Secondary Mathematics Teachers' Pedagogy Through

the Tool of Computer Algebra Systems

by

Candace Pearl Terry

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Dissertation Committee:

Dr. Angela Barlow, Co-Chair

Dr. Michaele Chappell, Co-Chair

Dr. Nancy Caukin

Dr. Mary Martin

Dr. Jeremy Strayer

This dissertation project is dedicated to my three brilliant children, Leah, Patsy, and Ruth Ann, and to my husband, Alton, the love of my life. These four have given me love, advice, support, challenge, joy, encouragement, and the best family life imagined.

As a special memorial to Leah Christine Terry, it is with great honor that I acknowledge the first philosophy degree in the family to you. Your literary piece inspired me to continue without ceasing to research, write, and complete. You are missed.

We Were Writers

We want immortality, so we try for it. Better if we can do it by not dying, but we'll take what we can get. So we have to write, as insurance that we will continue forever and ever or until the world stops because that's the only way we will.

We want to be original. We want to write something new, we must. Because when we write, we are translating ourselves onto paper. We pour the contents of our being into every stroke of every pen and every tap of every key. We believe that we are wasting ourselves, and nothing we write is ever enough. But we want it to be. Because it is the vessel in which we hold ourselves, so it is the only way we can know if we are enough.

We were writers, we thought, but writing is not an occupation for the living. It is a speaker from the grave. We were writers.

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Foremost, I give glory to the God of all creation. With Him all things are possible. I placed this verse where I could see it as a daily reminder. "Do not be anxious about anything, but in every situation, by prayer and petition, with thanksgiving, present your requests to God. And the peace of God, which transcends all understanding, will guard your hearts and your minds in Christ Jesus" Phillipians 4:6-7.

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ABSTRACT

Computer algebra systems (CAS) have been available for over 20 years and yet minimal CAS-rich opportunities present themselves formally to high school students. CAS tools have become readily accessible through free or inexpensive versions. Educators are emboldened to integrate essential mathematical tools in the reasoning and sense making of mathematical knowledge for students. It is the teacher that is at the heart of technology instruction, creating authentic environments for all learners.

This study investigated two secondary teachers pedagogy in classes that exploited CAS in the development of mathematical knowledge. A qualitative within-site case study design was used to explore each teacher's instructional practices. Teachers that exemplified qualities of CAS-infused instruction were purposively selected. Rich descriptive lesson vignettes as captured from classroom observations, written reflections, and interviews revealed participants' pedagogy. The pedagogical map framework guided the identification of participant pedagogical affordances of the utilization of CAS. Eight opportunities were observed as exploited by the participants that included subject level adjustments; classroom interpersonal dynamics with students; and mathematical tasks. Data revealed several emergent themes in operation as the teacher participants oriented their mathematics instruction: viewing CAS as a mathematical consultant, verifying answers, applying multiple representations, regulating access, providing guidance, and outsourcing procedures. The components interlock with one another to form a cohesive depiction of pedagogical decisions in the presence of CAS-rich classroom instruction. The schema of CAS-oriented instruction serves as a methodology for educators to create opportunities that enrich the development of mathematical content knowledge.

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Introduction

Teachers' instruction affords opportunities to exploit computer algebra systems (CAS) in the development of mathematical knowledge. Consideration of CAS technologies and pedagogical practices from the perspective of secondary education teachers was investigated in this research using a qualitative case study. United States high school teachers who currently integrate CAS technology in their teaching practices were selected as cases to illuminate the pedagogical practices utilized. The study examined the ways two teachers integrated CAS into their classes to develop mathematical knowledge and understandings. Through interviews, observations, surveys, and the collection of lesson plan artifacts from two teachers I analyzed the pedagogical opportunities afforded by CAS technology and sought the perceived motivations of these individuals.

This chapter presents an overview of the significance of addressing technology in mathematics teaching, in particular, teaching with CAS as a cognitive tool. Background of technology position statements, mathematical cognitive tools, and CAS technology are explicated. Considerable research on CAS technology has materialized in the last 20 years; so a brief history of these seminal inquiries will be summarized. However, in considering teaching methodology, the pedagogical map (P-Map) first presented by Pierce and Stacey (2008) will be offered as a way to specify individual teacher choices afforded by CAS.

Background of Study

"The stage has been set for the systematic examination of the impact of technology on the teaching and learning of mathematics" (Heid & Blume, 2008, p. vii). Scientific, graphic, and numeric calculators are technological devices widespread in use by K-12 teachers as a result of researchers and teachers who explored pedagogy with these tools to expand student understanding of mathematical concepts (Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003; Guin, Ruthven & Trouche, 2005). CAS is another technology that emerged in the late 20th century with capabilities of symbolic manipulation, a feature that can factor, expand, and solve equations as well as other characteristics. This technology "provides both an aid and a challenge to students as they develop solid understandings of mathematical concepts and processes" (Heid, 2003, p. 50). However, the teacher is at the center of change through pedagogical decisions of technology integration (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Pugalee, 2001).

The teacher determines both when and how to implement CAS (Heid & Blume, 2008; Heid, Thomas, & Zbiek, 2013; Simonsen & Dick, 1997; Zbiek & Hollebrands, 2008). "To use CAS in teaching to its full potential requires a particular set of skills and attitudes on the part of teachers, and so addressing teacher-related issues is crucial" (Heid et al., 2013, p. 631). Teachers have the complex duty to determine the specific learning goals, select a problem or task to meet these goals, develop questioning strategies to pose to students, and assist the development of mathematical understandings from student work with the task (Hiebert, 2003; Heid & Blume, 2008; Zbiek & Hollebrands, 2008). Future studies are needed to delve into the activities of teachers as they make these decisions and implement technologies (Fey, 2006; Pierce & Stacey, 2008; Zbiek & Hollebrands, 2008).

Technology Position Statements

Position statements claiming the need for effective instruction regarding technology tools are readily available from both the mathematics education community and technology associations (e.g., Common Core State Standards Initiative [CCSSI], 2010; International Society for Technology in Education [ISTE], 2008a, 2008b; National Council of Supervisors of Mathematics [NCSM], 2011; National Council of Teachers of Mathematics [NCTM], 2011, 2014). These statements recommend integration of technology tools into teaching practices and regard technology as a support to sound educational practice by teachers. The tools should be accessible to students and used to develop, create, and reinforce mathematical connections (NCTM, 2014). Rather than teach the mechanics of the tool in separate coursework, technology should serve to sustain the goals of the curriculum (ISTE, 2008b). NCSM and ISTE reiterate the need for fully integrated technologies as opposed to isolated elemental tools (ISTE, 2008b; NSCM, 2011).

NCTM (2014) indicated that technology is a rapidly changing landscape that currently encompasses interactive whiteboards, handheld devices, tablets, and desktop devices, all of which are used to assist in the learning of mathematics. The technologies that are mathematics-specific include CAS along with applications of graphing tools, dynamic geometry, spreadsheets, three-dimensional modeling, and data analysis applications. “An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (NCTM, 2014, p. 78).

The value of the educational tool depends on these factors: (a) the actions of the teacher as the initiator of implementation; (b) its utilization as a means to develop mathematical reasoning and sense making for learners; and (c) the development of creative and authentic learning environments (ISTE, 2008b; NCTM, 2014). It is not the luxury of the tool alone, but how the teacher effectively uses technology to build mathematical understandings for students (NCSM, 2014; NCTM, 2011, 2014). To strengthen understandings, teachers develop approaches to problems that may involve mental mathematics, paper-and-pencil, or a technology application (NCSM, 2011). As learners form familiarity with the available tools, they generate new ways to utilize these tools (Kaput, 1992; Pea, 1985). Masterful teachers possess significant pedagogical content knowledge (PCK) (Shulman, 1986) with regard to instructional technology tools and that not only serves as an impetus to effective integration of technology tools but also inspires creativity in learning through authentic environments that utilize technology (ISTE, 2008b). "Effective teachers model and apply the ISTE Standards for Students as they design, implement, and assess learning experiences to engage students and improve learning" (ISTE, 2008a, p. 1).

Technology Concerns

NCTM posited that no adverse affects have been attributed to the introduction of calculator technology (e.g., scientific, graphing, or CAS); however, not all in the mathematics education community view calculators as a positive addition to teaching (NCTM, 2011, 2014). Merriweather and Tharpe (1999) revealed that eighth-grade students had negative attitudes regarding graphing calculators. Given access to graphing technology for two weeks, students found the tool complex thus making the mathematics problems more confusing. Students preferred to rely on traditional paper-and-pencil methods. As well, teachers preferred to teach paper-and-pencil techniques first, followed by technology utilization (Brown et al., 2007; Ivy & Franz, 2016; Lee & McDougall, 2010). Students and teachers alike choose by-hand procedures as precursory to learning automated technologies.

In a meta-analysis on the use of graphing calculators in high schools, Kastberg and Leatham (2005) looked at the access to three key factors: graphing calculators, curriculum, and pedagogy. First, teachers limited student access to mathematics technology, even when graphing calculators were available. Second, the "disconnect between graphing calculators and the curriculum impeded students' ability to integrate various techniques learned" (p. 30). Curriculum often included technology as supplemental, rather than integrated. Finally, when teachers had technological tools available, pedagogy and teacher approaches to problems impacted the way students used the tools.

The NCTM Research Council recommended additional research regarding practitioner use of technology in the teaching and learning of mathematics and inquiry into the effectiveness of mathematics-specific technical tools. "Technology integration is a complex phenomenon that involves understanding teachers' motivations, perceptions, and beliefs about learning and technology" (Keengwe, Onchwari, & Wachira, 2008, p. 560). Additional concerns include the lack of professional learning experiences for teachers and the need for sustained opportunities to support teachers in their use of technological tools (NCSM, 2014; NCTM, 2014). These views are fortified by ongoing research that benefits classroom instruction for meaningful learning in mathematics (NCSM, 2014).

Cognitive Tools and Technologies

Mathematical objects that are used in quantification, computation, and organization and also "facilitate the technical dimension of mathematical activity" (Pierce & Stacey, 2010, p. 3) are referred to using multiple labels: cognitive tools (Zbiek, Heid, Blume, & Dick, 2007); cognitive technologies (Pea, 1985); cognitive technological tools (Drijvers & Trouche, 2008); symbolic tools (Artigue & Diderot, 2002); digital technologies (Weigand, 2014); handheld technology (Trouche & Drijvers, 2010); mathematical action technologies (Dick & Hollebrands, 2011; NCTM 2014); mathematics analysis software (Pierce & Stacey, 2010); or simply tools and technologies (Maschietto & Trouche, 2010). Tools have been used to develop understanding of mathematical concepts beginning with the use of compass and slide rules, physical manipulative materials (e.g., Diene's base number blocks, pattern blocks, and geometric figures), numerical tables (e.g., trigonometric and logarithmic function values), and electronic tools of many varieties (Fey, 2006). "Seen in a historical perspective, handheld tools have a long tradition of being at the heart of mathematical and scientific practice" (Trouche & Drijvers, 2010, p. 667).

The benefit of cognitive technologies is the potential to reshape how we think about things (Kaput, 1992; Pea, 1985; Trouche & Drijvers, 2010). Initially, novices adapt to new tools while learning content knowledge, a theory referred to as instrumental genesis (Artigue & Diderot, 2002; Drijvers, 2015). Once beyond the complexities, learners become flexible with the device (Doerr & Zangor, 2000). The tools provide access to multiple ways of representation and variations of notation (Kaput, 1992). "Technology should not be used as a replacement for basic understandings and intuitions" (NCTM, 2000, p. 25). Rather the tools assist in the process of knowledge acquisition by providing inventive ways to perceive information. As Pea (1985) stated, "A cognitive technology is provided by any medium that helps transcend the limitations of the mind, such as memory, in activities of thinking, learning, and problem solving" (p. 168).

CAS Technology Background

The genesis of CAS began in the 1970s with computer programs that had rudimentary abilities to solve mathematical problems through symbolic algebra in response to the scientific community's need to work complex calculations (Hamrick, 2007; Pierce & Stacey, 2008; Roberts, Leung, & Lin, 2012). Prior to the personal computer age, these algebra programs were not available to the general public, required a large complex computer system, and were limited in manipulations of inputs and outputs requiring the user to have strong knowledge about programming and mathematics (Demana & Waits, 1990; Lagrange, 2003). In more recent years, CAS technology has advanced to user handheld devices, notably the Texas Instruments TI-Nspire™ released in 2007 and Hewlett Packard Prime released in 2013 (Heid et al., 2013). These devices provided easy access and opportunity for daily use of CAS in classroom settings (Heid et al., 2013). As well, computer programs have advanced and given way to free or inexpensive online access and user-friendly interfaces, such as Wolfram Alpha, Mathematica, FX Math Pack, Desmos, and GeoGebra. "This rapid development of technology, from four function calculators to multi-representational connected devices, for a discipline that has evolved over thousands of years with a classroom tradition of teacher-centered exposition raises many questions and challenges for mathematics teachers" (Pierce & Stacey, 2013, p. 324).

