a) = F(a)$

2.) For all $x_i \in V(G)$, let $\phi_V^{(F)}(x_i) = F(x_i) = F(v_G(i)) = v_H(F(i))$

Theorem 3.37 *The pair $\phi^{(F)}$ is a quiver morphism.*

Proof: Let $a \in A(G)$. Then

$$\phi_V^{(F)}(s_1(a)) = y_j \iff F(C_P(a)) = j = C_Q(F(a)) \iff s_2(\phi_A^{(F)}(a)) = y_j$$

and

$$\phi_V^{(F)}(t_1(a)) = y_j \iff F(c_P(a)) = y_j = c_Q(F(a)) \iff t_2(\phi_A^{(F)}(a)) = y_j$$

So, $\phi^{(F)}(s_1(a)) = s_2(\phi_A^{(F)}(a))$ and $\phi^{(F)}(t_1(a)) = t_2(\phi_A^{(F)}(a))$ for all $a \in A(G)$, making $\phi^{(F)}$ a quiver morphism.

Definition 3.38 *Suppose* (P, v) *is a quiver poset and that* $P' = (P[G((P, v))], v)$ *Then we may define a pair of mutually inverse QP-morphisms* λ_P *and* ξ_P *as follows:*

1.) $\lambda_P : P \to P'$ *with these conditions*

a.) For all $a \in A(P)$ *let* $\lambda_P(a) = a$

b.) For all $i \in Max(P)$ *let* $\lambda_P(i) = i'$

c.) For all $x_i \in Min(P)$ *let* $\lambda_P(x_i) = x_i'$

2.) $\xi_P : P' \to P$ *with these conditions*

a.) For all $a \in A(P')$ *let* $\xi_P(a) = a$

b.) For all $i' \in Max(P')$ *let* $\xi_P(i') = i$

c.) For all $x_i' \in Min(P')$ *let* $\xi_P(x_i') = x_i$

Just as in Definition 3.33 these are the same mappings from Lemma 3.27 and form a quiver poset isomorphism on P and P'. We now define our morphism mapping for the functor Q.

Definition 3.39 *The Morphism Mapping* Q_M*: Suppose that* $F \in Hom(QP)$ *is a* QP*-morphism from a quiver poset* P *to a quiver poset* Q*. Then we define* $Q_O[F] = \phi^{\lambda_Q \circ F \circ \xi_P}$

Since we now have our object and morphism mappings, we are able to formally define our functors.

Definition 3.40 *The Functor* P*:*

1.) Define a mapping $P_O : Obj(Quiv) \to Obj(QP)$ *as follows. For each quiver* $G = (V(G), A(G), s_1, t_1)$*, let* $P_O[G] = (P(G), v_G)$*, as defined in Theorem 3.24.*

2.) Define a mapping $P_M : Hom(Quiv) \to Hom(QP)$ as follows. Suppose that $\phi \in Hom(Quiv)$ is a morphism from a quiver G to a quiver H. Then $P_M[\phi] = F^{(\rho_H \circ \phi \circ \vartheta_G)}$

Definition 3.41 *The Functor Q:*

1.) Define a mapping $Q_O : Obj(QP) \to Obj(Quiv)$ as follows: For each quiver poset (P, v), let $Q_O[(P, v)] = G[(P, v)]$ as defined in Theorem 3.23

2.) Define a mapping $Q_M : Hom(QP) \to Obj(Quiv)$ as follows. Suppose that $F \in Hom(QP)$ is a QP-morphism from a quiver poset P to a quiver poset Q. Then $Q_M[F] = \phi^{(\lambda_Q \circ F \circ \xi_P)}$

In order to prove these pairs of mappings are indeed functors, we first provide two lemmas.

