RESONANCE GRAPH

OF PERFECT MATCHINGS

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by

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DEDICATION

I dedicated this to my extremely supportive Dream Team and, the love of my life, Rashida.

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ABSTRACT

Let G be a graph with perfect matchings and let $\mathcal{C}$ be a set of linearly independent even cycles of G of width at most 2. The resonance graph $\mathcal{R}(G,\mathcal{C})$ is a graph with the vertex set $M \subseteq \mathcal{M}(G)$ such that two vertices M_i and M_j are adjacent if and only if $M_i \oplus M_j = E(c)$ for some cycle $c \in \mathcal{C}$.

In this paper, we extend the results obtained by Tratnik and Ye [22] to general graphs. Particulary, we show that the resonance graph of every graph with perfect matchings with respect to a set of linearly independent even cycles is bipartite and each connected component of the resonance graph is an induced cubical graph.

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CHAPTER 1

INTRODUCTION

In this chapter, we lay the foundation with basic definitions and explore the motivation of studying resonance graphs.

1.1 Basic Definitions

In this section, we define the basic concepts of order theory, linear algebra, and graph theory that will be used throughout this paper. We will be basing our terminology primarily on Davey & Priestly [4] and Birkhoff [2] for order theory, Strang [19] for linear algebra, and Diestel [5] for graph theory.

Let P be a set. A *partial order* is a binary relation $\leq$ on the set P satisfying the following conditions:

1. For all $x \in P$, $x \leq x$. (Reflexive)

2. If $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in P$. (Antisymmetric)

3. If $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in P$. (Transitive)

Definition 1.1. A set P equipped with a partial order $\leq$ is called a *partially ordered set*, poset for short.

For $x, y \in P$, x and y are *comparable* if $x \leq y$ or $y \leq x$.

Definition 1.2. A poset P is a *chain* if any two elements of P are comparable.

Let P be a poset and $X \subseteq P$. An element $a \in P$ is an *upper bound* of X if $x \leq a$ for all $x \in X$. Similarly, a *lower bound* of X is an element $b \in P$ such that $b \leq x$ for all $x \in X$. Now, for $a \in P$, a is the *least upper bound* of X if

(i) a is an upper bound of X, and

(ii) $a \leq y$ for all upper bounds y of X.

Similarly, for $b \in P$, b is the *greatest lower bound* of X if

(i) b is a lower bound of X, and

(ii) $z \leq b$ for all lower bounds z of X.

Note that the least upper bound of the entire poset P is referred to as the *top element* of P. This is denoted by $\top$. Similarly, the greatest lower bound of the entire poset P is referred to as the *bottom element* of P which is denoted by $\bot$.

For $x, y \in P$, if there exists a least upper bound of $\{x, y\}$ we denote it as $x \vee y$ which is called the *join* of x and y. Similarly, for $x, y \in P$, if there exists a greatest lower bound of $\{x, y\}$ we denote it as $x \wedge y$ which is called the *meet* of x and y.

Definition 1.3. A *lattice*, then, is a poset L whereby $x \vee y$ and $x \wedge y$ exists for all $x, y \in L$.

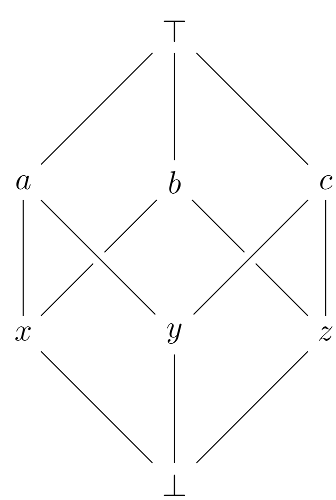


Figure 1: Hasse diagram of a lattice

Figure 1 gives an example of a lattice in order theory. We can see each pair of elements has a join and meet. More specifically, $\top = a \vee b = a \vee c = b \vee c$, $a = x \vee y$, $b = x \vee z$, $c = y \vee z$, $x = a \wedge b$, $y = a \wedge c$, $z = b \wedge c$, and $\bot = x \wedge y = x \wedge z = y \wedge z$.

Definition 1.4. A lattice L is *modular*, if, for all $x, y, z \in L$, $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 1.5. A lattice L is *distributive*, if, for all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

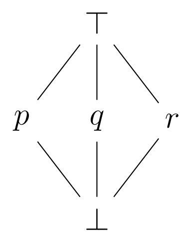


Figure 2: Non-distributive modular lattice

Though every distributive lattice is modular, the converse of that statement is not always true. Figure 2 is an example of a modular lattice that fails to be distributive. More specifically, $p \wedge (q \vee r) = p \neq \bot = (p \wedge q) \vee (p \wedge r)$.

Definition 1.6. Let V be a vector space over a field $\mathbb{F}$. For a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$ with scalars $c_1, \ldots, c_k \in \mathbb{F}$, we say $\vec{v}_1, \ldots, \vec{v}_k$ are *linearly independent* if $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ only when $c_1 = \cdots = c_k = 0$.

Definition 1.7. Let V be a vector space over a field $\mathbb{F}$. We say $w_1, \ldots, w_l$ *spans* V, if, for every vector $v \in V$, $v = c_1 w_1 + \cdots + c_l w_l$ whereby $c_i \in \mathbb{F}$.

Definition 1.8. A set of vectors B is a *basis* of vector space V, if B is a linearly independent subset of V that spans V.

The number of vectors in a basis is called the *dimension* of the vector space.

