ISOPERIMETRIC CONSTANTS IN PLANAR GRAPHS WITH HYPERBOLIC PROPERTIES

by

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A Thesis Submitted in Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematical Sciences

Middle Tennessee State University

2014

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ABSTRACT

Isoperimetric inequalities date back to ancient Greece where figures with equal perimeters but different shapes were compared (Zenodorus, On Isoperimetric Figures). The original problem was to maximize the area contained within a curve of specified length. In Euclidean geometry the result is a circle. This can be generalized to shapes on non-Euclidean surfaces as well as for higher dimensions where we seek to maximize the hyperdimensional volume respective to the hyperdimensional surface area.

The subject of this thesis is isoperimetric constants in planar graphs with hyperbolic properties. We first analyze isoperimetric constants in the flat plane which has curvature 0 and then give an overview of two isoperimetric constants that give a hyperbolicity criterion for infinite vertex-regular, face-regular planar graphs. Finally we extend the concept to general planar graphs with hyperbolic properties and establish that the same constants serve as a lower bound.

DEDICATION

This thesis is dedicated to my parents, who have taught me, encouraged me and supported me in my life. Thank you for all your patience, love and unconditional support.

ACKNOWLEDGMENTS

This thesis would not have been possible without the support and guidance of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of this study.

First and foremost, my utmost gratitude to Dr. Xiaoya Zha whose encouragement, guidance and support from the initial to the final level enabled me to complete this project. To Dr. James Hart, Dr. Zachariah Sinkala and Dr. Chris Stephens who taught me how to express my arguments using the language of mathematics. To Dr. Rebecca Calahan and Dr. Don Nelson who took a chance in granting me an assistantship and have supported me ever since.

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CHAPTER 1

INTRODUCTION

1.1 Isoperimetric Problems

The original isoperimetric problem dates back to ancient Greece and can be stated in the following manner: Among all simple closed curves with given length, which one bounds the largest area? Equivalently one could ask: Among all planar shapes with same area which one has the smallest perimeter? Zenodorus (an ancient Greek mathematician) gave us our first insight into proving that it was in fact the circle that minimized the area, however the complete problem was not solved until 1841 by Jacob Steiner (who also left a flaw in his proof which was later amended). This leads to the following inequalities:

Let R be any planar shape with fixed area: $Area(R)$, then

$$\frac{Area(R)}{Perimeter(R)} \leq c$$

$$\frac{Perimeter(R)}{Area(R)} \geq \frac{1}{c}$$

where the equality holds precisely when R is a circle. Here we say that $\frac{1}{c}$ is the isoperimetric constant.

Definition 1.1 *For a planar shape R we say that $\frac{1}{c} = \frac{Perimeter(R)}{Area(R)}$ is the isoperimetric number of R.*

In this thesis we will study the discrete version of this inequality. To this purpose we will present an equivalent definition for it at end of this chapter after having established some necessary definitions. In Chapter 3 we will further elaborate on this definition so that we may discuss isoperimetric inequalities of graphs with hyperbolic properties, that is, those which can be embedded in a hyperbolic plane.

1.2 Graph Theory Concepts and Definitions

A *graph* is a pair $G = (V, E)$ such that the elements of V are the *vertices* of the graph G and the elements of E are the *edges* of the graph G. The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information of which pairs of vertices form an edge and which do not. [2]

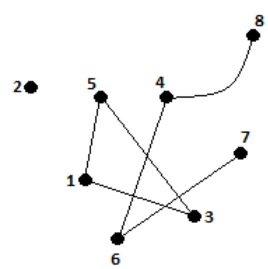

Figure 1: $V = \{1, ..., 8\}$, $E = \{\{1, 3\}, \{1, 5\}, \{3, 5\}, \{4, 6\}, \{4, 8\}, \{6, 7\}\}$

The vertex set of a graph is denoted $V(G)$ and its edge set $E(G)$. The order of a graph G, denoted $|G|$, represents the number of vertices in the graph. Similarly the number of edges is denoted by $||G||$. Graphs therefore may be finite or infinite according to their order.

A vertex v is said to be *incident* with an edge e if $v \in V(e)$, hence an edge may be defined by its endpoints $x, y \in V(G)$ and may be written as xy (or yx). If xy is an edge of G, we say vertices $x, y \in V(G)$ are neighbors. Two edges $e, f \in E(G)$ are said to be *adjacent* if they have an endpoint in common.

The union of two graphs G and K represents the union of their respective vertex and edge sets and is denoted $G \cup K := \{V(G) \cup V(K), E(G) \cup E(K)\}$. Similarly the intersect of two graphs is $G \cap K := \{V(G) \cap V(K), E(G) \cap E(K)\}$. If $V(K) \subseteq V(G)$ and $E(K) \subseteq E(G)$ then K is a *subgraph* of G and we write $K \subseteq G$. If $K \subseteq G$ and $K \neq G$ then K is a *proper subgraph* of G. If $K \subseteq G$ contains each edge $xy \in E(G)$ for $x, y \in V(K)$, then K is an *induced subgraph* of G.

The set of neighbors of a vertex $v \in V(G)$ is denoted $N_G(v)$. The *degree* of a vertex $v \in V(G)$, denoted $deg_G(v)$, is the number of edges in $E(G)$ which are incident to v. If all the vertices of a graph G have the same degree d, then G is said to be vertex-d-regular, or briefly d-regular.

A graph which can be drawn on the (Euclidean) plane in a manner such that it edges intersect only at their endpoints is said to be *planar*. An *empty* region which is completely bounded by a set of edges in a planar graph is referred to as a *face*. The set of faces of a graph G is denoted $F(G)$ and the number of edges which bound a face $R \in F(G)$ is the degree of a face, denoted by $d_G(R)$. If all the faces of a graph G have the same degree f, then G is said to be face-f-regular, or briefly f-regular. A planar graph which is both vertex-d-regular and face-f-regular is denoted as a (d, f)-regular graph, or briefly (d, f)-graph.

For a connected simple graph G embedded into the plane we define $\partial(G)$ to be the (smallest) set of edges which encloses the entire set of faces $F(G)$ of the graph (in this definition we do not consider the exterior face to be an element of $F(G)$).

A *directed graph* is a graph where each edge is given a direction by selecting an initial vertex and a terminal vertex. If the initial vertex and the terminal vertex are the same vertex, say x_0, then edge x_0x_0 is said to be a *loop.* If a graph has multiple edges connecting a pair of vertices then the graph is said to be a multigraph. A graph that does not have any loops or multiple edges is said to be *simple.* A graph is *connected* if any two vertices in the graph can be linked by a path of alternating vertices and edges.

Definition 1.2 *For a finite, simple, connected graph G embedded into the plane we define $\varphi = \frac{|\partial(G)|}{|F(G)|}$ to be the isoperimetric number of the graph.*

In Chapter 3 we will offer some similar definitions for isoperimetric numbers of infinite graphs.

CHAPTER 2

SURFACES AND THEIR TESSELLATIONS

2.1 Introduction

A *tessellation* (or tiling) is the result of covering a surface with polygons or other geometric shapes so that there are no overlaps or gaps.

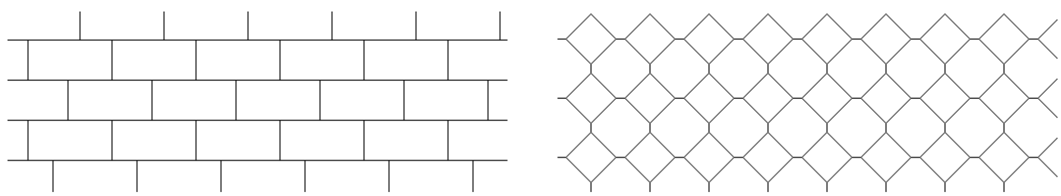

Figure 2: Two plane tessellations.

A *regular tessellation* is the result of covering a surface with congruent regular polygon. In such a tessellation each (internal) vertex in the tessellation will have the same degree. A tessellation with a f-gon (a polygon with f sides) will have face degree f. Hence, a regular tessellation may be defined mathematically by a pair (d, f) representing its vertex and face degrees, respectively.