CAS as an Essential Tool

Some mathematics educators believe that CAS technology must be recognized and utilized by all mathematics teachers and learners (Roschelle & Leinwand, 2011; Usiskin, 2006; Waits & Demana, 1998, 2000). Justification for the use of CAS resides in three key areas. First, CAS has the ability for non-standard numerical answers. Demana and Waits (1990) claimed that although CAS can provide exact answers, many times in real-life situations, the exact number furnishes little insight into the mathematical model of the problem under consideration. Technology provides flexibility in the answer format, allowing the user to perceive the output as approximate or precise. Likewise, problems need not be limited by the parameters provided in real-life situations, since calculations are not reliant on student proficiency of procedures. Supporting that, the NCTM Technology Principle (2000) asserted that the study of algebra need not be limited to certain simple types of problems since CAS tools allow students access to a variety of unordinary solution sets.

A second justification is to outsource higher cognitive-demand procedures to the handheld device (Drijvers, 2015; Heid, 2003; Pierce & Stacey, 2010). Traditional by-hand complex procedures can be confounding to the learner especially when arriving at unordinary solution sets. Therefore, outsourcing the procedures to the CAS can provide a more direct pathway to mathematical understandings (Heid, 2003). Oftentimes, errors occurred as students executed sub-procedures incorrectly, subsequently learners rely on CAS for correct calculations (Heid, 2003).

Finally, CAS is a tool in which learners conceived new ways of developing an understanding of mathematics (Heid & Blume, 2008; Heid et al., 2013; Kutzler, 2003; Pierce & Stacey, 2010; Zbiek & Hollebrands, 2008). CAS offers the opportunity to investigate concepts (e.g., variable, function, expression, and equation) more deeply and emphasize concepts that might not otherwise be prominent (Heid et al., 2013). Pierce and Stacey (2010) suggested that students were more active learners and, therefore, developed more robust understandings as each student reflected on his mathematical task in terms of outputs, representations, and multiple solution paths the CAS affords. These new pathways for teaching and learning were realized through the use of CAS. Usiskin (2006) claimed that it is each teacher's obligation to provide instruction with the use of CAS, as the tool elicits the need for new pedagogy. "If we truly wish to improve the use of mathematics in society, we have a moral obligation to further the use of instruments that can give so much power to people" (Usiskin, 2006, p. 5).

CAS Literature Overview

After the onset of CAS systems, many educators discovered and explored possibilities of instruction utilizing CAS, forming a new type of discussion about teaching with technologies (Fey et al., 2003; Stacey, Chick, & Kendal, 2004). Seminal research involved multiple representations, reorganization of curricula, the functional opportunities CAS provides, and dynamic features that interplay in those contexts using MuMath or Derive (Heid, 1988, 2003; Kutzler, 2010; Pierce & Stacey, 2002, 2008). As a result of this research, the need for additional organizations and venues for sharing this work emerged, along with volumes of compilations regarding these studies. A selection of these pieces of literature is listed in Table 1.

Table 1

Compilations of Technology Research and Perspectives in Mathematics

Year	Location	Editor(s)	Title
2003	United States	Fey, J., Cuoco, A., Kieran, C., McMullin, L., & Zbiek, R. M.	Computer Algebra Systems in Secondary Education Mathematics Education
2004	Melbourne, Australia	Stacey, K., Chick, H., & Kendal, M.	The Future of Teaching and Learning of Algebra: The 12th ICMI Study
2005	United Kingdom	Guin, D., Ruthven, K., & Trouche, L.	The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument
2008	United States	Heid, M. K. & Blume, G. W.	Research on Technology and the Teaching and Learning of Mathematics: Volume 1 Research Synthesis
2008	United States	Blume, G. W. & Heid, M. K.	Research on Technology and the Teaching and Learning of Mathematics: Volume 2 Cases and Perspectives
2011	United States	Dick, T. P. & Hollebrands, K. F.	Focus in High School Mathematics: Technology to Support Reasoning and Sense Making

The NCTM publication *Computer Algebra Systems in Secondary Education*, edited by Fey et al. (2003), was the first to release and publish a series of monographs that represented research from the inception of CAS up to publication. This handbook consolidated research conducted throughout the world, providing perspectives of the implications and potential usage for CAS in mathematics education. The comprehensive nature of the book provides a static and timeless resource for educators to consider pedagogical opportunities. One chapter written by Ball and Stacey of Australia, University of Melbourne's principal investigators in Victoria's large-scale implementation of CAS in secondary schools, considered the written work of students when they used CAS as a tool in learning. Stacey, Ball, and Pierce (2002-2013) have generated extensive work on CAS in secondary mathematics education. Additional ideas funneled through these Australia educators were elucidated extensively through many research publications, including the one that follows.

Stacey et al. (2004) served as editor for an additional volume published as proceedings from the Twelfth International Commission on Mathematical Instruction (ICMI), through the University of Melbourne, Victoria, Australia. In January 2000, the international program committee commenced to determine the members of the study conference on the topic of the future of teaching and learning algebra as new technologies have materialized. Through a large response to a call for papers, the ICMI realized the widespread interest; hence, it selected a core group of international representatives to participate. The reports from the various study groups formed the basis for the volume, *The Future of Teaching and Learning of Algebra*.

Finally, *Research on Technology and the Teaching and Learning of Mathematics: Research Synthesis*, edited by Heid and Blume (2008) synthesized discussion from researchers, teachers, educators, policy makers, and software designers that participated in the two international conferences sponsored by the National Science Foundation held at Pennsylvania State University. NCTM released this work in two volumes: *Research Synthesis* and *Cases and Perspectives*. The first volume (Heid & Blume, 2008) contained research on instruction in a technological environment, illuminating specific mathematical topics and the nature of mathematics learning. The second volume (Blume & Heid, 2008) provided detailed results from cases of research with technology in mathematics teaching and also overarching perspectives from leading researchers. The last of these volumes published in 2008 exacted 10 years from current research. Heid, Thomas, and Zbiek (2013)—while recognizing both empirical and theoretical findings—stressed the need for research focused in the teaching and learning of algebra with consideration of CAS' functional capabilities. Several recent dissertation studies have investigated aspects of CAS technologies (Fonger, 2012; Hicks, 2010; Ivy, 2011; Tokpah, 2008), but additional studies leave much to be developed.

Summary

A growing concern among some mathematics educators is the potential change in curriculum and culture of mathematics instruction due to access to CAS technology (Heid et al., 2013; Usiskin, 2006). Keeping in mind "that CAS technology can play a role in the conceptualization of models rather than simply being a tool that is used to solve a mathematical problem" (Heid et al., 2013, p. 633), the need for research continues. The landscape of mathematical technologies is vast and actively advancing, perpetuating the challenge of timeliness of studies (Pierce & Stacey, 2010). Leading national organizations (eg., ISTE, NCSM, NCTM) recognize the importance of continuing to address technology in teaching, but the literature is gradually becoming dated. "The rapid pace at which technology is evolving necessitates ongoing reexamination of the priorities of effective mathematics programs" (NCTM, 2014, p. 88).

Theoretical Framework

The limited use of CAS in U.S. classrooms (Heid et al., 2013; Pierce & Stacey, 2010; Schultz, 2003; Zbiek & Hollebrands, 2008) may not be fully understood without considering teachers' conception and usage of CAS (Fey, 2006; Zbiek & Hollebrands, 2008), as well as how teachers create opportunities in the classroom to instruct using CAS as a cognitive tool (Pierce & Stacey, 2008; Zbiek et al., 2007). The route toward a new style of mathematics teaching, especially by teachers that may be hesitant to embrace CAS, is "more likely to be carried through an evolutionary process rather than a revolutionary one" (Beaudin & Bowers, 1997, p. 129). Focusing on the spectrum of teacher usage and teaching craft will help to delineate developmental aspects of teacher transformations.

As teachers integrate CAS as an instructional tool, new prospects for the teaching and learning of mathematics may arise, shifting a perspective towards a rebalancing of skills, concepts, and applications (Pierce & Stacey, 2008). After years of research on students and teachers in Australia who utilized CAS consistently, Pierce and Stacey (2008, 2013) came to realize the key opportunities CAS affords for pedagogical change. Stacey developed the P-Map (see

Figure *1*) that encompassed all the opportunities noted through a multitude of observations (Pierce & Stacey, 2010). This map originates at the bottom with functional opportunities, curriculum and assessment change. The functional opportunities permit the 10 pedagogical opportunities the span around the center in gray blocks. The blocks are coded with the red typeset for cross-referencing in evidence tables.

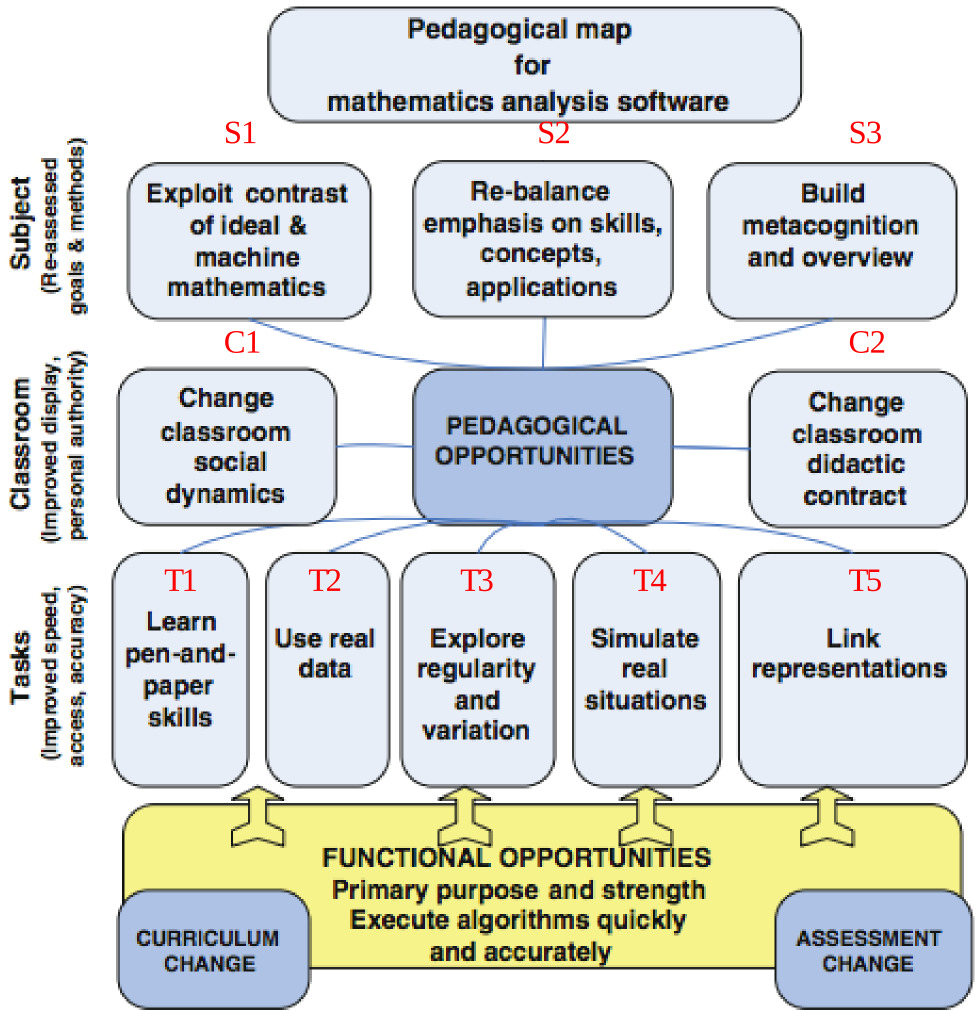

Figure 1. Pedagogical Map (P-Map) Adapted from "Mapping Pedagogical Opportunities Provided by Mathematics Analysis Software," by R. Pierce and K. Stacey, 2010, *International Journal of Computers for Mathematical Learning*, *15*, p. 6. Copyright 2010 by Springer International Publishing AG.

Pierce and Stacey's pedagogical framework (2010) was used to illuminate the opportunities in which teachers situated their utilization of CAS as a pedagogical tool and is referred to as the P-Map. Pierce and Stacey (2010) originally considered just CAS but expanded their viewpoint to include all varieties of mathematical analysis software (MAS). MAS is the "umbrella term to describe software with which the user can perform algorithmic processes required when working in one of more branches of mathematics" (Pierce & Stacey, 2010, p. 2). MAS varieties include all cognitive tools that operate in the algebraic realm of symbolic manipulation, dynamic functions, statistical packages, and graphic capabilities. "They facilitate the technical dimension of mathematical activity and allow the user to take action on mathematical objects or representations of those objects" (p. 2). A brief overview of the P-Map follows and is substantiated in the literature review. MAS as a functional machine is at the base of the map, illustrating its primary purpose to support users in the computation and manipulation of mathematical expressions and equations (Pierce & Stacey, 2008, 2010). The capability to consider curricula and assessment arises from the presence of this technology. Pierce and Stacey (2008, 2010) identified 10 unique affordances of technology put into instructional practice. These opportunities for pedagogy are grouped into three types: tasks, classroom, and subject.

The P-Map captured the technological skills as applied during the teachers' planning and instruction. The P-Map created by Pierce and Stacey (2010) was intended to reveal the ways that CAS can be put into practice in the classroom.

Statement of the Problem

The mathematics education community is ripe for technology infusion with the teacher at the heart of implementation (NCTM, 2014; Zbiek & Hollebrands, 2008). CAS tools have continued to advance in their simplicity, functionality, and availability but are still not a part of standard mathematics education practice (Heid et al., 2013). The U.S. education system leaves the implementation of technology in control of the individual teacher, guided by the NCTM standards (Heid & Blume, 2008). The language in *Principles to Action* (NCTM, 2014) strongly encourages teachers to consider putting research into practice, with one of the eight principles highlighting tools and technology. This technology standard stated, "An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (NCTM, 2014, p. 5).

