Operational Symmetry on Functions

by
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Abstract:

All functions possess symmetries over their input with certain operators. So-called *Symmetry-Sets* over a given function and operator are the sets of objects that can be operated with the input to the function under that operator without effecting the output of the function. This work shows that when the *domain* of a function forms algebraic structure – e.g. a Monoid, Group, Ring, etc. – with a given operator or pair of operators, the Symmetry-Sets over the same operator(s) have many nice properties. The work develops and enumerates many interesting results on so-called *Tessellations* – functions from the integers to some at-least cancellative-algebra – using the structure of Symmetry-Sets on them, i.e. *Period-Sets* when speaking of Tessellations. The behavior of the principal period of any given Tesselation is detailed, as well as how they interact with each-other when Tesselations are operated together using generalized *function-operators*. Briefly, a venue is developed for studying these Symmetry-Sets more thoroughly, by introducing the notion of *Allgebras*, an element set paired with the set of all definable operators on the element set. In this context, algebraic structures are *relations* between subsets of elements and operators.

Contents

Prologue

First, we will take the time to note a number of writing conventions that have been adopted in an effort to ease comprehension. These conventions are characteristic of what the author has found useful or – in the case of their absence – sought after when engaging with other texts of this genre – i.e. certainly technical and (at least attempting to be) on the border of explanatory.
These conventions are as follows:

1. Terms and Phrases that are used in a technical and precise way will be:
   (a) FULLY CAPITALIZED
   (b) **Bolded Inside the Section and Subsection they are Introduced**
   (c) Every Word In A Phrase Will Have The First Letter Capitalized
2. Terms and Phrases that are used as a precursor to a technical formalization of the same concept – i.e. if they are representing the fuzzy idea of the soon to be introduced technical term – will be:
   (a) 'Inside Single Quotations'
3. Terms and Phrases that are meant to be emphasized will be:
   (a) *Italicized*

For example, if we suppose that the word 'technical' were being used in accordance with the first convention, it would appear in its introductory section or subsection as: **TECHNICAL**. Prior or subsequent appearances would be rendered: TECHNICAL.

This work is – to the best of the author's ability – a relatively self-contained text. A result of this is that there will often be times when notation – familiar to some, but unfamiliar to others – will need to be introduced using previously defined or bound notation – i.e. we will create an ABBREVIATION. Similarly, there will be times when we will want the *idea* denoted by a given collection of notation – supposedly previously defined and/or bound in some way – to be *assumed* as commensurate with a given – supposedly previously understood – *value* – i.e. we will create an ASSIGNMENT. To this end, we will introduce two symbols that we will use with great frequency to accomplish exactly these two situations.

In the case that we are creating an ABBREVIATION for a collection of known notation with new notation, we will write:

$$\text{Free Notation} \ \diamond\!\!\Rightarrow \ \text{Bound Notation}$$

Very similarly, in the case that we are looking to ASSIGN some bound notation to 'have' the some value, we will write:

$$\text{Bound Notation} \ \square\!\!\Rightarrow \ \text{Fixed Value}$$

PART 0

FOUNDATIONS

CHAPTER I

LOGIC AND SET-THEORY

I.1 Propositional Calculus

Objective We will introduce the reader to the – somewhat formal – definition we will be using for **Propositions** as well as an entire battery of **Logical Symbols** and how they relate to each other, as well as the notion that 'compound' **Propositions** can be formed from simpler 'atomic' ones using these **Logical Symbols**.

Strategy Here we will develop a concept of a so-called **Zeroth-Order Logic**. This will differ from the First-Order Logic we will develop later to aid us in fully utilizing the subsequent Set-Theory we describe. We will begin by introducing notions common to **Zeroth-Order Logic**, namely, **Alphabets**, **Logical Connectives** – also called **Logical Symbols** – and **Rules Of Inference**.

I.1.1 Alphabets and Truth-Tables

What is an 'Alphabet'? An **Alphabet** – roughly speaking – is a collection of so-called 'atomic constants'. This means that these are **Propositions** that are indivisible and, so in a sense, can be understood as simple statements about the nature of things. The other kind of **Proposition** is a so-called Compound Proposition which will be introduced in the next sub-section. These statements can exclusively be either True or False. Often one speaks of 'supposing' each **Letter** in the **Alphabet** has one of these values in order to draw conclusions using the tools of **Propositional Calculus**; we will speak more on this in the next subsection.

For this first section we will only consider a finite **Alphabet** for simplicity, but it is worth noting that ordinarily an infinite alphabet is considered. We hold off on making the jump to an infinite alphabet till we begin our discussion of **First-Order Logic** primarily to motivate the differences between **Propositional Logic** and

QUANTIFICATIONAL LOGIC – that is, FIRST-ORDER LOGIC.

What do 'Letters' in an Alphabet look like? Ordinarily the **LETTERS** – atomic constants – in a Propositional Logic's alphabet are represented using the characters from the Latin Alphabet starting from P – so as to allude to the word **PROPOSITION** – and we will do the same here, although it will be relatively inconsequential as we will have little reason to return to this form of logic once we move on. It is included here primarily as motivation for why one would build up its successor, as well as a convenient separating mechanism to introduce LOGICAL CONNECTIVES before QUANTIFIERS. In fact, we will – for the purposes of explanation – only have need to use a *two* letter **ALPHABET** for the majority of our explanation, as we will not need any more distinct symbols than this. Despite this, we will officially designate – for this section, at the very least – four **LETTERS** to reside in our **ALPHABET** in service of a definition in the final subsection.

Definition I.1.1 (Propositional Alphabet)**.** The **PROPOSITIONAL ALPHABET** for a given **PROPOSITIONAL CALCULUS** contains all of the symbols that are used to each ABBREVIATE **ATOMIC PROPOSITIONS**. The two letters we will be using as for the remainder of the section are formally ABBREVIATED below:

$$\left.\begin{matrix} P \\ Q \\ R \\ S \end{matrix}\right\} \Leftrightarrow \textbf{ATOMIC PROPOSITIONS} \tag{I.1}$$

Truth-Tables In the next subsection we will consider LOGICAL CONNECTIVES but in order to do so we must first establish precisely what the notation associated with a **TRUTH-TABLE** means to convey. This will also be the first properly robust usage of our ABBREVIATION and ASSIGNMENT notation – and in concert, no less.

Definition I.1.2 (Truth-Table)**.** A **TRUTH-TABLE** is an ABBREVIATION for a handful of systematic ASSIGNMENTS. We will ABBREVIATE two sizes of **TRUTH-TABLES** as those are the sizes we will be using for constructing the majority of our **Logic**:

$$
\begin{array}{|c|c|}
\hline
* & \\
\hline
A & C \\
B & D \\
\hline
\end{array}
\;\Diamond\!\!\Rightarrow
\begin{cases}
*A & \square\!\!\Rightarrow\; C \\
*B & \square\!\!\Rightarrow\; D
\end{cases}
\tag{I.2}
$$

$$
\begin{array}{|c|cc|}
\hline
* & C & D \\
\hline
A & E & F \\
B & G & H \\
\hline
\end{array}
\;\Diamond\!\!\Rightarrow
\begin{cases}
A * C & \square\!\!\Rightarrow\; E \\
A * D & \square\!\!\Rightarrow\; F \\
B * C & \square\!\!\Rightarrow\; G \\
B * D & \square\!\!\Rightarrow\; H
\end{cases}
\tag{I.3}
$$

I.1.2 Basic Logical Symbols

Tautologies are Not Falsehoods In the case that some Proposition is **True** we have a particular symbol that we use to indicate it:

Definition I.1.3 (Logical Truth and Tautologies)**.** Going forward we will use the symbol $\top$ to refer to the logical notion of **True**, i.e. something that *is* the case. Formally:

$$\top \;\square\!\!\Rightarrow\; \textbf{True} \tag{I.4}$$

When something is *always* $\top$, we say it is a **Tautology**.

Similarly, we have a particular symbol for **False**:

Definition I.1.4 (Logical False and Falsehoods)**.** Going forward we will use the symbol $\perp$ to refer to the logical notion of **False**, i.e. something that *isn't* the case. Again, formally:

$$\perp \;\square\!\!\Rightarrow\; \textbf{False} \tag{I.5}$$

When something is *always* $\perp$, we say it is a **Falsehood**.

We will often use Truth-Tables – as defined in the previous subsection – in the following definitions to further elucidate what each logical symbol means about the logical quality of compound Propositions involving them, and we will use these two symbols extensively. Before we introduce the first of several **Logical Symbols** and **Logical Connectives**, a relevant definition:

Definition I.1.5 (Compound Proposition)**.** We say that a Proposition is a **Compound Proposition** if it contains any **Logical Symbols** or **Logical Connectives**.

Not the Excluded Middle To make our discussion explicit we define what we mean when we say **Logical Quantity** below.

Definition I.1.6 (Logical Quantity)**.** Every Proposition is understood as having a **Logical Quantity** – either $\top$ or $\bot$ in this case – that describes the 'accuracy' of the circumstances it is considered as asserting. That is:

$$P \;\Box\!\!\Rightarrow\; \top \text{ or } P \;\Box\!\!\Rightarrow\; \bot \tag{I.6}$$

$$\text{and} \tag{I.7}$$

$$Q \;\Box\!\!\Rightarrow\; \top \text{ or } Q \;\Box\!\!\Rightarrow\; \bot \tag{I.8}$$

$$\text{and} \tag{I.9}$$

$$R \;\Box\!\!\Rightarrow\; \top \text{ or } R \;\Box\!\!\Rightarrow\; \bot \tag{I.10}$$

$$\text{and} \tag{I.11}$$

$$S \;\Box\!\!\Rightarrow\; \top \text{ or } S \;\Box\!\!\Rightarrow\; \bot \tag{I.12}$$

We have a symbol that refers to the notion of changing the **Logical Quantity** of a Proposition, P, to its 'opposite'. If it has a **Logical Quantity** of $\top$ this process will instead give us $\bot$, and vice versa.

Definition I.1.7 (Logical Negation)**.** For any given Proposition P, if its **Logical Quantity** is $\top$ in a given circumstance, then we adjoin $\neg$ to its left side – like so: $\neg P$ – to indicate the Proposition which has a **Logical Quantity** of $\bot$ in that same circumstance. It also functions in the opposite direction; if a Proposition Q happens to have a **Logical Quantity** of $\bot$ then $\neg Q$ has a **Logical Quantity** of $\top$.

The following is the first Truth-Table we will use of many:

$\neg$	
$\top$	$\bot$
$\bot$	$\top$

(I.13)

There is an important notion that is associated with the – seemingly rather simple – concept we have just stated. As a reminder, the **Truth-Table** above abbreviates the two assignments:

$$\neg\top \;\Rrightarrow\; \bot$$

$$\neg\bot \;\Rrightarrow\; \top$$

That is we are *assuming* the The Law Of Excluded Middle; this refers to an implicit assumption we took in the previous subsection. That is that every Proposition has a **Logical Quantity** that is *either* $\top$ *or* $\bot$. It can not be both, and it can be no other value, such as 'half-true' or any other variation on a 'mixed' **Logical Quantity**. One of the consequences of such an assumption is known as Double Negation Elimination. This is one of our Rules Of Inference, so we wait to discuss it more directly in the course of this section's final subsection.

This and/or That Now for the first couple of interesting logical processes and symbols that accompany a discussion of them; namely, **Disjunctions** and **Conjunctions**.

Definition I.1.8 (Logical Disjunctions)**.** If we wish to express the Proposition that is true when *P or Q* (or indeed both) have a **Logical Quantity** of $\top$, we refer to the **Disjunction** of P and Q. Such a Proposition can be written like so:

$$P \vee Q$$

$$Q \vee P$$

The above two lines are read "P or Q" and "Q or P" respectively. Often, the process of **Disjunction** is also called 'Logical-Or' because of its intended interpretation as well as the Assignments made in the Truth-Table below:

$$\begin{array}{|c|cc|} \hline \vee & \top & \bot \\ \hline \top & \top & \top \\ \bot & \top & \bot \\ \hline \end{array} \tag{I.14}$$

Definition I.1.9 (Logical Conjunctions)**.** If we wish to express the Proposition that has a **Logical Quantity** of $\top$ only when *P and Q* each also have a **Logical Quantity** of $\top$, we refer to the **Conjunction** of P and Q

which can be written like so:

$$P \wedge Q$$

$$Q \wedge P$$

Similarly to before, these two lines can each be read as "P and Q" and "Q and P", respectively. Again similarly, the process of **Conjunction** may also be called 'Logical-And' owing to its intended interpretation and the Truth-Table responsible for its Assignments:

$$\begin{array}{|c|cc|} \hline \wedge & \top & \bot \\ \hline \top & \top & \bot \\ \bot & \bot & \bot \\ \hline \end{array} \tag{I.15}$$

If Equivalence Then Material Implication We have a symbol to account for the cases when two seemingly disparate Propositions are $\top$ and $\bot$ at exactly the same times, despite their seemingly distinct formulations. This is the symbol of **Logical Equivalence**, often understood as suggesting the phrase 'if and only if':

Definition I.1.10 (Logical Equivalence)**.** When two distinct Propositions are **Logically Equivalent**, and so have the exact same **Logical Quantity**, then we write:

$$P \Leftrightarrow Q$$

$$Q \Leftrightarrow P$$

These are read as "P if and only if Q" or "P is logically equivalent to Q" and "Q if and only if P" or "Q is logically equivalent to P". That is P's **Logical Quantity** must be $\top$ *if* Q's **Loggcial Quantity** is $\top$ and must be $\bot$ otherwise – i.e. Q's **Logical Quantity** is $\bot$ – hence "and only if". If this is not the case, then the Proposition describing the **Logical Equivalence** of P and Q must have a **Logical Quantity** of $\bot$. This is all formalized in the following Truth-Table:

$$\begin{array}{|c|cc|} \hline \Leftrightarrow & \top & \bot \\ \hline \top & \top & \bot \\ \bot & \bot & \top \\ \hline \end{array} \tag{I.16}$$

We also have a symbol when the previous relationship is more 'one-sided'. What do we mean by this? Well, suppose that P is *sufficient* for Q but *not necessary*. That is, if P has a **Logical Quantity** of $\top$, then we can say confidently that Q must also, but if instead P has a **Logical Quantity** of $\bot$, we can't say anything about Q one way or the other – i.e. it could have a **Logical Quantity** of $\top$ or $\bot$. This describes the notion of **Material Implication**.

Definition I.1.11 (Logical Material Implication)**.** If we want to talk about a Proposition such that P is sufficient but unnecessary for Q, we arrive at the concept of **Material Implication**. We may write such a Proposition as:

$$P \Rightarrow Q$$

$$Q \Leftarrow P$$

Such a Proposition is read as "If P, then Q" or "P Implies Q" and "Q if P" or "Q is implied by P", respectively (although they are **Logically Equivalent** since P is still 'pointing' to Q). We may also call P the **Antecedent** of Q and Q the **Consequent** of P. We have the relevant Assignments made by these two Truth-Tables:

$\Rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

(I.17)

$\Leftarrow$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$

(I.18)

The second is simply a mirror along the diagonal of the previous Truth-Table, but it is included here to demonstrate the utility of **Logical Connectives** that treat each side differently – i.e. are *not* Commutative. That is, we are able to capture two unique Truth-Tables – which as a collection themselves serve to enumerate *all* possible **Logical Connectives** in a sense – through simply flipping the direction a given symbol is pointing by virtue of its asymmetry. This also rather straightforwardly demonstrates the fact that:

$$(P \Leftrightarrow Q) \;\Leftrightarrow\; (P \Rightarrow Q \;\wedge\; P \Leftarrow Q) \tag{I.19}$$

The above is one of the most complicated **Compound Propositions** we have constructed – and so this might serve to illustrate the relative utility of having multiple ways to read **Compound Propositions** – as it can be read as "P is logically equivalent to Q if and only if P implies Q and Q implies P." One can become convinced of this by comparing the results of taking the **Conjunction** of each cell in the first Truth-Table with the cells in the same position in the second Truth-Table and comparing the result to the Truth-Table for **Logical Equivalence**. It is for this reason that **Logical Equivalence** is sometimes referred to as **Bi-Implication** or **Mutual Implication**.

Logic is Like a Good Friend One might be mildly perturbed – as I once was – by the Assignment in the bottom-right corner of each of our most recent Truth-Tables, that is: when both P and Q have a **Logical Quantity** of $\bot$, why is $P \Rightarrow Q$ still given a **Logical Quantity** of $\top$? This is a result of how we have described Q's relationship to P. In order to be confident that P *implies* Q, we need only make sure that Q behaves appropriately when P actually has a **Logical Quantity** of $\top$.

One can also think of **Material Implication** as something like a 'conditional promise'. Consider, I tell you that "I will go on a picnic with you if it doesn't rain." Logically, this is equivalent to $Q \Leftarrow P$ where P is understood as "It doesn't rain.", and Q is "I will go on a picnic with you." It is clear that if P has a **Logical Quantity** of $\top$ and Q does also, I have kept my promise to you since it supposedly did not rain and we went on a picnic together. If P does not, and we consequently don't go on a picnic – i.e. the **Consequent** Q has a **Logical Quantity** of $\bot$ – it would be inaccurate to say I had broken my promise, merely that the weather did not permit my keeping it, as it was a promise conditional on some *material circumstance* – the weather in this case. If P does not have a **Logical Quantity** of $\top$ – so it rains – but somehow Q does – i.e. we manage to go on a picnic still (say we opt for an indoor picnic instead) – then I have not exactly kept my promise, but I have rather gone above and beyond it and so I have certainly not broken it. However, if it does not rain, but I fail to go on our planned picnic, I *will* have broken my promise – this is commensurate with the only instance that the **Logical Quantity** of **Material Implication** is $\bot$: when the **Antecedent** P has a **Logical Quantity** of $\top$ while the **Consequent** Q has a **Logical Quantity** of $\bot$. It is in this sense, that the **Logical Quantity** of a **Material Implication** describes a promise being kept.

I.1.3 Negated Logical Symbols

To accompany each of our previous Logical Symbols, we have a whole collection of **Negated Logical Symbols**. These form those **Compound Propositions** that have a Logical Quantity of $\top$ precisely when their 'un-negated' counterparts have one of $\bot$, and vice versa. We have already seen the Negations of $\top$ and $\bot$, namely each other.

In the course of this subsection we will often give somewhat non-standard names to these processes in the pursuit of making them out to be processes in their own right, rather than 'merely' the Negation of previous processes (despite their being precisely that). This is to encourage the reader to develop an understanding of their own independence as logical processes worthy of consideration into and of themselves. The names are chosen also in service of this goal in an attempt to convey where they might indeed be applied.

Neither nor Exclusion? In One Case, Yes! The negation of a Disjunction is – aptly – known as a **Nondisjunction**, but we will instead consider it – hopefully more intuitively – as **Logical Nor**:

Definition I.1.12 (Logical Nor)**.** When we want to know when both P and Q have a Logical Quantity of $\perp$, we consider the process of **Nondisjunction** or **Logical Nor**:

$$P \downarrow Q \tag{I.20}$$

$$Q \downarrow P \tag{I.21}$$

These can each be read as "Neither P nor Q" and "Neither Q nor P." As the beginning of this subsection indicated, the Truth-Table for this process is totally equivalent to the Negation of our Truth-Table for a Disjunction:

$$\begin{array}{|c|cc|} \hline \downarrow & \top & \perp \\ \hline \top & \perp & \perp \\ \perp & \perp & \top \\ \hline \end{array} \tag{I.22}$$

The negation of a Conjunction is again – aptly – known as a *Nonconjunction*, but we will instead consider the naming scheme of **Exclusion**, so-named to suggest at least one Proposition being disallowed, i.e. each Proposition must 'exclude' the other when 'present':

Definition I.1.13 (Logical Exclusion)**.** Similarly, when we want to know when P and Q have a combination of Logical Quantities that are anything *except* both $\top$, we consider the process of **Nonconjunction** or **Exclusion**:

$$P \uparrow Q \tag{I.23}$$

$$Q \uparrow P \tag{I.24}$$

These may be read as "P and Q exclude each other" and "Q and P exclude each other" or – more verbosely

– "exclusively P or exclusively Q". This second linguistic construction hints at another Rule Of Inference: De'Morgan's Law. We save that discussion for the appropriate subsection, however. The Truth-Table for this process is – as was the case previously – is totally equivalent to the negation of our Truth-Table for a Conjunction:

$$\begin{array}{|c|cc|} \hline \uparrow & \top & \bot \\ \hline \top & \bot & \top \\ \bot & \top & \top \\ \hline \end{array} \tag{I.25}$$

The Contradiction of Unrequited Extant Preclusion We now consider the Negations of Logical Equivalence and Material Implication, to arrive at the notions of **Exclusive Disjunctions** – which we will rename **Logical Dissension** – and the unexcitingly named **Negated Implication**, which we will take the liberty of naming **Extant Preclusion**.

Definition I.1.14 (Logical Contradiction)**.** When we want the Proposition that has a Logical Quantity of $\top$ exactly when two distinct propositions, P and Q, have the opposite Logical Quantity, then we speak of the **Logical Contradiction** of P and Q. This is commonly known as an **Exclusive Disjunction**, and we preserve the symbolic consequence of such a naming, in hopes of preserving the insight it provides:

$$P \veebar Q \tag{I.26}$$

$$Q \veebar P \tag{I.27}$$

These are read as "P contradicts Q" and "Q contradicts P". The associated Truth-Table:

$$\begin{array}{|c|cc|} \hline \veebar & \top & \bot \\ \hline \top & \bot & \top \\ \bot & \top & \bot \\ \hline \end{array} \tag{I.28}$$

This naming scheme is adopted to convey the notion of mutually exclusive Logical Quantities.

Definition I.1.15 (Logical Extant Preclusion)**.** If we want to talk about a Proposition that has a Logical Quantity of $\top$ precisely when P also does but Q does not, we arrive at the concept of **Extant Preclusion**.

We may write:

$$P \multimap Q \tag{I.29}$$

$$Q \multimapinv P \tag{I.30}$$

Such a PROPOSITION is read as "P Precludes Q", and "Q is precluded by P". We now refer to P as the **PRECEDENT** and say Q is the **INCONSEQUENT**. Although, the $\not\Rightarrow$ symbol is often used for such a process, in keeping with disambiguating these processes more thoroughly, we use the above symbol pulling from Logical Circuit Diagrams the notion of a concluding circle on a gate being used to indicate NEGATION. The two associated TRUTH-TABLES for **EXTANT PRECLUSION** are as follows:

$\multimap$	$\top$	$\bot$
$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$

(I.31)

$\multimapinv$	$\top$	$\bot$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$

(I.32)

Unprecedented Consequences Similarly, as with MATERIAL IMPLICATION, **EXTANT PRECLUSION** might seem to have slightly odd assignments in our table, but for the opposite reason this time. If P has the LOGICAL QUANTITY of $\bot$, why do we not – as in the case of MATERIAL IMPLICATION – render the LOGICAL QUANTITY of $P \multimap Q$ as $\top$? Well, one answer might be that **EXTANT PRECLUSION** is the NEGATION of MATERIAL IMPLICATION – so we treat it oppositely – but that answer is somewhat unsatisfying, to say the least. Rather, consider that MATERIAL IMPLICATION describes an 'implication' that is contingent on "material circumstances", it is the opposite for **EXTANT PRECLUSION**. That is, **EXTANT PRECLUSION** describes – aptly – some 'preclusion' that is *extant* regardless. It speaks in a *positive* manner, and so can only have a LOGICAL QUANTITY of $\top$ when P does also meaning the necessary **PRECEDENT** is in place to be even capable of 'precluding' the **INCONSEQUENT** Q in the first place. That is to say, MATERIAL IMPLICATION admits the possibility of P's LOGICAL QUANTITY being $\bot$ as a vacuous case since P is 'not around' to speak to Q's LOGICAL QUANTITY, whereas for **EXTANT PRECLUSION** there is the assertion that P has LOGICAL QUANTITY of $\top$ and – ideally specifically because of this– Q can not. If either P has a LOGICAL QUANTITY of $\bot$, or Q does not, this assertion can't hold up, hence $P \multimap Q$ must have a LOGICAL QUANTITY of $\bot$. One can think of **EXTANT PRECLUSION** also as a kind of 'Conditional Exclusion', an

Exclusion that only wants to exclude Q and 'fails' otherwise.

I.1.4 Rules of Inference

This section follows – in a sense – from our definitions of the previous Logical Connectives. This is because the Truth-Tables that we provided cause Compound Propositions to have predictable and definite behaviors given known – or often assumed – **Premises**. A Logical Rule is something that is assumed and serves as the **Premises** upon which a **Conclusions** is **Proven**. We have done this somewhat implicitly, by first defining our Logical Symbols and Assigning how they interact with varieties of Logical Quantities of Propositions. We will now introduce a notation that we can use to write down a series of Logical Rules that these behaviors will follow from.

There is an established notation for Logical Rules known as Conditional Proofs that we will adopt a similar – albeit briefer – form of here. In essence, one states, line by line, a series of **Premises** or **Conditions**, and then below a dividing line, a series of Substitutions enabled by previous Logical Rules, and punctuated by a **Conclusion** or **Consequence** of the penultimate Substitution. The relevant abbreviation for a full Conditional Proof are reserved for the appendix, and instead the more brief format – which we will call a **Definitional Proof** – we will use to simply state **Rules Of Inference** is presented below:

Definition I.1.16 (Definitional Proof)**.** Given some Propositions, $P, Q, \ldots R$, as **Premises** or **Conditions**, and a Proposition S as a **Conclusion** that *definitionally* follows from them, one denotes a **Definitional Proof** of such a **Rule Of Inference** as follows:

$$\begin{array}{|c|} P \\ Q \\ \vdots \\ R \\ \hline S \\ \hline \end{array} \Leftrightarrow (P \wedge Q \wedge \ldots \wedge R) \Rightarrow S \tag{I.33}$$

Over the remainder of this section, we will be building a logical system of **Natural Deduction**, meaning that we will state most of the foundational **Rules Of Inference** that follow from the Assignments of our Logical Symbols. The expanded Conditional Proof format is included in the appendix primarily for the curious reader that might wonder how other Logical Rules – that we may end up using later – are derived from the ones we will now state below.