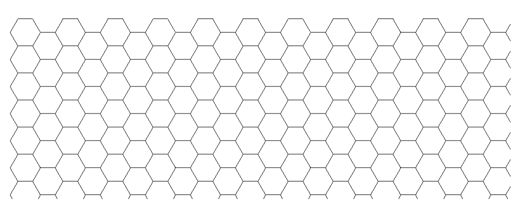

Figure 3: The $(3, 6)$ plane tessellation.

It should be noted that while the flat plane may be tessellated by $(3,6)$, as well as by $(4,4)$ and $(6,3)$, it cannot be tessellated by $(3,7)$ or $(3,5)$, or any other regular tessellation. The following function, which we will discuss in the next section, will help put this curious property into perspective. For $d \geq 3$ and $f \geq 3$:

$$H(d,f) = 4 - (d-2)(f-2)$$

and

$$H'(d,f) = \frac{1}{2} - \left(\frac{1}{d} + \frac{1}{f}\right)$$

2.2 Platonic Solids

A *platonic solid* is a regular, convex polyhedron with congruent faces of regular polygons. A proof that there are precisely 5 platonic solids was given by Theaetetus (a classical Greek mathematician), these solids are the Tetrahedron (four faces), the Hexahedron (six faces), the Octahedron (eight faces), the Dodecahedron (twelve faces), and the Icosahedron (twenty faces).

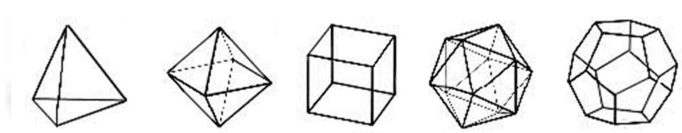

Figure 4: The Platonic Solids

Claim 1 *If R is a platonic solid with vertex degree d and face-degree f then $H(d,f) > 0$ and $H'(d,f) < 0$*

Proof: Let R be a platonic solid with V vertices, E edges and F faces, and let d, f be the respective degrees of the vertices and faces. Observe that each edge joins precisely 2 vertices and lies on precisely two faces, hence $fF = 2E = dV$. By Euler's formula [2] we have: $V - E + F = 2$. It follows that $V = \frac{2E}{d}$, $F = \frac{2E}{f}$ and therefore

$$\frac{2E}{d} - E + \frac{2E}{f} = 2$$

$$\frac{2E}{d(-2E)} - \frac{E}{(-2E)} + \frac{2E}{f(-2E)} = \frac{2}{(-2E)}$$

$$-\frac{1}{d} + \frac{1}{2} - \frac{1}{f} = -\frac{1}{E}$$

$$\frac{1}{2} - \left(\frac{1}{d} + \frac{1}{f}\right) < 0$$

which gives $H'(d,f) < 0$.

Now observe the following equivalent statements:

$$\frac{1}{2} - \left(\frac{1}{d} + \frac{1}{f}\right) < 0$$

$$\frac{1}{2} - \frac{d+f}{df} < 0$$

$$\frac{1}{2} < \frac{d+f}{df}$$

$$0 < 2d + 2f - df$$

$$-4 < -4 + 2d + 2f - df$$

$$-4 < -(d-2)(f-2)$$

$$4 - (d-2)(f-2) > 0$$

which gives $H(d,f) > 0$. $\square$

Claim 2 *If R is a (d, f)-tessellated surface with $H(d, f) > 0$ then R is a platonic solid.*

Proof: Assume, by way of contradiction, that there exists a (d, f)-tessellated surface R such that $4-(d-2)(f-2) > 0$ ($d \geq 3$ and $f \geq 3$) and $(d, f) \notin \{(3,3), (4,3), (3,4), (5,3), (3,5)\}$.

Then $(d, f) = (4,4), (d, f) = (4,5), (d, f) = (5,4), (d, f) = (5,5), d \geq 6$, or $f \geq 6$.

Observe that $H(4,4) = 0, H(4,5) = H(5,4) = -2, H(5,5) = -5$. Thus it must be the case that either $d \geq 6$, or $f \geq 6$.

Since $d \geq 3$ and $f \geq 3$ we have $(d-2) \geq 1, (f-2) \geq 1$ and therefore $(d-2)(f-2) \geq \max\{(d-2), (f-2)\}$. Suppose that $d \geq 6$, then $(d-2)(f-2) \geq (d-2) \geq 4$ which isn't possible. Thus we must conclude that $f \geq 6$. But then $(d-2)(f-2) \geq 4$, a contradiction.

We now must show that $\{(3,3), (4,3), (3,4), (5,3), (3,5)\}$ correspond uniquely to the platonic solids. Recall from Claim 1 that

$$fF = 2E = dV,$$

and from Euler's formula we have

$$V - E + F = 2.$$

It follows:

$$
\begin{aligned}
V - \frac{dV}{2} + \frac{dV}{f} &= 2; \\
V\left(1 - \frac{d}{2} + \frac{d}{f}\right) &= 2; \\
V\left(\frac{2f - fd + 2d}{2f}\right) &= 2; \\
V &= \frac{4f}{2f - fd + 2d}; \\
V &= \frac{4f}{4 - (d-2)(f-2)}.
\end{aligned}
$$

Similarly, by observing that $\frac{2E}{d} - E + \frac{dE}{f} = 2$ and $\frac{fF}{d} - \frac{fF}{2} + F = 2$ we get

$$E = \frac{2df}{4 - (d-2)(f-2)}$$

and

$$F = \frac{4d}{4 - (d-2)(f-2)}.$$

Therefore we have that E, F, and V are uniquely defined by d and f. All we have left to verify is that each pair (d, f) corresponds to a distinct platonic solid. Observe the following:

$(d, f) = (3, 3)$ implies $E = 6$, $F = 4$, and $V = 4$ which corresponds to the tetrahedron.

$(d, f) = (4, 3)$ implies $E = 12$, $F = 8$, and $V = 6$ which corresponds to the octahedron.

$(d, f) = (3, 4)$ implies $E = 12$, $F = 6$, and $V = 8$ which corresponds to the hexahedron.

$(d, f) = (3, 5)$ implies $E = 30$, $F = 12$, and $V = 20$ which corresponds to the dodecahedron.

$(d, f) = (5, 3)$ implies $E = 30$, $F = 20$, and $V = 12$ which corresponds to the icosahedron.

$\square$

2.3 The Flat Plane

2.3.1 Tessellations of the Flat Plane

In the previous section we discussed the five regular tessellations with $H(d, f) > 0$ and demonstrated that they correspond uniquely to the Platonic Solids. We now proceed to investigate tessellations of the Flat Plane with an aim to demonstrate that there are exactly 3 regular tessellations.

Claim 3 *If (d, f) is a regular tessellation of the Flat Plane then $H(d, f) = 0$ and $H'(d, f) = 0$.*

Proof:

Let G be the graph of the Flat Plane tessellated with regular polygons, where each polygon has $f \geq 3$ sides and each vertex lies at the intersection of $d \geq 3$ polygons. Note that, while we are not restricting which polygon we are using, it must be the case that each polygon is an f-gon in the regular tessellation.

Observe that since we are in the Flat Plane then the polygons must obey Euclidean geometry. Thus the interior angle must sum up to $(f-2)180°$ and therefore each interior angle measures $\frac{(f-2)180°}{f}$. Furthermore, since each vertex lies at the intersection of d polygons which leave no gaps in between, then the d angles of each face must sum up to $360°$ and therefore

$$d = \frac{360°}{\frac{(f-2)180°}{f}}$$

or simply $d(f) = \frac{2f}{f-2}$.

We note that $d(3) = 6, d(4) = 4, d(5) = \frac{10}{3}, d(6) = 3$, thus we can tessellate the Flat Plane with (regular) triangles, squares and hexagons, but not with pentagons since we may not have $3\frac{1}{3}$ pentagons meeting at a vertex.

