NON-SEPARATING CYCLES THROUGH PRESCRIBED

VERTICES IN PLANE GRAPHS

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ABSTRACT

A graph G is *k-connected* if it has at least $k+1$ vertices and remains connected after deleting $k-1$ vertices of G. A *plane graph* is a graph G with an embedding on a plane $\mathbb{R}^2$ such that the interior of an edge contains no vertex and no point of any other edge. When G is a plane graph, we call the regions of $\mathbb{R}^2 \backslash G$ the *faces* of G. A planar graph is a graph isomorphic to a plane graph. A plane triangulation is a plane graph in which each face is bounded by a 3-cycle. A graph G is *k-linked* if, for any given $2k$ vertices there are k disjoint paths joining each pair of them. For a given graph H, a graph G is *H-linked* if, for every injective map from the vertices of H to the vertices of G, G contains a subdivision of H. Let $(K_4 - e)$ be the graph obtained from K_4 by removing one edge. A graph G is $(K_4 - e)$-*linked* if, for every injective map from the vertices of $K_4 - e$ to the vertices of G, G contains a subdivision of $K_4 - e$. A graph is said to be *k-cyclable* if given any set of k vertices there is a cycle that contains the k vertices. We say that a graph is *k-ordered* or C_k-*linked* if given any set of k vertices there is a cycle through the k vertices in any specified order. A graph is $(K_2 \cup K_3)$-*linked* if for every set of two vertices and every set of three vertices there exists a path joining the two vertices and a cycle on the three vertices.

Seymour and Thomassen's 2-linkage theorem characterizes all graphs which have two disjoint paths connecting any given two pair of vertices. Goddard proved that every 4-connected maximal planar graph is 4-ordered. Ellingham, Plummer, and Yu proved that any 4-connected planar triangulation is $(K_4 - e)$-linked.

In this thesis we completely characterize the obstructions to $(K_2 \cup K_3)$-linkage in 4-connected plane triangulations. From these obstructions we can see that no 4-connected nor 5-connected plane triangulation is $(K_2 \cup K_3)$-linked.

CONTENTS

CHAPTER 1

INTRODUCTION

A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, ..., x_k\}$ $E = \{x_0 x_1, x_1 x_2, ..., x_{k-1} x_k\}$. If $P = x_0...x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1} x_0$ is called a *cycle*. The *length* of a cycle is its number of edges (or vertices; the cycle of length k is called a *k-cycle* and denoted by C_k [2]. A graph G is *k-connected* if it has at least $k + 1$ vertices and remains connected after deleting $k - 1$ vertices of G. Such sets of k vertices are referred to as *k-cuts*. Every k-connected graph has a cycle through any given k-vertices. A graph G is *k-linked* if, for any given $2k$ distinct vertices $s_1, t_1, s_2, t_2, ..., s_k, t_k$, there are k disjoint paths $P_1, ..., P_k$ such that each P_i joins s_i and t_i for $i \in [k]$. Clearly, a k-linked graph is k-connected, but not vice versa. For example, a non-maximal 4-connected graph is not 2-linked [9]. A necessary condition for G to be k-linked is that G is $(2k - 1)$-connected. This condition is not sufficient unless $k = 1$ [9]. Jung, [11] proved that every 6-connected graph is 2-linked.

An (x_1, x_2, y_1, y_2)-*linkage* is a set of two disjoint paths P_1, P_2 such that P_1 joins x_1 to y_1 and P_2 joins x_2 to y_2.

A *plane graph* is a pair (V, E) of finite sets with the following properties[2]:

  (i) $V \subset \mathbb{R}^2$;

  (ii) every edge is an arc between two vertices;

  (iii) different edges have different sets of endpoints;

  (iv) the interior of an edge contains no vertex and no point of any other edge.

When G is a plane graph, we call the connected components of $\mathbb{R}^2 \backslash G$ the *faces* of G. A planar graph is an abstract graph isomorphic to a plane graph [24]. A *maximal* planar graph is a graph which is edge maximal such that the addition of an edge

would contradict its planarity. A plane triangulation is a maximal planar graph in which each face is bounded by a 3-cycle. A graph G is a near-triangulation if G is planar and all faces of G except at most one are bounded by a 3-cycle, except one face bounded by a 4-cycle. A near-triangulation can be obtained from a plane triangulation by means of the removal of any vertex from the plane triangulation. An (x_1, x_2, y_1, y_2)-*web* consists of a plane graph G_0 in which the outer face is bounded by a 4-cycle $S_0 : x_1 x_2 y_1 y_2 x_1$ and is such that every other face is bounded by a 3-cycle [24]. An (x_1, x_2, y_1, y_2)-web also requires that G_0 has no separating 3-cycle [24]. For each 3-cycle S of G_0 we add K^S, a possible empty complete graph disjoint from G_0, and we join all vertices of K^S to all vertices of S [24]. S_0 is called the *frame* of the (x_1, x_2, y_1, y_2)-web G and G_0 is called the *rib* of G [24]. These definitions lead us to the statement of the following Theorem:

