CONNECTING LOGIC AND PROOF TECHNIQUES: IDENTIFYING LEARNING IN AN INTRODUCTION TO PROOFS COURSE

By

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Dedicated to my wife, whom I could not have done this without, and to our son, our greatest creation.

To my grandfathers, Dr. Mark S. Reed & Dr. J. Richard Zerby

And for Alan Cherkin, Cameron Herr, and all of my friends who died due to the vicious disease of addiction. I love you, and I think of you often.

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"If I have seen further, it is by standing upon the shoulders of giants"

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Abstract

Proof and proving are a core component of the discipline of mathematics and proving is a required exercise in many upper-level courses in mathematics at the undergraduate level. Writing proofs remains difficult for many students (Moore, 1994; Stylianides et al., 2017). To address this difficulty many universities have began offering Introduction to Proof courses. These courses typically cover three main areas, logic, proof techniques, and sets and functions (David & Zazkis, 2020). With this course's importance in students' transition to upper-level mathematics, it is worthwhile to investigate the connections that students make between the subcomponents of such a course. As such, in this dissertation study I sought to understand the connections that students make between logic and the techniques of proof in and Introduction to proofs course. In the first chapter I state the broad issues related to students learning of logic and proof techniques, to set the stage for the remainder of the manuscript. In the second chapter I present a research study on the connections that students make between logic, direct, and indirect modes of proof. In the third chapter I present a research study on the struggles that students face as they learn to write proofs with mathematical induction. In the fourth chapter I present a practitioner-minded piece where I highlight the typical issues that students face throughout an Introduction to Proofs course. Finally, in the fifth chapter I share some broad conclusions across these three manuscripts and reflect on students' learning throughout an Introduction to Proofs course.

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CHAPTER 1 INTRODUCTION

Proof and proving are core practices of mathematicians and are their primary means of communicating new mathematical results or ideas (De Villiers, 1990; Lakatos, 1963). Mathematical proofs are a combination of nuanced language and logical steps and deductions (Burton & Morgan, 2000; Lakatos, 1963; Raman, 2003). For undergraduates, transitioning to proving is not an easy endeavor (Moore, 1994), and thus many universities have adopted Introduction to Proof courses to help students successfully make this transition in their mathematical studies (Cook et al., 2019; David & Zazkis, 2020; Moore, 1994; Savic, 2017). These Introduction to Proof (ITP) courses cover a variety of topics (David & Zazkis, 2020), but their intention is to introduce students to the language and technique of proof, such as the rules of logical inference and the various techniques of proof (e.g., direct proof, proof by contradiction). Despite this desire, undergraduate students continue to struggle with proving after these courses in their proof-based mathematics coursework (e.g., Abstract Algebra, Real Analysis, Number Theory). Scholars in cognitive science have demonstrated humans have difficulty in utilizing and understanding arguments involving the conditional statement (e.g., Espino & Ramírez, 2013; Johnson-Laird, 1983), which is a key component of the various techniques of proof (e.g., direct proof: $P \rightarrow Q$; proof by contraposition: $\sim Q \rightarrow \sim P$). Though there have been studies on students' understanding of individual techniques of proof (e.g., Hub & Dawkins, 2018; Zandieh et al., 2014), few scholars have studied the possible threads or connections across students' understanding of these techniques of proof, particularly while they are learning these core techniques for the first time. This study, then, investigates students' understanding of one potential thread (i.e., the conditional statement), which has a unique role within each proof technique and could be a common source of difficulty across students' learning of proof techniques (Brown 2013; 2018; Hub & Dawkins, 2018). This study intends to help explain students' understanding of the conditional statement (as a concept of logic) in its relation to the various techniques of proof (i.e., direct proof, proof by contradiction, proof by contraposition, and mathematical induction) as students are learning these ideas in an ITP course.

Stating the Problem

Mathematical proof is an essential component of the discipline of mathematics (De Villiers, 1990; Dawkins & Weber, 2017; Stylianides et al., 2017). Proof has many roles within mathematics (De Villiers, 1990), and proofs are a mathematician's primary source of communicating new mathematical results or conjectures (Burton, 1998; De Villiers, 1990). This highlights the importance of mathematical proof within the community of mathematicians and proof's many uses within mathematics. However, scholars note that proof is largely absent from the K-16 curriculum in the United States (Stylianou et al., 2010).

With proof's importance within the community of mathematics, mathematics education scholars (e.g., Ball & Bass, 2003; Schoenfeld, 1994; Stylianou et al., 2010; Yackel & Hanna, 2003) and reform documents (e.g., The National Council of Teachers of Mathematics [NCTM], 2000) have called for proof to have more of an emphasis in the K-16 curriculum. The transition to proof is difficult for many (Moore, 1994). Indeed, high school students have difficulties when writing proofs (Herbst & Brach, 2006), or fail to see the need for a proof of a conjecture or theorem (Kunimine et al., 2009). These issues persist with undergraduates in proof-based mathematics courses (Weber, 2001). Consequently, there is a warrant to investigate the ways students develop the language of mathematical proof and adopt the techniques of proof, particularly with proof's mixture of complex language and syntax as well as mathematical content.

Many universities across the United States offer Introduction to Proof (ITP) courses in order to combat students' difficulty in the transition to proving in mathematics, and such courses have grown in prevalence throughout the years (Cook et al., 2019; David & Zazkis, 2020; Moore, 1994). Consequently, more research has been done on this population of students as they learn the elements of mathematical proof writing (e.g., Bleiler-Baxter & Pair, 2017; Brown, 2018). However, there is less research on the ways in which students develop the language of mathematical proof within these ITP courses or on how they incorporate logical rules and proof's complex language. Researchers have demonstrated that students face unique challenges when adopting individual proof techniques (e.g., direct proof, proof by contradiction) (Brown, 2018; Hub & Dawkins, 2018) and experience difficulty in learning and applying rules of logic and inference (e.g., understanding the conditional statement, using modus ponens/tollens) (Inglis & Simpson, 2008; 2009), though few studies have documented how students' progress through such a course.

When proving, often mathematical conjectures or theorems are of the form, *if P then Q*, referred to as conditional statements (e.g., if an integer is even, then its square is even). Cognitive psychologists have examined the issue of human beings reasoning about conditionals (e.g., Espino & Ramírez, 2013; Johnson-Laird, 1983). . Reasoning theorists such as Espino and Ramírez (2013) showed that human beings often make incorrect logical inferences when examining arguments in the form of a conditional and have similar difficulty when assessing the validity of symbolic logical arguments.

Conditional statements in mathematics, *if P then Q*, are really statements of the form, *if P(x) then Q(x)*, which is more precisely expressed as, *for every x such that P(x) is true, Q(x) is true*. Expressed in this latter form, it is easier to see that *x* must satisfy the assumptions of *P(x)*, and it is the responsibility of the prover to demonstrate that these assumptions directly imply *Q(x)*. For mathematicians, the process of assuming *P(x)* is likely a trivial one (Raman, 2003). However, students who are learning to write proofs with inquiry-based instruction often have difficulties in making the necessary linguistic choices (i.e., selecting modes of argument representation) or understanding the set-based logical structure of a conditional statement (Hub & Dawkins, 2018). That is, given a mathematical conjecture of the form, *if P then* Q, students (i.e., novices) are not as adept as mathematicians (i.e., experts) when translating the assumptions of *P(x)* into operable language to prove *Q(x)* or in understanding when conditional statements are true or false (Hub & Dawkins, 2018). For instance, in Hub and Dawkins' study, an undergraduate student, Hugo, was asked to think about conditional statements such as "if a number is not a multiple of 3, then it is not a multiple of 6," or "if a triangle is obtuse, then it is not acute." Hugo was less adept at understanding the connections between set-based reasoning about conditionals (Hub & Dawkins, 2018). (Hub & Dawkins, 2018). Moreover, it is not simply a matter of restating *P(x)* in the above conditional statements to prove the claim(s). For instance, to prove the result "if a number is not a multiple of three, then it is not a multiple of six," it is unlikely that a mathematician would begin their proof by simply reproducing the antecedent, "if a number is not a multiple of three..." (Morgan, 2002; Rotman, 2006). In mathematics, particularly when proving, the use of imperatives (e.g., Suppose, Let, Assume) is essential in conveying a message to the reader about the mathematical objects in play (Morgan, 2002; Rotman, 2006); and what follows the imperative often sets up the writer with mathematical objects to operate on. As such, a mathematician when proving the statement "if a number is not a multiple of three then it is not a multiple of six," would more likely begin their proof with an imperative such as, "Let x be an integer which is not a multiple of three," and then assert an operable form for x by writing something such as, "then, x can be written as 3n+1 or 3n+2 where n is a an integer." The field would benefit from understanding whether and how novices to proof writing are aware of mathematicians' conventions for proof writing, such as the use of imperatives and the transition to operable statements, as these linguistic conventions are essential to successful proof writing (Burton & Morgan, 2000; Morgan, 1998)

In an ITP course, students learn various techniques for proving conditional statements and statements of other logical forms (David & Zazkis, 2020). These proving techniques include proof by contradiction, proof by contraposition, mathematical induction, disproof by counterexample, and direct proof. Each of these proof techniques require the prover to utilize their understanding of the conditional statement in various ways. For instance, proving by contraposition requires the prover to take their original conjecture (*if P then Q*), and negate both *P* and *Q* to prove the statement *if ~Q then ~P*. Proof by contradiction requires the prover to assume both *P* and ~*Q* in order to find a logical contradiction which is a direct deduction from assuming both *P and* ~*Q*. Mathematical induction contains an embedded conditional after the basis case, requiring the prover to show $P(N) \rightarrow P(N+1)$. Finding a counterexample to a mathematical conjecture in the form of a conditional requires the prover to find an example which meets the assumptions of *P* but fails to meet the assumptions of *Q*. Finally, direct proof requires the prover to assume P_0, then deduce $P_{0.} \rightarrow P_1 \rightarrow \ldots \rightarrow P_{n.} \rightarrow Q$, where each P_i is a direct consequence of P_{i-1}.

Mathematical proof, however, is not done exclusively by manipulating symbols and making logical deductions (Fischbein, 1982; Lakatos, 1963). Indeed, mathematicians and students invoke many methods to complete or make sense of a proof, such as using examples or other less formal means (Sandefur et al., 2013; Weber et al., 2014). Scholars have theorized how mathematicians utilize intuition when solving proofs (Lakatos, 1963; Mariotti & Pedemonte, 2019), and often rely on ‘gut feelings’ (Fischbein, 1982). Burton (1998) found that mathematicians move back and forth between syntactic reasoning (e.g., manipulating symbols, making logical deductions) and semantic reasoning (e.g., using examples, graphs, pictures) when completing proofs or doing their research (Weber & Alcock, 2004). Moreover, mathematicians and students use examples in differing ways in order to make sense of the underlying structure of the mathematical phenomena at hand (Sandefur et al., 2013; Weber et al., 2014). These results suggest that there is more to learning about proof and proving than manipulating syntax and learning the necessary techniques or modes of argumentation; there is also a semantic (Weber & Alcock, 2004) component which is not as reliant on formal systems and logical deductions. This semantic component is related to what Stylianides (2007) refers to as modes of argument representation which are the various forms of expression that students use while proving, such as diagrams or pictures, and linguistic choices.

In an ITP course, students are learning the linguistic nuances of mathematical proof (Lew & Mejia-Ramos, 2019), as well as learning various proving techniques which are foundational to engaging in upper-level mathematics, together with other aspects of proving. Indeed, Lew and Mejia-Ramos (2019) found that students, even after they have completed an ITP course, do not fully understand the “nuances involved in the careful ways in which mathematicians introduce new mathematics objects in proofs” (p. 130). This is consistent with students expressing that they do not know how to start or begin a proof (Weber, 2001). Moreover, students may be unaware that proofs are written with academic language, and that mathematicians are also following general rules of academic language when writing their own proofs (Lew & Mejia-Ramos, 2019). However, research suggests that there are some commonalities in the ways that mathematicians and novices write proofs (Lynch & Lockwood, 2019). For instance, mathematicians and students use examples to understand conjectures and notice patterns, but novices view examples as more substantial and relevant to a proof (Lynch & Lockwood, 2019). ITP courses are designed to help students overcome these aforementioned difficulties and align themselves with the proof-writing standards of the community of mathematicians. However, students continue to struggle with proof writing in post-ITP courses (Lew & Mejia-Ramos, 2019; Zazkis et al., 2016).

As mentioned previously, undergraduate students also struggle with making use of the logical structure of conditionals and the intricacies involved in negating logical arguments (Hub & Dawkins, 2018; Inglis & Simpson, 2008; 2009). Logic and logical arguments are a key component of the Standard ITP course, namely phase I (David & Zazkis, 2020); and there is evidence that this logical understanding develops over time throughout one's mathematical training, particularly at the university level (Inglis & Simpson, 2008; 2009). Finally, phase II of the Standard ITP course involves students learning the various techniques of proof (e.g., direct proof, proof by contradiction) (David & Zazkis, 2020). Students have been known to have unique struggles relating to particular methods of proof. For instance, students are often unconvinced of the validity of indirect proofs (Antonini & Mariotti, 2008; Brown, 2018), or may fail to recognize the significance of the base case in proof by mathematical induction (Stylianides et al., 2007). Thus, in a Standard ITP course, particularly in the first two phases, students are learning the essential mechanics of mathematical proof. Students have difficulties within each phase, but less is known on potentially how these students overcome these misunderstandings and/or how their understanding in the first phase translates to their success in the second phase of the course. Thus, in such a foundational course, it is worthwhile to examine not only how students develop the nuanced language of proof, but also how they make sense of logic and logical statements, and whether or not this has any relationship with their adoption of, or struggle with particular proof techniques.

The structure of the remainder of this dissertation is as follows: The next chapter (Chapter 2), is a research study which sought to understand the connections that undergraduate students make between logic and direct and indirect modes of proof. The next chapter (Chapter 3) is a research study on how students learn to write proofs by mathematical induction. This study was separate from the first study for two main reasons. First, historically studies in mathematics education on induction have been solely focused on this proof method, as opposed to comparing it to other techniques of proof. Second, and most importantly for this study, students were introduced to mathematical induction separately from the other techniques of proof in the classroom context under study. The next chapter (Chapter 4) is a manuscript intended for practitioners of proof-based mathematics courses, and particularly ITP courses, where I describe how students develop their understanding of the conditional statement. Finally, in my conclusion (Chapter 5) I describe the overarching lessons I have learned throughout these two research manuscripts and reflecting on the practitioner-piece.

CHAPTER 2 LOGIC, DIRECT, AND INDIRECT PROOF: STUDENTS' LEARNING TO WRITE PROOFS IN AN INTRODUCTION TO PROOFS COURSE

Introduction

Students across the K-16 curriculum have shown difficulty in writing proofs in their mathematics courses (Moore, 1994; Harel & Sowder, 2007; Stylianides et al., 2017). There are many potential reasons for students to have this trouble. Students express that they fail to see the need for a proof, as certain results appear obvious (Dimmel & Herbst, 2018). When they feel an intellectual need for a proof, students have also expressed that they don't know how to begin or start a proof (Weber, 2001). Students show difficulty understanding the mechanics of proof writing in their upper-division or proof-based mathematics courses, such as Analysis and Number Theory (Zazkis et al., 2016).