Yet, teachers lack pedagogical examples that reflect CAS-enhanced instruction and, hence, utilize technology both infrequently and in limited methodologies (NCTM, 2014). Furthermore, as a result teachers limit student access to multiple forms of technology (Kastberg & Leatham, 2005). Obstacles to teaching with technology are numerous (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Hicks, 2010; Kaput, 1992). Ertmer (1999) referred to obstacles as first- and second-order barriers to change and included both extrinsic and intrinsic concerns. For example, limited equipment is a first-order barrier and is external to the teacher. Pedagogical skills are classified in the latter as a second-order barrier to change. Barriers will be examined further in the literature review. In summary, with the reduction in barrier issues, the problem of pedagogy can be centralized and examined.

Statement of Purpose and Research Questions

The purpose of this study was to understand: (a) what pedagogical opportunities mathematics teachers exploited with the presence of CAS; (b) how teachers aligned lessons to develop mathematical understandings; and (c) why these teachers wanted to orient their focus to exploit CAS in the development of mathematical knowledge. The following research question guided the study: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems (CAS) in the development of mathematical knowledge?

Significance of Study

Technology integration to instruction is both inevitable and expected (CCSSI, 2010; ISTE, 2008b; NCSM, 2011; NCTM, 2011, 2014), yet CAS' incorporation into U.S. classrooms has been slow and non-existent in most cases, despite its availability (Garner & Pierce, 2016; Heid et al., 2013; Robert et al., 2012). Furthermore, the ways teachers integrate technology effectively to promote mathematical understandings was not transparent. "Future practitioner questions about calculator use for mathematics teaching and learning should advance from questions of whether or not they are effective to questions of what effective practices with calculators entail" (NCTM, 2011, para. 7). Therefore, this study investigated secondary school teachers' educational practice.

The results of this study emphasized the pedagogical opportunities of teachers who utilized CAS to facilitate students' learning mathematics. This study contributed evidence of functional usage of the technology and provided testimonies to regard CAS as an essential tool in learning mathematics. Potential barriers to use, by either teacher or student, were intentionally minimized through research design, so that the teacher deliberately employed CAS through multiple prospects. These events were recorded with rich descriptions that can be dispatched as potential uses to the mathematics education community. Furthermore, in the presence of innovative CAS use, teacher motivations for these pedagogies were illuminated and contributed to the literature. Finally, Pierce and Stacey's (2010) P-Map framework was a lens to provide reflection on technological and pedagogical content knowledge in the utilization of CAS technology in secondary mathematics classrooms, giving utility to this contemporary instrument.

Definition of Terms

The following are terms that are used to inform the study. These are the definitions that will be used throughout this report. They are sequenced in the order in which they appear in the text and also as one definition relates to another.

Computer Algebra Systems (CAS)

CAS is defined as "software that enhances numeric and graphic operations with tools for formal manipulation of symbolic expressions . . . [and that] perform a wide variety of the numeric, graphic, symbolic, and logical operations that form the core components of algebra" (Fey et al., 2003, pp. 1-2). CAS can be used for: (a) exact numeric calculation; (b) exact symbolic calculation; (c) symbolic algebra; (d) symbolic manipulation; (e) dynamic representation of two- and three-dimensional graphs; (f) dynamic spreadsheet data tables; and (g) dynamic freelance geometric drawings. Additionally, calculus procedures of differentiation and integration can be performed on complex functions.

Symbolic Algebra

Symbolic algebra refers specifically to the algebraic procedures of simplifying, expanding, manipulating, and solving of algebraic expressions and equations. It may be helpful to think of symbolic procedures as the traditional procedures taught in high school algebra classes: factor, solve, simplify, and function operations.

Dynamic

Often with CAS systems a change is made in one value, and all other values linked to that value change simultaneously. A change in value is one example of the dynamic features of CAS but it is not limited to values alone. Differences may be a change to a geometric figure (e.g., a translation of a vertex of a polygon) or a graph of a function (e.g., a reflection). Likewise, these dynamic changes are not limited to one page or place in the digital document. A change can update simultaneously whenever the function or relation was identified symbolically elsewhere.

CAS Platforms

CAS platforms are widespread and may be devices produced by any of the following companies: Texas Instruments TI series, TI-Nspire™, Wolfram (e.g., alpha and Mathematica), Geogebra, Desmos, Casio, HP Prime series, and others that are less common. Often these platforms are thought of as handheld technology, but this study was not limited to any particular device and so encompassed CAS as a handheld, a tablet, or a computer device.

Orientation

One meaning of *orientation* according to Collins English Dictionary (2012) is the state of ones' philosophical beliefs, decisions, and choices. Orientation also can have a positional meaning that may involve seeking the relationship of a current position to surroundings. Orientation in this study used both interpretations; orientation is founded on a person's basic preferences and that position may adjust when connected to outside influences (e.g., teacher experience, student prior experience, distinct mathematics courses, or teacher pedagogical content knowledge).

Pedagogical Opportunities

Pedagogical opportunities refer to the multiple instructional choices and strategies that teachers use to teach mathematics. Pierce and Stacey (2008) claimed there is much variety in teaching when CAS is available and "different teachers make different choices about the changes they wish to make in their teaching style and approach to mathematics" (Pierce & Stacey, 2008, p. 6).

P-Map

The pedagogical opportunities map will be referenced in this study using the abbreviation P-Map. The focus of the P-Map (see Figure 1) developed by Pierce and Stacey (2010) is to identify, organize, and highlight those opportunities using this taxonomy as teachers integrate any type of MAS or technological cognitive tool.

Task

The term *task* described any type of activity, problem, or action students or teachers used to elicit mathematical content knowledge in the classroom.

Interpersonal Dimension of the Classroom

The dynamics of a mathematics CAS-driven classroom can have varying interpersonal dimensions. The confines pertain to student-to-teacher and student-to-student relations. No longer is the teacher the soul authority; CAS can act as an expert in mathematical and algebraic computations, thereby shifting the roles of teacher and student to consultant and fellow investigator (Pierce & Stacey, 2010).

Mathematics Analysis Software (MAS)

MAS is the "umbrella term [used] to describe software with which the user can perform algorithmic processes required when working in one of more branches of mathematics" (Pierce & Stacey, 2010, p. 2). Furthermore, "[The cognitive tools] facilitate the technical dimension of mathematical activity and allow the user to take action on mathematical objects or representations of those objects" (p. 2).

Black Box versus White Box Technology

The phrases black box and white box, introduced by Buchberger (1990), associated known and unknown calculations of technological devices. Black box is used to describe lack of insight into the inner workings of calculations and computations of a technological tool when an input is acted upon by the technology and an output is returned (Cedillo & Kieran, 2003). In contrast white box technologies show step-by-step actions upon a mathematical expression, providing awareness to the procedures acted upon by the technology.

Screencast

The ability to video capture a computer screen with an audio recording simultaneously is called a screencast. The audio recording is similar to a podcast, but the benefit of a screencast is the ability to see the activities on the computer that accompany the voice narrative.

Mathematical Authority

The interpretation for *mathematical authority* used in this research was the reference to Amit and Fried's definition, (2005) but paraphrased by Langer-Osuna (2017): "the most relevant type of authority is that of the expert who possesses mathematical knowledge that is taken as true" (p. 238).

Chapter Summary

CAS is a cognitive tool that can be utilized in mathematics classrooms to develop understanding of mathematics (Fey at al., 2003; Heid et al., 2013). A vast amount of research has already investigated some of the value for CAS, but much of it is international (Fey et al., 2003; Heid & Blume, 2008). NCTM (2014) recommended technological tools as essential resources in the classroom to support students in the visualization and conceptualization of mathematical knowledge. Teachers are the initiators of such new instruction. Considering teachers' moves and what those moves entail is the first step in addressing CAS technology integration.

CHAPTER II: REVIEW OF LITERATURE

Introduction

CAS technologies have become readily available tools in an ever-changing landscape of digital technology (NCTM, 2014). These tools have the potential to reshape the ways that learners can approach mathematical knowledge (Blume & Heid, 2008; Fey et al., 2003; Pea, 1995; Usiskin, Anderson, & Zotto, 2010). The teacher is viewed as the agent of change introducing novel pedagogies that integrate CAS technology in the teaching and learning of mathematics (Ertmer & Ottenbreit-Leftwich, 2010; Judson, 2006; Kastberg & Leatham, 2005; Pugalee, 2001). This study investigated two teachers' pedagogy as they utilized CAS technology to construct mathematical knowledge.

The purpose of this study was to understand: (a) what pedagogical opportunities mathematics teachers exploited with the presence of CAS; (b) how teachers aligned lessons to develop mathematical understandings; and (c) why these teachers wanted to orient their focus to exploit CAS in the development of mathematical knowledge. The following research question guided the study: How do secondary mathematics teachers orient their instructional practices to exploit CAS in the development of mathematical knowledge?

A discussion of teachers that utilize technology in their classrooms helps to garner a perspective in terms of the teacher as the primary agent of change (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Pugalee, 2001) and, also, teacher beliefs regarding mathematics instruction (Ball, Thames, & Phelps, 2008; Ernest, 2012; Shulman, 1986; Thompson, 1992). A look at some of the obstacles for the successful implementation of CAS technology will assist in understanding the lack of acceptance of CAS instructional practices (Brickner, 1995; Ertmer, 1999; Ivy & Franz, 2016; Wachira & Keengwe, 2011). Awareness of deterrents to technology utilization hinged on teacher beliefs about CAS and its place in curricula for secondary education. The P-Map created by Pierce and Stacey (2008) will be explicated as a framework in the research for this study.

The following pieces will be explained in the chapter. First, a background of CAS technology in education will be shared. Second, will be an account on the culture of mathematics education in terms of curriculum, teachers, and standards. Third, research literature regarding utilization of technology, teacher beliefs about technology, barriers and obstacles in technology integration will be described. Finally, I will present a summary of several theoretical perspectives that emerged from CAS utilization by educators.

Global Background of CAS

The integration of CAS into educational arenas around the world is riveting. Its place merely serves as a backdrop and cultivation of CAS as a tool for educational purposes. Although CAS was invented in the 1970s and then experimented with in classrooms in the 1980s, CAS has struggled to find a place in education amongst the debates on the spectrum of the extremes of stark use to incessant use (Tokpah, 2008).

The alphabetic listing below in Table 2 highlights the seminal research of CAS utilization in countries around the globe. Difficulty arises in describing the chronological implications of the research, since pieces were conducted simultaneously beginning in the late 1980s and they continue to update to present times. Table 2 provides a summary of these studies with a partial chronology including geographical location, researcher, and a general topic. The upcoming sections provide a brief overview highlighting the presence and impact of CAS for the geographical regions.

Table 2

Summary of Global Background of CAS

Year	Location	Researcher(s)	Topic
1984	United States	Heid, M. K.	Effects of re-sequencing skills and concepts in a calculus class at college level
1992	United States	Zbiek, R. M.	Prospective teachers mathematical understanding in a CAS-rich environment
2001	United States	Edwards, M. T.	Comparison of two secondary algebra classes using CAS as a tool to develop mathematical understanding
1984	Austria	Aspetsberger, K. & RISC Institute, Johannes Kepler University of Linz	MuMath in secondary schools: train in programming and usage for mathematics
1990's	Austria	Bohm, J., Buchberger, B., Kutzler, B. & Heugl, H.	Austrian Center for Didactics of CAS white box/black box principles
1994	France	Artigue, M. & Lagrange, J.	Instrumental Genesis
1990's	Australia	University of Melbourne Ball, L. Flynn, P., Pierce, R. & Stacey, K.	CAS as a tool in secondary education
2000-2005	Australia	CAS-CAT Project	Integration of CAS Use at the Secondary School
2004	New Zealand	Thomas, M. O. J., & Hong, Y. Y.	One week college freshman
2005	New Zealand	Neill, A., & Maguire , T.	One year secondary school

Australia

The University of Melbourne initiated a research project investigating CAS in the instruction of calculus in secondary education in 1998 (Kendal, Stacey, & Pierce, 2005). The basis of the study was to understand the development of conceptual understandings through multiple representations afforded by CAS technology. In their observations, researchers noted that teachers chose different approaches to teaching concepts (Kendal et al., 2005). As researchers, Stacey and Pierce continued to synthesize their understandings of teaching utilizing CAS. They noted that "CAS affords a range of key opportunities to change and improve the teaching of mathematics" (Pierce & Stacey, 2008, p. 6), formulating a framework referred to as a pedagogical map. These pedagogical opportunities will be explained in detail later in the *CAS theoretical perspectives* section of this chapter.

Educational leaders in Victoria, Australia became notable for their decision to require CAS on the high school senior exam for one part of the graduation assessment. The effort was a partnership between the University of Melbourne, the Victorian Curriculum and Assessment Authority, and three calculator companies, Casio, Hewlett-Packard, and Texas Instruments (Garner, 2004). The title of the project was Computer Algebra Systems in Schools: Curriculum, Assessment, and Teaching (CAS-CAT) and extended through the years 2000-2005 (Garner, 2004; Heid et. al., 2013; Pierce, Ball, & Stacey, 2009). This large-scale project mandated CAS instruction in secondary schools in preparation for the exams and later included all of the states of Australia that independently had included CAS for their instruction and comprehensive exams.