Introducing and Eliminating Negation As Well As Not Negation We will begin with some of the simplest **Rules Of Inference**, all having to do with Negation. These are **Negation Introduction**, **Negation Elimination**, and **Double Negation Elimination**.

Definition I.1.17 (Negation Introduction)**.**

$$\begin{array}{|c|} \hline P \Rightarrow Q \\ P \Rightarrow \neg Q \\ \hline \neg P \\ \hline \end{array} \tag{I.34}$$

This rule follows from the recognition that the only case where $P \Rightarrow Q$ and $P \Rightarrow \neg Q$ both have a Logical Quantity of $\top$ is when P has one of $\bot$, thus we can infer that $\neg P$ must have a Logical Quantity of $\top$.

Definition I.1.18 (Negation Elimination)**.**

$$\begin{array}{|c|} \hline \neg P \\ \hline P \Rightarrow Q \\ \hline \end{array} \tag{I.35}$$

This rule follows from realizing that when P has a Logical Quantity of $\bot$ – i.e. $\neg P$ has a Logical Quantity of $\top$ – then $P \Rightarrow Q$ must have a Logical Quantity of $\top$, as discussed in subsection **0.I.1.2**.

Definition I.1.19 (Double Negation Elimination)**.**

$$\begin{array}{|c|} \hline \neg\neg P \\ \hline P \\ \hline \end{array} \tag{I.36}$$

This is the rule we spoke of previously in reference to **The Law Of Excluded Middle**. This relies on the implicit assumption that P can either have a Logical Quantity of exactly $\top$ or $\bot$ and nothing else. So if the negation of $\neg P$ has a Logical Quantity of $\top$, then $\neg P$ must have one of $\bot$, meaning that P must have a Logical Quantity of $\top$ just as $\neg\neg P$.

Introducing and/or Eliminating Disjunctions and/or Conjunctions The next four **Rules Of Inference** we will discuss have to do with 'adding' or 'removing' a Disjunction or Conjunction while still being confident the overall Logical Quantity of the Proposition being considered is $\top$. We start with the **Disjunction Introduction** followed by the **Conjunction Introduction**.

Definition I.1.20 (Disjunction Introduction)**.**

$$\begin{array}{|c|}\hline P \\ \hline P \vee Q \\ \hline \end{array} \tag{I.37}$$

$$\begin{array}{|c|}\hline Q \\ \hline P \vee Q \\ \hline \end{array} \tag{I.38}$$

This rule follows from the recognition that if we have that P has a LOGICAL QUANTITY of $\top$ – or indeed Q does – then $P \vee Q$ must also. One can gain confidence in this by observing the previous TRUTH-TABLE for DISJUNCTIONS.

CONJUNCTION INTRODUCTION is exceptionally similar:

Definition I.1.21 (Conjunction Introduction)**.**

$$\begin{array}{|c|}\hline P \\ Q \\ \hline P \wedge Q \\ \hline \end{array} \tag{I.39}$$

This rule follows nearly straight from the definition of a CONJUNCTION. If we have that Q has a LOGICAL QUANTITY of $\top$ as well as P, then $P \wedge Q$ must also. One can gain confidence in this by again observing the TRUTH-TABLE for CONJUNCTIONS.

We will now swap our order slightly, as **CONJUNCTION ELIMINATION** is rather more straightforward than **DISJUNCTION ELIMINATION**.

Definition I.1.22 (Conjunction Elimination)**.**

$$\begin{array}{|c|}\hline P \wedge Q \\ \hline P \\ \hline \end{array} \tag{I.40}$$

$$\begin{array}{|c|}\hline P \wedge Q \\ \hline Q \\ \hline \end{array} \tag{I.41}$$

This rule follows – again – straight from the definition of a CONJUNCTION. If we have that $P \wedge Q$ has a LOGICAL QUANTITY of $\top$ then P will and Q also will.

Now for the more complicated DISJUNCTION ELIMINATION:

Definition I.1.23 (Disjunction Elimination)**.**

$$\begin{array}{|c|} \hline P \Rightarrow Q \\ R \Rightarrow Q \\ P \vee R \\ \hline Q \\ \hline \end{array} \tag{I.42}$$

This rule follows from considering that when $P \Rightarrow Q$, $R \Rightarrow Q$, and $P \vee R$ all have a Logical Quantity of $\top$ then either P must also, and so $P \Rightarrow Q$ demands that Q does, or R does and so $R \Rightarrow Q$ demands that Q does. One can gain confidence in this **Rule Of Inference** by observing the previous Truth-Table for both Material Implications and Disjunctions, though Material Implication is the more likely of the two to yield insight, as it is primarily what mechanically enables this **Rule Of Inference**.

Introducing and Eliminating Logical Equivalence Now we will consider the ways we might **Introduce Logical Equivalence**:

Definition I.1.24 (Logical Equivalence Introduction)**.**

$$\begin{array}{|c|} \hline P \Rightarrow Q \\ Q \Rightarrow P \\ \hline P \Leftrightarrow Q \\ \hline \end{array} \tag{I.43}$$

This rule follows from considering that $P \Rightarrow Q$ can have a Logical Quantity of $\top$ if P does not or both P and Q do. Similarly, $Q \Rightarrow P$ can have a Logical Quantity of $\top$ if Q does not or both Q and P do. In order for both of these to have a Logical Quantity of $\top$, however, we must have that either both P and Q do not or they both do. This is precisely how we have defined Logical Equivalence according to our previous Truth-Table.

Logical Equivalence Elimination is the converse of this rule:

Definition I.1.25 (Logical Equivalence Elimination)**.**

$$\begin{array}{|c|} \hline P \Leftrightarrow Q \\ \hline P \Rightarrow Q \\ \hline \end{array} \tag{I.44}$$

$$\begin{array}{|c|} \hline P \Leftrightarrow Q \\ \hline Q \Rightarrow P \\ \hline \end{array} \tag{I.45}$$

As previously stated, this is functionally the converse of the previous **Rule Of Inference**, so we will forgo justifying it here and trust that the reader can find a comfortable level of confidence using their own reasoning.

Modus Ponens and Implication Conjunction This is one of the most classic **Rules Of Inference**. **Modus Ponens** translates to 'mode that by affirming affirms'. The reason for this naming will be apparent once we have written the rule itself. One can also think of it as **Implication Elimination**.

Definition I.1.26 (Modus Ponens)**.**

$$\begin{array}{|c|} \hline P \Rightarrow Q \\ P \\ \hline Q \\ \hline \end{array} \tag{I.46}$$

This rule follows from considering that $P \Rightarrow Q$ can have a Logical Quality of $\top$ if P does not or both P and Q do. So, when $P \Rightarrow Q$ and P have a Logical Quality of $\top$, then so must be Q.

The last **Rule Of Inference** we will describe – before moving on to **Rules Of Replacement** – is called **Implication Conjunction**:

Definition I.1.27 (Implication Conjunction)**.**

$$\begin{array}{|c|} \hline P \Rightarrow Q \\ R \Rightarrow S \\ \hline (P \wedge R) \Rightarrow (Q \wedge S) \\ \hline \end{array} \tag{I.47}$$

Rules of Replacement Briefly we will speak of so-called **Rules Of Replacement**. These follow from our Truth-Tables as well, though are still worth noting momentarily. They include the notions about Conjunctions and Disjunctions of Associativity – the property that enables us to evaluate them in any temporal order we please – Commutativity – the property that enables us to evaluate them without regard for which side of the symbol the propositions occur on – and Distributivity, which we will discuss more thoroughly in the following section. Also included are the important notions of Transposition and De Morgan's Law. We will go ahead and state – formally – **Transposition** and De'Morgan's Law before proceeding to develop First-Order Logic in the next section.

Definition I.1.28 (Transposition)**.**

$$\frac{P \Rightarrow Q}{\neg Q \Rightarrow \neg P} \tag{I.48}$$

$$\frac{\neg Q \Rightarrow \neg P}{P \Rightarrow Q} \tag{I.49}$$

This is the property of MATERIAL IMPLICATION that $P \Rightarrow Q$ is LOGICALLY EQUIVALENT to $\neg Q \Rightarrow \neg P$. It follows from considering that $P \Rightarrow Q$ is true in the exact same cases as $\neg Q \Rightarrow \neg P$ since, if P is **True** then Q must also be, but if Q is **False** then P must also be. As a result, if Q has a LOGICAL QUANTITY of $\perp$ – necessitating that P does not – then both $\neg Q$ and $\neg P$ will have LOGICAL QUANTITIES of $\top$. Similarly, if P has a LOGICAL QUANTITY of $\top$ then Q must also, so $\neg Q$ and $\neg P$ must both not. As we can see, they share their LOGICAL QUANTITIES in all cases and so are LOGICALLY EQUIVALENT.

Definition I.1.29 (De Morgan's Law)**.**

$$\frac{\neg(P \wedge Q)}{\neg P \vee \neg Q} \tag{I.50}$$

$$\frac{\neg(P \vee Q)}{\neg P \wedge \neg Q} \tag{I.51}$$

There are many ways to arrive at this Law, but one might consider that NOR and EXCLUSION each are the NEGATED version of DISJUNCTIONS and CONJUNCTIONS, respectively. Then, consider that NEGATING both input PROPOSITIONS, is akin to flipping the output of a TRUTH-TABLE along its top-left to bottom-right diagonal, and it becomes clear that these COMPOUND PROPOSITIONS are LOGICALLY EQUIVALENT. There are more robust ways of reasoning one's way to **DE MORGAN'S LAW**, but we entrust such a task to the reader in favor of brevity.

I.2 First-Order Logic

Objective We wish to familiarize the reader with the more broad notion of a **FIRST-ORDER LOGIC** – i.e. **QUANTIFICATIONAL LOGIC**. We want to introduce the notion of **VARIABLES** as objects whose assignment varies for nearly the entire duration of their consideration. We also want to introduce the notion of **FORMULAE** as PROPOSITIONS that take an 'input' – namely: **VARIABLES** – that then go on to determine their LOGICAL QUANTITY for any given evaluation. We will then conclude by introducing several symbols to achieve **QUANTIFICATION**.

Strategy We will attempt to motivate **Variables** and **Formulae** by first considering **Formulae** as 'questions' we are asking about 'things' – i.e. **Variables**. We will then define several **Logical Quantifiers**. We will then introduce notation for **Bound Quantifiers** that will be used to aid in constructing Set-Theory in the next section.

I.2.1 Variables and Formulae

This subsection will be rather brief. We will define **Variables** and **Formulae** as counterparts to one another that provide something of an 'interface' to Propositional Calculus. In this conception, **Variables** are symbols that represent an entire range of potential *things* – which *things* is left to be decided, but ostensibly, the range of *all things*, potentially. Their counterparts are **Formulae** which take an **Input** *thing* and return a Logical Quantity. Presumably, each **Formula** is asking some kind of 'logical question' about its **Input** and then assigning a Logical Quantity to indicate in what way the **Input** answers the question – in the positive, yielding $\top$, or in the negative, yielding $\perp$.

Definition I.2.1 (Logical Formula)**.** A **Logical Formula** takes any valid **Input** to either Logical Quantity, i.e. $\top$ or $\perp$,. We will choose to represent **Logical Formulae** using the lowercase letter of the Greek alphabet:

$$\left.\begin{matrix}\alpha\\ \beta\\ \gamma\\ \vdots\end{matrix}\right\} \diamond\!\!\Rightarrow \textbf{Atomic Formulae} \tag{I.52}$$

Definition I.2.2 (Formulaic Variable)**.** A **Formulaic Variable** is a symbol that represents an **Input** to a **Formula**. A **Free Variable** is a **Variable** that is not Bound. **Variables** are Bound – if not through some given Abbreviation or Assignment – through the process of iterated Quantification, so we wait till the next subsection to discuss Bound Variables more thoroughly. We will choose to represent **Formulaic Variables** using lowercase letters of the Latin alphabet, often – but not always – starting nearer the end:

$$\left.\begin{matrix}\vdots\\ x\\ y\\ z\end{matrix}\right\} \diamond\!\!\Rightarrow \textbf{Formulaic Variables} \tag{I.53}$$

We will discuss two of our most basic **Formulae**.

The Trivial Formulae The two trivial **Formulae** – the **Universal Tautology** and the **Universal Falsehood** – reuse rather appropriate notation from Propositional Logic:

Definition I.2.3 (Universal Tautology)**.** This **Formulae** always has a Logical Quantity of $\top$ for any variable x and so is written and Assigned:

$$\top(x) \;\square\!\!\Rightarrow\; \top \tag{I.54}$$

Definition I.2.4 (Universal Falsehood)**.** This **Formulae** always has a Logical Quantity of $\bot$ for any variable x and so is written and Assigned:

$$\bot(x) \;\square\!\!\Rightarrow\; \bot \tag{I.55}$$

I.2.2 Quantification and Equality of Variables

Here we will introduce the notion of **Variable Quantification** and **Variable Equality**. In **First-Order Logic**, this is most usually done by introducing new symbols, namely: $\exists$, $\exists!$, $\nexists$, $\forall$, and – of course – the well known $=$ and $\neq$. This is the route we will be taking, albeit somewhat reluctantly.

All of Them but Also Not All of Them We will first define the **Universal Quantifier** as it can be used as a *somewhat* firm foundation to assign meaning to the remainder of our other **Logical Quantifiers** via Abbreviation.

Definition I.2.5 (Universal Quantifier)**.** The **Universal Quantifier** is understood as **Binding** a variable to **Span** *all* Variables that meet a certain condition. That is, we would like the **Universal Quantifier** to **Span** a smaller collection – than that of the 'totality' – of Variables that satisfy some **Binding Formula**, ψ. We write such a 'bounded binding' like so:

$$\forall x\, \psi(x) \;\square\!\!\Rightarrow\; \text{For All Formulaic Variables such that the Logical Quantity of } \psi(x) \text{ is } \top \ldots \tag{I.56}$$

A 'genuinely universal' **Universal Quantifier** may be written substituting $\psi(x)$ for $\top(x)$ since $\top(x)$ has a Logical Quantity of $\top$ regardless of the variable.

$$\forall x \;\diamond\!\!\Rightarrow\; \forall x\, \top(x) \tag{I.57}$$

Definition I.2.6 (Existential Quantifier)**.** The **Existential Quantifier** is an Abbreviation for the Negation of the **Universal Quantifier** and its **Binding Formula**:

$$\exists x\, \psi(x) \;\; \Leftrightarrow \;\; \neg\forall x\, \neg\psi(x) \tag{I.58}$$

This can be read as "There exists some x such that the Logical Quantity of $\psi(x)$ is $\top$ ..." or "There is at least one x such that the Logical Quantity of $\psi(x)$ is $\top$ ..."

This means the so-called 'Inexistential Quantifier' would be the Negation of the 'genuinely universal' **Universal Quantifier**:

$$\nexists x \;\; \Leftrightarrow \;\; \neg\forall x\, \neg\top(x) \tag{I.59}$$

This Abbreviation reads as "There does not exist x ..." We may write these Abbreviations and be confident they actually represent the notion of **Existential Quantifiers** by virtue of the semantic equivalence of the notation they are Abbreviating. Consider that if we said – in the case of the first Abbreviation – "For *not* all Formulaic Variables such that the Logical Quantity of $\neg\psi(x)$ is $\top$ ..." we would have conveyed the same meaning that 'some' – but not all – of the Formulaic Variables give $\psi(x)$ the Logical Quantity of $\bot$, meaning 'some' others must give $\psi(x)$ the Logical Quantity of $\top$. Similarly, what the 'Inexistential Quantifier' Abbreviates can be read as "For not all x such that the Logical Quantity of $\neg\top(x)$ is $\top$ ..." but of course $\neg\top(x) \Leftrightarrow \bot(x)$, which will always have a Logical Quantity of $\bot$.

Indiscernible is Basically Identical, Right? We will define **Variable Equality** according to three qualities that it must exhibit, which will allow us to define our final **Variable Quantifier**.

Definition I.2.7 (Variable Equality)**.** We say that two Variables are **Equal** if they are joined by a Logical Connective, denoted $=$, that makes the following three Formulae into Tautologies – i.e. makes each **Universal Quantifier** 'genuinely universal'.

1. Reflexivity: $\forall x\, (x = x)$
2. Symmetry: $\forall x \forall y\, [(x = y) \Leftrightarrow (y = x)]$
3. Transitivity: $\forall x \forall y \forall z\, [[(x = y) \wedge (y = z)] \Leftrightarrow (x = z)]$

In words, each of these mean:

1. Reflexivity: For all x, x is **Equal** to itself.

2. Symmetry: For all x and y, x is **Equal** to y if and only if y is **Equal** to x.

3. Transitivity: For all x, y, and z, x is **Equal** to y and y is **Equal** to z if and only if x is also **Equal** to z.

We also have the following Abbreviation:

$$x \neq y \iff \neg(x = y) \tag{I.60}$$

The One and Only We use the **Unique Existential Quantifier** to Abbreviate a specific combination of an **Existential Quantifier** and **Universal Quantifier** with a **Binding Formula** that utilizes **Variable Equality**. In this way, the **Unique Existential Quantifier** seems an appropriate culmination to conclude this section.

Definition I.2.8 (Unique Existential Quantifier)**.** We refer to the following abbreviation as a **Universal Existential Quantifier**

$$\exists! x\, \psi(x) \iff \exists x \forall y\, [\psi(y) \Leftrightarrow (x = y)] \tag{I.61}$$

This can be read as "There exists a unique x such that $\psi(x)$ has a Logical Quantity of $\top \ldots$" The notation being Abbreviated can be interpreted as saying "There exists some x such that for all y the Logical Quantity of $\psi(y)$ is Logically Equivalent to $x = y$." So, in the case that $\psi(y)$ has a Logical Quantity of $\top$, it must be that $x = y$ does also, otherwise both $\psi(y)$ and $x = y$ will have a Logical Quantity of $\bot$, i.e. $\neg\psi(y)$ and $x \neq y$ will have Logical Quantities of $\top$.

I.3 Subsets and Set-Building

Objective We wish to familiarize the reader with the common notion of **Set-Builder Notation** as well as common **Set Operations**. We seek to demonstrate how the Logical Symbols from the first section can be used to create **Sets** rather intuitively, and then how these may be combined to form new **Sets** also. Notably, the Axioms Of Set-Theory are absent save for allusions to which of them allow us to perform the relevant **Set Operations**, in order to ensure confidence in the reader. The common **ZFC** list of axioms are made available in the appendix, for the curious reader.

Strategy This section will Abbreviate collections of notation in order to reference many important **Sets** as well as motivate the usage of the previous Logical Symbols in how one might form more complex **Sets**.

I.3.1 Set-Builder Notation

Everything in Set-Theory Was a Set . . . Before we begin, the understanding that *almost* everything in **Set-Theory** is a **Set**, is somewhat technical, but important. We will speak as if we consider *everything* a **Set**, in order to simplify our language, and this is a result of this acknowledgement and the intention to steer clear of so-called Proper Classes. We will go about showing throughout this section how one can create bigger **Sets** from smaller **Sets** and vice versa.

Sets in Sets Sets are defined by their **Membership**, and only their **Membership**. What this means is that two sets are *completely identical* – i.e. properly Equal – if they each have the same **Elements** in them. As a result, **Sets** defined in seemingly different ways might still be the same **Set** if we can show that all of the **Elements** in one is in the other and vice versa. For the curious, this is commonly known as The Axiom Of Extensionality.

We will first define somewhat rigorously what we mean by **Membership** and the notion that some element is **Contained** in some **Set**.

Definition I.3.1 (Membership)**.** We utilize a so-called Impredicative Definition so that one might be provided at all. For something more robust, one might consider a definition more along the lines of how we defined Equality in the previous section, i.e. one could consult the Axioms Of ZF and let a Logical Connective $\in$ represent **Memebership** if and only if it makes all of the Axioms into Tautologies. We will, instead submit an Impredicative Definition – i.e. a definition that references the object being defined (but not necessarily the definition itself) – so that the reader might have an intuitive grasp of **Membership** without consulting the – at times – somewhat dense Axioms Of ZF.

$$x \in X \iff \forall Y \left[(Y \neq X \wedge \forall y \left[y \in X \Rightarrow y \in Y \right]) \Rightarrow x \in Y \right] \tag{I.62}$$

This left-hand side of this Abbreviation can be read as “x in X”, where x and X are understood as being distinct **Sets**. The right-hand side of this is admittedly a bit longer when linguistically reconstructed. It reads “For all Y distinct from X and such that for every y whose **Membership** in X implies their **Membership** in Y it is implied that x has **Membership** in Y.” Well. . . what does that mean? In essence, it says that every **Set** that isn't X but *at least* shares all the **Membership** of X – i.e. it could have a larger **Membership** – must also contain x.

We will now notate the Abbreviation for $x \notin X$, which is – in truth – simply $\neg(x \in X)$, however, it is worthwhile to be able to easily inspect what such a Formula would look like so we can determine if it meets our expectations for what $\notin$ should mean:

$$x \notin X \iff \exists Y \left[x \notin Y \multimap (Y = X \vee \exists y \left[y \in X \multimap y \in Y \right]) \right] \tag{I.63}$$

Now we can read it as "There must exist a **Set** that x is not in that Precludes either being equal to X or the Existence of an **Element** y whose **Membership** in X Precludes their membership in that **Set**.

We will now create some notation, that communicates so called **Set-Inclusion**, that will later allow us to show that **Membership Equivalence** gives us **Set Equivalence** in the case for certain **Sets** we are interested in. Central to this notion is the definition of a **Subset**.

Definition I.3.2 (Subsets/Supersets & Proper Subsets/Supersets)**.** We say that one **Set** X is a **Subset** of another **Set** Y if all **Members** of X are also **Members** of Y. The relevant Abbreviation is as follows:

$$X \subseteq Y \iff \forall a \left[a \in X \Rightarrow a \in Y \right] \tag{I.64}$$

We say X is a **Proper Subset** of Y if there is some **Element** in Y that is *not* in X:

$$X \subset Y \iff \forall a \left[a \in X \Rightarrow a \in Y \right] \wedge \exists b \left[b \in X \multimapinv b \in Y \right] \tag{I.65}$$

We say that one **Set** X is a **Superset** of another **Set** Y if all **Members** of Y are also **Members** of X. The relevant Abbreviation is simple:

$$X \supseteq Y \iff Y \subseteq X \tag{I.66}$$

Similarly, for a **Proper Superset**:

$$X \supset Y \iff Y \subset X \tag{I.67}$$

Building Subsets Out of Supersets A **Set** can be constructed from one of its **Supersets** using nearly any Logical Formula with this notation:

Definition I.3.3 (Set-Builder Notation)**.** For some **Formula** ψ we say that there is also a **Set**, X, that can be retrieved by ψ 'acting on' the **Elements** of another **Set**, Y. We use the notation below to describe such a **Set**:

$$X \;\square\!\!\Rightarrow\; \left\{ x \in Y \;:\; \psi(x) \right\} \tag{I.68}$$

We interpret the previous assignment as "The **Membership** of Y such that $\psi(x)$ has a Logical Quantity of $\top$ is the **Memebership** of X." As previously noted, we assume a priori that x must be a **Set**.

The Trivial Subsets Built by the Trivial Formulae We will conclude this subsection by noting two kinds of **Subsets** that can always vacuously be built from any given **Set**; namely, those defined by the trivial Logical Formulae $\top(x)$ and $\bot(x)$.

Definition I.3.4 (The Trivial Subsets)**.** The first **Subset** is built by the Formula $\top(x)$ 'acting on' some set X:

$$X = \left\{ x \in X \;:\; \top(x) \right\} \tag{I.69}$$

One can readily see this **Set** is simply itself over again, which introduces the important idea that a **Set** is always a **Subset** of itself. It is for this reason that we define **Proper Subsets**. This is also an excellent demonstration of **Set Equality**, which can be interpreted as the case when any two **Sets** are **Subsets** of eachother.
The other **Subset** is built by the Formula $\bot(x)$ 'acting on' some set X:

$$\varnothing = \left\{ x \in X \;:\; \bot(x) \right\} \tag{I.70}$$

This is a **Set** named – appropriately – **The Empty-Set**. One can readily see this **Set** is named aptly, as no **Set** will ever be able to be in it, regardless of which **Set** X we have started with.

I.3.2 Intersections, Unions, Complements, and Differences (Oh My!)

The Similarities Between Sets We will now introduce several interactions between Sets that will be helpful for building other Sets. The first of these will be the **Intersection** of two or more Sets:

Definition I.3.5 (Intersection)**.** We say that the **Intersection** of two Sets X, Y is a Set, that has the Membership of only those Sets in X and Y:

$$x \in \left(X \bigcap Y \right) \;\Leftrightarrow\; x \in X \wedge x \in Y \tag{I.71}$$

Similarly, this process can be repeated an arbitrary amount of times by **Indexing** by some set I:

$$x \in \bigcap_{i \in I} X_i \iff \forall i \left[i \in I \right] \Rightarrow x \in X_i \tag{I.72}$$

What Sets Can Learn From Each-Other The related notion is the **Union** of two or more Sets:

Definition I.3.6 (Union)**.** We say that the **Union** of two Sets X, Y is the Set whose Membership are those Sets that are in X or Y:

$$x \in \left(X \bigcup Y \right) \iff x \in X \vee x \in Y \tag{I.73}$$

Similarly, this process can be repeated an arbitrary amount of times by **Indexing** by some set I:

$$x \in \bigcup_{i \in I} X_i \iff \exists i \left[i \in I \right] \Rightarrow x \in X_i \tag{I.74}$$

Set Theory Out of Logic The attentive reader will notice that in the sense that these are the 'set versions' of Conjunctions and Disjunctions, that Set Inclusion is the 'set version' of Material Implication. It then also becomes clear that Set Equality is the 'set version' of Logical Equivalence.