Theorem 1.1 (Thomassen, [24]). *Let x_1, x_2, y_1, y_2 be vertices of a graph G. If G has no (x_1, x_2, y_1, y_2)-linkage and the addition of any edge to G results in a graph containing an (x_1, x_2, y_1, y_2)-linkage, then G is an (x_1, x_2, y_1, y_2)-web. Conversely, any (x_1, x_2, y_1, y_2)-web is maximal with respect to the property of not containing an (x_1, x_2, y_1, y_2)-linkage.*

For a given graph H, a graph G is H-*linked* if, for every injection $\phi : V(H) \to V(G)$, the graph G contains a subdivision of H with $\phi(v)$ corresponding to v for each $v \in V(H)$. The definition of H-linkage generalizes k-linkage. An H-linked graph is k-linked if H is the union of k independent edges, i.e., k disjoint copies of K_2 [7]. A graph on four vertices is called a *kite* if it is obtained from K_4 by deleting two adjacent edges. *Kite-linkage* is a type of H-linkage in which H is a kite. For a given graph H, let $f(H)$ be the minimum integer α such that every α-connected graph is H-linked. Liu, Rolek, Stephens, Ye, and Yu [7] proved the following:

Theorem 1.2 ([16]). *Every 7-connected graph is kite-linked.*

Jung, [11] proved that every 6-connected graph is 2-linked. Let $(K_4 - e)$ be the graph obtained from K_4 by removing one edge. This graph is referred to as the

diamond. Hence, a $(K_4 - e)$-linked graph can also be referred to as *diamond*-linked. For graphs on surfaces Ellingham, Plummer, and Yu [6] investigated $(K_4 - e)$-linked graphs and proved the following result:

Theorem 1.3 (Ellingham, Plummer, Yu [6]). *Any 4-connected planar triangulation is $(K_4 - e)$-linked.*

A graph H is a *minor* of a graph G if a copy of H can be obtained from G via repeated edge deletion and/or edge contraction. A facial cycle of a plane graph is a cycle whose interior or exterior does not intersect the graph, and a facial cycle in a planar graph is a cycle which is facial in some plane representation of the graph [9]. If the graph is 3-connected, a facial cycle is facial in any plane representation of the graph [9]. Let G and H be graphs. An *H-minor* in G is a set $\{G_x : x \in V(H)\}$ of pairwise disjoint connected subgraphs of G indexed by the vertices of H, such that if $xy \in E(H)$ then some vertex in G_x is adjacent to some vertex in G_y [7]. Each subgraph G_x is called a *branch set* of the minor [7]. A complete graph K_t-minor in G is *rooted* at distinct vertices $v_1, \ldots, v_t \in V(G)$ if $v_1, \ldots, v_t$ are in distinct branch sets [7]. Hence, a graph's containing of a rooted minor is a stronger condition than a graph's containing of the same minor, while a graph's property of linkage of the same type is a stronger condition than both containing a minor and a rooted minor. For example, the property that a graph G is K_4-linked is a stronger condition than the graph G containing K_4 as a rooted minor, and the property that G contains K_4 as a rooted minor is a stronger condition than the property that G contains K_4 as a minor. Consider graphs G and G'. If $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ then G' is a *subgraph* of G. Now, let $x, y \in V(G)$ and let $xy \in E(G)$. G' is an *induced subgraph* of G if $x, y \in V(G')$ implies $xy \in E(G')$. An *induced path* of a graph G is a path which constitutes an induced subgraph of G.

A graph is said to be *k-cyclable* if given any set of k vertices there is a cycle that contains the k vertices. This property was introduced by Watkins and Mesner [8], who characterized 3-cyclable graphs and gave sufficient conditions for a graph to be

k-cyclable. It is well known that being 2-connected is equivalent to being 2-cyclable, and that in general being k-connected implies k-cyclable. Also, a Hamiltonian graph is one that is k-cyclable for all k [5]. We say that a graph is *k-ordered* or C_k-*linked* if given any set of k vertices there is a cycle through the k vertices in any specified order. McCarty, Wang, and Yu [18] proved the following theorem.

Theorem 1.4 ([18]). *Every 7-connected graph is 4-ordered.*

Being 3-ordered is equivalent to being 3-cyclable, but for $k \geq 4$ being k-ordered is stronger than being k-cyclable [5]. Goddard [9] proved the following theorem.

Theorem 1.5 ([9]). *A 4-connected maximal planar graph is 4-ordered.*

Mukae and Ozeki [20] proved the following stronger result.

Theorem 1.6 ([20]). *Let G be a 4-connected triangulation of any surface. Then G is 4-ordered.*

The following is the well-known Lovász Path Removal Conjecture.