Austria

Klaus, Aspetsberger, and Buchberger (i.e., researchers from Johanns Kepler University of Linz) became pioneers in the development of pedagogy using CAS as a cognitive tool. The researchers took leadership in 1984 integrating lessons teaching how to use the MuMath program (i.e., Derive) and performing symbolic calculations. Soon after, the Austrian Center for Didactics of Computer Algebra (ACDCA) was formed, thus engaging many more researchers (Kutzler, 2010). The intent was to develop pedagogical tools for the instructors in Austria and to supply teachers with support for utilizing CAS tools in meaningful and inventive ways (Bohm, 2007). Simultaneously, in the early 1990s, Austria's government purchased a general license of Derive (e.g., CAS product) for use with the general population of teachers and students (Heugl, 1996). The ACDCA initiated and supervised six projects in secondary schools in Austria through the years 1993 and 2006 (Bohm, 2007) with the integration of updated CAS tools and continued research on the didactical practices.

France

Studies about the conceptual dimensions of teaching and learning using Derive (i.e., CAS) began in secondary classrooms in 1994 in conjunction with the University of Paris (Lagrange, 2003). Since France had a centralized curriculum, researchers were quick to integrate CAS in mathematics education. The focus of interest was "toward the changes produced by the introduction of CAS in teaching and learning mathematics in everyday situations and toward the search for conditions that produce satisfactory results" (Lagrange, 2003, p. 270). The findings from this first study did not show any enhancement in conceptual understanding and neither did they conclude a reduced cognitive struggle occurred for learners. Paper-and-pencil work challenges had been replaced by complications in syntax and other technical facilities. Furthermore, through observations of teachers in their classrooms, researchers noted that little reflection time or developed questioning strategies were offered to assist in making mathematical connections.

The second phase of research commenced with the release of the TI-92 symbolic handheld calculators in 1996. Widespread daily use of CAS became possible in French schools. Using knowledge from the first study, researchers provided flexibility to choose paper-and-pencil methods or CAS techniques (Lagrange, 2003). "We learned that students should have time to build CAS techniques for successful integration to take place" (Lagrange, 2003, p. 274). Tasks were designed to support reflection on the mathematics and commands afforded by CAS. Rich mathematical discussions were a significant part of the pedagogy.

Furthermore, the theory of instrumental genesis was developed through France's transition to CAS as a way to understand cognitive tools (Artigue & Diderot, 2002). The tool as an object by itself had sometimes been referred to as the artifact. However, as CAS was utilized to build tasks one upon another by human gestures it becomes an instrument (Artigue & Diderot, 2002; Drijvers & Trouche, 2008; Maschietto & Trouche, 2010). The scheme created by the user of performing actions, operations, and evaluation of constraints on the device to create meaning from the input or output made the tool an instrument. This scheme was referred to as the theoretical construct of *instrumental genesis* (Artigue & Diderot, 2002).

New Zealand

The University of Auckland initiated a one-week study with first-year college students to begin to understand how new users of CAS employ the technology when solving mathematics problems (Thomas & Hong, 2004). This short-term study revealed results consistent with the view that instrumentation of a tool was a lengthy process in the development of skills and schemes that enable effective utilization of CAS (Thomas & Hong, 2004). The schemes that these participants utilized were direct calculations, checks in work, investigations of a concept, and finally, direct complex procedures for either reducing cognitive load or reducing difficulty in by-hand computation.

A much larger scale study followed in secondary education a few years later conducted by the New Zealand Council for Educational Research. The student participants were enrolled in one of the six schools scattered between the north and south islands, with two teachers at each school receiving extensive, quality professional development in the use of CAS as an exploratory tool (Neill & Maguire, 2005). A major aim was to capture the stories of elements of effective practice in hopes of replication. The report concluded that teachers' pedagogy was enhanced through a student-discovery approach developing mathematical understandings, but that assessments must reflect the change in pedagogy (Neill & Maguire, 2005).

United States

During the period from the 1980s to the present, attention to CAS was influenced by three occurrences. First, James Fey (University of Maryland) posed questions about CAS to his mathematics education doctoral students, engaging Kathleen Heid's interest and seminal research on re-sequencing of concepts and skills in a college calculus class

(Heid, 1988). Second, the NCTM's publication edited by Fey, *Computer Algebra Systems in Secondary Education*, was released in 2003, and consolidated much of the research from CAS' inception up to publication. This handbook summarized research conducted throughout the world, providing perspectives for the potential of CAS in mathematics education. Third, Mathematics Educators Exploring Computer Algebra Systems (MEECAS) was formed to advocate for CAS with partnership to the leadership of USACAS conferences. The stated purposes for MEECAS are to improve the learning of mathematics through CAS tools, encourage experimentation with CAS, support research, and network with colleagues that share a similar vision (https://www.meecas.org/). Additional dissertation work has considered CAS and graphing utilities (Edwards, 2001; Fonger, 2012; Hicks, 2010; Ivy, 2011; Tokpah, 2008; Zbiek, 1992).

Summary of Global Background

Global studies were limited at the commencement of this study. This list was not comprehensive; rather it highlighted some impactful research due to CAS availability. Primarily the studies were selected for review because the focus was on secondary school students and mathematics coursework at that level. Australia, Austria, and France all addressed large-scale studies with the consequence of research related to pedagogy: Australia created the P-Map; Austria investigated didactics (i.e., pedagogy); and France developed the construct, *theory of instrumental genesis*. In spite of the research, CAS continued to be an ambiguous tool (Heid & Blume, 2008; Fey et al., 2003).

Culture of Mathematics Instruction in the United States

The teaching and learning culture can further help frame the pervasive situation of mathematics education in the United States. McCloskey (2014) developed a framework with the analytical lens of *ritual* to help describe the persistence of conventional pedagogical practice that remains in American classrooms. She argued, "The concept of ritual may be a particularly promising way for researchers in mathematics education to notice, describe, and explain the enduring and cultural nature of mathematics classroom practices" (McCloskey, 2014, p. 20). This work was generated as a response to the dilemma that Stigler and Hiebert (1999) discovered through the TIMMS video study.

United States mathematics teaching practices captured by videotapes in the 1990s (National Center for Education Statistics, 1995) reflected an abundance of valued methodology in which the student is a learner of algorithms, memorizer of formulas, and emitter of facts (Stigler & Hiebert, 1999). These illustrations of American classrooms were not randomly chosen, but self-selected by the teacher or principal to represent the best teaching practices in their respective schools. The sample educational settings showed the routines of each teacher's classroom structure, tasks, questions, and explanations, which are all a part of the culture of teaching (Stigler & Hiebert, 1999). The classroom structure lies within a larger culture that must take into consideration not only global characteristics of education policies, centralization of standards, organization and types of schools, access and equity to education, but also teacher, parent, and student beliefs about hard work and learning (National Research Council [NRC], 2001). "In every country, the complex system of school mathematics is situated in a cultural matrix"

(NRC, 2001, p. 31). The foundational contribution to education is the curriculum, the teacher, and standards and will be discussed in the following sections.

Curriculum

The United States curriculum traditionally "has been characterized as superficial" (NRC, 2001, p. 37). Its tendency was to focus on a few critical topics for each grade and provide an overview of other topics, leaving insufficient time to master the new ideas (NRC, 2001). Within the critical topics, only a narrow perspective was discussed (Hiebert, 2003). The need for remediation superseded the advancement into new content. Additionally, "much of the curriculum deals with calculating and defining" (Hiebert, 2003, p. 11). This continual review of the material, lack of depth or concentrated study on new topics, and crowding of content has been the typical pattern in American education (NRC, 2001).

However, recent state legislation has adopted the CCSSI (2010) and NCTM (2014) released *Principles to Actions*, a guide to educators and policymakers that describe the essential elements of teaching and learning mathematics. The intent for the design of CCSSI was to provide focus and coherency for teacher's curricula (CCSSI, 2010). Additionally, NCTM's *Principles to Actions* in its teaching and learning guiding principle states that "an excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically" (NCTM, 2014, p. 5). The goals of these documents are rigorous, growing out of the demand for research-based evidence in the field of mathematics education.

Teacher Practices

Teaching practices are inherited, not invented by teachers (Philipp, 2007; Stigler & Hiebert, 1999). As observers have emerged with descriptions of classroom instruction, reports all reflect that at "the core of teaching– the way in which the teacher and students interact about the subject being taught– has changed very little over time" (NRC, 2001, p. 48). The activities, tasks, and assessments that occur in a classroom are clearly teacher choice (Stigler & Hiebert, 1999). Often observed is "the traditional approach to solving problems— to teach a procedure and then assign students to practice the procedure" (Hiebert, 2003, p. 17). This is the belief that teachers uphold to be fair to the learners (Hiebert, 2003; Philipp, 2007).

Standards

NCTM released the first set of national mathematics educational standards in 1989 marking the start of the standards-based education movement (NCTM, 2014). States followed with the adoption of similar state standards. Yet, it was the CCSSI (2010) that bonded and created a near-consensus of the collective U.S. to develop consistency in standards throughout the nation (NCTM, 2014). Gill and Boote (2012) interpreted reform practice as one in which the primary activity in mathematics classes is problem solving as a means to develop deep conceptual understandings. However, Gill and Boote (2012) recognized that the culture in schools and classrooms do not reflect those standards. "Despite the tremendous amount of effort devoted by many mathematics educators to promote, defend, and implement reform-based mathematics education, procedural mathematics persists" (Gill & Boote, 2012, p. 3). Thus, Gill and

Boote (2012) considered aspects of educational culture in the U.S. as both procedural and conceptual knowledge.

Procedural and Conceptual Understandings

Two inherent types of knowledge in the learning of mathematics are procedural and conceptual understandings (Brownell, 1947; Erlwanger, 1973; NRC, 2001; Skemp, 1977; Stigler & Hiebert, 1999; NCTM, 2014). Knowledge of the two ideas along with teacher goals characterize the variety of pedagogical practices in mathematics educations. Teachers choose one type of knowledge or both combined to determine focus of instruction. A brief description of these historical perspectives and a review of terminology that frame the current outlook of mathematical teaching is provided in the next sections.

Meanings Of and Meanings For

Brownell (1947) defined meaning in the teaching of mathematics in an effort to improve instruction. He differentiated between learning arithmetic in connection with real-life examples and learning mathematics for the sake of having knowledge about arithmetic and its connections to other mathematics (Brownell, 1947). The first was described as meanings for; the latter meanings of. Moreover, Brownell depicted a continuum of learning with various degrees of meanings, relaying a notion that "meanings are relative, not absolute" (Brownell, 1947, p. 9).

Rules without Reason

Erlwanger's case study of Benny revealed misunderstandings of mathematics through interviews, although Benny reported correct answers during his practice and assessment (Erlwanger, 1973). Benny participated in a sixth-grade, independent-study curriculum. The study involved a series of one-on-one interviews in which the researcher sought to understand Benny's mathematical knowledge. Extensive questioning revealed a weakness in Benny's conceptions, "Benny emphasizes rules rather than reasons in his work" (p. 57). This classic study gave rise to the concern of "mastery of content and skill" (p. 51) void of understanding.

Relational versus Instrumental Understanding

Skemp (1977) expressed grave concern with the use of the term *understanding* purporting that *relational understanding* had deeper value in that it included both "knowing what to do and why" (p. 21). However, most regard *instrumental understandings* as a worthy type of understanding, when in fact, this has been described as "rules without reason" (Skemp, 1977, p. 21). Teachers and their students may have possessed a rule and applied it but not have understood the validity of the rule (Skemp, 1977).

Procedural Fluency and Conceptual Understanding

A more recent perspective by NCTM (2014) provided a mathematics teaching practice (MTP) to the effort of teachers aiding students in achieving procedural fluency while endorsing conceptual understanding. NCTM recognized that to achieve fluency practice strategies are employed to that effort but not in the absence of creating an overview and connection to mathematical concepts through multiple strategies. "This idea supports students in developing the ability to understand and explain their procedures, choose flexibly among methods and strategies . . . and produce accurate answers efficiently" (NCTM, 2014, p. 46).

Summary of Procedural and Conceptual Understanding

The two perspectives of procedural and conceptual understanding embody a continuum (Brownell, 1947) or layering (Skemp, 1977) of understandings. "Knowing how to execute procedures does not ensure that students know what they are doing" (Hiebert & Wearne, 2003, p. 3). In support of that notion, the PISA report (OCED, 2012) found that students in the United States showed particular weakness in cognitively challenging demands such as formulating a problem from a text, but they showed strength in extracting values from formulas and also in the interpretation of results. Students demonstrated how to execute procedures and find results but were not always able to justify their solution, echoing Skemp's (1977) rules without reason ideology. Yet, "instruction can be designed to promote deeper conceptual understanding" (Hiebert, 2003, p. 16). The ideology of developing mathematical understandings rests at this critical juncture between traditional paper-and-pencil calculations and automated CAS calculations. Teachers have been exploring the utilization of graphing calculators as a tool in their instructional practice, primarily performing both paper-and-pencil and CAS-automated calculations, to ensure that deep conceptual connections were developed in the learner (Ivy & Franz, 2016; Lee & McDougall, 2010).