However the following argument demonstrates that $\{d(f)\}_{f=3}^{\infty}$ is a monotone decreasing sequence:

$$\begin{aligned} f &\geq 3 \\ 2f(f-1) &> 2f(f-1)-4 \\ (2f)(f-1) &> 2(f+1)(f-2) \\ \frac{2f}{f-2} &> \frac{2(f+1)}{(f+1)-2} \\ d(f) &> d(f+1) \end{aligned}$$

It follows that if $f = 6+n$, $n \in \mathcal{Z}^+$, then $d(f) < d(6) = 3$ which is not possible. We conclude that the only regular tessellations of the Flat Plane are:

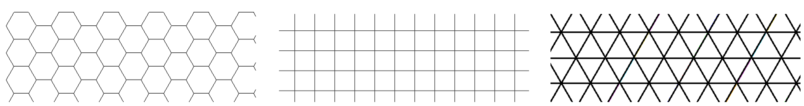

Figure 5: $(3,6), (4,4), (6,3)$.

Furthermore, each of these tessellations correspond to

$$H(d,f) = H'(d,f) = 0.$$

2.3.2 Isoperimetric Constant of the Flat Plane

Having categorized the tessellations of the Flat Plane we are now interested in calculating the isoperimetric number of a flat plane tessellation as defined in Definition 1.2. However a graph embedded in the Flat Plane may be finite or infinite, yet our definition applies only to graphs with finite boundaries and finite discrete area. We will address this in depth in chapter 3 where we give a definition that encompasses planar graphs of any order. In this chapter we will continue to work with Definition 1.2 for finite graphs and will avoid defining isoperimetric constants for infinite graphs. Instead we will discuss a related constant which we refer to as the *isoperimetric limit* of a graph. This constant will serve as a lower bound for the isoperimetric constant when define it in Chapter 3.

Definition 2.3 *For a planar (d, f)-graph G and a face $R_0 \in F(G)$ we define a lens set centered at R_0 to be the set of subgraphs $\{L_0, L_1, L_2, ...\}$ where $L_0 = G[R_0]$ and $L_n = \{R \in F(G) - F(L_{n-1}) \mid V(R) \cap V(L_{n-1}) \neq \emptyset\}$, $n = 1, 2, 3, ...$*

Definition 2.4 *For a planar (d, f)-graph G and for a lens set center at a face R_0 we define W_n to be the n-ball subgraph centered at R_0 by $W_n = G[\bigcup_{i=0}^{n} L_n]$.*

Definition 2.5 *For a planar (d, f)-graph G and for finite subgraphs $W_0, W_1, W_2, ...$ of G we define the isoperimetric limit $\phi(W_n) = \lim_{n \to \infty} \varphi(W_n)$*

Claim 4 *The isoperimetric limit of a regular Flat Plane tessellation is $\phi = 0.$.*

Observe that φ compares the cardinality of a non-empty set of edges to the cardinality of a nonempty set of faces, hence ϕ is a non-negative number.

Proof that $\phi(3, 6) = 0$:

Let G be the graph of the Flat Plane tessellated with regular hexagons, that is the $(3, 6)$ tessellation. We will show that the isoperimetrical constant of the $(3, 6)$ tessellated flat plane is 0. Select a face $R_0 \in F(G)$ and set $L_0 = G[R_0]$, an induced subgraph of G.

Let $L_n = \{R \in F(G) - F(L_{n-1}) \mid V(R) \cap V(L_{n-1}) \neq \emptyset\}$, $n = 1, 2, 3, ...$ Here we use L to represent the different lenses of an induced subgraph of G centered at face R_0.

Further, let $W_n = G[\bigcup_{i=0}^{n} L_n]$

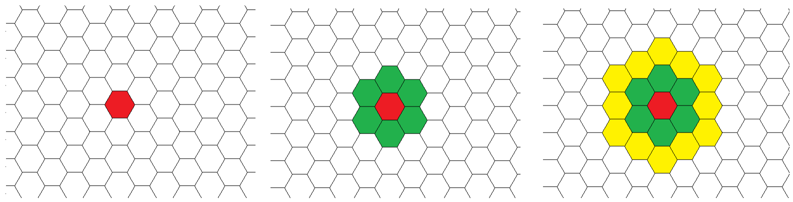

Figure 6: W_0, W_1, W_2

Set $\partial(W_0) = E(W_0)$ and let $\partial(W_n) = \{e \in E(G) \mid e \in E(W_n) \cap E(W_{n+1})\}$. Finally set $P_n = |\partial(W_n)|$. That is, P_n is the number of edges on the boundary of induced graph W_n.

Then $P_0 = 6$, $P_1 = 18$ and in general $P_n = 12n + 6$.

We are now interested in how many faces are bounded by each subgraph W_n, thus set $A_n = |F(W_n)|$

Then $A_0 = 1$, $A_1 = 7$ and in general $A_n = 1 + \sum_{i=0}^{n} 6i$. Having established these we are now interested in $\varphi_n = \frac{P_n}{A_n}$, or more interestingly $\varphi(G) = \lim_{n\to\infty} \frac{P_n}{A_n}$.

$$\begin{aligned}
\phi(G) = \lim_{n\to\infty} \varphi_n &= \lim_{n\to\infty} \frac{12n+6}{1+\sum_{i=0}^{n} 6i} = \lim_{n\to\infty} \frac{12n+6}{1+6\sum_{i=0}^{n} i} = \lim_{n\to\infty} \frac{12n+6}{1+6\sum_{i=1}^{n} i} \\
&= \lim_{n\to\infty} \frac{12n+6}{1+6\frac{(n)(n+1)}{2}} = \lim_{n\to\infty} \frac{12n+6}{1+6\frac{n^2+n}{2}} = \lim_{n\to\infty} \frac{12n+6}{1+3(n^2+n)} \\
&= \lim_{n\to\infty} \frac{12n+6}{3n^2+3n+1} = 0.
\end{aligned}$$

$\square$

We have shown that the flat plane has isoperimetric limit 0 under the $(3,6)$ tessellation. We are interested in verifying that flat plane's isoperimetric properties are preserved under other tessellations.

Proof that $\phi(4,4) = 0$:

Let G be the graph of the Flat Plane tessellated with squares, that is the $(4,4)$ tessellation. We will show that the isoperimetric limit of the $(4,4)$ tessellated flat plane is also 0.

We define P_n and A_n as before to demonstrate that $\varphi(G) = \lim\limits_{n\to\infty} \frac{P_n}{A_n} = 0$

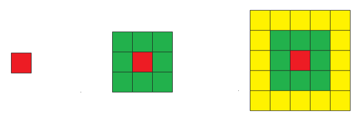

Figure 7: W_0, W_1, W_2

Observe that $P_n = 8n + 4$ and $A_n = 1 + \sum\limits_{i=0}^{n} 8i$

$$\begin{aligned}
\phi(G) = \lim_{n\to\infty} \varphi_n &= \lim_{n\to\infty} \frac{8n+4}{1+\sum\limits_{i=0}^{n} 8i} = \lim_{n\to\infty} \frac{8n+4}{1+8\sum\limits_{i=0}^{n} i} = \lim_{n\to\infty} \frac{8n+4}{1+8\sum\limits_{i=1}^{n} i} \\
&= \lim_{n\to\infty} \frac{8n+4}{1+8\frac{(n)(n+1)}{2}} = \lim_{n\to\infty} \frac{8n+4}{1+8\frac{n^2+n}{2}} = \lim_{n\to\infty} \frac{8n+4}{1+4(n^2+n)} \\
&= \lim_{n\to\infty} \frac{8n+4}{4n^2+4n+1} = 0.
\end{aligned}$$

□

Thus we have verified that isoperimetric properties are conserved under this tessellation as well. We proceed to the remaining Flat Plane tessellation, $(6,3)$.

Proof that $\phi(6,3) = 0$:

Let G be the graph of the Flat Plane tessellated with equilateral triangles, that is the $(6,3)$ tessellation. We will show that the isoperimetrical constant of the $(6,3)$

tessellated flat plane is also 0.