Conjecture 1.1 (Lovász [17]). *There exists a function $f = f(k)$ such that the following holds. For every $f(k)$ connected graph G and two vertices s and t in G, there exists a path P with endpoints s and t such that $G - V(P)$ is k-connected.*

Let G be any connected graph containing a cycle and let m and n be two non-negative integers such that $1 \leq m + n \leq |V(G)|$. Then the graph G is said to satisfy property $C(m, n)$ (or simply, G is $C(m, n)$), if for any two disjoint sets S_1 and S_2 contained in $V(G)$ with $|S_1| = m$ and $|S_2| = n$, there is a cycle C in G such that $S_1 \subset V(C)$, but $S_2 \cap V(C) = \varnothing$ [10]. A graph is *claw*-free if it contains no induced subgraph isomorphic to $K_{1,3}$ [10]. Győri, et al. [10] proved the following theorem for claw-free graphs:

Theorem 1.7 ([10]). *Let G be a 3-connected claw-free graph. Then G satisfies property $C(4, 1)$.*

CHAPTER 2

PRELIMINARY RESULTS

In this section we state results and prove them as needed for use in later sections.

Theorem 2.8 (Kuratowski, [15]). *Let G be a graph. G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

Let λ denote the connectivity of a finite simple graph G. Let ζ be the largest integer z not exceeding $|V(G)|$ such that for any set $U \subset V(G)$ with $|U| = z$, there is a cycle in G which contains U [26]. Let S be a set of vertices in G. Let $i(S)$ denote the component index of S which is the number of components in $G(V - S)$.

Theorem 2.9 (Watkins and Mesner, [26]). *Let G be a graph with $\lambda = 2$. A necessary and sufficient condition that $\zeta = \lambda$ is that there exists a set $S \subset V$ such that one of the following three (sets of conditions) holds:*

  I. *$S = 2$ and $i(S) \geq 3$.*

  II. *(a) $S = \{s^1, s^2, s^3, s\}$*

      (b) Each set $S^m = \{s^m, s\}$ separates G ($m = 1, 2, 3$).

      (c) Each pair of elements of S is joined by an arc in G having no interior vertex in S.

  III. *(a) $S = \{s_n^m : m = 1, 2, 3; n = 1, 2\}$*

      (b) Each set $S^m = \{s_1^m, s_2^m\}$ separates G ($m = 1, 2, 3$).

      (c) There is an arc in G joining s_n^m to $s^p - q$ with no interior vertex in S if and only if $m = p$ or $n = q$

Theorem 2.10 (Munkres, [21]). *Let C be a simple closed curve in S^2. Then C separates S^2 into precisely two components W_1 and W_2. Each of the sets W_1 and W_2 has C as its boundary; that is, $C = \overline{W_i} - W_i$ for $i = 1, 2$.*

Theorem 2.11 ([2])**.** *Every maximal planar graph with at least four vertices is 3-connected.*

Every plane triangulation is a maximal planar graph. Therefore, by Theorem 2.11, we have that every plane triangulation is 3-connected.

Theorem 2.12 (Menger's Theorem, [19])**.** *A graph is k-connected if and only if it contains k independent paths between vertices.*

Proposition 2.2. *Every 4-cut in a 4-connected plane triangulation G separates G into exactly two components.*

Proof. Suppose G is a 4-connected plane triangulation and suppose to the contrary that the 4-cut separates G into three components. Each vertex of each of the three components must necessarily be connected to each vertex of the 4-cut. Thus, by contracting these components into vertices, we can see that G contains $K_{3,4}$ as a minor. Thus, G contains $K_{3,3}$ as a minor. Hence, by Kuratowski's Theorem, G is non-planar. This contradicts our assumption that G is a plane triangulation. □

Proposition 2.3. *Every plane triangulation is 3-cyclable.*

Proof. Suppose G is a plane triangulation. We wish to show that G is 3-cyclable. Since G is a plane triangulation, by Theorem 2.11, G must be at least 3-connected. Thus, since k-connectedness implies k-cyclability [9], we can conclude that G is 3-cyclable. □

Theorem 2.13 (Dirac [3])**.** *If a graph is k-connected for $k \geq 2$, then for every set of k vertices in the graph there is a cycle that passes through all the vertices of the set.*

Let G be a 4-connected plane triangulation, and let v be a vertex of G. Then all neighbors of v together with v induce a wheel, denoted by $W(v)$. So $W(v) - v$ is a cycle of G, induced by all neighbors of v.

Proposition 2.4. *Let G be a 4-connected plane triangulation and let v be a vertex of G. Then $N[v]$ induces a wheel.*

Proposition 2.5. *Let G be a 2-connected near-triangulation with a 2-cut $S = \{u, v\}$. Then $uv \in E(G)$ and $G - S$ consists of exactly two components.*

Proof. Firstly, we wish to show that $uv \in E(G)$. Thus, suppose that $uv \notin E(G)$. Then, since $S = \{uv\}$ is a 2-cut, there must be at least one vertex in each component of $G - S$ which joins $G - S$ to S. Hence, if $uv \notin E(G)$, then G is not a near-triangulation. Hence, $uv \in E(G)$. Now we wish to show that $G - S$ consists of exactly two components. Thus, suppose to the contrary that $G - S$ consists of $n \geq 3$ components. If we contract each vertex of each of these n components to a single vertex, since at least one vertex in each component of $G - S$ joins $G - S$ to S, then G contains $K_{2,3}$ as a minor. Thus, since $uv \in E(G)$, the contraction of uv shows that G contains $K_{1,3}$ $\qquad\square$

An embedding of a graph in a surface is *polyhedral* if the boundaries of every two faces meet properly, i.e., their intersection is either empty, or a single vertex or an edge [10]. A graph is a *polyhedral map* if it is polyhedrally embedded.