Definition 1.9. An *incidence vector* of a subset T of a set S is the vector $x_T := (x_s)_{s \in S}$ such that

$$x_s = \begin{cases} 1 & s \in T \\ 0 & s \notin T \end{cases}$$

.

Definition 1.10. Let $\mathbb{Z}$ denote the set of all integers and let $\mathbb{R}^m$ denote the m-dimensional real space. Given n linearly independent vectors $b_1, b_2, \ldots, b_n \in \mathbb{R}^m$, the *lattice* generated by $b_1, b_2, \ldots, b_n$ is defined as $L\left(b_1, b_2, \ldots, b_n\right) = \left\{\sum x_i b_i : x_i \in \mathbb{Z}\right\}$.

Definition 1.3 defines a lattice structure using concepts based from the field of order theory. However, like many terms in mathematics, lattice is not a unique name. Regev [18] defines a lattice structure algebraically, Definition 1.10. The applications of this form of a lattice has many uses in cryptography. Recently, Lai, Yang, Yu, Chen, and Bai [11] have implemented this form of lattices as a foundation for hash proof systems.

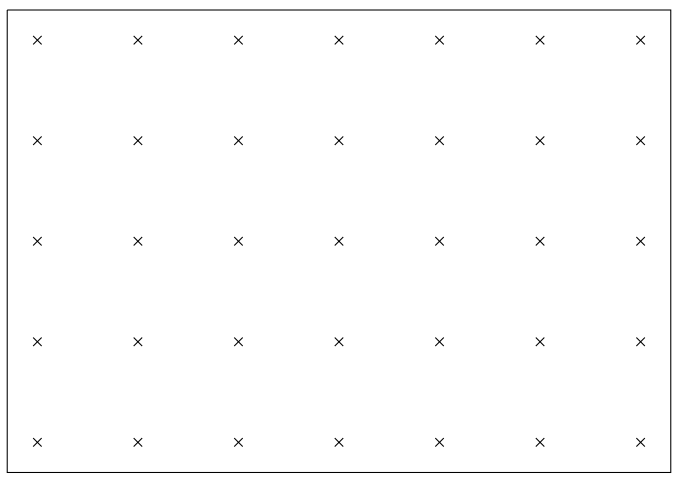


Figure 3: A lattice in $\mathbb{R}^2$

Figure 3 is an example of a lattice in $\mathbb{R}^2$ [18].

Definition 1.11. A *graph* is a pair $G = (V, E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. For example, x and y are elements of V so then we denote the element of E that connects x and y as xy.

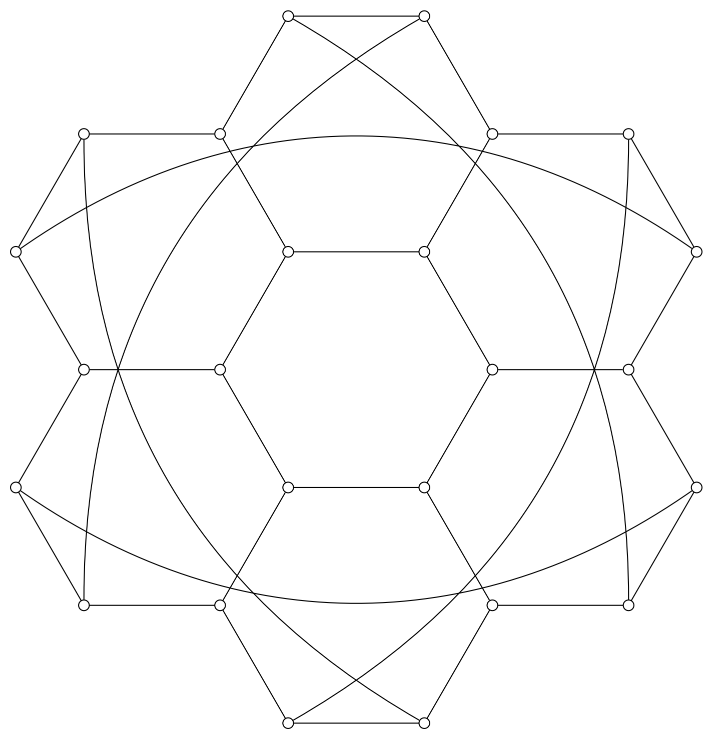


Figure 4: An example of a graph

The elements of V are the vertices of the graph G and the elements of E are its edges. We denote the set of vertices of G as $V(G)$ and the set of edges of G as $E(G)$. A graph $G'(V', E')$ is a *subgraph* of graph $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. An *induced subgraph* is a subgraph G' of graph G that contains all the edges $xy \in E$ with $x, y \in V'$. $G' \subseteq G$ is a *spanning subgraph* of G if $V' = V$.

The number of vertices of a graph G is its *order* which is denoted as $|G|$. A graph of order 0 or 1 is called a *trivial graph*. A graph is a *k-dimensional hypercube* with $k \geq 1$ if its vertices are all binary strings of length k and a pair of adjacent vertices' strings differ in only one position. We denoted a *k-dimensional hypercube* as Q_k. A *directed graph* is a pair (V, E) of disjoint sets of vertices and edges where every edge has an initial vertex and a terminal vertex.

