EXISTENCE OF SOLUTIONS TO THE INFINITE DIMENSIONAL KURAMOTO MODEL

by

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ABSTRACT

The Kuramoto model is a kinetic model of phase-coupled oscillators. This model has a long legacy, starting from Kuramoto's text, *Oscillations, Waves, and Turbulence*, which provided the foundation and initial analysis of the model. Kuramoto defined convenient measures of the average phase and variance to describe the population's macroscopic state. Since Kuramoto's original publication, researchers have worked to better characterize solution dynamics. In a much-cited paper, Ott and Antonsen showed that in some cases solutions adhere to a convenient ansatz, which greatly simplifies model analysis. Here we combine the method of characteristics with an iterative technique to prove existence and uniqueness of solutions to the infinite dimensional Kuramoto model. Our result differs from the result of Ott and Antonsen in that we require the initial data is twice continuously differentiable and show existence of continuously differentiable solutions, whereas the ansatz of Ott and Antonsen requires the solution and initial data are twice continuously differentiable and belong to a special class of Fourier series. We believe the alternative approach to model analysis developed here has the potential to open doors to future results on the extensively studied Kuramoto model.

DEDICATION

This thesis is dedicated to the two men who raised me. My father, Brent Krueger, who supported me and constantly incites passion in me to continue my studies. My stepfather, Troy Jones, who gave me the morals and the passion for hard work that I have used to grow as a student, a researcher, and a teacher.

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Contents

CHAPTER 1

Background

The Kuramoto model was proposed in 1975 by Dr. Yoshiki Kuramoto to give a more concrete definition to intuition developed by Dr. Arthur Winfree on phase models and oscillators [5]. This model of entrainment can be modified to describe the behavior of either a discrete or continuum population of oscillators. More specifically, the Kuramoto model considers the synchronization of globally coupled phase oscillators where the coupling between oscillator pairs is given as the sine of their phase difference. This model is built with uniform coupling strength. In addition, the population is characterized by a symmetric distribution of natural velocities, that is, the oscillators need not be identical. Much of Kuramoto's work for this model was on a finite number of oscillators. Kuramoto's work towards the continuum limit was an attempt to discern the conditions for the synchronization-desynchronization transition. He stated a critical condition on the coupling strength for the continuum limit of the model, and also considered the nature of the bifurcation at the critical value for the coupling strength. However the analysis proved difficult, so he used intuition and experimentation to posit further results. Kuramoto postured that for the continuum limit of the Kuramoto model we would see that many problems would remain open, as it was difficult to approach analytically [5]. Additional results were obtained by Dr. John Crawford. Through the use of center manifold theory and linear analysis, Crawford studied the stability of solutions to the Kuramoto model. The bulk of his research was released in the 1990's [1], [2]. Then in 2008, Dr. Edward Ott and Dr. Thomas Antonsen developed an ansatz that can be generated given the initial condition of a specific geometric Fourier series. They also developed an exact, closed form solution to the Kuramoto model when considering a Lorentzian oscillator frequency distribution, thus proving existence with this distribution. The ansatz is well-studied and used to develop numerical results in physical mathematics due its closed form solution given the common physics distribution of the Lorentzian [6].

Most current research in this area focuses on variations and/or control of the Kuramoto model. A prominent model variant is the Kuramoto-Sakaguchi model. The Kuramoto-Sakaguchi model includes a phase-lag on the angle between oscillators which impacts the attractive or repulsive behavior of the oscillators [8]. With the assumption of the initial conditions and a Lorentzian distribution of natural frequencies, researchers have adopted the Ott-Antonsen ansatz to develop results for this model.

Here we study the existence and uniqueness of global solutions to the infinite dimensional Kuramoto model from a different perspective that is based on the method of characteristics for solving first order partial differential equations. We derive a new result on the existence of global solutions, however the primary contribution of the paper is the application of the characteristics to the study of this model. We believe this alternative approach has the potential to open the door for future discoveries about this model.

CHAPTER 2

Model exposition

In this paper we consider the infinite dimensional Kuramoto model,

$$\frac{\partial \rho}{\partial t}(t,\theta,\omega) = -\frac{\partial}{\partial \theta}\left(\rho\omega + k\rho(t,\theta,\omega)\int_0^{2\pi}\int_{-\infty}^{\infty}\rho(t,\hat{\theta},\hat{\omega})\gamma(\hat{\omega})\sin(\hat{\theta}-\theta)\ d\hat{\omega}\ d\hat{\theta}\right); t>0, \theta\in\mathbb{R}, \omega\in\mathbb{R} \quad (1)$$

$$\rho(0,\theta,\omega) = \rho_0(\theta,\omega); \theta\in\mathbb{R}, \omega\in\mathbb{R} \quad (2)$$

where $\rho_0(\theta,\omega)$ is the initial 2π-periodic density of a population of oscillators. Here ω is a parameter describing the natural velocity, which varies according to a density $\gamma(\omega)$, θ is the angle, and t is time. In particular, $\int_a^b \gamma(\omega)\ d\omega$ gives the probability an oscillator has natural velocity between a and b and

$$\int_{-\infty}^{\infty} \gamma(\omega)\ d\omega = 1. \quad (3)$$

We seek solutions $\rho(t,\theta,\omega)$ that are 2π-periodic in θ and represent probability distributions as,

$$\int_{-\infty}^{\infty} \rho(t,\theta,\omega)\gamma(\omega)\ d\omega$$

gives the probability density of oscillator angles (θ) at time t, so that for $0 \le a \le b$ $\int_a^b \int_{-\infty}^{\infty} \rho(t,\theta,\omega)g(\omega)\ d\omega\ d\theta$ gives the probability an oscillator has an angle between a and b at time t, and

$$\int_0^{2\pi}\int_{-\infty}^{\infty} \rho(t,\theta,\omega)\gamma(\omega)\ d\omega\ d\theta = 1; \quad t>0. \quad (4)$$

Note in addition, assuming the solution is 2π-periodic, the coupling term,

$$\int_0^{2\pi}\int_{-\infty}^{\infty} \rho(t,\hat{\theta},\omega)\gamma(\omega)\sin(\hat{\theta}-\theta)\ d\omega\ d\hat{\theta} \quad (5)$$

is equal to

$$\int_{j2\pi}^{(j+1)2\pi}\int_{-\infty}^{\infty} \rho(t,\hat{\theta},\omega)\gamma(\omega)\sin(\hat{\theta}-\theta)\ d\omega\ d\hat{\theta}, \quad (6)$$

where $j2\pi \leq \theta \leq (j+1)2\pi$. So, the model behaves as if oscillators are only coupled to those near them.

In this work, we consider a simplified model where all oscillators have a single natural velocity, i.e. $\omega \equiv 0$. Under these simplified assumptions, the model becomes

$$\frac{\partial \rho}{\partial t}(t,\theta) = -\frac{\partial}{\partial \theta}\left(k\rho(t,\theta)\int_0^{2\pi} \rho(t,\hat{\theta})\sin(\hat{\theta}-\theta)\ d\hat{\theta}\right); \quad t>0, \theta \in \mathbb{R} \tag{7}$$

$$\rho(0,\theta) = \rho_0(\theta); \quad \theta \in \mathbb{R}, \tag{8}$$

where ρ_0 is non-negative, 2π-periodic in θ and satisfies

$$\int_0^{2\pi} \rho_0(\theta)\ d\theta = 1; \quad \theta \in \mathbb{R}. \tag{9}$$

It will be useful to write (7) as

$$\frac{\partial \rho}{\partial t}(t,\theta) = -\frac{\partial \rho}{\partial \theta}(t,\theta)\left(k\int_0^{2\pi} \rho(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta}\right) + \rho(t,\theta)\left(k\int_0^{2\pi} \rho(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta}\right), \tag{10}$$

where we have differentiated through the integral in (7) with respect to θ, as ρ is assumed to be continuous in all variables and sine is smooth.