Proposition 2.6. *Every polyhedral map is $C(3, 1)$.*

A plane triangulation is indeed a polyhedral map. The above proposition tells us that a plane triangulation has a cycle through any given three vertices but avoiding the fourth given vertex.

Proposition 2.7. *Every 2-connected plane near-triangulation is 3-cyclable.*

Proof. Let G be a 2-connected near triangulation. If G has a face which is not bounded by a triangle, choose a point from the face and make it to be a new vertex z, and then connect z to all vertices on the face boundary. Then the resulting graph G' is a plane triangulation. By Proposition 2.6, it follows that G' has a cycle through any given three vertices of G but avoiding z. Therefore, G is 3-cyclable. This completes the proof. $\qquad\square$

Note that the cyclability for 2-connected plane near-triangulation is the best possible. The following example shows that there are 2-connected plane near-triangulations which are not 4-cyclable.
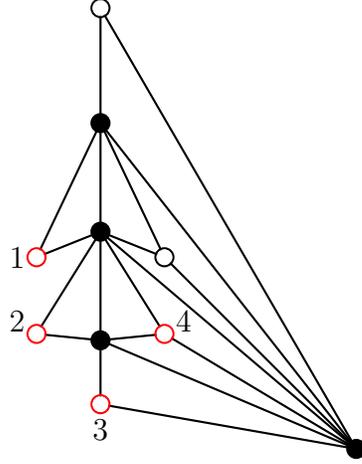

Figure 1: A 2-connected near triangulation without a cycle through vertices $1, 2, 3, 4$.

Let $W(v)$ be a wheel and x and y be two non-adjacent vertices on the rim C of $W(v)$. Let H be a connected subgraph of $G - \{x, y\}$ such that $H \cap C(x, y) \neq \emptyset$, $H \cap C(y, x) \neq \emptyset$, and $v \in H$. If v is not a cut-vertex of H, then H has a cycle D containing v, which contains exactly one vertex from each of $H \cap C(x, y)$ and $H \cap C(y, x)$. Since G is a plane triangulation, G is K_5-minor free and therefore G does not contain a path from x to y which is disjoint from D. It follows immediately that H is an $\{x, y\}$-separating subgraph. We state this property as follows.

Proposition 2.8. *Let G be a plane triangulation and let $W(v)$ be a wheel centered at v such that its rim $C = W(v) - v$ contains two non-adjacent vertices x and y. Assume that H is a subgraph of $G - \{x, y\}$ such that $v \in H$, $H \cap C(x, y) \neq \emptyset$, and $H \cap C(y, x) \neq \emptyset$. If v is not a cut-vertex of H, then H is an $\{x, y\}$-separating subgraph.*

CHAPTER 3

OBSTRUCTIONS

Let G be a 4-connected plane triangulation, and suppose v_1, v_2, v_3, x, y are five given vertices of G. In this section we prove that if G contains the following obstructions, it does not have a cycle C through v_1, v_2, v_3 such that $G - V(C)$ has a path joining x and y.

A k-wheel is a graph with $k + 1$ vertices constructed by joining a vertex x to all vertices on a k-cycle with ordered vertices $v_1, v_2, \ldots, v_k$, denoted by $W(x; v_1, \ldots, v_k)$. Note that Obstructions I, III, and IV are proven the same way.