Teachers Utilizing Technology in Teaching Practice

Technological tools, such as graphing calculator technology, computer applications, and CAS, have been integrated into mathematics teaching practices steadily as new devices were invented (Ertmer & Ottenbreit-Leftwich, 2010; Pea, 1985; Ronau et al., 2014). Graphing calculator technology includes the features of scientific numeric calculation, graphing capabilities, programming capabilities, and basic statistical tools.

Studies involving teachers' usage of these tools underpin that integration. Selection criteria for studies in this section included those studies (a) conducted in fairly recent years (i.e., 1999-2016), (b) focused on secondary school teachers, (c) administered in the United States or Canada, and (d) represented teacher implementation of mathematics technology. A review of studies involving teachers utilizing technology indicated these fundamental points: the teacher is the primary agent of change (Ertmer & Ottenbreit-Leftwich, 2010; Judson, 2006; Kastberg & Leatham, 2005); the teacher beliefs about mathematics instruction with technology is central to technological pedagogical practice (Lee & McDougall, 2010; Wachira & Keengwe, 2011); multiple barriers exist that prevent teachers from shifting to technology usage (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Ivy & Franz, 2016; Wachira & Keengwe, 2011); the mathematics curriculum has changed in some ways to support instruction with technology (Kastberg & Leatham, 2005; Milou, 1999); and a need still exists for educational models that reflect technology-infused instruction (Bitner & Bitner, 2002: Dewey, Singletary, & Kinzel, 2009; Lee & McDougall, 2010; Ostler & Grandgenett, 2001). This section serves to support those points, although these studies focused on the use of graphing calculators unless otherwise noted.

Teacher as an Agent of Change

Teachers are the agent of change for technology use in classrooms (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Pugalee, 2001), and those changes can be viewed in terms of student access to technology (Judson, 2006; Kastberg & Leatham, 2005; Özgün-Koca, Meagher, & Edwards, 2011). Consideration of teacher demographics and teacher attributes inform the creation of technology pre-service training and professional development. Judson (2006) asserted that teachers harbor the authority of class reform and the potential for implementation of adopted policy, including technology-infused instructional practices. "Issues of teacher change are central to any discussion of technology integration" (Ertmer & Ottenbreit-Leftwich, 2010, p. 258). It is not merely the presence of technology, but the instructional moves by the teacher that incites a powerful effect on student learning (Ertmer & Ottenbreit-Leftwich, 2010).

Student access to technology. Through an analysis of literature on research of graphing calculators at the secondary level, Kastberg and Leatham discovered with consistency that access to graphing calculators was being "mediated by the teacher" (Kastberg & Leatham, 2005, p. 26). Regardless of the technology availability to students, the teacher made the decision as to when, how, and under what conditions the technology would be brought into classroom instruction. Furthermore, the teacher moderated the type of activities: checking algebraic solutions, graphing related functions, and comparing the results with hand calculations (Kastberg & Leatham, 2005).

Similar findings revealed that teachers were the gatekeepers for instruction that utilized technology (Lee & MacDougall, 2010; Wachira & Keengwe, 2011). The Lee and MacDougall (2010) multiple case study noted that for two of the three teachers that were observed, each brought out graphing calculators for use at special times with certain activities, thereby limiting access to technology. Likewise, in a study that explored the teacher perspective of technology integration barriers, Wachira and Keengwe (2011) confirmed that decisions of using technology ultimately depended on the teacher in terms of availability of technology and the teacher's beliefs about technology use. Given the importance of the teacher, identification of teacher consistency using technology and the characteristics of these teachers will be discussed.

Teacher consistency of technology use. Studies have revealed that experienced teachers and those teaching more advanced mathematical topics, such as advanced coursework in secondary schools, used graphing calculator technology with greater frequency (Dewey et al., 2009; Milou, 1999). Milou (1999) conducted a self-report survey study of grades 7-12 teachers, and it revealed algebra II teachers using technology more regularly than algebra I teachers. Furthermore, Milou noted that high school teachers had greater frequency than middle school teachers in accessing technology as a teaching tool. Dewey's replicated survey study of 109 teachers provided an "indication that usage is higher among older teachers with more experience" (Dewey et al., 2009, p. 390). This was justified by the notion that senior teachers were often assigned to teach more advanced mathematics subjects to which the concepts lend to graphing calculator technology more readily than foundational mathematics concepts. However, another likely argument was that senior teachers had developed confidence in their content knowledge and comfort with a variety of instructional practices (Dewey et al., 2009). Therefore, it was the seasoned teacher with a broader repertoire of experience that enabled the teacher to have more flexibility to engage students in using technology as a tool in his teaching practice (Dewey et al., 2009).

The same efficacious use of the graphing calculator resembled Doerr and Zangor's Netherlands study (2000) beholding keen reflection from a single-teacher, two-class case study. The 20-year veteran teacher possessed graphing calculator expertise and was observed teaching precalculus classes. It was ascertained that the teacher's confidence in content knowledge and skill led to flexible use of the graphing calculator during instruction (Doerr & Zangor, 2000). The teacher portrayed a willingness to encourage students to adopt solutions through multiple methods with graphing calculator technology, as well as visibly share those methods on a projection device for a whole classroom discussion (Doerr & Zanger, 2000).

More advanced mathematics courses appear to involve more advanced uses of calculator technology (Dewey et. al.; 2008; Ivy & Franz, 2016; Milou, 1999). Ivy and Franz's (2016) multiple case study explored technological pedagogical content knowledge (TPACK) with contrasting data from two veteran teachers' self-reported surveys and interviews, alongside the researchers' lesson observations. One teacher was observed teaching an algebra I class and the other a precalculus class. The survey data indicated that both teachers regarded their technology integration as exemplary (Ivy & Franz, 2016). The observation data, however, provided evidence of multiple factors that influenced the researchers' decision to rank the algebra I teacher at the lowest TPACK level and the precalculus teacher at the top, the fourth of five levels (Ivy & Franz, 2016). The teacher of the advanced mathematics class was observed instructing at a higher level of TPACK rank, a parallel to Milou's finding (1999). Teachers that integrated technology with more advanced usage appeared to have a higher level of TPACK and also teach higher level of mathematics courses (Ivy & Franz, 2016; Milou, 1999).

In summary, these studies (i.e., Dewey et al., 2009; Doerr & Zangor, 2000; Ivy & Franz, 2016; Milou, 1999) reflect experienced teacher and greater TPACK with larger consistency of technology utilization. Furthermore, teachers of advanced mathematics courses were more frequently using technology in their instructional approach to teaching mathematics. It followed that attributes of teachers who use technology need consideration to further understand the technology integration into teaching practice.

Teacher attributes. Limited research exists on attributes for teachers who use MAS. Often these studies rely on self-report data. However, the following characteristics were themes in several case studies, literature reviews, and survey data: experience with personal use of technology (Doerr & Zangor, 2000; Lee & MacDougall, 2010); practice of constructivist instruction (Judson, 2006; Lee & MacDougall, 2010); expertise in student inquiry (Doerr & Zangor, 2000; Ivy & Franz, 2016); and strength of PCK (Chamblee, Slough, & Wunsch, 2008; Ertmer & Ottenbreit-Leftwich, 2010; Ivy & Franz, 2016; Shulman, 1986; Wachira & Keengwe, 2011). These four common characteristics for teachers who implement technology with fidelity will be examined in the next sections.

Experience with personal use. "Teachers who use technology for their personal use are more comfortable using technology in their classrooms" (Lee & MacDougall, 2010, p. 858). The inverse was also found to be true for a teacher utilizing CAS (Zbiek, 2002). In the analysis of a multiple case study, Zbiek noted one of the teachers had minimal CAS experience and was observed avoiding the use of CAS in her instruction, although the teacher was expected to teach with CAS. Zbiek (2002) noted that the teacher exposed students first to by-hand skills and followed with CAS symbolic manipulations, justifying the need for students to understand the concept first.

Traditional versus constructivist approach. Another distinguishing characteristic of teachers was their philosophical approach to teaching as either traditional or constructivist, and it paralleled teacher's pedagogical uses of technology (Ertmer &

Ottenbreit-Leftwich, 2010; Judson, 2006). "In general, teachers with more traditional beliefs will implement more traditional or low-level uses, whereas teachers with more constructivist beliefs will implement more student-centered or high-level technology uses" (Ertmer & Ottenbreit-Leftwich, 2010, p. 262). A case study by Lee and McDougall (2010) disclosed that the teacher created "an environment in her classroom where constructivist-learning opportunities are possible" (p. 864). As the teacher accessed utility of the graphing calculator he promoted exploration, analyzed comparisons of several handheld screens, and generated student discourse regarding those differences in outputs. Constructivist instruction was evident throughout his instruction.

Questioning strategies. Teachers who have expertise in student inquiry use questioning strategies while adapting technology tools for instructional goals (Doerr & Zangor, 2000; Ivy & Franz, 2016). Doerr and Zangor (2000) cited the teacher as a mediator in the interpretation and explanation of calculator computations. The mediator role often occurred through viewing outputs on the device and realizing the limitations of the calculator solutions that also prompted classroom discourse (Doerr & Zangor, 2000). Another example, a case study by Ivy and Franz (2016), cited a precalculus teacher who fostered a student-centered classroom with the teacher as a facilitator. Students were provided preprinted guided questions on a task to be completed with the use of a handheld graphing device. The teacher encouraged students to discuss technical issues on the device with one another, to allow for student-to-student discourse. However, mathematical content questions connecting the task to the underlying concepts were generally addressed by large-group discussions that demonstrated teacher-to-student discourse through the use of technology (Ivy & Franz, 2016). In both studies, the teacher relied on his expertise of student inquiry combined with hands-on technology use to facilitate students' development of mathematical understanding of concepts.

Level of PCK. PCK describes a teacher's ability to develop instruction that exposes concepts to students through the activities, questions, and lessons surrounding the content (Shulman, 1986). Educators expect teachers to have a strong PCK base and the elemental technology skills in order to integrate technology both regularly and seamlessly into instruction (Chamblee et. al., 2008; Ertmer & Ottenbreit-Leftwich, 2010; Ivy & Franz, 2016; Wachira & Keengwe, 2011). All of the teachers in these studies lacked exemplary mathematics teaching practices that demonstrated high-quality technology integration. The research provided ample evidence of weak PCK paired with low levels of technology integration.

Ivy and Franz (2016) compared two teachers that appeared to have similar PCK, as determined through interviews. However, they each exhibited fundamentally different pedagogies. The classroom observations provided data that contradicted the use of technology as described by one case. "Through examination of these two participants . . . it is suggested that significant PCK serves as an impetus to effective technology integration" (Ivy & Franz, 2016, p. 14). Similarly, Wachira and Keengwe (2011) interviewed both mathematics coaches and teachers. Consequently, they gleaned that teachers not only lacked skills to effectively utilize technology but also lacked pedagogy and expertise to create appropriate technology-infused activities. Ertmer and Ottenbreit-Leftwich (2010) concluded that teachers are challenged to develop instruction. They contended that good examples of practitioners' pedagogy were a necessary component not only to facilitate PCK but also to form beliefs about technology in education.

Summary of teacher attributes. Personal use of technology, constructivist instruction, expertise in student inquiry, and PCK all contributed to the fidelity of teacher technology utilization in instructional practices. Teachers who had experience in personal use of technology were more likely to use technology in their classroom instruction (Lee & MacDougall, 2010; Zbiek, 2002). Likewise, PCK, teacher inquiry experience, and other constructivist methodologies led to more advanced types of technological uses in the classroom (Doerr & Zangor, 2000; Ertmer & Ottenbreit-Leftwich, 2010; Ivy & Franz, 2016).

Summary of teacher as an agent of change. Teachers are the key to advancement of technology use in mathematics education (Chamblee et al., 2008; Ertmer & Ottenbreit-Leftwich, 2010; Wachira & Keengwe, 2011). Studies showed that teacher background and experience demographics were linked to teachers' technology usage in the classroom (Dewey et al, 2009; Doerr & Zangor, 2000; Ivy & Franz, 2016; Milou, 1999). Furthermore, some teacher attributes were a catalyst for teacher instruction that utilized teaching technology with fidelity. Few studies acknowledged exemplary teacher utilization of technology; indeed, many teachers lacked the knowledge and expertise to integrate technology. "Teachers did not know how to take advantage of technology as powerful tools to strengthen students' understanding of mathematics" (Wachira & Keengwe, 2011, p. 23). Repeatedly, studies revealed that teachers needed additional training and support to develop mathematics instructional practice with technology (Chamblee et al., 2008; Ertmer & Ottenbreit-Leftwich, 2010; Kastberg & Leatham, 2005; Simonsen & Dick, 1997; Wachira & Keengwe, 2011). Therefore, without such training, teachers will not gain the TPACK necessary to adopt instruction to ISTE and NCTM standards (ISTE, 2008b; NCTM, 2016).

Teacher Beliefs about Mathematics Instruction

In the effort to understand teacher beliefs about mathematics instruction with technology one must consider three conceptions: (a) the nature of content knowledge for teaching (Ball et al., 2008; Ernest, 2012; Shulman, 1986; Thompson, 1992); (b) the nature of teacher beliefs about teaching and learning (Ernest, 2012, 2016; Garegae, 2016; Gill & Boote, 2012; Thompson, 1992); and (c) the interplay between teacher beliefs and teaching practice (NCTM, 2014). Shulman (1986) conceived the notion of PCK, the unique knowledge of subject area and teaching, along with the representations and instructional conceptions for the subject area. It is Ball et al. (2008) that parsed out the content knowledge for the teaching element of Shulman's conception, specifically in terms of mathematical knowledge for teaching (MKT). Yet, the value in understanding the nature of content knowledge for teaching and the nature of teacher beliefs about teaching and learning is that a strong correlation between teacher beliefs and pedagogical practice exists (Ravitz, Becker, & Wong, 2000; Thompson, 1992). The distinguishing characteristics of content knowledge, beliefs about teaching, and the interplay between the two are shared in the sections that follow.