To address these issues many universities across the country have developed Introduction to Proof (ITP) courses to give students a foundation in the mechanics of proof writing. These ITP courses typically (David & Zazkis, 2020) cover three main content areas: logic, proof techniques, and sets and functions. There are many reasons for students who are transitioning from their computation-based mathematics courses (e.g., Calculus) to proof-based courses (e.g., Abstract Algebra) to learn the underpinning of logic in tandem with or as a precursor to the techniques of proof. Having a robust understanding of logic, in particular understanding the logical connectives (i.e., conjunction, disjunction, implication, biconditional, and negation) is essential in learning and utilizing the various techniques of proof. In particular, each technique of proof (e.g., direct proof, proof by contradiction) relies on the implication statement ($P \rightarrow Q$) in a unique way relative to the other techniques. See Table 2.1 for the ways in which the techniques of proof utilize the conditional.

Table 2.1

Techniques of proof and their use of the implication statement.

Proof Technique	Conditional Use
Direct Proof	$P \rightarrow Q$
Proof by Contraposition	$\sim Q \rightarrow \sim P$
Proof by Contradiction	$(P \text{ and } \sim Q) \rightarrow F$
Mathematical Induction	$[P(1)] \text{ and } [P(n) \rightarrow P(n+1)]$

Often in mathematics conjectures are stated in the form of $P \rightarrow Q$, which reads P implies Q. This is more accurately stated as, if P(x) then Q(x), or even more precisely as, for all x such that P(x) is true, then Q(x) is true. This reliance on the conditional has the potential to compound students' difficulties in proof writing. As scholars in cognitive science have noted, human beings generally have difficulty in understanding and interpreting arguments involving the conditional statement. For instance, Wason (1966; 1967) found when engaging participants in the 4-card problem (or Wason Selection Task) that nearly 96% of the participants responded inaccurately as to which cards needed to be turned over to guarantee the truth of the given conditional claim. Further, studies have shown that mathematics students analyze conditional arguments more accurately than non-mathematics majoring university students (Inglis & Simpson, 2008; 2009). University students have also demonstrated that studying logic can improve students' conditional reasoning skills (Attridge et al., 2016).

Students have also demonstrated some difficulty in using the various modes of proof which rely on manipulating the conditional, such as proof by contradiction or proof by contrapositive. When proving directly, students have expressed that they do not know how to begin or start a proof (Weber, 2001). Findings by Wu Yu et al., (2003) suggest that students can successfully utilize proof by contradiction only after they have learned the syntactic handle of negating the components of their claim and understand the method's connection to proof by contraposition. Though there is a lack of research in how students understand and utilize proof by contraposition (Stylianides et al., 2017), results from Yopp (2020) suggest that students can make the connection between the truth of the contrapositive's claim ($\sim Q \rightarrow \sim P$) and the truth of the original claim ($P \rightarrow Q$). As students in ITP courses are expected to learn the rules of logic and utilize this knowledge while they learn the nuances of proof writing, it is worthwhile to investigate the connections that they make between their understandings of logic, their understandings of the methods of proof, and what it looks like to 'learn' how to write proofs in mathematics.

This study, then, is guided by the following research question:

1) What are the connections that students make between logic and the techniques of proof in an ITP course?

Review of Literature

In the following section I review various literature related to students learning logic and modes of mathematical proof. However, I have not included a review of literature on students' understanding of mathematical induction, as the scope of this study did not include students' learning of induction. Historically, studies on students' learning of induction have been solely focused on their adoption of this proof technique, as it is often taught separately from the other techniques of proof, and induction is a unique proof method with its reliance on the Natural numbers. Thus, I present literature in the following three areas: on students' learning of logic and direct proof, students' proving with contradiction, and students' proving with contraposition.

Direct Proof and Logic

Research on students' understanding of logic in undergraduate mathematics is limited, particularly as it pertains to the connections between logic and the techniques of proof. Epp (2003) had some suggestions on why students need instruction in logic to promote students' learning of proof, such as how one arbitrary element can represent an entire set, or how definitions utilize biconditional statements and have an "if" and an "only if" component. With these suggestions in mind, it is again relevant to mention the conditional statement's importance in proving, in utilizing definitions, in understanding argument forms, and in utilizing the various techniques of proof. Weber (2001) found that students often struggle in how to begin a proof. Students' issues with beginning a proof likely are related to their lack of knowledge of the nuanced language that mathematicians use when proving (Burton & Morgan, 2000), such as the use of imperatives (e.g., Let, Suppose, Assume).

Research suggests that mathematics majors perform well in recognizing valid and invalid conditional arguments (Inglis & Simpson, 2008; 2009). Similarly, training students in logic significantly improves their performance on logical reasoning tasks (Attridge et al., 2016). However, mathematics education researchers are largely unaware of whether improving logical reasoning translates to successful proof writing, or whether students can use logic accurately and meaningfully as they prove.

Proof by Contradiction

When proving a mathematical conjecture of the form *if P then Q* (a conditional statement), it is often useful or more advantageous to prove the result by assuming the truth of P and $\sim Q$ in order to find a logical deduction which contradicts a previously derived logical assumption (i.e., proof by contradiction). When undergraduates utilize this proof technique, or read proofs which involve contradiction, they are often unconvinced of the result (Antonini & Mariotti, 2008; Brown, 2018). Similarly, though preservice mathematics teachers correctly assessed various proofs by contradiction, these preservice teachers misunderstood the assumptions necessary for proof by contradiction (Demiray & Bostan, 2017). Antonini (2003) noted that students can use indirect methods of proof (such as contradiction) spontaneously, as they often do so in everyday circumstances (e.g., If it rained, then the ground is wet. The ground is not wet, therefore it did not rain). Antonini (2003) suggests that for students to learn about and successfully utilize proof by contradiction, students must first productively use examples to convince themselves of the results in which they are trying to prove.

Reading indirect proofs, such as proof by contradiction, is also not an easy task for students (Antonini & Marioti, 2008; Bleiler et al., 2014; Brown, 2018). Bleiler and colleagues (2014) found that preservice secondary mathematics teachers (PSMTs) inaccurately judged proofs by contradiction. These were PSMTs assessing various arguments, as a part of a course designed to improve their ability to read, comprehend, and critique mathematics (Bleiler et al., 2014). In the study the PSMTs often did not recognize proof by contradiction versus a direct proof of the same conjecture, and they had a superficial understanding of proof by contradiction (Bleiler et al., 2014). This lack of understanding was related to these PSMTs' incorrect syntactic reasoning, where they correctly assumed the negation, but failed to correctly operate on their initial assumptions. These PSMTs were able to successfully categorize valid arguments and could also recognize the limitations of inductive reasoning. In another study (Wu Yu et al., 2003) it was found that high school and university students' understanding of the parts of proof by contradiction (such as negating a premise) and understanding the law of contraposition present challenges for students. These authors found that it is not simply negating the premise which presents students with difficulty, but that negating statements with varying quantifiers presents problems (Wu Yu et al., 2003). That is, negating a statement with quantifiers “some”, negating “all”, and negating “only one” are qualitatively different from one another. Lin and colleagues note that negating a statement is a separate skill which can develop without understanding proof by contradiction.

Proof by Contrapositive

Proving a mathematical conjecture by contraposition (or contrapositive) involves the prover taking a conditional statement (*If P then Q*) and negating both *P* and *Q* to prove the statement *if ~Q then ~P*. Though related to proving by contradiction, there is a marked distinction within this proving technique from proof by contradiction. Namely, contradiction requires a prover to find a logical deduction which is in contradiction to a previous logical assumption from an *and* statement (i.e., *P* and ~*Q*). Instead, proving by contraposition requires the prover to maintain the structure of the conditional, while negating and switching the subject and object of the conditional. Hence, proving by contrapositive may have differing cognitive components than other methods of proof.

Proving by contraposition (or by contradiction) can be difficult for students as these proofs are often concerned with the *not-conclusion* of the desired result (Yopp, 2017). One benefit of indirect proofs, particularly proof by contrapositive, is that they can give the prover some conceptual insight into the presence of examples and counterexamples, or what characteristics a counterexample would need to have in order to prove a claim wrong (Yopp, 2017).

Brown (2013) found that upper-level mathematics students productively utilized indirect proof methods when they were able to explicitly connect the mathematical statements to various logical forms which they knew to be equivalent to their original conjecture in the form of a conditional (e.g., $P \rightarrow Q$; $\sim P \vee Q$; $\sim Q \rightarrow \sim P$). Brown (2013) suggests that explicit instruction in ITP courses should highlight the logical structure within proof, such as the negating of statements as well as the logical rules to make deductions (such a Modus Ponens). Arzarello and Sabena (2011) claim that students should be given insight into the teleological function of indirect proof. That is, students should understand the purposes of indirect proof. When students write proofs, they might convince themselves of the truth given their original conjecture (*if P then Q*), but *prove* the conjecture using an indirect method (Arzarello & Sabena, 2011). To this end Harel and Sowder (1998) reported that students dislike indirect proof methods (e.g., proof by contradiction, proof by contraposition) because they do not help explain the truth of the conjecture.

Theoretical Perspective

In the recent Compendium chapter and review of the literature on proof and proving, Stylianides and colleagues (2017) assert that research has "tended to focus on what older students cannot do" which leads to the faulty conclusion that "mathematics majors cannot write proofs" (p. 243). In this research study I attempt to counter this narrative and describe how students learn to write proofs in ITP courses, particularly highlighting the connections that they make between logic and the techniques of proof, while highlighting their skills as advanced mathematics learners and growing proof writers.

Recall that this study seeks to describe the connections that students make between logic and the techniques of proof as they learn in an ITP course. To understand and analyze students' learning, I utilized a modified version of the theoretical frame used by Sandefur and colleagues (2013), which they used in describing how students' example use impacts their proof production.

Sandefur et al., utilized a theoretical frame consisting of several components and subcomponents. For the first layer, they adopt a heuristic from Mason (1980) about how one makes sense of a mathematical problem, describing this process as: *Manipulating - Getting a Sense Of - Articulating* (MGA). Each component of MGA has unique subcomponents. According to Sandefur et al. (2013) *Manipulating* mathematical objects can be done either by using syntax (e.g., logical symbols, algebraic procedures) or semantics (e.g., diagrams, examples) (Alcock & Weber, 2004). Sandefur and colleagues describe *Getting a Sense Of* a mathematical conjecture or problem with two subcomponents: Technical Handle (TH) and Conceptual Insight (CI). Conceptual Insight (CI) refers to when one attempts to understand the "structural relationship pertinent to the phenomenon of interest that indicates why the statement is likely to be true" (Sandefur et al., 2013, p. 328), and Technical Handle refers to "ways of manipulating or making use of the structural relations that support the conversion of CI into acceptable proofs" (Sandefur et al., 2013, p. 328). Finally, *Articulating* refers to physical, verbal, or written actions of a student (Sandefur et al., 2013). Sandefur and colleagues (2013) describe Articulating as a manifestation of TH and CI.

Sandefur and colleagues utilized this theoretical perspective to describe how undergraduate students' example use potentially impacts their writing of proofs by mathematical induction. Thus I chose to use Sandefur et al.'s (2013) frame, as their study was how undergraduate students produce proofs in mathematics. In order to answer my research question, I adjust Sandefur et al.'s original framework slightly to attend more closely to the particular phenomenon under study within my research; that is, the connections students make between mathematical logic and the various techniques of proof. When students are faced with a claim to prove, and they decide to prove indirectly, they typically will need to consider how to reformat the claim so that it fits within their indirect proof structure. Therefore, in this study, Manipulation is redefined as *Discussion Surrounding the Claim*. This added specificity helps to clarify what students are manipulating (i.e., the conjecture/claim) and allows me to focus on how students' manipulation of the conjecture/claim is potentially connected to logic.

For instance, when proving a statement by contrapositive, one needs to take the original implication ($P \rightarrow Q$) and negate both the claims and restate the claim as $\sim Q \rightarrow \sim P$. Symbolically this appears trivial, but when dealing with mathematical conjectures this is often not the case. It is during this phase that students need to rely on their understanding of logic and logical connectives (i.e., and ($\wedge$), or ($\vee$), implication ($\rightarrow$), biconditional ($\leftrightarrow$), negation ($\sim$)). They also need to use and understand logical quantifiers (e.g., for all ($\forall$), there exists ($\exists$)) amongst other mathematical symbols and language that are potentially foreign or new to them. These sorts of discussions would be categorized as syntactic discussion around the conjecture or claim. Students could also have semantic discussions about the conjecture. As I have found previously (i.e., Reed et al., accepted) when students discuss proofs by mathematical induction, they often need to manipulate or restate the conjecture in terms of semantics. For instance, in this study one student analyzed a sample proof of the claim $1 + 3 + \ldots + (2N - 1) = N^2 + 3$ (a false claim), and asked his small-group members the following as he attempted to make sense of the given claim:

> Or, I guess. Well, I guess 2N-1. Well...I'm not even following what it's trying to do because, like this 2N -1 is just kind of out there. Like, I don't... Are we plugging in a number from before? Because if you plug 3 in you get 5. If you plug 5 you get 9.

In this excerpt, this student is attempting to make sense of the conjecture, specifically the left hand side, which is the sum of the first N odd integers. Largely, his sensemaking is with semantic reasoning, that is ‘plugging’ in numbers and testing various examples. Thus, we see that *Discussion Surrounding the Claim* can be described using the same subcomponents as *Manipulation*, namely, semantic and syntactic reasoning. More often than not, it is the syntactic manipulation which helps me to describe the connections that students make between logic and proof;when students symbolize an argument their discussion naturally shifts to discussing the symbols and how to utilize them productively.

Getting a Sense Of also is in need of a modification to better answer my research question. I redefine Getting a Sense Of as *Discussion Surrounding the Argument, or Argument Form.* Sandefur and colleagues (2013) describe this (Getting a Sense Of) as having two components, Conceptual Insight (CI) and Technical Handle (TH). I hasten to note that terms remain useful in my redefined component of the framework. As a case-in-point, when students are proving in an ITP course their discussion is often centered around finding the key idea (Raman, 2003) or conceptual insight, and developing the technical capabilities to describe this insight. For instance, when proving the conjecture, “an integer is even if and only if its square is even,” one demonstrates their *technical handle* in recognizing that this conjecture is in the form of a biconditional and stating its conditional components. The key idea or *conceptual insight* that students need in order to prove this result is to understand that when multiplying even integers a factor of 2 can always be removed. So, technical handle and conceptual insight allowed me to describe the connections that students make between logic and proof by allowing me to hone in on the places where they have productively made use of key ideas and demonstrate the technical language necessary to write them in a proof.

Finally, *Articulation* is slightly modified from Sandefur and colleagues’ (2013) description. Although I previously mentioned that this component of the frame is not a direct answer to my research question, this component still provides some experimental value, as Sandefur et al., have described this component as the manifestation of TH and CI. Thus, I adopt this term as well for these theoretical components, but modify them slightly to adapt for online learning due to Covid-19. I use *Articulation* as the final written (typed), shared, product of the students. There are exceptions to when Articulation is coded outside of this description, such as when students 'share' their screen and use the communal Zoom whiteboard, or perform a similar action. This adopted framework is summarized in Table 2.2 below.

Table 2.2

Adapted Theoretical Framework from Sandefur et al. (2013).