Obstruction I. (a) The wheel $W(v_1; x, \ldots, v_2, \ldots, y, \ldots, v_3)$. That is, any wheel which consists of the vertices v_1, x, v_2, y, v_3 and is such that one of $\{v_1, v_2, v_3\}$ serves as the hub.
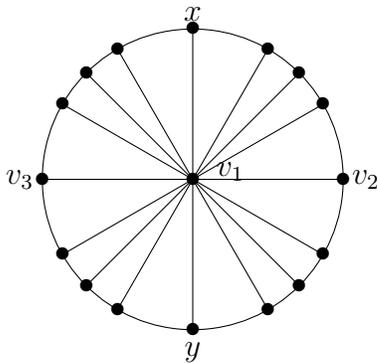
Obstruction I. (b) The wheel $W(v_1; x, \ldots, y, \ldots v_3)$ in which v_2 and v_3 lie on opposite sides of W as depicted in the graph below. Note that the region containing v_2 is an arbitrary triangulated component.
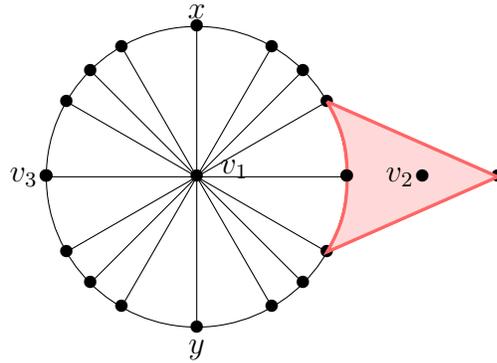
Obstruction I. (c) The wheel $W(v_1; x, ..., y, ...)$ which separates v_2 and v_3 which lie on opposite sides of W as depicted in the graph below. Note that the region containing v_2 and the region containing v_3 are arbitrary triangulated components.
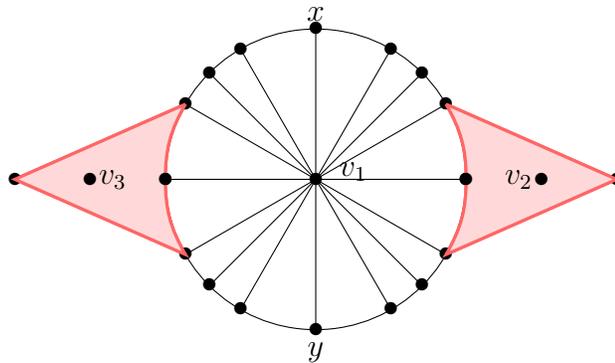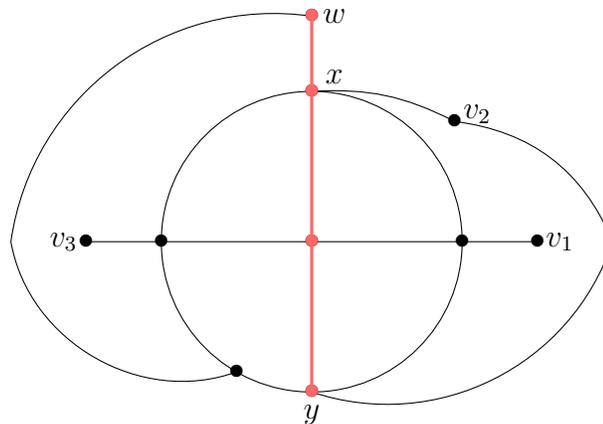

Proof. Let D be a cycle containing the vertices v_1, v_2, v_3 which is disjoint to the vertices x and y. Using Proposition 2.8, let $H = D$ and let $v = v_1$. Then D is an $\{x, y\}$-separating subgraph. Thus, $G - V(D)$ has no path joining x and y. $\qquad \square$

Obstruction II. The Diamond K_4^-. That is, the complete graph on four vertices minus one edge connecting the prescribed vertices x and y. This obstruction is such that it consists of any two vertices of the set $\{v_1, v_2, v_3\}$ as well as x and y such that x and y are not adjacent.
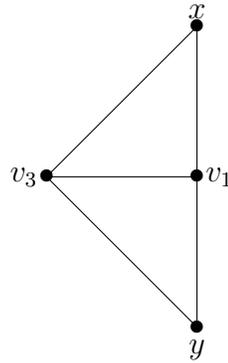
Proof. As previously stated, since G is a plane triangulation, every face is bounded by a 3-cycle. Without loss of generality, suppose the prescribed vertices lie on the Diamond as above. Note that v_2 is not contained in the Diamond. Now, by Theorem 2.12, since G is 4-connected there exists a path P_1 from v_2 to v_1 which is disjoint to y, x, and v_3. Also, by Theorem 2.12, there exists a path P_2 from v_3 to v_2 which is disjoint to y, x, and v_1. The paths P_1 and P_2 are such that $P_1 \cup P_2$ is a cycle C through the vertices v_1, v_2, and v_3 which avoids x and y. Hence, one of $\{x, y\}$ is on the interior of the cycle, while the other is on the outside. Therefore, by Theorem 2.10, $G - V(C)$ does not have a path joining x and y. □

Obstruction III. A 4-cut S in which $\{x, y\} \subset S$, $xy \notin E(G)$, and S separates any two of $\{v_1, v_2, v_3\}$.

Proof. Let D be a cycle containing the vertices v_1, v_2, v_3 which is disjoint to the vertices x and y. Using Proposition 2.8, let $H = D$ and let v be the unlabelled vertex of the 4-cut. Then D is an $\{x, y\}$-separating subgraph. Thus, $G - V(D)$ has no path joining x and y. $\qquad\square$

Obstruction IV. $K_{2,3}$ with prescribed vertices $\{x, y, v_1, v_2, v_3\}$ in which each path of distance two joining x and y within $K_{2,3}$ is obstructed by some $v_i, v_j, v_h \in \{v_1, v_2, v_3\}$. Note that each face of the graph is an arbitrary triangulated component.
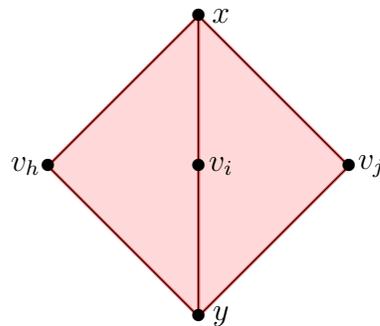
Proof. Let D be a cycle containing the vertices v_1, v_2, v_3 which is disjoint to the vertices x and y. Using Proposition 2.8, let $H = D$ and let $v = v_i$. Then D is an $\{x, y\}$-separating subgraph. Thus, $G - V(D)$ has no path joining x and y. $\qquad\square$