The nature of mathematical content knowledge for teaching. Mathematical knowledge has traditionally been difficult to define, in particular to epistemology and ontology of mathematics for teaching (Ernest, 2012, 2016; Thompson, 1992). "Historically, mathematics has long been viewed as the paradigm of infallibly secure knowledge" (Ernest, 1998, p. 1), but Ernest began a 20-year discourse asserting differences between this perspective of absolutism versus fallibilism philosophies of mathematical knowledge, introducing the notion of social constructivism as a construct for mathematical knowledge (Ernest, 1998). The fallibilist view perceives mathematics knowledge as socially and humanly constructed, therefore, making it subject to fault (Ernest, 2012). Ernest makes clear that reforming the definition of mathematics provided "an underpinning for the central focus of mathematics education, namely the teaching and learning of mathematics" (Ernest, 2012, p. 9). In that quest Thompson (1992), Shulman (1986), and Ball et al. (2008) provide more detail about mathematical knowledge for teaching. Thompson (1992) proposed, from a conglomeration of standards documents at the time, that mathematics knowledge for teaching is the activity of students engaging in mathematical problem solving, exploring, discovering, and creating that elicits learners to use reasoning, argumentation, and critical thinking skills. Ernest' construct (1998) aligns with Thompson's constructivist view (1992) and contrasts with the instrumentalist perspective of mastery of concepts, procedures, and algorithms. However, Ernest' social constructivist view does not deny the value as a place in mathematics curriculum (Ernest, 2012; Garegae, 2016; Thompson, 1992).

A more general point of view about content knowledge from Shulman (1986) required the teacher to both understand that something exists and also know why it exists.

> The teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened or even denied. Moreover, we expect the teacher to understand why a given topic is

particularly central to a discipline whereas another may be somewhat peripheral. (Shulman, 1986, p. 9)

Shulman's PCK framework combines content knowledge and pedagogical knowledge to form a unique type of knowledge specific to teachers. Although embracing Shulman's PCK framework, Ball et al. (2008) claimed that the distinction of mathematical knowledge for teaching remained unclear, so advanced the discussion further by shifting toward how teachers used and applied knowledge in the work of teaching. Ball et al. (2008) considered domains of mathematical knowledge through careful consideration of the specific activities of teachers, such as lesson planning, implementing, explaining, evaluating, and attending to other classroom concerns. These are known as the *Domains of MKT* and are provided in Table 3. These domains serve to map out teacher knowledge for individual teachers, pre-service teacher programs, and professional development.

Table 3

Domains of MKT

Subject Matter Knowledge	PCK
Common content knowledge	Knowledge of content and students
Specialized content knowledge	Knowledge of content and curriculum
Horizon content knowledge	Knowledge of content and teaching

Note. Adapted from "Content knowledge for teaching: What makes it special?" by Ball et al., 2008, *Journal of Teacher Education*, *59*, p. 402-404.

The nature of teacher beliefs about teaching and learning mathematics.

Understanding teachers' utilization of technology and the instructional moves that the teacher makes is related to teacher beliefs about teaching and learning (Thompson, 1992). Literature shows that teachers form their beliefs primarily from their individual experiences as a student and those beliefs are part of teachers' conscious and subconscious thoughts (Thompson, 1992). Furthermore, teachers' epistemological understanding about the nature of mathematics inferred connections to the development of knowledge, the methods for instruction, and the audience to be taught (Garegae, 2016). As a basis, Ernest (2012, 2016) conceived philosophies about the nature of mathematics. The following discussion follows from that foundation.

Ernest's (2012, 2016) three philosophical perspectives facilitate an understanding of teacher beliefs: problem solving social-constructivist, Platonist, and instrumentalist views (Garegae, 2016). The first view, social-constructivist, deduces mathematics as constructed through solving problems, and its fallibilistic nature of potentially possessing inaccuracies suggested that what was once verified may later prove false. The second, Platonist, deems mathematics as a static content that is discovered. Finally, the instrumentalist perspective regarded mathematics as rules, algorithms, and disconnected facts but none-the-less was regarded as truths. Similar to Gill and Boote's (2012) description of procedural mathematics, teacher beliefs can contain one or more of Ernest's philosophies simultaneously (Garegae, 2016). Ernest (2012, 2016) approached the teaching of mathematics from a philosophical position of intertwining learning theories with content knowledge. He, therefore, anticipated that the learner constructed his knowledge through perspective content. The argument builds to how teachers develop their conceptions, and ends with the NCTM *Principles to Actions* (2014) discussion about productive and unproductive beliefs.

Foremost are the two views: beliefs that are productive and those that are unproductive (NCTM, 2014); and beliefs that center on teachers' views of active engagement of students through social constructivist moves (Ernest, 2012). Teachers often perceive that they are already implementing reform-based practices of a constructivist nature (Gill & Boote, 2012). NCTM (2014) asserts that teachers should focus on developing conceptual knowledge through procedural knowledge, contextual problems, and also through engagement of students in explorative activities that foster perseverance through productive struggle. This view of productive beliefs should not be viewed as good or bad, rather it promotes reflection by teachers to support and encourage student opportunity to learn (NCTM, 2014).

The interplay of teacher beliefs and teaching practice. Implicit in NCTM's (2014) productive and unproductive beliefs about teaching and learning mathematics was that no one theory existed as the absolute fixed belief that must be put into practice. Furthermore, the determination as to which begets the other, belief or practice, is not yet decided (Cobb, Wood, & Yackel, 1990; Ernest, 2016). As a philosopher and mathematician, Ernest (2016) claimed that the aims of mathematics instruction arise through an organized social activity; therefore, purposes for teaching and learning mathematics contain many divergent views. Likewise, knowing mathematics materialized through both a social and cognitive aspect (Cobb, Yackel, & Wood, 1992). Taking perspectives from Ernest (2016) and Cobb et al. (1990, 1992), knowing, learning, and teaching mathematics depend on social constructs to form beliefs that, in turn, forge mathematical instructional practices. Thompson's approach was the converse; the covert beliefs were recognized through teacher decisions (Thompson, 1992). Thompson defined a teacher's conceptions of teaching and learning mathematics by the elements that the teacher considered as goals, roles of teacher and student, instructional approaches, appropriate activities, procedures, and sufficient aftermaths. It is both convenient and appropriate to gather these artifacts, scrutinize each one, and draw conclusions about teacher beliefs. Thompson's primary concern was the link between what a teacher believed about mathematical knowledge and the way that knowledge was situated in the teaching and learning context (Thompson, 1992). Cobb et al. (1990) considered teacher beliefs as they informed practice, and then how that impacted student learning through a linear relationship. Cobb et al. suggested that "beliefs are expressed in practice, and problems or surprises encountered in practice give rise to opportunities to reorganize beliefs" (p. 145). The interplay between beliefs and practice continually evolve, one informing the other (Cobb et al., 1990; Ernest, 2012, 2016).

Summary of teacher beliefs about mathematics instruction. First, a highly rated teacher would be knowledgeable in his field of study (Ball et al., 2008). "What constitutes understanding of the content is only loosely defined" (p. 389). Second, the expression *teacher belief* has been referred to with multiple terms: "Words like conceptions, perceptions, feelings, inferences, preferences, and attributions are used interchangeably in the literature" (Garegae, 2016, p. 2). Third, teacher beliefs and content knowledge of mathematics have long posed the challenge of delineating the two (Ball et al., 2008; Shulman, 1986; Thompson, 1992). Knowledge must satisfy a truth conviction (Thompson, 1992). Each informs the other and their differences are subtle. Beliefs fall on a continuum of the degree of conviction and also carry the notion of disputability (Thompson, 1992). However, beliefs and knowledge about mathematics influence the aim of teaching, another complex domain (Ball et al., 2008; Ernest, 2016; Ravitz et al., 2000; Thompson, 1992).

Teacher Beliefs about Mathematics Technology Utilization

Technology is on the forefront in educational practice, and somewhat on the leading edge is discussion regarding teacher beliefs about teaching and learning mathematics when technology is used as a tool to develop mathematical understandings (Dewey et al., 2009; Ertmer & Ottenbreit-Leftwich, 2010; Lee & McDougall, 2010). As computer technologies dawned in the educational arena, Pea (1985) astutely pointed out that the "cognitive technologies we invent serve as instruments of cultural redefinition" (p. 167). Pea continued to exhort the need for reorientation of teaching and learning, but also embracing new means of mental functioning that occurred because of technology. This idea of reorientation originated in teachers' beliefs about technology as a cognitive tool and the choices the teacher made towards technology-integrated curriculum (Heid & Blume, 2008; Thompson, 1992).

"Mathematics teachers should judiciously adopt technology that supports effective instruction but not simply for the sake of using more technology in the classroom" (NCTM, 2014, p. 80). The authors stated that teachers often tend to student issues with technical procedures with calculators and then do not provide opportunities for students to connect the problems to the mathematical content. Additionally, Ivy and Franz (2016) presented this finding regarding teacher's PCK: "participants demonstrated inconsistences between their perceptions of their instructional practices and observed instructional practice" (p. 12). Furthermore, Kastberg and Leatham (2005) found that graphing calculators were not accessed for advanced operations. Obstacles to effective teaching with technology stem from unproductive beliefs and are listed in Table 4. NCTM recognized that the value of technology was dependent on the method and purpose to which the tools are being used. The shift towards technology utilization must be rooted in mathematical reasoning and sense making (NCTM, 2014).

Table 4

Beliefs about Tools and Technology in Learning Mathematics

Unproductive Beliefs	Productive Beliefs
Calculators and other tools are at best a frill or distraction and at worst a crutch that keeps students from learning mathematics. Students should use these tools only after they have learned how to do procedures with paper-and-pencil.	Technology is an inescapable fact of life in the world in which we live and should be embraced as a powerful tool for doing mathematics. Using technology can assist students in visualizing and understanding important mathematical concepts and support students' mathematical reasoning and problem solving.
School mathematics is static. What students need to know about mathematics is unchanged (or maybe even threatened) by the presence of technology.	Technology and other tools not only change how to teach but also affect what can be taught. They can assist students in investigating mathematical ideas and problems that might otherwise be too difficult or time-consuming to explore.
Physical and virtual manipulatives should be used only with very young children who need visuals and opportunities to explore by moving objects.	Students at all grade levels can benefit from the use of physical and virtual manipulative materials to provide visual models of a range of mathematical ideas.
Technology should be used primarily as a quick way to get correct answers to computations.	Finding answers to a mathematical computation is not sufficient. Students need to understand whether an answer is reasonable and how the results apply to a given context. They also need to be able to consider the relative usefulness of a range of tools in particular contexts.
Only select individuals, such as the most advanced students or students who reside in districts that choose technology as a budgetary priority, should have access to technology and tools, since these are optional supplements to mathematics learning.	All students should have access to technology and other tools that support the teaching and learning of mathematics.
Using technology and other tools to teach is easy. Just launch the app or website, or hand out the manipulatives, and let the students work on their own.	Effective use of technology and other tools requires careful planning. Teachers need appropriate professional development to learn how to use them effectively.
Online instructional videos can replace classroom instruction.	Online instructional videos must be judiciously adopted and used to support, not replace, effective instruction.

Note. Adapted from Principles to actions: Ensuring mathematical success for all, by

National Council of Teachers of Mathematics, 2014, Reston, VA: Author, p. 82. Copyright 2014 by the National Council of Teachers of Mathematics, Inc.

Ivy and Franz (2016) expressed concern about teachers with low PCK and, hence, unproductive beliefs such as "teaching procedures and memorization over reasoning and conceptual understanding, mastering a set of basic skills prior to exploring and solving contextual problems, and a focus on step-by-step procedures to minimize classroom struggle" (p. 13). In a similar way, Wachira and Keengwe (2011) referenced the fact that teachers in their study "did not know how to take advantage of technology as powerful tools to strengthen students' understanding of mathematics" (p. 23). As with Ivy and Franz (2016) report, teachers lacked the expertise of integrating technology with limited technological pedagogical content knowledge. Unproductive beliefs constitute one type of barrier to teaching with technology.

Barriers to Teachers Implementing Technology in the Classroom

Often classified as first-and second-order barriers to implementation, teaching with technology has presented challenges to teachers (Brickner, 1995; Ertmer, 1999). First-order barriers are viewed as external to the teacher; whereas second-order are considered internal (Brickner, 1995; Ertmer, 1999). Some examples of first-order barriers are the lack of equipment, technical support, teacher release time, and professional development involving utilization (Ertmer, 1999). Internal or second-order barriers are more difficult to isolate as they are interconnected to teacher beliefs about instruction with technology (Ertmer, 1999). "These barriers relate to teachers' beliefs about teacher-student roles as well as their traditional classroom practices including teaching methods, organizational and management styles, and assessment procedures" (Ertmer, 1999, p. 51). Overcoming these barriers "requires teachers to restructure their belief systems about computer implementation and their identity therein" (Brickner, 1999, p. 274). In trying to capture this perspective, selected studies (Ivy & Franz, 2016; Wachira & Keengwe, 2011) reflect concerns regarding Pea's conception of a cultural redefinition and Brickner's view of second-order barriers. Through such lens, both Ivy and Franz (2016) and Wachira and Keengwe (2011) perceived teachers' bearing minimal facility of the affordances of graphing calculator technology. The teachers in these studies lacked provisions for student access and low-effectiveness in the utilization of the tools.