Lemma 3.42 *Suppose $(P, v), (Q, u)$, and (R, w) are QP's. If ϕ is a quiver morphism from $G(P)$ to $G(Q)$ and ψ is a quiver morphism from $G(Q)$ to $G(R)$, then $F^{(\psi \circ \phi)} = F^{(\psi)} \circ F^{(\phi)}$.*

Proof: Let P, Q, and R be as above. Then $F^{(\psi \circ \phi)}$ is a mapping from P to R. We seek to show that $F^{(\psi \circ \phi)} = F^{(\psi)} \circ F^{(\phi)}$. Let $a \in A(P)$. Then by **Definition 3.31**,

$$F^{(\psi \circ \phi)}(a) = (\psi \circ \phi)_A(a) = (\psi_A \circ \phi_A)(a) = \psi_A(\phi_A(a)) = F^{(\psi)}(\phi_A(a)) = F^{(\psi)} \circ F^{(\phi)}(a)$$

Now suppose that $i \in Max(P)$. Also suppose that $\phi_V(i, x_i) = (j, y_j)$ and $\psi_V(j, y_j) = (k, z_k)$. Then we have that $F^{(\psi)} \circ F^{(\phi)}(i) = F^{(\psi)}(j) = k$. Since $(\psi \circ \phi)_V((i, v(x_i)) = (k, w(z_k))$, we also have that $F^{(\psi \circ \phi)}(x_i) = z_k$.

Finally suppose that $x_i \in Min(P)$. Also suppose that $\phi_V(i, x_i) = (j, y_j)$ and $\psi_V(j, y_j) = (k, z_k)$ as before. Then $F^{(\psi)} \circ F^{(\phi)}(x_i) = F^{(\psi)}(y_j) = z_k$. Since $(\psi \circ \phi)_V((i, x_i) = (k, z_k)$, we also have that $F^{(\psi\circ\phi)}(x_i) = z_k$.

Thus, $F^{(\psi\circ\phi)} = F^{(\psi)} \circ F^{(\phi)}$, as desired.

Lemma 3.43 *Suppose G, H, and R are quivers. If F is a QP-morphism from $P(G)$ to $P(H)$ and S is a QP-morphism from $P(H)$ to $P(R)$, then $\phi^{(S\circ F)} = \phi^{(S)} \circ \phi^{(F)}$.*

Proof: Let G, H and R be as above. We seek to show that $\phi^{(S\circ F)} = \phi^{(S)} \circ \phi^{(F)}$. Suppose that $a \in A(G)$. Then

$$\phi_A^{(S\circ F)}(a) = (S \circ F)(a) = S(F(a)) = \phi_A^{(S)}(F(a)) = (\phi_A^{(S)} \circ \phi_A^{(F)})(a)$$

Now suppose that $x_i \in V(G)$. Then

$$\phi_V^{(S\circ F)}(x_i) = (S \circ F)(x_i) = S(F(x_i)) = \phi_V^{(S)}(F(x_i)) = \phi_V^{(S)} \circ \phi_V^{(F)}(x_i).$$

Thus, $\phi^{(G\circ F)} = \phi^{(G)} \circ \phi^{(F)}$.

We are now ready to prove the functorial properties of Q and P.

Theorem 3.44 *The pair $Q = (Q_O, Q_M)$ is a covariant functor.*

Proof: Clearly Q_O and Q_M are mappings with appropriate domain and codomain.

Let $P \in Obj(QP)$ with $I_P \in Hom(QP)$ denoting the identity morphism on P. Then $Q_M[I_P] = \phi^{(\lambda_P \circ I_P \circ \xi_P)} = \phi^{(\lambda_P \circ \xi_P)} = \phi^{(I_{P'})} = \phi^{(I_P)} = (\phi_A^{(I_P)}, \phi_V^{(I_P)})$.

Let $a \in A(Q_O[P]) = A(G(P))$. Then $\phi_A^{(I_P)}(a) = I_P(a) = a$. Now let $(i, x_i) \in V(G(P))$. Then $\phi_V^{(I_P)}((i, x_i)) = (I_P(i), I_P(x_i)) = (i, x_i)$. So, $Q_M[I_P] = I_{Q_O[P]}$. This shows Q_M preserves the identity morphism on any given object of QP.