We define P_n and A_n as before and demonstrate that $\phi(G) = \lim_{n\to\infty} \frac{P_n}{A_n} = 0$

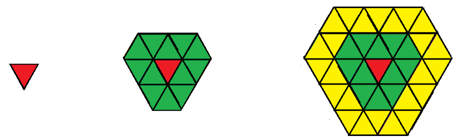

Figure 8: W_0, W_1, W_2

Observe that $P_n = 6n + 3$ and $A_n = 1 + \sum_{i=0}^{n} 12i$

$$\begin{aligned}
\phi(G) = \lim_{n\to\infty} \varphi_n &= \lim_{n\to\infty} \frac{6n+3}{1 + \sum_{i=0}^{n} 12i} = \lim_{n\to\infty} \frac{6n+3}{1 + 12\sum_{i=0}^{n} i} = \lim_{n\to\infty} \frac{6n+3}{1 + 12\sum_{i=1}^{n} i} \\
&= \lim_{n\to\infty} \frac{6n+3}{1 + 12\frac{(n)(n+1)}{2}} = \lim_{n\to\infty} \frac{6n+3}{1 + 12\frac{n^2+n}{2}} = \lim_{n\to\infty} \frac{6n+3}{1 + 6(n^2+n)} \\
&= \lim_{n\to\infty} \frac{6n+3}{6n^2 + 6n + 1} = 0.
\end{aligned}$$

□

2.4 The Hyperbolic Plane

2.4.1 Generating Functions in Hyperbolic Tessellations

In the previous section we demonstrated that each tessellation of the Flat Plane yields the same isoperimetric limit $\phi = 0$. This will not be the case in hyperbolic tessellations, a fact that we will discuss in the next chapter. Before doing so we will find the isoperimetric limit of one such tessellation with the aim to gain some understanding on the general formula.

Consider the graph G of a regular $(7, 3)$ tessellation which corresponds to a Hyperbolic Plane. Select a vertex x_0 and set $L_0 = G[x_0]$. For $n = 1, 2, 3, ...$ set $L_n = \{R \in F(G) \mid V(R) \cap V(L_{n-1}) \neq \emptyset\}$. Finally set $W_n = G[\bigcup_{i=0}^{n} L_n]$.

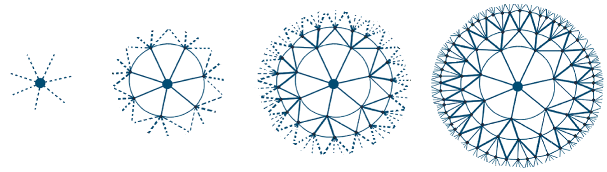

Figure 9: W_0, W_1, W_2, W_3

For $n = 1, 2, 3, ...$, set $A_n = \{x \in V(W_n - W_{n-1}) \ : \ |N_{W_n}(x) \cap V(K_{n-1})| = 2\}$ and set $B_n = \{x \in V(W_n - W_{n-1}) \ : \ |N_{W_n}(x) \cap V(W_{n-1})| = 1\}$. Then at each lens we are separating the vertices into two sets, the set A_n of vertices on the n^{th} lens which have two interior edges incident to it and the set B_n of vertices on the n^{th} lens which have one interior edge incident to it. Since our graph is a triangulation ($f = 3$) each vertex has precisely one or two interior incident edges. [4] [5]

Further, we set $a_n = |A_n|$ and $b_n = |B_n|$. Observe that $a_1 = 0, b_1 = 7$ and $a_{n+1} = a_n + b_n$, $b_{n+1} = 2b_n + a_n$, $n = 1, 2, 3, ...$ Since a lens L_n has precisely $a_n + b_n$ vertices on the boundary, this gives us that there are $a_n + b_n$ edges on the boundary of W_n.

Although this method is effective for calculating the boundary of small subgraphs it becomes quite cumbersome when estimating large subgraphs. For example, to calculate the boundary of B_4 we would first need to calculate $a_1, b_1, a_2, b_2, a_3, b_3$. To avoid this we will convert this recursive relation into a generating function by linear algebra:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$
$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$$
$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$
$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 0 \\ 7 \end{pmatrix}$$

By diagonalizing the matrix we have:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n = (PDP^{-1})^n = PD^nP^{-1}$$

where $P = \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \\ \frac{-\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{pmatrix}$.

Observe that $D^n = \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix}$, $\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}$. Here λ_1, λ_2 correspond to the eigenvalues of the original 2×2 matrix.

By expanding the product we have:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 7\frac{\sqrt{5}}{5}({\lambda_1}^n - {\lambda_2}^n) \\ 7(\frac{5+\sqrt{5}}{10}){\lambda_1}^n + 7(\frac{5-\sqrt{5}}{10}){\lambda_2}^n \end{pmatrix}, \lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

Therefore we have

$$|\partial(W_{n+1})| = 7\frac{\sqrt{5}}{5}({\lambda_1}^n - {\lambda_2}^n) + 7(\frac{5+\sqrt{5}}{10}){\lambda_1}^n + 7(\frac{5-\sqrt{5}}{10}){\lambda_2}^n$$

Or simply:

$$|\partial(W_{n+1})| = \frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right){\lambda_1}^n + \frac{7}{5}\left(\frac{5-3\sqrt{5}}{2}\right){\lambda_2}^n.$$

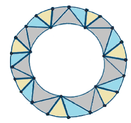

Figure 10: Right triangle edges corresponding to vertices

Having shown that we can efficiently calculate the length of the boundary for an arbitrary ball graph W_n we are now interested in how many triangles it encloses, that is what is its discrete area?

Observe that some triangles point inwards and others point outwards, thus we cannot simply count the bases of the triangles. However each triangle in L_n has a rightmost edge (right $\approx$ clockwise) which is incident to a vertex on the boundary of L_n. Conversely, each vertex on the boundary is incident to either one or two of these edges. In fact, we can declare that a vertex in A_n is incident to exactly two of these rightmost (interior) edges and a vertex in B_n is incident to exactly one of these rightmost (interior) edges. This leads us to:

$$|F(L_n)| = 2a_n + b_n$$

and therefore

$$|F(W_n)| = \sum_{k=1}^{n} 2a_k + b_k,$$

equivalently

$$|F(W_n)| = \sum_{k=1}^{n} 2(7\frac{\sqrt{5}}{5}({\lambda_1}^{k-1} - {\lambda_2}^{k-1})) + 7(\frac{5+\sqrt{5}}{10}){\lambda_1}^{k-1} + 7(\frac{5-\sqrt{5}}{10}){\lambda_2}^{k-1}$$

Equivalently,

$$|F(W_n)| = \sum_{k=1}^{n} \frac{7(1+\sqrt{5})}{2} {\lambda_1}^{k-1} + \sum_{k=1}^{n} \frac{7(1-\sqrt{5})}{2} {\lambda_2}^{k-1}$$

However we can generalize this statement by observing that it is the sum of a finite geometric sequence [1]:

Let $S_n = \sum_{k=1}^{n} \{\lambda_{i\in\{1,2\}}\}^{k-1}$. Then

$$S_n = \frac{{\lambda_i}^n - 1}{\lambda_i - 1}$$

Therefore we have

$$|F(W_n)| = \frac{7(1+\sqrt{5})}{2} \frac{({\lambda_1}^n - 1)}{(\lambda_1 - 1)} + \frac{7(1-\sqrt{5})}{2} \frac{({\lambda_2}^n - 1)}{(\lambda_2 - 1)}.$$

Recall from the previous section that we define φ_n to be the ratio of the boundary to the area of a graph W_n with n lenses, that is,

$$\varphi_n = \frac{|\partial(W_n)|}{|F(W_n)|} = \frac{\frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right){\lambda_1}^{n-1} + \frac{7}{5}\left(\frac{5-3\sqrt{5}}{2}\right){\lambda_2}^{n-1}}{\frac{7(1+\sqrt{5})}{2}\frac{({\lambda_1}^n-1)}{(\lambda_1-1)} + \frac{7(1-\sqrt{5})}{2}\frac{({\lambda_2}^n-1)}{(\lambda_2-1)}}$$

Since the numerator and denominator share the same power of λ it is conceivable that $\lim_{n\to\infty} \varphi_n \neq 0$ for our $(7,3)$ tessellation.