Wachira and Keengwe (2011) collected K-12 teacher perceptions on the integration of technology in the teaching of mathematics during a masters' level graduate course entitled *Teaching Mathematics with Technology*. The primary evidence cited internal barriers as lack of time, lack of pedagogical knowledge, and lack of confidence in teaching with technology (Wachira & Keengwe, 2011). Teachers did not have sufficient training on technological tools and did not know how to use the tools to advance student understandings of mathematics concepts. Teacher anxiety about making mistakes or failing to trouble-shoot problems during teaching surfaced in discussions. Also, teachers expressed a lack of creativity in regards to TPACK and using pedagogies to reshape how students learn mathematics.

A similar study framed with the TPACK Development Model (Niess et al., 2009) suggested a potential solution to overcome second-order barriers. Ivy and Franz (2016) surveyed, interviewed, and observed seven secondary mathematics teachers who volunteered to participate in a teacher-perception-of-technology study. Ivy and Franz acknowledged teachers' PCK self-report data revealed perceptions that teachers had positive and progressive beliefs, yet the actual teaching practice appeared as a mismatch rendering low-level uses of technology (Ivy & Franz, 2016). One teacher maintained alignment at higher levels, managing to break through second-order barriers, thereby implementing more advanced levels of mathematics technology. "It is suggested that significant PCK serves as an impetus to effective instructional technology integration" (Ivy & Franz, 2016, p. 14).

A primary barrier to teaching mathematics with technology is an issue regarding teacher beliefs about teaching and learning with technology (Ivy & Franz, 2016; Wachira & Keengwe, 2011). However, it was suggested that improving access to software and equipment, teachers need training to strengthen TPACK, and conversing about instructional practices, teachers can overcome barriers to technology and find new roles to use mathematical technologies in the classroom (Ivy & Franz, 2016; Wachira & Keengwe, 2011).

Roles of Technology in the Classroom

CAS can afford various pedagogical opportunities such as investigations (Brown, et al., 2007; Kastberg & Leatham, 2005), access to real-world problem contexts (Drijvers, 2000; Kastberg & Leatham, 2005), multiple representational forms (Fonger, 2012; Zbiek & Hollebrands, 2008), and mathematical authority (Langer-Osuna, 2017; Schoenfeld, 2016). Teachers should consider multiple ways to utilize CAS in order to teach utility of the tool; to engage students in problem solving; to extend thinking for complex problems; and to consider models of problems through the use of the tool (Heid, 2003). Various roles of technology are discussed in the sections that follow.

Investigations. The meta-analysis by Kastberg and Leatham (2005), a review of research on graphing calculator technology, found that many teachers developed pedagogy that encouraged student exploration and investigation of mathematics problems with technology. Kastberg and Leatham reported technology provided students a multiple representational context that benefitted their opportunity to explore problems in multiple contexts and also to construct cognitive links between representational models. The tool's efficiency to transition from one representation to another provided the context for students to easily access the concepts. Similarly, Brown et al. (2007) noticed that teachers valued investigations perhaps because students had a more positive attitude towards mathematics. The tool galvanized students to move to higher-level thinking processes and enhanced their learning experience (Brown et al., 2007). Furthermore, Brown et al. (2007) reported that teachers' perception of calculator use "led to better understanding, provided a stimulus, generated interest, and enhanced student performance" (p. 112).

Real-world problem contexts. Drijvers (2000) chose a research approach of obstacles to learning in a secondary school CAS environment. Through real world optimization problems and investigating solutions, students found challenges in the variation of outputs. Those challenges came primarily through utilization of the device especially in terms of syntax. Drijvers (2000) claimed that CAS introduced more issues than resolutions. "An obstacle, now, is a barrier provided by the CAS that prevents the student from carrying out the [utilization] scheme that s/he has in mind. As a result, the obstacle stops the process of shifting between the 'pure' mathematics and the problem situation" (p.195).

In contrast, Kastberg and Leatham's (2005) meta-analysis revealed that students with CAS exposure outperformed students with traditional instruction. In particular, the University of Chicago School Mathematics Project (UCSMP) curriculum was developed to include application problems and the expectation that graphing calculators would be used as a tool in problem contexts. Students who were instructed with this curriculum outperformed those students who were taught with traditional curricula (Kastberg & Leatham, 2005).

Drijvers (2000) and Kastberg and Leathams (2005) contrasting projects of real-world problem contexts revealed elemental differences in student competency of the cognitive tool. Technical activity with the tool created a challenge in the first study and curriculum written with technology integration benefitted learners in the second study. Given the theory of instrumental genesis (Artigue & Diderot, 2002), it is befitting that the teacher's role is to assist in real-world problem contexts by first addressing instruction of CAS tools.

Multiple representations. "The ability to shift between different representations of a problem can help students develop a deeper understanding of mathematical concepts" (NCTM, 2014, p. 84). Zbiek and Hollebrands (2008) claimed a technology affordance is the opportunity to collect data and view linked representations to that data. They reported results from a SimCalc study with prospective teachers of the graphic representations and changes in the expression as features that were motivational to deepen conceptual understandings.

The idea that accessing different representations of mathematics problems can assist learners in developing conceptions in meaningful ways is not unfamiliar (Fonger, 2012; Heid, 2003; Kaput, 1992; Kutzler, 2010; Pierce & Stacey, 2002, 2008). In fact, seminal research involved multiple representations and dynamic features that interplay in contexts using MuMath or Derive (Heid, 1998, 2001; Kutzler, 2010). Instruction with multiple representations benefited students beyond their ability to just create the representation; those students had multiple methods and pathways to understand problems (Fonger, 2012; Heid, 1998, 2001). CAS was an efficient tool to formulate multiple representations.

Authority shift. The location of authority in the mathematics classroom—with the teacher, textbook, discipline of mathematics, or across students—has implications for sense-making opportunities (Hamm & Perry, 2002; O'Donnell, 2006). "Classrooms in which authority is shared between the teacher and the students offer students opportunities to take ownership of their ideas, leading to greater conceptual understanding and greater identification with mathematics" (Langer-Osuna, 2017, p. 238). The mathematics community references other domains in the authority frameworks: intellectual (Langer-Osuna, 2017); anthropogogical (i.e., pedagogical authority) (Gerson & Bateman, 2010); and didactic contract (Pierce & Stacey, 2010). Pierce and Stacey suggested that CAS could change student perspective of mathematical authority in the classroom. "MAS technology in the classroom introduces an 'authority' other than the teacher, and students may gain a new sense of personal authority" (Pierce & Stacey, 2010, p. 8).

Summary of the roles of technology. Many roles for CAS technology have been employed; some with successful results (Fonger, 2012; Heid & Blume, 2008: Kastberg & Leatham, 2005; Pierce & Stacey, 2004, 2008, 2010, 2013). The tool can introduce new concerns of learning how to use it prior to accepting it as a viable helpful technology (Drijvers, 2000). Artigue and Diderot (2002) theory of instrumental genesis may resolve concerns that evolve around the introduction of new technologies. Access to technologies can advantage learners through investigations, explorations, multiple representation and real-world contexts (Fonger, 2012; Kastberg & Leatham, 2005; Zbiek & Hollebrands, 2008). Finally, CAS technology can take on a new role of authority in the classroom (Langer-Osuna, 2017; Pierce & Stacey, 2010).

CAS Theoretical Perspectives

Mathematics education researchers have generated multiple frameworks regarding cognitive technologies that assist in understanding CAS' functional properties, its usefulness, and potential learning outcomes. Mewborn (2005) presented a general use for frameworks (e.g., model, construct, theory, paradigm, framework). First, a framework can guide a study by allowing a researcher to notice particular events or observations. Second, a framework can allow the researcher to perceive similarities and differences between observations. Third, a framework can provide orientation or perspective to the occurrences of a study within a context. Lastly, words of caution are commissioned to the researcher to avoid confinement to an ideology. Keeping Mewborn's perspective in mind, I selected three frameworks to review: Black Box and White Box (Buchberger, 1990), Heugl's Competence Model (2005), and the P-Map (Pierce & Stacey, 2010). In the following sections, each framework will be described. The purpose for selection of the P-Map framework for this study will be shared.

Black Box and White Box

Theoretical perspectives indicate CAS as a black box technology (Cedillo & Kieran, 2003; Drijvers, 2000; Heid & Edwards, 2001; Ozgun-Koca, 2009) upon which individuals operate with no comprehension or transparency of actions taken by the device. This challenged mathematicians' consideration of methods to reveal internal actions of CAS technology. The user necessitated viewing the actions or steps in a sequential order, conceiving the viewpoint of white box technology. Ozgun-Koca (2009) discussed a third prospect offering a resolution to the problem by accessing a symbolic mathematics guide (SMG). The device software (e.g., SMG) contained a menu of choices in which the user chooses a decision to act upon the mathematics, thereby establishing a chain of procedures.

Two potential utilizations for the purpose of learning and doing mathematics were: (a) generating results quickly for pattern seeking (black box), and (b) performing step-by-step procedures (white box or SMG) (Ozgun-Koca, 2009). The first purpose illuminates symbolic outputs for explorative purposes (Heid, 2003); the second affords the development of procedural fluency (Heid, 2003; Ozgun-Koca, 2009). The three perspectives (i.e., black box, white box, and SMG) facilitate an understanding of CAS as well as pedagogy that can utilize CAS.

Huegl's Competence Model

The competence model for standards in mathematics education was adopted to include an emphasis on technology as a means by which learners engage in mathematical learning (Heugl, 2005). The model included four performance classes and was presented to delineate different types of uses of CAS technology by students and teachers: (a) modeling and representing, (b) operating and calculating, (c) interpreting and documenting, and (d) arguing and reasoning. Huegl's classes aligned with mathematics teaching practices (NCTM, 2014). Furthermore, Heugl expected changes to standards and assessments to ensue concurrently with an influx of curricula that included technology use. "We are sure that the use of technology will increase the joy and interest of the students and they will experience the learning of mathematics in a more meaningful way because we can offer them a more meaningful mathematics" (Heugl, 2005, p. 11).

Pedagogical-Map (P-Map) Framework

CAS offers multiple opportunities for use, unmistakably computational purposes and functional use, but also pedagogical applications (Pierce & Stacey, 2010). The functional uses are the foundation for inventing MAS devices and have fulfilled many computational tasks. "Pedagogical opportunities and their actualization are less evident to teachers" (Pierce & Stacey, 2010, p. 2). The focus of the P-Map (Figure 1) developed by Pierce and Stacey (2010) is to identify, organize, and highlight those opportunities using this taxonomy as teachers integrate MAS or mathematical cognitive tools. The map is oriented from the base as the functional opportunities of CAS elicit opportunities for education implications in curriculum, assessment, and pedagogical change; yet only pedagogical affordances continue to be exploited with the map.

The three classifications of the type of affordances on the P-Map are tasks, classroom, and subject. Five different affordances make up category of tasks: scaffold by-hand skills; use of real data; exploration of regularity and variation; simulation of real situations; and the links to multiple representations. The classroom category refers to changes in both the social dynamics and didactic contracts that students and teachers experience. The subject or content matter has the potential to affect pedagogy in three ways: exploit the contrast between machine and ideal results; shift the balance of skills, concepts, and applications; and emphasize metacognition of the thinking processes in which learners are engaged during the lesson. More details and examples for each of the ten pedagogical opportunities are provided in Table 5, an adaptation from Pierce and Stacey (2010).

Table 5

Detailed Description of Pedagogical Opportunities

Type	Opportunity	Description	Example
Subject	Exploit Contrast of Ideal and Machine Mathematics	Teachers deliberately use ‘unexpected’ error messages, format of expressions, graphical displays as catalyst for rich mathematical discussion	Syntax in the device provides an unexpected output, different from pen-and-paper solutions.
	Re-balance Emphasis on Skills, Concepts, and Applications	Teacher adjusts goals: spend less time on routine skills; more time on concepts and applications. Increase on mathematical thinking.	Heid’s seminal research on re-sequencing of concepts and skills in a calculus course Dynamic geometry can shift from memorization of facts to conjecturing and proving through visual arguments
	Build Metacognition and Overview	Teachers give overview as introduction or summation: link concepts through manipulation of symbolic expressions and use of multiple representations	Promote curiosity or instill a question, questioning strategies for reflection on the mathematical concept(s)
Class-room	Change Classroom Social Dynamics	Teachers facilitate rather than dictate. Encourage group work. Encourage students to initiate discussion and share their learning with the class.	Linking action with mathematical reflection Constructivist approach to instruction
	Change Classroom Didactic Contract	Teachers allow technology to become a new authority. Change what is expected of students/teachers. Permit or constrain explosion of available methods.	Role changes for both teacher and student, possibly teacher as facilitator and student as consultant.