To motivate (7), first consider the scenario where oscillators are subject to an initial density, ρ_0, with compact support on $\mathbb{R}$. Let $\rho_1(t,\theta)$ be the density of these oscillators at time t, so that $\rho_1(t,\theta)$ is a solution of

$$\frac{\partial \rho}{\partial t}(t,\theta) = -\frac{\partial \rho}{\partial \theta}(t,\theta)\left(k\int_{-\infty}^{\infty} \rho(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta}\right) + \rho(t,\theta)\left(k\int_{-\infty}^{\infty} \rho(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta}\right),$$

for $t>0$ and $\theta \in \mathbb{R}$. Also ρ_1 is subject to initial condition, $\rho_1(0,\theta) = \rho_0(\theta)$, for $\theta \in \mathbb{R}$, where $\rho_0(\theta)$ is C^2 on $\mathbb{R}$ with finite support, and

$$\int_{-\infty}^{\infty} \rho_0(\theta)d\theta = 1.$$

Since we naturally identify θ with $\theta + 2k\pi$, for any $k \in \mathbb{Z}$, we define

$$\rho_2(t,\theta) := \sum_{j\in\mathbb{Z}} \rho_1(t,\theta+2\pi j)\ ;\ \theta \in [0,2\pi).$$

Assuming that ρ_1 has finite support, so the sum in the definition of ρ_2 is finite,

$$
\begin{aligned}
\frac{\partial \rho_2}{\partial t}(t,\theta) &= \sum_{j\in\mathbb{Z}} \frac{\partial \rho_1}{\partial t}(t,\theta+2\pi j) \\
&= -\sum_{j\in\mathbb{Z}} \frac{\partial \rho_1}{\partial \theta}(t,\theta+2\pi j)k\int_{-\infty}^{\infty} \rho_1(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta} + \sum_{j\in\mathbb{Z}} \rho_1(t,\theta+2\pi j)k\int_{-\infty}^{\infty} \rho_1(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta} \\
&= -\frac{\partial \rho_2}{\partial \theta}(t,\theta)k\sum_{j\in\mathbb{Z}} \int_{2\pi j}^{2\pi(j+1)} \rho_1(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta} + \rho_2(t,\theta)k\sum_{j\in\mathbb{Z}} \int_{2\pi j}^{2\pi(j+1)} \rho_1(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta} \\
&= -\frac{\partial \rho_2}{\partial \theta}(t,\theta)k\int_0^{2\pi} \sum_{j\in\mathbb{Z}} \rho_1(t,\hat{\theta}+2\pi j)\sin(\hat{\theta}-\theta)d\hat{\theta} + \rho_2(t,\theta)k\int_0^{2\pi} \sum_{j\in\mathbb{Z}} \rho_1(t,\hat{\theta}+2\pi j)\cos(\hat{\theta}-\theta)d\hat{\theta} \\
&= -\frac{\partial \rho_2}{\partial \theta}(t,\theta)k\int_0^{2\pi} \rho_2(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta} + \rho_2(t,\theta)k\int_0^{2\pi} \rho_2(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta}
\end{aligned}
$$

Thus, ρ_2 is a 2π-periodic solution of (7).

CHAPTER 3

Existence of Solutions

We seek to show the existence of nonnegative, global, 2π-periodic, C^1 solutions of (7)-(8) subject to the normalization condition,

$$\int_0^{2\pi} \rho(t,\theta)\ d\theta = 1,; \quad t \geq 0, \theta \in \mathbb{R}. \tag{11}$$

In addition, the initial data, $\rho_0(\theta,\omega)$, is C^2 in θ, 2π-periodic, and nonnegative. We employ an iterative method involving an approximating sequence, characterize the elements of the sequence, and establish convergence to a solution. Specifically, we consider the following approximating sequence

$$\frac{\partial \rho_{n+1}}{\partial t}(t,\theta) = -\frac{\partial \rho_{n+1}}{\partial \theta}(t,\theta)\left(k\int_0^{2\pi} \rho_n(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta}\right) + \rho_{n+1}(t,\theta)\left(k\int_0^{2\pi} \rho_n(t,\hat{\theta})\cos(\hat{\theta}-\theta)d\hat{\theta}\right), \tag{12}$$

where ρ_{n+1} satisfies the initial condition

$$\rho_{n+1}(0,\theta) = \rho_0(\theta). \tag{13}$$

First we show that if ρ_n is C^2, non-negative, and 2π-periodic, there exists a global solution to equation (12). Note that (12) is a first order, linear, differential equation, hence we will employ the method of characteristics.

Let

$$F_n(x,y,z,p,q) := p + q\left(k\int_0^{2\pi} \rho_n(x,\hat{\theta})\sin(\hat{\theta}-y)d\hat{\theta}\right) - z\left(k\int_0^{2\pi} \rho_n(x,\hat{\theta})\cos(\hat{\theta}-y)d\hat{\theta}\right), \tag{14}$$

so (12) can be expressed as

$$F_n\left(t,\theta,\rho_{n+1},\frac{\partial \rho_{n+1}}{\partial t},\frac{\partial \rho_{n+1}}{\partial \theta}\right) = 0.$$

The characteristic equations associated with (12) are

$$\begin{aligned}
\dot{x}_{n+1}(s) &= 1 && (15)\\
\dot{y}_{n+1}(s) &= k\int_0^{2\pi} \rho_n(x_{n+1}(s),\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s))d\hat{\theta} && (16)\\
\dot{z}_{n+1}(s) &= z_{n+1}(s)k\int_0^{2\pi} \rho_n(x_{n+1}(s),\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))d\hat{\theta}. && (17)
\end{aligned}$$

together with

$$\begin{aligned}
\dot{p}_{n+1}(s) &= -q_{n+1}(s)k\int_0^{2\pi} \frac{\partial}{\partial x}\rho_n(x_{n+1}(s),\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s))d\hat{\theta}\\
&+ z_{n+1}(s)k\int_0^{2\pi} \frac{\partial}{\partial x}\rho_n(x_{n+1}(s),\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))d\hat{\theta} && (18)\\
&+ p_{n+1}(s)k\int_0^{2\pi} \rho_n(x_{n+1}(s),\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))d\hat{\theta},\\
\dot{q}_{n+1}(s) &= 2q_{n+1}(s)k\int_0^{2\pi} \rho_n(x_{n+1}(s),\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))d\hat{\theta}\\
&+ z_{n+1}(s)k\int_0^{2\pi} \rho_n(x_{n+1}(s),\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s))d\hat{\theta}, && (19)
\end{aligned}$$

where in (15)-(19) we have used the following expressions for the partial derivatives of F with respect to x, y, and z. In deriving these expressions, we use the fact that ρ_n is C^2 to differentiate through the integrals with respect to x and y.

