Lattice Structures in Finite Graph Topologies

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This is dedicated to my family and my friends, without whom none of this would be possible.

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ABSTRACT

In his 2005 dissertation, Antoine Vella studied the relationships between hypergraphs and a topological space defined using graph-theoretical concepts called the classical topology. In this paper, we take this process a step further to open set lattices constructed from these topological spaces. In this paper, we will characterize the structure of these lattices entirely for finite simple graphs. We then use these results to conjecture on a possible relationship between hypergraphs and open set lattices.

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CHAPTER 1

INTRODUCTION

In a 2005 Ph.D. dissertation, Antoine Vella described a transition from graphs to topological spaces. The goal of this thesis will be to move one step further to establish a relationship between graphs and lattices. We will utilize tools from a variety of discrete mathematics, namely graph theory, topology and order theory. The first part of the body will be to explore the topological space introduced in Vella's paper. The benefit of such a topology is that it is constructed using graph-theoretical properties and, since isomorphism translates into the topological space, all combinatorial concepts should be expressible in topological language. We will only use a few of these properties in our transition. The second part of this thesis will be to use these topological spaces to construct open set lattices. The structure of these lattices and characterizations of their ideals are covered in detail. The third part will discuss considerations for future research. For example, how to start with an open set lattice and determine whether it corresponds to a graph and to which graph it corresponds (if it does). This direction is far less obvious than the work from part two, and so we will discuss in a little detail which lattice properties allow for correspondence to graphs. Some questions naturally arise from this transition from graph theory to order theory, the primary of which being whether or not we can continue from the topological spaces and conclude that combinatorial concepts are preserved in lattices. This question, and questions like it, will be discussed in the conclusion. We begin the preliminary discussion with an introduction to graph theory, from which we move to topology and then order theory.

1.1 Graph Theory

The basis of most of this thesis comes from graph theory. Everything we will do begins with a graph. This section will cover the concepts necessary to prove the main results. We begin with definitions. The first concept we will discuss is that of a graph. Any definitions we will use will be taken from Diestel's book on graph theory [2]. A graph, $G = (V, E)$, is an ordered pair of sets where V is called the set of vertices (also nodes and points) and $E \subseteq [V]^2$ is called the set of edges (also lines). We can see from this that the edges are represented by a pair of vertices. We will assume that $V \cap E = \emptyset$. The usual way to construct a graph is to draw a dot to represent each vertex and draw a line between two dots if they form an edge of G. The method you use to draw a graph is not important as long as the information provided from V and E is consistent with the drawing. For the rest of this paper, we will use V_G to represent the vertex set of some graph G and E_G to represent the edge set of G.

It is beneficial to consider a way to quantify the graph as a whole. The concept we use for this is called the order of G, denoted $|G|$. The order of a graph is defined to be the number of vertices of G. We note that the typical notation used for the order of a graph is the notation commonly associated with the size (or cardinality) of a set. If $|V_G|$ represents the size of the vertex set of G, then $|G| = |V_G|$ according to the definition of order. A graph of order 0 or 1 is called trivial (i.e. no vertices and thus no edges, or a single vertex with no edges). With few exceptions when we deal with topological spaces, the graphs in this paper will all be finite unless explicitly stated otherwise.

A vertex v is said to be incident to an edge e if v is a coordinate of e. We then call v an endvertex (or end) of e and say that e joins its end vertices. Two vertices are adjacent if they are endvertices of the same edge. Two edges are adjacent if they share an endvertex. An edge of a graph with endvertices u and v is given by $\{u, v\}$ since an edge is represented by a pair of vertices. To simplify the notation, we will often use a single letter, i.e. $e = \{u, v\}$, or, if it is important to specify endvertices, we will write uv instead of $\{u, v\}$. If $u \in U$ and $v \in V$, then $e = uv$ is said to be a UV-edge.

Definition 1.1 (Edge Neighborhood / Degree)**.** The set of all edges incident to a vertex v, denoted by $N(v)$, is called the edge neighborhood of v. The order of this set is called the degree of v, and is denoted $d(v)$.

The degree of vertices will be very important in our investigation of topological spaces. The topology we consider uses edge neighborhoods to define open sets, and so the sizes of these will be determinable by the degrees of the vertices.

There are a few different types of graphs. The first we will talk about is the hypergraph, which are the generalization of graphs. With hypergraphs, we can remove the requirement that $E \subseteq [V]^2$ and instead have that $E \subseteq Su(V)$, or the powerset of V. Any graph that is not a hypergraph is simply called a graph. A graph contains a loop if it contains an edge of the form vv, i.e an edge whose endvertices are the same vertex. A graph has multiple edges between vertices if there are at least two distinct edges e and f that share the same endvertices. A graph with a loop and/or multiple edges is called a multigraph. A graph is called simple if it does not contain any loops or multiple edges between any two vertices. The most commonly encountered graphs are simple graphs. With few exceptions, every graph in this paper will be assumed to be simple.

If we consider an arbitrary simple graph and pick a vertex, v, we can traverse the graph by picking an edge that is incident to v (if any) and move to another vertex, say u. If we continue in this fashion, being careful not to repeat edges, then we have created a path on the graph. Intuitively, we can think of it as walking from one place to another without using the same road twice. A trip like this is often seen as a route from one place to another. The concept of paths in graphs is very much the same.

Definition 1.2 (Paths / Cycles)**.** A path is a graph $P = (V, E)$ where

$$V = \{v_1, \cdots, v_n\},\ E = \{v_1v_2, \cdots, v_{n-1}v_n\}$$

with each v_i distinct. The path is created by beginning at v_1 and traversing the edge with v_1 as an endvertex and moving through the edges that pass through each vertex in V_P in order. If v_nv_1 is an edge of G, adding v_nv_1 to P will create a cycle of G.

Cycles and paths can be graphs on their own, but they are often singled out of graphs and, in this sense, are examples of subgraphs. To define subgraphs in general, let $G = (V, E)$ and $G' = (V', E')$ be graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is said to be a subgraph of G. We use the typical subset notation to indicate subgraphs, i.e. $G' \subseteq G$. If $G' \neq G$, then G' is called a proper subgraph of G, denoted $G' \subset G$. If, for every $u, v \in V'$, each edge between them in E is in E', then G' is said to be an induced subgraph of G (in particular, induced by the vertex set V').

Definition 1.3 (Connected Graph)**.** A graph G is said to be connected provided, for each pair of vertices $u, v \in V_G$, there is a path P joining u and v in G.

A graph that is not connected is called disconnected. Of course, a graph can be forcefully disconnected by removing edges and/or vertices. Once the graph is disconnected, the remaining pieces are components that are still connected. We can cover all possible disconnections and find the largest components. These are the maximal connected subgraphs of the graph we started with. They are, then, induced subgraphs (if they weren't, then you could add an edge and get a bigger connected subgraph) and their vertex sets partition the vertex set of the graph they are induced from.

Another way to classify graphs is by the degrees of its vertices. For example, in the graph of a triangle, every vertex has degree 2. A graph G is said to be regular if every vertex in G has the same degree, i.e. the edge neighborhoods of every vertex are the same size. Graphs of this type are well-studied.

Definition 1.4 (Complete Graph)**.** A graph G is said to be complete if every vertex in G is adjacent to every other vertex in G, i.e. there is an edge between every pair of vertices in G.

The complete graphs are special, in that they have quite a few graph properties without having any edges or vertices added or removed, and are denoted by K_n where n is the number of vertices of the graph. First, every complete graph is also regular, in particular, the degree of each vertex is the number of edges in the graph minus one. Also, every complete graph is connected. Every vertex is adjacent to the others, and so there is a one edge path between every pair vertices.

Definition 1.5 (Isomorphism / Automorphism)**.** Let G and G' be graphs. We call G and G' isomorphic and write $G \simeq G'$ if there exists a bijection $\phi : V \to V_{G'}$ with $uv \in E_G \iff \phi(u)\phi(v) \in E_{G'}$ for all $u, v \in V$. Such a map ϕ is called an isomorphism; if $G = G'$, then it is called an automorphism.