The Set Theoretic Negation Continuing in this vein of creating 'set versions' of our Logical Symbols, we might consider what arises when we give the same treatment to Logical Negation. Well, we receive our definition of the **Complement** of a Set:

Definition I.3.7 (Complement)**.** We say that the **Complement** of a Set X is the Set with Membership of those Sets not in X:

$$x \in X^C \iff x \notin X \tag{I.75}$$

There is a mild subtly here, that we opt to not cover, however.

How Sets Can Exclude Each-Other We will now define the so-called 'negation' of the Subset relationship between Sets – i.e. the 'set-version' of Extant Preclusion – which we call the **Difference**. We will then consider the 'negation' of Set Equality – i.e. the 'set-version' of Logical Contradiction – known as the **Symmetric Difference**.

Definition I.3.8 (Difference)**.** We say that the **Difference** of two Sets X, Y is the Set which has the Membership of X except for the component of Membership shared with Y:

$$x \in X \setminus Y \iff x \in X \multimap x \in Y \tag{I.76}$$

Definition I.3.9 (Symmetric Difference)**.** We say that the **Symmetric Difference** of two Sets X, Y is the Set with Membership from X but not Y and Y but not X, i.e. the Membership of X and Y that Contradict eachother are not included in the new Membership:

$$x \in \left(X \bigtriangleup Y\right) \iff x \in X \veebar x \in Y \tag{I.77}$$

I.3.3 Powersets, and Cartesian Products

All of the Subsets We will have cause to speak of every single Subset of a given Set X. This is done using the **Powerset** construction:

Definition I.3.10 (Powerset)**.** The **Powerset** of a set X is the set who has Membership of exactly all of the Subsets of X:

$$x \in \mathcal{P}(X) \iff x \subseteq X \tag{I.78}$$

Order Matters Sometimes If a Set is purely identified by its Membership then how are we meant to denote – say – an **Ordered Pair**? A Set containing both of the Elements will not do, because Sets are totally unordered, meaning we could not say which was 'first' like we would desire.

Definition I.3.11 (Ordered Pair)**.** An **Ordered Pair** of two Sets a, b such that $a \in A$ and $b \in B$, is an Abbreviation for another Set that manages to Order them:

$$(a, b) \iff \begin{cases} \Big\{ x \in \mathcal{P}\left(\mathcal{P}\left(A \cup B\right)\right) : \\ \exists! f \left[f \in x \wedge f \in \mathcal{P}(A) \wedge \forall g \left[g \in f \Leftrightarrow g = a \right]\right] \\ \wedge \\ \exists! s \left[s \in x \wedge s \in \mathcal{P}\left(\mathcal{P}\left(A \cup B\right)\right) \wedge \forall t \left[t \in s \Leftrightarrow (t = a \vee t = b)\right]\right] \Big\} \end{cases} \tag{I.79}$$

Definition I.3.12 (Cartesian Product)**.** The **Cartesian Product** of two Sets A and B is the set of all **Ordered**

Pairs with the first Element in A and the second Element in B:

$$x \in A \times B \;\Leftrightarrow\!\!\Rightarrow\; \exists a \exists b \left[a \in A \wedge b \in B \Leftrightarrow \; x = (a, b) \right] \tag{I.80}$$

We also use so-called **Product Notation** in order to describe iterated **Cartesian Products**. In the case of n applications of the **Cartesian Product** on a single Set X, called the $(n+1)^{\text{th}}$ **Cartesian-Product**, we write the resulting Set of such an iterated process as:

$$\prod^{n} X \;\Leftrightarrow\!\!\Rightarrow\; \underbrace{X \times X \times \ldots \times X}_{n \text{ Times}} \tag{I.81}$$

Occasionally, we will use *In-Line Product Notation* which is mildly different; the equivalent in-line notation for the Set described above would be: $\prod^{n} X$. In the case of infinite iteration – which is allowable – one replaces the n with ∞.

Alternatively, if one wishes to describe a more general iterated **Cartesian Products** between Sets that are not equivalent – as will be the case in the following chapter for Heterogeneous Relations and the like – we will often speak of an Index Set I – as before with Unions and Intersections. This is to be understood as a Set that assigns each of its elements to some fixed set X_i – or indeed in many cases each element i is responsible for somehow 'determining' the Set X_i itself. We write the iterated **Cartesian Product** of such a family of Sets, said to be "indexed by the Set I" as follows:

$$x \in \prod_{i \in I} X_i \;\Leftrightarrow\!\!\Rightarrow\; \forall i \left[i \in I \Rightarrow \exists ! x_i \left[x_i \in X_i \Leftrightarrow \; x_i \in x \right] \right] \tag{I.82}$$

The equivalent in-line notation is $\prod_{i \in I} X_i$.

CHAPTER II

RELATIONS AND FUNCTIONS

II.1 Kinds of Relations

Objective We will explore several different ways to define and categorize **Relations** so that we may study them later.

Strategy We will establish the foundational notion of **Relations** as Subsets of Cartesian-Products in order to make sense of them and so we might define certain properties on them.

II.1.1 Relations as Subsets

We will formalize the notion of a **Relation** as a Subset of the Cartesian-Product of two or more Sets.

Definition II.1.1 (Relation)**.** A **Relation**, R, between a family of Sets – X_i, indexed by the Set I, where each Set X_i is referred to as the **Domain** of R – is a Subset of the Cartesian-Product of those Sets:

$$R \subseteq \prod_{i \in I} X_i \tag{II.1}$$

We say that when a Tuple $(x_1, x_2, \ldots x_i)$ is a Member of R – that is $(x_1, x_2, \ldots x_i) \in R$ – then each element of the Tuple relates to the other Elements based on its position – i.e. its originating set if $X_i \neq X_j$ for each Ordered Pair (i, j) – in the Tuple.

Arity The **Arity** of a **Relation** is the number of Sets that the Cartesian-Product it is a Subset of has in it. So, a Relation over two Sets X, Y is a **Binary Relation**. Similarly, a **Relation** over three Sets X, Y, Z is a **Ternary Relation**. In general, a Relation of n Sets is said to be an **n-Ary Relation**. We will restrict our focus in future sections to largely only **Binary-Relations**, but will speak in general of **n-Ary Relations** for the remainder of this section.

In the case that we are discussing a **Binary Relation**, however, we will utilize **Infix Notation**. Consider a **Binary Relation**, $\sim$, that relates a **Domain** X to a **Co-Domain** Y – only in the case of a **Binary Relation** do we distinguish the **Co-Domain** – and specifically **Relates** the fixed Elements $x \in X$ and $y \in Y$ the Abbreviation that describes **Infix Notation** is:

$$x \sim y \iff (x, y) \in R \tag{II.2}$$

II.1.2 Genus

The **Genus** of a Relation qualifies which Sets the Cartesian-Product it is a Subset of is over. There are primarily two broad **Genera** – both of which will be familiar to the reader from other areas – that is: **Heterogenous** and **Homogenous**.

Definition II.1.2 (Heterogenous Relations)**.** A **Heterogeneous Relation** is one between a family of sets X_i indexed by K such that:

$$\exists i \exists j \left[i \in K \wedge j \in K \Rightarrow X_i \neq X_j \right] \tag{II.3}$$

In words, this means that a **Heterogenous Relation** is one between *distinct* Sets, i.e. not all of the Sets that are being Related are the same Set. A **Homogenous Relation** is – predictably – the opposite of this:

Definition II.1.3 (Homogenous Relations)**.** A **Homogenous Relation** is one between a family of Sets X_i indexed by K such that:

$$\forall i \forall j \left[i \in K \wedge j \in K \Leftrightarrow X_i = X_j \right] \tag{II.4}$$

In the next section we will describe many properties on **Homogenous Relations**, but will turn to considering both **Genera** of Relations in the section after that when discussing Functions.

II.2 Properties of Homogenous Relations

Objective We will explore several common properties of RELATIONS on a single SET – i.e. HOMOGENOUS RELATIONS – as well as the names we give to RELATIONS that express them.

Strategy We will use the distinction we created in the previous section about the GENUS OF A RELATION and focus on HOMOGENOUS RELATIONS for this section. Notably, we restrict our focus to BINARY-RELATIONS for the remainder of the work – including this section – as indicated in the previous section. Although we will be making this restriction, a later discussion elucidates a way by which a conversation about BINARY-RELATIONS serves to facilitate a conversation about all n-ARY RELATIONS, as well as describing how this process can be done for UNARY-FUNCTIONS and BINARY-OPERATORS.

II.2.1 Reflexive and Irreflexive

Definition II.2.1 (Reflexive Relations)**.** A **REFLEXIVE RELATION**, $\sim$, is a HOMOGENOUS RELATION on X such that:

$$\forall x[x \in X \implies x \sim x] \tag{II.5}$$

In word, this means that **REFLEXIVE RELATIONS** must relate every ELEMENT in the DOMAIN to itself.

Definition II.2.2 (Irreflexive Relations)**.** An **IRREFLEXIVE RELATION**, $\sim$, is a HOMOGENOUS RELATION on X such that:

$$\forall x[x \in X \implies x \nsim x] \tag{II.6}$$

Similarly, in words, this means that **IRREFLEXIVE RELATIONS** must not relate any ELEMENT in the DOMAIN to itself.

II.2.2 Symmetric and Anti-Symmetric

Definition II.2.3 (Symmetric Relations)**.** A **SYMMETRIC RELATION**, $\sim$, is a HOMOGENOUS RELATION on X such that:

$$\forall x \forall y[x \in X \wedge y \in X \implies x \sim y \Leftrightarrow y \sim x] \tag{II.7}$$

In words, this means that **Symmetric Relations** that relate a Pair of Elements must relate the mirror of that Pair also.

Definition II.2.4 (Anti-Symmetric Relations)**.** An **Anti-Symmetric Relation**, $\sim$, is a homogenous relation on X such that:

$$\forall x \forall y [x \in X \wedge y \in X \;\Rightarrow\; ((x \sim y \wedge y \sim x) \Rightarrow (x = y))] \tag{II.8}$$

In words, this means that **Anti-Symmetric Relations** that Relate both a Pair and its Mirror only does so for a Pair of Equal Elements.

II.2.3 Transitive and Anti-Transitive

Definition II.2.5 (Transitive Relations)**.** A **Transitive Relation**, $\sim$, is a Homogenous Relation on X such that:

$$\forall x \forall y \forall z [(x \in X \wedge y \in X \wedge z \in X) \Rightarrow ((x \sim y \;\wedge\; y \sim z) \;\Rightarrow\; x \sim z)] \tag{II.9}$$

In words, this means that **Transitive Relations** that Relate a Pair, and the right of that Pair to another Element, must also Relate the left of the initial Pair to the new Element.

Definition II.2.6 (Anti-Transitive Relations)**.** An **Anti-Transitive Relation**, $\sim$, is a Homogenous Relation on X such that:

$$\forall x \forall y \forall z [(x \in X \wedge y \in X \wedge z \in X) \Rightarrow ((x \sim y \;\wedge\; y \sim z) \;\Rightarrow\; x \nsim z)] \tag{II.10}$$

In words, this means that **Anti-Transitive Relations** that Relate a Pair, and the right of that Pair to another Element, must never Relate the left of the initial Pair to the new Element.

II.2.4 Connex, Semi-Connex, and Trichotomous

Definition II.2.7 (Connex Relations)**.** A **Connex Relation**, $\sim$, is a Homogenous Relation on X such that:

$$\forall x \forall y [(x \in X \wedge y \in X) \;\Rightarrow\; \big(x \sim y \;\vee\; y \sim x\big)] \tag{II.11}$$

In words, this means that **Connex Relations** must Relate every Pair of Elements, its mirror Pair, or both.

Definition II.2.8 (Semi-Connex Relations)**.** A **Semi-Connex Relation**, $\sim$, is a Homogenous Relation on X such that for $x, y \in X$:

$$\forall x \forall y[(x \in X \wedge y \in X) \Rightarrow (x \neq y \;\Rightarrow\; \left(x \sim y \;\vee\; y \sim x\right))] \tag{II.12}$$

In words, this means that **Semi-Connex Relations** must Relate every Pair of Unequal Elements or the respective mirror Pair.

Definition II.2.9 (Trichotomous Relations)**.** A **Trichotomous Relation**, $\sim$, is a Homogenous Relation on X such that:

$$\forall x \forall y[(x \in X \wedge y \in Y) \Rightarrow \begin{cases} [(x \sim y) \multimap & [(y \sim x) \vee (x = y)]] \\ & \vee \\ [(y \sim x) \multimap & [(x \sim y) \vee (x = y)]] \\ & \vee \\ [(x = y) \multimap & [(x \sim y) \vee (y \sim x)]] \end{cases} \Bigg] \tag{II.13}$$

In words, this means that for **Trichotomous Relations** exactly one of the following is true: a Pair is Related; a Pair's mirror is Related; the Pair is Equal.

II.2.5 Order Relations

We will now give standard names to Relations that convey a sense of 'order' on the Elements in X.

Definition II.2.10 (Preorder)**.** A Homogenous Relation that is Reflexive and Transitive is a **Preorder**.

Definition II.2.11 (Total Preorder)**.** A **Preorder** that is Connex is a **Total Pre-Order**.

Definition II.2.12 (Partial Order)**.** A **Preorder** that is Anti-Symmetric is a **Partial Order**.

Definition II.2.13 (Total Order)**.** A **Partial Order** that is Connex is a **Total Order**.

Definition II.2.14 (Strict Preorder)**.** A Homogenous Relation that is Irreflexive and Transitive is a **Strict Preorder**.

Definition II.2.15 (Strict Total Preorder)**.** A **Strict Preorder** that is Semi-Connex is a **Strict Total Pre-Order**.

Definition II.2.16 (Strict Partial Order)**.** A **Strict Preorder** that is Anti-Symmetric is a **Strict Partial Order**.

Definition II.2.17 (Strict Total Order)**.** A **Strict Partial Order** that is Semi-Connex is a **Strict Total Order**.

II.2.6 Equivalence Relations

We will now give standard names to Relations that convey a sense of 'equivalence' on the Elements in X.

Definition II.2.18 (Partial Equivalence Relation)**.** A Homogenous Relation that is Symmetric and Transitive is a **Partial Equivalence Relation**.

Definition II.2.19 (Equivalence Relation)**.** A **Partial Equivalence Relation** that is Reflexive is an **Equivalence Relation**.

II.3 Properties of Heterogenous Relations

Objective We will now explore several different ways to categorize all relations with an emphasis on the more general case of Heterogenous Relations.

Strategy We will use the foundation we created in one of the previous sections about Relations as subsets of Cartesian-Products to define certain properties by discussing conditions on member pairs of the relation.

II.3.1 Uniqueness Properties

There are a number properties that deal with the **Uniqueness** of a Relation between a given Pair of Elements.

Definition II.3.1 (Injective Relations)**.** An **Injective Relation**, $\sim$, is a Homogenous Relation on X or a Heterogeneous Relation on X and Y such that:

$$\forall x \forall y \forall z[(x \in X \land y \in X \land z \in X) \Rightarrow ((x \sim y \ \land \ z \sim y) \Rightarrow x = z)] \tag{II.14}$$

$$\text{or} \tag{II.15}$$

$$\forall x \forall y \forall z[(x \in X \land y \in Y \land z \in X) \Rightarrow ((x \sim y \ \land \ z \sim y) \Rightarrow x = z)] \tag{II.16}$$

This property is also called **Left-Unique**, and in words means that for every Pair of Elements, the left Element must be the *only* Element that Relates to the right Element.

Definition II.3.2 (Functional Relations)**.** A **Functional Relation**, $\sim$, is a Homogenous Relation on X or a Heterogeneous Relation on X and Y such that:

$$\forall x \forall y \forall z [(x \in X \land y \in X \land z \in X) \Rightarrow ((x \sim y \ \land \ x \sim z) \Rightarrow y = z)] \tag{II.17}$$

$$\text{or} \tag{II.18}$$

$$\forall x \forall y \forall z [(x \in X \land y \in Y \land z \in Y) \Rightarrow ((x \sim y \ \land \ x \sim z) \Rightarrow y = z)] \tag{II.19}$$

This property is also called **Right-Unique**, and in words means that for every Pair of Elements, the right Element must be the *only* Element that Relates to the left Element.

II.3.2 Totality Properties

There are a number properties that deal with the **Totality** of a Relation on its Domain and Co-Domain.

Definition II.3.3 (Serial Relations)**.** A **Serial Relation**, $\sim$, is a Homogenous Relation on X or a Heterogeneous Relation on X and Y such that:

$$\forall x [(x \in X \land \exists y [y \in X]) \Rightarrow (x \sim y)] \tag{II.20}$$

$$\text{or} \tag{II.21}$$

$$\forall x [(x \in X \land \exists y [y \in Y]) \Rightarrow (x \sim y)] \tag{II.22}$$

This property is also called **Left-Total**, and in words means that for every Element in the Domain, there is a Pair in the Relation that has it as the left Element.

Definition II.3.4 (Surjective Relations)**.** A **Surjective Relation**, $\sim$, is a Homogenous Relation on X or a Heterogeneous Relation on X and Y such that: for every $y \in X$ or every $y \in Y$ there is at least one $x \in X$:

$$\forall y [(y \in X \land \exists x [x \in X]) \Rightarrow (x \sim y)] \tag{II.23}$$

$$\text{or} \tag{II.24}$$

$$\forall y [(y \in Y \land \exists x [x \in X]) \Rightarrow (x \sim y)] \tag{II.25}$$

This property is also called **Right-Total**, and in words means that for every element in the Co-Domain, there is a Pair in the Relation that has it as the right Element.

II.3.3 Kinds of Functions

We now have enough definitions to construct a notion of a **Function**:

Definition II.3.5 (Function)**.** A **Function** is a Homogenous Relation or Heterogeneous Relation, f, that is both Serial and Functional.

Important to note is that because a **Function** distinguishes Input from Output, if one has an n-Ary Relation, where $n \neq 2$, that they wish to distinguish as a **Function**, they must specify where such a split in the Elements in Member Tuples takes place. That is, one must specify some Equation $n = i + o$, that is interpreted to mean that the first i Elements in a Tuple are the Input and the final o Elements are the Output. This will also us to simply talk about Input and Output Elements from the respective i-Ary and o-Ary Sets, and we would call such a **Function** an i-Ary Function (we ordinarily suppress the number of Outputs as $o \neq 1$ tends to rarely be the case; as a result, most of the time, an appropriate n-Ary Relation is an $(n-1)$-Ary Function). Most commonly we are speaking of Binary Relations, which would be **Unary Functions**, as the only choices for i and o are each 1; so most **Functions** we discuss simply Map one Input to one Output.

Ordinarily, instead of the Infix Notation we often use in the case of Binary Relations. We instead adopt **Function Notation**, which will be reminiscent of our Formula Notation from Chapter 1. So, for a function f with Domain X and Co-Domain Y that assigns some fixed Element $x \in X$ to some other fixed Element $y \in Y$, we will denote such a Mapping as:

$$f(x) \;\; \Box\!\!\Rightarrow \;\; y \tag{II.26}$$

This also holds in the case were $i \neq 1$, and we separate Inputs by commas. Consider an i-Ary Function f, from $\prod_1^i X$ to Y such that it Assigns some fixed $(x_1, x_2 \ldots x_i) \in \prod_1^i X$ to some fixed $y \in Y$, we denote such a Mapping like so:

$$f(x_1, x_2, \ldots x_i) \;\; \Box\!\!\Rightarrow \;\; y \tag{II.27}$$

In the case that $o \neq 1$, we simply replace y with the appropriately sized Tuple.

This definition is derived from the idea that a **Function** ought be able to map every Element from its Domain, and only map each Element in the Domain to one Element in the Co-Domain. The other Uniqueness and Totality properties yield us the other kinds of **Functions** with which we are familiar:

Definition II.3.6 (Injection)**.** An **Injection** is a **Function** that is Injective.

This kind of function is also often said to be ‘into’.

Definition II.3.7 (Surjection)**.** A **Surjection** is a **Function** that is Surjective.

This kind of **Function** is also often said to be ‘onto’.

Definition II.3.8 (Bijection)**.** A **Bijection** is a **Function** that is both Injective and Surjective.

Aptly, this kind of **Function** will be frequently be called ‘into and onto’.

The Set of Functions We use the notation Y^X to denote the Set of all **Functions** with X as the Domain, and Y as the Co-Domain. We further denote the Subsets of this Set as follows: the **Injections** from X to Y as $I[Y^X]$; the **Surjections** from X to Y as $S[Y^X]$; and the **Bijections** from X to Y as $B[Y^X]$. It is worth noting that:

$$B[Y^X] \iff \left(I[Y^X] \bigcap S[Y^X]\right) \tag{II.28}$$

II.4 Function Composition

Objective We will explore the notion of **Function Compositon** and get an understanding of what it means to **Compose** two Functions.

Strategy We will use the terminology introduced from the previous section, to try to better understand what kind of Functions are the result of specific **Function Compositions**.

II.4.1 Composing Functions

When composing Homogenous Functions there are not many special cases to consider and so we are able to compose any Homogenous Function on a given Set X with any other Function on that same Set. That being said, we still need to introduce the notion of **Function Composition** in the first place:

Definition II.4.1 (Homogenous Function Composition)**.** For two Homogenous Functions $f, g \in X^X$, we can **Compose** these Functions in either direction, to receive two potentially distinct Homogenous Functions on

X, $f \cdot g$ and $g \cdot f$, respectively. We do this by using the OUTPUT of one as the INPUT to the other. For every ELEMENT in X:

$$[f \cdot g](x) \;\square\!\Rightarrow\; f(g(x)) \tag{II.29}$$

$$[g \cdot f](x) \;\square\!\Rightarrow\; g(f(x)) \tag{II.30}$$

When composing HETEROGENOUS FUNCTIONS there are several things to consider:

Definition II.4.2 (Heterogenous Function Composition)**.** For two HETEROGENOUS FUNCTIONS $g \in Y^X$, and $f \in Z^Y$, we can **COMPOSE** these FUNCTIONS in just one direction, to receive a distinct HETEROGENOUS FUNCTION in the SET Z^X, namely $f \cdot g$. We do this by using the OUTPUT of g as the INPUT to f. For every ELEMENT in X:

$$[f \cdot g](x) \;\square\!\Rightarrow\; f(g(x)) \tag{II.31}$$

Because $f \cdot g \in Z^X$ it is a FUNCTION of a completely different flavor to f or g. It will take an INPUT from X but produce an OUTPUT in Z, something that neither f or g can do independently.

II.4.2 Composition's Preservations

When considering our previous kinds of FUNCTIONS and the associated SETS, one might wonder if these properties are maintained in FUNCTION COMPOSITION, and indeed they are. We will go about proving each of these preservations, and consider only HETEROGENOUS FUNCTIONS as HOMOGENOUS FUNCTIONS are a special case of that broader class.

Theorem II.4.1 (The Composition of Injections is an Injection)**.** *For two* INJECTIONS*, $g \in I[Y^X]$ and $f \in I[Z^Y]$, we have that $f \cdot g \in I[Z^X]$.*

Proof. Since f, g are INJECTIVE:

$$\forall a \forall b[(g(a) \in Y \wedge g(b) \in Y) \Rightarrow [f(g(a)) = f(g(b)) \Leftrightarrow \; g(a) = g(b)]] \tag{II.32}$$

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow [g(a) = g(b) \Leftrightarrow \; a = b]] \tag{II.33}$$

Thus, $f \cdot g \in I[Z^X]$. □

Theorem II.4.2 (The Composition of Surjections is a Surjection)**.** *For two* SURJECTIONS*, $g \in S[Y^X]$ and $f \in S[Z^Y]$, we have that $f \cdot g \in S[Z^X]$.*

Proof. Since f, g are SURJECTIVE, for every $y \in Y$, there exists an $x \in X$ such that:

$$\forall y \exists x[(y \in Y \wedge x \in X) \Rightarrow (g(x) = y)] \tag{II.34}$$

Similarly, for every $z \in Z$ there exists $y \in Y$ such that:

$$\forall z \exists y[(z \in Z \wedge y \in Y) \Rightarrow (f(y) = z)] \tag{II.35}$$

So as a result:

$$\forall z \exists y[(z \in Z \wedge y \in Y) \Rightarrow (\exists x[(y \in Y \wedge x \in X) \Rightarrow (g(x) = y)] \Rightarrow f(g(x)) = z)] \tag{II.36}$$

Thus, for every $z \in Z$, there exists an $x \in X$ such that $[f \cdot g](x) = z$, so $f \cdot g \in S[Z^X]$. □

Theorem II.4.3 (The Composition of Bijections is a Bijection)**.** *For two* BIJECTIONS*, $g \in B[Y^X]$ and $f \in B[Z^Y]$, we have that $f \cdot g \in B[Z^X]$.*

Proof. This follows directly as a consequence of the two previous proofs.

$$\forall f \forall g[(f \in I[Z^Y] \wedge g \in I[Y^X]) \Rightarrow (f \cdot g \in I[Z^X])] \tag{II.37}$$

Which follows from the first proof,

$$\forall f \forall g[(f \in S[Z^Y] \wedge g \in S[Y^X]) \Rightarrow (f \cdot g \in S[Z^X])] \tag{II.38}$$

Which follows from the second proof. Thus – because of the IMPLICATION CONJUNCTION RULE OF INFERENCE – we are able to conclude that:

$$\forall f \forall g[((f \in I[Z^Y] \wedge f \in S[Z^Y]) \wedge (g \in I[Y^X] \wedge g \in S[Y^X])) \Rightarrow (f \cdot g \in I[Z^X] \wedge f \cdot g \in S[Z^X])] \tag{II.39}$$

Of course, because of how INTERSECTIONS of SETS are defined, we have that the line above simplifies to:

$$\forall f \forall g[(f \in B[Z^Y] \wedge g \in B[Y^X]) \Rightarrow (f \cdot g \in B[Z^X])] \tag{II.40}$$

□

CHAPTER III

OPERATORS AND ALGEBRAS

III.1 Functions as Operators

Objective We will now explore how **Operators** are defined as Multi-Variate Functions.