$$\begin{aligned}
(F_n)_x &= q_{n+1}k\int_0^{2\pi} \frac{\partial}{\partial x}\rho_n(x_{n+1},\hat{\theta})\sin(\hat{\theta}-y_{n+1})d\hat{\theta}\\
&- z_{n+1}k\int_0^{2\pi} \frac{\partial}{\partial x}\rho_n(x_{n+1},\hat{\theta})\cos(\hat{\theta}-y_{n+1})d\hat{\theta}, && (20)\\
(F_n)_y &= -q_{n+1}k\int_0^{2\pi} \rho_n(x_{n+1},\hat{\theta})\cos(\hat{\theta}-y_{n+1})d\hat{\theta}\\
&- z_{n+1}k\int_0^{2\pi} \rho_n(x_{n+1},\hat{\theta})\sin(\hat{\theta}-y_{n+1})d\hat{\theta}, && (21)\\
(F_n)_z &= -k\int_0^{2\pi} \rho_n(x_{n+1},\hat{\theta})\cos(\hat{\theta}-y_{n+1})d\hat{\theta}. && (22)
\end{aligned}$$

We show that the system of equations (15)-(17) has a unique global solution under any initial data. However, for the purpose of solving (12) subject to (13), the following initial data, are of interest:

$$(x_{n+1}(0), y_{n+1}(0), z_{n+1}(0)) = (0, \theta_0, \rho_0(\theta_0)), \tag{23}$$

since (23) is the initial data for the x, y and z components of the characteristic equations for (12) together with (13). Later we will specify initial conditions for p_{n+1} and q_{n+1} corresponding to (12) and (13). For convenience we will denote solutions of (15)-(17) subject to general initial data,

$$(x_{n+1}(s_0), y_{n+1}(s_0), z_{n+1}(s_0)) = (x_0, y_0, z_0) := X_0, \tag{24}$$

as

$$X_{n+1}(s; s_0, X_0) = (x_{n+1}(s; s_0, X_0), y_{n+1}(s; s_0, X_0), z_{n+1}(s; s_0, X_0)),$$

and solutions subject to (23) by

$$X_{n+1}(s; \theta_0) = (x_{n+1}(s; \theta_0), y_{n+1}(s; \theta_0), z_{n+1}(s; \theta_0)).$$

Lemma 3.1. *If $\rho_0(\theta)$ and $\rho_n(t, \theta)$ are C^2 in all arguments, then there exists a unique, global solution, $X_{n+1}(s; s_0, X_0) = (x_{n+1}(s; s_0, X_0), y_{n+1}(s; s_0, X_0), z_{n+1}(s; s_0, X_0))$, to (15)-(17) subject to the initial conditions of (24). Moreover, $X_{n+1}(s; s_0, X_0)$ is C^2 with respect to all arguments.*

Proof. Let $G_{n+1}(s, x, y, z)$ be defined by the right hand side of (15)-(17). We will refer to the components of G_{n+1} as $[G_{n+1}]_i$, $i = 1, 2, 3$, and we will restrict the values of s to a compact interval $[0, T]$. Note that the first two characteristics equations can be decoupled and solved individually. The unique, global solution to (15) satisfying the first component of (24) is

$$x_{n+1} = s + x_0,$$

which is clearly C^2 in all arguments. Substituting $x_{n+1}(s) = s + x_0$ into (16) gives a differential equation for y_{n+1} in s and y_{n+1} alone. Since sine is smooth and its derivative is bounded by one, $[G_{n+1}]_i$ is continuously differentiable with respect y and its derivative with respect to y is uniformly bounded for $(s, y) \in [0, T] \times \mathbb{R}$.

In particular, $[G_{n+1}]_i$ is uniformly Lipschitz continuous with respect to y on $[0, T] \times \mathbb{R}$ and therefore, there exist a unique, global (i.e. defined for all $s \in \mathbb{R}$) solution of $(15) - (16)$ satisfying the first two components of (24) (See Theorem 2.2, p.38 and Corollary 2.6, p.41 of [7]) that is continuously differentiable with respect to s.
With x_{n+1} and y_{n+1} in hand, (17) can be solved by separation of variables. This gives a global solution:

$$z_{n+1}(s; s_0, X_0) = z_0 e^{k \int_{s_0}^{s} \int_0^{2\pi} \rho_n(\tau,\hat{\theta}) cos(\hat{\theta} - y_{n+1}(\tau; s_0, X_0)) d\hat{\theta}\ d\tau}, \tag{25}$$

continuously differentiable in s. Note that $z_{n+1}(s, s_0, X_0) \neq 0$ provided $z_0 \neq 0$ and $z_{n+1}(s, s_0, X_0) \equiv 0$, provided $z_0 = 0$.

Thus unique functions of s $x_{n+1}(s; s_0, X_0), y_{n+1}(s; s_0, X_0)$ and $z_{n+1}(s; s_0, X_0)$ satisfying $(15) - (17)$, subject to (24), exist are continuously differentiable for all $s \in \mathbb{R}$. Moreover, since our assumptions on ρ_n make $G_{n+1}(s, x, y, z)$ C^2 in all arguments, the solution,

$$X_{n+1}(s; s_0, X_0) = (x_{n+1}(s; s_0, X_0), y_{n+1}(s; s_0, X_0), z_{n+1}(s; s_0, X_0)),$$

of $(15) - (17)$, subject to (24) is, in fact, C^2 with respect to all arguments (See theorem 2.10, p.46 of [7]). That is, $x_{n+1}(s; s_0, X_0), y_{n+1}(s; s_0, X_0)$ and $z_{n+1}(s; s_0, X_0)$ are twice continuously differentiable with respect to s, the initial conditions, s_0 and X_0. □

Note, that the solution $X_{n+1}(s, X_0) = (x_{n+1}(s, X_0), y_{n+1}(s, X_0), z_{n+1}(s, X_0))$ of $(15) - (17)$, subject to (23), satisfies $x(s) = s$, and

$$z_{n+1}(s; \theta_0) = \rho_0(\theta_0) e^{k \int_0^s \int_0^{2\pi} \rho_n(\tau,\hat{\theta}) cos(\hat{\theta} - y_{n+1}(\tau; \theta_0)) d\hat{\theta}\ d\tau}. \tag{26}$$

In particular, since ρ_0 is nonnegative, so is $z_{n+1}(s;\theta_0)$.

By the existence and uniqueness established in Lemma 3.1, if we define $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi_n(s,\theta_0) = (s, y_{n+1}(s;\theta_0))$ then ϕ_n is onto and invertible with $\phi_n^{-1}(s,\theta) = (s, y_{n+1}(0;s,\theta))$. In fact, we have the following corollary:

Corollary 3.2. *If $\rho_0(\theta)$ and $\rho_n(t,\theta)$ are C^2 in all arguments then there exists a global C^2 solution $\rho_{n+1}(t,\theta)$ of (12) together with (13).*

Proof. As we have just noted the mapping $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ which carries (s,θ_0) to $(s, y_{n+1}(s;\theta_0))$, where $y_{n+1}(s;\theta_0)$ is the unique solution of (16) subject to $y_{n+1}(0) = \theta_0$, is onto $\mathbb{R}^2$ and one-to-one, and hence invertible. Moreover, by Lemma 3.1, ϕ_n is C^2. In fact,

$$\frac{\partial \phi_n}{\partial s}(s,\theta_0) = (1, [G_{n+1}]_2(s, y_{n+1}(s;\theta_0))),$$

and

$$\frac{\partial \phi_n}{\partial \theta_0}(s,\theta_0) = (0, \exp\left(\int_0^s \frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right)).$$