CHAPTER 4

MAIN RESULT

Theorem 4.14. *Let G be a 4-connected plane triangulation. For two vertices x and y, and three other vertices v_1, v_2, and v_3, the graph G has a cycle C through v_1, v_2, and v_3 such that $G - V(C)$ has a path joining x and y unless G contains Obstruction I, II, III, or IV.*

In this section, we will show that if G does not contain Obstruction I, II, III, or IV, then the graph G has a cycle C through v_1, v_2, and v_3 such that $G - V(C)$ has a path joining x and y.

Suppose then that G does not contain Obstruction I, II, III, or IV. Suppose $xy \in E(G)$. Thus, $G - \{x, y\}$ is a 2-connected near triangulation. Hence, by Proposition 2.7, $G - \{x, y\}$ is 3-cyclable. That is, $G - \{x, y\}$ contains a cycle C through the vertices v_1, v_2, and v_3. Hence, the graph G has a cycle C through v_1, v_2, and v_3 such that $G - V(C)$ has a path joining x and y.

Now, suppose $xy \notin E(G)$. We then have two cases to consider. Case 1: $dist(x, y) = 2$. Case 2: $dist(x, y) \geq 3$.

Let G be a 4-connected plane triangulation. Let C be a cycle of G, and denote the subgraph obtained from G by removing all vertices and edges on the left of C along the clockwise direction of C by int$[C]$.

Theorem 4.15. *Let G be a 4-connected plane triangulation. For any given five vertices x, y, v_1, v_2 and v_3, if $\mathrm{dist}(x, y) \geq 3$, then G has a cycle C through v_1, v_2, v_3 such that $G - V(C)$ has a connected component containing x and y.*

Proof. Let G be a 4-connected plane triangulation. By Theorem 2.12, G has four internally disjoint induced (x, y)-paths P_1, P_2, P_3 and P_4.

It follows from $\mathrm{dist}(x, y) \geq 3$ that $N_G(x) \cap N_G(y) = \emptyset$. Let W_x and W_y be the wheels of x and y, respectively. Without loss of generality, assume that P_i joins

$x_i \in N_G(x)$ to $y_i \in N_G(y)$ such that x_1, x_2, x_3, x_4 appear in clockwise order on the cycle of W_x and y_1, y_2, y_3, y_4 appear in anti-clockwise order on the cycle of W_y.

Let $C_{ij} = P_i \cup P_j$ and $R[i,j] = \text{int}[C_{ij}] - \{x, y\}$ where $i, j \in [4]$. Then $R[i,j]$ is a 2-connected near-triangulation because $N_G(x) \cap N_G(y) = \emptyset$.

An induced (x, y)-path P_k is *non-removable* to a vertex v_i if either P_k contains v_i or $G - V(P_k)$ has a cut-vertex separating v_i from $R[k+1, k-1]$ where $k-1, k+1 \in [4]$ modulo 4. Otherwise, P_k is *removable* to v_i. Note that, if P_k is non-removable to a vertex v_i, then v_i belongs to either $R[k, k+1]$ or $R[k-1, k]$.

Claim. *For a prescribed vertex v_i, if P_k is non-removable, then all other three (x, y)-paths are removable to v_i.*

Proof of Claim. Without loss of generality, assume that P_1 is non-removable to v_i. Then $v_i \in R[4, 1]$ or $R[1, 2]$. So $v_i \in R[4, 1] \cup R[1, 2] = R[4, 2]$, which means that P_3 is removable to v_i. Therefore, it suffices to show that both P_2 and P_4 are also removable to v_i.

If $v_i \in V(P_1)$, then $v_i \in R[1, 3]$ and $v_i \in R[3, 1]$ which means both P_4 and P_2 are removable to v_i. So assume that $v_i \notin P_1$. Then either $v_i \in R[4, 1] - V(P_1 \cup P_4)$ or $v_i \in R[1, 2] - V(P_1 \cup P_2)$. By symmetry, assume that $v_i \in R[1, 2] - V(P_1 \cup P_2)$. Then $v_i \in R[1, 3]$ and P_4 is removable to v_i. In the following we show that P_2 is also removable to v_i.

Since P_1 is non-removable and $v_i \notin V(P_1)$, then $G - V(P_1)$ has a cut-vertex w separates v_i and $R[2, 4]$. Since G is plane triangulation, both faces incident with w containing edges from both components separated by w in $G - V(P_1)$ must be a triangle, which implies that w has at least two neighbors in P_2. Let w_1, w_2 be two neighbors of w on P_1 such that the two subpaths of $P_1(x, w_1)$ and $P_1(w_2, y)$ have no neighbors of w. Therefore, $\text{int}[w_1 w w_2 \cup P_1[w_1, w_2]] \cup R[3, 1]$ is a 2-connected near-triangulation containing v_i. Therefore, P_2 is removable to v_i. This completes the proof of Claim.