Recall that $\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2} < 1$, hence $\lim_{n\to\infty} {\lambda_2}^n = 0$. Observe that $\lambda_1 = \frac{1}{\lambda_2}$,

[1] Thank you Dr. Stephens.

then it follows:

$$\begin{aligned}
\lim_{n\to\infty}\varphi_n &= \lim_{n\to\infty}\frac{|\partial(W_n)|}{|F(W_n)|} = \lim_{n\to\infty}\frac{\frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right){\lambda_1}^{n-1}+\frac{7}{5}\left(\frac{5-3\sqrt{5}}{2}\right){\lambda_2}^{n-1}}{\frac{7(1+\sqrt{5})}{2}\frac{({\lambda_1}^n-1)}{(\lambda_1-1)}+\frac{7(1-\sqrt{5})}{2}\frac{({\lambda_2}^n-1)}{(\lambda_2-1)}} \\
&= \lim_{n\to\infty}\frac{\frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right)\lambda_2{\lambda_1}^{n}+\frac{7}{5}\left(\frac{5-3\sqrt{5}}{2}\right)0}{\frac{7(1+\sqrt{5})}{2}\frac{({\lambda_1}^n-1)}{(\lambda_1-1)}+\frac{7(1-\sqrt{5})}{2}\frac{(0-1)}{(\lambda_2-1)}} \\
&= \lim_{n\to\infty}\frac{\frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right)\frac{3-\sqrt{5}}{2}{\lambda_1}^{n}}{\frac{7(1+\sqrt{5})}{2}\frac{({\lambda_1}^n)}{(\frac{3+\sqrt{5}}{2}-1)}} = \lim_{n\to\infty}\frac{\frac{7}{5}\left(\frac{5+3\sqrt{5}}{2}\right)\frac{3-\sqrt{5}}{2}{\lambda_1}^{n}}{7{\lambda_1}^n} \\
&= \lim_{n\to\infty}\frac{\frac{7(4\sqrt{5}){\lambda_1}^n}{20}}{7{\lambda_1}^n} = \frac{\sqrt{5}}{5}
\end{aligned}$$

It is worth noting that

$$\frac{\sqrt{5}}{5} = (3-2)\sqrt{1-\frac{4}{(7-2)(3-2)}} = (f-2)\sqrt{1-\frac{4}{(d-2)(f-2)}}; f=3, d=7.$$

We will discuss this in more detail in the following chapter.

CHAPTER 3

MAIN THEOREM AND OTHER RESULTS

3.1 Introduction

In this chapter we will investigate isoperimetric constants for graphs with hyperbolic properties, that is, graphs satisfying in some manner $H(d, f) \leq 0$. First we will take a look at some preliminary results for (d, f)-graphs, following which we will introduce the main theorem which gives a lower bound for a general hyperbolic tessellation.

Let G be a connected undirected graph without loops and multiple edges, where $V(G)$ is the set of vertices of G, $E(G)$ is the set of edges in G and $F(G)$ is the set of faces in G. For $x \in V(G)$, the degree of x in G, denoted by $deg_G(x)$, implies the number of edges incident with x. The neighborhood of x in G, denoted $N_G(x)$, implies the set of vertices adjacent to x in G. For $R \in F(G)$, the degree of R in G, denoted $d(R)$, implies the number of edges of the boundary of R.

A graph G is said to be a (d, f)-graph if it satisfies the following:

(1) G is planar and already embedded in the plane;

(2) G is regular in the ordinary sense, that is, $deg_G(x) = d$ for every vertex $x \in V(G)$ and $d \geq 3$;

(3) Every face in R is an f-gon, that is, $d(R) = f$ for every face $R \in F(G)$, where $F(G)$ is the set of faces of G, $d(R)$ is the number of edges of the boundary of R and $f \geq 3$.

A graph G is said to be a (d^+, f^+)-graph if it satisfies the following:

(1) G is planar and already embedded in the plane;

(2) G is not necessarily vertex-regular and $deg_G(x) \geq d$ for every vertex $x \in V(G)$ and $d \geq 3$;

(3) G is not necessarily face-regular and $d(R) \geq f$ for every face $R \in F(G)$, where $F(G)$ is the set of faces of G, $d(R)$ is the number of edges of the boundary of R and $f \geq 3$.

Observe that a (d, f)-graph is always a (d^+, f^+)-graph but the converse is not generally true. We will refer to (d^+, f^+)-graphs broadly.

Set $H(d, f) = 4 - (d - 2)(f - 2)$. It is well known that, if G is a (d, f)-regular planar graph and if $H(d, f) = 0$, then G is one of the platonic graphs, which are finite regular polyhedra. If G is a (d, f)-regular planar graph with $H(d, f) \leq 0$, then G is an infinite graph [4]. In this paper we deal only with (d^+, f^+)-graphs G satisfying that $(d - 2)(f - 2) \geq 4$ where $d = min\{deg_G(x)|x \in V(G)\}, f = min\{d(R)|R \in G\}$.

The following are two isoperimetric constants discussed in [4]. The first, $\alpha(\cdot)$, is an analogue of Cheeger's constant, and the other, $\alpha^*(\cdot)$, gives a "hyperbolicity criterion for an infinite planar graph" [4]. These $\alpha(\cdot)$ and $\alpha^*(\cdot)$ are defined as follows:

Definition 3.6

$$\alpha(G) = \mathit{inf}\{|E(\partial_v K)|/Area(K) \mid K \textit{ is a finite subgraph of } G\}$$

where

$$Area(K) = \sum_{x \in V(K)} deg_G(x)$$

and

$$E(\partial_v K) = \{xy \in E(G) \mid x \in V(K), y \in V(G) \backslash V(K)\};$$

Definition 3.7

$$\alpha^*(G) = \mathit{inf}\{|E(\partial_f K)|/|F(K)| \mid K \textit{ is a finite subgraph of } G\}$$

where

$$E(\partial_f K) = \{xy \in E(K) \cap E(F') \mid F' \in F(G) \backslash F(K)\}.$$

The following theorem declares exact values of $\alpha(\cdot)$ and $\alpha^*(\cdot)$ for a (d, f)-regular graph.

Theorem 3.8 *For a (d, f)-regular planar graph G satisfying $H(d, f) = 4 - (d-2)(f-2) \leq 0$, we have*

$$\alpha(G) = \frac{d-2}{d}\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

and

$$\alpha^*(G) = (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}.$$

A proof of this Theorem is given in [4]. The first equality was also given independently by [3].

3.2 Main Theorem

In this section we will discuss the main theorem of this paper which gives a lower bound for general planar graphs with hyperbolic properties. The remainder of this chapter will be devoted to developing necessary definitions and lemmas so that we may prove this theorem in Section 3.5.