Discussion Surrounding the Claim		**Discussion Surrounding the Argument**		Articulation
Semantic Reasoning	Syntactic Reasoning	Conceptual Insight	Technical Handle	
example use, diagrams, pictures, graphs (Weber & Alcock, 2004)	logical symbols, algebraic reasoning and manipulation, logical reasoning (Weber & Alcock, 2004)	refers to when one attempts to understand the "structural relationship pertinent to the phenomenon of interest that indicates why the statement is likely to be true" (Sandefur et al., 2013, p. 328	"ways of manipulating or making use of the structural relations that support the conversion of CI into acceptable proofs" (Sandefur et al., 2013, p. 328).	The final (typed) product of the students' group proof
Formerly Manipulating from Sandefur et al. (2013)		Formerly Getting-a-Sense-Of from Sandefur et al. (2013)		

Methodology

Participants & Setting

This study took place at a large public institution in the southeastern United States during the Fall of 2020 within an Introduction to Proofs course. The course occurred during the first full semester of online-learning due to the Covid-19 pandemic. Students and the instructor met together synchronously twice a week on the Zoom platform, with a mixture of students doing group work in Zoom Breakout rooms and having a whole class lecture and discussion. Overall, 13 students elected to participate in the study. The participants of the study were 3 women and 10 men, predominantly white (11/13). All students were either majoring or minoring in mathematics at the institution. This course had a course structure which I outline below. Students in this course would typically engage in a number of group-sensemaking exercises on topics related to the course content and goals. Often class began with a discussion of the 'big ideas' from the day prior. The instructor would then launch whatever task the students would be working on in groups for the day and assign students to various breakout rooms. These tasks ranged from engaging in logical reasoning tasks, discussing definitions, and proving conjectures. Students would then collaborate in their group's Google Doc, and class would often end with a discussion and presentation of various students' group work. For my analysis, I focused only on when students were engaging in group-proving (as opposed to discussing definitions or engaging in logical reasoning tasks) in order to understand how their knowledge of logic potentially translates or manifests as they write proofs.

Data Collection & Procedures.

Upon IRB approval, all course activities and instruction were recorded via Zoom, including 'breakout' room discussions where the groups were made up of 3-4 students working on proof-based tasks. As this class was structured with the intent of inquiry-based instruction, group work was often a component of the course. Due to many students not participating in the study, when there were not times of group proving the instructor often mixed participants and nonparticipants to promote robust discussion among the students. These days were recorded for the purposes of context, but not included in the analysis. In order to answer my research question, I focused on group-proving episodes composed entirely of research participants, which were dispersed throughout a number of days throughout the semester. Each group consisted of no more than 4 individuals.

Group Proving & Artifacts

As mentioned previously, the main data source for this manuscript is Zoom recordings of students proving in groups during class. Group work was a core component of the class, but group performance was not the focus of the study. Rather, by studying the students' interactions within the group, students were able to discuss their ideas in a naturalistic setting and provide the researcher with insights into potential connections between proof techniques and logic. Any claims discussed are not in reference to the group, but rather they describe how individuals in an ITP course may behave in a group under similar conditions. Along with the video and audio of each research group, I also collected students' final written (typed) artifacts from the group proving exercises (i.e., their proofs). To organize the data, and keep track of group proving episodes, I took daily observation protocols of each group and noted who the members of the group(s) were, what problem or task they were working on that given day, if proving - what techniques they used, and any key moments during the episode.

Data Analysis

To answer my research question (i.e., What are the connections that students make between logic and the techniques of proof in an ITP course?) I identified nine group proving episodes for an analysis of students' discussion during those episodes together with the group's final written proof. Originally, I planned to identify three episodes each of groups using the various techniques of proof (i.e., direct, contrapositive, and contradiction) however, throughout the semester students only utilized proof by contraposition on two occasions. Thus, I supplemented one episode where they proved by contradiction. Thus in total, there were three episodes of direct proof, two episodes of proof by contrapositive, and four episodes of proof by contradiction chosen for analysis. Once the episodes were selected, they were watched and transcribed by each talk-turn (i.e., each time the speaker switched). The transcripts were then coded with the modified MGA framework and its subcomponents. To do this coding, first I decided in each talk-turn whether their discussion was focused on the claim or conjecture (component 1 of my Theoretical Frame), or on the argument or argument form (component 2 of my Theoretical Frame). If I decided that they were discussing the conjecture, I then identified various portions of their speech which aligned with the description in my theoretical framework of either semantic or syntactic reasoning. If I decided that they were discussing the argument, I then identified portions of their speech as either them demonstrating their technical handle or expressing conceptual insight. I took these coded portions of their text and copied them into an Excel sheet which was overlain with my theoretical frame (shown in Table 2.3) for further analysis and to identify themes amongst the coded portions. For clarity and trustworthiness I provide a sample codebook below in Table 2.3 which highlights my coding scheme and analysis process.

Table 2.3

Sample codebook using the modified MGA framework

Discussion Surrounding the Claim	Discussion Surrounding the Argument

Semantic Reasoning	Syntactic Reasoning	Conceptual Insight	Technical Handle
I'm going to try and do a little drawing so we can better visualize it.	Is a negation of an implication an implication itself?	[while discussing a diagram] No, it's not. Wherever [the set] B is not, is the white outside of the bigger orange circle, right?	The reason why B complimentary is a subset of A complimentary is because A complimentary includes B... B complementary does not. [describing their symbolic proof]
Formerly Manipulating from Sandefur et al. (2013)		Formerly Getting-a-Sense-Of from Sandefur et al. (2013)	

Once each transcript was transcribed and coded, I met with my research team of fellow proof researchers (a graduate student peer and my academic advisor) where they then checked and verified the coded portions of the text relative to my theoretical frame described above. Each particular meeting lasted an hour, and we would discuss three coded transcripts at a time. I took notes of the emerging themes we discussed so that I could return to the transcripts later and describe in detail the prevalence of the themes. It is from these discussions that the three themes emerged, which I describe later in the results section.

As in many qualitative research studies, the line between theoretical framing and results are often blurred (Sandefur et al., 2013). As such, one coding term which became useful to me in describing students' learning to write proofs was "Stuck." Again, as I mentioned above, I hope to describe students' proof learning without deficit language. To this end, the following quote is perhaps useful. Andrew Wiles, who famously proved Fermat's Last theorem, described in an interview towards the end of his career what it means to do mathematics, "What you have to handle when you start doing mathematics

… is accepting this state of being stuck" (plusmathsorg, 2016, 1:00). As such, I have also added a category to this frame which I refer to as Stuck, to capture instances when students demonstrate a lack of technical or linguistic handle when composing their group proofs. For my research, a student exhibited being stuck whenever they expressed verbally to the group an indication that they were aware that they lacked direction in the proof. I see students' awareness of being Stuck as an asset that students possess while proof writing. Indeed, we do not want students to absentmindedly push symbols, or not follow logical rules while writing proofs. It is a good thing when students recognize how and when they are stuck in the proving process, much like Andrew Wiles (plusmathsorg, 2016) asserts. This term Stuck is also useful in describing that students can, at times, be unaware of how to translate a conjecture ($P \rightarrow Q$) into its contrapositive ($\sim Q \rightarrow \sim P$) or a contradictory ($P$ and $\sim Q$) statement to be proved. Examples of when students demonstrate they are stuck are shown below in the results.

Theoretical Perspective and Emergence of Themes

As I have described above, the theoretical framework which guided my study and data analysis was modified from Sandefur and colleagues (2013) Manipulating-Getting-a-Sense-Of-Articulating (MGA) frame. The key modifications I made to analyze my data were changing the Manipulating stage to [students'] Discussion surrounding the Claim. Again, this component has two sub-components, namely semantic and syntactic reasoning (Weber & Alcock, 2004). To answer my research questions about the connections that students make between logic and the techniques of proof, I was first able to hone in on when students were discussing the claim, and focus on the portions of their speech when they were using syntactic reasoning. It is from these instances when students were using this syntactic reasoning that they were making use of and demonstrating their knowledge of logic. Within these instances I was able to look across the episodes and notice and describe themes with my research team. The second modification I made to Sandefur and colleague's (2013) frame was changing the Getting-a-sense-of phase to [students'] Discussion surrounding the Argument. Again, this component had two subcomponents, conceptual insight and technical handle. When students were making use of their technical handle when writing a proof, I was able to hone in on these instances and describe what connections (if any) students were making to the rules of logic which they had previously learned. By coding and utilizing the modified framework in this way, I was also able to notice and describe when students lacked a technical handle. As I have described above, rather than describe students' learning with deficit language, I saw this as an asset and form of learning. This led to the emergence of the stuck code which I described above and for which I present results in the following section.

Results

As I have previously described, research on undergraduate students' proof writing tends to focus on their limitations rather than the things they can accomplish. Thus, it is my hope that I can share the results while highlighting how students learn to connect their understanding of logic as they learn to write proofs in an ITP course. To do this, I present the results in two sections aligned with my theoretical framework. First, I will share findings from the (M) phase or the *discussion surrounding the conjecture,* then share findings from the (G) phase or the *discussion surrounding the argument*.

Discussion Surrounding the Claim

- *Theme 1: Students understand that before they can prove a result indirectly, they must first accurately manipulate the conjecture using their knowledge of logic.*

As students were engaging in group proving during class, when utilizing indirect modes of proof, students used their understanding of logic to manipulate the claim (in the form of a conditional statement) into its contrapositive or to prove the result by contradiction. The connections they were making between logic and the modes of proof were particularly noticeable during their early proof-writings. Table 2.4 presents several examples of students discussing how to manipulate the conjecture to be proved indirectly in group-proving exercises.

Table 2.4

Students discussing the logical manipulation of the given conjecture

Speaker (Pseudonym)	Student Quote
Dennis	"Personally, I think the first thing that we'll need to do is break it [the biconditional claim] up into individual implication statements."
Greg	"Last off we had just figured out the negation [of the claim]. That was it."
Zoren	"Yeah, if you can write the proof, I will write the implication [in the form of its contrapositive]. OK. So I'm writing the implication and you can go ahead and write that proof."

In these excerpts, I note that the students are all having distinct discussions about manipulating the conjecture, which is separate from their writing of the proof or even making sense of the claim itself (i.e., whether the claim is true or false). In all of these early instances, taking the components of the claim of a conditional statement, if P then

Q, and translating that statement to $\sim Q \rightarrow \sim P$ (its contrapositive) or P and $\sim Q$ (to prove by contradiction) is its own separate exercise, which students have demonstrated requires a separate conversation than writing the proof itself.

A promising result from Dennis (quote 1 in Table 2.4), is that he recognized the underlying logic of mathematical conjectures. In this instance, Dennis identified the biconditional conjecture as having two parts to be proved (i.e., $P \rightarrow Q$ and $Q \rightarrow P$) and that proving this claim required them to "break it up into individual implication statements."

In the second student quote, Greg recognized that moving forward with the proof indirectly required him (and his group) to accurately recognize the components of the implication statement to be proved, and to accurately word and manipulate these using their logical knowledge. In the third Zoren demonstrates that early on in the semester, he believed recognizing the logical form of a conjecture and breaking it down into its symbolic components (P and Q) and switching and negating them was of equal intellectual merit as writing the actual proof. It would have perhaps been more fruitful for the entire group to engage in this exercise of restating the conditional claim in its contrapositive form with Zoren. So, when proving indirectly, students can and do recognize the logical components of mathematical conjectures.

While Zoren's quote above does highlight the theme (Theme 1), this quote came from an illuminating exchange as students were discussing how to prove the conjecture, "If L and M are odd integers, then L*M is an odd integer." In this exchange, students have a detailed discussion about how they may go about proving this conjecture indirectly. This is shown in Table 2.5 below.

Table 2.5

Students' extended discussion of logically manipulating the conjecture

Speaker (Pseudonym)	Student Quote / Exchange
Zoren	I mean, it's right. Everything is good about this proof. And I hear[d] her say … we're going to prove by contrapositive... She said it like in the beginning. But I mean, that's what she wants us to do. Like, not what she wants us [to do]. Maybe it's easier or something because the sentence said, "if L and M are odd, then L times M is odd." So if we say...we tried to prove it by contrapositive, then we can say if L times N is even, then L and N is even, too. So they're both even
John	So. All right. Yeah. That works too. Let's do it that way. That sounds like a good...
Zoren	2K times 2K is like 4K. 4K is 2 times 2K. So 2 times the definition of even numbers is gonna give us even numbers
John	Yeah. OK. And I like that one better. That seems a lot simpler than what happened to it, because [my proof] would require me to prove that an even number multiplied by any number was even. And it also required me to prove that an even number plus an even number is an even number. So let's do it your way. I think mine's a little too complicated. In a sense of, like, you have to prove other stuff inside of my own proof.
...	...
John	So, so, do you want to go ahead and write down the contrapositive Ibraham?
Zoren	Yeah, if you can write the proof, I will write the implication. OK. So I'm writing the implication and you can go ahead and write that proof.
John	Sounds good. I'm going to go ahead and do it and then I might copy and paste it into the document. So we're saying that both L and M are now even in this case for the contrapositive?.
Zoren	Mmmhmm
John	Sounds good….Oh did you see.. if L is odd and M is odd apply DeMorgan's Laws. That what she [said]
Zoren	uhh
John	I mean, what we're doing is not bad.
Zoren	Yeah. We just have L or M.

John	Do you want me to do it again with DeMorgans Laws?
Zoren	Yeah, and I did it with the DeMorgan law. So basically just say L or M instead of L
John	oh okay
Zoren	and M
John	OK, let me do it that way, then. Cause I just did the contrapositive. So if L.
Zoren	Times... times M is even number then L or M is even number
John	Perfect. So L is equal to 2K... M is equal to 2K. So
Zoren	So do we have to move like 3 cases? in this case, since L or M has to be an even number, so maybe we have [to] prove if they're both even. Or one. I mean, two cases or one [is] even.

Most noteworthy about this exchange is the length of the conversation that these students have on how to prove the result using its contrapositive statement. These students understand that in order to prove their original result using contraposition requires a precise manipulation of symbols and language. Encouragingly, John mentions that they need to use DeMorgan's Laws in order to negate the original conjecture (with an 'and' statement) to an 'or' statement. This detailed and lengthy exchange fully demonstrates the first theme, that before students prove indirectly, they recognize that they must first manipulate the conjecture using their knowledge of logic.

Theme 2. When proving indirectly, students offload the work to the symbols (syntax) when manipulating the conjecture.

Another theme which presented itself as students were discussing the conjecture was that they offloaded the work of manipulating the conjecture to the symbols when using indirect modes of proof. Table 2.6 presents an emblematic theme that students, more so in the beginning of the term, utilize their logical knowledge in tandem with rephrasing the proof or conjecture.

Table 2.6

Anthony & Greg's Conversation on Negating an Implication (If p is a rational number and q is an irrational number, then p-q is an irrational number).

Speaker (Pseudonym)	Student Quote
Greg	Is a negation of an implication an implication itself?
Anthony	I don't think it is, is it? Because it's A and not B?
Greg	Yeah. How do you word that?

In this exchange, Greg and Anthony are discussing how to negate the conjecture (If p is a rational number and q is an irrational number, then p-q is an irrational number) to prove it by contradiction. Indeed, to prove a statement by contradiction, providing yourself with the correct assumptions is crucial to having a valid proof. The dialogue show that these students need to look back at the definition of implication (A→B) to confirm its negation has the equivalent form (A and ~B).

In another excerpt students were proving the claim "If L and M are odd integers, then L * M is an odd integer." Their exchange is highlighted in Table 2.7.

Table 2.7

*Zoren & John's use of logical rules to negate a statement (If L and M are odd integers, then L * M is an odd integer).*

Speaker (Pseudonym)	Student Quote / Exchange
John	So we're saying that both L and M are now even in this case for the contrapositive?
Zoren	Mmmhmm

John	Sounds good. Oh did you see… if L is odd and M is odd apply DeMorgan's Laws.
Zoren	I mean, what we're doing is not bad.
John	Yeah. We just have L or M.
Zoren	Do you want me to do it again with DeMorgans Laws?