Hence, the Jacobian of ϕ_n satisfies

$$\begin{vmatrix} 1 & 0 \\ [G_{n+1}]_2(s, y_{n+1}(s;\theta_0)) & \exp\left(\int_0^s \frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right) \end{vmatrix} = \exp\left(\int_0^s \frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right) \neq 0,$$

since $\frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}v(\nu;\theta_0))$ is bounded for $\nu \in [0,s]$. By the inverse function theorem (See Theorem 7, page 632, appendix C of [3]), ϕ_n^{-1} is C^2 at each point in $\mathbb{R}^2$. Finally, we can calculate the total derivative of ϕ_n^{-1} at (s,θ) as

$$D\phi_n^{-1} = \frac{1}{\exp\left(\int_0^s \frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right)} \begin{bmatrix} \exp\left(\int_0^s \frac{\partial [G_{n+1}]_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right) & 0 \\ -[G_{n+1}]_2(s, y_{n+1}(s;\theta_0)) & 1 \end{bmatrix} \tag{27}$$

where $\theta_0 = y_{n+1}(0;s,\theta)$. Using (27), we see that $\rho_{n+1}(s,\theta) := z_{n+1}(\phi_n^{-1}(s,\theta))$ is a C^2 global solution of (12). Moreover, $\rho_{n+1}(0,\theta_0) = z_{n+1}(\phi_n^{-1}(0,\theta_0)) = z_{n+1}(0;\theta_0) = \rho_0(\theta_0)$. $\square$

Corollary 3.3. *If ρ_n is a C^2 global solution of (12) together with (13) then $\frac{\partial \rho_{n+1}}{\partial t}(\phi_n(t,\theta_0))$ and $\frac{\partial \rho_{n+1}}{\partial \theta}(\phi_n(t,\theta_0))$ satisfy the characteristic equations (18) and (19) together with the initial data*

$$p_{n+1}(0;\theta_0) = -\frac{\partial \rho_0}{\partial \theta}(\theta_0)\left(k\int_0^{2\pi}\rho_0(\hat{\theta})\sin(\hat{\theta}-\theta_0)\ d\hat{\theta}\right)+\rho_0(\theta_0)\left(k\int_0^{2\pi}\rho_0(\hat{\theta})\cos(\hat{\theta}-\theta_0)\ d\hat{\theta}\right), \quad (28)$$

$$q_{n+1}(0;\theta_0) = \rho_0'(\theta_0), \quad (29)$$

respectively. That is $\frac{\partial \rho_{n+1}}{\partial t}(t,\theta) = p_{n+1}(\phi_n^{-1}(t,\theta))$ *and* $\frac{\partial \rho_{n+1}}{\partial \theta}(t,\theta) = q_{n+1}(\phi_n^{-1}(t,\theta))$

Proof. See Theorem 1, page 99, section 3.2 and (30), page 104 of [3]. □

Lemma 3.4. *If ρ_n and ρ_0 are nonnegative and 2π-periodic in θ, and $\int_0^{2\pi}\rho_n(t,\theta)d\theta =$ 1, then*

i $\rho_{n+1}(t,\theta) \geq 0$.

ii $\rho_{n+1}(t,\theta)$ *is 2π-periodic in θ.*

iii $\int_0^{2\pi}\rho_{n+1}(t,\theta)d\theta = 1$.

Proof.

i Since $\rho_{n+1}(t,\theta) = z_{n+1}(\phi_n^{-1}(t,\theta))$, where z_{n+1} denotes the third component of the solution of $(15)-(17)$ under the initial data $(0, y_{n+1}(0), z_{n+1}(0)) = (0,\theta_0,\rho_0(\theta_0))$, by (26), $\rho_{n+1}(t,\theta) \geq 0$, provided $\rho_0(\theta) \geq 0$.

ii Given any t and θ, we evaluate $\rho_{n+1}(t,\theta)$ and $\rho_{n+1}(t,\theta+2\pi)$ using characteristics. First note that if $(s, y_{n+1}(s;\theta_0))$ is the unique characteristic curve through (t,θ), i.e. if $y_{n+1}(t;\theta_0) = \theta$, then $(s, y_{n+1}(s;\theta_0)+2\pi)$ is the unique characteristic curve through $(t, \theta + 2\pi)$. In fact, $(s, y_{n+1}(s; \theta_0 + 2\pi)) = (s, y_{n+1}(s; \theta_0) + 2\pi)$. Indeed,

$$\begin{aligned}
\frac{d}{ds}(y_{n+1}(s;\theta_0+2\pi)) &= k\int_0^{2\pi} \rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s;\theta_0+2\pi))d\hat{\theta} \\
\frac{d}{ds}(y_{n+1}(s;\theta_0)+2\pi) &= k\int_0^{2\pi} \rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s;\theta_0))d\hat{\theta} \\
&= k\int_0^{2\pi} \rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s;\theta_0)+2\pi)d\hat{\theta}
\end{aligned}$$

That is, $y_{n+1}(s; \theta_0 + 2\pi)$ and $y_{n+1}(s; \theta_0) + 2\pi$ solve the same initial value problem, so

$$y_{n+1}(s;\theta_0+2\pi) = y_{n+1}(s;\theta_0)+2\pi, \tag{30}$$

by uniqueness. In other words, $y_{n+1}(s; \theta_0)$ *mod* 2π is 2π-periodic in θ. Since cosine and ρ_0 are 2π-periodic in θ and $y_{n+1}(s; \theta_0 + 2\pi) = y_{n+1}(s; \theta_0) + 2\pi$ we see from (26) that $z_{n+1}(s; \theta_0) = z_{n+1}(s; \theta_0 + 2\pi)$ for all s. Finally,

$$\begin{aligned}
\rho_{n+1}(t,\theta) &= z_{n+1}(t;\theta_0) \\
&= z_{n+1}(t;\theta_0+2\pi) \\
&= \rho_{n+1}(\phi_n(t;\theta_0+2\pi)) \\
&= \rho_{n+1}(t, y_{n+1}(t;\theta_0+2\pi)) \\
&= \rho_{n+1}(t, y_{n+1}(t;\theta_0)+2\pi) \\
&= \rho_{n+1}(t,\theta+2\pi)
\end{aligned}$$

iii Note that (12) can be simplified as

$$\frac{\partial\rho_{n+1}}{\partial t}(t,\theta) = -\frac{\partial}{\partial\theta}\left(k\rho_{n+1}(t,\theta)\int_0^{2\pi}\rho_n(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta}\right). \tag{31}$$

Integrating both sides with respect to θ from 0 to 2π, we get

$$\frac{\partial}{\partial t}\int_0^{2\pi}\rho_{n+1}(t,\theta)d\theta = -\left(k\rho_{n+1}(t,\theta)\int_0^{2\pi}\rho_n(t,\hat{\theta})\sin(\hat{\theta}-\theta)d\hat{\theta}\right)\Bigg|_{\theta=0}^{\theta=2\pi} = 0. \quad (32)$$

Thus $\int_0^{2\pi}\rho_{n+1}(t,\theta)d\theta$ is constant with respect to t, and so by (11) iii holds.