Theorem 3.9 *For a planar graph (d^+, f^+)-graph G satisfying $H(d, f) = 4 - (d - 2)(f - 2) \leq 0$ we have*

$$\alpha(G) \geq \frac{d-2}{d}\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

and

$$\alpha^*(G) \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

That is, for any finite subgraph of $K \subset G$:

$$\inf\left\{\frac{|E(\partial_v K)|}{Area(K)}\right\} \geq \frac{d-2}{d}\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

and

$$\inf\left\{\frac{|E(\partial_f K)|}{|F(K)|}\right\} \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

3.3 Definitions

We introduce the concept of two n-balls in a similar manner to the ones discussed in [4] but with the condition of regularity removed:

Definition 3.10 *For a planar graph G, we pick and fix a vertex $x_0 \in V(G)$ and define the (vertex-centered) n-ball $B_n = B_n(G, x_0)$ in G as follows:*

$$B_0 = G[\{x_0\}] \text{ and } B_n = G[\{R \in F(G) \mid V(R) \cap V(B_{n-1}) \neq \emptyset\}], n = 1, 2, ...$$

For this n-ball B_n, we also set

$$V_n = \{x \in V(B_n) \backslash V(B_{n-1})\} \text{ for } n = 1, 2, ...;\ V_0 = x_0$$

Definition 3.11 *For a planar graph G, we pick and fix a face $R_0 \in F(G)$ and define the (face-centered) n-ball $B_n^* = B_n^*(G, R_0)$ in G as follows:*

$$B_0^* = G[\{R_0\}] \text{ and } B_n^* = G[\{R \in F(G) \mid V(R) \cap V(B_{n-1}^*) \neq \emptyset\}], n = 1, 2, ...$$

For this n-ball B_n^*, we also set

$$V_n^* = \{x \in V(B_n^*) \backslash V(B_{n-1}^*)\} \text{ for } n = 1, 2, ...;\ V_0^* = V(R_0)$$

Definition 3.12 *[4] For any finite subgraph K of G, we pick and fix a vertex $x_0 \in V(K)$; we set $B_n = B_n(G, x_0)$ in G as defined in Definition 3.4. and we set $B_n^* = B_n^*(g, x_0)$ in G as defined in Definition 3.5.*

We set

$$N = N(K, x_0) = max\{n \mid V(K) \cap V_n \neq \emptyset\}$$

For $n = 0, 1, ..., N$, we denote

$$W_n \text{ to be the subgraph of } K \text{ induced by } F(K) \cap F(B_n)$$

and

$$F_n = F(W_n)\backslash F(W_{n-1})$$

Similarly we set

$$N^* = N^*(K, R_0) = max\{n \mid V(K) \cap V_n^* \neq \emptyset\}$$

For $n = 0, 1, ..., N^*$, we denote

$$W_n^* \text{ to be the subgraph of } K \text{ induced by } F(K) \cap F(B_n^*)$$

and

$$F_n^* = F(W_n^*)\backslash F(W_{n-1}^*)$$

Definition 3.13 *For a finite planar graph W_N (or equivalently for W_N^*) induced from a planar subgraph $K \subset G$ and a vertex $v \in V_k$ (V_K^*) as defined in Definition 3.4. (Definition 3.5.), we say $P = P(K, v)$ is an outward path rooted at v given that*

(*i*) $|V(P)| \geq 2$;

(*ii*) v is a start vertex for P, that is $|N_P(v)| = 1$;

(*iii*) If x is an internal vertex in $P \cap V_n$ and $deg_G(x) > 3$
then $|N_P(x) \cap V_{n-1}| = 1$
and $|N_P(x) \cap V_{n+1}| = 1$;

(*iv*) If x is an internal vertex in $P \cap V_n$ and $deg_G(x) = 3$
then $|N_P(x) \cap V_n| = 1$
and $|N_P(x) \cap V_{n-1}| + |N_P(x) \cap V_{n+1}| = 1$.

(*v*) P has an end vertex $z \in V_N$, and $|V(P) \cap V_N| = 1$;

That is P is a finite path with the properties that

(1) P begins at $v \in V_k$ (equivalently V_k^*) and ends at $z \in V_N$ (V_N^*);

(2) P travels outwards in a most efficient manner.

Definition 3.14 *For an outward path $P(K,v) = v, x_1, x_2, ..., x_n, x_{n+1}, ..., z$, we say P is a **right-leaning** path rooted at v if:*

(1) $v \in V_k$ and $x_1 \in N_G(v) \cap V_{k+1}$;

(2) If $min\{deg_G(x_i) : x_i \in V(P)\} > 3$, then:

$x_{n+1} \in N_P(x_n) \cap V_{n+1}$ is the rightmost vertex of $N_G(x_n) \cap V_{n+1},\ (n \geq 1)$;

(3) If $min\{deg_G(x_i) : x_i \in V(P)\} = 3$, then:

$x_{2n+1} \in N_P(x_{2n}) \cap V_n$ is the rightmost vertex of $N_G(x_{2n}) \cap V_{2n+1},\ (n \geq 0), and$

$x_{2n+2} \in N_P(x_{2n+1}) \cap V_{n+1}$ is the rightmost vertex of $N_G(x_{2n+1}) \cap V_{n+1},\ (n \geq 0)$.

The preceding definition defines paths that begin at a vertex $v \in V_k$ (equivalently V_k^*), continue through a neighboring vertex of v in V_{k+1} (V_{k+1}^*) and then extends outwards (whenever possible) choosing the rightmost vertex each time.

Observe that for $v \in V_k$ (V_k^*) there exist $\lambda = |N_K(v) \cap V_{k+1}|$ (V_{k+1}^*) disjoint right-leaning paths rooted at v. For simplicity we will index them $P_1, P_2, ..., P_\lambda$ such that

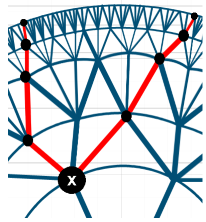

Figure 11: Two right-leaning paths at x

P_1 is the leftmost path and P_λ is the rightmost path.

Definition 3.15 *For a graph K and a vertex $v \in V_n = V_n(K)$ (similarly for V_n^*) we define an outward V-wedge rooted at vertex v, denoted $\vee(v)$, to be the induced (smaller) subgraph bounded by $\partial(K)$ and by two neighboring outward, right-leaning paths P_i, P_{i+1} $(1 \leq i \leq \lambda - 1)$ rooted at v.*

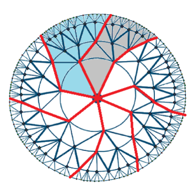

Figure 12: V-wedges

Definition 3.16 *For a graph K, a face $R \in F_n = F_n(K)$ and an edge $e = [xy] \in E(R) \cap F_{n+1}$ we define an outward U-wedge rooted at edge e, denoted $\sqcup(e)$, to be the induced (smaller) subgraph bounded by $\partial(K)$, by the edge e and by two outward, right-leaning paths $P_i(x), P_i(y)$ $(1 \leq i \leq \lambda - 1)$ rooted at x and y respectively.*

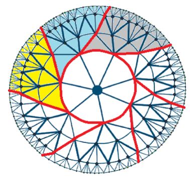

Figure 13: U-wedges

3.4 Assumptions and Lemmas

Assumption 3.1 [4] We may make the following assumptions on finite subgraph K of G:

(a) for every $e \in E(K)$ there exists a face $R \in F(K)$ such that $e \in E(R)$;

(b) K is connected and has no cut-vertex [2];

(c) $\partial_f K$, the subgraph of K induced by $E(\partial_f K)$, is connected and a cycle.

The proof of the validity of these assumptions is given in [4] and is therefore omitted here.

Lemma 3.17 *Let G be a (d^+, f^+)-non-regular planar graph satisfying $H(d, f) = 4-(d-2)(f-2) \leq 0$ and K be a subgraph of G satisfying Assumption 3.1. Let $\bigvee(x)$ be an outward V-wedge contained between paths P_i and P_{i+1}.*

If $|P_i| = |P_{i+1}|$ and if $\bigvee(x)$ is (d, f)-regular then

$$\frac{|E(\partial_f \bigvee(x))|}{|F(\bigvee(x))|} \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

where $\partial_f \bigvee(x) = \partial_f K \cap \bigvee(x)$.

[2] A cut-vertex is a vertex which if removed would separate the graph into two disjoint subgraphs.

Proof: Since $|P_i| = |P_{i+1}|$ we identify corresponding vertices and edges and glue d copies of $\bigvee(x)$ to form a (d,f)-regular ball, say B. Then by Theorem 1.1.:

$$\begin{aligned}\frac{|E(\partial_f \bigvee(x))|}{|F(\bigvee(x))|} &= \frac{d|E(\partial_f \bigvee(x))|}{d|F(\bigvee(x))|}\\ &= \frac{|E(\partial_f B)|}{|F(B)|}\\ &\geq (f-2)\sqrt{1-\frac{4}{(d-2)(f-2)}}.\end{aligned}$$

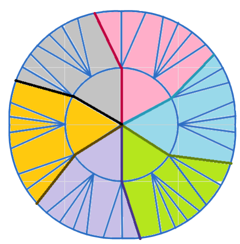

Figure 14: Gluing of V-wedges

Lemma 3.18 *Let G be a (d^+, f^+)-non-regular planar graph G satisfying $H(d,f) = 4-(d-2)(f-2) \leq 0$ and K be a subgraph of G satisfying Assumption 3.1. Let $e = [xy]$ be an edge in K and $\bigsqcup(e)$ be an outward U-wedge contained between paths $P_i(x)$ and $P_i(y)$.*

If $|P_i(x)| = |P_i(y)|$ and $\bigsqcup(e)$ is (d,f)-regular then

$$\frac{|E(\partial_f \bigsqcup(e))|}{|F(\bigsqcup(e))|} \geq (f-2)\sqrt{1-\frac{4}{(d-2)(f-2)}}$$

Proof: same argument as with $\bigvee(x)$.