Again in this student vignette, these students offload the work of restating the conjecture by first breaking the statement down into symbols and using their understanding of the contrapositive and ability to apply DeMorgan's Law when negating an "and" statement. This, again, is encouraging that students indeed make these connections between logic and proof. One thing which this group of students struggled with, though, was moving back from their symbolized argument form into English. John wondered whether "Not Odd" could be expressed as even. The students eventually came to the conclusion that they were equivalent, but it is perhaps worthwhile in instruction to have pointed conversations about translating conjectures from symbols back into mathematical language.

An extended example of Theme 2 is further exemplified below in Table 2.8. In this excerpt the students are proving the claim, "If $A \subseteq B$, then $A^C \subseteq B^C$."

Table 2.8

Students' extended discussion on offloading the work to syntax

Speaker (Pseudonym)	Student Quote / Exchange
Greg	So I'm pretty sure you're just saying [that] statement and then just being, like, because it's the contrapositive... Is that. No, it can't be it. Cause that's just way too simple.

Anthony	It's way too easy
Anthony	If we can prove that ~A is B complimentary, then yeah. I don't see a problem with it. They're... sorry, not B is B complimentary. If we can prove that I'm pretty sure we're golden.
Taylor	No we had the if...the second part was flipped, like if it said, then the complement of A is a subset of complement of B, that would be false though right?
Greg	Yeah
Taylor	Okay
Anthony	Oh, wait. Can we do a contrapositive with...? Is that a logical statement?
Greg	Yeah. Both of these are logical statements. And If we assume the first logical statement to be true. Cause that's like how you prove an if-then. It's like two “if-thens” inside of a bigger “if-then” thing. So if you assume the first logical statement to be true, then its contrapositive would also be true. And that second logical statement is the contrapositive. Right?
Taylor	Oh wait
Anthony	Oh, well, that sounds right.
Taylor	Do we need to swap A and B on the graph or did you already say that?
Anthony	No, I did that correctly. You're good.
Taylor	Wait A is ..
Anthony	A is a subset of B
Taylor	Oh ok I got it. I don't know why I keep thinking it's the other way around
Anthony	I know right, It's confusing. But anyway, I think Will is pretty much got it done. We just need to find a way to write it out and prove that A. Or not A is a complement of not B.
Taylor	Ok. Then how would we show the graph from the complement of B and the complement of A. Would it be the same as that? But with A and B swapped?
Greg	I don't think you need a graph
Anthony	Uh no I just think you'd have to describe that A-not is every A is not. And just say that B...
Taylor	Well then wouldn't that be false then.

Again, these students spend a considerable amount of time discussing how to symbolically relate the conjecture and its logical components. Although they switch between semantic and syntactic reasoning (i.e., discussing logical manipulations versus discussing their drawn pictures), these students recognize the underlying logical structure of the proof and claim. The group also seems to understand that this logical structure of the claim lends itself to be proved indirectly. Indeed, when Greg realizes and expresses that this claim and its contrapositive are related by their nested "if-thens" is encouraging, as recognizing the logical structure is an important aspect of proving.

Discussion Surrounding the Proof or Argument

- *Theme 3. Students recognize when they are limited or are in this state of* ***Stuck***

The main theme that I identified as students were discussing how to write up their proofs was that they recognized when they were stuck. Again, as I have described above, I wish to highlight how this is an asset that we want students to possess. Indeed, they do realize when they lack a technical handle. In one exchange, a conversation continued from Table 2.6 about negating the conjecture to be proved by contradiction. Anthony and Greg have the exchange shown in Table 2.9.

Table 2.9

Anthony & Greg discuss how to word the negation

Speaker (Pseudonym)	Student Quote / Exchange
Anthony	So, the negation of "if A then B" is equivalent to "A and not B."
Greg	Yeah, but how do you say that like, English words with this, in this specific scenario?

In this later discussion, although the students have successfully negated the implication, Greg recognizes that he does not have the language to reword the conjecture in its new logical form. So, although students can successfully utilize their logical knowledge, they may need some guidance into how to re-word these types of statements. As I mentioned above, a different group of students also had a brief conversation as to whether "Not Odd" was the same as Even. These are potentially related issues with students misunderstanding the connection between the conditional and indirect proof methods proving their original conjecture.

In another episode, students are proving the conjecture "An integer is even if and only if its square is even." Promisingly, the students recognize this statement as a biconditional that must be separated into its "if" and "only if" parts, demonstrating their technical handle. As they attempt to prove one direction, the students have the exchange shown in Table 2.8 below.

Table 2.10

Eliza & Dennis recognize their algebraic limitations

Speaker (Pseudonym)	Student Quote
Eliza	I have the first part and not the, um, prove that "if the square of integer is even then integer is even."
Dennis	I think the first thing that we'll need to do is break it up into, like individual implication statements.
Eliza	[The issue] is that you get to the point where you have... If the square of some integer has to... it's even right? So we let it equal 2K, where K is an integer. Right? You end up with this issue where you have the square root of two times the square root of K.

Again, it is important that these students were successfully able to recognize and dissect the biconditional statement into its two components. Eliza recognizes that when proving one component directly, they run into an issue with the square root of k. Here it is also crucial that she recognize this, and not continue on with her proof. She (and her group) must recognize that this algebraic manipulation gets them stuck and they must prove this claim indirectly.

Finally, when proving a result directly about Sets and Functions, "If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$", students expressed their conceptual insight into the conjecture but recognized that this insight did not suffice as proof. This is highlighted below in Table 2.11.

Table 2.11

Eliza recognizes she is stuck with wording her conceptual insight

Speaker (Pseudonym)	Student Quote
Eliza	So, like, logically like if you... because you can represent this visually, it's very easy to understand that if A is a subset of B and B is a subset of C, then there is no other. There's no other conclusion to draw than A is a subset of C. You can draw it out visually. It's very easy to understand. It's just that I think that is not proof.

Here Eliza expresses her insight and the key idea behind this proof through a visual aid. She describes the conclusion as the only one you can draw given this semantic argument. Importantly, she recognizes that though this result is visually "obvious" her visual aids does not suffice as proof, moreover, she recognizes that she does not quite possess the language to translate this picture into a proof.

So, again, students are adept in recognizing their lack of technical handle, or when they are stuck when proving. It is also important to recognize that students (and mathematicians) get stuck on different aspects of proof. As I describe above, students recognize they were stuck in several contexts, such as translating statements into a new logical structure (i.e., to prove indirectly), and also get stuck while looking for a key idea within a proof.

Discussion & Limitations

Recall that this study sought to describe the connections that students make between logic and the techniques of proof, and while doing so I attempted to describe the assets that students possess or the learning that takes place in ITP courses. I found three main themes across the group-proving episodes which are promising results for demonstrating student learning. Before I describe those themes and their benefits, another addition to the field that this paper makes is the modification of Sandefur and colleagues' (2013) MGA Framework. It was fruitful to have this framework in mind as I first analyzed my data, but the frame was too loose in its categories for my purpose. However, it was viewing my data through this lens which allowed me to see that students were indeed having distinct conversations about the conjecture, and then about writing the proof. Thus, the framework was modified from the data from *Manipulation* to *Discussion around the Claim*, and *Getting a Sense Of* to the *Discussion surrounding the Proof*. Analyzing students' discussions in this way allowed me to identify the three themes within these categories.

The first two themes were from the (M) or Discussion Surrounding the Claim phase. These themes are important enough to be offset as bullet items for greater clarity and emphasis.

- *Theme 1: Students understand that before they can prove a result indirectly, they must first accurately manipulate the conjecture using their knowledge of logic.*
- *Theme 2. When proving indirectly, students offload the work to the symbols (syntax) when manipulating the conjecture.*
- *Theme 3. Students recognize when they are limited or are in this state of **Stuck***

These results are promising for many reasons. First, learning to write proofs is a complex process. So, the fact that students utilized their syntactic knowledge to ease the burden of proving is not surprising, and should be encouraged. This result is consistent with other literature and recommendations for students learning to write (indirect) proofs (Brown, 2018). Similarly, it is critical that students recognize the underlying logical structure of mathematical conjectures, so that they can then apply their knowledge of logic to translate the conjecture and write the proof.

Indeed, these students had spent the previous several class periods learning about the truth-tables of the various logical connectives and specifically learning about the conditional through various tasks (e.g, Wason, 1968). Negating a compound sentence is often not trivial. These findings are important as the intent of this course and other ITP courses is to allow students to make organic connections between their knowledge of logic (learned early on in the semester) and their knowledge of the techniques of proof (learned later in the semester), and indeed they did make these connections. So, the notion that they can recognize the underlying logical structure of mathematics is promising; this course seemed to alleviate some predictable difficulties to novices in proof writing.

The third theme I identified was that students were aware when they were stuck in writing a proof. There are two ways that students expressed being stuck on their proof or lacking a technical handle to progress in their proof. . First, students expressed that at times they lacked the mathematical language to describe their conceptual insight into a conjecture. That is, they had a picture, or a key idea behind the proof but could not articulate this insight in a mathematically rigorous way (i.e., a proof). Second, they recognized that they were stuck in translating their original conjecture, often in the form of a conditional, to be proved indirectly. Although they could often manipulate the logical structure correctly, they recognized they were stuck in translating this into an English sentence or statement. Again, I highlight as an asset that students do not blindly push symbols or write illogical conclusions. Recognizing when you are limited in mathematics is essential because you must be able to know when you have utilized all of your available resources, which is particularly relevant to proof writing. Instructors of proof-based mathematics courses should encourage students to begin to feel comfortable in this state of being stuck, as Andrew Wiles mentioned and is quoted above.

This study makes two significant contributions to the field. First, in these data I have shown that students can and do make connections between logic and the techniques of proof in an ITP course. This is significant as this is (at least partially) the intent of the course. These data suggest that students understand that proving requires some knowledge of and utilization of their understanding of logic. Second, students recognize when they are limited (or stuck) when writing a proof. Many studies on students' learning of proof describe what they cannot do. In this study I was able to show that students recognize when they need more sophisticated language or theoretical tools to prove their result. Recognizing this limitation is indeed a form of learning, especially as these students are beginning to learn the mechanics of writing mathematical proofs.

This study was not without limitations. First, this study took place during the Fall of 2020, the first full-time semester for many universities completely online due to Covid-19. This came with a number of challenges which are worth mentioning. First, research on sociomathematical norms (Yackel & Cobb, 1996) are all relevant when dealing with in-person learning. To date, there have been few studies on how online learning affects mathematics group work, particularly in advanced-mathematics courses. This challenge brought about a unique feature to this course, namely that students were working collaboratively in a Google Doc as opposed to working on a whiteboard or chart paper. This allowed students to modify their group-proofs in real time, line by line, and perhaps use language which they would not normally write down on paper. A delimitation of this study was that recording both the students' discussion and their modifications to their proof in real-time presented too great a challenge. As such, I was only able to collect their final written product, and unable to see what changes they may have made throughout their conversation. Finally, another delimitation of this study was I chose not to interview any students and ask what they meant regarding particular exchanges. As such, I had to infer meaning from conversations and their final typed product which is perhaps not aligned with the speaker’s intended meaning.

Conclusions & Future Directions

This study sought to find the connections that students make between logic and proof. Indeed, students in this ITP course did make rich connections between the logic content portion and proof writing portion of these courses. As I have described above, students were aware of how to use logic to their advantage when writing proofs and making sense of arguments. Future research would benefit from exploring how students can learn to manipulate these symbols properly and then translate this symbolic statement back into an English mathematical statement. Students should also be encouraged to explicitly use their syntactic or symbolic knowledge to help them in writing proofs.

Another finding was that students recognized their limitations in proof writing. Further studies should investigate what sorts of things students rely on or fall back to when they are stuck in proof writing. As a practitioner of proof-based mathematics courses, I would like my students to productively rely on their rich background in mathematics and mathematical knowledge. As an experienced prover and doer of mathematics, I know that getting unstuck in a proof can be due to finding a key idea through examples or pictures (e.g., semantic reasoning) as well as making logical deductions and manipulating symbols (e.g., syntactic reasoning). Future studies should investigate what types of reasoning students use when they encounter difficulties when proving, and whether they utilize semantic or syntactic reasoning. Similarly, students seem to get stuck on different types of actions or mathematical objects when proving. Future studies should parse out the different ways that students recognize when they cannot progress in their proofs.

CHAPTER 3 STUDENTS' STRUGGLES IN WRITING PROOFS BY MATHEMATICAL INDUCTION

Introduction

Students studying university mathematics have demonstrated certain difficulties when transitioning from their computation-based mathematics courses (e.g., Calculus) to their upper-level, proof-based, mathematics courses (Moore, 1994; Harel & Sowder, 2007; Stylianides et al., 2017). In particular, students have shown that there are unique difficulties to adopting and utilizing various techniques of proof (e.g., proof by contradiction, proof by contrapositive) in their upper-level mathematics courses (Zazkis & Mills, 2017). Amongst these techniques of proof which are troublesome to students is proof by mathematical induction (Palla et al,. 2012; Norton & Arnold, 2019; Stylianides et al., 2007; 2016).

Mathematical Induction (MI) plays an important role in proving results in discrete mathematics and number theory, and is one proof method which students often learn for the first time in Introduction to Proof courses. Hammock (2013) in his textbook *Book of Proof*, describes MI in the following way (p.154):

Proposition The statements $S_1, S_2, S_3, \ldots$ are all true.

Proof. (Induction)

(1) Prove that the first statement S_1 is true.

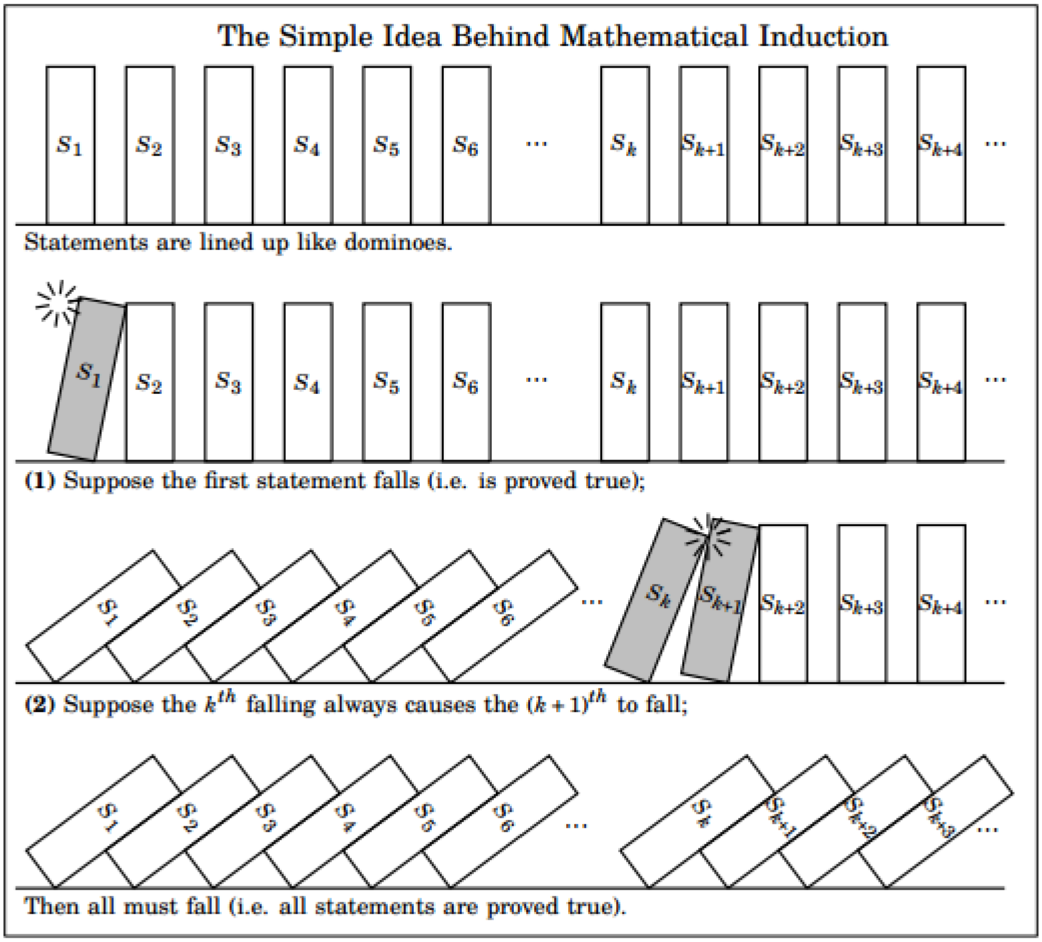
(2) Given any integer $k \geq 1$, prove that the statement $S_k \rightarrow S_{k+1}$ is true.