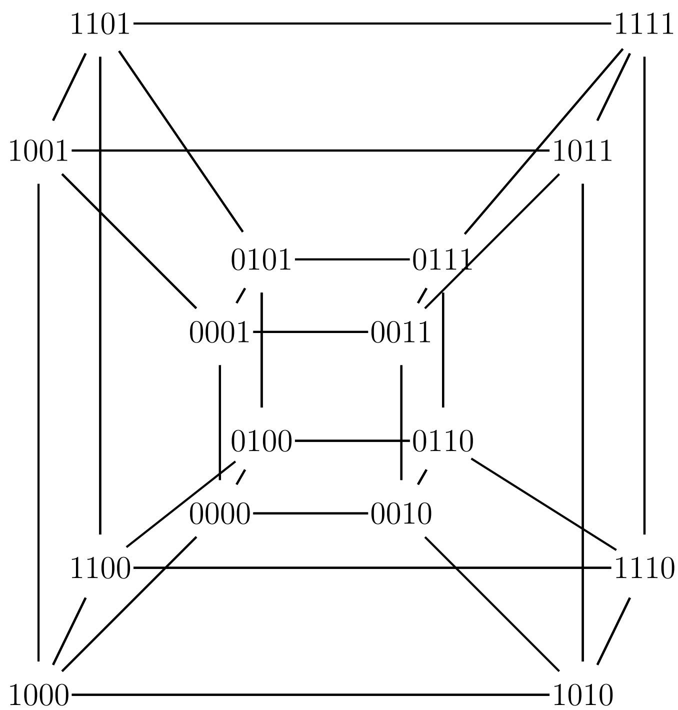


Figure 5: 4-dimensional hypercube

The previous figure, Figure 5, is an example of a 4-dimensional hypercube. The vertices are binary strings indicating their position and relationship to adjacent vertices.

Definition 1.12. Let G and H be graphs. An injective function $f : V(H) \to V(G)$ is called an *embedding* if $xy \in E(H)$ implies $f(x) f(y) \in E(G)$ for any $x, y \in V(H)$.

We note that if there exists an embedding $f : V(H) \to V(G)$ then H is isomorphic to a subgraph of G. Also, if H is embedded into a k-dimensional hypercube as a induced subgraph then H is an *induced cubical*.

A *path* is a non-empty graph $P = (V, E)$ such that $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$ where $x_i \neq x_j$ for $i \neq j$.

Definition 1.13. A *cycle* is a non-empty graph $C = (V, E)$ such that $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k, x_kx_0\}$ where $x_i \neq x_j$ for $i \neq j$.

A non-empty graph G is called *connected* if any two of its vertices are joined by a path in G. A *connected component*, or component, of a graph is a maximal connected subgraph of the previously mentioned graph. A graph not containing any cycles, known as acyclic, is called a *forest*. A connected forest is called a *tree*. So then, a *spanning tree* is subgraph which has all vertices connected with the minimal number of edges.

Definition 1.14. A graph $G = (V, E)$ is called a *bipartite graph* if V admits a partition into 2 classes such that every edge has its ends in different classes.

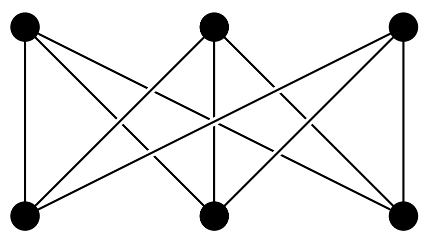


Figure 6: Bipartite graph

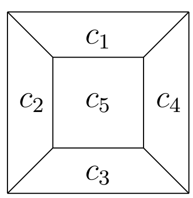
A *matching* M is a set of independent edges in a graph G.

Definition 1.15. A *perfect matching* of a graph G is a set of independent edges of G which covers all vertices of G. We denote the set of all perfect matchings of G as $\mathcal{M}(G)$.

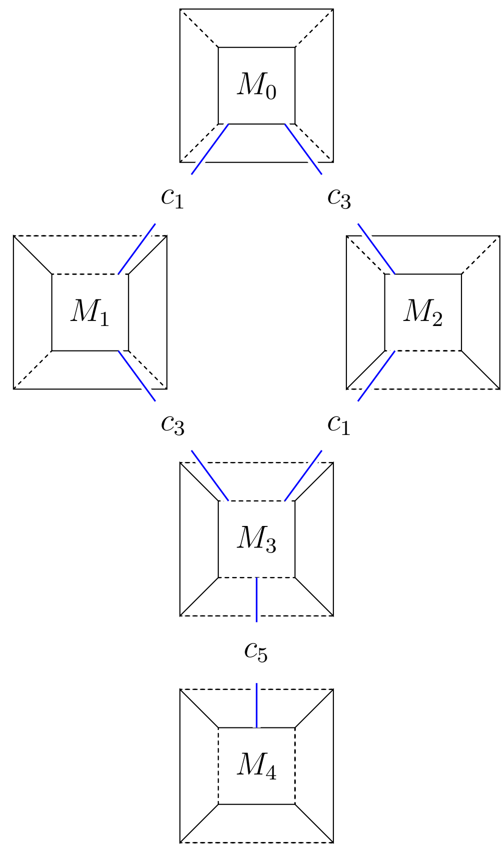
Lovász [13] defines a *matching lattice* as the lattice generated by the set $\mathcal{M}(G)$ of incidence vectors of perfect matchings of the graph G.