Strategy We will use the foundation we created in the previous section about Functions to describe **Operators** and a brief discussion on limiting our consideration to **Binary-Operators**

III.1.1 Functions with n Arguments and 1 Output

If we consider a Function with n Arguments, and only one Output, we will have arrived at the notion of an **Operator** – an **n-Ary Operator**, to be specific. The primary difference between an n-Ary Function and an **n-Ary Operator** is really one of convention and intended application. As a result, we will abstain from providing a definition distinct from that of a function, as no notation is introduced additionally for the general n-Ary case.

It is worth noting that – while not terribly standard – an **Operator** need not be Homogenous necessarily, despite this ordinarily being the case. It does need to be Homogenous to exhibit many of the properties we will discuss in the next section, but a **Heterogenous Operator** is possible and is termed an External Operator. Such **Operators** will be vital to the discussion to the brief discussion we will have in the next subsection that deals with justifying our limiting consideration to only **Binary-Operators**.

III.1.2 A Justification for Restriction to Binary-Operators

It might seem overly limiting to restrict focus to only BINARY-OPERATORS, as we will choose to do for the remainder of this work. The reason for this is primarily a result of where research in the field of ABSTRACT-ALGEBRA decided to look, and this will always suffer from selection-bias – e.g. are BINARY-OPERATOR's genuinely as much more interesting than TERNARY or any other n-ARY OPERATOR as the difference in research would suggest? – but it will also tend to favor that which is easiest to work with while still managing to seemingly escape triviality. We wish to offer an alternative – if not all-encompassing – justification for why one might be willing to restrict consideration to only **Binary-Operators**.

The crux of this justification is that while one may not have the same level of descriptive power on the characteristics of the OPERATOR, any n-ARY OPERATOR can be *represented* as a **Binary-Operator**. How is this the case?

Consider our previous distinction in the definition of a FUNCTION, where one is required to stipulate an equation $n = i + o$ that defines the number of INPUTS i and the number of OUTPUTS o. This split, already turns all FUNCTIONS into *Binary Relations*, as we then consider SUBSETS of the CARTESIAN-PRODUCT of two SETS, namely the iterated CARTESIAN-PRODUCT of a family of DOMAINS INDEXED by a SET I having exactly i MEMBERS, and the ITERATED CARTESIAN-PRODUCT of a family of CO-DOMAINS INDEXED by a SET O having exactly o MEMBERS. From this lense, *Unary* FUNCTIONS, which are BINARY-RELATIONS, are also UNARY-OPERATORS. As a result, BINARY-OPERATORS are BINARY-FUNCTIONS, and so TERNARY RELATIONS.

For an n-ARY OPERATOR – which is already an n-ARY FUNCTION with $o = 1$ (which is a SERIAL and FUNCTIONAL $(n + 1)$-ARY RELATION) – consider that it is a SUBSET of the CARTESIAN-PRODUCT that has been iterated $n + 1$ times. By virtue of our naming, we have that $n + 1 = i + o$ such that $i = n$ and $o = 1$. We can now perform an additional split, this time on i. Considering an EQUATION that prescribes a split of i – say something like $i = l + r$ – we already have a way of converting any $(n + 1)$-ARY RELATION – that is, an n-ARY OPERATOR – into a TERNARY-RELATION, i.e. a BINARY-OPERATOR by considering the appropriate SUBSET of the CARTESIAN-PRODUCT of the ITERATED CARTESIAN-PRODUCT INDEXED by a SET L with l MEMBERS and another ITERATED CARTESIAN-PRODUCT INDEXED by a SET R with r MEMBERS.

Symbolically, such an equivalence could be represented like so:

$$\prod_{j \in I \cup \{\omega\}} X_j = \prod_{j \in I} X_j \times X_\omega \tag{III.1}$$

$$\prod_{j \in I} X_j \times X_\omega = \left(\prod_{a \in L} X_a \times \prod_{b \in R} X_b \right) \times X_\omega \tag{III.2}$$

The first line represents the initial split of the $(n+1)$-ARY RELATION into an n-ARY FUNCTION with one OUTPUT set, represented by X_ω. The second line represents the split of the DOMAIN SETS on that FUNCTION into two distinct categories that will represent the *Left* and *Right* ARGUMENTS respectively to the resulting BINARY-OPERATOR.

One may notice that this process is not *strictly* limited to the BINARY and TERNARY case – those are simply the most helpful ones for facilitating conversations about UNARY-FUNCTIONS and BINARY-OPERATORS, respectively – and indeed it is the case that any n-ARY RELATION could be re-expressed by any m such that $2 \leqslant m < n$. As mentioned at the beginning of this 'justification', however, it is worth noting that this is a somewhat 'lossy' way to represent the RELATION, as one loses the ability to describe properties about the OPERATOR that are strictly reserved for any ARITY greater than m.

III.2 Properties of Binary-Operators

Objective We will explore a number of the most common properties that BINARY-OPERATORS can posses, and will henceforth restrict our attention to almost exclusively HOMOGENOUS BINARY-OPERATORS, or **INTERNAL OPERATORS**.

Strategy We will first describe the most commonly assumed property **CLOSURE**, before moving to the second most commonly assumed: **ASSOCIATIVITY**. In the same subsection in which we discuss **ASSOCIATIVITY** we will discuss the similar concept of **COMMUTATIVITY**. We will then turn our attention to properties that – while still truly global on the DOMAIN/CO-DOMAIN in nature – will have to do more directly with the existence and behavior of specific types of ELEMENTS.

III.2.1 Closure

In almost all conversations of ALGEBRAS, we will want for our OPERATOR to be **CLOSED**, which means the following:

Definition III.2.1 (Closure)**.** In order for an INTERNAL OPERATOR, $\cdot$, on a SET X, to exhibit the property of **CLOSURE** it must be the case that:

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow \exists c \exists d[(c \in X \wedge c = (a \cdot b)) \wedge (d \in X \wedge d = (b \cdot a))]] \tag{III.3}$$

This property follows from considering only strictly HOMOGENOUS OPERATORS, but in the case that the Co-

Domain is a Set distinct to the Domain set(s) – perhaps a Superset, as is the case for Division on the Integers – it is important to verify this property.

III.2.2 Associativity and Commutativity

The two likely most commonly discussed properties in a Group-Theory class are **Associativity** and **Commutativity**. This makes sense as **Associativity** is required in order for a Group to be formed in the first place, wheras **Commutativity** 'upgrades' a Group into an Abelian Group, which have many nice properties and are extensively studied.

Definition III.2.2 (Associativity)**.** In order for an Internal Operator, $\cdot$, on a Set X, to exhibit the property of **Associativity** it must be the case that:

$$\forall a \forall b \forall c[(a \in X \land b \in X \land c \in X) \Rightarrow [((a \cdot b) \cdot c) = (a \cdot (b \cdot c))]] \tag{III.4}$$

So, restated in other words: $a \cdot b$ Operated with c must be the same as a Operated with $b \cdot c$.

This property is exceptionally important for many properties of studied Algebras as without it many Equations quickly become totally intractable. **Commutativity**, on the other hand, is much more 'optional'; while it enables a great deal more manipulations to be entertained in the course of doing Algebra, its absence does not preclude having a very detailed and thorough discussion of many Algebras.

Definition III.2.3 (Commutativity)**.** In order for an Internal Operator, $\cdot$, on a Set X, to exhibit the property of **Commutativity** it must be the case that:

$$\forall a \forall b[(a \in X \land b \in X) \Rightarrow (a \cdot b = b \cdot a)] \tag{III.5}$$

I.e. a Operated with b must be the same as b Operated with a.

III.2.3 Identity and Inverse Elements

These are two of the most commonly discussed types of Elements discussed in a Group-Theory class as well: **Identity Elements** and **Inverse Elements**. This, too, makes sense as each are required in order for a Group to be formed. Though, unlike Associativity removing these Elements, may still yield somewhat interesting Algebraic Structures.

Definition III.2.4 (Identity Element)**.** In order for an INTERNAL OPERATOR, $\cdot$, on a SET on X, to IDENTIFY an **IDENTITY ELEMENT** it must be the case that:

$$\forall x[x \in X \Rightarrow \exists \varepsilon[\varepsilon \in X \wedge (\varepsilon \cdot x = x)]] \tag{III.6}$$

$$\forall x[x \in X \Rightarrow \exists \varepsilon[\varepsilon \in X \wedge (x \cdot \varepsilon = x)]] \tag{III.7}$$

If an ELEMENT ε only succeeds at satisfying one of the two above lines, then it is called a **RIGHT IDENTITY ELEMENT** or a **LEFT IDENTITY ELEMENT**, respectively.

Many things are notable about the concept of an **Identity Element**, but we note two here as particularly important. First, the **IDENTITY ELEMENT** is UNIQUE for any OPERATOR that has one. The proof for this is simple and left for the reader (hint: it is often rendered as a PROOF BY CONTRADICTION). Secondly, the presence of an **IDENTITY ELEMENT** is required for **INVERSE ELEMENTS** to even be defined. Why will become obvious in the following definition.

Definition III.2.5 (Inverse Elements)**.** In order for an INTERNAL OPERATOR, $\cdot$, on a SET on X, to IDENTIFY **INVERSE ELEMENTS** it must first posses an **IDENTITY ELEMENT**, and also have it be the case that:

$$\forall x[x \in X \Rightarrow \exists x^{-1}[x^{-1} \in X \wedge (x^{-1} \cdot x = \varepsilon)]] \tag{III.8}$$

$$\forall x[x \in X \Rightarrow \exists x^{-1}[x^{-1} \in X \wedge (x \cdot x^{-1} = \varepsilon)]] \tag{III.9}$$

If such an ELEMENT only satisfies one of the previous two lines, then it is termed a **RIGHT INVERSE ELEMENT** or **LEFT INVERSE ELEMENT**, respectively. If an OPERATOR does not identify two-sided **INVERSE ELEMENTS**, it can possess *distinct* **RIGHT INVERSE ELEMENTS** and **LEFT INVERSE ELEMENTS**.

III.2.4 Absorbing Elements

The following type of ELEMENT often is rarely described specifically, as it is most commonly a consequence of introducing on the properties that we will describe later that we term COLLABORATIVE. It will be discussed more particularly then.

Definition III.2.6 (Absorbing Element)**.** In order for an INTERNAL OPERATOR, $\cdot$, on a SET X, to IDENTIFY an

Absorbing Element it must be the case that:

$$\forall x[x \in X \Rightarrow \exists \mu[\mu \in X \wedge (\mu \cdot x = \mu)]] \tag{III.10}$$

$$\forall x[x \in X \Rightarrow \exists \mu[\mu \in X \wedge (x \cdot \mu = \mu)]] \tag{III.11}$$

If an Element μ only succeeds at satisfying one of the two above lines, then it is called a **Right Absorbing Element** or a **Left Absorbing Element**, respectively.

III.3 Algebras with One Operator

Objective We will list the names of **Algebras** that exhibit different combinations of the previous five properties.

Strategy We will rely on the definitions provided in the previous section to almost exclusively state definitions over the course of this section so that we may use the terminology appropriate for discussing certain kinds of **Algebras**.

III.3.1 Algebras with One Property

Definition III.3.1 (Magma)**.** An **Algebra** $\mathcal{X} \Leftrightarrow (X, \cdot)$ such that $\cdot$ is Closed, is called a **Magma** or **Closed-Algebra**.

Definition III.3.2 (Semi-Groupoid)**.** An **Algebra** $\mathcal{X} \Leftrightarrow (X, \cdot)$ such that $\cdot$ is Associative, is called a **Semi-Groupoid** or an **Associative Algebra**.

Definition III.3.3 (Commutative Algebra)**.** An **Algebra** $\mathcal{X} \Leftrightarrow (X, \cdot)$ such that $\cdot$ Commutative, is called a **Commutative Algebra**.

Definition III.3.4 (Unital Algebra)**.** An **Algebra** $\mathcal{X} \Leftrightarrow (X, \cdot)$ such that there is an Identity Element on $\cdot$, is called a **Unital Algebra**.

Definition III.3.5 (Invertible Algebra)**.** An **Algebra** $\mathcal{X} \Leftrightarrow (X, \cdot)$ such that every Non-Identity Element is an Inverse Element on $\cdot$, is called an **Invertible Algebra**.

The latter three of these will seem... well, rather obvious, and that's because there aren't special names for them as they don't have structure enough for them to be deemed interesting to study in and of themselves.

The ALGEBRAS that *would* belong in the following subsection – however, are unnamed – have been omitted for brevity and clarity.

III.3.2 Algebras with Two Properties

Definition III.3.6 (Semi-Group)**.** An ALGEBRA that is CLOSED and ASSOCIATIVE is called a **SEMI-GROUP**.

Definition III.3.7 (Small Category)**.** An ALGEBRA that is ASSOCIATIVE and UNITAL is called a **SMALL CATEGORY**.

Definition III.3.8 (Unital Magma)**.** An ALGEBRA that is CLOSED and UNITAL is called a **UNITAL MAGMA**.

Definition III.3.9 (Quasi-Group)**.** An ALGEBRA that is UNITAL and INVERTIBLE is called a **QUASI-GROUP**.

III.3.3 Algebras with Three Properties

Definition III.3.10 (Monoid)**.** An ALGEBRA that is CLOSED, ASSOCIATIVE, and UNITAL is called a **MONOID**.

Definition III.3.11 (Inverse Semi-Group)**.** An ALGEBRA that is CLOSED, ASSOCIATIVE, and INVERTIBLE is called an **INVERSE SEMI-GROUP**.

Definition III.3.12 (Commutative Semi-Group)**.** An ALGEBRA that is CLOSED, ASSOCIATIVE, and COMMUTATIVE is called a **COMMUTATIVE SEMI-GROUP**.

Definition III.3.13 (Loop)**.** An ALGEBRA that is CLOSED, UNITAL, and INVERTIBLE is called a **LOOP**.

Definition III.3.14 (Groupoid)**.** An ALGEBRA that is ASSOCIATIVE, UNITAL, and INVERTIBLE is called a **GROUPOID**.

III.3.4 Algebras with Four Properties

Definition III.3.15 (Commutative Monoid)**.** An ALGEBRA that is CLOSED, ASSOCIATIVE, UNITAL, and COMMUTATIVE is called a **COMMUTATIVE MONOID**.

Definition III.3.16 (Group)**.** An ALGEBRA that is CLOSED, ASSOCIATIVE, UNITAL, and INVERTIBLE is called a **GROUP**.

III.3.5 Abelian Groups

The algebraic structure that possesses all five properties described in the previous section is called an **ABELIAN GROUP** which is a very important type of ALGEBRA and will be important in the section after the next. These structures are very extensively studied and serve as the foundation upon which many of the TWO-OPERATOR ALGEBRAIC STRUCTURES that have been studied rest upon.

III.4 Collaborative Binary-Operators

Objective We will explore what it means for two Binary-Operators to be **Collaborative**.

Strategy We will discuss the two most common kinds of **Collaboration**.

III.4.1 Distributive Collaboration

One is probably already exceptionally familiar with this kind of Collaboration between two Operators, but first: what is meant by Collaboration between Operators? We will abstain from providing a technical definition, instead favoring a more intuitive understanding of the notion. We say that two – or indeed more, but we will only consider the case of two – Operators **Collaborate** if there is some Identity that can link the two together over the same Domain Set(s). It is hoped that this notion will be made more concrete with the following two important examples of the property discussed here and in the next subsection.

Definition III.4.1 (Distributive Collaboration)**.** We say that two Operators – say $+$ and $*$ defined on X, for familiarity – have **Distributive Collaboration** if:

$$\forall a \forall b \forall c[(a \in X \wedge b \in X \wedge c \in X) \Rightarrow [a * (b + c) = (a * b) + (a * c)]] \tag{III.12}$$

$$\forall a \forall b \forall c[(a \in X \wedge b \in X \wedge c \in X) \Rightarrow [(b + c) * a = (b * a) + (c * a)]] \tag{III.13}$$

(III.14)

Without loss of generality, we have supposed that $*$ **Distributes Over** $+$, and we have supposed that $*$ is the first operator and $+$ the second, in our following description. In words – though it is a dense property symbolically *and* linguistically – for any three elements $a, b, c \in X$, the result of an Operation between one Element and the result of the other operation on the other two Elements is Equivalent to the latter Operation joining the results of two applications of the first Operator between the first Element and each of other two Elements individually. That is hard to parse, to say the least, so we encourage understanding to come from the equivalent symbolic description. If two Operators only express this kind of **Collaboration** in ways described by one of the previous lines, it is said to have **Left-Distributivity** or **Right-Distributivity**, respectively. We almost always consider the two-sided version of this **Collaboration**, though, in practice.

III.4.2 Absorptive Collaboration

One might be understandably less familiar with this kind of Collaboration between two Operators.

Definition III.4.2 (Absorptive Collaboration)**.** We say that two operators – say & and || defined on X, in an attempt to preserve some familiarity for those who have it – have **Absorptive Collaboration** if:

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow (a \,\&\, (a \,||\, b) = a)] \tag{III.15}$$

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow (a \,||\, (a \,\&\, b) = a)] \tag{III.16}$$

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow ((a \,||\, b) \,\&\, a = a)] \tag{III.17}$$

$$\forall a \forall b[(a \in X \wedge b \in X) \Rightarrow ((a \,\&\, b) \,||\, a = a)] \tag{III.18}$$

$$\tag{III.19}$$

In words, for any two Elements $a, b \in X$, the result of an Operation between one Element and the result of the other Operation on both Elements is Equivalent to first Element. We need not specify which Operator is 'first', as this property is usually required for both Operators to express on each other. If two Operators only express this kind of Collaboration in ways described by either the top two or bottom two of the previous lines, it is said to have **Left-Absorption** or **Right-Absorption**, respectively. Further, if it only expresses one of the top two lines and the matching line from the bottom pair, it is said to have **Partial-Absorption**. In the case that both of these are the case, we encourage the reader to disregard the structure altogether but nevertheless reluctantly suggest the terminology of **Partial-Left-Absorption** or **Partial-Right-Absorption**, respectively. A structure that exhibits only the top and bottom lines or the interior two lines we have chosen to suggest describing as possessing **Antagonistic-Absorption** – as both a descriptor for the property as well as one wishing to study it.

III.5 Collaborative Algebras

Objective We will give names to the structures that possess two Operators that Collaborate together to exhibit one of the properties described in the previous section. This will be the final section of Part 0, as it will provide us the final definitions necessary to have a complete and informed conversation about the results presented in Part 1.

Strategy We will rely on the definitions provided in the previous section to state definitions over the course of this section so that we may use the terminology appropriate for discussing certain kinds of Algebras. We will use terminology relating both to single operator Algebras, as well as the properties that define them in addition to the Collaborative properties we described in the previous section. Unlike one of the previous sections that was on ALGEBRAS with only one OPERATOR, the later structures we will define – largely in the first subsection – are not characterized exclusively by their possession of previously described properties. Instead, for these structures, we will describe additional structure on them that give them an even richer theory. These ALGEBRAS and their additional structure will prove essential to describing several results in Part 1, and particularly Chapter II.

III.5.1 Rngs, Rings, Commutative Rings, and More

We will now describe a whole host of structures that possess DISTRIBUTIVE COLLABORATION.

All Kinds of R[i]ngs!

Definition III.5.1 (Rng)**.** An ALGEBRA $\mathcal{R} \Leftrightarrow (R, +, *)$ such that $(R, +)$ is an ABELIAN GROUP, $(R, *)$ is a SEMI-GROUP, and $+, *$ share DISTRIBUTIVE COLLABORATION where $*$ distributes over $+$, is a **RNG**.

Definition III.5.2 (Ring)**.** An ALGEBRA $\mathcal{X} \Leftrightarrow (R, +, *)$ that is otherwise a **RNG** except that $(R, *)$ now forms a MONOID is a **RING**.

Definition III.5.3 (Commutative Ring)**.** An ALGEBRA $\mathcal{X} \Leftrightarrow (R, +, *)$ that is otherwise a **RING** except that $(R, *)$ now forms a COMMUTATIVE MONOID is a **COMMUTATIVE RING**.

Consequences of Collaboration An important consequence of the above definitions is that each of the structures will have an ABSORBING ELEMENT identified by $*$. This is because there is always an IDENTITY ELEMENT identified by $+$, and DISTRIBUTIVE COLLABORATION thus forces that same element to become the ABSORBING ELEMENT identified by $*$. We provide a brief informal PROOF to this claim below; for any two ELEMENTS $a, b \in R$ and the IDENTITY ELEMENT IDENTIFIED by $+$, which will be named 0 going forward:

$$a = a + 0 \tag{III.20}$$

$$b * a = b * (a + 0) \tag{III.21}$$

$$b * a = (b * a) + (b * 0) \tag{III.22}$$

$$0 = b * 0 \tag{III.23}$$

Briefly: the first line relies on the definition of an IDENTITY ELEMENT; the second line simply OPERATES the second ELEMENT b with both EQUIVALENT sides of the EQUATION; the third line substitutes the characterizing

Identity in Distributive Collaboration; the fourth line is implicitly applying the existence of Inverse Elements existing in $(R,+)$. As a result, we arrive at the defining Equation for an Absorbing Element, thus proving the necessity of the Identity Element identified by $+$ – i.e. 0 – becoming an Absorbing Element in $(R,*)$.

Additionally, there is terminology that relates to the 'Divisibility' of Elements in these structures. We will go through the trouble writing definitions for these abbreviations as they are rather important ideas:

Definition III.5.4 (Irreducible Elements and Indivisibility)**.** The notion of **Irreducible Elements**, is that of those Elements that can not be represented as the result of any $a, b \in R$ being Operated using $*$. Formally, we write the Set of **Irreducible Elements** of R as $Ir[R]$:

$$\forall p[(p \in R \wedge \forall x \nexists y[x \in R \wedge y \in R \wedge ((x * y = p) \vee (y * x = p))]) \Rightarrow p \in Ir[R]] \tag{III.24}$$

We may also say that **Irreducible Elements** have *no* **Divisors**.

Definition III.5.5 (Reducible Elements and Divisibility)**.** In the case that an Element is indeed a **Reducible Element**, we term any Element d that occurs in any of the possible ways the Element can be represented using $*$ a **Divisor** of it. Formally, we write the Set of **Reducible Elements** of R as $Re[R]$ and the **Divisors** of a given Element c in R as $D_R[c]$:

$$\forall c[(c \in R \wedge \exists x \exists y[(x \in R \wedge y \in R) \Rightarrow (x * y = c)]) \Rightarrow (c \in Re[R] \wedge x \in D_R[c] \wedge y \in D_R[c])] \tag{III.25}$$

Using this terminology, the notion of **Irreducible Elements** is Equivalent to the notion of **Indivisible Elements**. Further, a **Unit** is simply the name given to an Element in $(R,*)$ that possesses an **Inverse Element**. We will also define **Annihilators**:

Definition III.5.6 (Annihilators of an Element)**.** The **Set Of Annihilators** of a given Element are abbreviated like so:

$$Ann_{\mathcal{R}}(x) \iff \left\{ y \in R \ : \ x * y = 0 \vee y * x = 0] \right\} \tag{III.26}$$

A natural consequence of this definition is that Elements that have **Annihilators** other than 0 are themselves an **Annihilator** to the Elements that are **Non-Zero Annihilators** for it.

What is an Ideal? We will briefly describe the notion of an **Ideal** as it is necessary to describe the third structure below, and, as a result, to our conversation in Chapter II in Part 1. An **Ideal** is a Subset of these most recently described Algebraic Structures such that it is Closed under $*$. This is – in truth – the extent of the definition; however, it does not fully articulate the importance of these structures in revealing the structure of **R[i]ngs**. Rather than go on at length about this importance, we choose to wait and demonstrate it in Part 1.

Commutative Rings with More Structure

Definition III.5.7 (Integral Domain)**.** A **Commutative Ring**, $\mathcal{R} \Leftrightarrow (R, +, *)$ – with an Absorbing Element notated a 0 – is an **Integral Domain**, if:

$$\forall a \forall b[(a \in R \wedge b \in R) \Rightarrow (a * b \neq 0)] \tag{III.27}$$

Any element in a **Commutative Ring** that fails this condition – thus precluding the algebra from being an **Integral Domain** – is termed a **Zero-Divisor**, since it would be in the Set $D_R[0]$. So an Equivalent formulation of the previous definition is something like "A **Commutative Ring** that has *no* **Zero-Divisors**" or "A **Commutative Ring** such that $D_R[0] = \{0\}$."