□

Corollary 3.5. *If $\rho_0(\theta)$ and $\rho_n(t,\theta)$ are C^2 in all arguments, then a global solution of (12) together with (13) is unique.*

Proof. Suppose $u_1(t,\theta)$ and $u_2(t,\theta)$ are C^2, 2π-periodic global solutions of (12). Consider the difference of these solutions squared and integrate with respect to θ such that for all t,

$$\begin{aligned}
\frac{d}{dt}\int_0^{2\pi}(u_1-u_2)^2d\theta &= \int_0^{2\pi}2(u_1-u_2)\frac{\partial(u_1-u_2)}{\partial t}d\theta \\
&= \int_0^{2\pi}2(u_1-u_2)\left[-\frac{\partial(u_1-u_2)}{\partial\theta}k\int_0^{2\pi}\rho_n(t,\hat{\theta})sin(\hat{\theta}-\theta)d\hat{\theta} + k(u_1-u_2)\int_0^{2\pi}\rho_n(t,\hat{\theta})cos(\hat{\theta}-\theta)d\hat{\theta}\right]d\theta \\
&= \int_0^{2\pi}-\frac{\partial(u_1-u_2)^2}{\partial\theta}k\int_0^{2\pi}\rho_n(t,\hat{\theta})sin(\hat{\theta}-\theta)d\hat{\theta}d\theta + k\int_0^{2\pi}(u_1-u_2)^2\int_0^{2\pi}\rho_n(t,\hat{\theta})cos(\hat{\theta}-\theta)d\hat{\theta}d\theta
\end{aligned}$$

now we use integration by parts on the first integral,

$$\mu = k\int_0^{2\pi}\rho_n(t,\hat{\theta})sin(\hat{\theta}-\theta)d\hat{\theta}; \quad d\mu = -k\int_0^{2\pi}\rho_n(t,\hat{\theta})cos(\hat{\theta}-\theta)d\hat{\theta}d\theta$$

$$d\lambda = -\frac{\partial(u_1-u_2)^2}{\partial\theta}; \quad \lambda = -(u_1-u_2)^2$$

Since sine, ρ_n, u_1, and u_2 are 2π-periodic, $\mu\lambda\big|_0^{2\pi} = 0$ and first integral reduces to

$$-\int\lambda d\mu = -k\int_0^{2\pi}(u_1-u_2)^2\int_0^{2\pi}\rho_n(t,\hat{\theta})cos(\hat{\theta}-\theta)d\hat{\theta}d\theta,$$

so

$$\frac{d}{dt}\int_0^{2\pi}(u_1-u_2)^2d\theta = 0.$$

Therefore $u_1 = u_2$ almost everywhere. Thus we have uniqueness of global, C^2 solutions to (12) and (13).

□

Now we consider the existence and uniqueness of solutions p_{n+1} and q_{n+1} of (18)-(19). For this we reintroduce the following initial data:

$$p_{n+1}(0;\theta_0) = -\frac{\partial\rho_0}{\partial\theta}(\theta_0)\left(k\int_0^{2\pi}\rho_0(\hat{\theta})\sin(\hat{\theta}-\theta_0)\ d\hat{\theta}\right)+\rho_0(\theta_0)\left(k\int_0^{2\pi}\rho_0(\hat{\theta})\cos(\hat{\theta}-\theta_0)\ d\hat{\theta}\right), \quad (33)$$

$$q_{n+1}(0;\theta_0) = \rho_0'(\theta_0). \quad (34)$$

Lemma 3.6. *If $\rho_0(\theta)$ and $\rho_n(t,\theta)$ are C^2 in all arguments, there exists a unique, global solution to (18)-(19) subject to initial conditions (28)-(29), continuously differentiable with respect to s and θ_0.*

Proof. Let $[G_{n+1}]_4(s,p,q)$ and $[G_{n+1}]_5(s,p,q)$ be defined by the right-hand side of (18) and (19), respectively. Since, $x_{n+1}(s;s_0,X_0)$, $y_{n+1}(s;s_0,X_0)$, and $z_{n+1}(s;s_0,X_0)$ are C^2 by Lemma 3.1 and ρ_n is C^2 in each of its arguments by hypothesis, we see that $[G_{n+1}]_5$ is C^2 in all variables by the Leibniz Rule. To see that $[G_{n+1}]_4$ is C^2 despite its dependence on $\frac{\partial\rho_n}{\partial s}$, note that in case $n=0$, $\rho_n(x,\hat{\theta})=\rho_0(\hat{\theta})$, so $\frac{\partial\rho_n}{\partial s}\equiv 0$, and $G_4(s,p_{n+1},q_{n+1})$ is C^2 is all variables. In case $n\geq 1$,

$$\frac{\partial}{\partial s}\rho_n(s,\hat{\theta}) = \frac{\partial}{\partial\hat{\theta}}\left(\rho_n(s,\hat{\theta})k\int_0^{2\pi}\rho_{n-1}(s,\hat{\hat{\theta}})\sin(\hat{\hat{\theta}}-\hat{\theta})d\hat{\hat{\theta}}\right). \quad (35)$$

Substituting the right hand side of (35) into (18) and integrating by parts we have,

$$\begin{aligned}\dot{p}_{n+1}(s) &= -q_{n+1}(s)k\int_0^{2\pi}\rho_n(s,\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))\int_0^{2\pi}\rho_{n-1}(s,\hat{\hat{\theta}})\sin(\hat{\hat{\theta}}-\hat{\theta})d\hat{\hat{\theta}}d\hat{\theta} \quad (36)\\ &- z_{n+1}(s)k\int_0^{2\pi}\rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s))\int_0^{2\pi}\rho_{n-1}(s,\hat{\hat{\theta}})\sin(\hat{\hat{\theta}}-\hat{\theta})d\hat{\hat{\theta}}d\hat{\theta}\\ &+ p_{n+1}(s)k\int_0^{2\pi}\rho_n(s,\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s))d\hat{\theta}.\end{aligned}$$

So that $[G_{n+1}]_4$ is also C^2. In particular, $[G_{n+1}]_4$ and $[G_{n+1}]_5$ are linear in p_{n+1} and q_{n+1} and hence uniformly Lipschitz continuous on $[0,T]\times\mathbb{R}^2$ with respect to p_{n+1}

and q_{n+1}. It follows that there exists a unique global solution of (18)-(19) subject to (28)-(29) (See corollary 2.6, p.41 of [7]). Moreover a solution of (18)-(19) is C^2 with respect to s and its initial data, $p_{n+1}(0), q_{n+1}(0)$. Since this data is a continuously differentiable function of $\rho_0(\theta_0)$ and $\frac{\partial \rho_0}{\partial \theta}(\theta_0)$, which are continuously differentiable with respect to θ_0, we see that the solution of (18)-(19) subject to (28)-(29) is continuously differentiable with respect to s and θ_0. □

Alternatively, we can give an explicit formula for the solution of (18)-(19) subject to (28)-(29). In particular, the solution of (19) subject to (29) is

$$q_{n+1}(t;\theta_0) = e^{\int_0^t 2f_{n+1}(s;\theta_0)ds}\left[\int_0^t g_{n+1}(s;\theta_0)z_{n+1}(s;\theta_0)e^{-\int_0^s 2f_{n+1}(\nu;\theta_0)d\nu}ds + \rho_0'(\theta_0)\right] \quad (37)$$

$$= z_{n+1}(t;\theta_0)^2\left[\int_0^t \frac{g_{n+1}(s;\theta_0)}{z_{n+1}(s;\theta_0)}ds + \frac{\rho_0'(\theta_0)}{\rho_0(\theta_0)^2}\right], \quad (38)$$

provided $z_{n+1}(0;\theta_0) = \rho_0(\theta_0) \neq 0$, where

$$f_{n+1}(s;\theta_0) := k\int_0^{2\pi} \rho_n(s,\hat{\theta})\cos(\hat{\theta} - y_{n+1}(s;\theta_0))d\hat{\theta} = \frac{\partial [G_{n+1}]_2}{\partial y}(s, y_{n+1}(s,\theta))$$

and

$$g_{n+1}(s;\theta_0) = k\int_0^{2\pi} \rho_n(s,\hat{\theta})\sin(\hat{\theta} - y_{n+1}(s;\theta_0))d\hat{\theta} := [G_{n+1}]_2(s, y_{n+1}(s,\theta)).$$