Now suppose that $F_1, F_2 \in Hom(QP)$ with $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_3$. Then

$$\begin{aligned} Q_M[F_2 \circ F_1] &= \phi^{(\lambda_{P3} \circ F_2 \circ F_1 \circ \xi_{P1})} = \phi^{(\lambda_{P3} \circ F_2 \circ I_{P2} \circ F_1 \circ \xi_{P1})} \\ &= \phi^{(\lambda_{P3} \circ F_2 \circ \xi_{P2} \circ \lambda_{P2} \circ F_1 \circ \xi_{P1})} \\ &= \phi^{(\lambda_{P3} \circ F_2 \circ \xi_{P2})} \circ \phi^{\lambda_{P2} \circ F_1 \circ \xi_{P1})} \text{ (lemma 3.42)} \\ &= Q_M[F_2] \circ Q_M[F_1] \end{aligned} \tag{1}$$

So, Q_M preserves composition of morphisms. Thus, our pair of mappings is indeed a functor.

Theorem 3.45 *The pair $P = (P_O, P_M)$ is a covariant functor.*

Proof: Clearly P_O and P_M are mappings with appropriate domain and codomain.

Let $G \in Obj(Quiv)$ with I_G denoting the identity morphism on G. Then $P_M[I_G] = F^{(\rho_G \circ I_G \circ \vartheta_G)} = F^{(\rho_G \circ \vartheta_G)} = F^{(I_{G'})} = F^{(I_G)}$.

Let $a \in A(P_O[G]) = A(P(G), v)$. Then $F^{(I_G)}(a) = I_G(a) = a$. Now let $i \in Max(P(G))$. Then $F^{(I_G)}(i) = I_G(i) = i$. Finally, let $x_i \in Min(P(G))$. Then $F^{(I_G)}(x_i) = I_G(x_i) = x_i$. This shows P_M preserves the identity morphism on any given object of Quiv.

Now suppose $\phi, \gamma \in Hom(Quiv)$ with $\gamma : G_1 \to G_2$ and $\phi : G_2 \to G_3$. Then

$$\begin{aligned} P_M[\phi \circ \gamma] &= F^{(\rho_{G_3} \circ \phi \circ \gamma \circ \vartheta_{G_1})} = F^{(\rho_{G_3} \circ \phi \circ I_{G_2} \circ \gamma \circ \vartheta_{G_1})} \\ &= F^{(\rho_{G3} \circ \phi \circ \vartheta_{G_2} \circ \rho_{G_2} \circ \gamma \circ \vartheta_{G1})} \\ &= F^{(\rho_{G3} \circ \phi \circ \vartheta_{G_2})} \circ F^{(\rho_{G_2} \circ \gamma \circ \vartheta_{G1})} \text{ (lemma3.41)} \\ &= P_M[\phi] \circ P_M[\gamma] \end{aligned} \tag{2}$$

So, P_M preserves composition of morphisms. Thus, our pair of mappings in indeed a functor.

Next we will show that our functors are both full and faithful.

Suppose (P, v) is a quiver poset and suppose $G = (V(G), A(G), s, t)$ is a quiver. Suppose further that there exists a QP-morphism f from $(P(G), v_G)$ to (P, v). We will show there exists a unique quiver morphism $\phi = (\phi_A, \phi_V)$ from G to $G[(P, v)]$ such that $\xi_P \circ P_M[\phi] = f$.

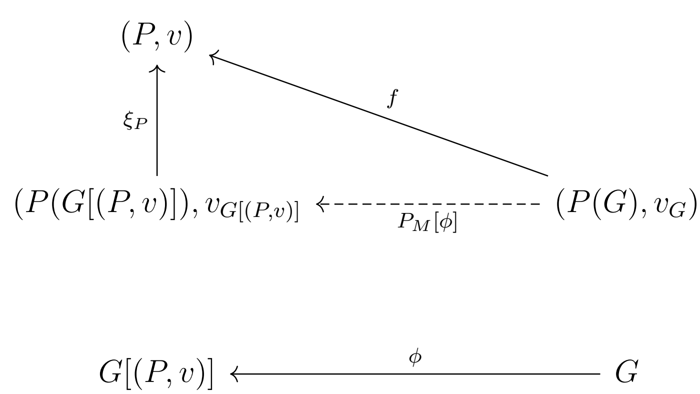

Figure 12: Diagram for Theorem 3.46

For all $a \in A(G)$, let $\phi_A(a) = F(a)$. Now suppose that $x_i \in V(G)$. Then $i \in Max(P(G)), x_i \in Min(P(G))$, and $v(f(i)) = f(v_G(i)) = f(x_i)$. With this in mind let $\phi_V(x_i) = (f(i), f(x_i))$.