As is typical with isomorphisms, if two graphs are isomorphic, then they have the same structure (i.e. can be drawn to look identical to each other). We will see later that isomorphisms are preserved when endowing a graph with a particular topology. We will discuss definitions from topology next.

1.2 Topology

We now move to topology. We want to be able to use graphs to create topological spaces as opposed to embedding graphs onto a surface first. Later in this paper, we will discuss a topological space that preserves graph isomorphism, which in turn preserves many graph theoretical properties under the "action" of applying a topological space. This section will provide the definitions we will need to explore the topological space discussed in this thesis. Any definitions used for this section are taken from Munkres' book on topology [4].

Definition 1.6 (Topology)**.** Let X be any set and let Ω be a family of subsets from X. We say that Ω is a topology on X provided the following conditions hold:

1. The empty set is a member of Ω.

2. The set X is a member of Ω.

3. If $A_1, A_2 \in \Omega$, then $A_1 \cap A_2 \in \Omega$.

4. If $\mathcal{F}$ is any nonempty family of Ω, then $\bigcup\{F : F \in \mathcal{F}\}$ is a member of Ω.

Given a set X and a topology Ω on X, we will call the pair (Ω, X) a topological space and refer to the members of Ω as opens. Using this type of setup, we can define a family to accept certain sets as members. If the family satisfies the conditions in the definition, then it is a topology on a set. The members are then considered open. Let (X, Ω) be a topological space. A subset C of X is closed relative to Ω provided $X - C \in \Omega$ and we will denote the set of closed sets by $\kappa(\Omega)$. We can see from this that a set being closed is not as simple as not being a member of Ω. It is possible that an open set is also closed. Since the closed sets are the complements of the open sets, we can see that the set of all closed sets satisfies the negation of conditions (3) and (4) in the definition of topology, i.e. the collection is closed under finite union and arbitrary intersection.

Definition 1.7 (Basis)**.** Let X be any set and let Ω be a topology on X. A basis for Ω is a family $\mathcal{B}_\Omega \subseteq \Omega$ such that every member of Ω is the union of a subcollection from $\mathcal{B}_\Omega$.

A basis set for a topology can be considered to be a representative family (or subcollection or subset) of that topology. We could then say that every open set is represented by the union of a collection of elements from the basis set. A basis is a collection that can generate the entire topology via unions. Some collections of sets are contained in the topological space, but are not basic collections. In order to think about how one topological space could be contained in another, we use the following proposition without proof.

Proposition 1.8. Let (X, Ω) be a topological space. If $Y \subset X$, then $\Omega_Y = \{Y \cap U : U \in \Omega\}$ is also a topological space.

Collections of the form of Ω_Y are called subspace topologies on Y. We note first that it does not matter whether or not Y is open, but it is true that every nonempty set in Ω_Y has nonempty intersection with at least one open set in Ω from X.

Definition 1.9 (Continuity)**.** Let (X, Ω) and (Y, Θ) be topological spaces. A function $f : X \to Y$ is continuous relative to Ω and Θ provided $f^{-1}(U) \in \Omega$ for every $U \in \Theta$.

With continuity of functions defined, we can talk about the topological equivalent of isomorphism in graphs, homeomorphism. Let (X, Ω) and (Y, Θ) be topological spaces. Then a function $f : X \to Y$ is a topological homeomorphism provided f is a bijection, f is continuous relative to Ω and Θ and f^{-1} is continuous relative to Θ and Ω. Just like with the typical idea of isomorphism, if a homeomorphism exists between two topological spaces, they are structurally identical, i.e. they can be made or arranged to be indistinguishable from each other.

The last concept we will cover is connectivity in topological terms. A topological space (X, Ω) is said to be connected provided X cannot be represented as a union of disjoint members of Ω. If such a representation can be found, the space is called disconnected or separated. Such sets are called a separation of X relative to Ω.

1.3 Order Theory

The last piece needed to discuss the results in this thesis is order theory, in particular, partially ordered sets (posets) and lattices. We will begin with a graph, create a topological space from it, and now we wish to use this topological space as a partially ordered set to create a lattice, which is what the majority of this thesis will focus on. Any definitions used are taken and adapted from Birkhoff's book on lattice theory [1].

Definition 1.10 (Partially Ordered Set)**.** A partially ordered set (or poset for short) is a system $\mathcal{P} = (P, \leq)$ consisting of a set P and a binary relation $\leq \subseteq P \times P$ satisfying the following conditions:

1. For all $x \in P$, we have $x \leq x$ (reflexivity).

2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).

3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

From this point on, as long as no confusion results, we will refer to a poset by its underlying set. The binary relation in posets is referred to as the partial ordering on the set. If, for some partial ordering $\leq$, and some x, y in a set P, $x \leq y$ or $y \leq x$, then y and x are said to be comparable. Otherwise, they are said to be incomparable, and we write $x||y$. A poset is called a chain provided every pair of elements are comparable, and an antichain if any pair of elements is comparable only if they are equal.

Definition 1.11 (Order Dual)**.** Let $\mathcal{P} = (P, \leq)$ be any poset. The order dual of $\mathcal{P}$ is defined to be the system $\mathcal{P}^{op} = (P, \leq_{op})$ where $x \leq_{op} y \iff y \leq x$.

With this definition, we can obtain the dual of any statement by replacing the partial ordering with its order dual. This gives rise to the well-known duality principle, which states that a statement is true of all posets if and only if its order dual is true of all posets.

Any subset Q of P may be regarded as a poset in its own right under the restriction to Q of the partial ordering of P and call such sets subposets of P.

Definition 1.12 (Lowerset / Upperset)**.** Let P be a poset and let $L \subseteq P$. We say that L is a lowerset (or order ideal) of P provided

$$(p \in P) \wedge [(\exists x \in L)(p \leq x)] \Rightarrow p \in L$$

An upperset (or order filter) of P is defined to be a lowerset of P^{op}. We let $\mathcal{L}(P)$ and $\mathcal{U}(P)$ be the set of all lowersets and uppersets, respectively, of P.

If $X \subset P$, then we will denote the lowerset generated by X by $\downarrow X$ and the upperset generated by X by $\uparrow X$. If $X = \{x\}$ is a singleton, then $\downarrow X$ is called a principal lowerset generated by X. A common use of lowersets and uppersets is in intervals. If we let $x, y \in P$ for some poset P, then $[x, y] = \downarrow x \cap \uparrow y$ is called the interval of x and y. We will use these extensively in our decomposition of lattices.

Definition 1.13 (Order Homomorphism)**.** Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be posets. A mapping $f : P \to Q$ is called an order homomorphism or an isotone function provided, $\forall x, y \in P$:

$$x \leq y \Rightarrow f(x) \preceq f(y)$$

We say that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic provided there exists a bijective order homomorphism whose inverse is also an order homomorphism.

Now that we have established what partially ordered sets are, we are ready to begin talking about lattices. We will begin by introducing two binary operations, called meet and join. The meet of a set is defined to be the infimum of that set (if it exists) and is denoted by $\wedge$. The join of a set is defined to be the supremum of that set (if it exists) and is denoted by $\vee$. A lattice is a poset where each pair of elements has a meet and a join. The greatest element of the lattice (if it exists) is called the top of the lattice, and is denoted by $\top$. The least element (if it exists is called the bottom and is denoted by $\bot$. A lattice is said to be complete if every subset of the underlying poset has a meet and a join that is also in the lattice.

Now, let L be a lattice and let $x, y \in L$. Then, if $x \leq y$ but there exists no element, z, in L such that $x \leq z \leq y$, we say that y covers x in L and denote this by $x \prec y$.

Definition 1.14 (Atoms / Co-atoms)**.** Let L be a lattice and $x \in L$. Then x is said to be an atom of L provided it covers the bottom of L. The element x is said to be a co-atom of L if it is an atom of L^{op}.