Definition III.5.8 (Unique Factorization Domain)**.** An **Integral Domain**, $\mathcal{R} \Leftrightarrow (R, +, *)$ such that every Element is representable as the result of a finite and unique – up to order and inclusion of Units – Operation by $*$ of **Irreducible Elements**, is a **Unique Factorization Domain** or **UFD** for short. Formally:

$$\forall x \left[(x \in R) \Rightarrow \exists! \mathbb{P}_{\mathcal{R}}[x] \left[\left(\begin{array}{c} (\mathbb{P}_{\mathcal{R}}[x] \subseteq Ir[R] \times \mathbb{N}) \\ \Downarrow \\ \forall a \exists b \left[\begin{array}{c} a \in Ir[R] \\ \wedge \\ b \in \mathbb{N} \end{array} \Rightarrow (a, b) \in \mathbb{P}_{\mathcal{R}}[x] \right] \\ \wedge \\ \prod_{(p, n_p) \in \mathbb{P}_{\mathcal{R}}[x]} p^{n_p} = x \end{array} \right) \right] \right] \tag{III.28}$$

We call such a Set $\mathbb{P}_{\mathcal{R}}[x]$ the **Prime Decomposition** of X.

This is the Algebraic Structure that generalizes the **Fundamental Theorem Of Arithmetic**. In this way, any **Integral Domain** that permits an analogue to the **Fundamental Theorem Of Arithmetic** is a **UFD**. This notion will become central to our discussion in Chapter II of Part 1.

Definition III.5.9 (Principal Ideal Domain)**.** An **Integral Domain**, $\mathcal{R} \Longleftrightarrow (R, +, *)$ such that every **Ideal** on $\mathcal{R}$ is generated by exactly one Element, i.e. every **Ideal** is **Prinicipal**, is called a **Principal Ideal Domain** or **PID** for short.

A common – and important to the results soon to be discussed – example of a **PID** is $\mathcal{Z}$, i.e. the Integers! This structure on $\mathcal{Z}$ will be heavily leveraged in Chapter II of Part 1.

III.5.2 Fields and Skew-Fields

We will now briefly describe **Skew-Fields** and **Fields**.

Definition III.5.10 (Skew-Field)**.** A Ring, $\mathcal{R} = (R, +, *)$, such that every Element has an Inverse Element Identified by $*$ is a **Skew-Field**. Equivalently, $(R, *)$ must form a Group.

One may notice that the only difference in a **Skew-Field** between $(R, +)$ and $(R, *)$ is Commutativity, besides the asymmetry in their Distributive Collaboration. Even this difference is removed in the case of **Fields**:

Definition III.5.11 (Field)**.** A Commutative Ring, $\mathcal{R} = (R, +, *)$, such that every Element has an Inverse Element Identified by $*$ is a **Field**. Equivalently, $(R, *)$ must form an Abelian Group.

While **Fields** do genuinely have an exceptionally broad and rich theory, we forgo discussing it in more detail here, as our focus primarily has to do with Ring-Theory, and as such, Rngs, Rings, and Commutative Rings. More particularly, the Algebraic Structures that arise when added requirements are imposed as indicated in the previous subsection.

III.5.3 Lattices

We will merely state the definition for **Lattices** before moving on, despite exceptional interest in them being well-earned. It is notable that **Lattices** very naturally appear in **Order-Theory**.

Definition III.5.12 (Lattice)**.** An algebra $\mathcal{L} = (L, ||, \&)$ such that $(L, ||)$ and $(L, \&)$ both form Commutative Semi-Groups and the two Operators share Absorptive Collaboration.

III.5.4 Collaborative External Operators

We will briefly note the kinds of structures that arise when one considers adjoining an **External Operator** to a variety of other Algebraic Structures, namely Rings and Fields, that enjoy sharing a 'multiplication' with a version of Distributive Collaboration with the structure they are joined to.

Definition III.5.13 (Module)**.** A Ring, $\mathcal{R} = (R, +, *)$ that has a **Scalar Multiplication**, $\cdot$, defined between its Elements and the Elements of an Abelian Group, $\mathcal{S} = (S, \oplus)$ such that it possesses Distributive Collaboration on both $\oplus$ and $+$, as well as is 'compatible' with $*$, is named a **Module** or more particularly an **R-Module**. Formally:

$$\forall a \forall b \forall x \forall y[((a \in S \ \wedge b \in S) \wedge (x \in R \wedge y \in R)) \Rightarrow (r \cdot (x + y) = (r \cdot x) + (r \cdot y))] \tag{III.29}$$

$$\forall a \forall b \forall x \forall y[((a \in S \ \wedge b \in S) \wedge (x \in R \wedge y \in R)) \Rightarrow ((r \oplus s) \cdot x = (r \cdot x) + (s \cdot x))] \tag{III.30}$$

$$\forall a \forall b \forall x \forall y[((a \in S \ \wedge b \in S) \wedge (x \in R \wedge y \in R)) \Rightarrow ((r * s) \cdot x = r \cdot (s \cdot x))] \tag{III.31}$$

Definition III.5.14 (Vector-Space)**.** A **Module** such that the defining Ring is actually also a Field is termed a **Vector-Space**.

PART 1

RESULTS

CHAPTER I

FUNCTION-ALGEBRAS

I.1 Preserved Algebras

Objective In this section we will first discuss the notion of a **Function-Algebra** before then describing its relationship to Algebras as characterized in the previous part. Specifically, we will be showing what properties are preserved from Algebras into the appropriate **Function-Algebra** analogue.

Strategy We will accomplish this by first defining the creation of a **Function-Algebra** from any Algebra. We will then demonstrate that when one supposedly has a particular Algebra to start, that the associated **Function-Algebra** preserves many of its properties using Algebraic manipulations of the definitions.

I.1.1 Preserved Properties of Operators

It can be shown that Y^X paired with **Term-Wise Extensions** of Operators already defined on Y form similarly-classed Algebraic Structures to those formed on Y. We call this pairing of a Function-Set with Extended Operators a **Function-Algebra**. We will show that this is the case first for characteristics of Operators, and then for Identity Element, Inverse Element, and Absorbing Elements. First, we will define what we mean by **Term-Wise Extensions**.

Definition I.1.1 (Function Operator Extension)**.** For some Algebra $\mathcal{Y} \Leftrightarrow (Y, \cdot)$, Functions $\alpha \in Y^X$ and $\beta \in Y^X$:

$$\alpha \odot \beta \Leftrightarrow \forall x[(x \in X) \Rightarrow ([\alpha \odot \beta](x) = \alpha(x) \cdot \beta(x))] \tag{I.1}$$

Further, $\forall\alpha\forall\beta\forall\gamma[\alpha \in Y^X \wedge \beta \in Y^X \wedge \gamma \in Y^X]$:

$$\alpha \odot \beta = \gamma \;\Leftrightarrow\; \forall x[(x \in X) \Rightarrow ([\alpha \odot \beta](x) = \gamma(x))] \tag{I.2}$$

Definition I.1.2 (Function-Algebra)**.** For some Co-Domain Y that forms an Algebra $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ – or in the case of a Collaborative Algebra with two Operators, i.e. if $\mathcal{Y} \Leftrightarrow (Y, +, *)$ – and *non-empty* Domain X, we denote the **Function-Algebra** created by the **Extended Operator**, $\odot$ – or $\oplus$, $\circledast$ respectively:

$$\mathcal{Y}^X \;\Leftrightarrow\; (Y^X, \odot) \tag{I.3}$$

$$\mathcal{Y}^X \;\Leftrightarrow\; (Y^X, \oplus, \circledast) \tag{I.4}$$

We will now demonstrate that Associativity and Commutativity are preserved into our **Function-Algebras**.

Lemma I.1.1 (Function-Algebras Preserve Associativity)**.** *If $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ is an* Associative-Algebra*, then $\mathcal{Y}^X$ is also an* Associative-Algebra.

Proof. Consider $\forall\alpha\forall\beta\forall\gamma[\alpha \in Y^X \wedge \beta \in Y^X \wedge \gamma \in Y^X]$:

$$\forall x[(x \in X) \wedge ([(\alpha \odot \beta) \odot \gamma](x) = [\alpha \odot \beta](x) \cdot \gamma(x))] \tag{I.5}$$

$$\Updownarrow \tag{I.6}$$

$$\forall x[(x \in X) \wedge ([\alpha \odot \beta](x) \cdot \gamma(x) = (\alpha(x) \cdot \beta(x)) \cdot \gamma(x))] \tag{I.7}$$

$$\Updownarrow \tag{I.8}$$

$$\forall x[(x \in X) \wedge ((\alpha(x) \cdot \beta(x)) \cdot \gamma(x) = \alpha(x) \cdot (\beta(x) \cdot \gamma(x)))] \tag{I.9}$$

$$\Updownarrow \tag{I.10}$$

$$\forall x[(x \in X) \wedge (\alpha(x) \cdot (\beta(x) \cdot \gamma(x)) = \alpha(x) \cdot [\beta \odot \gamma](x))] \tag{I.11}$$

$$\Updownarrow \tag{I.12}$$

$$\forall x[(x \in X) \wedge (\alpha(x) \cdot [\beta \odot \gamma](x) = [\alpha \odot (\beta \odot \gamma)](x))] \tag{I.13}$$

Thus, we may conclude:

$$\forall x[(x \in X) \wedge ([(\alpha \odot \beta) \odot \gamma](x) = [\alpha \odot (\beta \odot \gamma)](x))] \;\Leftrightarrow\; (\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma) \tag{I.14}$$

$\square$

Lemma I.1.2 (Function-Algebras Preserve Commutativity)**.** *If $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ is a* COMMUTATIVE-ALGEBRA*, then $\mathcal{Y}^X$ is also a* COMMUTATIVE-ALGEBRA.

Proof. Consider $\forall\alpha\forall\beta\forall\gamma[\alpha \in Y^X \wedge \beta \in Y^X \wedge \gamma \in Y^X]$:

$$\forall x[(x \in X) \wedge ([\alpha \odot \beta](x) = \alpha(x) \cdot \beta(x))] \tag{I.15}$$

$$\Updownarrow \tag{I.16}$$

$$\forall x[(x \in X) \wedge (\alpha(x) \cdot \beta(x) = \beta(x) \cdot \alpha(x))] \tag{I.17}$$

$$\Updownarrow \tag{I.18}$$

$$\forall x[(x \in X) \wedge (\beta(x) \cdot \alpha(x) = [\beta \odot \alpha](x))] \tag{I.19}$$

Thus, we may conclude:

$$\forall x[(x \in X) \wedge ([\alpha \odot \beta](x) = [\beta \odot \alpha](x))] \Leftrightarrow \alpha \odot \beta = \beta \odot \alpha \tag{I.20}$$

□

We will now demonstrate that IDENTITY ELEMENTS, INVERSE ELEMENTS, and ABSORBING ELEMENTS are preserved into our **FUNCTION-ALGEBRAS**.

We first define a specific SET of FUNCTIONS that makes the following proofs somewhat trivial:

Definition I.1.3 (Trivial Functions)**.** The **SUBSET OF TRIVIAL FUNCTIONS** – notated $Tr[Y^X]$ – of a FUNCTION-SET Y^X are those FUNCTIONS that MAP every ELEMENT in X to exactly the same ELEMENT in Y:

$$y_{Y^X} \in Tr[Y^X] \Leftrightarrow \exists! y_Y \exists! y_{Y^X}[(y_Y \in Y \wedge y_{Y^X} \in Y^X) \Rightarrow (\forall x[y_{Y^X}(x) = y_Y])] \tag{I.21}$$

Lemma I.1.3 (Function-Algebras Preserve Identity Elements)**.** *If $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ is an* ALGEBRA *with an* IDENTITY ELEMENT *ε_Y, then $\mathcal{Y}^X$ is also an* ALGEBRA *with an* IDENTITY ELEMENT *ε_{Y^X}.*

Proof. Consider $\forall\alpha[\alpha \in Y^X]$ and the FUNCTION $\varepsilon_{Y^X} \in Tr[Y^X]$

$$\forall x[(x \in X) \Rightarrow ([\varepsilon_{Y^X} \odot \alpha](x) = \varepsilon_{Y^X}(x) \cdot \alpha(x))] \tag{I.22}$$

$$\Updownarrow \tag{I.23}$$

$$\forall x[(x \in X) \Rightarrow (\varepsilon_{Y^X}(x) \cdot \alpha(x) = \varepsilon_Y \cdot \alpha(x))] \tag{I.24}$$

$$\Updownarrow \tag{I.25}$$

$$\forall x[(x \in X) \Rightarrow (\varepsilon_Y \cdot \alpha(x) = \alpha(x))] \tag{I.26}$$

Thus, we may conclude:

$$\forall x[(x \in X) \Rightarrow ([\varepsilon_{Y^X} \odot \alpha](x) = \alpha(x))] \;\Leftrightarrow\; \varepsilon_{Y^X} \odot \alpha = \alpha \tag{I.27}$$

□

Lemma I.1.4 (Function-Algebras Preserve Inverse Elements)**.** *If* $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ *is an* INVERTIBLE-ALGEBRA*, then* $\mathcal{Y}^X$ *is also an* INVERTIBLE-ALGEBRA*.*

Proof. Consider FUNCTIONS $\alpha, \beta \in Y^X$, and let $\chi, \psi \in Y^X$ be such that $\alpha(x) \cdot \chi(x) = \beta(x)$ and $\psi(x) \cdot \alpha(x) = \beta(x)$. It immediately follows that $\chi(x), \psi(x)$ exist for all x, because $\mathcal{Y}$ is an INVERTIBLE-ALGEBRA – so χ, ψ themselves must also exist – making $\mathcal{Y}^X$ an INVERTIBLE ALGEBRA. □

Lemma I.1.5 (Function-Algebras Preserve Absorbing Elements)**.** *If* $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ *is an* ALGEBRA *with an* ABSORBING ELEMENT μ_Y*, then the* ALGEBRA $\mathcal{Y}^X$ *has an* ABSORBING ELEMENT μ_{Y^X}*.*

Proof. Consider $\forall\alpha[\alpha \in Y^X]$ and the FUNCTION $\mu_{Y^X} \in Tr[Y^X]$

$$\forall x[(x \in X) \Rightarrow ([\mu_{Y^X} \odot \alpha](x) = \mu_{Y^X}(x) \cdot \alpha(x))] \tag{I.28}$$

$$\Updownarrow \tag{I.29}$$

$$\forall x[(x \in X) \Rightarrow (\mu_{Y^X}(x) \cdot \alpha(x) = \mu_Y \cdot \alpha(x))] \tag{I.30}$$

$$\Updownarrow \tag{I.31}$$

$$\forall x[(x \in X) \Rightarrow (\mu_Y \cdot \alpha(x) = \mu_Y)] \tag{I.32}$$

Thus, we may conclude:

$$\forall x[(x \in X) \Rightarrow ([\mu_{Y^X} \odot \alpha](i) = \mu_Y)] \;\Leftrightarrow\; \mu_{Y^X} \odot \alpha = \mu_{Y^X} \tag{I.33}$$

□

I.1.2 Preserved Collaborative Properties

We now will show that several Collaborative properties are also maintained, such as Distributivity and Absorption.

Lemma I.1.6 (Function-Algebras Preserve Distributivity)**.** *If $\mathcal{Y} \Leftrightarrow (Y, +, *)$ is a* Distributive-Algebra, *(without loss of generality, suppose $*$* Distributes *over $+$), then $\mathcal{Y}^X$ is also a* Distributive-Algebra.

Proof. Consider $\forall\alpha\forall\beta\forall\gamma[\alpha \in Y^X \wedge \beta \in Y^X \wedge \gamma \in Y^X]$:

$$\forall x[(x \in X) \Rightarrow ([\alpha \circledast (\beta \oplus \gamma)](x) = \alpha(x) * [\beta \oplus \gamma](x))] \tag{I.34}$$

$$\Updownarrow \tag{I.35}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) * [\beta \oplus \gamma](x) = \alpha(x) * (\beta(x) + \gamma(x)))] \tag{I.36}$$

$$\Updownarrow \tag{I.37}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) * (\beta(x) + \gamma(x)) = \alpha(x) * \beta(x) + \alpha(x) * \gamma(x))] \tag{I.38}$$

$$\Updownarrow \tag{I.39}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) * \beta(x) + \alpha(x) * \gamma(x) = [\alpha \circledast \beta](x) + [\alpha \circledast \gamma](x))] \tag{I.40}$$

$$\Updownarrow \tag{I.41}$$

$$\forall x[(x \in X) \Rightarrow ([\alpha \circledast \beta](x) + [\alpha \circledast \gamma](x) = [(\alpha \circledast \beta) \oplus (\alpha \circledast \gamma)](x))] \tag{I.42}$$

Thus, we may conclude:

$$\forall x[(x \in X) \Rightarrow ([\alpha \circledast (\beta \oplus \gamma)](x) = [(\alpha \circledast \beta) \oplus (\alpha \circledast \gamma)](x))] \;\Leftrightarrow\; \alpha \circledast (\beta \oplus \gamma) = (\alpha \circledast \beta) \oplus (\alpha \circledast \gamma) \tag{I.43}$$

□

Lemma I.1.7 (Function-Algebras Preserve Absorption)**.** *If $\mathcal{Y} \Leftrightarrow (Y, ||, \&)$ is an* Absorbent-Algebra, *then $\mathcal{Y}^X \Leftrightarrow (Y^X, \ovee, \owedge)$ – excuse the slightly different* Operator *appearance – is also an* Absorbent-Algebra.

Proof. Consider $\forall\alpha\forall\beta\forall\gamma[\alpha \in Y^X \wedge \beta \in Y^X \wedge \gamma \in Y^X]$:

$$\forall x[(x \in X) \Rightarrow ([\alpha \ovee (\alpha \owedge \beta)](x) = \alpha(x) \mid\mid [\alpha \owedge \beta](x))] \tag{I.44}$$

$$\Updownarrow \tag{I.45}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) \mid\mid [\alpha \owedge \beta](x) = \alpha(x) \mid\mid (\alpha(x) \,\&\, \beta(x)))] \tag{I.46}$$

$$\Updownarrow \tag{I.47}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) \mid\mid (\alpha(x) \,\&\, \beta(x)) = \alpha(x))] \tag{I.48}$$

Thus, we may conclude:

$$\forall x[(x \in X) \Rightarrow ([\alpha \ovee (\alpha \owedge \beta)](x) = \alpha(x))] \;\Leftrightarrow\; \alpha \ovee (\alpha \owedge \beta) = \alpha \tag{I.49}$$

Similarly:

$$\forall x[(x \in X) \Rightarrow ([\alpha \owedge (\alpha \ovee \beta)](x) = \alpha(x) \,\&\, [\alpha \ovee \beta](x))] \tag{I.50}$$

$$\Updownarrow \tag{I.51}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) \,\&\, [\alpha \ovee \beta](x) = \alpha(x) \,\&\, (\alpha(x) \mid\mid \beta(x)))] \tag{I.52}$$

$$\Updownarrow \tag{I.53}$$

$$\forall x[(x \in X) \Rightarrow (\alpha(x) \,\&\, (\alpha(x) \mid\mid \beta(x)) = \alpha(x))] \tag{I.54}$$

$$\tag{I.55}$$

And, likewise, we may conclude:

$$\forall x[(x \in X) \Rightarrow ([\alpha \owedge (\alpha \ovee \beta)](x) = \alpha(x))] \;\Leftrightarrow\; \alpha \owedge (\alpha \ovee \beta) = \alpha \tag{I.56}$$

□

I.1.3 Resulting Preservations

With all of this, we have shown that many of the most basic Algebraic-Properties we have described are preserved over into our Function-Algebras. In fact, we can now conclude the following Theorems with no additional Proof necessary.

One-Property Algebra Preservations

Theorem I.1.8 (Function-Algebras Preserve Magma)**.** *If $\mathcal{M} \Leftrightarrow (M, \cdot)$ is a* MAGMA*, then $\mathcal{M}^X$ is a* MAGMA.

Theorem I.1.9 (Function-Algebras Preserve Semi-Groupoids)**.** *If $\mathcal{S} \Leftrightarrow (S, \cdot)$ is a* SEMI-GROUPOID*, then $\mathcal{S}^X$ is a* SEMI-GROUPOID.

Theorem I.1.10 (Function-Algebras Preserve Commutative Algebras)**.** *If $C \Leftrightarrow (C, \cdot)$ is a* COMMUTATIVE ALGEBRA*, then C^X is a* COMMUTATIVE ALGEBRA.

Theorem I.1.11 (Function-Algebras Preserve Unital Algebras)**.** *If $\mathcal{U} \Leftrightarrow (U, \cdot)$ is a* UNITAL ALGEBRA*, then $\mathcal{U}^X$ is a* UNITAL ALGEBRA.

Theorem I.1.12 (Function-Algebras Preserve Invertible Algebras)**.** *If $\mathcal{I} \Leftrightarrow (I, \cdot)$ is an* INVERTIBLE ALGEBRA*, then $\mathcal{I}^X$ is an* INVERTIBLE ALGEBRA.

Two-Property Algebra Preservations

Theorem I.1.13 (Function-Algebras Preserve Semi-Groups)**.** *If $\mathcal{S} \Leftrightarrow (S, \cdot)$ is a* SEMI-GROUP*, then $\mathcal{S}^X$ is a* SEMI-GROUP.

Theorem I.1.14 (Function-Algebras Preserve Small Categories)**.** *If $\mathcal{S} \Leftrightarrow (S, \cdot)$ is a* SMALL CATEGORY*, then $\mathcal{S}^X$ is a* SMALL CATEGORY.

Theorem I.1.15 (Function-Algebras Preserve Unital Magmas)**.** *If $\mathcal{U} \Leftrightarrow (U, \cdot)$ is a* UNITAL MAGMA*, then $\mathcal{U}^X$ is a* UNITAL MAGMA.

Theorem I.1.16 (Function-Algebras Preserve Quasi Groups)**.** *If $\mathcal{Q} \Leftrightarrow (Q, \cdot)$ is a* QUASI GROUP*, then $\mathcal{Q}^X$ is a* QUASI GROUP.

Three-Property Algebra Preservations

Theorem I.1.17 (Function-Algebras Preserve Monoids)**.** *If $\mathcal{M} \Leftrightarrow (M, \cdot)$ is a* MONOID*, then $\mathcal{M}^X$ is a* MONOID.

Theorem I.1.18 (Function-Algebras Preserve Inverse Semi-Groups)**.** *If $\mathcal{I} \Leftrightarrow (I, \cdot)$ is an* INVERSE SEMI-GROUP*, then $\mathcal{I}^X$ is an* INVERSE SEMI-GROUP.

Theorem I.1.19 (Function-Algebras Preserve Commutative Semi-Groups)**.** *If $C \Leftrightarrow (C, \cdot)$ is a* COMMUTATIVE SEMI-GROUP*, then C^X is a* COMMUTATIVE SEMI-GROUP.

Theorem I.1.20 (Function-Algebras Preserve Loops)**.** *If $\mathcal{Y} \Leftrightarrow (Y, \cdot)$ is a* LOOP*, then $\mathcal{Y}^X$ is a* LOOP.

Theorem I.1.21 (Function-Algebras Preserve Groupoids)**.** *If $\mathcal{Y} \Longleftrightarrow (Y, \cdot)$ is a* Groupoid*, then $\mathcal{Y}^X$ is a* Groupoid.

Four-Property Algebra Preservations

Theorem I.1.22 (Function-Algebras Preserve Commutative Monoids)**.** *If $C \Longleftrightarrow (C, \cdot)$ is a* Commutative Monoid*, then C^X is a* Commutative Monoid.

Theorem I.1.23 (Function-Algebras Preserve Groups)**.** *If $\mathcal{G} \Longleftrightarrow (G, \cdot)$ is a* Group*, then $\mathcal{G}^X$ is a* Group.

The Five-Property Algebra and Collaborative-Algebra Preservations

Theorem I.1.24 (Function-Algebras Preserve Abelian Groups)**.** *If $\mathcal{A} \Longleftrightarrow (A, +)$ is an* Abelian Group*, then $\mathcal{A}^X$ is an* Abelian Group.

Theorem I.1.25 (Function-Algebras Preserve Rngs)**.** *If $\mathcal{R} \Longleftrightarrow (R, +, *)$ is a* Rng*, then $\mathcal{R}^X$ is a* Rng.

Theorem I.1.26 (Function-Algebras Preserve Rings)**.** *If $\mathcal{R} \Longleftrightarrow (R, +, *)$ is a* Ring*, then $\mathcal{R}^X$ is a* Ring.

Theorem I.1.27 (Function-Algebras Preserve Commutative Rings)**.** *If $C \Longleftrightarrow (C, +, *)$ is a* Commutative Ring*, then C^X is a* Commutative Ring.

Theorem I.1.28 (Function-Algebras Preserve Lattices)**.** *If $\mathcal{L} \Longleftrightarrow (L, ||, \&)$ is a* Lattice*, then $\mathcal{L}^X$ is a* Lattice.

I.2 Unpreserved Algebras

Objective In contrast to the last section we will demonstrate those properties of Algebras that are not trivially preserved by Function-Algebras.

Strategy We will primarily be looking at the concept of Zero-Divisors and their inevitable appearance in Function-Algebras.

I.2.1 Functions Have Zero-Divisors

Notably absent from this list is the preservation of Fields – as well as our more structured Ring-Like structures – why? We have yet to show that for some Algebra $\mathcal{R} \Longleftrightarrow (R, +, *)$ – such that $(R, +)$ and $(R \setminus \{0\}, *)$ form Abelian-Groups – that $(R^X / \{0_{R^X}\}, \circledast)$ also forms an Abelian Group, and in fact, it explicitly never does. In fact, we can make an even stronger statement:

Theorem I.2.1 (Function-Algebras Have Zero-Divisors)**.** *For any commutative ring $\mathcal{R} \Leftrightarrow (R, +, *)$, $\mathcal{R}^X$ cannot be an* INTEGRAL DOMAIN.