It follows from Claim and the Pigeon Hole principle that there exists one from the

four induced (x, y)-paths P_i's which is removable to all v_1, v_2 and v_3. Without loss of generality, assume that P_1 is removable to all v_1, v_2 and v_3. Therefore, $G - V(P_1)$ has a 2-connected near-triangulation containing all v_1, v_2 and v_3. By Proposition 2.7, G has a cycle C through v_1, v_2 and v_3 which is disjoint from P_1. Therefore, $G - V(C)$ has a component containing P_1 and hence x and y. This completes the proof. $\qquad\square$

Let G be a connected graph. A maximal 2-connected subgraph of G is called a *block* of G. Let w be a cut vertex of G. We say w *separates* connected subgraphs Q and Q^c if $Q \cup Q^c = G$ and $Q \cap Q^c = \{w\}$. The subgraph Q^c is called the complement of Q in G, and Q is also the complement of Q^c in G.

Proof of Theorem 4.14. Assume that G does not contain the four obstructions. For any given five vertices v_1, v_2, v_3, x, and y, we need to prove that G has a cycle through any given three vertices v_1, v_2 and v_3 which does not separate the two remaining vertices x and y.

If $xy \in E(G)$, then $G - \{x, y\}$ is 2-connected which has only one face which is not a triangle. So $G - \{x, y\}$ is a 2-connected near triangulation. It follows that G does have a cycle containing v_1, v_2 and v_3, which cannot separate x and y. By Theorem 4.15, we assume that $\operatorname{dist}(x, y) = 2$.

Claim 1. *The graph G does not have a 3-path xvy such that $v \notin \{v_1, v_2, v_3\}$.*

Proof of Claim 1. Suppose to the contrary that G has a path xvy such that $v \notin \{v_1, v_2, v_3\}$. Since G is 4-connected, $G - V(P)$ is connected. If $G - V(P)$ is 2-connected, then $G - V(P)$ is a 2-connected plane near triangulation. Therefore, $G - V(P)$ has a cycle through v_1, v_2 and v_3. Now, assume that $G - V(P)$ has a cut vertex w. Then $S := \{x, v, y, w\}$ is a 4-cut. Since G does not contain Obstruction III, it follows that v_1, v_2, v_3 belong to a block of $G - V(P)$ which is a 2-connected near triangulation which has a cycle through v_1, v_2 and v_3. This completes the proof of Claim 1.

From Claim 1, every 3-path joining x and y must contain a vertex from $\{v_1, v_2, v_3\}$. Then $N(x) \cap N(y) \subseteq \{v_1, v_2, v_3\}$. Since G does not contain Obstructions II and IV, it follows that $N(x) \cap N(y)$ has at most two non-adjacent prescribed vertices.

Since $\text{dist}(x, y) = 2$, without loss of generality, assume that $v_1 \in N(x) \cap N(y)$. Let $W(v_1)$ be the wheel centered at v_1. Then x and y separates the rim-cycle C of $W(v_1)$ into two paths $C[x, y]$ and $C[y, x]$. Since G does not contain Obstruction II, it follows that both $C[x, y]$ and $C[y, x]$ contain at least four vertices.

Claim 2. *Both $G - C[x, y]$ and $G - C[y, x]$ are connected.*

Proof of Claim 2. By symmetry, it suffices to show $G - C[x, y]$ is connected. Suppose to the contrary that $G - C[x, y]$ has at least two components Q_1 and Q_2. Since G is a 4-connected triangulation, it follows from Proposition 2.1 that C has no chords. Then G has a face f which is incident with both Q_1 and Q_2. Then $\partial f \cap Q_i$ is a trail connecting two vertices of $Q_i \cap C[x, y]$ with a length of at least two for $i \in [2]$. So ∂f has at least four edges, which contradicts that G is a plane triangulation. This completes the proof of Claim 2.

Since G does not contain Obstruction I. (a), it follows that one of $C[x, y]$ and $C[y, x]$ does not contain v_2 or v_3. Without loss of generality, assume that $C[x, y]$ contains neither v_2 nor v_3. So $|V(C) \cap \{v_2, v_3\}| = |C[y, x] \cap \{v_2, v_3\}|$. If $\{v_2, v_3\} \subseteq C[y, x]$, without loss of generality, assume y, v_2, v_3, x appear on $C[y, x]$ in order. Then $D := C[v_2, v_3] \cup v_2 v_1 v_3$ is a desired cycle containing the three prescribed vertices and is disjoint from $C[x, y]$. So, in the following, assume that $|C[y, x] \cap \{v_2, v_3\}| \leq 1$.

Case 1. The path $C[y, x]$ contains exactly one prescribed vertex. Without loss of generality, assume that $v_3 \in C[y, x]$ and $v_2 \in G - W(v_1)$. Note that v_3 and v_1 belong to the same block of $H := G - C[x, y]$. If H has no cut-vertex separating v_2 from v_1 and v_3, then H contains a 2-connected near triangulation containing v_1, v_2 and v_3. Then it follows from Proposition 2.7 that H and hence G has a desired cycle through v_1, v_2 and v_3, which is disjoint from $C[x, y]$. So assume that H has a cut-vertex w separating v_2 from the block Q containing v_1 and v_3.