Atoms are an example of a type of elements that are called join irreducible, or an element a such that if $a = b \vee c$, then $a = b$ or $a = c$. Dually, co-atoms are examples of meet irreducible elements. A lattice is called atomic if every lowerset contains an atom. Atoms and co-atoms will be important when we explore lattices created from graphs, especially with regards to prime ideals.

Definition 1.15 (Ideal)**.** Let P be a poset. A subposet D of P is directed every finite subset of D has an upper bound in D. A directed lowerset of P is called an ideal of P.

If we recall that lattices are simply posets along with the operations meet and join, this definition is the same for lattices. It is worth noting that, while every nonempty lattice is a directed set, not every subset of the lattice will be directed. In general, all ideals are lowersets, but in finite lattices, a lowerset is an ideal if and only if it is principle.

Definition 1.16 (Prime Ideal)**.** Let L be a lattice. A proper ideal, P, of L is a prime ideal provided, for $x, y \in P$, $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$.

If we let L be a lattice, then if the meet distributes over the join, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, or, dually, if the join distributes over the meet, then L is said to be distributive. In addition, if, for $x \in L$, there exists at least one $y \in L$ such that $x \wedge y = \bot$ and $x \vee y = \top$, then L is said to be complemented. A lattice is called Boolean provided it is distributive and complemented. A simple example of a Boolean lattice is the powerset of a set ordered by subset inclusion. In general, only the existence of prime ideals in some classes of lattices is proven. In particular, prime ideals are guaranteed to exist in Boolean lattices.

There are classes of Boolean lattices that are special. A Boolean lattice that is complete and atomic is isomorphic to a field of sets (i.e. it's order is a power of two). It is actually for complete, atomic Boolean lattices that prime ideals are guaranteed to exist. The Boolean Prime Ideal Theorem [cite] states that every Boolean algebra (complete, atomic Boolean lattice) has a prime ideal. The results of this theorem have been extended for other classes of lattices (distributive, for example). We will only be dealing with finite lattices, so it will not require as much rigor to characterize the prime ideals.

CHAPTER 2

THE CLASSICAL GRAPH TOPOLOGY

2.1 Defining the Topology

In his dissertation, Antoine Vella explored a topological space constructed from graph-theoretical properties, specifically edge neighborhoods. He calls it the 'classical topology' on graphs.

Definition 2.17 (The Classical Topology)**.** Let $G = (V, E)$ be a graph. Let $\Omega_G \subseteq Su(V \cup E)$ be defined by $U \in \Omega_G \iff v \in U$ implies every edge with endvertex v is also a member of U. This collection forms a topology and is called the classical topology on G.

The set $V \cup E$ is called the ground set of Ω_G since it is the set that all the members of the topology come from. The motivation for studying this topology is that, since topological homeomorphism follows from graph isomorphism, other, if not all, combinatorial properties should be expressible in a topological language. If we think about the set of closed sets, it is easy to verify that they also form a topological space with the same ground set. This allows the classical topology to satisfy more than the four basic conditions of topological spaces (See Definition 1.6), in that we can add closure under arbitrary intersection for open sets and closure under arbitrary union for closed sets.

It turns out there is a straightforward basis for the classical topology. If we take the empty set, the set of singleton edges and the sets containing a single vertex and it's edge neighborhood, then every open set of the topology can be represented by a union of these sets. The intersections of these will be at most a singleton edge since our graphs are simple (i.e. the intersection of two sets with a single vertex must be empty or a single edge, otherwise we have a multigraph) and the intersections of the singleton edges are always empty. This basis will prove useful in later results.

2.2 Important Results

A good first question is to ask what, exactly, do we know is expressible in topological language. Vella references many properties, but we will provide only those properties relevant to this project. We refer the reader to Vella's paper for the remainder.

The first set of results will be about the connectivity of graphs, subgraphs and graph isomorphisms. This proposition was stated without proof in Vella's paper, and it will stated without proof here as well.

Proposition 2.18 (Vella)**.** A graph is graph-theoretically connected if and only if it is connected with respect to the classical topology. The relative topology inherited from the classical topology by a subgraph is its classical topology. A bijection between the vertex sets of two simple graphs is a graph-theoretic isomorphism if and only if it extends to a homeomorphism between the two ground sets.

From this proposition we can see that connectivity in graphs is equivalent to connectivity in the classical topology. This is not very surprising, seeing as disconnections in graphs alter edge neighborhoods in such a way that each component can themselves be open sets and thus form a separation in the classical topology. The statement for subgraphs, though, is only about construction. In his paper, Vella goes on to say that a graph H is a subgraph of a graph G if and only if the ground set of H is a closed set in the classical topology on G.

Once the topology is constructed, a logical question would then be how do graph theoretical concepts translate into a topological language. A proposition from Vella's paper covers quite a few topological properties.

Proposition 2.19 (Vella)**.** Let G be a graph and Ω_G the topological space obtained when $V(G) \cup E(G)$ is equipped with the classical topology. Then:

- Every vertex is closed, and the set of vertices is discrete;
- Every edge is open, and the set of edges is discrete;

- G is finite if and only if Ω_G is compact;
- The set of edges is dense in Ω_G if and only if no vertex of G has degree zero;
- The vertices incident with an edge e are precisely the boundary points of $\{e\}$;
- Ω_G is locally connected, locally compact and metacompact;
- G is locally finite if and only if Ω_G is rim-compact;
- Ω_G has Lebesgue dimension 1 if and only if G has at least one edge, and Lebesgue dimension 0 otherwise.

This proposition covers many terms from both graph theoretical and topological standpoints. In this thesis, we will only use the first five results from this proposition, the others are presented to the reader for completeness.

We will first talk about the second property: that every edge is open and the set of edges is discrete. That every edge is open is not immediately obvious, as there is no specific mention of it in the definition of the classical topology. However, a set that consists of edges has no vertices, and so is a subset of the ground set that satisfies the definition of the classical topology, and is hence open. From this, the discrete property arises immediately. We now turn our attention to the first property. That a set of only vertices is closed is obvious, unless the graph has no edges. We will assume for the rest of this paper that, for all graphs, every vertex has at least one incident edge. Under this assumption, that the vertices are closed is clear, and, similarly, discreteness follows. We must be careful to note that the edge set is discrete in the classical topology, while the vertex set is discrete in the closed set topology.

Since we have assumed that every vertex is incident to at least one edge, then the fourth property can be reduced to state that the set of edges is always dense in the corresponding classical topology. This property is strengthened by the fifth property, or that the boundary points of an edge are precisely the vertices incident to it. Hence, the boundary of the edge set must be all vertices, and the closure of the edge set is the ground set as desired. This property, and the first two, tell us that the closure of sets depends solely on the edges in the set. If the set is only vertices, then the set is automatically closed. If the set has only edges, then the closure of that set is all of the vertices that represent those edges. We can then see that this allows for the possibility of many sets that are both closed and open.

The third property is interesting to note. A graph is finite if and only if the corresponding classical topology is compact. Although this is difficult to see in the converse, given a topology of this form that is compact, a finite cover may imply a contraction or series of contractions that would reduce the graph to a finite one (if the topology wasn't already finite). I conjecture that the compact condition can be considered weak, and can be strengthened to require that the topology also be finite, although I do not attempt a proof in this thesis.

CHAPTER 3

MOVING TO LATTICES

3.1 Open Set Lattices

Now that we have explored the classical topology for graphs, we are now ready to proceed to lattices. As always, we will start with a graph and construct its classical topology. From this, we will then take the topology and apply the partial ordering of subset inclusion. This type of construction is not new, as it has many applications in category and representation theory.

Definition 3.20 (Open Set Lattice)**.** Let Ω be a topological space. Then, the open sets of Ω partially ordered under subset inclusion form a lattice, called the open set lattice of Ω. The meet for these lattices is set intersection, if finite, and the join is set union.