Theorem 3.46 *The mapping ϕ defined above is a quiver morphism and has the property that $\xi_P \circ P[\phi] = f$. Furthermore, if $\psi = (\psi_A, \psi_V)$ is any quiver morphism with this property, then $\phi = \psi$.*

Proof: First we show that ϕ is indeed a quiver morphism. Let $a \in A(G)$. Suppose that $s_2(\phi_A(a)) = (j, y_j)$. Then $s_2(\phi_A(a)) = s_2(f(a)) \iff C(f(a)) = j = f(C(a)) \iff f(s_1(a)) = (j, y_j)$. Similarly, suppose that $t_2(\phi_A(a)) = (j, y_j)$. Then $t_2(\phi_A(a)) = t_2(f(a)) \iff c(f(a)) = j = f(c(a)) \iff f(t_1(a)) = (j, y_j)$.

To show that $\xi_P \circ P_M[\phi] = f$, suppose $a \in P(G)$. Then $\xi_P \circ P_M[\phi](a) = \xi_P \circ \phi_A(a) = \xi_P \circ f(a) = f(a)$. Now suppose that $i \in Max(P(G)))$, $f(i) = j$, and $f(x_i) = y_j$. Then $P_M[\phi](i) = j \iff \phi_v(x_i) = (f(i), F(x_i)) = (j, y_j)$,which is true by assumption. So, $\xi_P \circ P_M[\phi](i) = \xi_P(j) = j = f(i)$. Now suppose that $x_i \in Min(P(G)))$. Then $\xi_P \circ P_M[\phi](x_i) = \xi_P \circ \phi_V(x_i) = \xi_P(f(i), f(x_i)) = \xi_P(j, y_j) = y_j = f(x_i)$. Thus, $\xi_P \circ P_M[\phi] = f$.

This shows that the functor P is full.

Next, we show that if there exists a quiver morphism $\psi = (\psi_V, \psi_A)$ such that $\xi_P \circ P_M[\psi] = f$, then $\psi = \phi$. Suppose that $a \in A(G)$. Then $\phi_A(a) = f(a) = \xi_P \circ P_M[\psi](a) = \xi_P \circ \psi_A(a) = \psi_A(a)$. Now suppose that $f(i) = j$ and $f(x_i) = y_j$. Then

$$\phi_V(x_i) = (f(i), f(x_i)) = (j, y_j) = (\xi_P \circ P_M[\psi](i), \xi_P \circ P_M[\psi](x_i))$$
$$\Updownarrow$$
$$\xi_P \circ P_M[\psi](i) = j \text{ and } \xi_P \circ P_M[\psi](x_i) = y_j$$

$$\Updownarrow$$

$$P_M[\psi](i) = \lambda_P(j) = j \text{ and } P_M[\psi](x_i) = \lambda_P(y_j) = (j, y_j)$$

$$\Updownarrow$$

$$\psi_V(x_i) = (j, y_j) = \phi_V(x_i)$$

Therefore, $\phi = \psi$, as desired. This shows that the functor P is faithful and proves the result.

We will show that Q is both full and faithful in a similar manner.

Suppose $G = (V(G), A(G), s, t)$ is a quiver and suppose (P, v) is a quiver poset. Suppose further that there exists a quiver morphism $\phi = (\phi_A, \phi_V)$ from G to $G[(P, v)]$. We will show there exists a unique QP-morphism f from $(P(G), v_G)$ to (P, v) such that $Q_M[f] \circ \rho_G = \phi$.