Lemma 3.19 *Let H be a finite, planar graph and let H_1, H_2 be subgraphs of H such that $F(H_1), F(H_2) \neq \emptyset$ and $F(H_1) \cap F(H_2) = \emptyset$ and $F(H_1) \cup F(H_2) = F(H)$. Then*

$$\frac{|E(\partial_f H)|}{|F(H)|} \geq min\left\{\frac{|E(\partial_f H \cap H_1)|}{|F(H_1)|}, \frac{|E(\partial_f H \cap H_2)|}{|F(H_2)|}\right\}$$

Proof: Let

$$\varphi = min\left\{\frac{p}{q}, \frac{s}{t}\right\},$$

where

$$p = |E(\partial_f H \cap H_1)|,\ q = |F(H_1)|,$$
$$s = |E(\partial_f H \cap H_2)|,\ t = |F(H_2)|.$$

It follows:

$$\frac{p}{q} \geq \varphi, \quad \frac{s}{t} \geq \varphi;$$
$$p \geq q\varphi, \quad s \geq t\varphi;$$
$$\frac{p+s}{q+t} \geq \frac{q\varphi + t\varphi}{q+t} = \varphi.$$

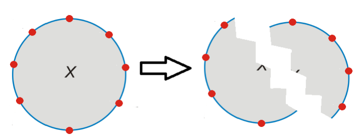

Figure 15: Separating a graph into two subgraphs

We have demonstrated, in the preceding lemmas, that regions $\bigvee(\cdot)$ and $\sqcup(\cdot)$ can be contracted and that their contractions do not increase the isoperimetric ratio. A restriction of these ratios is that their outward bounding paths must be of equal length. However if the paths aren't of the same length then the region cannot be contracted in general. The following lemma allows us to deal with this occurrence by demonstrating that the outermost layer of a graph, if it is (d, f)-regular, may be contracted and the resulting graph will have a smaller (or equal) isoperimetric ratio.

Lemma 3.20 *For a finite subgraph K of G and induced subgraph $W_N = G[F(K) \cap F(B_N)]$. If $W_N \neq B_N$ and W_N, W_{N-1} satisfy the following conditions:*

$$(1) \quad deg(v) \geq d, d(R) \geq f, \text{ for all } v \in V(W_{N-1}), R \in F(W_{N-1})$$

$$(2) \quad deg(v) = d, d(R) = f, \text{ for all } v \in V(W_N \backslash W_{N-1}), R \in F(W_N \backslash W_{N-1}).$$

then

$$\frac{|E(\partial_f W_N)|}{|F(W_N)|} \geq min\left\{\frac{|E(\partial_f W_{N-1})|}{|F(W_{N-1})|}, (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}\right\}.$$

Proof:

Let

$$S = F(W_N) \setminus F(W_{N-1}).$$

Observe that induced graph $G[S]$ may be disconnected. Thus S can be expressed as the union of edge disconnected sets $S_1, ..., S_k$ such that

$$S = S_1 \cup ... \cup S_k,$$
$$E(S_i) \cap E(S_j) = \emptyset, 1 \leq i \neq j \leq k$$

Let

$$A = E(\partial_f W_N) \cap E(\partial_f W_{N-1})$$
$$B_i = \big(E(\partial_f W_N) \setminus A\big) \cap E(S_i), 1 \leq i \leq k$$
$$C_i = \big(E(\partial_f W_{N-1}) \setminus A\big) \cap E(S_i), 1 \leq i \leq k$$

Observe that

$$\frac{|E(\partial_f W_N)|}{|F(W_N)|} = \frac{|E(\partial_f W_{N-1})| + |B| - |C|}{|F(W_{N-1})| + |S|}.$$

Therefore, by Lemma 3.16, it suffices to show that

$$\frac{|B_i| - |C_i|}{|S_i|} \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}},$$

for each $1 \leq i \leq k$.

Assume, by way of contradiction, that there exists a counterexample. Since this counterexample must be finite there must exist a minimal counterexample. Let $K \subset G$ be a minimal counterexample, let S_i be the smallest subgraph of K such that

$$\frac{|B_i| - |C_i|}{|S_i|} < (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}},$$

and let $s = |\{R \in F(S_i)\}|$.

Let

$$F_N = F(W_N) \backslash F(W_{N-1}) \text{ and } f_N = |F_N|$$

$$A_{N-1}^{+} = V(G[F_N]) \cap V_{N-1},$$

$$A_{N-1} = \{x \in A_{N-1}^{+} \mid xy \in Z_N \text{ for any } y \in N_K(x) \cap V_N\},$$

$$a_{N-1} = |A_{N-1}|,$$

where Z_N is the set of edges xy for which there exists two faces $R_1, R_2 \in F_N$ such that $xy \in E(R_1) \cap E(R_2)$.

Further, we inductively define $\{a_k\}_{k=0}^{N-1}$ by

$$F_k = F(W_k)\backslash F(W_{k-1}) \text{ and } f_k = |F_k|$$

$$A_{k-1}^{+} = V(G[F_k]) \cap V_{k-1},$$

$$A_{k-1} = \{x \in A_{k-1}^{+} \mid xy \in Z_k \text{ for any } y \in N_K(x) \cap V_k\},$$

$$a_{k-1} = |A_{k-1}|,$$

where Z_k is the set of edges xy for which there exists two faces $R_1, R_2 \in F_k$ such that $xy \in E(R_1) \cap E(R_2)$.

By assumption $W_N \neq B_N$ and therefore $a_0 = 0$. Thus we may set

$$m = max\{0 \leq k \leq N-1 | a_k = 0\}.$$

Now consider the induced graph $\mathcal{F} = \bigcup_{k=m}^{N}\{F_k\}$. If $\mathcal{F}$ is (d, f)-regular then we know by [4] that

$$|E(\partial_f W_N)| - |E(\partial_f W_{N-1})| \geq \sqrt{\frac{(f-2)((d-2)(f-2)-4)}{d-2}}(|F(W_N)| - |F(W_{N-1})|$$

and therefore

$$\frac{|B_i| - |C_i|}{|S_i|} \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}.$$

Thus we may conclude that there exists a face $R \in F(\mathcal{F})$ with $d(R) > f$ or there exists a vertex $x \in V(\mathcal{F})$ with $deg(x) > d$.

Set

$$l_F = max\{m \leq k \leq N-1 | \exists R \in F(F_k) \text{ satisfying } d(R) > f\}$$

and set

$$l_V = max\{m \leq k \leq N-1 | \exists x \in V(F_k) \text{ satisfying } deg(x) > d\}$$

and let

$$l = max\{l_F, l_V\}.$$

If $l_F > l_V$ then let $Q \in F(F_l)$ satisfying $d(Q) > f$ and let $e = st \in E(Q) \cap E(F_{l+1})$. Since $l_F > l_V$ and F_l represents the largest indexed non-regular F_k we have that $\sqcup(e)$ is (d, f)-regular. Furthermore we have that $|P_i(s)| = |P_i(t)|$ thus we may contract region $\sqcup(e) \subset \mathcal{F}$ and obtain a smaller graph. We denote said graph as $\overline{\sqcup(e)} \subset \mathcal{F}$.