The definition for these lattices is deceptively simple, and, in fact, it does appear that these lattices are a natural extension of all topological spaces. In 1914, Felix Hausdorff was the first to formally associate topological spaces with a lattice via the open sets. However, these lattices were not used by many until they were used by Marshall Stone [3] in the 1930s in his work on topological representations of Boolean algebras. In lattice theory, open set lattices have been shown to be a class of complete Heyting algebras. In category theory, these lattices are called locales. Both of these labels are beyond the scope of this thesis, but are worth mentioning for reference, as it is widely known that these lattices have a lot of interesting properties:

Proposition 3.21. Let L be any open set lattice. Then:

- L is complete.
- L is distributive.
- For all elements $x \in L$ and all subsets S of L, we have that:

$$x \wedge \bigvee_{s \in S} s = \bigvee_{s \in S}(x \wedge s)$$

The completeness is given by the properties of topological spaces. The meet is set intersection when the lattice is finite because open sets are only closed under finite intersection, and when the lattice is infinite we only need to make a small adjustment and we then have the generalized meet for the lattice. Since open sets are closed under arbitrary union, then it follows immediately that every subset of the lattice has a supremum. The remaining two properties are results of the set theoretical properties of union and intersection and the completeness of the lattice. It is worth noting, however, that since the classical topology from graphs extends the closure of open sets to arbitrary intersections, then the meet is set intersection even if the lattice is infinite.

3.2 From the Classical Topology

Given a graph endowed with its classical topology, we can now construct the open set lattice that corresponds to it. The goal of this section will be to understand the structure of these lattices in general. As we characterize pieces of these lattices, we will also attempt to translate everything into order-theoretical terms only.

We will note first that the open set lattices constructed from these topologies satisfy all of the properties of the previous section with the added property that the meet is set intersection even if the lattice is infinite. We will begin the characterization of these lattices by describing the atoms and co-atoms.

Lemma 3.22. Let G be a graph with n vertices endowed with the classical topology and let L be the corresponding open set lattice. If $x \in L$ has $n-1$ vertices as elements, then x contains $E(G)$. Furthermore, the number of such sets is n.

Proof. The proof of this is clear from the incidence of the edges. Every vertex is incident to at least one edge and every edge is incident to exactly two vertices. Hence, if x has $n-1$ vertices, then even the edge incident to the vertex not in x is in x by the definition of the classical topology. Hence $E(G) \subseteq x$ as desired. Each of these sets is missing exactly one vertex, so we get unique open sets by omitting one vertex from the ground set. We can do this at most n times, and so there are exactly n sets of this type. □

Proposition 3.23. Let G be a graph with n vertices and k edges endowed with the classical topology and let L be the corresponding open set lattice. Then the atoms of L are the singleton edges and the co-atoms of L are the sets with exactly $n-1$ vertices

Proof. The singleton edges are atoms since they are the only sets that can cover the empty set, or the bottom of L. If another set covered the empty set, then it could not contain an edge and hence only has vertices. These sets are closed, which is a contradiction. By the previous lemma, any set in L with $n-1$ vertices contains $E(G)$, and so these sets must be covered by $V(G) \cup E(G)$, or the ground set of the topology. If another set was covered by this, then that set would have at most $n-2$ vertices and at most k edges. In all cases, an element of the same form as x will cover that set, which is a contradiction. Hence, only the sets with $n-1$ vertices are co-atoms of L. □

Corollary 3.24. L is an atomic lattice.

Proof. By the previous proposition, we have that the atoms are the singleton edges. Every nonempty set in L contains at least one edge, and thus must contain at least one singleton edge in their lowersets. Hence L is atomic. □

That these sets are the atoms and co-atoms should not be surprising. A quick observation will show that these are the smallest and largest proper sets that can be considered open in general. Not only that, but their unions and intersections eventually boil down to the singleton edges or work up to the sets with $n-1$ elements. That L is atomic, though, is a nice property. From these results, we can see that the set of all atoms is the edge set of the graph and the co-atoms have a one-to-one correspondence with the vertex set (let each co-atom be mapped to the vertex that it doesn't contain).

If we recall from chapter 2 that the basis for the classical topology consists of the singleton edges and the sets containing exactly one vertex and its edge neighborhood, then we can translate these sets into lattice theoretical terms. We already know that the singleton edges are the atoms, so we only need to translate the basic sets with vertices.

Definition 3.25 (Lattice Basis)**.** Let L be an open set lattice and let $\mathcal{A}$ be the set of all the atoms of L. The basis of the lattice is defined to be the atoms of L and any set $S \in L$ such that $S - \bigcup \mathcal{A} \neq \emptyset$ and S covers some $X \subseteq \bigcup \mathcal{A}$.

We note that since the basic elements in graphs have a single vertex and its edge neighborhood, then, for any basic element $B \in L$ we have that $|B - \bigcup \mathcal{A}| = 1$, which is enough to guarantee that B will cover a subset of $\bigcup \mathcal{A} = E(G)$. With this definition, we now have a general definition for basic elements that agrees with what we know are the basic elements of the classical topology.

We will now turn our attention to ideals, or, more specifically, prime ideals. As covered in chapter 1, since our lattices are always finite, a collection of subsets from the lattice will be an ideal if and only if this collection forms a principal lowerset of the lattice. The question is then which, if any, of these ideals are prime. It is easy to see that the maximal ideals are principally generated by the co-atoms. That these are prime is not as obvious. Let L be an open set lattice constructed from the classical topology on some graph, G, with n vertices and let I be the lowerset generated by some $x \in L$ where x has $n-1$ vertices from G. We consider $y \cap z \in I$ for some $y, z \in L$. If neither y nor z are in I, then both y and z contain the one vertex from G that x does not contain, and so $y \cap z \notin I$, which is a contradiction. Hence I is a prime ideal. We have now shown that, since all of our graphs have at least one co-atom, all of the graphs we will consider in this thesis are guaranteed to have at least one prime ideal. But are these the only prime ideals of this type of lattice? It turns out that the answer is no, but the ideals that are prime but not maximal turn out to be generated by some of the elements of the lattice that are covered by the co-atoms.

Lemma 3.26. Let L be the open set lattice constructed from a graph G endowed with the classical topology. Then no edge-only set from L generates a prime ideal.

Proof. Let $x \in L$ be an edge-only set and suppose by way of contradiction that $\downarrow x$ is a prime ideal of L. Since x is an edge-only set, $\downarrow x$ must also be an edge-only set, and thus contains at least one singleton edge as a member, say, $\{e\}$. To see that this lowerset is not prime, we will consider basic elements with single vertices. Let $y, z \in L$ be basic elements with a single vertex such that $y \cap z = \{e\}$ (we know such elements exist since every edge is incident to exactly two vertices and we are not dealing with multigraphs). Then $y \cap z \in \downarrow x$ but neither y nor z are in $\downarrow x$, which contradicts the primality of $\downarrow x$. Hence, $\downarrow x$ cannot be a prime ideal. □

Theorem 3.27. Let L be the open set lattice constructed from a graph G with n vertices and k edges endowed with the classical topology. Then every principal, non-trivial lowerset P of L is a prime ideal of L if and only if P is generated by a co-atom or by an element of L with exactly $n-2$ vertices and $k-1$ edges from G.

Proof. First, suppose that $n < 2$ and $k < 1$. Then G is either empty or a single vertex, which are trivial graphs and hence have no prime ideals but the trivial ideal. Now, we will suppose that $n = 2$ and $k = 1$, or that $G = K_2$, and let $\{u, v\}$ be the vertex set of G and $\{e\}$ be the edge set of G. We will show that P is a prime ideal if and only if P is generated by a co-atom of L. Let P be a prime ideal of L. Then P could not possibly be generated by an element of L with $n-2$ vertices and $k-1$ edges, since the only element in L satisfying this property is the empty set. By the previous lemma, the lowerset generated by the only edge set in L, $\{e\}$, cannot be prime either. Also, the ground set, $\{u, v, e\}$, cannot generate a prime ideal since prime ideals are proper. This leaves the elements $\{u, e\}$ and $\{v, e\}$ (The open sets with u and v and their edge neighborhoods). So P must be generated by one of these sets, both of which are co-atoms of L. Hence P is generated by a co-atom. Now, let $x \in L$ be a co-atom. Then the lowerset generated by x is a maximal proper ideal and is thus prime by an argument similar to the one presented prior to lemma 3.25.