And, in case $z_{n+1}(0;\theta_0) = \rho_0(\theta_0) = 0$,

$$q_{n+1}(t;\theta_0) = \rho_0'(\theta_0)e^{\int_0^t 2f_{n+1}(s;\theta_0)ds}. \quad (39)$$

Hence,

$$p_{n+1}(t;\theta_0) = e^{\int_0^t f_{n+1}(s;\theta_0)ds}\left[\int_0^t c_{n+1}(s;\theta_0)e^{-\int_0^s f_{n+1}(\nu;\theta_0)d\nu}ds + p_{n+1}(0;\theta_0)\right] \quad (40)$$

$$= z_{n+1}(t;\theta_0)\left[\int_0^t \frac{c_{n+1}(s;\theta_0)}{z_{n+1}(s;\theta_0)}ds + \frac{p_{n+1}(0;\theta_0)}{\rho_0(\theta_0)}\right]. \quad (41)$$

Where (41) holds provided $z_{n+1}(0;\theta_0) = \rho_0(\theta_0) \neq 0$, and

$$\begin{aligned} c_{n+1}(s;\theta_0) = \; & -q_{n+1}(s;\theta_0)k\int_0^{2\pi}\rho_n(s;\hat{\theta})\cos(\hat{\theta}-y_{n+1}(s;\theta_0))\int_0^{2\pi}\rho_{n-1}(s,\hat{\hat{\theta}})\sin(\hat{\hat{\theta}}-\hat{\theta})d\hat{\hat{\theta}}d\hat{\theta} \\ - \; & z_{n+1}(s;\theta_0)k\int_0^{2\pi}\rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s;\theta_0))\int_0^{2\pi}\rho_{n-1}(s,\hat{\hat{\theta}})\sin(\hat{\hat{\theta}}-\hat{\theta})d\hat{\hat{\theta}}d\hat{\theta}. \end{aligned} \tag{42}$$

Now, from (26) and Lemma 3.5 parts i and iii we see that for all θ_0.

$$|z_{n+1}(t;\theta_0)| \;\leq\; \rho_0(\theta_0)\mathrm{e}^{kt} \tag{43}$$

$$|z_{n+1}(t;\theta_0)| \;\geq\; \rho_0(\theta_0)\mathrm{e}^{-kt} \tag{44}$$

$$|f_{n+1}(t;\theta_0)| \;\leq\; k \tag{45}$$

$$|g_{n+1}(t;\theta_0)| \;\leq\; k \tag{46}$$

$$\begin{aligned} |q_{n+1}(t;\theta_0)| \;&\leq\; |z_{n+1}(t;\theta_0)^2|\left(\left|\int_0^t \frac{g_{n+1}(s;\theta_0)}{z_{n+1}(s;\theta_0)}ds\right| + \left|\frac{\rho_0'(\theta_0)}{\rho_0(\theta_0)^2}\right|\right) \\ &=\; e^{\int_0^t 2f_{n+1}(s;\theta_0)\,ds}\left(\rho_0(\theta_0)\int_0^t\left|\frac{g_{n+1}(s;\theta_0)}{e^{\int_0^s f_{n+1}(\tau;\theta_0)\,d\tau}}\right|ds + |\rho_0'(\theta_0)|\right) \\ &\leq\; e^{2kt}\left(\rho_0(\theta_0)e^{kt} + |\rho_0'(\theta_0)|\right) \quad z_{n+1}(0) \neq (0) \end{aligned} \tag{47}$$

$$|q_{n+1}(t;\theta_0)| \;\leq\; \rho_0'(\theta_0)\mathrm{e}^{2kt}, \quad z_{n+1}(0) = (0). \tag{48}$$

Note that since ρ_0 is C^2 and 2π-periodic in θ, (43) provides a uniform bound, $B_z(T)$, on $\rho_{n+1}(t,\theta) = z_{n+1}(\phi_n^{-1}(t,\theta))$. Also, by Corollary 3.3, (47)-(48) provide a uniform bound $B_q(T)$ on $\frac{\partial\rho_{n+1}}{\partial\theta}(t,\theta) = q_{n+1}(\phi_n^{-1}(t,\theta))$. That is, $\{\rho_n\}_{n=1}^{\infty}$ and $\left\{\frac{\partial\rho_n}{\partial\theta}\right\}_{n=1}^{\infty}$ are uniformly bounded on $[0,T]\times[0,2\pi]$. It follows from the PDE (12) that

$$\left|\frac{\partial\rho_n}{\partial t}(t,\theta)\right| \;\leq\; k(B_q(T)+B_z(T)), \tag{49}$$

for all $n \in \mathbb{N}$ and $(t,\theta) \in [0,T]\times\mathbb{R}$. Hence, the sequence $\{\rho_n\}_{n=0}^{\infty}$ is uniformly bounded and equicontinuous. We have the following lemma:

Lemma 3.7. *The sequence $\{\rho_n\}_{n=0}^{\infty}$ is uniformly bounded and equicontinuous on $[0,T]\times[0,2\pi]$.*

In establishing a global solution of (10), we would like to show also that $\frac{\partial \rho_n}{\partial t}$ and $\frac{\partial \rho_n}{\partial \theta}$ are uniformly bounded and equicontinuous on $[0, T] \times [0, 2\pi]$. Toward this goal, we have already established uniform bounds. For equicontinuity, it suffices to show $D\frac{\partial \rho_n}{\partial t} = \left[\frac{\partial^2 \rho_n}{\partial t^2}, \frac{\partial^2 \rho_n}{\partial t \partial \theta}\right]$ and $D\frac{\partial \rho_n}{\partial \theta} = \left[\frac{\partial^2 \rho_n}{\partial t \partial \theta}, \frac{\partial^2 \rho_n}{\partial \theta^2}\right]$ are uniformly bounded on $[0, T] \times [0, 2\pi]$, for all $T > 0$. We have established the existence of a global C^2 solution $\rho_n(t, \theta)$ of (12). Moreover, by Corollary 3.3, it can be verified that along the projected characteristics (15)-(16),

$$\frac{\partial \rho_{n+1}}{\partial t}(t, y_{n+1}(t; \theta_0)) = p_{n+1}(t; \theta_0) \tag{50}$$

$$\frac{\partial \rho_{n+1}}{\partial \theta}(t, y_{n+1}(t; \theta_0)) = q_{n+1}(t; \theta_0) \tag{51}$$