It follows by mathematical induction that every S_n is true.

Often along with this definition, students are shown the following diagram (Figure 3.1) with the accompanying domino analogy.

Figure 3.1

Hammack's Domino Figure and Analogy



Some of the confusion students face around induction is that the *inductive method* is a heuristic method which produces a generalizable claim from a finite set of examples, whereas *mathematical induction* is a rigorous form of deductive proof (Ernest, 1984, p. 181). There are many other potential reasons for students' confusion on the method of mathematical induction. Indeed, the Mathematical Association of America (Newman, 1958) noted that the form of mathematical induction is difficult to understand. To this end, researchers have shown that undergraduate students demonstrate a lack of understanding about the logical form of mathematical induction and see assuming the inductive hypothesis as a form of circular reasoning (Palla et al., 2012).

I first started thinking about the teaching and learning of mathematical induction as a doctoral student several years ago while engaging in a teaching internship with an experienced instructor of the university's ITP course. During this time, we co-designed and co-implemented a task based on Stylianides et al. (2007) study where students were asked to evaluate two sample induction arguments (one valid, one invalid) and describe whether or not these arguments proved the claims. This activity (in Appendix A) was intended to provide students with an opportunity to make sense of the definition of mathematical induction and connect this definition to the format of induction prior to writing induction proofs themselves. In this way, students could unpack the definition of induction prior to engaging in proofs using mathematical induction. Other scholars (e.g., Norton & Arnold, 2019; Harel, 2001) have recommended other tasks to introduce induction to students. The students under study were introduced to induction using a more standard approach. So, this study seeks the answer to the following research question:

> What are the struggles that students in an Introduction to Proofs course face as they learn how to prove results using mathematical induction?

Literature Review

The main sources of difficulty for students learning mathematical induction are the technical notation and the processes of the inductive step $P(n) \rightarrow P(n+1)$ (Carotenuto et al., 2018; Palla et al., 2012). In a large quantitative study, Palla et al. (2012) found that secondary students hold many misconceptions regarding the form and function of mathematical induction. Only a few students recognized the density of the Real numbers being an obstacle in utilizing mathematical induction, and the majority of students faced difficulty in deriving P(n+1) from P(n) (Palla et al., 2012).

Ed Dubinsky (1986; 1989) laid the foundation for what is largely known about students' understanding of mathematical induction. Indeed, he mentioned that "if you question [undergraduate] students - … although almost all of them will have heard of induction, not many of them will be able to say anything intelligent about what it is" (Dubinsky, 1986, p. 305). Harel (2001) built upon this work and utilizing a novel instructional approach (i.e., promoting quasi-induction) found that students made more rich connections to the definition of induction when using this instructional technique versus "hastily introduc[ing] the definition" (Brown, 2008, p. 1). Harel described, "In quasi-induction the conviction that P(n) is true for any given natural number *n* stems from one's ability to imagine starting from P(1) and going through the inference steps, P(1) → P(2), P(2) → P(3), … , P(n-1) → P(n)" (Harel, 2001, p.26). This does not mean that one actually runs through many steps, but that he or she realizes that in principle this can be done for any given natural number *n*. Indeed, Harel (2001) utilized the framework which he and Sowder (Harel & Sowder, 1997) identified as *proof schemes*, and that students who learned induction via this novel teaching strategy of promoting quasi-induction have a transformational proof scheme with mathematical induction. Harel (2001) also found that the sequencing of the tasks impacted students' adoption of induction as a transformational proof scheme. Indeed, it seems that authors of proof-based textbooks sequence tasks based on perceived difficulty and not the intellectual need of the students (Harel, 2001). When comparing university students' and 6th grade students' learning of induction via pattern tasks, Brown (2008) found that the nature of the task had an impact on their success in proving their claims, and that "the development of mathematical induction as a means to solve a class of problems necessarily entails shift in students' ways of knowing infinite processes – in particular, their ways of thinking about iterative processes and ways of understanding implications" (Brown, 2008, p. 17). More recent work by Arnold and Norton (2019) have demonstrated similarly promising results that quasi-induction is a promising instructional technique to alleviate some of the difficulties of university students learning mathematical induction, though many cognitive challenges remain.

Undergraduate university students in an ITP course recognized mathematical induction as an appropriate technique to prove a number theoretic conjecture (Sandefur et al., 2016). In Sandefur and colleagues' study students were asked to find any patterns that arose when dividing 5^n by 3, with the expectation that this pattern would be accompanied by a proof as this was an ITP course (Sandefur et al., 2016). Sandefur and colleagues (2016) found that giving students the opportunity to first explore a conjecture which lent itself to be proved by induction allowed the students to understand the form and function of induction more robustly. This is consistent with suggestions by Harel (2001) and Norton and Arnold (2019) on promoting quasi-induction as a productive method for introducing induction. These findings are promising as Smith (2006) found that students often dislike the method of mathematical induction due to not understanding how or why the process works. These findings suggest that students lack conceptual insight as it pertains to mathematical induction, but that exploration of conjectures which lend themselves to be proved by induction can aid in their understanding.

Theoretical Perspective

To situate this study and theoretically frame students' writing of proofs by mathematical induction, I use an adapted framework from Stylianides and colleagues (2013), which they describe as MGA - Manipulating - Getting a Sense Of - Articulating. The components of this framework also have subcomponents that translate to the adapted frame which I describe in the paragraphs below. Sandefur and colleagues (2013) sought to understand how students productively use examples during the proving process, whereas this study seeks to discover how students learn to compose proofs by induction. Thus, the framework proposed by Sandefur et al. (2013) requires some modifications to better frame this activity of composing proofs by induction.

First, Manipulation "means using familiar mathematical objects as worked examples, including acting with the symbols, examples, and representations, for a specific purpose" (Sandefur et al., 2013, p. 327). Sandefur and colleague's (2013) study, again, was focused on how students' example use led to successful proof production. I have observed through initial rounds of coding that students have distinct discussions about the conjecture and whether or not it is true and how they might prove it, and distinct discussions about the specifics of their proof (i.e., they discuss the conjecture and the proof separately). To make better use of Sandefur and colleagues (2013) framework, I modify the Manipulation category to refer to when students are engaged in *Discussion Surrounding the Claim*. Within this category, Sandefur and colleagues describe this component as having two distinct parts: semantic and syntactic reasoning (Weber & Alcock, 2004). Semantic reasoning refers to when individuals use informal reasoning, such as examples, charts, or diagrams (cite). Syntactic reasoning refers to when an individual reasons using more formal means, such as logical deductions or algebraic manipulations (Weber & Alcock, 2004).

Second, Sandefur et al. (2013) indicates the intent of Manipulation is to "get-a-sense-of (G) some underlying structure, pattern, or relationship by experiencing the effects of various actions and forming conjectures" (Sandefur et al., 2013, p. 327). As I have described above, early rounds of coding indicated that students were having separate conversations about the truth of the conjecture and the proof of the conjecture. Thus, instead of Getting-a-Sense-Of - I refer to when students are *Discussing the Argument*, either in its form or in specific details within the proof. Within this category Sandefur et al. (2013) refer to this portion of argumentation as having two distinct components: Conceptual Insight and Technical Handle (Sandefur et al., 2013). Conceptual insight refers to when one searches for a key idea (Raman, 2003) within a proof, as to why the claim is true. Technical Handle refers to when one uses algebraic manipulations, understands proof methods, and other more 'technical' forms of argumentation.

Finally, for Articulating, Sandefur and colleagues (2013) describe how one articulates or describes their understanding of the two previous components. Articulation, as described by Sandefur et al (2013) is the manifestation of students' technical handle and conceptual insight. This component remains relatively unchanged, with the exception of including student groups' final written product as a means of analysis.

Thus, this modified framework allowed me to hone in on important parts of student conversations surrounding their group proofs. First, I was able to discern whether or not students were discussing the conjecture or the claim itself. If the student were discussing the claim, I could then describe this discussion in terms of semantic and syntactic reasoning, whether they were using examples or less formal means or using logic and making deductions. If it was the case that their discussion was around the proof or their argument, I could then apply this framework to discern whether or not this discussion was around the technical handle of writing up the proof, or if they were seeking some conceptual insight or key idea (Raman, 2003).

Methodology

Participants & Setting

This study took place in the Fall of 2020, at a large public state university, during the first full time semester entirely online via Zoom due to Covid-19 safety protocols. Overall, 13 students elected to participate in the study. Of the 13 students, 3 were female and the remaining were male, and all but two (male) students were white. All students in the study were either majoring or minoring in mathematics at the university.

The instructor of this course, an expert ITP course instructor, sought to engage students in many group-learning exercises, particularly group-proving. While the structure of the course generally followed the Standard ITP sequence (David & Zazkis, 2020), it also had a daily structure which lent itself to have students engaging in sensemaking exercises with their peers.

Data Collection & Procedures

The data from this study came from my dissertation study as I followed 13 student participants throughout a Standard (David & Zazkis, 2020) ITP course as they engaged in typical course exercises, such as discussing definitions, analyzing arguments, and writing group-proofs. Students in this study were introduced to the definition and method of mathematical induction after they had first had time to make sense of sample proofs by induction modified from Stylianides et al. (2007) study on preservice teachers' understanding of mathematical induction. After having had an opportunity to unpack the definition of induction alongside these sample proofs, students then spent the following two class periods engaging in group-proving exercises of conjectures which lent themselves to be proved by induction.The data gathered here from these two class periods is used to provide insight into how students make sense of MI as they write these types of proofs for the first time.

Context

As I mentioned above, students in this study learned the method of induction separately from the other techniques of proof, with an instructional routine modified from the task presented to students in Stylianides and colleagues' (2007) study on preservice mathematics teachers' understanding of induction. The data from the current study stemmed from the two class days after this adaptation of Stylianides et al. (2007) instructional routine. During these two class days, students engaged in group-proving exercises on conjectures that students had been asked to informally explore for homework before the class sessions. During the two days of practicing proving with induction in groups, students worked for approximately 30 minutes each day and were expected to have some mathematical argument that their group could defend to the class.

Data Analysis

The data for this study was gathered as a part of my dissertation. Students engaged in group-proofs in conjectures which lent themselves to be proved by MI. With 3 research-groups, over 2 days of the course, I analyzed 6 episodes of students engaging in group-proofs with MI. Each group-proving session was transcribed for further analysis, so that I could apply the MGA framework as an analytical tool to describe students' learning how to write proofs with MI. Each transcription was separated by talk-turn, that is each time the speaker changed. I then applied the proposed MGA framework to each talk turn. First, I decided whether the student/speaker was referring to discussing the claim (M) or discussing the proof or argument of the claim (G). I then decided which portion of the talk-turn fit into the subcategories described above as components of either (M) with subcomponents of semantic and syntactic reasoning, and (G) with subcomponents of Technical Handle and Conceptual Insight. By doing this, I was able to hone in on portions of students' speech where they discussed the logistical aspects of proving results with induction, and when they were convincing themselves of the truth of the claim. This allowed me to describe how students learn to adopt the mode of induction as a viable method of proof.

As an example, during the study a group of students were proving the claim, "If n is a natural number, then $n! \geq 2^{n-1}$." During the group-proving exercise a student, Brad, stated the following:

> So I think we have. I just want to settle a couple cases. Like 1! is greater than or equal to $2^0 = 1$. 2! is gonna be 2, [which] is greater than or equal to 2. So at least settling those and then. So if we assume the case is, well, we'll write out the base case or whatever. I'd like to do a couple of them, actually, but then if we assume that $n!$ is greater than or equal to 2^{n-1}, then that implies that $(n + 1)!$ is greater than or equal to 2^n. And we can show that's true because all we're doing is multiplying one side by (n + 1) and one side by 2. And in every one of these cases (n + 1) is greater than or equal to 2.

The specific codes assigned to Brad's talk-turn are laid out below in Table 3.1. In the above from Brad, I only coded a semantic component (within the Discourse Surrounding the Claim) and technical handle (within the Discourse Surrounding the Proof). For clarity, I provide sample codes from other group discussions within the syntactic component and conceptual insight component as a sample codebook in Table 3.1 Note that throughout the 6 episodes, no portions of students' discussion were coded as syntactic when discussing the claim. For further clarity of what I describe as syntactic reasoning when discussing the conjecture, I provide a sample quote from students as they proved the claim “If p is a rational number and q is an irrational number, then p-q is an irrational number” using proof by contradiction (an indirect proof method).

Table 3.1

Coding of Brad’s talk-turn with modified MGA Framework and supplemental codes

Discourse Surrounding the Claim (M)		Discourse Surrounding the Argument or Proof (G)	
Semantic	Syntactic	Conceptual Insight	Technical Handle
I just want to settle a couple cases. Like 1! Is greater than or equal to $2^0 = 1$. 2! is gonna be 2, [which] is greater than or equal to 2.	Is a negation of an implication an implication itself? ** No student discussion was coded for syntactic reasoning while discussing the conjecture	And then once it's 4 then this case is correct for what you were talking about with the exponential take overtakes the linear function.	[T]hen if we assume that $n!$ Is greater than or equal to 2^{n-1}, then that implies that $(n + 1)!$ Is greater than or equal to 2^n. And we can show that's true because all we're doing is multiplying one side by $(n + 1)$ and one side by 2

Thus, rather than coding one individual talk-turn with binary codes (i.e., 0 for a lack of a code or 1 for a talk-turn possessing a code), I was able to pinpoint the portions of a student's speech that were in various categories. This allowed me to then have my research team (a fellow graduate student, and my dissertation advisor and instructor of the course under study) discuss the various codes I had given to each of the talk-turns in the transcript. Then, we discussed as a team the themes I identified across the 6 episodes, which my coding schema brought about.

The lines between analytic framing and results can be somewhat blurry in qualitative research (Sandefur et al., 2013). To this end, as I was analyzing my data, I noticed that students voiced when they lacked a technical handle when proving. Thus, one coding term that became useful outside of the MGA framework is the term *Stuck*. Indeed, it is a useful skill to know when one is stuck in proof writing. Andrew Wiles, who famously proved Fermat's Last Theorem said later in his career about the matter, "What you have to handle when you start doing mathematics … is accepting this state of being stuck" (Wiles, 2016, 1:00). So, I use this term to describe when students demonstrate an awareness that they are stuck while proving.