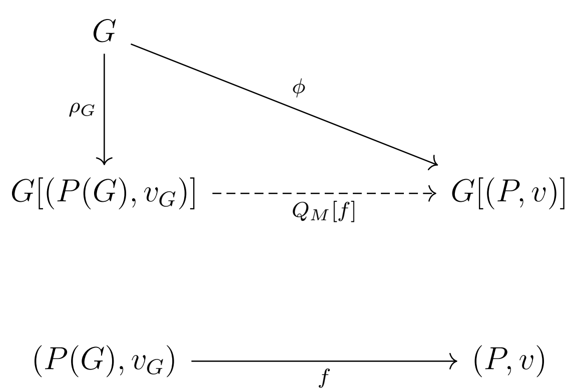

Figure 13: Diagram for Theorem 3.47

For all $a \in A(P(G))$, let $f(a) = \phi_A(a)$. Suppose $i \in Max(P(G)$ and consider $x_i \in V(G)$. We know $\phi_V(x_i) = (j, x_j)$ for some $j \in Max(P(G))$ and some $x_j \in Min(P(G))$. With this in mind, let $f(i) = j$ and $f(x_i) = x_j$.

Theorem 3.47 *The mapping f defined above is a QP-morphism and has the property that $Q_M[f]\circ\rho_G = \phi$. Furthermore, if f is any QP-morphism with this property, then $f = g$.*

Proof: First, we show that f is a QP-morphism. Let $a \in A(P(G)$ and suppose that $s_2(\phi_A(a)) = (j, y_j)$. Then

$$C(f(a)) = j$$
$$\Updownarrow$$
$$s_2(\phi_A(a)) = (j, y_j) = \phi_A(s_1(a)), \text{ which is true by hypothesis}$$
$$\Updownarrow$$
$$f(C(a)) = j$$

So, f is a QP-morphism.

Now we show that $Q_M[f] \circ \rho_G = \phi$. Let $a \in A(G)$. Then $(Q_M[f] \circ \rho_G)(a) = (f_A \circ \rho_A)(a) = f_A(a) = f(a) = \phi_A(a)$. Now let $x_i \in V(G)$ and suppose that $\phi_V(x_i) = (j, y_j)$. Then $(Q_M[f] \circ \rho_G)(x_i) = (f_V \circ \rho_V)(x_i) = f_V(i, x_i) = ((f(i), f(x_i)) = (j, y_j) = \phi_V(x_i)$. So, $\phi_A = (f_A \circ \rho_A)$ and $\phi_V = (f_V \circ \rho_V)$. Hence, $Q_M[f] \circ \rho_G = \phi$.

This shows that the functor Q is full.

To show the uniqueness of f, suppose that g is another QP-morphism such that $Q_M[g] \circ \rho_G = \phi$, in other words, $(g_A \circ \rho_A) = \phi_A$ and $(g_V \circ \rho_V) = \phi_V$. Let $a \in A(P(G))$. Then $f(a) = \phi_A(a) = (g_A \circ \rho_A)(a) = g_A(a) = g(a)$. Now suppose that $\phi_V(x_i) = (j, y_j)$. Then,

$$f(i) = j \text{ and } f(x_i) = y_j$$

$$\Updownarrow$$

$$\phi_V(x_i) = (j, y_j) = (g_V \circ \rho_V)(x_i) = g_V(i, x_i) = (g(i), g(x_i)), \text{ which is true by assumption}$$

$$\Updownarrow$$

$$(j, y_j) = (g(i), g(x_i))$$

$$\Updownarrow$$

$$g(i) = j \text{ and } g(x_i) = y_j$$

So, $f(i) = g(i)$ and $f(x_i) = g(x_i)$. Thus, $f = g$, as desired. This shows that the functor Q is faithful and proves the result.

Next, we will show that our pair of functors are each essentially surjective.

Theorem 3.48 *The functor P is essentially surjective.*

Let $(P, v) \in Obj(QP)$. We seek to show there exists a $G \in Obj(Quiv)$ such that $P_O[G]$ is isomorphic to (P, v). Let $G = Q_O[(P, v)]$. Then $P_O[Q_O[(P, v)]] = P'$, which is isomoprhic to P, as shown previously. Thus, P is essentially surjective.

Theorem 3.49 *The functor Q is essentially surjective.*

Let $G = (V(G), A(G), s, t) \in Obj(Quiv)$. We seek to show there exists a $(P, v) \in Obj(QP)$ such that $Q_O[(P, v)]$ is isomorphic to G. Let $(P, v) = P_O[G]$. Then $Q_O[P_O[G]] = G'$, which is isomorphic to G, as previously shown. Thus, Q is essentially surjective.

It follows that both Q and P form a categorical equivalence on the categories $Quiv$ and QP.

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