Definition 1.16. Let G be a graph and $\mathcal{M}(G)$ be the set of all perfect matchings of G. Also let a set $\mathcal{C}$ be a set of linearly independent even cycles of G. A *resonance graph* $\mathcal{R}(G, \mathcal{C})$ is a graph with the vertex set $M \subseteq \mathcal{M}(G)$ such that two vertices M_i and M_j are adjacent if and only if $M_i \oplus M_j = E(c)$ for some $c \in \mathcal{C}$.

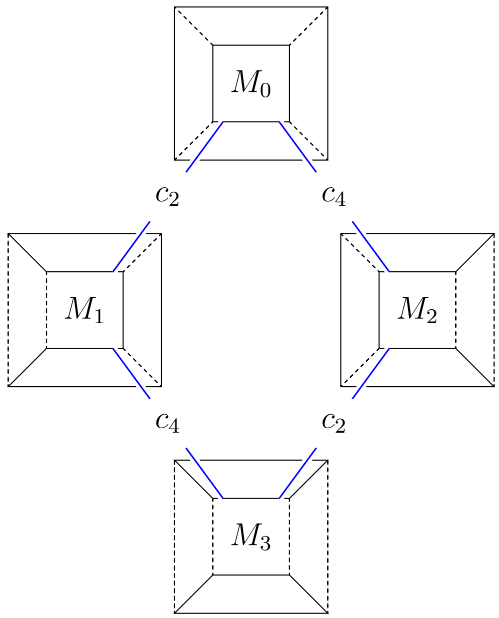
Table 1: Resonance graphs of Q_3 with respect to distinct subsets of $\mathcal{C}(Q_3)$ where dashed lines indicate perfect matchings



Q_3



Resonance graph $\mathcal{R}(Q_3, \{c_1, c_3, c_5\})$



Resonance graph $\mathcal{R}(Q_3, \{c_2, c_4\})$

In Table 1, we have examples of resonance graphs of the 3-dimensional hypercube Q_3. These examples highlight how the form of a resonance graphs is subject to the set of linearly independent even cycles.

1.2 Motivations

Resonance graphs were first introduced by chemists in connection with benezoid systems which are finite connected plane bipartite graphs whereby every interior region is bounded by a regular hexagon of unit length [9]. Resonance graphs were later reintroduced by mathematicians as Z-transformation graphs [8]. Further study has resulted in extending concepts of the resonance graphs of a catacondensed benzenoid graphs [10] to resonance graphs of plane weakly-elementary bipartite graphs [23]. Zhang shows that the concept continues to hold on with resonance graphs of plane bipartite graphs [12, 25].

Let G be a graph, for each two vertices $u, v \in V(G)$, the minimal number of edges between them is called their *distance*, denoted by $d_G(u, v)$. A set of vertices that are on the shortest path between u and v is called the *interval* which is defined as $I(u, v) = \{x : d_G(u, v) = d_G(u, x) + d_G(x, v)\}$. A graph is a *median graph* if the intervals of every three vertices intersect at a single point, i.e.,

$$|I(u, v) \cap I(u, w) \cap I(v, w)| = 1 \text{ for all } u, v, w \in V(G)$$

Note that one of the three vertices could be that point. Median graphs first appeared in work by Nebeskỳ [15]. However, the concepts were studied earlier by Birkhoff & Kiss [3] and Avann [1]. It is known that the covering graph of a distributive lattice is a median graph, and a median graph can always be embedded in a hypercube as an induced subgraph, i.e., an induced cubical graph.

We define a *surface* Σ to be a compact and connected 2-dimensional manifold without boundary. A *face* of a graph embedded in Σ is the closure of a connected component of the surface having removed the graph. A face is *even* if it is contained in an even cycle. It follows that the set of even faces is called an *even-face set* [22]. A graph G is *elementary* [14] if the edges of G contained in a perfect matching induce a connected subgraph. Furthermore, a graph is *weakly elementary* [24] if every inner face of evey elementary component of G is still a face of the original G.

Theorem 1.17 (Klavžar, Žigert, and Brinkmann, [10])**.** *The resonance graph of a catacondensed even ring system is a median graph.*

Zhang, Lam, and Shiu obtained a general result for plane weakly elementary bipartite graphs as follows.

Theorem 1.18 (Zhang, Lam, and Shiu, [23])**.** *The resonance graph of a plane weakly-elementary bipartite graph is a median graph.*

A stronger result for plane elementary bipartite graphs has been obatined by Lam and Zhang [12].

Theorem 1.19 (Lam & Zhang, [12]). *Let G be a plane elementary bipartite graph and $\mathcal{F}$ be the set of all inner faces of G. Then $\mathcal{R}(G, \mathcal{F})$ is the covering graph of a distributive lattice.*

This result has been reached independently by Felsner [6] and Propp [17]. Other structures such as, orientations and flows and spanning trees, have been shown to share the properties as the resonance graph of a plane elementary bipartite graph [7, 17]. However, if the graph is not plane bipartite, it may not be the covering graph of a distributive lattice. For example, see [21].

Conjecture 1.20 (Tratnik & Žigert Pleteršek, [21]). *Every connected component of the resonance graph a fullerene is a median graph.*

Note Ovchinnikov [16] shows that the class of median graphs is a proper subclass of the class of parital cubes. We also know that the class of partial cubes are a subclass of the class of induced cubical graphs [16]. It is weaker problem to prove that every connected component of the resonance graph of a fullerene is a partial cube. In [22], Tratnik and Ye proved that every connected component of the resonance graph of a fullerene is an induced cubical graph by the following general result.