Proof. Showing the existence of NON-INTEGRAL ELEMENTS – that is, NON-ZERO ELEMENTS that are in $D_R[0_R]$ – regardless of any additional structure on $\mathcal{R}$ will suffice. We have that for $0_{R^X} \in Tr[R^X]$, let:

$$\psi(\sigma, S) \Leftrightarrow (\sigma \in R^X \multimap \sigma = 0_{R^X}) \Rightarrow \left(\begin{bmatrix} S \neq \varnothing \\ \wedge \\ S \subseteq X \end{bmatrix} \Rightarrow \forall s \begin{bmatrix} ((s \in S) \Rightarrow (\sigma(s) \neq 0_R)) \\ \wedge \\ ((s \in X \setminus S) \Rightarrow (\sigma(s) = 0_R)) \end{bmatrix}\right) \tag{I.57}$$

$$\phi(\sigma, S) \Leftrightarrow (\sigma \in R^X \multimap \sigma = 0_{R^X}) \Rightarrow \left(\begin{bmatrix} S \neq \varnothing \\ \wedge \\ S \subseteq X \end{bmatrix} \Rightarrow \forall s \begin{bmatrix} ((s \in S) \Rightarrow (\sigma(s) = 0_R)) \\ \wedge \\ ((s \in X \setminus S) \Rightarrow (\sigma(s) \neq 0_R)) \end{bmatrix}\right) \tag{I.58}$$

$$\tag{I.59}$$

Worth noting is that because of how these are defined, we obviously have that $(\psi(\sigma, S) \Leftrightarrow \neg\phi(\sigma, S))$. For $\alpha \in R^X$ and $\beta \in R^X$ that are each NON-ZERO, it follows that there exists $A \subseteq X$ and $B \subseteq X$ such that $\psi(\alpha, A)$ and $\psi(\beta, B)$ have LOGICAL QUANTITIES of $\top$. However, we do not have that $(\psi(\alpha, A) \wedge \psi(\beta, B)) \Rightarrow (A \cap B \neq \varnothing)$ will always have a LOGICAL QUANTITY of $\top$, and this means:

$$\exists\alpha\exists\beta\exists A\exists B[((\alpha \in R^X \wedge \beta \in R^X) \wedge (A \subseteq X \wedge B \subseteq X) \wedge (\psi(\alpha, A) \wedge \psi(\beta, B)) \wedge (A \cap B = \varnothing)] \tag{I.60}$$

$$\Downarrow \tag{I.61}$$

$$(\phi(\alpha, B) \wedge \phi(\beta, A)) \tag{I.62}$$

Thus, for two such SETS A and B we have that $A \cup B = X$, so it must be the case that:

$$\forall x[(x \in X) \Leftrightarrow (x \in A \veebar x \in B)] \tag{I.63}$$

It follows from this as well as $(\psi(\alpha, A) \Leftrightarrow \neg\phi(\alpha, A))$ and $(\psi(\beta, B) \Leftrightarrow \neg\phi(\beta, B))$ that:

$$\forall x[(x \in X) \Rightarrow ([\alpha \circledast \beta](x) = 0_{R^X}(x))] \tag{I.64}$$

$$\Updownarrow \tag{I.65}$$

$$\alpha \circledast \beta = 0_{R^X} \tag{I.66}$$

This tells us that $\alpha \in D_R[0_R]$ and $\beta \in D_R[0_R]$, but because $\alpha \neq 0_{R^X}$ and $\beta \neq 0_{R^X}$, we have that $D_R[0_R] \neq \{0\}$ and so $\mathcal{R}^X$ can not possibly be an INTEGRAL DOMAIN. In fact, we have showed that in the COMMUTATIVE RING $\mathcal{R}^X$, there exists *many* NON-ZERO ELEMENTS α, β for each PAIR of DISJOINT SUBSETS A, B of X, so that $\alpha \circledast \beta = 0_{R^X}$. □

Is There Something to be Done? Can we define a concept analogous to INTEGRAL DOMAINS that is preserved by FUNCTION-ALGEBRAS, that may allow us to inherit analogous properties? Rather than the SET of all NON-ZERO ELEMENTS in R^X, for some INTEGRAL DOMAIN $\mathcal{R}$, we want to consider the set of **NEVER-ZERO ELEMENTS**:

$$_{\neq 0}R^X \iff \left\{ \lambda \in R^X \;:\; \forall x[(x \in X) \Rightarrow (\lambda(x) \neq 0_R)] \right\} \tag{I.67}$$

This SET will behave in much the way that we would like for NON-ZERO ELEMENTS to behave.

Lemma I.2.2 (All Function-Algebra Annihilators Have 0's)**.** *For some* INTEGRAL DOMAIN*,* $\mathcal{R} \iff (R, +, *)$*, the* SET *of* NEVER-ZERO ELEMENTS *in* R^X *is exactly the* COMPLEMENT *of the* UNION *of all* SETS OF ANNIHILATORS *for each* NON-ZERO FUNCTION*:*

$$_{\neq 0}R^X = R^X \setminus \left(\bigcap_{\lambda \in R^X \setminus \{0_{R^X}\}} Ann_{\mathcal{R}^X} \right)(\lambda) \tag{I.68}$$

Proof. It will suffice to show that:

$$\sigma \in \left(\bigcap_{\lambda \in R^X \setminus \{0_{R^X}\}} Ann_{\mathcal{R}^X}(\lambda) \right) \Leftrightarrow \exists x[x \in X \wedge \sigma(x) = 0_R] \tag{I.69}$$

That is to say that σ must have 'at least one zero'. We know that $\exists \lambda[(\lambda \in R^X \setminus \{0_{R^X}\}) \Rightarrow (\lambda \circledast \sigma = 0_{R^X})]$. Because $\mathcal{R}$ is an INTEGRAL DOMAIN, we know that $[\lambda \circledast \sigma](x) = 0_R$ means that either $\lambda(x) = 0_R$ or $\sigma(x) = 0_R$. Because λ is non-zero, it must be that $\exists w[(w \in X) \wedge (\lambda(w) \neq 0_R)]$. This means that $\sigma(w) = 0_R$, meaning that σ cannot be a **NEVER-ZERO FUNCTION**. □

A corollary of this that we will also prove for the insight it provides:

Lemma I.2.3 (Never-Zero Functions are Never Annihilators)**.** *For some* INTEGRAL DOMAIN*,* $\mathcal{R} \iff (R, +, *)$*, the set of* **NEVER-ZERO FUNCTIONS***,* $_{\neq 0}R^X$*, is exactly the* SET S*, such that:*

$$\bigcup_{\lambda \in S} Ann_{\mathcal{R}^X}(\lambda) = \{0_{R^X}\} \tag{I.70}$$

Proof. We will show – by contradiction – that all FUNCTIONS in S must be NEVER-ZERO. Suppose that $\sigma \in S$, is not NEVER-ZERO, which is to say $\exists w[w \in X \wedge \sigma(x) = 0_R]$. Now consider:

$$\exists\lambda[(\lambda \in R^X \wedge \lambda(w) \neq 0) \wedge \forall x[x \in X \setminus \{w\} \wedge \lambda(x) = 0_R]] \tag{I.71}$$

All such λ thus should be in $Ann_{\mathcal{R}^X}(S)$ – because they would ANNIHILATE σ – but they are clearly not, indicating that there can be no such w for σ, meaning it must be **NEVER-ZERO** – a contradiction. □

Now we are ready to show that these **NEVER-ZERO FUNCTIONS**, are indeed the analogue to **INTEGRAL DOMAINS** that we sought.

Lemma I.2.4 (Never-Zero Elements Behave Nicely)**.** *If $\mathcal{R} \Leftrightarrow (R, +, *)$ is an* INTEGRAL DOMAIN*, then* $({}_{\neq 0}R^X, \circledast)$ *is a* COMMUTATIVE MONOID *that obeys the* CANCELLATION PROPERTY.

Proof. CLOSURE is trivial; every OUTPUT in each **NEVER-ZERO FUNCTION** is – aptly – not zero, and because $\mathcal{R}$ is an INTEGRAL DOMAIN, then each OUTPUT in the EXTENDED PRODUCT of two FUNCTIONS cannot be zero. Hence, the EXTENDED PRODUCT of two **NEVER-ZERO FUNCTIONS** is also **NEVER-ZERO**. ASSOCIATIVITY and COMMUTATIVITY are preserved, as previously shown. Now we want to show that:

$$\forall\alpha\forall\beta\forall\gamma[(\alpha \in {}_{\neq 0}R^X \wedge \beta \in {}_{\neq 0}R^X \wedge \gamma \in {}_{\neq 0}R^X) \Rightarrow ((\alpha \circledast \beta = \alpha \circledast \gamma) \Leftrightarrow \beta = \gamma)] \tag{I.72}$$

We will do so below; $\forall\alpha\forall\beta\forall\gamma[\alpha \in {}_{\neq 0}R^X \wedge \beta \in {}_{\neq 0}R^X \wedge \gamma \in {}_{\neq 0}R^X]$:

$$\forall x[(x \in X) \wedge ([\alpha \circledast \beta](x) = [\alpha \circledast \gamma](x))] \tag{I.73}$$

$$\Updownarrow \tag{I.74}$$

$$\forall x[(x \in X) \wedge (\alpha(x) * \beta(x) = \alpha(x) * \gamma(x))] \tag{I.75}$$

$$\Updownarrow \tag{I.76}$$

$$\forall x[(x \in X) \wedge (\beta(x) = \gamma(x))] \tag{I.77}$$

Thus, we may conclude:

$$\forall x[(x \in X) \wedge (\beta(x) = \gamma(x))] \Leftrightarrow \beta = \gamma \tag{I.78}$$

□

CHAPTER II

TESSELLATIONS

II.1 Period-Sets

Objective We will take a more particular look at a specific FUNCTION-ALGEBRA: $\mathcal{Y}^{\mathcal{Z}} \Leftrightarrow (Y^{\mathbb{Z}}, \cdot)$ such that $\mathcal{Z} \Leftrightarrow (\mathbb{Z}, +, *)$ – i.e. the INTEGERS. We will start by discussing the SETS that will reveal the structure of this FUNCTION-ALGEBRA, referred to as **PERIOD-SETS**.

Strategy We will introduce the basic definitions of what a **PERIOD-SET** is and then demonstrate some of its basic properties to get a sense of why it reveals the structure of our FUNCTION-ALGEBRA $\mathcal{Y}^{\mathcal{Z}}$ that we will hence-forth refer to as a **TESSELLATION-ALGEBRA** or just **TESSELLATIONS** for short.

II.1.1 Basic Properties

As motivation, we will frame our **TESSELLATION-ALGEBRA** as considering a slight modification to the INFINITE CARTESIAN PRODUCT of some NON-EMPTY SET Y, i.e. $\prod^{\infty} Y$. This is the SET of so-called **INFINITE SEQUENCES** $\sigma : \mathbb{N} \to Y$. We will instead consider the SET of 'double-ended' **INFINITE SEQUENCES**, $\tau : \mathbb{Z} \to Y$, henceforth referred to as **TESSELLATIONS**. We will adopt the standard convention of referring to the SET of all of these FUNCTIONS as $Y^{\mathbb{Z}}$.

Definition II.1.1 (Y-Tessellations)**.** The set of **TESSELLATIONS** that MAP to ELEMENTS in some NON-EMPTY SET Y are notated as $Y^{\mathbb{Z}}$.

We will also refer to an INPUT to **TESSELLATION** functions as an **INDEX** and multiple as **INDICES**. Similarly, we will refer to the OUTPUTS of **TESSELLATIONS** as **TERMS**.

We will see that – if one is previously familiar with them, of course – ordinary **Sequences** behave almost identically to **Tessellations**; however, **Tessellations** have a couple of additional properties that allow us to have a more complete conversation. Namely, they differ from an ordinary **Sequence** in their ability to accept **Negative Indices**. This additional property will go on to be clearly desirable, because otherwise our next definition would require many special-considerations. The difference can be formally characterized by the structure that a particular collection of Sets over **Tessellations** have in comparison to the analogous collection of Sets over ordinary **Sequences**. These Sets are each **Tessellation**'s **Period-Set**. A **Period** is meant as some **Idempotent Shift** of every **Index**, i.e. a **Finite Shift** to all **Indices** such that each respective **Term** remains the same.

Definition II.1.2 (Period-Set)**.** The set of **Idempotent Shifts** on a **Tessellation** τ are denoted as follows:

$$P[\tau] \iff \Big\{ p \in \mathbb{Z} \ : \ \forall i[(i \in \mathbb{Z}) \Rightarrow \tau(i) = \tau(i+p)] \Big\} \tag{II.1}$$

Importantly, this definition includes 0 as a **Period**, which is vital. All **Tessellations** have 0 as a **Period**, even those that have no **Periodic Recurrence**; that is to say, all Functions in $Y^{\mathbb{Z}}$ that are not **Periodic** have exactly a single **Period** of 0. This is fundamentally what allows us to assert that all **Tessellations** – all Functions from $\mathbb{Z}$ to Y – have at least one **Period**. We will prove this claim, then we will go about describing two other characteristics of $P[\tau]$, for arbitrary τ.

Lemma II.1.1 (Universal Periodicity)**.** *$P[\tau]$ is* Non-Empty *for all $\tau \in Y^{\mathbb{Z}}$ if Y is* Non-Empty*:*

$$Y \neq \varnothing \ \Rightarrow \ \forall\tau[\tau \in Y^{\mathbb{Z}} \wedge P[\tau] \neq \varnothing] \tag{II.2}$$

Proof. By the definition of 0 as the Additive Identity of $\mathbb{Z}$, we have that:

$$\forall i[(i \in \mathbb{Z}) \Leftrightarrow \ (i = i + 0)] \tag{II.3}$$

Thus, we may say that:

$$\forall\tau\forall i[(\tau \in Y^{\mathbb{Z}} \wedge i \in \mathbb{Z}) \Rightarrow \tau(i) = \tau(i+0)] \tag{II.4}$$

Thus, from this we can conclude that $0 \in P[\tau]$ which gives us $P[\tau] \neq \varnothing$. □

We will now move on to proving a lemma that will allow us to conclude something rather remarkable about **PERIOD-SETS**.

Lemma II.1.2 (Absorption of Period-Sets)**.** *For $x \in P[\tau]$, it is the case that $nx \in P[\tau], \forall n \in \mathbb{Z}$:*

$$\forall\tau\forall x\forall n[(\tau \in Y^{\mathbb{Z}} \wedge x \in P[\tau] \wedge n \in \mathbb{Z}) \Rightarrow (nx \in P[\tau])] \tag{II.5}$$

Proof. We will use INDUCTION with the definition of **PERIOD SETS** as our BASE CASE, assuming the GENERAL CASE, i.e. $\tau(i) = \tau(i+nx)$, to hold for all POSITIVE INTEGERS n. It remains to be shown that $\tau(i) = \tau(i+(n+1)x)$ follows. Consider $\forall\tau\forall x\forall n[(\tau \in Y^{\mathbb{Z}} \wedge x \in P[\tau] \wedge n \in \mathbb{Z}^{+})]$

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i) = \tau(i + nx))] \tag{II.6}$$

$$\Downarrow \tag{II.7}$$

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i) = \tau((i + nx) + x))] \tag{II.8}$$

$$\Updownarrow \tag{II.9}$$

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i) = \tau(i + (n + 1)x))] \tag{II.10}$$

The second of the above lines is made possible because of the definition of **PERIODS**. That is, we know that the **TERM** of a **TESSELLATION** does not change under a shift of **INDEX** by a **PERIOD**, thus $\tau(i) = \tau(i + nx) \Rightarrow \tau(i) = \tau((i + nx) + x)$. The lines above then allow us to conclude:

$$\forall\tau\forall x\forall n[(\tau \in Y^{\mathbb{Z}} \wedge x \in P[\tau] \wedge n \in \mathbb{Z}^{+})] \;\Rightarrow\; \forall i[i \in \mathbb{Z} \Rightarrow (\tau(i) = \tau(i + nx))] \tag{II.11}$$

Now, all that we must do to finish the proof for all INTEGERS, is to show that the $n = -1$ case is also true:

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i - x) = \tau(i - x))] \tag{II.12}$$

$$\Downarrow \tag{II.13}$$

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i - x) = \tau((i - x) + x))] \tag{II.14}$$

$$\Updownarrow \tag{II.15}$$

$$\forall i[i \in \mathbb{Z} \Rightarrow (\tau(i - x) = \tau(i))] \tag{II.16}$$

From these lines we are able to conclude:

$$\forall\tau\forall x\forall n[(\tau \in Y^{\mathbb{Z}} \wedge x \in P[\tau] \wedge n \in \mathbb{Z})] \;\Rightarrow\; \forall i[i \in \mathbb{Z} \Rightarrow (\tau(i) = \tau(i + nx))] \tag{II.17}$$

□

This is all we need to conclude the following corollary:

Theorem II.1.3 (Period-Sets are Ideals)**.** *Every* **Period-Set** $P[\tau]$ *is an* Ideal *of* $\mathcal{Z} = (\mathbb{Z}, +, *)$.

Because this is the case, and every Ideal of $\mathcal{Z}$ is Principal, we can assign a Canonical Representation to our **Period-Sets**. Specifically we set our representation to be the Generator of the Ideal in $\mathbb{Z}$, which is the Smallest Positive Non-Zero Element of the Ideal in this case, or 0 in the case of the 0 Ideal. Going forward, we will speak of ***The*** **Period** of some **Tessellation** τ both in reference to this Representation Element, as well as the Ideal it is representing, depending on the context. It is only when we consider 'double-ended sequences' – **Tessellations** –as opposed to ordinary **Infinite Sequences**, that our **Period-Sets** form this structure, hence our modification made initially. To study the structure of **Tessellations** further, we will look at the properties on certain Subsets of $Y^{\mathbb{Z}}$ in a the next sections; we will define definitions for this discussion now so that we may state a corollary of this theorem.

Definition II.1.3 (Period-Set of a Set)**.** For some $S \subseteq Y^{\mathbb{Z}}$:

$$P[S] \;\Leftrightarrow\; \bigcap_{\tau \in S} P[\tau] \tag{II.18}$$

Definition II.1.4 (Tessellations with Specific Period)**.** We will refer to all $\tau \in Y^{\mathbb{Z}}$ with **Period** n as follows:

$$Y^{\mathbb{Z}\backslash n\mathbb{Z}} \;\Leftrightarrow\; \left\{\tau \in Y^{\mathbb{Z}} \;:\; \forall i[(i \in \mathbb{Z}) \Rightarrow \forall x[x \in n\mathbb{Z} \Leftrightarrow \; \tau(i) = \tau(i + x)]]\right\} \tag{II.19}$$

With these definitions, we have the tools to state a corollary of our previous theorem – that will be somewhat obvious now due to our choice of notation, at least to those familiar with Modular Arithmetic – that will highlight structure in $Y^{\mathbb{Z}}$.

Corollary II.1.3.1 (Periods Partition Tessellations)**.** *All distinct* Sets *of* **Tessellations** *with* **Specific Period***,* $Y^{\mathbb{Z}/n\mathbb{Z}}$*, are* Pairwise Disjoint*, and their* Union *is exactly* $Y^{\mathbb{Z}}$.

$$\forall Y \left[Y \neq \varnothing \Rightarrow \left(\bigcup_{n,m \,\in\, \mathbb{Z}^{+} \,\wedge\, n \neq m} \left(Y^{\mathbb{Z}\backslash n\mathbb{Z}} \cap Y^{\mathbb{Z}\backslash m\mathbb{Z}}\right) = \varnothing \;\wedge\; \bigcup_{n \in \mathbb{Z}^{+}} Y^{\mathbb{Z}/n\mathbb{Z}} = Y^{\mathbb{Z}} \right) \right] \tag{II.20}$$

II.2 Resultant-Period Sets

Objective We will consider the SETS that contain all of the possible PERIODS of the TESSELLATION that results when two TESSELLATIONS are OPERATED only knowing what the PERIOD-SET for each OPERAND TESSELLATION is.

Strategy We will motivate the **RESULTANT-PERIOD SETS** piece by piece. We will – after the first subsection – suppress the more formal notation we have been using throughout the rest of the document. It is the belief of the author that otherwise many of the discussions would become even more cumbersome than the content already necessitates it to be.

II.2.1 Some Brief Number-Theory

We will often find it necessary to make reference to the **PRIMES** on $\mathcal{Z}$, and to be precise:

Definition II.2.1 (Primes and Prime Ideals)**.** We say that a $\mathcal{R}_P$ is a PRIME IDEAL of a COMMUTATIVE RING, $\mathcal{R} \Leftrightarrow (R, +, *)$ if:

$$\forall a \forall b[(a \in R \wedge b \in R \wedge a * b \in \mathcal{R}_P) \Rightarrow (a \in \mathcal{R}_P \vee b \in \mathcal{R}_P)] \tag{II.21}$$

We then define the **SET OF PRIMES**, $\mathbb{P}_{\mathcal{R}}$, of the same COMMUTATIVE RING $\mathcal{R}$ like so:

$$\forall p[((p \in R) \Rightarrow \forall a \forall b[(a \in R \wedge b \in R \wedge a * b \in \mathcal{R}_P) \Rightarrow (a \in \mathcal{R}_P \vee b \in \mathcal{R}_P)]) \Rightarrow p \in \mathbb{P}_{\mathcal{R}}] \tag{II.22}$$

Also important is the notion of **PRIME-POWERS** which are always NATURAL NUMBERS: $\mathbb{N}$. They are the right ELEMENT in each PAIR of elements in $\mathbb{P}_{\mathcal{R}}[x]$ for each ELEMENT in R. In the case of $\mathbb{Z}$ these are of course notated as $\mathbb{P}_{\mathcal{Z}}[n]$ for some $n \in \mathbb{Z}$. It will be worth noting a particular interpretation of MULTIPLICATION and DIVISION:

Definition II.2.2 (Multiplication and Division as Addition and Subtraction)**.** We have that MULTIPLICATION and DIVISION are actually essentially ABBREVIATIONS for ADDITION and SUBTRACTION of INTEGERS PRIME-POWERS.

Consider $\forall a \forall b[a \in \mathbb{Z} \wedge b \in \mathbb{Z}]$:

$$n * m \ \Leftrightarrow \prod_{\substack{(p,n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \\ (p,m_p) \in \mathbb{P}_{\mathcal{Z}}[m]}} p^{n_p+m_p} \tag{II.23}$$

$$\frac{n}{m} \ \Leftrightarrow \prod_{\substack{(p,n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \\ (p,m_p) \in \mathbb{P}_{\mathcal{Z}}[m]}} p^{n_p-m_p} \tag{II.24}$$

It is worth noting – of course – that:

$$\forall p \forall n_p \forall m_p \left[(p \in \mathbb{P}_{\mathcal{Z}} \wedge (p,n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \wedge (p,m_p) \in \mathbb{P}_{\mathcal{Z}}[m]) \Rightarrow \left(\frac{n}{m} \in \mathbb{Z} \Leftrightarrow \ (n_p \geqslant m_p) \right) \right] \tag{II.25}$$

That is, in order for $\frac{n}{m}$ to be an Integer, it must be the case that that all **Prime-Powers** of n are greater than or equal to m's.

We now define several **Helper Functions**:

Definition II.2.3 (MIN, MAX, and EQ)**.** The three Functions listed below will allow us to express several more Functions in our next definition.

$$MIN(x,y) \Leftrightarrow \begin{cases} x & x \leqslant y \\ y & y \leqslant x \end{cases} \tag{II.26}$$

$$MAX(x,y) \Leftrightarrow \begin{cases} y & x \leqslant y \\ x & y \leqslant x \end{cases} \tag{II.27}$$

$$EQ(x,y) \Leftrightarrow \begin{cases} x & x = y \\ 0 & x \neq y \end{cases} \tag{II.28}$$

Definition II.2.4 (GCD, LCM, and $GCUD$)**.** The following Functions can be thought of as applying the previous three just defined to the PRIME DECOMPOSITION of two POSITIVE INTEGERS.

$$GCD(n,m) \iff \prod_{\substack{(p,n_p)\in\mathbb{P}_{\mathbb{Z}}[n] \\ (p,m_p)\in\mathbb{P}_{\mathbb{Z}}[m]}} p^{MIN(n_p,m_p)} \tag{II.29}$$

$$LCM(n,m) \iff \prod_{\substack{(p,n_p)\in\mathbb{P}_{\mathbb{Z}}[n] \\ (p,m_p)\in\mathbb{P}_{\mathbb{Z}}[m]}} p^{MAX(n_p,m_p)} \tag{II.30}$$

$$GCUD(n,m) \iff \prod_{\substack{(p,n_p)\in\mathbb{P}_{\mathbb{Z}}[n] \\ (p,m_p)\in\mathbb{P}_{\mathbb{Z}}[m]}} p^{EQ(n_p,m_p)} \tag{II.31}$$

The first two FUNCTIONS will be the familiar **GREATEST COMMON DIVISOR** and **LEAST COMMON MULTIPLE** that are rather frequently used. The third is somewhat more exotic; it can be shown that it is in fact the **GREATEST COMMON *Unitary* DIVISOR** FUNCTION.

Definition II.2.5 (Divisors and Unitary Divisors)**.** Given the construction of the **GREATEST COMMON DIVISOR** and **GREATEST COMMON UNITARY DIVISOR** FUNCTIONS, we may write a more concise definition for the SET of DIVISORS for a given $n \in \mathbb{Z}$, and soon a definition for the SET of **UNITARY DIVISORS** of n.

$$D[n] \iff \left\{ d \in \mathbb{Z}^+ \ : \ GCD(n,d) = d \right\} \tag{II.32}$$

A **UNITARY DIVISOR** of some POSITIVE INTEGER n is a DIVISOR m of n such that $GCD(m, \frac{n}{m}) = 1$. This equation tells us that once m has been divided out of n – represented by the fraction $\frac{n}{m}$ – the result has no common FACTORS with m anymore. This can intuitively be thought of as a DIVISOR that removes 'every copy' of any PRIMES it possesses in common with the DIVIDEND. We can use the **GREATEST COMMON UNITARY DIVISOR** FUNCTION to a write a more succint definition of **UNITARY DIVISORS** however:

$$U[n] \iff \left\{ u \in D[n] \ : \ GCUD(n,u) = u \right\} \tag{II.33}$$

It is worth noting that $U[n] \subseteq D[n]$ for all n, naturally.