By the assumption of minimality of K we must have that:

$$\frac{|E(\partial_f W_N \ \cap \ \overline{\sqcup(e)})| - |E(\partial_f W_{N-1} \ \cap \ \overline{\sqcup(e)})|}{(|F(W_N \ \cap \ \overline{\sqcup(e)})| - |F(W_{N-1} \ \cap \ \overline{\sqcup(e)})|} \geq \sqrt{\frac{(f-2)((d-2)(f-2)-4)}{d-2}}$$

And since $\sqcup(e)$ is (d, f)-regular we have by [4]:

$$\frac{|E(\partial_f W_N \ \cap \ \sqcup(e))| - |E(\partial_f W_{N-1} \ \cap \ \sqcup(e))|}{(|F(W_N \ \cap \ \sqcup(e))| - |F(W_{N-1} \ \cap \ \sqcup(e))|} \geq \sqrt{\frac{(f-2)((d-2)(f-2)-4)}{d-2}}$$

But then by Lemma 3.16

$$\frac{|E(\partial_f W_N)| - |E(\partial_f W_{N-1})|}{(|F(W_N)| - |F(W_{N-1})|} \geq \sqrt{\frac{(f-2)((d-2)(f-2)-4)}{d-2}}$$

which is a contradiction.

If $l_F \leq l_V$ then let $v \in V(F_l) \cap V_l$ be a vertex satisfying $d(Q) > f$. Then $\bigvee(v)$ is (d, f)-regular with $|P_i(v)| = |P_{i+1}(v)|$. By the same argument as above we conclude that

$$\begin{array}{cc} \frac{|E(\partial_f W_N)|-|E(\partial_f W_{N-1})|}{(|F(W_N)|-|F(W_{N-1})|} & \geq \\ min\{\frac{|E(\partial_f W_N \cap \overline{\bigvee(v)})|-|E(\partial_f W_{N-1} \cap \overline{\bigvee(v)})|}{(|F(W_N \cap \overline{\bigvee(v)})|-|F(W_{N-1} \cap \overline{\bigvee(v)})|}, \frac{|E(\partial_f W_N \cap \bigvee(v))|-|E(\partial_f W_{N-1} \cap \bigvee(v))|}{(|F(W_N \cap \bigvee(v))|-|F(W_{N-1} \cap \bigvee(v))|}\} & \geq \\ \sqrt{\frac{(f-2)((d-2)(f-2)-4)}{d-2}} & . \end{array}$$

Once again a contradiction. $\square$

3.5 Proof of Main Theorem

Let G be a (d^+, f^+)-non-regular planar graph satisfying $H(d, f) = 4-(d-2)(f-2) \leq 0$ and let K be a finite subgraph of G. We assume that K is such as described in Assumption 3.1. Furthermore, let $W_N = G[F(K) \cap F(B_N)]$ where $N = N(K, x_0) = max\{n \mid V(K) \cap V_n \neq \emptyset\}$.

Let $\varphi(K) = \frac{|E(\partial(K))|}{|F(K)|}$. We will show that $\varphi \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$.

Let $\delta_N = max\{deg_G(x)|v \in V_N\}$ and let $x \in V_N$ be a vertex with $deg_G(x) = \delta$. If $\delta = d$ we have nothing left to reduce and we may continue onto the next procedure. If $\delta > d$, let $e = [xy]$ be an edge where $x \in V_N$ and $y \in V(G \setminus K)$. Since $\alpha * (\cdot)$ does not measure said edges we may delete e and $\varphi(K \setminus e) = \varphi(K)$. We continue by deleting outward edges until $\delta = d$ and let the resulting be $K^{(\delta_N)}$.

$$\varphi(K^{(\delta_N)}) = \varphi(K)$$

Let $\gamma_N = max\{d_{K^{(\delta_N)}}(R)|R \in F_{\delta_N}\}$ and let $Q \in F_{\delta_N}$ be a face with $d(Q) = \gamma_N$. If $\gamma_N = f$ we have nothing left to reduce (in F_{δ_N}) and we may continue. If $\gamma_N > f$, let $e = [uv] \in E(Q) \cap E(\partial(K^{(\delta_N)}))$. We contract edge e to a vertex $u = v$. Observe that

$$\varphi(K^{(\delta_N)}/\{e\}) = \frac{|E(\partial(K^{(\delta_N)}))| - 1}{|F(K^{(\delta_N)}/\{e\})|} = \frac{|E(\partial(K^{(\delta_N)}))| - 1}{|F(K^{(\delta_N)})|} < \frac{|E(\partial(\delta_N))|}{|F(K^{(\delta_N)})|} = \varphi(K^{(\delta_N)})$$

Continuing in like manner we contract edges in $E(Q) \cap E(\partial(K^{(\delta_N)})$ until each face in F_{δ_N} has degree f. Let the resulting graph be $K^{(\gamma_N)}$.

$$\varphi(K^{(\gamma_N)}) \leq \varphi(K)$$

Observe that F_{γ_N} is (d, f)-regular. Applying Lemma 3.5 we contract region F_{γ_N}, let $K^{(N')}$ be the resulting graph. By Lemmas 3.4. and 3.5. we have

$$\varphi(K^{(\gamma_N)}) \geq min\left\{\varphi(K^{(N')}) \ , \ (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}\right\}$$

and hence

$$\varphi(K) \geq min\left\{\varphi(K^{(N')}) \ , \ (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}\right\}$$

Now observe that $K^{(N')}$ is actually $W_{N-1} = G[F(K) \cap F(B_{N-1})]$ and $N - 1 = N'$ where $N' = N'(K, x_0) = max\{n \mid V(K^{(N')}) \cap V_n \neq \emptyset\}$. Thus we may iterate the process $N - 1$ times and each time

$$\varphi(K) \geq min\left\{\varphi(K^{(N')}) \ , \ (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}\right\}$$

where $K^{(N')}$ is the resulting graph. Let K_0 be the result of $N - 1$ iterations. Observe that $K_0 = W_1$ is a 1-ball centered at x_o with $\lambda = (x_0)$ faces. Let $F(K_0) =$

$\{R_1, R_2, ..., R_\lambda\}$. It follows that

$$\varphi(K_0) = \frac{|E(\partial(K_0))|}{|F(K_0)|} = \frac{\sum_{i=1}^{\lambda}(|E(R_i)| - 2)}{\lambda} \geq \frac{\sum_{i=1}^{\lambda}(f-2)}{\lambda} = f - 2$$

Therefore

$$\varphi(K) \geq min\left\{f - 2\ ,\ (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}\right\} \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}.$$

□

Thus we have shown that

$$\alpha^*(G) \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

by reducing a graph K to a (d, f)-graph. An argument in [4] is given to demonstrate that this graph must also satisfy

$$\alpha(G) \geq \frac{d-2}{d}\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

3.6 Summary

For a planar (d^+, f^+)-graph G satisfying $H(d,f) = 4 - (d-2)(f-2) \leq 0$ we have

$$\alpha(G) \geq \frac{d-2}{d}\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

and

$$\alpha^*(G) \geq (f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$$

Conjecture 3.21 *For a planar (d^+, f^+)-graph G satisfying $H(d,f) = 4 - (d-2)(f-2) \leq 0$. If $D = \max\{deg_G(x) \mid x \in V(G)\}$ exists and if $F = \max\{d_G(R) \mid R \in F(G)\}$ exists, then*

$$\alpha^*(G) \leq (F-2)\sqrt{1 - \frac{4}{(D-2)(F-2)}}$$

It is worth noting that while this would give an upper bound for the isoperimetric number for the general graph it would not necessarily be a good bound, for if a graph G is locally (d,f)-regular but tends to (D,F)-regularity elsewhere then the constant $\alpha^*(G)$ would tend towards $(f-2)\sqrt{1 - \frac{4}{(d-2)(f-2)}}$ since $\alpha^*(G)$ is defined to be the infimum ratio, $\inf\left\{\frac{|E(\partial_f K)|}{|F(K)|}\right\}$, of all subgraphs $K \subset G$. Thus while this conjecture may be interesting to some (at least one) and likely achievable, it is the author's opinion that a new ratio must be defined to give an effective upper bound.

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