Suppose now that $n > 2$ and $k > 1$ and that P is a prime ideal of L. We need to show that P is generated by a co-atom or by an element of L with $n-2$ vertices and

$k-1$ edges. Suppose to the contrary that P is not generated by a co-atom and P is not generated by an element of L that has $n-2$ vertices and $k-1$ edges. Then P must be generated by an edge-only set, an element that has $0 < l < n-2$ vertices and k edges, or an element that has $0 < l < n-2$ vertices and $0 < m < k-1$ edges. By lemma 3.25, no edge only set can generate a prime ideal and we get a contradiction, so suppose that $P = \downarrow x$ for some $x \in L$ with at least one vertex. If x has k edges and at most $n-3$ vertices, then there are three vertices in G that are not in x, say $\{v_1, v_2, v_3\}$. Let $y = x \cup \{v_1\}$ and $z = x \cup \{v_2\}$, which are both open since x contains every edge. Then $y \cap z = x \in \downarrow x$ but neither y nor z are in $\downarrow x$. Hence, the lowerset generated by x, P, cannot be prime, which is a contradiction. Now suppose x has at most $k-2$ edges, then there are two edges in G that are not in x, call them e_1 and e_2. Now, let $y = x \cup \{e_1\}$ and $z = x \cup \{e_2\}$ (which are both open since these are unions of open sets) and we get a similar contradiction. Hence, x must be a co-atom or have exactly $n-2$ vertices and $k-1$ edges.

Finally, suppose that $x \in L$ is either a co-atom or an element with exactly $n-2$ vertices and $k-1$ edges. We need to show that $P = \downarrow x$ is a prime ideal. If x is a co-atom, then P is a maximal proper ideal and thus prime, so suppose that x has exactly $n-2$ vertices and $k-1$ edges and let $y, z \in L$ such that $y \cap z \in P$. Suppose by way of contradiction that neither y nor z are in P. Then y and z have either an extra vertex or an extra edge that x does not have. We note that, either way, y and z will have the edge that x does not contain, call it e. Hence, $e \in y \cap z$, but $e \notin x$. This contradicts the fact that $y \cap z \in P$. Hence, at least one of y or z must be in P and P is a prime ideal of L. □

We have now characterized the prime ideals of the open set lattices constructed from graphs. Just as the basic elements are reasonably close to the bottom of the lattice, the elements that generate the prime ideals are covered by the the top of the lattice or are covered by co-atoms. In chapter four, when we consider future directions, we will see that this may be part of a larger pattern.

One issue that comes with this conversion to lattices is that, like finite topologies in general, we cannot count the number of elements in the lattice itself. Given that the set of edges and the set of vertices are both discrete (see Proposition 2.19), we know that these lattices will have at least $2^{|V|} + 2^{|E|}$ elements, where $|V|$ and $|E|$ are the sizes of the vertex and edge sets, respectively. When the graph in question has a small number of vertices and edges, then the number of elements generated stays relatively small, but as we add more vertices and edges, the number grows exponentially. This is a problem, since the graphs have fewer pieces (i.e. vertices, edges, subgraphs, etc.) than their corresponding lattices it would not be very interesting to move to a much larger system unless an immediate benefit were clear. Two intuitive approaches to this problem would be to find a usage for the larger system that renders the size unimportant or to remove pieces of the system to make the size more reasonable while retaining the pertinent information that the larger system provided. The solution we will use in this thesis attempts the latter approach. It turns out that, since the vertices and edges are discrete, every graph with the same number of edges and vertices will have similar discrete substructures as part of their lattices. Hence, these discrete structures (which arguably are the reason there are many elements in the lattices) can be removed without obscuring any information that makes one graph distinguishible from another.

Proposition 3.28. Let L be the open set lattice constructed from a graph G endowed with the classical topology. Then L has two Boolean sublattices, B_E and B_V, such that $\bot_{B_V} = \top_{B_E} = E(G)$.

Proof. By Proposition 2.18, the set of edges, $E(G)$ is discrete. Hence, every element in the powerset of $E(G)$ is open in the lattice. The set $B_E = Su(E(G))$, ordered by subset inclusion, is then a Boolean lattice. Again, by proposition 2.18, the set of vertices, $V(G)$, is discrete. However, this set is discrete in the closed sets of the classical topology. Then, $B_V = \{X \cup E(G) : X \in Su(V(G))\}$ is a set of open sets of L that is isomorphic to $Su(V(G))$. Thus B_V ordered under subset inclusion is also a Boolean lattice. We note that the top of B_E is $E(G)$ and, since the bottom of $Su(V(G))$ is the empty set, the bottom of B_V is $\emptyset \cup E(G) = E(G)$ as desired. □

As was said before, these Boolean sublattices are determined only by the number of edges and vertices of the original graph. Any two graphs with the same number of edges and vertices (even if they are not structurally similar) will yield these same Boolean sublattices. This means that any uniqueness information from the graph, if such information exists, is not contained in either of these sublattices, otherwise they would not be the same for all graphs with the same number of vertices and edges. Hence, we can remove them completely and not deprive ourselves of any useful information. We will, however, use the element from the lattice that these two Boolean lattices share.

3.3 Removing the Clutter

Now that we know that removing the two Boolean sublattices we found in the previous chapter is not a problem with regards to the graph's uniqueness, we will remove them from the lattices. In this section we will characterize the remaining structure in preparation for the next section where we prove the first part of our main theorem. We will note that proposition 3.28 tells us that removing these Boolean sublattices removes all of the edge-only sets and any set with vertices that also has the entire edge set as a subset. Hence, the remaining sets must have at least one vertex and one edge and cannot have the entire edge set as a subset. If we recall from definition 3.25 and theorem 3.27, the only sets that will remain from the basic elements are those with only one vertex and the only sets that will remain from the prime ideals are those that have $n-1$ vertices and $k-2$ edges. We note, then, that these must be the extrema of the remaining piece(s) of the lattice.

Definition 3.29 (Pseudo-Vertices and -Edges)**.** Let L be an open set lattice. Then, if $X \in L$ is a basic element of L that is not an atom such that X covers some $S \subset \bigcup \mathcal{A}$, then X is called a pseudo-vertex of L. If X is covered by a co-atom of L, but $\bigcup \mathcal{A} \not\subseteq X$, then X is called a pseudo-edge of L.

Using terms we've developed so far, pseudo-vertices are non-atom basic elements that do not contain the edge set as a subset and pseudo-edges are elements that are covered by the co-atoms but do not contain the edge set as a subset. We will note that, when defined in this way, the pseudo-vertices and pseudo-edges have an interesting lattice property.

Proposition 3.30. Let L be the open set lattice corresponding to a graph G with n vertices and k edges. Then every pseudo-vertex is join irreducible in L and every pseudo-edge is meet irreducible L.