Theorem 1.21 (Tratnik & Ye, [22]). *Let G be a graph embedded in a surface Σ and let $\mathcal{F} \neq \mathscr{F}(G)$ be an even-face set. Then every connected component of the resonance graph is an induced cubical graph.*

1.3 Main Result

In this thesis, we consider the resonance graphs of graphs without surface embedding information. We define resonance graph based on linearly independent sets of the cycle space of a graph. Hence, we are be able to extend Tratnik and Ye's result from embedded graphs to graphs without any restrictions. The following is the main result.

Theorem 1.22. *Let G be a graph with perfect matchings and $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycles of width at most 2. Then every connected component of the resonance graph $\mathcal{R}(G, \mathcal{C})$ is an induced cubical graph.*

The paper proceeds as follows: some propositions and definitions focusing on the cycle space are given in Chapter 2, the proof for Theorem 1.22 is given in Chapter 3.

CHAPTER 2

CYCLE SPACE

Let $G = (V, E)$ be a graph and $\mathbb{F}_2 = \{0, 1\}$. The incidence vectors of all subsets of $E(G)$ form a vector space over $\mathbb{F}_2$, which is called the *edge space* $\mathcal{E}(G)$ of G. Every vector of $\mathcal{E}(G)$ corresponds to a subset of $E(G)$, and the addition of two vectors in $\mathcal{E}(G)$ corresponds the symmetric difference of two subsets of $E(G)$. The empty set $\emptyset \subseteq E$ is the zero element. Scalar multiplication is defined as the following:

- $0 \cdot F := \emptyset \qquad F \in \mathcal{E}(G)$
- $1 \cdot F := F \qquad F \in \mathcal{E}(G)$

Definition 2.1. The *cycle space* $\mathfrak{C}(G)$ is a subspace of $\mathcal{E}(G)$ generated by the set of all cycles of G. We denote the set of all cycles of G as $\mathscr{C}(G)$.

Note that the cycle space $\mathfrak{C}(G)$ and the set of cycles $\mathscr{C}(G)$ of graph G have a one-to-one correspondence.

Proposition 2.2. *The function $f : \mathfrak{C}(G) \to \mathscr{C}(G)$ is bijective.*

Proof. Let $f : \mathfrak{C}(G) \to \mathscr{C}(G)$ be a function such that $x_C \mapsto C$ for $x_C \in \mathfrak{C}(G)$ such that $x_C = (x_e)_{e \in E(G)}$ whereby $x_e = 1$ if $e \in C$ else $x_e = 0$. Suppose, for $C, D \in \mathscr{C}(G)$, $C = D$. So then, $E(C) = E(D)$. Since the edge sets of C and D are equal, each component of their incidence vectors are equivalent. Thus, $x_C = x_D$ which shows that f is injective.

Now, for some $B \in \mathscr{C}(G)$, there is $x_B \in \mathfrak{C}(G)$ such that $x_B = (x_e)_{e \in E(G)}$ whereby $x_e = 1$ if $e \in B$ else $x_e = 0$. So then, $f(x_B) = B$ which shows that f is surjective. Therefore, f is bijective. $\square$

The following propositions, Proposition 2.3 and Proposition 2.4, are known results. Their proofs are included for completeness of the paper.

Proposition 2.3. *Every connected graph contains a spanning tree.*

Proof. Let G be a connected graph. Suppose G has no cycles. It follows then that G is its own spanning tree. So then, suppose G contains cycles. Now, let T be a minimal connected spanning subgraph of G. Assume T has a cycle c. Then, for some edge $e \in E(c)$, $T \setminus e$ is still connected and spanning. So then, $T \setminus e \subsetneq T$. However, this is a contradiction to the minimality of T. Thus, T is a spanning tree of G. $\square$

Proposition 2.4. *The cycle space* $\mathfrak{C}(G)$ *has a basis* $\{c_1, c_2, \ldots, c_k\}$.

Proof. Let G be a connected graph and T be the spanning tree contained in G. Note $|E(G) \setminus E(T)| = m - n + 1$ where $|E(G)| = m$ and $|V(G)| = n$ since $|E(T)| = |V(G)| - 1$. So then, $E(G) \setminus E(T) = \{e_1, e_2, \ldots, e_k\}$ where $k = m - n + 1$. Now, for some $e_i \in (E(G) \setminus E(T))$, $e_i \cup T$ contains a cycle c_i. Also note if $e_i \neq e_j$, then $c_i \neq c_j$. Suppose $\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k = 0$ where $\alpha_i \in \mathbb{F}_2$. Since each c_i contains a distinct edge e_i, then $\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k = 0$ only when $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Thus, $\{c_1, c_2, \ldots, c_k\}$ is linearly independent.

Now suppose, for some cycle $c \in \mathscr{C}(G)$, $c \neq \sum \alpha_i c_i$ for some $\alpha_i \in \mathbb{F}_2$ and $c_i \in \{c_1, c_2, \ldots, c_k\}$. So then c is a cycle where all edges originate from $E(T)$. However, by definition, T contains no cycles. Thus, for all $c \in \mathscr{C}(G)$, $c = \sum \alpha_i c_i$. Therefore, $\{c_1, c_2, \ldots, c_k\}$ is the basis of the cycle space of G. $\square$

Note the dimension of the cycle space of G is $|\{c_1, c_2, \ldots, c_k\}| = k = m - n + 1$. A set of cycles of G is *linearly independent* if their incidence vectors are linearly independent in the cycle space of G. The *width* of a set of cycles $\mathcal{C}$, denoted by $\operatorname{width}(\mathcal{C})$, is the maximum cardinality of cycles which contains a common edge. For example, the set of all facial cycles of a graph embedded in a surface has width 2.