Claim 3. *Every cut vertex of H has at least two neighbors on $C[x, y]$.*

Proof of Claim 3. The boundary of H is a closed walk and w appears at least twice on the closed walk. Every appearance of w on the closed walk implies a distinct neighbor on $C[x, y]$ since G is a simple plane triangulation. This completes the proof of Claim 3.

By Claim 3, the cut-vertex w has at least two neighbors on $C[x, y]$. Choose two neighbors w_1 and w_2 of w on $C[x, y]$ such that $C[x, w_1]$ and $C[w_2, y]$ contain no other neighbors of w. Then $\{w, w_2, v_1, w_1\}$ is a 4-cut separating v_2 from v_3. Note that $w \in Q$ but $v_2 \notin Q$. Since $N(x) \cap N(y) \subseteq \{v_1, v_2, v_3\}$, it follows that $\{w_1, w_2\} \neq \{x, y\}$ (otherwise, $w \in N(x) \cap N(y)$). Therefore, G contains the Obstruction I (b), a contradiction.

Case 2. The path $C[y, x]$ does not contain a prescribed vertex. So both $C[x, y]$ and $C[y, x]$ do not contain any prescribed vertex. Let $H = G - C[x, y]$ and $H' = G - C[y, x]$. If one of H and H' is 2-connected, then G has a desired cycle thought v_1, v_2 and v_3 which does not separate x and y by Proposition 2.7. By Claim 2, both H and H' are connected and hence have cut-vertices.

Let Q be the block of H containing $C(y, x)$ which has at least two vertices. Hence $v_1 \in Q$. Let Q' be the block of H' containing $C(x, y)$. Similarly, $v_1 \in Q'$. Let w be the cut-vertex of H separates Q and Q^c, and let w' be the cut-vertex of H' separates Q' and $(Q')^c$. By Claim 3, let w_1 and w_2 be the two neighbors of w on $C[x, y]$ such that $C[x, w_1]$ and $C[w_2, y]$ contains no other neighbors of w. Similarly, let w_1' and w_2' be the two neighbors of w' on $C[y, x]$ such that $C[y, w_1']$ and $C[w_2', x]$ contains no other neighbors of w'.

Claim 4. *It holds that $Q^c \subseteq Q'$ and $(Q')^c \subseteq Q$.*

Proof of Claim 4. First, we are going to show that $Q^c \subseteq Q'$. If $w_1 = x$, let x_1 be the neighbor of x on $C[x, y]$ and let P_x be the path on the rim-cycle of $W(x)$ connecting w and x_1, which belongs to the interior of the 4-cycle $ww_1v_1w_2w$. If $w_1 \neq x$, let

$P_x = ww_1$. Similarly, let P_y be the path on the rim-cycle of $W(y)$ connecting w and y_1 (the neighbor of y on $C[x,y]$) inside of the 4-cycle $ww_1v_1w_2w$ if $w_2 = y$, and $P_y = ww_1$ if $w_2 \neq y$. Then let $D := wP_xv_1P_yw$, a cycle of G. Based on the choice of w_1 and w_2, we have $Q^c \subseteq \text{int}[D]$. Since $\text{int}[D] \subseteq Q'$, it holds that $Q^c \subseteq Q'$.

By the symmetry of $C(x,y)$ and $C(y,x)$ (both of them do not contain prescribed vertices), a similar argument as above validates $(Q')^c \subseteq Q$. This completes the proof of Claim 4.

If Q contains both v_2 and v_3, then it follows from Proposition 2.7 that Q has a desired cycle containing v_1, v_2 and v_3, which is disjoint from $C[x,y]$. So assume Q contains at most one prescribed vertex, which means that $Q^c - w = H - V(Q)$ contains a prescribed vertex. Without loss of generality, assume that $v_2 \in Q^c - w$. If $w = v_3$, then the cycle $D := wP_xx_1v_1y_1w$ and its interior induce a graph $\text{int}[D]$ containing Q^c and hence all three prescribed vertices. So $\text{int}[D]$ has a desired cycle through v_1, v_2 and v_3, which is disjoint from $C[y,x]$. So in the following assume $v_3 \neq w$.

By symmetry of Q and Q', we can derived that $v_3 \in (Q')^c - w'$ and w' is not a prescribed vertex. By Claim 4, it follows that $Q^c \cap (Q')^c = \emptyset$. Therefore, v_2 is included inside of the 4-cycle $ww_1v_1w_3w$ and v_3 belongs to the interior of the 4-cycle $w'w_1'w_2'v_1$. Note that both w and w' are not prescribed vertices. By Claim 1, $\{w_1, w_2\} \neq \{x,y\}$ and $\{w_1', w_2'\} \neq \{x,y\}$. Therefore, G contains Obstruction I (c), a contradiction. This completes the proof. $\square$

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