Proof. Let L be an open set lattice. Suppose $X \in L$ is a pseudo-vertex of L. Then X is a non-atom basic element of L that covers some $S \subset E(G)$. Since X is a basic element of L, we know that $|X - E(G)| = 1$. Suppose to the contrary that there exist $X_1, X_2 \in L$ distinct from X such that $X_1 \cup X_2 = X$. If both X_1 and X_2 are not subsets of $E(G)$, then $X_1 \cup X_2$ could not be X since no two basic elements contain the same vertex. If only one of X_1 and X_2 was not a subset of $E(G)$, then that set must be X by similar reasoning. If X_1 and X_2 are both subsets of $E(G)$, then their union could not contain a vertex and so could not be X. If only one of X_1 and X_2

were a subset of $E(G)$, then one of them must be S, otherwise S would cover that set since X covers S. But the only union including S that yields X is $X \cup S$. Hence X must be join irreducible. To see that pseudo-edges are meet irreducible we note that pseudo-edges are only covered by co-atoms of L. But, since every co-atom contains $E(G)$ as a subset, no intersection of co-atoms could yield a pseudo-edge (which has one less edge). Hence pseudo-edges must be meet irreducible. □

This helps us ensure that the pseudo-vertices and pseudo-edges are the relative extrema after removing the Boolean lattices described in proposition 3.28.

Proposition 3.31. Let L be the open set lattice constructed from a graph G with n vertices and k edges endowed with the classical topology. Then, if $x \in L-(B_E \cup B_V)$, $x \in [B, P]$ where B is a pseudo-vertex of L and P is a pseudo-edge of L. In other words, $L-(B_E \cup B_V)$ is a collection of intervals of the form $[B, P]$.

Proof. Let $x \in L - (B_E \cup B_V)$ and suppose by way of contradiction that there is no $[B, P]$ that contains x. Then x must be an edge only set (since, if x had a vertex, $\downarrow x$ would contain a basic element since L is atomic), or a co-atom (again, since L is atomic). In both cases, $x \in B_E \cup B_V$, which is a contradiction. Hence such an interval must exist. □

Note that these intervals need not be disjoint from each other. Now we have characterized the structure of the lattice entirely. Each of these open set lattices can be decomposed into two Boolean sublattices that share the edge set as a common extremum and a collection of intervals determined by elements that generate prime ideals and basic elements. A good question would then be how we could determine data about graphs from the lattices themselves. Intuitively, the collections of intervals contain all of this data, if it exists, but, as they currently are, I have not been able to determine any usable pattern. So, we will try to restructure the remaining pieces of the lattice into something that we can say is "nicer".

Definition 3.32 (Component)**.** Let L be an open set lattice and let $\mathcal{P}$ be the set of all pseudo-edges of L that generate prime ideals. Then $\bigcup_{P \in \mathcal{P}} [B, P]$, where B is any pseudo-vertex of L, is called a component of L.

By taking components, we are allowing all of the intervals with common basic elements to become one independent piece of the lattice. We will address the use of components in chapter four.

3.4 From Graphs to Lattices

In this section we will talk about the properties that the lattices have once we have arrived at one from a graph. First, we will address a particular graph that does not agree with every other graph in terms of how their lattices decompose. K_2 is the smallest graph allowed based on what we determined a graph was (at least one edge and every vertex must be incident to at least one edge). However, if we construct the lattice from K_2 and remove the Boolean lattices guaranteed from proposition 3.28, it turns out that there is nothing left to work with.

Lemma 3.33. K_2 is the only graph whose open set lattice can be represented as the union of the two Boolean sublattices guaranteed by proposition 3.28.

Proof. Let G be any graph and let L be the open set lattice constructed from G. First, we will show that if $G = K_2$, then L can be represented as the union of Boolean sublattices of L. Let $V(K_2) = \{u, v\}$ and $E(K_2) = \{e\}$. Then the classical topology on K_2 is $\Omega_{K_2} = \{\emptyset, \{e\}, \{u, e\}, \{v, e\}, \{u, v, e\}\}$. Thus, the subsets $\{\emptyset, \{e\}\}$ and $\{\{e\}, \{u, e\}, \{v, e\}, \{u, v, e\}\}$ form the Boolean sublattices from L guaranteed by proposition 3.28 whose union is L. Let G be any graph other than K_2. Then G must have at least one more vertex and one more edge than K_2 (since, by our standards of what a graph is, there is no graph smaller than K_2). So G must have at least three vertices and at least two edges. Then, by proposition 3.28, the sets $B_V = \{X \cup E(G) : X \in Su(V(G))\}$ and $B_E = Su(E(G))$ form Boolean sublattices of L. Since there exists a vertex $v \in V(G)$ such that v is not incident to every edge in G, then the basic open set containing v cannot be a member of either B_V or B_E, and so $L - (B_E \cup B_V)$ cannot be empty. Hence L cannot be represented as the union of the Boolean lattices guaranteed by proposition 3.28. $\square$

We have now confirmed our suspicion that the lattice constructed from K_2 does indeed have nothing left when the Boolean sublattices are removed. We also managed to show that, for any other lattice constructed from a graph, there will always be at least one element left over after the Boolean sublattices are removed. We want to show now that only one vertex of a graph can be incident to every edge in the graph. The reason we do this is because if a vertex is incident to every edge, the basic element that contains that vertex cannot be a pseudo-vertex and thus pose a risk to the existence of pseudo-vertices.

Proposition 3.34. Let G be any graph that is not K_2. If G has a vertex that is incident to every edge in G, then no other vertex has that property. In other words, at most one vertex of G can be incident to every edge in G.

Proof. Suppose that there are two vertices in G that are incident to every edge in G, u and v. Let w be any other vertex from G adjacent to u. Then v must be incident to the edge uw as well. This is not possible, since an edge can only have two end-vertices. Hence, by contradiction, only one of u or v can be incident to every vertex. □

We exclude K_2 since the only two vertices are adjacent to each other. In every other case, though, we cannot have that two vertices are incident to every edge. Now, we wish to show that the pseudo-vertices and pseudo-edges have vertex- and edge-like properties.

Proposition 3.35. Let L be an open set lattice constructed from a graph G with n vertices and k edges endowed with the classical topology. Then one of the following is true:

1. If L has a non-atom basic element with $E(G)$ as a subset, call it B_e, then every other basic element is a pseudo-vertex of L and $B_i \cap B_j = \emptyset$ for all pseudo-vertices $B_i, B_j \in L$.

2. If L has no non-atom basic elements with $E(G)$ as a subset, then for each pseudo-vertex $B_i \in L$, there exists another pseudo-vertex $B_j \in L$ such that $B = B_i \cap B_j$ consists of a single edge from G. Furthermore, there exists a unique pseudo edge $P \in L$ such that $B_1 \not\subset P$ and $B_2 \not\subset P$.

Proof. Let L be an open set lattice constructed from a graph. By the previous proposition, then, the edge set is either a subset of a single non-atom basic element of L or none of them.

1. If there is a non-atom basic element that contains $E(G)$ as a subset, then clearly that element is the only non-atom basic element that is not a pseudo-vertex. Let

$B_i, B_j \in L$ be pseudo-vertices of L and suppose to the contrary that $B_i \cap B_j \neq \emptyset$. Then the single vertices in B_i and B_j, call them v_i and v_j, respectively, must be adjacent in G and $B_i \cap B_j = e_{ij}$ (the edge joining v_i and v_j). If this edge was not the intersection, then either vertices are adjacent over two or more distinct edges, which is a contradiction, or these two elements have a vertex in common, which is also a contradiction. If we let B represent the non-atom basic element that contains $E(G)$ as a subset, then we will note that $e_{ij} \in B \cap B_i$ and $e_{ij} \in B \cap B_j$. And so the vertex in B is adjacent to v_i and v_j via the same edge, which contradicts the fact that all edges can only have two endvertices.

2. Now suppose that L has no non-atom basic elements with $E(G)$ as a subset and let $B_1 \in L$ be a pseudo-vertex (We note now that all non-atom basic elements are also pseudo-vertices in this case). Since all vertices in G must be adjacent to at least one other vertex in G, there must be an edge joining the vertex from B_1 to some other vertex in G, we will call the pseudo-vertex containing this vertex B_2. Then $B_1 \cap B_2$ must consist of only the edge joining the two vertices from B_1 and B_2, otherwise we arrive at the contradictions presented in (1). Since pseudo-edges have $k-1$ edges from G, there can only be one edge not present. There are exactly two pseudo-vertices with that edge in it (if there was a third, there'd have to be a three vertices adjacent to each other via the same edge, a contradiction), and hence they cannot be a subset of that pseudo-edge. Two pseudo-edges cannot be missing the same edge, otherwise they would have to have the same vertices as well and would be equal. Hence there is a unique pseudo-edge that doesn't contain the two pseudo-vertices B_1 and B_2 as subsets.