Thus,

$$\begin{aligned} D\left[\frac{\partial \rho_n}{\partial t}(t, y_{n+1}(t; \theta_0))\right] &= \left[\frac{\partial^2 \rho_n}{\partial t^2}(t, y_{n+1}(t; \theta_0)), \frac{\partial^2 \rho_n}{\partial t \partial \theta}(t, y_{n+1}(t; \theta_0))\right] \begin{bmatrix} 1 & 0 \\ G_2(s, y_{n+1}(s; \theta_0) & \exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu; \theta_0)) d\nu\right) \end{bmatrix} \\ &= \left[\dot{p}_{n+1}(t; \theta_0), \frac{\partial p_{n+1}}{\partial \theta_0}(t; \theta_0)\right], \end{aligned} \tag{52}$$

that is,

$$\begin{aligned} &\left[\frac{\partial^2 \rho_n}{\partial t^2}(t, y_{n+1}(t; \theta_0)), \frac{\partial^2 \rho_n}{\partial t \partial \theta}(t, y_{n+1}(t; \theta_0))\right] = \\ &\qquad \frac{1}{\exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu; \theta_0)) d\nu\right)} \left[\dot{p}_{n+1}(t; \theta_0), \frac{\partial p_{n+1}}{\partial \theta_0}(t; \theta_0)\right] \begin{bmatrix} \exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu; \theta_0)) d\nu\right) & 0 \\ -G_2(s, y_{n+1}(s; \theta_0)) & 1 \end{bmatrix} \end{aligned} \tag{53}$$

Also,

$$\begin{aligned} &D\left[\frac{\partial \rho_n}{\partial \theta}(t, y_{n+1}(t; \theta_0))\right] \\ &\qquad = \left[\frac{\partial \rho_n}{\partial t \partial \theta}(t, y_{n+1}(t; \theta_0)), \frac{\partial^2 \rho_n}{\partial \theta^2}(t, y_{n+1}(t; \theta_0))\right] \begin{bmatrix} 1 & 0 \\ G_2(s, y_{n+1}(s; \theta_0) & \exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu; \theta_0)) d\nu\right) \end{bmatrix} \\ &\qquad = \left[\dot{q}_{n+1}(t; \theta_0), \frac{\partial q_{n+1}}{\partial \theta_0}(t; \theta_0)\right], \end{aligned} \tag{54}$$

that is,

$$\left[\frac{\partial \rho_n}{\partial t \partial \theta}(t, y_{n+1}(t;\theta_0)),\ \frac{\partial^2 \rho_n}{\partial \theta^2}(t, y_{n+1}(t;\theta_0))\right] = \frac{1}{\exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right)}\left[\dot{q}_{n+1}(t;\theta_0),\ \frac{\partial q_{n+1}}{\partial \theta_0}(t;\theta_0)\right]\begin{bmatrix}\exp\left(\int_0^s \frac{\partial G_2}{\partial y_{n+1}}(\nu, y_{n+1}(\nu;\theta_0))d\nu\right) & 0 \\ -G_2(s, y_{n+1}(s;\theta_0)) & 1\end{bmatrix} \tag{55}$$

We will proceed by showing the the total derivative of $\frac{\partial \rho_n}{\partial \theta}$ (that is, (55)) is uniformly bounded, independent of n, on $[0,T] \times [0, 2\pi]$. It will then follow that the sequence $\left\{\frac{\partial \rho_n}{\partial \theta}\right\}_{n=0}^{\infty}$, is equicontinuous. We have already seen that $D\phi_n^{-1}$ is uniformly bounded independent of n, so it remains to show that the partial derivatives $\frac{\partial q_{n+1}}{\partial \theta_0}(t;\theta_0)$ and $\dot{q}_{n+1}(t;\theta_0)$ are uniformly bounded for $(t,\theta) \in [0,T] \times \mathbb{R}$.

If we differentiate (16) in terms of the initial condition θ_0, we generate

$$\begin{aligned}\frac{\partial \dot{y}_{n+1}}{\partial \theta_0}(s;\theta_0) &= \frac{\partial}{\partial \theta_0}\left(k\int_0^{2\pi} \rho_n(s,\hat{\theta}) sin(\hat{\theta} - y_{n+1}(s;\theta_0))d\hat{\theta}\right) \\ &= -\frac{\partial y_{n+1}}{\partial \theta_0}(s;\theta_0)k\int_0^{2\pi} \rho_n(s,\hat{\theta}) cos(\hat{\theta} - y_{n+1}(s;\theta_0))d\hat{\theta},\end{aligned} \tag{56}$$

where we have used $\frac{\partial}{\partial \theta_0}x_{n+1}(t;\theta_0) \equiv 0$.

Solving for $\frac{\partial \dot{y}_{n+1}}{\partial \theta_0}(s;\theta_0)$, then we find that

$$\frac{\partial \dot{y}_{n+1}}{\partial \theta_0}(s;\theta_0) = e^{-\int_0^t f(s;\theta_0)ds} \tag{57}$$

Since $-k \le f_{n+1}(s;\theta_0) \le k$,

$$\left|\frac{\partial y_{n+1}}{\partial \theta_0}(s;\theta_0)\right| \le e^{kT}. \tag{58}$$

Now

$$z_{n+1}(s;\theta_0) = \rho_0(\theta_0)\exp\left(\int_0^s f_{n+1}(\tau;\theta_0)\ d\tau\right),$$

so

$$\frac{\partial z_{n+1}}{\partial\theta_0}(s;\theta_0)=\rho_0'(\theta_0)\exp\left(\int_0^s f_{n+1}(\tau;\theta_0)\ d\tau\right)+z_{n+1}(s;\theta_0)\frac{\partial y_{n+1}}{\partial\theta_0}(s;\theta_0)(s;\theta_0)\int_0^s g_{n+1}(\tau;\theta_0)\ d\tau\ , \tag{59}$$

and

$$\left|\frac{\partial z_{n+1}}{\partial\theta_0}(s;\theta_0)\right|\leq|\rho_0'(\theta_0)|e^{kT}+kT\rho_0(\theta_0)e^{2kT}, \tag{60}$$

where we have previously defined $g_{n+1}(s;\theta_0)=k\int_0^{2\pi}\rho_n(s,\hat{\theta})\sin(\hat{\theta}-y_{n+1}(s;\theta_0))d\hat{\theta}$. Now we look to uniformly bound $\frac{\partial q_{n+1}}{\partial\theta_0}(t;\theta_0)$ with respect to time. From (38), if $z_{n+1}(0;\theta_0)=\rho_0(\theta_0)\neq 0$,

$$q_{n+1}(t;\theta_0)=z_{n+1}(t;\theta_0)^2\left(\int_0^t\frac{g_{n+1}(s;\theta_0)}{z_{n+1}(s;\theta_0)}ds+\frac{\rho_0'(\theta_0)}{\rho_0(\theta_0)^2}\right) \tag{61}$$

$$=e^{\int_0^t 2f_{n+1}(s;\theta_0)\ ds}\left[\rho_0(\theta_0)\int_0^t\frac{g_{n+1}(s;\theta_0)}{e^{\int_0^s f_{n+1}(\tau;\theta_0)\ d\tau}}+\rho_0'(\theta_0)\right] \tag{62}$$

Since the derivatives of f_{n+1} and g_{n+1} with respect to θ_0 are continuous and bounded in magnitude,

$$\left|e^{\int_0^s f_{n+1}(\tau;\theta_0)\ d\tau}\right|\geq e^{-ks}, \tag{63}$$

and $\rho_0(\theta_0)$ is C^2 and 2π-periodic, we can see from the Leibniz and quotient rules that $\frac{\partial q_{n+1}}{\partial\theta_0}(s;\theta_0)$ is bounded independent of n, on $[0,T]\times\mathbb{R}$.