Results

To highlight how my theoretical and analytic frame aided in my analysis, I present the result of this study in two sections: first around students' Discourse Surrounding the Claim or conjecture (originally M from Sandefur's MGA framework) and second around students' Discussion Surrounding the Argument (originally G from Sandefur's MGA framework). Finally, I demonstrate when students are stuck when writing proofs with MI.

Students' Discussing the Claim (M)

As students were engaging in group proof exercises on induction, the majority of their discussion was centered around composing the proof or the discourse surrounding the argument (i.e., G). Groups spent little time discussing the conjecture, convincing themselves and the group whether or not the claim was true, or any technical aspects of the initial proving process. These students largely were aware that their proofs should be done using the method of induction. If they had any discussion regarding the claim it was related to how to restate the conjecture in a more operable way or to put it into words. For instance, series $\sum_{i=1}^{n} i$ can be represented operably as 1 + 2 + … + n, which can be restated in words as the sum of the first *n* integers. These types of discussions were rare, but useful as they summarized their claims.

For instance, in the vignette I described above in my data analysis description, the student, Brad, restated the claim verbally as, “N factorial is greater than or equal to 2^{N-1}.” This verbal description was, perhaps, helpful in proving their claim by stating the key idea (Raman, 2003) that factorials grow faster than exponentials.

Theme 1: Students may not recognize the necessity of the base case

The main theme from students discussing the conjecture or claim (M) was their confusion on the necessity of the base case and its significance in the proving process. As researchers have described previously, students have some difficulty in understanding the need for and usefulness of the base case. For instance, one group was proving the claim, “If n is a natural number, then the sum of the first n even numbers is n*(n+1).” In the beginning of their group’s discussion, John stated, “I guess the first thing is to plug in for 1, then plug in for 2, and then do 1 through 5 and see what we get. And then keep going.” Indeed, some students struggle with understanding how the base case helps them to prove the result. As John stated, they need to check case after case, “and then keep going.” He does not recognize that the first case is indeed the base case. On a positive note, this is potentially an important sense-making moment for John, in understanding that proving results by induction is proving an infinite sequence of statements and in becoming personally convinced of the validity of the claim.

In another group-proving episode, students were proving the claim, “If n is a natural number, then $n! \geq 2^{n-1}$.” Again, early on in this group proving episode one student, Brad, stated to the group,

> I just want to settle a couple cases. Like 1! is greater than or equal to 2^0= 1. 2! is gonna be 2, [which] is greater than or equal to 2. So at least settling those and

then. So if we assume the case is, well, we'll write out the base case or whatever.

I'd like to do a couple of them

To Brad, the base case serves two purposes, to satisfy the rules of induction but also to convince himself of the claim. This is not problematic, but we want students to understand that the base case is the first statement of an infinite sequence of statements. There seemed to be some confusion among students about how many cases were necessary to state in a proof by MI to convince themselves and convince the potential reader of their proofs.

Students Discussing the Proof (G)

Theme 2: Students face difficulties in stating the inductive hypothesis and proving the inductive step

Research on students' learning of induction has shown that the largest obstacle to their understanding the method and completing proofs is their struggle with the inductive hypothesis (Dubinsky, 1989; Harel, 2001; Arnold & Norton, 2019). An example of this issue is highlighted in the following exchange shown in Table 3.2. This conversation is a continuation of students discussing the claim, "If n is a natural number, then the sum of the first n even numbers is n*(n+1)."

Table 3.2

Students discuss the inductive hypothesis

Speaker (Pseudonym)	Student Quote / Exchange
John	So N is equal to K. Do we have to do K plus one or K minus one? It's K minus one right?
Dennis	Yeah so it's K + 1. And since it's an even number we would have to do I think 2K + 2. I think?
	...

John	So we set that in there. I'd say, therefore, if the conjecture holds equals N= K, then it holds for N equals K plus one. Can we say that automatically?

First, John demonstrates his lack of understanding about the significance of the inductive hypothesis. From this quote I notice that he does not understand the domino analogy and the iterative nature of the proofs by induction and how the inductive hypothesis allows one to prove this iterative claim. He then demonstrates his incomplete understanding, asking if now that they have met the criteria for mathematical induction (the base case and proving P(k) → P(k+1)) whether they can assume automatically that their conjecture is true.

In the following class session, Dennis and John have a similar exchange shown in Table 3.3 below. During this exchange and group-proving episode, this group was proving the claim, "If n is a natural number and $n \geq 4$, then $3^n > 2n^2 + 3n$." Prior to this discussion the group had settled on whether they had satisfied the base case, and were trying to word how to prove the inductive step.

Table 3.3

John's struggle with the inductive hypothesis

Speaker (Pseudonym)	Student Quote / Exchange
John	Yeah. I don't know about you guys, but I think if you did three to the K plus one first before we did three to the N=K. And then if you notice that if you put the (K+1) into the N^2, that should make a quadratic formula...
Dennis	Yeah, but the thing about induction is that you have to start with N=K Otherwise you'd be working backwards.
John	Ok. You can't you can't work backwards at all. Is that a rule?
Dennis	It's just… It'd be counterproductive because you'd start with the base case and then you'd be trying to go below the base case.

This exchange shows the value in group proving as a sense-making exercise. By allowing students to compose induction proofs in groups, these students were able to have rich discussions and relay their current understandings of the mode of induction to each other. This group continues to grapple with their understanding of the inductive hypothesis, shown in the exchange in Table 3.4.

Table 3.4

Continued discussion of the Inductive Hypothesis

Speaker (Pseudonym)	Student Quote / Exchange
Dennis	The test for N equals K. Can we just... do we just substitute in K for N?
John	I think so, I mean , that's the one that's confusing me because it doesn't really help you in any way that much.
Taylor	Oh, I think she said the reason why it helps you is because that step is like the induction step.
John	Hmm
Taylor	So doing that allows you to do K plus one.

Again, in this exchange we see that these students do not fully understand the necessity of the inductive hypothesis and what this affords them. This exchange highlights that students do not see that stating the inductive hypothesis is the antecedent of the implication statement $P(k) \rightarrow P(k+1)$ (i.e., the inductive step). John is correct that the switching of variables is not necessary to prove results by induction. However, the method of induction requires you to assume $P(n)$ is true and use this assumption to demonstrate that $P(n+1)$ is true; it is this iterative process that allows you to make the claim about an infinite sequence of statements.

Another result from research which I feel is critical in promoting students' rich understanding of induction is that the sequence of tasks can impact students' understanding of induction (Harel, 2001). In the following quote shown in Table 3.5, this notion is highlighted.

Table 3.5

Greg's reliance on a previous 'trick' in proving by induction

Speaker (Pseudonym)	Student Quote
Greg	Are you supposed to add it to the other side? Or are you supposed to plug in K+1 for all the Ks?

As I mentioned above, Harel (2001) has noted that the sequencing of tasks in textbooks potentially impacts students' understanding of induction, particularly how to *state* or assume the inductive hypothesis. Indeed, there is a qualitative difference between proving results by induction which are summations and results which are statements about inequalities. In this excerpt, Greg is relying on his previous knowledge of how to produce induction proofs about statements with summations, namely that he should add or multiply the next term in the sequence.

Theme 3: When proving with induction students recognize when they are stuck

A key result from this study is that students recognize when they are stuck while writing induction proofs in ITP courses. I have described this above as an asset, and I continue to believe that recognizing this is beneficial when writing proofs. Typically when writing induction proofs, students recognized they were stuck in two distinct ways: (i) on how to state the inductive hypothesis and (ii) when manipulating the algebra of the stated inductive hypothesis to arrive at the desired conclusion (i.e., proving the inductive step).

In the following quote in Table 3.6, I note that Dennis questions how he (and the group) would go about assuming the inductive hypothesis and proving the inductive step. As I described above, the sequencing of tasks can impact how students attempt to prove results by induction.

Table 3.6

Dennis describes his hypothesis on proving the inductive hypothesis (G1 10/29)

Speaker (Pseudonym)	Student Quote
Dennis	And..would we prove the case for (K+1) by just multiplying both sides by 3? And then showing that that's....greater..[trails off]

I highlight this quote to again demonstrate how the sequencing of tasks may impact students' production of induction proofs, but also here that Dennis is stuck on how to move forward with proving the inductive step. Potentially due to his initial experiences in proving results by induction (i.e., The task in Appendix A), Dennis is stuck on how to move forward with his assumption in order to prove P(k+1).

In another exchange, students get stuck as they try to manipulate their way through the complex algebra of proving the inductive step. This is shown in Table 3.7 below as this group attempts their proof of the claim, "If n is a natural number and $n \geq 4$, then $3^n > 2n^2 + 3n$."

Table 3.7

Students discuss manipulating the algebra when proving the inductive step(G3 10/29)

Speaker (Pseudonym)	Student Quote / Exchange
Brad	Gotcha. Are you kinda stuck on where to go from here?
Asad	Um no I mean yeah kinda

Brad	Yeah I can't figure out what to do with that three on the other side since its multiplication and not...we just need to show that (4n + 3) is going to be greater than whatever 3^n is multiplied by 3. Could we divide by 3 and then just have a bunch of fractions on the right? I don't know if you could do that.

Here it is important to highlight that these students did not get stuck in stating the assumptions of the inductive hypothesis such as in Table 3.6, but rather they got stuck in how to now manipulate the algebra to arrive at their desired conclusion. This group's proof is shown in Figure 3.2 below.

Figure 3.2

Group 3's proof of the claim "If n is a natural number and $n \geq 4$, then $3^n > 2n^2 + 3n$"

Prove that P(1) is true:

The base case is $n = 4$ so:
$3^4 > 2(4)^2 + 3(4)$
$81 > 32 + 12$
$81 > 44$ which is true.
Therefore, the base case is true.

Assume:
$3^n > 2n^2 + 3n$

Then:
$3 \times 3^n > (2(n+1)^2 + 3(n+1))$
$2 \times 3^n + 3^n > 4n + 5 + (2n^2 + 3n)$
Now all we need to prove is that $2 \times 3^n > 4n + 5$
For values greater than or equal to 4, the inequality above is always true.

As this group describes above, the trouble with completing this proof which they describe is that a factor of 3 cannot be divided cleanly on both sides, leaving them with fractions. It is critical that these students recognize that this potential division by 3 does not help them in their proof, and they must find another way to productively move forward.

Discussion & Limitations

As I have highlighted above, students continue to grapple with the intricacies of proving by mathematical induction in typical ways highlighted by previous literature. Indeed, students struggle with understanding the necessity of the base case (Dubinsky, 1989; Palla et al., 2012) and face difficulties in both stating the inductive hypothesis and proving the inductive step (Dubisnky, 1986; 1989; Harel, 2001; Arnold & Norton, 2019). Nonetheless, many students were able to successfully complete induction proofs and perform the necessary algebra when stating the inductive hypothesis and proving the inductive step.

These results are important, as the intent of the (ITP) course is to alleviate students' struggles in proof writing. However, as I have demonstrated and described above, students continue to struggle with the conceptual and technical aspects of proving with mathematical induction. Thus, instructors of proof-based mathematics courses (i.e., courses past the ITP course) should keep in mind that students may still be learning how to productively utilize various modes of proof, such as induction. As induction is an important technique when proving results in Analysis and Number Theory, it is important to understand that students in these courses may not have yet mastered this technique of proof. Another key finding from this study and from my broader dissertation work is that students recognize when they are stuck in writing their proofs. This is important as we do not want students to force their way through a proof using incorrect reasoning or logic. When proving by induction there are many ways that students can get stuck.

This study was not without its limitations and delimitations. First, this study took place in the Fall of 2020, which was the 2nd semester for these students under Covid-19 precautions and the first full-time semester on Zoom. This has many potential unknown impacts on sociomathematical norms (Yackel & Cobb, 1996) and how authority is manifested in group-proving episodes (Bleiler et al., 2020). As this study was conducted with students using Zoom, all group proofs were done collaboratively via Google Doc. The field is unaware how this transition to online-learning impacts students' learning, particularly students' group learning. Finally, in order to preserve fairness amongst the learners in the class, no research-participants were asked to reflect on their learning experiences or were interviewed in order to further understand their group interactions.

Conclusions

This study sought to understand the struggles that students face as they learn to write proofs by mathematical induction in an ITP course. I found that students in this course faced typical challenges when composing proofs by induction. These struggles include not understanding the necessity of the base case (Harel, 2001; Stylianides et al., 2007) and how to state the inductive hypothesis and prove the inductive step (Brown, 2008; Stylianides et al., 2007).

Research has shown that the method of quasi-induction is promising for students (Harel, 2001; Arnold & Norton, 2019) in that it helps students to understand the iterative nature of proofs by induction. Students in this study demonstrated quasi-inductive, iterative thinking when describing the base case. This study also demonstrated that the teaching strategy outlined in this study did have some effective components, such as students attending to the precise definition of induction as they composed their proofs. Future studies should examine how quasi-induction and examining induction proofs can be productively combined to further students' understanding.

CHAPTER 4 UNDERSTANDING THE CONDITIONAL AND ITS ROLE IN PROOF

Introduction

Throughout the last several decades universities across the United States have begun offering Introduction to Proof (ITP) courses to help university students make the transition from their early mathematics coursework, such as Calculus or Linear Algebra, with proof-based mathematics courses, such as Abstract Algebra or Number Theory. Several recent studies (David & Zazkis, 2020; Dawkins et al., 2020) have illuminated the nature and scope of these ITP courses. The Standard (David & Zazkis, 2020) ITP course has three core topics which are designed to help students begin to develop as mathematical provers: (1) mathematical logic; (2) proof techniques; and (3) sets and functions. In this manuscript, I will attend specifically to the potential connections in student understanding between the first two core topics: mathematical logic and techniques of proof (e.g., proof by contraposition, proof by contradiction). The goal of this work is to provide practitioners of these ITP courses with a useful synthesis of the literature surrounding how students make sense of mathematical logic (c.f., Topic 1, David & Zazkis, 2020) as well as the techniques of proof (c.f., Topic 2, David & Zazkis, 2020) and to provide vignettes from a standard ITP course highlighting the themes that arose from my reading and synthesis of the literature

Cognitive scientists and reasoning theorists (Johnson-Laird, 1995; Evans et al., 2007) have long described humans' general difficulty with understanding and analyzing arguments that involve the conditional statement ($P \rightarrow Q$, read "If P, then Q"). To this end, scholars in mathematics education (Brown, 2013; 2018) have described how university students may inaccurately use techniques of proof that rely on the conditional statement. Mathematics education scholars have also engaged in numerous studies on how students utilize, and potentially misuse, the various techniques of proof (e.g., contradiction, contraposition, mathematical induction) (Brown, 2013; 2018). By and large these studies are focused on how students utilize a singular technique of proof for a given conjecture. To date, there have been few studies or analyses of the literature which describe the connections, if any, between how students utilize their understanding of the rules of logic and how they use the various techniques of proof. This is potentially important with each technique's unique reliance on the conditional statement. See Table 4.1 for the syntactic connection between the conditional and each technique of proof.

Table 4.1

Conditional Use Per Proof Technique

Proof Technique	Conditional Use
Direct Proof of P→Q	P → Q
Proof by Contraposition of P→Q	~Q → ~P
Proof by Contradiction of P→Q	(P and ~Q) → F
Proof by Mathematical Induction of P(n) for all natural numbers, *n*.	P(1) and [(P(n) → P(n+1)]

Data & Setting

As I have described above, this manuscript is intended as a resource for instructors of ITP courses that synthesizes the key challenges students face in logic and proof writing. To bring to life some of the broad findings from previous literature, I present vignettes from a recent research study that was situated within an inquiry-based ITP course. To this end, it is worth describing the general structure of the ITP course so that readers of this work can make more thoughtful connections to their classroom and students' thinking and learning. Offered at a large public university in the Southeastern United States, the course was inquiry-oriented, and students largely worked together in groups discussing definitions, theorems, and writing proofs. This ITP course falls into the Standard category outlined by David and Zazkis (2020) in that students first learn logic, followed by proof techniques, and then they apply this knowledge in proving results about sets and functions. The course took place in the Fall of 2020, so students all joined synchronous class sessions remotely via Zoom. Students regularly worked together in small groups (Zoom breakout rooms) where they recorded their ideas in a collective Google Doc.