We will briefly state a lemma related to **UNITARY DIVISORS** that will be very useful later.

Lemma II.2.1 (The Quotient of an Integer by a Unitary Divisor is Itself a Unitary Divisor)**.** *For an* INTEGER

n and $u \in U[n]$, it is the case that $\frac{n}{u} \in U[n]$.

$$\forall n \forall u \left[(n \in \mathbb{Z} \wedge u \in U[n]) \Rightarrow \left(\frac{n}{u} \in U[n] \right) \right] \tag{II.34}$$

Proof. Since $u \in U[n]$ we know that $GCD(n,u) = u$ and $GCUD(n,u) = u$. In essence this means that every **PRIME-POWER** of m is equal to the **PRIME-POWER** on the same PRIME in n or 0:

$$\forall n \forall u \left[(n \in \mathbb{Z} \wedge u \in U[n]) \Rightarrow \forall p \forall n_p \forall u_p \left[\begin{array}{c} (p \in \mathbb{P}_{\mathcal{Z}} \wedge (p, n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \wedge (p, u_p) \in \mathbb{P}_{\mathcal{Z}}[u]) \\ \Downarrow \\ (n_p = u_p \vee u_p = 0) \end{array} \right] \right] \tag{II.35}$$

Because of this we have that:

$$\forall n \forall u \left[(n \in \mathbb{Z} \wedge u \in U[n]) \Rightarrow \frac{n}{u} = \prod_{\substack{(p, n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \\ (p, u_p) \in \mathbb{P}_{\mathcal{Z}}[u]}} p^{n_p - u_p} \right] \tag{II.36}$$

$$\Downarrow \tag{II.37}$$

$$\forall n \forall u \left[(n \in \mathbb{Z} \wedge u \in U[n]) \Rightarrow \frac{n}{u} = \prod_{(p, n_p) \in \mathbb{P}_{\mathcal{Z}}[n] \setminus \mathbb{P}_{\mathcal{Z}}[u]} p^{n_p} \right] \tag{II.38}$$

And since we have that the only **PRIME-POWERS** that are left in $\frac{n}{u}$ are exactly equal to ones that are in $\mathbb{P}_{\mathcal{Z}}[n]$ or 0, we have that $GCUD(n, \frac{n}{u}) = \frac{n}{u}$. □

II.2.2 Properties of Tessellation Operation

Consider a CANCELLATIVE ALGEBRA $\mathcal{Y} \Longleftrightarrow (Y, \cdot)$, and suppose that $\alpha \in Y^{\mathbb{Z}/a\mathbb{Z}}$, $\beta \in Y^{\mathbb{Z}/b\mathbb{Z}}$, and $\gamma \in Y^{\mathbb{Z}/c\mathbb{Z}}$, such that $\alpha \odot \beta = \gamma$. We would like to know what SET $P_{a,b} \subseteq \mathbb{Z}$ contains all valid choices of c for fixed a, b. Note that $LCM(a,b) \in a\mathbb{Z}$ and $LCM(a,b) \in b\mathbb{Z}$, by necessity.

Lemma II.2.2 (The Least Common Multiple of the Periods of Two Operand Tessellations is in the Period-Set of the Resulting Operated Tessellation)**.** *Consider a* CANCELLATIVE ALGEBRA *$\mathcal{Y} = (Y, \cdot)$, and suppose that $\alpha \in Y^{\mathbb{Z}/a\mathbb{Z}}$, $\beta \in Y^{\mathbb{Z}/b\mathbb{Z}}$, and $\gamma \in Y^{\mathbb{Z}/c\mathbb{Z}}$, such that $\alpha \odot \beta = \gamma$. It must be the case that $LCM(a,b) \in c\mathbb{Z}$.*

Proof.

$$[\alpha \odot \beta](i) = \gamma(i) \tag{II.39}$$

$$[\alpha \odot \beta](i + LCM(a,b)) = \gamma(i + LCM(a,b)) \tag{II.40}$$

$$\alpha(i + LCM(a,b)) \cdot \beta(i + LCM(a,b)) = \gamma(i + LCM(a,b)) \tag{II.41}$$

$$\alpha(i) \cdot \beta(i) = \gamma(i + LCM(a,b)) \tag{II.42}$$

$$[\alpha \odot \beta](i)(i) = \gamma(i + LCM(a,b)) \tag{II.43}$$

$$\gamma(i) = \gamma(i + LCM(a,b)) \tag{II.44}$$

□

This lemma means we know that $\exists x \in \mathbb{Z} \ni LCM(a,b) = xc$. We know so far then that $c \in D[LCM(a,b)]$. Stated another way, we have found the Set that contains all possible choices for c, meaning we now must restrict this Set to only the genuinely valid choices. The next lemma will provide the criteria that allows us to do just that.

Lemma II.2.3 (Each Least Common Multiple of the Periods of Two Operand Tessellations with the Period of the Operated Tessellation Must be Equivalent To Each-Other)**.** *Consider $\alpha \in Y^{\mathbb{Z}/a\mathbb{Z}}$, $\beta \in Y^{\mathbb{Z}/b\mathbb{Z}}$, and $\gamma \in Y^{\mathbb{Z}/c\mathbb{Z}}$ such that $\alpha \odot \beta = \gamma$. It is the case that:*

$$LCM(a,c) = LCM(b,c) \tag{II.45}$$

Proof. First, we will show that $LCM(a,c) \in b\mathbb{Z}$:

$$[\alpha \odot \beta](i) = \gamma(i) \tag{II.46}$$

$$[\alpha \odot \beta](i + LCM(a,c)) = \gamma(i + LCM(a,c)) \tag{II.47}$$

$$\alpha(i + LCM(a,c)) \cdot \beta(i + LCM(a,c)) = \gamma(i + LCM(a,c)) \tag{II.48}$$

$$\alpha(i) \cdot \beta(i + LCM(a,c)) = \gamma(i) \tag{II.49}$$

$$\alpha(i) \cdot \beta(i + LCM(a,c)) = \alpha(i) \cdot \beta(i) \tag{II.50}$$

$$\beta(i + LCM(a,c)) = \beta(i) \tag{II.51}$$

And further, we will show that $LCM(b, c) \in a\mathbb{Z}$:

$$[\alpha \odot \beta](i) = \gamma(i) \tag{II.52}$$

$$[\alpha \odot \beta](i + LCM(b, c)) = \gamma(i + LCM(b, c)) \tag{II.53}$$

$$\alpha(i + LCM(b, c)) \cdot \beta(i + LCM(b, c)) = \gamma(i + LCM(b, c)) \tag{II.54}$$

$$\alpha(i + LCM(b, c)) \cdot \beta(i) = \gamma(i) \tag{II.55}$$

$$\alpha(i + LCM(b, c)) \cdot \beta(i) = \alpha(i) \cdot \beta(i) \tag{II.56}$$

$$\alpha(i + LCM(b, c)) = \alpha(i) \tag{II.57}$$

Each of these chains of Equalities allow us to conclude that $\exists n \in \mathbb{Z} \ni LCM(a, c) = nb$ and $\exists m \in \mathbb{Z} \ni LCM(b, c) = ma$. It must be the case that n and m are linked, somehow, through c.

$$nb = LCM(a, c) \tag{II.58}$$

$$LCM(b, nb) = LCM(b, LCM(a, c)) \tag{II.59}$$

$$nb = LCM(a, b, c) \tag{II.60}$$

$$ma = LCM(b, c) \tag{II.61}$$

$$LCM(a, ma) = LCM(a, LCM(b, c)) \tag{II.62}$$

$$ma = LCM(a, b, c) \tag{II.63}$$

$$nb = ma \tag{II.64}$$

$$\tag{II.65}$$

Because $LCM(a, c) = nb$, $LCM(b, c) = ma$ and $nb = ma$, we can conclude that $LCM(a, c) = LCM(b, c)$.

□

Thus, only those $d \in D[LCM(a, b)]$ such that $LCM(a, d) = LCM(b, d)$ are valid choices for c. Using our definition of LCM from earlier, we pry further; suppose that $LCM(a, d) = LCM(b, d)$ for some $d \in$

$D[LCM(a,b)]$:

$$LCM(a,d) = LCM(b,d) \tag{II.66}$$
$$\prod_{p\in\mathbb{P}_Z} p^{MAX(a_p,d_p)} = \prod_{p\in\mathbb{P}_Z} p^{MAX(b_p,d_p)} \tag{II.67}$$
$$MAX(a_p,d_p) = MAX(b_p,d_p) \tag{II.68}$$

This EQUATION tells us that:

$$\big(a_p, b_p \leqslant d_p \;\vee\; d_p \leqslant a_p = b_p\big) \tag{II.69}$$

But, because $d \in D[LCM(a,b)]$, we know that $LCM(LCM(a,b),d) = LCM(a,b)$. Using our definition of LCM previously again:

$$LCM(LCM(a,b),d) = LCM(a,b) \tag{II.70}$$
$$\prod_{p\in\mathbb{P}_Z} p^{MAX(MAX(a_p,b_p),d_p)} = \prod_{p\in\mathbb{P}_Z} p^{MAX(a_p,b_p)} \tag{II.71}$$
$$MAX(MAX(a_p,b_p),d_p) = MAX(a_p,b_p) \tag{II.72}$$

Thus, we know that $d_p \leqslant MAX(a_p,b_p)$. These are enough for us to state a theorem that governs how PERIODS interact when TESSELLATIONS are OPERATED.

Theorem II.2.4 (Resultant Period-Set After Tessellation Operation)**.** *Consider $\alpha \in Y^{\mathbb{Z}/a\mathbb{Z}}$, $\beta \in Y^{\mathbb{Z}/b\mathbb{Z}}$, and $\gamma \in Y^{\mathbb{Z}/c\mathbb{Z}}$ such that $\alpha \odot \beta = \gamma$. It is the case that c must belong to the* SET*:*

$$P_{a,b} = \left\{ \frac{LCM(a,b)}{d} \in \mathbb{Z} \;:\; d \in D[GCUD(a,b)] \right\} \tag{II.73}$$

Proof. This is because $d \in D[GCUD(a,b)]$ means that $MAX(d_p, EQ(a_p,b_p)) = EQ(a_p,b_p)$. Hence:

$$\big((d_p \leqslant a_p, b_p \;\wedge\; a_p = b_p) \;\vee\; (d_p = 0 \;\wedge\; a_p \neq b_p) \big) \tag{II.74}$$

Thus, any $c \in P_{a,b}$ – being EQUAL to $\frac{LCM(a,b)}{d}$ – is such that when $a_p \neq b_p$, $c_p = MAX(a_p,b_p)$ because $d_p = 0$, but when $a_p = b_p$, $c_p \leqslant a_p, b_p$ since $c_p = MAX(a_p,b_p) - d_p$ and $u_p \leqslant a_p, b_p$. Meaning that it is always the case that $MAX(a_p,c_p) = MAX(b_p,c_p)$. Hence, $LCM(a,c) = LCM(b,c)$ and $c \in D[LCM(a,b)]$. □

Lemma II.2.5 (The Resultant Period of Two Tessellations with the Same Period is a Divisor of that Period)**.** *For $\tau_a, \tau_b \in Y^{\mathbb{Z}\backslash n\mathbb{Z}}$ and $\tau_c \in Y^{\mathbb{Z}/m\mathbb{Z}}$ such that $\tau_a \odot \tau_b = \tau_c$, it must be that $c \in D[n]$.*

Proof. For this we need only simplify the Set we constructed earlier, and because both τ_a, τ_b have period n, we write $P_{n,n}$:

$$P_{n,n} = \left\{ \frac{LCM(n,n)}{d} \in \mathbb{Z} \; : \; d \in D\big[GCUD(n,n)\big] \right\} \tag{II.75}$$

$$P_{n,n} = \left\{ \frac{n}{d} \in \mathbb{Z} \; : \; d \in D\big[n\big] \right\} \tag{II.76}$$

$$P_{n,n} = \left\{ d \in \mathbb{Z} \; : \; d \in D\big[n\big] \right\} \tag{II.77}$$

$$P_{n,n} = D\big[n\big] \tag{II.78}$$

□

II.3 Identical Resultant-Periods

Objective In this section we will consider the pairs of Periods that produce the exact same Resultant-Period Set.

Strategy We will motivate this set by building up the requirements in individual sets before combining them into a single set that will have the property for which this section is named.

II.3.1 Pairs That Share a Greatest Common Unitary Divisor

We now turn to considering which distinct Pairs of Periods will yield us the same **Resultant Period-Set** once Tessellations of the respective Periods are Operated. We seek a Set $S \subseteq \mathbb{Z} \times \mathbb{Z}$ such that $\Big((a,b),(c,d) \in S\Big) \Rightarrow \Big(P_{a,b} = P_{c,d}\Big)$. When one considers the construction of $P_{n,m}$, it becomes clear that for two distinct Pairs $(a,b),(c,d)$ to generate the same Set, it must be that $LCM(a,b) = LCM(c,d) \wedge GCUD(a,b) = GCUD(c,d)$. First we will describe the Set of Pairs that share their $GCUD$, and to motivate this construction, suppose $GCUD(a,b) = x$. Then it must be that $a = nx$ and $b = mx$ for some $n,m \in \mathbb{Z}$. What restrictions may we place on n,m?

Lemma II.3.1 (Pairs that Share Their GCUD)**.** *The* Set *of* Pairs *that share the $GCUD$ of x is constructed* *like so:*

$$\left\{(xn, xm) \in \mathbb{Z} \times \mathbb{Z} \ : \ GCD(n, x) = GCD(m, x) = GCUD(n, m) = 1\right\} \tag{II.79}$$

Proof. We begin by using our Prime-Power definition of the $GCUD$ Function from earlier:

$$a = nGCUD(a, b) \tag{II.80}$$

$$b = mGCUD(a, b) \tag{II.81}$$

$$\prod_{p \in \mathbb{P}_Z} p^{a_p} = \prod_{p \in \mathbb{P}_Z} p^{n_p + EQ(a_p, b_p)} \tag{II.82}$$

$$\prod_{p \in \mathbb{P}_Z} p^{b_p} = \prod_{p \in \mathbb{P}_Z} p^{m_p + EQ(a_p, b_p)} \tag{II.83}$$

$$a_p = n_p + EQ(a_p, b_p) \tag{II.84}$$

$$b_p = m_p + EQ(a_p, b_p) \tag{II.85}$$

$$a_p - n_p = EQ(a_p, b_p) \tag{II.86}$$

$$b_p - m_p = EQ(a_p, b_p) \tag{II.87}$$

The Function $EQ(s, t)$ can only Equal either s or t if $s = t$ and 0 otherwise. So, these last two lines allow us to form the following Implications, applied to each Prime Power individually:

$$a_p = b_p \ \Rightarrow \ n_p = m_p = 0 \tag{II.88}$$

And similarly:

$$a_p \neq b_p \ \Rightarrow \ (a_p = n_p \ \wedge \ b_p = m_p) \tag{II.89}$$

From these we may conclude that $GCD(n, x) = GCD(m, x) = 1$ and $GCUD(n, m) = 1$. This is because our first Implication tells us that if $x_p = EQ(a_p, b_p) \neq 0$ then $n_p = m_p = 0$; our second Implication tells us that if $x_p = EQ(a_p, b_p) = 0$ then $EQ(n_p, m_p) = 0$, so with our first Implication, we know that every $EQ(n_p, m_p)$ calculation in $GCUD(n, m)$ must be 0, because either $x_p = 0$ or $x_p \neq 0$ i.e. $GCUD(n, m) = 1$. Similarly, these also tell us that $MIN(n_p, x_p) = MIN(m_p, x_p) = 0$, since $n_p = m_p = 0$ in the case that $x_p \neq 0$, i.e. $GCD(n, x) = GCD(m, x) = 1$. □

This lemma tells us that for a Pair to share a $GCUD, x$ each Element in the Pair must be a Product of n and m respectively with x such that they are each Co-Prime to x and do not share a $GCUD$ between each other. We will express these as Inequalities relating to the Prime-Powers of the respective Numbers for clarity:

$$MIN(n_p, x_p) = 0 \tag{II.90}$$

$$\left(0 = n_p < x_p \;\vee\; 0 = x_p < n_p\right) \tag{II.91}$$

$$MIN(m_p, x_p) = 0 \tag{II.92}$$

$$\left(0 = m_p < x_p \;\vee\; 0 = x_p < m_p\right) \tag{II.93}$$

$$EQ(n_p, m_p) = 0 \tag{II.94}$$

$$\left(n_p = m_p = 0 \;\vee\; n_p \neq m_p\right) \tag{II.95}$$

II.3.2 Pairs That Share a Least Common Multiple

Now we will use these conditions to determine which of these Pairs share their LCM, i.e. suppose $LCM(xn, xm) = z$ for some z. We will first complete a short lemma regarding Co-Prime Numbers and their LCM, however, to noticeably simplify our search.

Lemma II.3.2 (Numbers Co-Prime to Arguments in LCM Distribute Over Them)**.** *It is the case that if* $GCD(a, b) = GCD(a, c) = 1$ *then* $aLCM(b, c) = LCM(ab, ac)$.

Proof. We have that $MIN(a_p, b_p) = MIN(a_p, c_p) = 0$, so either $a_p = 0$ or $b_p = c_p = 0$. Now consider, $aLCM(b, c)$ and $LCM(ab, ac)$:

$$aLCM(b, c) = \prod_{p \in \mathbb{P}_Z} p^{a_p + MAX(b_p, c_p)} \tag{II.96}$$

$$LCM(ab, ac) = \prod_{p \in \mathbb{P}_Z} p^{MAX(a_p + b_p, a_p + b_p)} \tag{II.97}$$

So we would like to show that $a_p + MAX(b_p, c_p) = MAX(a_p + b_p, a_p + b_p)$. This must be the case, though, when one considers that either $a_p = 0$ or $b_p = c_p = 0$. If $a_p = 0$, then $a_p + MAX(b_p, c_p) = MAX(b_p, c_p)$ and $MAX(a_p + b_p, a_p + c_p) = MAX(b_p, c_p)$; alternatively, if $b_p = c_p = 0$ then $a_p + MAX(b_p, c_p) = a_p$ and $MAX(a_p + b_p, a_p + c_p) = MAX(a_p, a_p) = a_p$. □

Because of this previous lemma then we know that $LCM(xn, xm) = xLCM(n, m)$, so we know that if $LCM(xn, xm) = z = xLCM(n, m)$. Which is to say, $z = xy$ for $LCM(n, m) = y$. As a result, the restrictions we found on n, m will be the primary influence in our search for which of them will share a LCM. Namely, n, m must be Co-Prime to x and they can not share a $GCUD$ themselves. We will go on to state a theorem that describes the construction of the set we seek and then prove that indeed all Pairs share both $GCUD$ and LCM in the following subsection. First, to allow easier readability of said theorem, we will define the **Subordinate Function**:

Definition II.3.1 (Subordinate Function)**.** The **Subordinate** of an Integer n is defined like so:

$$S(n) \mathrel{\square\!\!\Rightarrow} \prod_{p \in \mathbb{P}_Z} p^{MIN(n_p, |n_p - 1|)} \tag{II.98}$$

We will now, briefly, prove three properties with varying relevance about the **Subordinate** of an Integer, each with respect to one of our Prime-Power Functions, LCM, GCD, and $GCUD$.

Lemma II.3.3 (The LCM of an Integer and its Subordinate is the Integer)**.** *It is the case that for all* Integers *n, that:*

$$LCM(n, S(n)) = n \tag{II.99}$$

Proof. (Technically, this follows from the fact that $(\mathbb{N}_0, LCM, GCD)$ forms a Lattice, and so the Absorptive Collaboration holds on Compositions of each Function, (which actually follows from the fact that $(\mathbb{N}_0, MAX, MIN)$ forms a Lattice; the inheritance of this property is a result of our ability to describe LCM and GCD using MAX and MIN) but we will show it without appealing to this fact.)
We will first express this as the relevant Prime-Decomposition, as usual:

$$LCM(n, S(n)) = \prod_{p \in \mathbb{P}_Z} p^{MAX(n_p, MIN(n_p, |n_p - 1|))} \tag{II.100}$$

So, we need to show that $MAX(n_p, MIN(n_p, |n_p - 1|)) = n_p$. This is easy to see when one considers that if $n_p \geqslant 1$ then $MIN(n_p, |n_p - 1|) = n_p - 1$, (we may drop the Absolute Value because we know it is Non-Negative), and clearly $MAX(n_p, n_p - 1) = n_p$. Similarly if one considers that $n_p = 0$, then $MIN(n_p, |n_p - 1|) = 0$ because $|0 - 1| = 1$, and we are left with $MAX(n_p, n_p) = MAX(0, 0)$. ◻

Lemma II.3.4 (The GCD of an Integer and its Subordinate is the Subordinate)**.** *It is the case that for all*

INTEGERS *n, that:*

$$GCD(n, S(n)) = S(n) \tag{II.101}$$

Proof. (This too follows from the fact that $(\mathbb{N}_0, LCM, GCD)$ forms a LATTICE, for the same reason as above, but – again – we will show it without appealing to this fact.)

We will express this as the relevant PRIME-DECOMPOSITION, as before:

$$GCD(n, S(n)) = \prod_{p \in \mathbb{P}_Z} p^{MIN(n_p, MIN(n_p, |n_p - 1|))} \tag{II.102}$$

So, we need to show that $MIN(n_p, MIN(n_p, |n_p - 1|)) = MIN(n_p, |n_p - 1|)$. This is rather obvious to see when one notes that when $n_p \geqslant 1$ then $MIN(n_p, |n_p - 1|) = n_p - 1$, , and clearly $MIN(n_p, n_p - 1) = n_p - 1$. Similarly if one considers that $n_p = 0$, then $MIN(n_p, |n_p - 1|) = 0$ because $|0 - 1| = 1$, and we are left with $MIN(n_p, n_p) = MIN(0, 0)$. □

Lemma II.3.5 (The GCUD of an Integer and its Subordinate is 1)**.** *It is the case that for all* INTEGERS *n, that:*

$$GCUD(n, S(n)) = 1 \tag{II.103}$$

Proof. This is easy to see if one remembers that $GCUD$ is defined using EQ:

$$GCUD(n, S(n)) = \prod_{p \in \mathbb{P}_Z} p^{EQ(n_p, MIN(n_p, |n_p - 1|))} \tag{II.104}$$

When one considers that $MIN(n_p, |n_p - 1|)$ is either equal to 0 or $n_p - 1$, it becomes clear that the surrounding EQ can only ever yield 0 since either $n_p = 0$ or $n_p \neq n_p - 1$. A 0 on every PRIME in a PRIME-DECOMPOSITION will always yield 1. □

With our definition of the **SUBORDINATE** of an INTEGER and a NUMBER of nice properties proven, we are now ready to state our theorem.

II.3.3 Pairs That Share a GCUD and an LCM

Theorem II.3.6 (Pairs of Periods that Share the Same Resultant Period Set)**.** *The set such that any two* PAIRS $(a, b), (c, d)$ *with* MEMBERSHIP *satisfy the condition,* $P_{a,b} = P_{c,d}$ *is determined by the choice of two* CO-PRIME NUMBERS x, y *such that* $GCUD(a, b) = GCUD(c, d) = x$ *and* $LCM(a, b) = LCM(c, d) = xy$*. Any set that* *has this property for fixed x, y is a* Subset *of the following* Set*:*

$$\pi_{x,y} \Longleftrightarrow \left\{(xvi, xwj) \in \mathbb{Z} \times \mathbb{Z} \ : \ \left(\left(v \in U[y] \ \wedge \ w = \frac{y}{v}\right) \ \wedge \ \left(i \in D[S(w)] \ \wedge \ j \in D[S(v)]\right)\right)\right\} \tag{II.105}$$

Proof. We will demonstrate that it is in fact always the case that $LCM(xvi, xwj) = xy$ and then we will show that $GCUD(xvi, xwj) = x$ as well. After that, we will finish the proof by showing that any other Set with this property must be this Set or a Subset of it.

Beginning with the LCM:

$$LCM(xvi, xwj) = xLCM(vi, wj) \tag{II.106}$$

$$xLCM(vi, wj) = \prod_{p \in \mathbb{P}_{\mathbb{Z}}} p^{x_p + MAX(v_p + i_p, w_p + j_p)} \tag{II.107}$$

We would like to show that $x_p + y_p = x_p + MAX(v_p + i_p, w_p + j_p)$, which is obviously reducible to showing that $y_p = MAX(v_p + i_p, w_p + j_p)$.

To start, note that $v \in U[y]$, means we know that either $v_p = y_p$ or $v_p = 0$.

First, consider $v_p = y_p$. It would be that $w_p = 0$ when $v_p = y_p$, and because $i \in D[S(w)]$ we know that $i_p \leqslant MIN(w_p, |w_p - 1|)$. So, when $v_p = y_p$, it must be that $i_p = 0$ and our calculation simplifies to $MAX(y_p + 0, 0 + j_p)$. Further since $j \in D[S(v)]$ it must be that $j_p \leqslant MIN(v_p, |v_p - 1|) = MIN(y_p, |y_p - 1|)$ which allows us to conclude that $MAX(y_p, j_p) = y_p$.