Or, if $z_{n+1}(0;\theta_0)=0$, then

$$q_{n+1}(t;\theta_0)=\rho_0'(\theta_0)\mathrm{e}^{\int_0^t 2f_{n+1}(s;\theta_0)ds},$$

so again we see that $\frac{\partial q_{n+1}}{\partial\theta_0}(s;\theta_0)$ is bounded on $[0,T]\times\mathbb{R}$ independent of n.

We will also notice that from (19) and the bounds (43)-(47), $\dot{q}_n(t)$ is uniformly bounded on $[0,T] \times \mathbb{R}$. Hence the sequence $\left\{\frac{\partial \rho_n}{\partial \theta}\right\}_{n=0}^{\infty}$ is uniformly bounded and equicontinuous on $[0,T] \times \mathbb{R}$. From equation (12), we see that $\left\{\frac{\partial \rho_n}{\partial t}\right\}_{n=0}^{\infty}$, is uniformly bounded and equicontinuous on $[0,T] \times \mathbb{R}$. We have established the following lemma:

Lemma 3.8. *The sequences* $\left\{\frac{\partial \rho_n}{\partial t}\right\}_{n=0}^{\infty}$ *and* $\left\{\frac{\partial \rho_n}{\partial \theta}\right\}_{n=0}^{\infty}$ *are uniformly bounded and equicontinuous on* $[0,T] \times [0,2\pi]$.

From Lemmas 3.8 and 3.9 and by the Arzela-Ascoli Theorem [4], we can choose a subsequence on which $\{\rho_n\}_{n=0}^{\infty}$, $\left\{\frac{\partial \rho_n}{\partial \theta}\right\}_{n=0}^{\infty}$, and $\left\{\frac{\partial \rho_n}{\partial t}\right\}_{n=0}^{\infty}$ converge uniformly on $[0,T] \times [0,2\pi]$. Furthermore, uniform convergence of the derivatives together with convergence of the function gives the limit of the derivative is the derivative of the limit. That is:

$$\lim_{n\to\infty} \rho_{n_k}(t,\theta) = \rho(t,\theta) \tag{64}$$

$$\lim_{n\to\infty} \frac{\partial}{\partial t}\rho_{n_k}(t,\theta) = \frac{\partial}{\partial t}\lim_{n\to\infty} \rho_{n_k}(t,\theta) = \frac{\partial \rho}{\partial t}(t,\theta) \tag{65}$$

$$\lim_{n\to\infty} \frac{\partial}{\partial \theta}\rho_{n_k}(t,\theta) = \frac{\partial}{\partial \theta}\lim_{n\to\infty} \rho_{n_k}(t,\theta) = \frac{\partial \rho}{\partial \theta}(t,\theta) \tag{66}$$

Uniform convergence enables us to pull limits through the integral in equation (10), so that this subsequence uniformly converges to a C^1 solution of equation (10) on $[0,T] \times [0,2\pi]$. Since $T > 0$ was arbitrary, there exists a global C^1 solution of equation (10) on $\mathbb{R} \times [0,2\pi]$. We have established the following theorem:

Theorem 1. *There exists a global* C^1 *solution of equation (10) on* $\mathbb{R} \times [0,2\pi]$.

CHAPTER 4

Conclusion

Here we have proven there exist global solutions to the infinite-dimensional Kuramoto model, subject to C^2 initial data. To the best of our knowledge, our approach to studying existence of solutions for this model is novel. Hence our existence result differs from the previous result of Ott and Antonsen, as their initial data and the solution ansatz is given by a special type of Fourier series.

We consider the method through which we developed these results to be as valuable as the results themselves. In future work we will show that this method also yields an alternative characterization of the solution in terms of a limiting projected characteristic. Moreover this method has the potential to yield new results about stability of steady states and can provide a basis for new methods of numerical simulation. It is important to note that this in only the first step in applying the theory of characteristics to the Kuramoto model. Whereas the Ott-Antonsen ansatz is applicable to models with distributed natural velocities, so far we have only considered the model without natural velocities. In the future, we plan to adapt the characteristic method to address the case where natural velocities vary between oscillators. Since previous results for the Kuramoto model with natural velocities include conditions on the distribution of natural velocities, we are excited to see if the method of analysis developed here can also yield new results for this more general version of the

Kuramoto model.

APPENDIX E

Theorem 5.9. *Leibniz Rule:*

Consider

$$\int_a^b f(t,x)dx \text{ where } t \in I \tag{E.67}$$

If f and $\frac{\partial f}{\partial t}$ are continuous with respect to x on the interval $[a,b]$ and continuous with respect to t on $\mathbb{R}$. Then

$$\frac{d}{dt}\int_a^b f(t,x)dx = \int_a^b \frac{\partial}{\partial t} f(t,x)dx \tag{E.68}$$

Proof. Consider the difference quotient

$$\frac{1}{k}\int_a^b f(t+k,x) - f(t,x)dx \tag{E.69}$$

Then by the Mean Value Theorem

$$\frac{1}{k}\int_a^b f(t+k,x) - f(t,x)dx = \int_a^b \frac{\partial}{\partial t} f(\phi(x,k),x)dx, \tag{E.70}$$

where $\phi(x,k)$ is between t and $t+k$. Notice that $\frac{\partial f}{\partial t}$ is continuous on $\mathbb{R} \times [a,b]$, so $\frac{\partial f}{\partial t}$ is uniformly continuous on $[t-1,t+1] \times [a,b]$. Therefore given any $\epsilon > 0$ there exists $\delta > 0$ so that if $1 > \delta > k > 0$ then

$$\left|\frac{\partial}{\partial t} f(\phi(k,x),x) - \frac{\partial}{\partial t} f(t,x)\right| < \frac{\epsilon}{b-a} \tag{E.71}$$

for all $x \in [a, b]$. Thus,

$$\left|\frac{1}{k}\int_a^b f(t+k,x) - f(t,x)dx - \int_a^b \frac{\partial}{\partial t} f(t,x)dx\right| \tag{E.72}$$

$$= \left|\int_a^b \frac{\partial}{\partial t} f(\phi(x,k),x) - \frac{\partial}{\partial t} f(t,x)dx\right| \tag{E.73}$$

$$\leq \int_a^b \left|\frac{\epsilon}{b-a}\right| dx \tag{E.74}$$

$$= \frac{\epsilon}{b-a}(b-a) = \epsilon \tag{E.75}$$

Since ϵ is arbitrary, we have proven Leibniz Rule. □

References

[1] J.D. Crawford. Scaling and singularities in the entrainment of globally coupled oscillators. *Physical Review Letters*, 74(21), 1995.

[2] J.D. Crawford and Davies K.T.R. Synchronization of globally coupled phase oscillators: singularities and scaling for general couplings. *Physica*, 125(1), 1999.

[3] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island, 2002.

[4] Avner Friedman. *Foundations of Modern Analysis*. Dover Publications, Mineola, New York, 1982.

[5] Yoshiki Kuramoto. *Chemical Oscillations, Waves, and Turbulence*. Dover Publications, Inc., Mineola, New York, 1984.

[6] Edward Ott and Thomas M. Antonsen. Low dimensional behavior of large systems of globally coupled oscillators. *Chaos*, 18, 2008.

[7] Gerald Teschl. *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, Providence, Rhode Island, 2012.

[8] G. A. Gottwald W. Yue, L. D. Smith. Model reduction for the Kuramoto-Sakaguchi model: The importance of non-entrained rogue oscillators. *Physical Review*, 101, 2020.