I transcribed, coded, and analyzed 15 episodes of group-proving to study the connections that students made between logic and the techniques of proof. I also analyzed three episodes of students engaging in the Wason selection task (Wason, 1966; 1968), and analyzed three groups of students examining and critiquing proofs by mathematical induction as they learned the definition and mode of argumentation. It is from these 21 episodes that I highlight issues from the literature surrounding students' understanding of logic and the techniques of proof. Largely, my analysis of students' work has been on their spoken word as they produced group-proofs. Thus, I use this manuscript as a platform to discuss students' final written--- in this case typed--- products alongside their discussions around these proofs, while also highlighting themes from the literature which scholars have shown students struggle with as they develop as proof writers. By examining students' final proofs, I will help practitioners of proof-based mathematics courses to make connections between their students' discussion around proof writing and how this is reflected in their written (or typed) proofs. Similarly, their written work gives further insight into how they are making sense of the course material and learning how to write proofs, which is the intent of the course. Thus, I use this manuscript to reflect upon common areas of struggle for students, which are informed by previous research literature, and brought to life by classroom vignettes and students' proofs.

Setting the Stage

As I have observed and reflected on how students learn in ITP courses, on their struggles and how to alleviate them, I have noticed three key themes:

1. The conditional statement plays an essential role in proving and students struggle with understanding its logical form.
2. Students struggle to 'begin' a proof (Weber, 2001).

Students struggle to manipulate conjectures in the form of the conditional ($P \rightarrow Q$) into other logical forms to be proved (i.e., its contrapositive or contradictory statement).

Thus, I structure this manuscript according to these three themes. Within each theme I will briefly present related literature surrounding these issues, as well as present vignettes and student work which highlight these issues, so that practitioners of ITP courses can be better prepared to help students to overcome these difficulties.

Theme 1: The conditional statement plays an essential role in proving and students struggle with understanding its logical form

In the following section, I first present vignettes and the final written solution to two student groups as they engaged in the Wason selection task (Wason, 1966; 1968) to illuminate the types of discussion that students have surrounding these conditional reasoning tasks and to show their groups' final solution. I then briefly summarize the literature on students engaging in similar sorts of reasoning tasks. Finally, I describe how these data shed a new light on some of these research findings.

Students' Engagement in the Four Card Problem

In the beginning of the course under study, students were introduced to various important terms and definitions in logic, such as logical quantifiers (i.e., for all, ∀; there exists, ∃) and logical connectives (i.e., and, ∧; or, ∨; conditional, → ; biconditional, ↔; negation, ~). As I have described above, the conditional statement (P→Q, read "P implies Q") plays an essential role in proving conjectures in mathematics, and is a demonstrated source of difficulty as students write proofs which rely on its' manipulation (Brown, 2013; 2018; Hub & Dawkins, 2018). In order to help students make the connection between the truth-table definition of the conditional (Table 4.2) and conditional arguments, students engaged in the Wason Selection Task (Wason, 1965), otherwise known as the Four-Card Problem.

Table 4.2

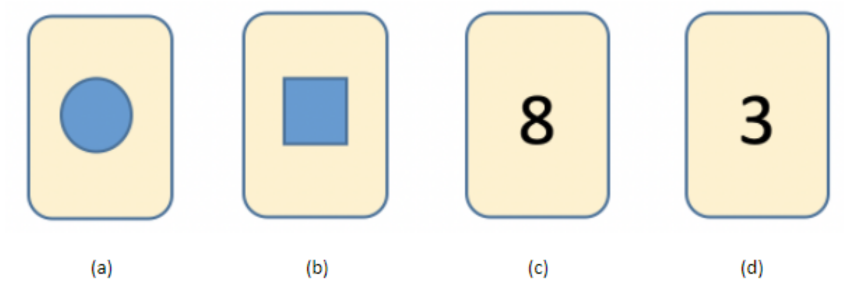
Truth-Table Definition of the Conditional Statement

P (antecedent)	Q (consequent)	P → Q (conditional)
T	T	T
T	F	F
F	T	T
F	F	T

In the following vignettes from my dissertation data and group-work exercises students engaged in the Four Card Problem (Wason, 1966; 1968) shown in Figure 4.1 below.

Figure 4.1

The Four-Card Problem.



Your friend claims that ALL four cards satisfy the following rule:

If one side of a card is a circle, then the other side is an even number.

You have doubts regarding your friend's claim. What is the minimum number of cards that you need to turn in order to know ***for sure*** whether all four cards satisfy this rule? Which cards do you need to turn? Justify your response clearly and completely.

The solution to this problem is to flip cards (a) and (d) to check the truth of the claim. Card (a) must be flipped because there is a circle on one side, and so there must be an even number on the back for the claim to be true or else it would be the case that $T \rightarrow F$, which results in a false implication. Similarly, Card (d) must be flipped because if there was an even number on the other side it would again be a case of $T \rightarrow F$.

As students were reasoning through this task, Group 1 (a small group of 3 students, Anthony, Eliza, and Greg) had the following exchange shown in Table 4.3:

Table 4.3

Student exchange about the Four Card Problem

Speaker (Pseudonym)	Student Quote / Exchange
Anthony	OK. I think we need to turn over all four because we need to see if each and every one has the rule, is true. Because if we turn (a), (b), and (d), it might be the case that (c), on the other end, instead of it being a circle, it's a square.
Eliza	This is the implication. Right? [*Pointing to the statement "If one side of a card is a circle, then the other side is even."*] I think you only need to turn (a), (c), and (d) because it's only regarding if one side of the card is a circle. So, we don't care if it's a square. So, we don't even need to turn (b). We only need to turn (a), (c), and (d). We need to turn (a) because if it's a circle we're expecting an even number on the back side.
Anthony	Hmm yeah you're right
Greg	Yeah, I would also argue you don't need to turn to (c) because it doesn't matter if [the other side of] (c) is a square or circle, you know?

In this student vignette there are several things worth highlighting. First, similar to Wason's (1966; 1968) original study, it is not uncommon for students to incorrectly solve this problem at first. Billy and Eliza's responses were typical of many students in this class. Second, Billy and Eliza's explanations of why it is that they must turn (incorrect) cards are particularly illuminating. In their explanation, both students emphasized the word *if* which has a significant meaning in this context. Scholars in cognitive science (e.g., Johnson-Laird, 1995; Evans et al., 2007) have described different modes of thinking which one can have about the conditional. Amongst these are the probabilistic view (Evans et al., 2007), which captures Billy's description that "it might be the case" that cards need to be flipped. Eliza, however, demonstrates a different type of reasoning about the conditional which is not aligned with the probabilistic paradigm

(Evans et al., 2007). From their discussion, there is no evidence that students have connected the truth-table definition of implication (or conditional) to this task yet. Greg, however, is able to correctly reason about only needing to turn cards (a) and (d) and convinces his peers of the same. When writing their final justification, the group does connect to the truth-table definition of implication, citing the truth values of the antecedent (the "if" part of the conditional) and the consequent (the "then" part of the conditional). The group comes to this final conclusion in written form:

> The cards that would need to be flipped are cards (a) and (c). It is unnecessary to flip card (b) because your friend's claim does not deal with what is on the other side of a square card, and it is unnecessary to flip card (c) because the claim is not false if the other side of the card is a square.

This response pattern was typical of the other research groups' responses to the task. That is, although all members of the group did not initially agree upon which cards needed to be flipped to prove the claim, given the opportunity to discuss with their peers led them to make either explicit or implicit connections to the truth-table and thus accurately respond to the prompt. As another example, see Group 2's description for why cards (a) and (d) must be flipped:

> We only need to flip 2 cards, the circle (i.e., card (a)) and the 3 (i.e., card (d)). If the circle were to have any number which is odd on it, the statement would not hold, so we must flip it. The statement would also not hold if the 3 had a circle on its back, so it must also be flipped. The square and the 8 could have anything on the other side without affecting the statement.

Again, these students described which cards needed to be flipped and how it would impact the statement. Although there is no explicit mention of the truth-table, these students give a clear description that provides an implicit nod to each row of the conditional truth-table and definition in their own words.

Literature on Students' Understanding of Conditional Reasoning Tasks

Several studies have shown that students who study mathematics past the compulsory level (i.e., past high school) perform better on conditional reasoning tasks than their non-mathematical counterparts. Inglis and Simpson (2008; 2009) utilized a quantitative approach to study undergraduates' understanding of the conditional statement using conditional reasoning tasks shown in Figure 4.2. These authors found that the more mathematics courses a student took, the better they performed on these conditional reasoning tasks.

Figure 4.2

Inglis & Simpson's Typical Reasoning Task

A typical reasoning task from Inglis & Simpson (2008, p.191) for the rule if ~p then q, with an explicitly negated premise (~q).	*This problem concerns an imaginary letter-number pair. Your task is to decide whether or not the conclusion necessarily follows from the rule and the premise.* *Rule*: If the letter is not H then the number is 1. *Premise*: The number is not 1 *Conclusion*: The letter is H. (YES – it follows) (NO – it does not follow)

Other studies have demonstrated that studying higher education does improve performance on conditional reasoning tasks, but that studying mathematics leads to further improvement than those studying, say, English Literature (Attridge & Inglis, 2013). These longitudinal results were replicated in a follow up study in the UK where Attridge and colleagues (2016) found that students studying high levels of mathematics improved in their conditional reasoning skills over time.

Connecting Research to the Classroom

It is perhaps tempting to instructors of ITP courses to assume that all of their students possess a rich understanding of logic and conditional arguments which the literature suggests (i.e., Inglis & Simpson, 2008; 2009; Attridge et al., 2016). However, the vignettes I provided above demonstrate that many students enter into these courses in need of further refinement of their conditional reasoning skills. Indeed, as these students were engaging in a reasoning task that research suggests they would perform well on, many students expressed difficulty in describing why certain cards needed to be flipped or not in the Four Card Problem. Again, as I have described above in the brief review of literature, all of the studies which I mentioned were done using a quantitative approach, comparing students with higher levels of mathematical training to those with less mathematical training. So, to say that mathematics majors perform better than others does not help to illuminate the potential misunderstandings that these students possess, and how to help them alleviate these misconceptions or gaps in understanding.

Theme 2: Students struggle to 'begin' a proof (Weber, 2001)

One issue that students have expressed in their struggles in proof writing is that they do not know where to begin or start their proof (Weber, 2001). As scholars have described, the language of mathematical proofs is complicated and foreign to those outside of the community (Burton & Morgan, 2000). Often when proving a conjecture in mathematics, it is useful to begin with an *imperative* (e.g., Let, Suppose, Assume) to make the stated conjecture operable. When proving a conjecture in the form of a conditional statement ($P \rightarrow Q$), a mathematician would not simply re-state "P" in its given form. For instance, when proving the conjecture "If an integer *n* is even then its square is even," a mathematician would not begin their proof with the phrase, "If an integer *n* is even." Rather, they would use an imperative to make the antecedent (the "if" part of the conditional) operable in order to arrive at the consequent (the "then" part of the conditional). So, a mathematician would instead say, "Let *n* be an even integer."

How Would a Novice Approach this Problem?

In the ITP course under study a group of 4 students (Lisa, Eliza, Dennis, and Rose) were attempting a proof of the stronger version of the example conjecture stated above, "An integer is even if and only if its square is even." Notice that this problem is stated as a biconditional (↔) statement, with the forward (→) direction being "If an integer is even, then its square is even," and the backward (←) component being "If an integer's square is even, then the integer is even." The students wrote the following proof for the forward direction shown in Figure 4.3.

Figure 4.3

Direct Proof of forward direction of the claim (if an integer is even, then its square is even)

Prove: An integer is even if and only if its square is even.

First, prove that if an integer, n, is even, then its square is even.

If n=2z, then $n^2 = (2z)^2$ Let z be an integer.
$n^2 = 4z^2$
$n^2 = 2(2z^2)$ Let $2z^2$ be equal to k where k is an integer (we can do this because integers are closed under multiplication)
$n^2 = 2k$ where n=2k is an even number.

First, what I would like to mention about this proof is that it is logically correct. However, their proof would benefit from some restructuring and more precise language. For instance, as I mentioned above, using imperatives (e.g,. Let) to make a claim operable is often useful in proof writing. While these students did indeed use such a term, it was not to restate their given conjecture, which they had reframed as P→Q, but rather it was 'letting' their variable *z* be an integer. Indeed, they began their proof with an "if-then" statement, restating the forward direction of the biconditional claim. Although these students were able to productively prove this result, one can imagine how this imprecise use of language and terms would inhibit their proof writing of more complex mathematical results. Additionally, their proof lacks a general structure and suffers from some grammatical issues, which the ITP course is intended to help alleviate. For instance, their proof contains no punctuation. This makes it difficult to tell what things they have inferred from previous lines of their proof and impacts the flow when reading it. I have observed throughout my experiences in ITP courses that the main barrier to students 'starting' a proof is is that they do not understand the usefulness of imperatives and how to state the given conjecture to be proved. This is one topic where instructors should make students aware of the subtleties of language when beginning a mathematical proof. When proving a conjecture in the form of $P \rightarrow Q$, novices of proof writing do not understand the difference between stating "*P*" and stating "*Let P be true*" in order to prove Q.

Thoughts on Proving by Induction

Research has shown that there are certain predictable ways that students struggle with understanding and adopting the method of mathematical induction.Largely, these issues are related to the definition and various components of proofs by mathematical induction, such as the base case and proving the inductive step. Indeed, students view the stating of the base case as an unnecessary step (Stylianides et al., 2007) and do not understand its importance in the structure of proofs by mathematical induction (Palla et al., 2012). Similarly, students have demonstrated a lack of understanding when proving the inductive step (Dubinsky 1986; 1990; Harel, 2001). Indeed, students have demonstrated that they view the inductive hypothesis as a form of circular reasoning (Palla et al., 2012), and they also struggle in the mechanics of proving the inductive step.

Another potential reason for students' demonstrated difficulties in adopting induction as a method of proving is that they struggle in *stating* the inductive hypothesis. Proofs using mathematical induction must first demonstrate the base case (i.e., The first step in the infinite chain of statements) is true, and then prove the inductive step, that the P(k) case implies that the P(k+1) case is true. Stating that P(k) is true is what mathematicians refer to as the *inductive hypothesis*. As I have described above, students struggle in 'beginning' a proof. This struggle appears in a different manner when students are proving with mathematical induction, as most students are familiar with the definition of induction and the necessity of the base case, though they might fail to recognize its importance (Palla et al., 2012). ather, I have observed that students struggle with stating the inductive hypothesis more than in actually proving the inductive step, and what language is necessary in order to assume the inductive hypothesis.

For instance, in one illuminating proof of the conjecture, "If n is a natural number and $n \geq 4$, then $3^n > 2n^2 + 3n$." students have the following exchange shown in Table 4.3.

Table 4.3

Students' discussion about replacing N for K in the inductive hypothesis

Speaker (Pseudonym)	Student Quote / Exchange
Dennis	The test for N equals K. Can we just... Do we just substitute in K for N?
John	I think so, I mean , that's the one that's confusing me because it doesn't really help you in any way that much.