Proposition 2.5. *Let G be a graph and $\mathcal{C}$ be a set of cycles in G. If $\mathcal{C}$ is linearly independent of width at most 2, then there exists an $e \in E(\mathcal{C})$ which is covered by exactly one cycle of $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be linearly independent. Suppose, to the contrary, that every edge $e \in E(\mathcal{C})$ is covered by at least two cycles of $\mathcal{C}$. Note that $\text{width}(\mathcal{C}) \leq 2$. It follows that every edge of $E(\mathcal{C})$ is contained by exactly two cycles of $\mathcal{C}$. So then, $\sum \alpha_i c_i = 0$ with $\alpha_i = 1$ for all $\alpha_i \in \mathbb{F}_2$. However, this is a contradicition since $\mathcal{C}$ is linearly independent. Therefore, there is some edge $e \in E(\mathcal{C})$ which is covered by exactly one cycle of $\mathcal{C}$. $\square$

CHAPTER 3

RESONANCE GRAPH

Let G be a graph and $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycle of width at most 2. We define a cycle sequence as a collection of cycles of G that generate cycles of the resonance graph $\mathcal{R}(G, \mathcal{C})$.

Lemma 3.1. *Let G be a graph with perfect matchings and let $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycles of widith at most 2. Assume that $C = M_0 M_1 \cdots M_{t-1} M_0$ is a cycle of the resonance graph $\mathcal{R}(G, \mathcal{C})$. Let cycle $c_i \in \mathcal{C}$ correspond to the edge $M_i M_{i+1}$ for $i \in \{0, 1, \ldots, t-1\}$. Then every cycle of $\mathcal{C}$ appears an even number of times in the cycle sequence $(c_0, c_1, \ldots, c_{t-1})$.*

Proof. Let c be a cycle of $\mathcal{C}$, and let $\delta(c)$ be the number of times c appears in the cycle sequence $(c_0, c_1, \ldots, c_{t-1})$. It suffices to show that $\delta(c) \equiv 0 \pmod 2$. Since $C = M_0 M_1 \cdots M_{t-1} M_0$ is a cycle of $\mathcal{R}(G, \mathcal{C})$ and c_i is the corresponding cycle of the edge $M_i M_{i+1}$, it follows that $M_i \oplus M_{i+1} = E(c_i)$ for $i \in \{0, 1, \ldots, t-1\}$. So

$$E(c_0) \oplus E(c_1) \oplus \cdots \oplus E(c_{t-1}) = \oplus_{i=0}^{t-1} (M_i \oplus M_{i+1}) = \emptyset \tag{1}$$

where all subscripts take modulo t.

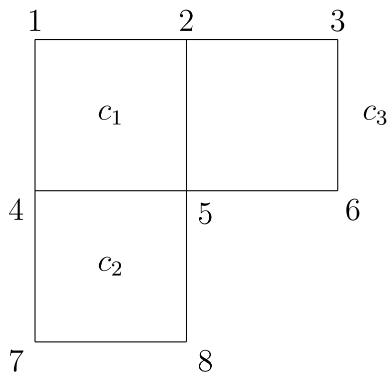
Let c and d be two cycles of $\mathcal{C}$ such that there exists an edge $e \in E(c) \cap E(d)$. Since $\text{width}(\mathcal{C}) \leq 2$, e is contained by only c and d from $\mathcal{C}$. The total number of cycles in the sequence $(c_0, c_1, \ldots, c_{t-1})$ containing e is even by (1). It follows that $\delta(c) + \delta(d) \equiv 0 \pmod 2$. So $\delta(c) \equiv \delta(d) \pmod 2$. Thus, all cycles of G have the same parity.

By Proposition 2.5, there exists an edge e which is covered by exactly one cycle $c \in \mathcal{C}$ since $\mathcal{C}$ is linearly independent and has width at most 2. By (1), $\delta(c) \equiv 0 \pmod 2$. Hence $\delta(c) \equiv \delta(d) \equiv 0 \pmod 2$ for any cycle d of $\mathcal{C}$. Thus, every cycle of $\mathcal{C}$ appears an even number of times in the cycle sequence $(c_0, c_1, \ldots, c_{t-1})$. $\square$

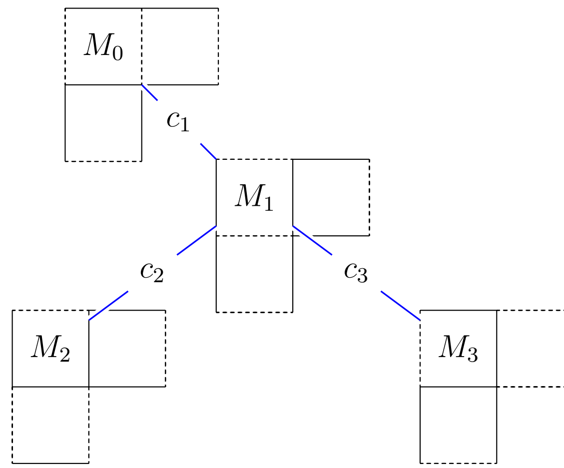
The following theorem follows immediately from Lemma 3.1.