Alternatively, if $v_p = 0$, because $w_p = y_p - v_p$ then $w_p = y_p$, so our calculation will simplify to $MAX(0 + i_p, y_p + j_p)$. Since $j \in D[S(v)]$, it is the case that $j_p \leqslant MIN(v_p, |v_p - 1|)$, but because $v_p = 0$ then $j_p = 0$. Again, our calculation simplifies to $MAX(i_p, y_p)$, but $i \in D[S(w)]$ gives us that $i \leqslant MIN(w_p, |w_p - 1|) = MIN(y_p, |y_p - 1|)$, and so it must be that $MAX(i_p, y_p) = y_p$.

With that we have shown that $y_p = MAX(v_p + i_p, w_p + j_p)$ and so we may conclude that $LCM(xvi, xwj) = xy$.

Now, we wish to demonstrate that $GCUD(xvi, xwj) = x$, which is notably easier since y is Co-Prime to x.

$$GCUD(xvi, xwj) = \prod_{p \in \mathbb{P}_{\mathbb{Z}}} p^{EQ(x_p + v_p + i_p, x_p + w_p + j_p)} \tag{II.108}$$

So, we would like to show that $EQ(x_p + v_p + i_p, x_p + w_p + j_p) = x_p$.

Since $v \in U[y]$, $w = \frac{y}{v}$, $i \in D[S(w)]$, and $j \in D[S(v)]$, it is easy to see that $v, w, i, j \in D[y]$. Since $\left(D[x] \bigcap D[y]\right) = \{1\}$ – which is to say they are Co-Prime – we may then conclude that if $x_p \neq 0$ it must be that $v_p = i_p = w_p = j_p = 0$. From this we know that when $x_p \neq 0$ then $EQ(x_p + v_p + i_p, x_p + w_p + j_p) = EQ(x_p + 0 + 0, x_p + 0 + 0) = EQ(x_p, x_p) = x_p$.

So, when $x_p = 0$, we have $EQ(x_p + v_p + i_p, x_p + w_p + j_p) = EQ(0 + v_p + i_p, 0 + w_p + j_p)$. Because we know that every EQ calculation in $GCUD(xvi, xwj)$, such that $x_p \neq 0$, is Equal to x_p, we wish to show that it is always the case that $EQ(v_p + i_p, w_p + j_p) = 0$ when $x_p = 0$. Consider that $\left(D[v] \bigcap D[w]\right) = \{1\}$ since $v \in U[y]$ and $w = \frac{y}{v}$. As a result, if $v_p \neq 0$ then $i_p = w_p = 0$ and our calculation simplifies to $EQ(v_p + 0, 0 + j_p)$. Though since $j \in D[S(v)]$, it must be that $j \leqslant MIN(v_p, |v_p - 1|)$. As a result $j_p \leqslant v_p - 1$ so $EQ(v_p, j_p) = 0$. Alternatively, if $v_p = 0$ then $j_p = 0$ as well, meaning $EQ(v_p + i_p, w_p + j_p) = EQ(i_p, w_p)$. But, for the same reason, since $i \in D[S(w)]$, then $i \leqslant MIN(w_p, |w_p - 1|)$. If $w_p \neq 0$ then $i_p \leqslant w_p - 1$ necessitating $EQ(i_p, w_p) = 0$; if instead $w_p = 0$, then $i_p = 0$ too, so $EQ(i_p, w_p) = 0$.

With that we have shown that $EQ(x_p + v_p + i_p, x_p + w_p + j_p) = x_p$, and thus shown $GCUD(xvi, xwj) = x$.

Finally, we will show that any set E of Pairs such that, for fixed x, y, all Pairs have LCM equal to xy and $GCUD$ equal to x, is a Subset of $\pi_{x,y}$. We will show this by Contradiction.

Suppose instead that there exists some Set E such that every Pair $(a, b) \in E$ has the property where $LCM(a, b) = xy, GCUD(a, b) = x$, but $E \nsubseteq \pi_{x,y}$. Then it would be the case that $E/\pi_{x,y} \neq \varnothing$. So, consider $(a, b) \in E/\pi_{x,y}$.

Since $GCUD(a, b) = x$, we know that $x \in U[a]$ and $x \in U[b]$. As a result, we may rewrite a and b as xn and xm for some specific $n \in U[a]$ and $m \in U[b]$, namely $\frac{a}{x}$ and $\frac{b}{x}$ respectively. We know that $n \in U[a]$ and $m \in U[b]$ from a previous lemma showing that the Quotient of an Integer by a Unitary Divisor is also a Unitary Divisor. Also from a previous lemma we know that it must be that $GCUD(n, m) = 1$, i.e. $\left(U[n] \bigcap U[m]\right) = \{1\}$, and $GCD(x, n) = GCD(x, m) = 1$ i.e. $\left(D[x] \bigcap D[n]\right) = \{1\}$ and $\left(D[x] \bigcap D[m]\right) = \{1\}$. From this it is also the case that $LCM(xn, xm) = xLCM(n, m) = xy$ and so $LCM(n, m) = y$.

Since $LCM(n, m) = y$, we know that $MAX(n_p, m_p) = y_p$, i.e. $\left(n_p \leqslant m_p = y_p \;\vee\; m_p \leqslant n_p = y_p\right)$. We know that there is no Common Unitary Divisor between n and m though, thus $n_p \neq m_p$ unless $n_p = m_p = 0$. This allows us to say then that $\left(n_p < m_p = y_p \;\vee\; m_p < n_p = y_p\right)$. This allows us to further say that $(\left(U[y] \bigcap U[n]\right) \neq \varnothing \;\vee\; \left(U[y] \bigcap U[m]\right) \neq \varnothing)$. Without loss of symmetry, suppose $\left(U[y] \bigcap U[n]\right) \neq \varnothing$ and consider some $s \in \left(U[y] \bigcap U[n]\right)$. We know that s must be such that each $s_p = y_p$ or $s_p = 0$. This allows us to rewrite $n = sf$ such that if $s_p = y_p$ then $f_p = 0$ and if $s_p = 0$ then $f_p < y_p$ since $\left(n_p < m_p = y_p \;\vee\; m_p < n_p = y_p\right)$. Similarly, as a result we may conclude that $\frac{y}{s} \in \left(U[y] \bigcap U[m]\right)$ since $\left(n_p < m_p = y_p \;\vee\; m_p < n_p = y_p\right)$ tells us that if $n_p = y_p$ as in the case when $s_p = y_p$, then it must be that

$m_p < n_p$, meaning that there must be some $t \in \left(U[y] \bigcap U[m]\right)$ such that $t_p = y_p$ when $s_p \neq y_p$, namely $t = \frac{y}{s}$. This allows us to rewrite $m = tg$ such that when $t_p = y_p$ then $g_p = 0$ and when $t_p = 0$ then $g_p < y_p$. We will now notice that necessarily $f \in D[S(t)]$ and $g \in D[S(s)]$. This is because exactly when $f_p = 0$ then $t_p = 0$ and when $g_p = 0$ then so does $s_p = 0$; similarly, when $t_p = y_p$ then $s_p = 0$ so since $f_p < y_p$ it follows in this case that $f_p < t_p$, and the same is true for g_p and s_p respectively. This shows that it indeed must be the case that $f \in D[S(t)]$ and $g \in D[S(s)]$.

With all of that we have shown that $a = xn = xsf$ and $b = xm = xtg$ where s, t, f, g are exactly as described in our constructed Set, hence $(a, b) \in \pi_{x,y}$, a Contradiction. □

This allows us to rewrite our definition of a Resultant Period Set – if we should so wish – such that it will now represent the unique such Set, and further we can derive every Pair of Periods that will yield it from this new definition.

$$P^{x,y} \Leftrightarrow \left\{dy \in \mathbb{Z} \ : \ d \in D[x]\right\} \tag{II.109}$$

$$\tag{II.110}$$

The associated Pairs are as our previous theorem constructed them.

CHAPTER III

ALLGEBRAS

III.1 Allgebras and Symmetry-Sets

Objective In this section we will introduce the notion of an **Allgebra** and formalize Algebraic Structures on it by defining the generalization of a Period-Set: a Symmetry-Set.

Strategy We will take the foundation we created relating to how we understand Algebraic-Structures coming out of the introduction of Operators – and those from Functions – and extend our intuition to *all* possible Operators. We will look for patterns among them, and how they collectively bundle together to create similarly classed Algebraic Structures over Subsets of our Domain.

III.1.1 What kinds of Symmetries are Interesting?

Consider two Non-Empty Sets each Paired with the Set of all possibly definable Binary Operators on each respectively – henceforth referred to as ***All*gebras** – denoted: $\mathcal{X} \Leftrightarrow (X, \Theta_X)$, $\mathcal{Y} \Leftrightarrow (Y, \Theta_Y)$. We denote the set of all functions from X to Y as Y^X.

Definition III.1.1 (Symmetry Points Functions)**.** For $\lambda \in Y^X$, $x, s \in X$, and $\vartheta_X \in \Theta_X$, we define the **Symmetry**

Points of ϑ_X on λ as follows.

$$\sigma_L(\lambda, \vartheta_X) \Leftrightarrow \left\{ s \ : \ \lambda(x) = \lambda(\vartheta_X(s, x)) \right\} \tag{III.1}$$

$$\sigma_R(\lambda, \vartheta_X) \Leftrightarrow \left\{ s \ : \ \lambda(x) = \lambda(\vartheta_X(x, s)) \right\} \tag{III.2}$$

$$\sigma_C(\lambda, \vartheta_X) \Leftrightarrow \left(\sigma_L(\lambda, \vartheta_X) \bigcap \sigma_R(\lambda, \vartheta_X) \right) \tag{III.3}$$

When subscripts are omitted in future definitions the original three indicated above, L, R, C – meant to denote 'Left', 'Right', and 'Commutative' – are Implied, resulting in three new definitions each time.

Consequence III.1.2 (Commutative Operator-Symmetry Implies Left and Right Operator-Symmetry)**.** Any Function from X to Y that has a **Commutative Operator-Symmetry**, must then have a **Left Operator-Symmetry** and **Right Operator-Symmetry** – namely, the previously mentioned **Commutative Operator-Symmetry**.

Definition III.1.3 (Operator-Symmetries of Functions)**.** Consider a Mapping Σ^Θ that will be from $\mathcal{P}(Y^X) \rightarrow \mathcal{P}(\Theta_X)$. For $\Lambda \subseteq Y^X$, and $\vartheta_X \in \Theta_X$, we define the **Operator-Symmetries** of the Set of Functions Λ as the Set of Operators that have **Symmetry Points** on at least one Function in said Set.

$$\Sigma^\Theta(\Lambda) \Leftrightarrow \left\{ \vartheta_X \ : \ \exists \lambda \in \Lambda \ni \sigma(\lambda, \vartheta_X) \neq \varnothing \right\} \tag{III.4}$$

Definition III.1.4 (Function-Coverage of Operators)**.** Consider a Mapping Σ^Λ that will be from $\mathcal{P}(Theta_X) \rightarrow \mathcal{P}(Y^X)$. For $\lambda \in Y^X$, and $\theta_X \subseteq \Theta_X$, we define the **Function-Coverage** of the Set of Operators θ_X as the Set of all Functions that have at least one **Symmetry Point** with at least one Operator in said Set.

$$\Sigma^\Lambda(\theta_X) \Leftrightarrow \left\{ \lambda \ : \ \exists \vartheta_X \in \theta_X \ni \sigma(\lambda, \theta_X) \neq \varnothing \right\} \tag{III.5}$$

Lemma III.1.1 (Σ^Θ & Σ^Λ are Functions)**.** *Each of these* Mappings *are defined for all* Inputs.

Lemma III.1.2 (Non-Empty Input to Σ^Θ Will Yield Non-Empty Output)**.** *The* Function Σ^Θ *will not* Map *to the* Empty-Set *for any argument except the* Empty-Set *itself.*

Proof. Take any fixed Element $x_0 \in X$ – note that we already assumed that X is non-empty – and consider

$\vartheta_{x_0} \in \Theta_X$ defined:

$$\vartheta_{x_0}(x, y) = \begin{cases} x & y = x_0 \\ y & x = x_0 \\ x_0 & \text{else} \end{cases} \tag{III.6}$$

For all $\lambda \in Y^X$, it is the case that $\lambda(\vartheta_{x_0}(x, x_0)) = \lambda(\vartheta_{x_0}(x_0, x)) = \lambda(x) \; \forall \; x \in X$, which means that $x_0 \in \sigma_L(\lambda, \vartheta_{x_0})$, and $x_0 \in \sigma_R(\lambda, \vartheta_{x_0})$, and so $x_0 \in \sigma_C(\lambda, \vartheta_{x_0})$ as a result. This allows us to conclude that $\Sigma^\Theta(\{\lambda\})$ is always NON-EMPTY, and because all SINGLETON INPUTS yield NON-EMPTY OUTPUT, any COMPOSITE SET will also be NON-EMPTY. □

Corollary III.1.2.1 ($\Sigma^\Theta(\Theta_X) = Y^X$)**.**

Lemma III.1.3 (Non-Empty Input to Σ^Λ Will Yield Non-Empty Output)**.** *The function Σ^Λ will not* MAP *to the* EMPTY-SET *for any* INPUT *except the* EMPTY-SET *itself.*

Proof. Take any fixed ELEMENT $y \in Y$ – note that we already assumed that Y is NON-EMPTY – and consider $y_\lambda \in Tr[Y^X]$. For some $\vartheta \in \Theta_X$, consider $x_L, x_R, x_\vartheta \in X$. Then, it must be that $y_\lambda(x_\vartheta) = y_\lambda(\vartheta(x_L, x_\vartheta)) = y_\lambda(x_L) = y_\lambda(\vartheta(x_\vartheta, x_R)) = y_\lambda(x_R)$. This means that $x_L, x_\vartheta \in \sigma_L(y_\lambda, \vartheta)$, $x_R, x_\vartheta \in \sigma_R(y_\lambda, \vartheta)$, and as a result $x_\vartheta \in \sigma_C(y_\lambda,)$. This means that we can conclude $\Sigma^\Lambda(\{\vartheta\})$ is always NON-EMPTY, and because all SINGLETON INPUTS yield NON-EMPTY OUTPUT, any COMPOSITE SET will also be NON-EMPTY. □

Corollary III.1.3.1 ($\Sigma^\Lambda(Y^X) = \Theta_X$)**.**

III.2 Symmetry-Set Relations on Functions

Objective We will define several RELATIONS that allow us to say more about our ALLGEBRAS.

Strategy We will use common relationships between SETS, specifically applied to SYMMETRY-SETS, in order to gain additional structure – and so insight – in the form of several ORDER RELATIONS and EQUIVALENCE RELATIONS on ALLGEBRAS.

III.2.1 Several Interesting Relations on Functions

Definition III.2.1 (Regularity & Similarity of Function Sets: $\lesssim, \sim$)**.** Suppose $\alpha, \beta \subseteq Y^X$; we say that α is **Less Regular** than β if and only if $\Sigma^{\Theta}(\alpha) \subseteq \Sigma^{\Theta}(\beta)$.

$$\alpha \lesssim \beta \Longleftrightarrow \Sigma^{\Theta}(\alpha) \subseteq \Sigma^{\Theta}(\beta) \tag{III.7}$$

Further we say that α and β are Similar if and only if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$, or Equivalently:

$$\alpha \sim \beta \Longleftrightarrow \Sigma^{\Theta}(\alpha) = \Sigma^{\Theta}(\beta) \tag{III.8}$$

Lemma III.2.1 (Regularity is a Pre-Order on $\mathcal{P}(Y^X)$)**.** *It is the case that $\lesssim_L, \lesssim_R$, and $\lesssim_C$ are all* Reflexive, *and* Transitive.

Proof. The proof is self-evident, as the definition of **Regularity** is entirely in terms of Set Inclusion which itself is Reflexive and Transitive. □

Lemma III.2.2 (Similarity is an Equivalence Relation on $\mathcal{P}(Y^X)$)**.** *It is the case that $\sim_L, \sim_R$, and $\sim_C$ are all* Reflexive, Symmetric, *and* Transitive.

Proof. The proof is self-evident, as the definition of **Similarity** is entirely in terms of Set Equality which itself is Reflexive, Symmetric, and Transitive. □

Lemma III.2.3 (Σ^{Θ} is an Order-Homomorphism from $(\mathcal{P}(Y^X), \lesssim)$ to $(\mathcal{P}(\Theta_X), \subseteq)$)**.** *This, too, is a direct result of* **Regularity** *being defined using* Set-Inclusion.

Corollary III.2.3.1 (Similarity Identifies a Kernel of Σ^{Θ})**.**

Definition III.2.2 (Coherence & Concurrence of Operators: $\lhd$, $\bowtie$)**.** Suppose $\theta_a, \theta_b \subseteq \Theta_X$; we say that θ_a is **Less Coherent** than θ_b if and only if $\Sigma^{\Lambda}(\theta_a) \subseteq \Sigma^{\Lambda}(\theta_b)$.

$$\theta_a \lhd \theta_b \Longleftrightarrow \Sigma^{\Lambda}(\theta_a) \subseteq \Sigma^{\Lambda}(\theta_b) \tag{III.9}$$

Further we say that θ_a and θ_b are **Concurrent** if and only if $\theta_a \lhd \theta_b$ and $\theta_b \lhd \theta_a$, or equivalently:

$$\theta_a \bowtie \theta_b \Longleftrightarrow \Sigma^{\Lambda}(\theta_a) = \Sigma^{\Lambda}(\theta_b) \tag{III.10}$$

Lemma III.2.4 (Coherence is a Pre-Order on $\mathcal{P}(\Theta_X)$)**.** *It is the case that* $\lhd_L, \lhd_R$, *and* $\lhd_C$ *are all* Reflexive, *and* Transitive.

Proof. The proof is self-evident, as the definition of **Coherence** is entirely in terms of Set Inclusion. $\square$

Lemma III.2.5 (Concurrence is an Equivalence Relation on $\mathcal{P}(\Theta_X)$)**.** *It is the case that* $\bowtie_L, \bowtie_R$, *and* $\bowtie_C$ *are all* Reflexive, Symmetric, *and* Transitive.

Proof. The proof is self-evident, as the definition of **Concurrence** is entirely in terms of Set Equality. $\square$

Lemma III.2.6 (Σ^Λ is an Order-Homomorphism from $(\mathcal{P}(\Theta_X, \lhd)$ to $(\mathcal{P}(Y^X), \subseteq)$)**.** *This, too, is a direct result of* **Coherence** *being defined using* Set Inclusion.

Corollary III.2.6.1 (Concurrence Identifies a Kernel of Σ^Λ)**.**

Definition III.2.3 (Resemblance & Correspondence of Function-Operator Pairs: $\vdash, \models$)**.** Let $\alpha, \beta \subseteq Y^X$, and suppose $\theta_a, \subseteq \Sigma^\Theta(\alpha)$ and $\theta_b \subseteq \Sigma^\Theta(\beta)$ we say that (α, θ_a) **Resembles** (β, θ_b) if and only if $\sigma(\alpha, \theta_a) \subseteq \sigma(\beta, \theta_b)$. (Note: We could equivalently take two Subsets of Θ_X and take α and β to be Subsets of the respective Σ^Λ Outputs.)

$$(\alpha, \theta_a) \multimapinv (\beta, \theta_b) \mathrel{\Diamond\!\!\Rightarrow} \sigma(\alpha, \theta_a) \subseteq \sigma(\beta, \theta_b) \tag{III.11}$$

Further, we say that (α, θ_a) and (β, θ_b) **Correspond** if and only if $(\alpha, \theta_a) \vdash (\beta, \theta_b)$ and $(\beta, \theta_b) \vdash (\alpha, \theta_a)$, or equivalently:

$$(\alpha, \theta_a) \multimapboth (\beta, \theta_b) \mathrel{\Diamond\!\!\Rightarrow} \sigma(\alpha, \theta_a) = \sigma(\beta, \theta_b) \tag{III.12}$$

Lemma III.2.7 (Resemblance is a Pre-Order on $\mathcal{P}(Y^X \times \Theta_X)$)**.** *The relation of* **Resemblance** *is* Reflexive, *and* Transitive.

Lemma III.2.8 (Correspondence is an Equivalence Relation on $\mathcal{P}(Y^X \times \Theta_X)$)**.** *The* **Correspondence** Relation *is* Reflexive, Symmetric, *and* Transitive.

Lemma III.2.9 (σ is an Order-Homomorphism from $(\mathcal{P}(Y^X \times \Theta_X), \multimapinv)$ to $(\mathcal{P}(X), \subseteq)$)**.** *This, too, is a direct result of* **Resemblance** *being defined using* Set-Inclusion.

Corollary III.2.9.1 (Correspondence Identifies a Kernel of σ)**.**

Appendices

.1 Conditional Proof

Given some *Premises* or *Conditions*, $P_1, P_2, \ldots P_n$, a series of *Substitutions*, $P_a, P_b \ldots P_x \vdash P_s; P_c, P_d \ldots P_y \vdash P_t; \ldots P_e, P_f \ldots P_z \vdash Q$ that follow from previously assumed or proven *Logical Rules* $L_i, L_j, \ldots L_k$, and a *Conclusion* Q that logically follows from the final substitution, one denotes a **Conditional Proof** of such a **Logical Rule** as follows:

$$
\begin{array}{r|cc}
P.1) & P_1 & \\
P.2) & P_2 & \\
\vdots & \vdots & \\
P.n) & P_n & \\
\hline
S.1) & P_a, P_b \ldots P_x \vdash P_s & \because L_i \\
S.2) & P_c, P_d \ldots P_y \vdash P_t & \because L_j \\
\vdots & \vdots & \vdots \\
S.m) & P_e, P_f \ldots P_z \vdash Q & \because L_k \\
\hline
C & \therefore Q &
\end{array}
\tag{13}
$$

The far left column is a labeling scheme: P for *Premises*, S for *Substitutions*, and C for *Consequence* or *Conclusion*. The middle column is where *Premises*, *Substitutions* and the *Conclusion* are placed. The right column is only used in the middle row for *Substitutions* in order to explain what previous **Logical Rule** enabled that substitution. The symbol $\therefore$ is interpreted as meaning 'therefore', and $\because$ as meaning 'because [of]'.

.2 Axioms of Zermelo-Fraenkel Set-Theory

Let the **Language Of ZFC** reference the First-Order Logic we establish in **0.I.1**, supplemented with the Logical Connective $\in$ and its Negation $\notin$ defined in **0.I.2.1**.

1. **Axiom Of Extensionality**:

$$\forall x \forall y \left[\forall z \left(z \in x \Leftrightarrow z \in y\right) \Rightarrow (x = y)\right] \tag{14}$$

2. **Axiom Of Regularity**:

$$\forall x\left[\exists a\left(a\in x\right)\Rightarrow\exists y\left(y\in x\wedge\nexists z\left(z\in y\wedge z\in x\right)\right)\right] \tag{15}$$

3. **Axiom Schema Of Restricted Comprehension**:

   Let ψ be a Formula in the **Language Of ZFC** such that all Free Variables are among $z,a,b,c\ldots w$ [insert footnote about not being limited by the length of the latin alphabet] with y explicitly not Free in ψ.

$$\forall z,\forall a,\forall b,\forall c\ldots\forall w\exists y\forall x\left[x\in y\Leftrightarrow\ \left[(x\in z)\wedge\psi(x)\right]\right] \tag{16}$$

4. **Axiom Of Pairing**:

$$\forall x\forall y\exists!z\left[\forall a\left(a\in z\Leftrightarrow\ \left(a=x\vee a=y\right)\right)\right] \tag{17}$$

5. **Axiom Of Union**:

$$\forall S\,\exists!A\forall Y\forall x\left[\left(x\in Y\wedge Y\in S\right)\Leftrightarrow\ x\in A\right] \tag{18}$$

6. **Axiom Schema Of Replacement**: Let ψ be a Formula in the **Language Of ZFC** such that all Free Variables are among $x,y,A,a,b,c\ldots w$ with B explicitly not Free in ψ.

$$\forall A,\forall a,\forall b,\forall c\ldots\forall w\left[\exists x\left(x\in A\Rightarrow\exists!y\psi(y)\right)\Rightarrow\exists!B\forall x\left(x\in A\Leftrightarrow\ \exists y\left(y\in B\right)\wedge\psi(y)\right)\right] \tag{19}$$

7. **Axiom Of Infinity**†:

$$\exists N \left[\begin{array}{c} \exists! e \left[\begin{array}{c} \forall x \left(x \notin e \right) \\ \Downarrow \\ e \in N \end{array} \right] \\ \wedge \\ \forall y \left[\begin{array}{c} y \in N \\ \Downarrow \\ \left(\begin{array}{c} \left(y = e \right) \\ \vee \\ \exists! Y \left[\begin{array}{c} Y \in N \\ \wedge \\ \exists! \hat{y} \forall z \left[\begin{array}{c} \left(z \in \hat{y} \Leftrightarrow \; z = y \right) \\ \wedge \\ \left(z \in Y \Leftrightarrow \; \left(z = y \vee z = \hat{y} \right) \right) \end{array} \right] \end{array} \right] \end{array} \right) \end{array} \right] \end{array} \right] \tag{20}$$

8. **Axiom Of Power Set**

$$\forall x \exists! y \forall z \left[\forall a \left(a \in z \Rightarrow z \in x \right) \Leftrightarrow \; z \in y \right] \tag{21}$$

†*: Rendered here as an abstract piece of art.*

.3 Axiom of Choice

$$\forall X \left[\forall a \forall b \left(a \in X \wedge b \in X \Rightarrow \nexists y \left(y \in A \wedge y \in B \right) \right) \Leftrightarrow \; \exists C \left(\forall x \left(x \in X \Leftrightarrow \; \exists! c \left(c \in x \wedge c \in C \right) \right) \right) \right] \tag{22}$$

$$\textbf{ZFC} \;\; \Leftrightarrow \;\; \textbf{Zermelo-Fraenkel Set-Theory} \text{ with the } \textbf{Axiom Of Choice} \tag{23}$$