To these students, stating the inductive hypothesis and replacing the variable *N* for *K* does not seemingly afford them anything. Combining this conversation with their written work leads me to believe that they possess a checklist mentality in terms of proving by mathematical induction. That is, students have been shown to view mathematical induction as a procedural proof method. Their final typed proof of this conjecture is shown in Figure 4.4.

Figure 4.4

Students' final typed proof of the conjecture

Problem 3.10. If n is a natural number and $n \geq 4$, then $3^n > 2n^2 + 3n$. [Note that the inequality is false if $n < 4$.]

Test for n=3

$3^{(3)} > 2(3)^2 + 3(3)$

$27 = 27$

Test for n=4

$3^{(4)} > 2(4)^2 + 3(4)$

81>44

Test for n=k

3^k > 2k^2+3k

Test for n= k+1

3^(k+1) > 2(k+1)^2 + 3(k+1)

3*3^k> 2k^2+4k+2+3k+3

3*3^k > 2k^2+2 +7k+3

ln(3)3^n > 4n +3

n=3 29 > 15

n=4 88>19

The exponent function will always increase more than the linear equation for any value of n > 3 as the exponent will be a positive number

If $n \geq 4$ and $3^n > 2n^2 + 3n$ is true then for any value of n > 4 will be true

There are several things worth noting from this proof. First, the group tested a case below the given claim. This highlights students' potential misunderstandings with the necessity of the base case in proving claims with induction and confusion over its importance in proving with this method. Second, the students ended their proof with a pseudo-induction proof, showing examples of a claim that they needed to be true in order to verify their original result. But most importantly, and relevant to this manuscript, is that these students, instead of using an imperative clause (e.g., let, suppose), used the word ‘test’ to describe the truth of an arbitrary case (i.e., stating the inductive hypothesis). When proving by induction you must assume or let the inductive hypothesis be true, rather than test for it. This issue again highlights the complex issue of what it means to *prove* a conjecture of the form *if* P *then* Q, to a novice. It would likely be beneficial for students to have the logical structure of proof methods explicitly pointed out to them, and they would also benefit from having pointed conversations about what it means, and how to prove that a conditional statement is *true* (Brown, 2013; 2018).

Theme 3: Students struggle to manipulate conjectures in the form of the conditional (P → Q) into other logical forms to be proved

The third and final theme I have noticed about students’ learning in an ITP course, is that when they prove indirectly, handling the logical manipulation of the conjecture has a unique difficulty. Again, as I have described above and have strewn throughout this manuscript, human beings struggle with conditional statements and arguments which rely on it, and mathematics majors are still susceptible to these difficulties. Given students’ demonstrated struggle with the conditional statement, it is no wonder that they have (predictable) struggles with adopting and utilizing indirect modes of proof which rely on manipulating a conditional statement into a new logical form and translating this form in the English language.

Does P → Q mean the same as ~Q → ~P ?

Indeed, scholars have demonstrated that students struggle in two predictable ways when proving indirectly. The first, which I highlight in the following section, is that students, though they are aware that P → Q is logically equivalent to ~Q → ~P (its contrapositive), may struggle in understanding that these statements are still logically equivalent. Studies have shown that mathematics majors and humans in general view implication statements as cause and effect

(Brown 2013; Harel, 2001). Consider for example a student using contraposition to prove the conjecture “An integer is even if and only if its square is even.” It is reasonable for students to be suspicious if they have actually proved the desired result. Indeed, how does proving a result about odd numbers relate to proving a biconditional statement about even numbers and their squares?

A Sample Case

As I have posited above, to novices of proof writing, it is reasonable to assume that students may fail to make the connection between the English translation of a conjecture in the form of a conditional (P → Q) to its contrapositive (~Q → ~P). In the ITP course under study, a group of 3 students (Greg, Anthony, and Zoren) were proving the claim, “If L and M are odd integers, then L * M is an odd integer.” During this group-proving exercise the students have the following illuminating exchange shown in Table 4.4

Table 4.4

Students’ conversation about transforming the contrapositive claim

Speaker (Pseudonym)	Student Quote / Exchange
Zoren	So we have to prove two implications. The first one is “if L or M is not odd then L times M is not odd.”
John	So, instead of saying not odd then you could write even.
Anthony	Are those the same thing?

Indeed, deciding whether a *not* even number is equivalent to an odd number is an important conversation for these students to have. Importantly, they do recognize that in order to prove their claim by contraposition they must negate both the antecedent (the “if”) and the consequent (the “then”), but do not immediately recognize that the negation of odd is even. And similarly, these students did not discuss whether or not this new claim, written as the contrapositive, proved their original claim. As I have described above and literature suggests, this can be another intellectual hurdle for students to face in proof writing.

Conclusion

In this paper I described the main challenges that students face as they learn throughout an Introduction to Proofs course. First, I described how the conditional statement ($P \rightarrow Q$) plays an essential role in proving and in the language of mathematics, and that undergraduate students in mathematics enter into ITP courses with a limited understanding of conditional statements and arguments which rely on its structure. Second, I described how students face difficulties in beginning a proof (Weber, 2001). I posit that this is due to their misunderstandings of how to *prove* a conditional statement, namely by assuming P and logically deducing Q. Novices in proof writing struggle due to their unfamiliarity with the particular language of mathematical proof, such as using imperatives. Finally, I described how students face difficulties when proving indirectly, as this requires them to manipulate a conditional statement ($P \rightarrow Q$) to its contrapositive ($\sim Q \rightarrow \sim P$) or its contradictory ($P$ and $\sim Q$) claim.

Instructors of ITP courses should highlight the conditional and its importance throughout the course. Indeed, tasks such as the Wason (Wason, 1966; 1968) Selection Task and other conditional reasoning problems are beneficial in helping students to make the connection between the truth-table definition of implication and arguments which rely on the conditional. One suggestion I have for instructors of ITP courses is to have discussions about how to prove conditional statements in mathematics and continuously point out the logical structure of mathematical claims which arise in the course. This will necessarily lead to the introduction of imperative clauses (e.g., let, suppose, assume) which are a crucial but seemingly under-emphasized part of proving (Burton & Morgan, 2000). Finally, when students are proving indirectly, scholars have suggested (Brown, 2013; 2018) for students to first identify the logical structure of the conjecture, and identify its components. Offloading some of the intellectual work of switching and negating a conjecture to the symbols allows students to focus more on proving in their argument. Students should be encouraged to utilize their logical knowledge which they have just learned early on in these typical ITP courses as they develop as provers.

CHAPTER 5 CONCLUSION

The purpose of this study was to gain an understanding into the ways that students in an Introduction to Proofs course develop as mathematical provers, in particular how they make connections between logic and the various techniques of mathematical proof. I use this conclusion to describe holistically what you have just read in the first four chapters, and to share some insights and anecdotes I gained as a researcher of students' learning mathematical proof.

In the first chapter, I described the complexity of mathematical proof, with its unique reliance on symbols and syntax (Burton & Morgan, 2000; Weber & Alcock, 2004), as well as relying on key ideas (Raman, 2003) or conceptual insight (Sandefur et al., 2013). Amongst these complex components of proving is that conjectures in mathematics are often stated in the form of a conditional ($P \rightarrow Q$), which scholars in cognitive science (e.g., Johnson-Laird, 1995; Evans et al., 2007) and scholars in mathematics education (e.g., Hub & Dawkins, 2018) have shown to present difficulties in proof and understanding conditional arguments. Due to the complex issues with proof and proving, many universities around the United States have begun offering Introduction to Proof courses to help students to overcome the complexities of learning mathematical proof. Recent studies (e.g., Cook et al., 2019; Dawkins et al., 2020; David & Zazkis, 2020) have illuminated the nature and scope of these courses. Typically (David & Zazkis, 2020) Intro to Proof courses cover three main topic areas: (1) logic; (2) techniques of proof; and (3) sets and functions. With students' demonstrated difficulties in logic and proving, I designed and implemented a study to investigate the connections that students make between logic and the techniques of proof. In this study, I followed a typical Introduction to Proof course throughout the semester as they engaged in typical course activities such as reading and understanding definitions, and writing group proofs.

In the second chapter, I described the first research study I engaged in, where I sought to understand the connections that students make between logic and both direct and indirect modes of proof (i.e., contrapositive and contradiction). To study this, I analyzed 9 episodes of students engaging in group-proofs where they utilized direct proof, proof by contradiction, and proof by contraposition. Mathematical induction was intentionally left out of this study as students learned induction separately from the other techniques of proof, and studies on induction are typically solely focused on this proof technique. In this study I found three major themes. First, students are aware when they must utilize their knowledge of logic to manipulate conjectures to be proved. The second theme I recognized is that when students are proving indirectly, they offload the work of manipulating the conjecture to the symbols or syntax. Finally, the third theme I found was that students recognize when they are stuck in writing a proof. I describe this as an asset, as we do not want students to absent mindedly manipulate symbols or make deductions which are not true. All of these findings are promising, as the intent of the course is to help students to make the connections between logic and the techniques of proof, and they indeed do.

In the third chapter, I described the second research study I engaged in where I sought to understand the struggles that students have when learning to write proofs by mathematical induction. To study this, I analyzed six episodes of students doing group-proofs with mathematical induction. Using the modified MGA framework from Sandefur and colleagues (2013), I found that students have typical struggles with adopting the mode of induction as a viable proof method. One such struggle is that students may lack understanding of the necessity of the base case, which is consistent with many other studies on induction (Dubinsky, 1986; 1990; Harel, 2001; Stylianides et al., 2007). Another typical struggle students have when learning about induction is how to state the inductive hypothesis in an operable way, and to prove the inductive step, which is again consistent with prior research on students' learning of induction (Harel, 2001; Palla et al., 2012). Harel (2001) found that quasi-induction was a productive teaching strategy to help alleviate students' struggles with proving by induction.

Finally, in the fourth chapter, I wrote a practitioner-minded piece where I described the three major themes which I viewed throughout the entire course which encapsulate students' learning. In my view, the role of the conditional statement (P → Q) cannot be overstated in this course. Indeed, the conditional statement is a main connecting piece between the logic and proof techniques section of the course. Thus, I used Chapter 4 to describe three main themes related to the conditional as students transition through an Introduction to Proofs course. First, I presented an insider view of students' reasoning through a typical conditional reasoning task in a small group. Most previous research utilizing this conditional task is performed on large quantitative scales, comparing mathematics majors to other groups of learners (e.g., Attridge & Inglis, 2013; Attridge et al., 2016; Inglis & Simpson, 2008; 2009). I also pointed out that students in Introduction to Proof courses do not all possess the same understanding of conditional arguments. Second, I described how students often struggle in starting a proof (Weber, 2001) and I posited and showed how using imperatives (i.e., Let, Suppose, Assume) can potentially help students to overcome this struggle. Finally, I described how students may struggle when proving results indirectly, either with contradiction or contraposition. . Here I showed that students have difficulty which stems around the conditional and its successful manipulation, and its translation back into a mathematical conjecture. This is a complicated process which should not be overlooked by practitioners of Introduction to Proof courses, and other courses where proving is a heavy component of the course.

As I have written conclusions in each research (Chapter 2 and 3) and practitioner-themed (Chapter 4) manuscript, I use these final paragraphs to reflect on lessons I have learned as a researcher of students in Introduction to Proof, and to offer some anecdotal musings having completed a dissertation during a pandemic and also having had experience as a proof-researcher prior to times of Covid (Bleiler-Baxter et al., 2019; 2020) and online learning of mathematical proof.

As I have described above, and in several other chapters, the importance of the conditional statement cannot be overstated. Although teaching the definitions of logical connectives and their truth-tables may occur in introductory-level philosophy courses, there is a reason that many Introduction to Proof courses cover this topic in detail early on in the semester (David & Zazkis, 2020). Again, this is due to the complex interplay between logic and the techniques of proof which rely on the conditional and its manipulation. As I have observed as an instructor (via teaching internship) and researcher of students in ITP courses, one key notion which is not often covered in the curriculum is the use of imperatives (e.g., Let, Suppose, Assume). As I have described in detail in Chapter 4, the lack of use of imperatives (or their more sophisticated use) can cause significant issues within students' proofs. Indeed, a part of this course's goals is to help students learn to write proofs. Understanding the affordances that imperatives can give novices offers some insight into why it is that mathematicians use such particular language. It is easy as an expert in mathematics and experienced proof writer to forget the nuanced language that was previously foreign to me as a mathematician. And so, it is no wonder that this material is largely absent from ITP courses, in spite of its importance. Indeed, it is easy to assume that providing students with logical tools, and knowledge of proof methods is enough to promote their growth as proof writers. However, these students demonstrated that the grammar of mathematical proof can have large impacts on the success of their proof-writing.

Lastly, as I have mentioned above, I have experience as a researcher of students learning in-person in ITP courses, and online via Zoom in the ITP course under study. I would like to offer some anecdotes as a researcher comparing both of these mediums. Students in this study benefited from their instructor having had to transition to the online-learning of proof earlier in the year when Covid protocols first began during the Spring of 2020 (one semester prior to this study). As such, and anecdotally to me as an observer, the beginning of the course under study largely resembled other ITP courses which I have previously studied (e.g., Bleiler-Baxter et al., 2019; 2020). However, again anecdotally, there was a stark contrast in students' group-proofs throughout the semester compared to in-person learning. This is, by my observation, due to students' collaborative capabilities via Zoom and their collective Google Doc. In previous semesters, students completed group proofs with pens or markers on large poster paper. Obviously, these different mediums afford different benefits and have different constraints. When students work in Zoom, it is easier for them to modify their proofs in real-time compared to on chart-paper. Also, while using individual computers and collaborating in a Google Doc, students could use features in Google Doc which are unavailable when writing on chart paper, such as using copy-and-paste, and editing grammar and tense without having to cross out or erase previously written work. This led to more visually appealing proofs, and allowed students to expand in their language more so than they would with markers and paper. An interesting future study could compare students' learning in an ITP course with computers as they write group-proofs, and compare their conversations and written final products to students proving on paper.

Appendix A

Sample Induction Arguments

Evaluate the following two induction arguments. If you believe the argument is a proof, explain to a skeptical classmate why they should also believe it is a proof. If you do not believe the argument is a proof, explain how you would modify it to make it a proof.

Consider the following statement:

For every $n \in \mathrm{N}$ the following is true: $1 + 3 + 5 + \ldots + (2n - 1) = n^2 + 3$

Proof:

First, I assume that the result is true for n=k.

This means that, $1 + 3 + 5 + \ldots + (2k - 1) = k^2 + 3$.

I check whether the result is true for n=k+1:

$1+3+5+\ldots+(2k-1)+(2k+1) = (k^2+3)+2k+1 = (k^2+2k+1)+3 = (k+1)^2+3$

True.

Therefore, $1 + 3 + 5 + \ldots + (2n - 1) = n^2 + 3$ for all $n \in \mathrm{N}$ by mathematical induction.

Consider the following statement:

For every natural number $n \geq 5$ the following result is true:

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n > 2^n$$

Proof:

I check whether the result is true for $n = 5$.

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120 > 2^5 = 32$$

So, I assume the result is true for n=k : $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k > 2^k$

Now, I check whether the result is true for n=k+1.

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k \cdot (k+1) > 2^k \cdot (k+1)$$

$$> 2^k \cdot 2 \text{ (because } k + 1 > 2)$$

$$= 2^{k+1}$$

Therefore, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n > 2^n$ for all $n \geq 5$.

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