Theorem 3.2. *Let G be a graph with perfect matchings and $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycles of width at most 2. Let $\mathcal{R}(G, \mathcal{C})$ be a resonance graph. Then $\mathcal{R}(G, \mathcal{C})$ is a bipartite graph.*

Proof. Let $C = M_0 M_1 \cdots M_{t-1} M_0$ be a cycle of $\mathcal{R}(G, \mathcal{C})$ and c_i be a cycle of G corresponding to the edge of $M_i M_{i+1}$ for $i \in \{0, 1, \ldots, t-1\}$. Since every cycle of G appears an even number of times in the cycle sequence $(c_0, c_1, \ldots, c_{t-1})$, C is an even cycle. Therefore, $\mathcal{R}(G, \mathcal{C})$ is a bipartite graph. □



(a) Graph G



(b) $\mathcal{R}(G, \mathcal{C})$

Figure 7: Bipartite resonance graph with width $(\mathcal{C}) = 3$ where $c_1 = 12541$, $c_2 = 45874$, and $c_3 = 1236541$

Theorem 3.2 shows that the resonance graph $\mathcal{R}(G, \mathcal{C})$ is bipartite given, if $\mathcal{C}$ is a set of linearly independent even cycles of width at most 2. However, there do exist bipartite resonance graphs $\mathcal{R}(G, \mathcal{C})$ whereby $\mathcal{C}$ has width greater than 2. Figure 7 gives an example of a resonance graph where $\mathcal{C} = \{c_1, c_2, c_3\}$ has width of 3 and is bipartite.

In [20], Tratnik and Pleteršek show an example of a resonance graph of a tubulene with perfect matchings not being connected. So then, it follows that our resonance graph $\mathcal{R}(G,\mathcal{C})$ may not be connected either. This motivates us to focus on the connected portion of our resonance graph. Let H be a connected component of $\mathcal{R}(G,\mathcal{C})$ such that $\{c_1, c_2, \ldots, c_k\} \subseteq \mathcal{C}$ is the set of cycles that correspond to the edges of H. For the author's convenience, we denote the set all elements of $E(H)$ that correspond to the cycle $c_i \in \mathcal{C}$ as E_i.

Proposition 3.3. *Let $\mathcal{R}(G,\mathcal{C})$ be the resonance graph of a graph G with perfect matchings with respect to a set of linearly independent even cycles $\mathcal{C} \subsetneq \mathscr{C}(G)$ of width at most 2 and H be a connected component of $\mathcal{R}(G,\mathcal{C})$. If $M_1M_2 \in E_i$, then M_1 and M_2 belong to different components of $H \setminus E_i$.*

Proof. Let $M_1M_2 \in E_i$. Assume M_1 and M_2 are in the same component of $H \setminus E_i$. So then, there exists a path P connecting M_1 and M_2 in $H \setminus E_i$. Note, since $P \in H \setminus E_i$, $E(P) \cap E_i = \emptyset$. Now, there is a cycle $C = P \cup \{M_1M_2\}$ of H. So then, c_i only appears once in the cycle sequence corresponding to the edges in C since $E(P) \cap E_i = \emptyset$. However, this contradicts Lemma 3.1. Therefore, M_1 and M_2 belong to different components of $H \setminus E_i$. □

So then, $H \setminus E_i$ is disconnected for any cycle $c_i \in \mathcal{C}$ that corresponds to edges in E_i. Define the quotient graph $\mathcal{H}_i$ of H with respect to c_i to be a graph constructed by contracting all the elements of $E(H) \setminus E_i$ and then replacing any multiple parallel edges with a single edge.

Lemma 3.4. *Let $\mathcal{R}(G,\mathcal{C})$ be the resonance graph of a graph G with perfect matchings with respect to a set of linearly independent even cycles $\mathcal{C} \subsetneq \mathscr{C}(G)$ of width at most 2. If c_i is a cycle of G that corresponds to some edge of a connected component H of $\mathcal{R}(G,\mathcal{C})$, then the quotient graph $\mathcal{H}_i$ with respect to c_i is bipartite.*

Proof. Let c_i be a cycle of G that corresponds to some edge of a connected component H of $\mathcal{R}(G,\mathcal{C})$. Suppose $\mathcal{H}_i$ has an odd cycle. We know vertices of $\mathcal{H}_i$ are connected components of $H \setminus E_i$. So then, there is a cycle C of H whereby c_i appears an odd number of times in the cycle sequence of C. Yet this contradicts Lemma 3.1, since we know every cycle of G appears an even number of times in the cycle sequence. So then, $\mathcal{H}_i$ contains no odd cycles. Thus, the quotient graph $\mathcal{H}_i$ with respect to c_i is bipartite. □

Since $\mathcal{H}_i$ with respect to c_i is bipartite, we shall let (A_i, B_i) be the bipartition of $\mathcal{H}_i$. Now let $\mathcal{M}_{A_i}$ and $\mathcal{M}_{B_i}$ be the sets of perfect matchings of G which correspond to vertices of $\mathcal{H}_i$ in A_i and B_i, respectively.