□

This gives us the framework that would justify whether or not vertices would be adjacent from a lattice-theoretical standpoint. We will see how this is important in the next chapter. It is reasonably clear that if we could find an open set lattice with similar properties, we could use the results from this proposition to create the graph that would correspond to it. We saw earlier that it was obvious that the pseudo-vertices correspond to vertices in the graph. We will now show that the pseudo-edges correspond to the edges, which we will use to prove an important property about components.

Lemma 3.36. Let L be an open set lattice created from a graph with n vertices and k edges endowed with the classical topology. Then there are k pseudo-edges in L.

Proof. Let X be a co-atom of L. If $v \in V(G)$ is the only vertex not in X, then removing an edge incident to v would require the removal of only one other vertex (since edges can only have two end-vertices). This would make a psuedo-edge of L. We can only do this for the k distinct edges of G and so there are exactly k pseudo-edges in L. □

Proposition 3.37. Let L be an open set lattice created from a graph with n vertices and k edges endowed with the classical topology. If C is a component of L with the pseudo-vertex $B \in L$ as its bottom, then $|max(C)| = k - deg(v)$ where v is the vertex from G such that $v \in B$. In other words, the number of maximal elements in C is the number of edges minus the degree of the vertex v.

Proof. Let C be a component in L with B as its bottom and let $v \in V(G)$ be the vertex from G in B. We need to show that $k - deg(v)$ is equal to the number of maximal elements in C. Let $\mathcal{P}_v$ be the set of all pseudo-edges of L that contain v as a memeber. These are the maximal elements of the component C, i.e. $|max(C)| = |\mathcal{P}_v|$. Let $deg(v) = i$. We need to see how many pseudo-edges contain v. By the previous lemma, there are exactly k pseudo-edges of L. We will show that the number of pseudo-edges that do not contain v is exactly i. Let X be the co-atom of L that does not contain v. If we remove a single edge incident to v, we only have to remove the other vertex incident to the same edge to get a pseudo-edge. We can only do this i times, one for each edge incident to v. The rest of the pseudo-edges must contain v, otherwise, by removing another vertex that is not adjacent to v, there are no edges we can remove that would not contradict the open-ness of the pseudo-edge. Hence there are i pseudo-edges that do not contain v and $|\mathcal{P}_v| = k - deg(v)$ as desired. □

From this proposition we can see that the maximal elements of the components are directly related to the degrees of the vertices from which the components are formed (specifically, the basic element that contains those vertices). This yields an interesting property for complete and regular graphs. For a complete graph, K_m, the number of maximal elements for every component C is then $|max(C)| = k - (k - 1) = 1$. For m-regular graphs, the number of maximal elements for every component C is $|max(C)| = k - m$. We arrive at these because complete graphs and regular graphs are graphs whose vertices all have the same degree.

We have now broken down the open set lattices constructed from graphs, and we understand most of the properties that these lattices hold. We determined which open sets were atoms or co-atoms and which open sets generate prime ideals. This opens up a question on its own, which we will discuss in the next chapter. We determined that the discreteness of the vertices and the edges correspond to two special Boolean sublattices, which will be crucial to a conjecture we will make in the next chapter. We determined that removing these Boolean sublattices does not remove any information that we cannot glean from the remaining pieces. Upon the removal of these Boolean lattices, we are guaranteed to have some number of interval sublattices between meet and join irreducible elements of the original lattice. These interval sublattices were used to form components, and information about the maximal elements of these components can be derived from the degree of the vertex that the component represents. This also opens a few questions, which we will discuss in the next chapter. This break down of lattices that we have done will hopefully allow for further study into the relationships between graphs and these types of lattices.

CHAPTER 4

DIRECTIONS FOR FUTURE RESEARCH

As we have seen in the last chapter, we have tried to understand the structure of the lattices that are constructed from simple finite graphs under the classical topology. From chapter two, we see that we can create a classical topology on any graph. However, attempts to understand the structure of the topology were largely unsuccessful. Aside from connectivity, compactness and discrete subspace topologies, there was not much we could say about what was left over. This could simply be because of a lack of understanding of finite topologies or perhaps the structure that we were looking for did not manage to make it to our research. Faced with this difficulty, we decided to look at the open set lattices constructed by making the open sets of these topologies ordered under subset inclusion. The structure of the open set lattices turned out to be much clearer, due in part to their ability to be represented pictographically. From this we were able to deconstruct the lattices as far as we could in the time we had, which is chapter three. In the last chapter, we only dealt with simple graphs, but there may be enough information from those that we can use to make a conjecture about finite hypergraphs in general. We will now discuss some directions and considerations that could be made for future research.

4.1 From the Perspective of Lattices

An interesting question we could ask at this point is whether or not there is a class of open set lattices that behaves similarly to the lattices we have described in chapter three. In this paper, we restricted ourselves to finite, simple, connected graphs. Trying to find such a class of lattices for these graphs proved to be too difficult. The number of edges and vertices had to be specified (and not violating graph rules), we had to maintain that edges were indeed proper (i.e. each edge has two end-vertices), and we had to guarantee that no more than one vertex could be incident to every edge (i.e. the graph is not a hypergraph). This made attempts at specifying a class of open set lattices difficult, and the resulting theorem was so descriptive that the fact that the lattices fit our descriptions was obvious (we had essentially made all of our descriptions of the lattice given from the graphs conditions for the lattices to have graph correspondence).

Despite this, we do believe that there is a class of lattices that will correspond to hypergraphs in general. When we loosen the restriction of our graphs, we allow ourselves to weaken or remove many conditions that we had to impose previously. Recall from chapter two, proposition 2.19, that, for every graph (even hypergraphs), every vertex is closed and the vertex set is discrete and every edge is open and the edge set is discrete. This is a common factor of all open set lattices constructed from graphs, and so it is conjectured that this is all that is required to say that an open set lattice will correspond to some type of graph (hypergraph, multigraph, simple graph, etc.)

Conjecture 4.38. Let L be an open set lattice. Then L is isomorphic to the open set lattice constructed by at least one hypergraph G with n vertices and k hyperedges endowed with the classical topology if and only if L contains two maximal Boolean sublattices B_V and B_E such that $|B_V| = 2^n$, $|B_E| = 2^k$, and $\bot_{B_V} = \top_{B_E} = \bigcup \mathcal{A}$ where $\mathcal{A}$ is the set of atoms from L.

This conjecture only makes use of the property we proved in proposition 3.28 about the Boolean sublattices. The information provided by these sublattices can be acquired elsewhere, but Boolean lattices have nice properties, and can be beneficial to use if available. This conjecture states that, as long as we have these Boolean sublattices (of the appropriate sizes), it doesn't matter what the other elements of the lattice as a whole are, we can still guarantee that there is some graph whose open set lattice is isomorphic to that lattice. We note that the basic elements of a lattice have a one-to-one correspondence to the vertices of the graph the lattice was constructed from. This could possibly help further determine which type of graph an arbitrary open set lattice could correspond to. For example, if more than one basic element in a lattice contains $\bigcup \mathcal{A}$ as a subset, that is the same as saying that more than one vertex is incident to every edge in the graph. This cannot happen in any type of graph but a hypergraph. If, for each $x \in \mathcal{A}$, there are exactly two basic elements that contain x as a subset, then this is the same as saying that all edges are proper. Hence we must be talking about a simple graph. Distinctions like this can be made, but identifying the exact graph that would construct a lattice isomorphic to the one we have requires far more distinctions. The question boils down to, "What types of open set lattices, if any, have the property of our conjecture?" Answering this question may lead to an understanding that will allow us to classify these open set lattices by which type of graph they correspond to. The hope is to be able to understand the structures that will allow us to determine which graph, exactly, that lattice corresponds to.

4.2 Prime Ideals and Basic Elements

There is a lot to be said about the prime ideals of a lattice. We certainly did not exhaust our options for their uses. A question we can ask here is whether or not there is more that we can do with prime ideals that would be useful from the perspective of graphs or lattices. In chapter three we characterized the prime ideals of these open set lattices completely. This is not common for most lattices where, usually, only the existence of prime ideals is guaranteed. The co-atoms are maximal proper ideals and, as discussed in chapter three, prime. However, when we remove the Boolean sublattices guaranteed from proposition 3.28, we can still have elements that generate prime ideals. We showed that these elements are meet irreducible in the lattice they originated from, and that the number of these is the same as the number of edges from the graph the lattice was constructed from (except K_2, see lemma 3.33).

From this we can conclude that, aside from K_2, there must be a one-to-one correspondence between elements that generate prime ideals (that are not co-atoms) and the edges of the graph. We showed, also in chapter three, that these elements have exactly $n-2$ vertices (when the graph has n vertices), and so it would be resonable to suppose that the structure of the graph itself could be obtained from these elements. The question then becomes which types of lattices would have this type of property. Could having an arbitrary open set lattice satisfying the previous conjecture with the added condition that their meet irreducible elements have certain properties be enough to construct the original graph?

In a similar fashion, we could say something similar about the basic elements of the lattice. Since these elements contain a single vertex and the edge(s) that are incident to that vertex, the structure of the graph could be gleaned from these elements. We have made a note several times of the one-to-one correspondence of the non-atom basic elements and the vertices of the graph, but we will also add that these elements are join irreducible in the lattice. If we added a similar condition as the meet irreducible elements, would we be able to reconstruct the original graph?

We believe that the answer may require both of these conditions. Since we have already shown that, in the case of simple graphs, the pseudo-vertices and pseudo-edges are join and meet irreducible (respectively), we will extend this to hypergraphs by using meet and join irreducible elements in general. Suppose we are given an arbitrary open set lattice that satisfies the conditions of the previous conjecture. Let X represent an arbitrary meet irreducible element of this lattice that is not a co-atom and let Y represent an arbitrary join irreducible element of this lattice that is not an atom. If, in addition to the conjecture we add the property that each Y is a vertex of the graph and that there is an edge incident to Y if and only if there is a meet irreducible element such that $Y \nsubseteq X$. Using this additional condition, we can see that two vertices are adjacent provided there is an X that contains neither of them as subsets. If more than one such X exists, then there are multiple edges between the vertices. If there is an X that does not contain three of the vertices as subsets, then X is a hyperedge. This allows for an expansion of the previous conjecture.

Conjecture 4.39. Let L be an open set lattice. Then L is isomorphic to the open set lattice constructed by a non-trivial hypergraph G with $n > 2$ vertices and $k > 2$ hyperedges endowed with the classical topology if and only if L contains contains two maximal Boolean sublattices B_V and B_E such that $|B_V| = 2^n$, $|B_E| = 2^k$, and $\bot_{B_V} = \top_{B_E} = \bigcup \mathcal{A}$ where $\mathcal{A}$ is the set of atoms from L. Specifically, the vertices of G are the non-atom join irreducible elements $\mathcal{J} \subseteq L$ and an edge is incident to some $J \in \mathcal{J}$ if and only if there is a meet irreducible element $M \in L$, that is not a co-atom of L, such that $J \not\subseteq M$. Furthermore:

1. The degree of J is the number of meet irreducible elements that does not contain it as a subset.

2. Let $J, K \in \mathcal{J}$. If there is a meet irreducible element that does not contain either J or K, then J and K are adjacent as vertices in G. If there is exactly one such meet irreducible element, then the edge is proper. If there are multiple such meet irreducible elements, then G must be a multigraph, and each such element represents a distinct edge between J and K.

3. If there is a meet irreducible element $M \in L$ that does not contain three or more elements from $\mathcal{J}$, then G is a hypergraph and the edge set representative of M is equal to the elements from $\mathcal{J}$ that M does not contain.

We will note that this will work for any hypergraph except for K_2, those graphs with two vertices and any number of edges between them, and any single vertex with a loop. These graphs will generate only the Boolean sublattices guaranteed by proposition 3.28. These graphs are trivial in the sense that the construction of the graph is obvious only by considering the Boolean sublattices from proposition 3.28. If we take K_2, for example, we get the Boolean sublattices of sizes $|B_E| = 2^1$ and $|B_V| = 2^2$. Hence, there is one edge and two vertices and the construction is trivial. If we add another edge to K_2 between the two vertices, then there are two edges connecting the same vertices. Hence, we get the Boolean sublattices with sizes $|B_E| = 2^2$ and $|B_V| = 2^2$, and so there are two edges and two vertices. The construction of this graph is, again, obvious. The only graphs that we cannot totally reconstruct with this conjecture (or some slightly modified conjecture) are those graphs that contain isolated vertices. This conjecture, if true, will provide a basis, at least, for the correspondence of open set lattices and hypergraphs.

4.3 Isolated Vertices

Isolated vertices (i.e. vertices with no incident edges) provide an interesting challenge to everything in this thesis. Since the vertex stands alone, the singleton vertex is an open set in the classical topology and hence an atom in the corresponding open set lattice. This means that a vertex would be in the atom set, and so taking the vertices to be the non-atom join irreducible elements would automatically exclude this vertex. Further, there is no join irreducible element that is not an atom that could be attributed to that vertex instead (since the singleton vertex is open, any edge set can be added to it to get another open set, hence any open set containing that vertex cannot be join irreducible).

After several attempts to address isolated vertices, we have determined that, under the current framework, there is no way to totally resolve the issue presented by isolated vertices. Using the previous conjecture, we would only construct the graph in which these vertices are excluded. To add to that, there would be no way of knowing whether or not there were supposed to be isolated vertices in any case. We conjecture that, if there is an atom of L that is contained in every meet irreducible element (that is not a co-atom of L), then that atom must be a vertex and its edge neighborhood will be empty by the previous conjecture. However, we have not had enough time to test this.

4.4 Components

In the previous chapter we showed that the components of open set lattices constructed from graphs had information about the degrees of the vertices of the graph. This is, in fact, a corollary of the previous conjecture as well, but perhaps there is a way to construct the original graph using only the components. If we suppose that a lattice satisfies the condition of conjecture 4.38, but only look at the components, we do get an interesting framework with which we can look at the relevant meet-irreducible and join-irreducible elements. The maximal elements are the edge representatives and the minimal elements are the vertex representatives.

Some questions would be whether or not the components can cover all hypergraphs. From initial observations, the answer is probably not. However, there may be certain graphs for which a component approach may be easier than the previous conjecture. If the graph is a connected simple graph where no vertices are incident to every edge, then knowing the components would streamline the process of reconstructing the original graph. We would only need to look at maximal elements, and that would probably be enough to construct the graph. The components seem to be a way to convey a lot of the information from (1) − (3) of the previous conjecture, especially for simple graphs.

4.5 Graph-Theoretical Properties in Order Theory

In his paper, Antoine Vella translated a lot of graph-theoretical properties into a topological language. An immediate question becomes whether or not the same is possible in an order-theoretical language. Would there be a way to use the open set lattice (once it's correspondence to a graph was established) to study the original graph it corresponds to? If we could, then there may be some problems which would be easier to address in order theory than in graph theory.

Because of time constraints, we were not really able to cover this in detail. However, this is and will be a very interesting direction for future research. If the previous conjectures are true, then perhaps there would be ways to begin translating graph-theoretical properties. Whether or not these translations would be useful, however, is another question. There's no guarantee that, once translated, the property would be any different (technically) from the graph perspective.

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