Define a function $\ell_i : V(H) \to \{0,1\}$ as follows, for any $M \in V(H)$,

$$\ell_i(M) = \begin{cases} 0 & M \in \mathcal{M}_{A_i} \\ \\ 1 & M \in \mathcal{M}_{B_i} \end{cases}$$

Furthermore, define a function $\ell : V(H) \to \{0,1\}^k$ such that, for any $M \in (H)$,

$$\ell(M) = (\ell_1(M), \ldots, \ell_k(M))$$

Proposition 3.5. *The function $\ell : V(H) \to \{0,1\}^k$ is injective.*

Proof. Let G be a graph with perfect matchings and H be a connected component of the resonance graph $\mathcal{R}(G, \mathcal{C})$ of G with respect to a set of linearly independent even cycles $\mathcal{C} \subsetneq \mathscr{C}(G)$ of width at most 2. Suppose $M_1 \neq M_2$ for any $M_1, M_2 \in V(H)$. Now let $P = M_1 X_1 \cdots X_{t-1} M_2$ be the shortest path joining M_1 and M_2 in H. Also let $\{d_1, d_2, \ldots, d_s\}$ be the cycles corresponding to the edges of P such that d_n corresponds to $X_{n-1}X_n$ for $n \in \{1, \ldots, s\}$ whereby $X_0 = M_1$ and $X_t = M_2$. Note d_i may be equal to d_j for some distinct $i, j \in \{1, \ldots, s\}$. Now if every cycle of G appears an even number of times in $(d_1, d_2, \ldots, d_s)$, then $M_2 = M_1 \oplus E(d_1) \oplus E(d_2) \oplus \cdots \oplus E(d_s) = M_1$. However, this contradicts our hypothesis that $M_1 \neq M_2$.

So then, there exists a cycle c_i of G that appears an odd number of times in $(d_1, d_2, \ldots, d_s)$. It follows that if we contract all elements of $E(P) \setminus E_i$, then the resulting path P' of $\mathcal{H}_i$, which joins M_1 and M_2, has an odd number of edges corresponding to c_i. Since $\mathcal{H}_i$ is bipartite, the end vertices of P', M_1 and M_2, belong to separate partitions. Thus, without loss of generality, M_1 belongs to $\mathcal{M}_{A_i}$ and M_2 belongs to $\mathcal{M}_{B_i}$. So $\ell_i(M_1) \neq \ell_i(M_2)$ which implies $\ell(M_1) \neq \ell(M_2)$. Therefore, $\ell : V(H) \to \{0,1\}^k$ is injective. $\square$

Lemma 3.6. *The function* $\ell : V(H) \to \{0,1\}^k$ *embeds* H *into a k-dimensional hypercube. Moreover,* ℓ *embeds* H *as an induced subgraph.*

Proof. Let G be a graph with perfect matchings and $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycles of width at most 2. Also let H be a connected component of the resonance graph $\mathcal{R}(G, \mathcal{C})$. Suppose M_1M_2 corresponds to a cycle $c_i \in \mathcal{C}$ for some $M_1M_2 \in E(H)$. It follows that $M_1M_2 \in E_i$. So then, for an arbitrary $j \in \{1, \ldots, k\}$ such that $j \neq i$, $M_1M_2 \in E(H) \backslash E_j$ since $E_i \cap E_j = \emptyset$. Furthermore, M_1 and M_2 are in the same connected component of $H \setminus E_j$. So $\ell_j(M_1) = \ell_j(M_2)$ for all $j \in \{1, \ldots, k\}$ such that $j \neq i$. By Proposition 3.5, we know $\ell(M_1) \neq \ell(M_2)$ if $M_1 \neq M_2$. So then, $\ell(M_1)$ and $\ell(M_2)$ differ only at ℓ_i. Thus, $\ell : V(H) \to \{0,1\}^k$ embeds H into a k-dimensional hypercube.

Since $\ell_i(M_1) \neq \ell_i(M_2)$ and $\ell_j(M_1) = \ell_j(M_2)$ for all $j \in \{1, \ldots, k\}$ such that $j \neq i$, without loss of generality, $M_1 \in \mathcal{M}_{A_i}$ and $M_2 \in \mathcal{M}_{B_i}$. Note there is a path P of H that connects M_1 and M_2 since H is a connected component of $\mathcal{R}(G, \mathcal{C})$. If we contract all elements of $E(P) \setminus E_i$, then the resulting path P' of $\mathcal{H}_i$, which joins M_1 and M_2, has an odd number of edges corresponding to $c_i \in \mathcal{C}$. Yet if we contract all elements of $E(P) \setminus E_j$, for any $j \in \{1, \ldots, k\}$ such that $j \neq i$, then the resulting path P'' of $\mathcal{H}_j$, which joins M_1 and M_2, has an even number of edges corresponding to $c_j \in \mathcal{C}$. So, for any edge $e \in E(G)$, e rotates along path P an even number of times if $e \notin E(c_i)$ and an odd number of times if $e \in E(c_i)$. It follows $E(c_i) = E(c_1) \oplus \cdots \oplus E(c_k) = M_1 \oplus M_2$. So then, $M_1M_2 \in E(\mathcal{R}(G, \mathcal{C}))$ which implies $M_1M_2 \in H$ since H is a connected component of $\mathcal{R}(G, \mathcal{C})$. Therefore, $\ell : V(H) \to \{0,1\}^k$ embeds H into a k-dimensional hypercube as an induced subgraph. $\square$

Theorem 3.7 follows directly from Lemma 3.6.

Theorem 3.7. *Let G be a graph with perfect matchings and $\mathcal{C} \subsetneq \mathscr{C}(G)$ be a set of linearly independent even cycles of width at most 2. Then every connected component of the resonance graph $\mathcal{R}(G, \mathcal{C})$ is an induced cubical graph.*

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