Type	Opportunity	Description	Example
	Learn Pen-and-paper Skills	Use instant ‘answers’ as feedback when learning processes.	Solve equations one step at a time. Use of a symbolic math guide (tutorial program within the device)
	Use Real Data	Work on real problems involving calculations that, done by hand, are error prone and time consuming.	Collect real data through the device, such as the height of a ball or the temperature of a cup of water.
Tasks	Explore Regularity and Variation	Strategically vary computations. Search for patterns. Observe effects of parameters. Use general forms.	Use of sliders to dynamically change the graph of a function. Alter a geometric shape with drag features. Expand or factor algebraic expressions and make observations.
	Simulate Real Situations	Use dynamic diagrams, drag, and collect data for analysis. Use technology generated statistical data sets.	Random function generator repeated times to create a histogram for 1000 tosses of two dice.
	Link Representations	Move fluidly between geometric, numeric, graphic, and symbolic representations.	Equation of a circle in symbolic sense, input numerical values, graphed, and drawn with geometry tools

Note. Adapted from "Mapping pedagogical opportunities provided by MAS," by R. Pierce, & K. Stacey, 2010, *International Journal of Computers for Mathematical Learning*, *15*, p. 6. Copyright 2010 by Springer International Publishing AG.

Summary of Theoretical Perspectives

Three theoretical frameworks provided perspectives to characterize CAS technologies. The Black Box and White Box (Buchberger, 1990) define particular usage of technologies with regard to symbolic computation. The user can either know the inner workings of the device by utilizing step-by-step procedures or alternatively, have no knowledge. The latter refers to a Black Box Technology. The competence model presented by Heugl (2005) outlined four classes in which teachers and students may utilize CAS. Finally, the P-Map framework represents 10 pedagogical affordances that teachers may use as they exploit MAS in the classroom. The P-Map was chosen as the theoretical framework for this study as the events were well defined for identification. Furthermore, the P-Map framework provided a context to describe the observed lessons and teacher pedagogies.

Chapter Summary

Teacher pedagogy is the logical first step to consider CAS as a useful cognitive tool to advance mathematical knowledge. Teachers are the agent for change (Chamblee et al., 2008; Ertmer & Ottenbreit-Leftwich, 2010; Wachira & Keengwe, 2011). Implementation begins through innovative teacher pedagogy impacting learners and continues to learners' receptibility for developing mathematical knowledge in a CAS-rich environment. General beliefs regarding teaching with technology effectuates teacher utilization of technological tools. In two studies, beliefs about teaching and learning mathematics impacted teacher's level of technology implementation (Ivy & Franz, 2006; Wachira and Keengwe, 2011). Teachers with low PCK implemented cognitive technologies with less advanced utilization.

Knowledge of potential barriers provides the opportunity to address concerns prior to technology implementation. Barriers or obstacles to teaching technology can be classified as extrinsic or intrinsic (Brickner, 1995; Ertmer, 1999). Extrinsic barriers are those outside the teacher's control, such as, equipment, professional development, and teacher release time for preparation. Intrinsic barriers include teacher beliefs about mathematical learning and technology interfacing with curriculum and assessment. Studies revealed that teachers needed additional training and support to develop mathematics instructional practice with technology (Chamblee et al., 2008; Ertmer & Ottenbreit-Leftwich, 2010; Kastberg & Leatham, 2005; Simonsen & Dick, 1997; Wachira & Keengwe, 2011).

Mathematical tools have generated roles in educational practice: investigations, access to real-world problems, multiple representation models, and the potential for a shift in mathematical authority. This study endeavored to reveal some roles through teacher observation in the classroom. The P-Map framework was used to illuminate opportunities.

CHAPTER III: RESEARCH METHODOLOGY

Introduction

The powerful functionality of CAS has the potential to affect mathematics instruction in secondary school mathematics (Heid & Blume, 2008; Pierce & Stacey, 2010). The multiple capabilities of symbolic algebra, the ease-of-use, and recent state-of-the-art technology give rise to opportunity for inventive pedagogy. Possibilities exist in which mathematics can be explored with new pedagogy and schemes (Fey et al., 2003; Guin et al., 2005). It begins with the teacher as the facilitator of questions but extends to learners as they consider new inquiries. Classroom activity on technological devices creates a story. The goal of this study was to uncover two teachers' decisions regarding their pedagogy, choices of adjustment to their instruction, and justifications to take opportunities for the design of CAS-oriented lessons. Knowledge of those findings will inform education leaders of potential teaching practices that promote students' development of mathematical knowledge.

This holistic (Creswell, 2007) qualitative study considered secondary teachers' pedagogy as they incorporated CAS technology in their classrooms. A multiple case study (Yin, 2009) was utilized to capture the essence of two teachers in their integration of CAS. The following elements of this chapter constitute the approach and rationale for the type of study. First, a research overview is provided, followed by a description of the research site, participants, instruments, data sources and the data analysis procedures. Finally, limitations, delimitations, ethical considerations, and trustworthiness will be discussed in this chapter.

Research Overview

The purpose of this study was to understand: (a) what pedagogical opportunities mathematics teachers exploited with the presence of CAS; (b) how teachers aligned lessons to develop mathematical understandings; and (c) why these teachers wanted to orient their focus to exploit CAS in the development of mathematical knowledge. The following research question guided the study: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems (CAS) in the development of mathematical knowledge?

The questions asked were by nature descriptive rather than experimental, characteristic of a qualitative study. "The case study is preferred in examining contemporary events" (Yin, 2009, p. 11). Additionally, this study did not manipulate any behavior of the teacher, students, or content. Rather, I was an outside observer, probing, inquiring, and inspecting the teachers' moves, seeking evidence of teachers' perceptions and actions of adaptation to teaching pedagogy in the context of relatively new technologies. A holistic analysis (Creswell, 2007; Yin, 2009) explicated the teacher pedagogy through multiple sources developing individual themes and interpretations for each case. The two cases were synthesized for the cross-case analysis providing more robust findings than for individual cases (Yin, 2009).

A multiple-case design (two cases) was selected deliberately to garner separate examples of CAS utilization. Yin (2009) claimed the potential for multiple cases outweighed the benefits of a single case for several reasons: (a) multiple cases always provide a more compelling study; (b) independent conclusions can corroborate one another; and (c) contrasting situations provide rich evidence. As well, this study employed a within-site scheme. Gay, Mills, and Airasian (2012) argued that multisite studies furnish stronger results, allowing for greater generalizability. However, this study was less interested in the reliability of replication. By keeping the study limited to one school, the cultural aspects remain fixed: multiple teachers with numerous lessons were varied.

The lead mathematics teacher at the school site identified three other teachers as potential participants; this kept static other factors and outside influences to the instruction at this location. However, after an on-site visit occurred, two participants were unable to provide additional data and, hence, were removed from the study. Data from this study provided more robust results due to intentional replication of conditions of the two participants (Yin, 2009). However, differences among teachers availed the opportunity for deeper analysis of the theoretical framework according to Yin (2009).

Research Context

I chose a high school that had teachers currently utilizing CAS technology in their classroom practice. The school was an independent, co-educational college preparatory day school with the reputation for excellence in teaching and learning. The setting location was on the outskirts of a large metropolitan area in the northern Great Lakes region of the United States. The school's two territories house separate units of early childhood, lower school, middle school, and high school, with an enrollment of approximately 1,100 students. The historic school recently underwent a renovation to incorporate a technology-rich environment and an open space concept. A proud heritage of the school was to create a culture where students can develop their character and intellect through attention to real-world activities. On several occasions the school hosted mathematics technology-focused conferences, demonstrating a dedication to the utilization of CAS.

Grandview, pseudonym for the research school site, assumed over a 100-year history for serving the surrounding population of youth. The day school educated approximately 1,082 students: 594 students in K-8 and 488 students in 9-12. The middle school, grades 6-8, employed three mathematics teachers, one for each grade level. Grandview's middle school location in the middle of a residential community with several businesses in close proximity provided access to many children. Endowment funds granted opportunities for students in need of financial assistance. The high school employed 12 math teachers; some teachers overlapped into STEM or science departments. The high school facility was five miles outside of the city with widespread space. Transportation was provided for students on an as-needed basis.

Students at Grandview were provided a laptop pre-loaded with TI-Nspire™ CAS software (along with many other general applications) for access at school and at home. Teachers used a binder application to organize work for the classes and as a management system for students to share documents with the teacher. Students took notes and completed assignments with a stylus pen through the touch-screen and keyboard that was organized in each individual's digital binder.

In pursuit of teachers to meet the original criteria, another study was initiated that required an Institutional Review Board application (protocol #16-2097, approved November 2015) for an information gathering survey (See Appendix A). This national search included sending and receiving anonymous surveys, analysis of open-ended responses, and follow-up phone interviews (see Appendix B). An online survey was created with questions regarding types of mathematics technology utilized and open-ended prompts about particular use of those technologies. Teachers were targeted who met one of the following criteria: recently attended or presented at a mathematics technology conference; or taught at a private secondary school.

The survey was sent electronically in November 2015 to approximately 305 teachers, with allowable acceptance of responses up through January 2016. Not all teachers proved to have authentic active email addresses. Exactly 56 teachers responded to the survey, which provided the option to contact the participant for additional information. Those that provided an email address became a sub-group of responders that were analyzed. I probed for three criteria: a current secondary school teaching assignment in the traditional mathematics course(s); a response that emulated a revelation of substantial usage of CAS technology; and a willingness to participate in educational research. Thirteen teachers were identified who indicated an adequate knowledge of integrating mathematics technology in their teaching practice. The teachers were ranked according to survey responses that revealed innovative lesson design using CAS. The top seven teachers were selected for initial contact and potential interview, leaving the option to consider others at a later time.

Four teachers were available to participate in follow-up phone interviews. Each was asked more specific questions regarding their online survey responses. By probing into detail of teaching assignments and willingness to serve as an in-depth research participant, potential candidates were culled. Very few of these four individuals were currently teaching the traditional mathematics classes, such as algebra 1, algebra 2, geometry, and precalculus. However, one teacher (identified below as Mr. Shasta) exhibited a high level of CAS use, taught an appropriate mathematics class, and conveyed a confidence that his school would accept such a research opportunity.

Therefore, out of the four phone surveys, Mr. Shasta was included as a potential research participant. In communicating with Mr. Shasta, he guided my decision to secure three additional candidates. The plan was to have all four participants employed at the same high school, exploiting the affordances of CAS, and teaching traditional college-preparatory mathematics classes. Data was collected from the four participants through the on-site phase of the study; however, limited participation by two teachers during the post-visit phase forced the issue to scale the four cases down to two cases.

Research Participants

Research participants were selected according to purposeful (Creswell, 2007) or purposive sampling (Gay et al., 2012). Teachers chosen for this study were depictive of skilled teachers who had utilized CAS technology in their classrooms. Purposive sampling best served the study because the type of individual selected was central to understanding the phenomenon (Creswell, 2007; Gay et al., 2012). In particular, it was the actions and perceptions of the participants that informed the study. Two secondary mathematics teachers who worked within the same work environment served as the units of analysis.

The two teachers, Ms. Springer and Mr. Shasta (pseudonyms), served as participants with unique dispositions in utilizing CAS technology. Their instructional practices were analyzed to reveal pedagogical affordances and justifications. Background information listed in Table 6 provides an introduction to the detailed descriptions of each participant.

Table 6

Overview of Research Participants

Teacher	Course Observed & Grade Level	Education	Teaching Experience	Years Experience	Years Using CAS
Springer	Calculus 12th grade	BS Mathematics Education (7-12) MEd Instructional Technology	Private School	9	5
Shasta	Algebra 1 8th grade	BS Mathematics & Philosophy MS Mathematics EdM Education EdD Education & Administration	Private Middle and High Schools	28	28

Ms. Springer

The Grandview mathematics department coordinator for grades 9-12 was Ms. Springer. She had been at the school for nine years; amidst the year of the study she taught algebra 1, algebra 2, and calculus. Her bachelor's degree was in mathematics education, and during her tenure at Grandview she earned a master's degree in educational technology. For the study, the calculus class was featured in Springer's lessons. Students selected enrollment in the calculus class for their senior year as preparation for college-level calculus. They chose the course over an advanced placement calculus class opting for extended time in the development of calculus concepts. Springer stated that she enjoyed the relaxed pace and reduced pressure of the non-AP classes.

Mr. Shasta

Mr. Shasta served as the mathematics department chair for Preschool-12 at Grandview. He had been at the school for five years, a portion of his 28-year teaching career. One of his first propositions at Grandview was the request to consider utilizing CAS technologies in all mathematics classes. Grandview's administration was supportive of this decision, working out resources for both faculty and students to receive full access to CAS. Shasta retained the role of onsite professional development expert, encouraging and sharing lesson ideas that were supported through CAS. Personally, he had used CAS from his first days of teaching and continued during his nearly 30-year teaching career. Generally, he had taught high school precalculus, statistics, and calculus courses, but during the year of this study he shifted to middle school filling an unexpected vacant mathematics teacher role. Shasta's eighth-grade algebra one classes contributed data to this research study. Although Shasta secured this new position as an eighth-grade instructor during the year of this study, he had previous experience teaching eighth-grade mathematics.

Instruments and Data Sources

The data collected for this study reflected instructional decisions and pedagogy that Springer and Shasta afforded when exploiting CAS in their teaching practices. Multiple sources of data aggregated during the pre-visit, on-site Visit, and post-visit provided the foundation for the research analysis. Three primary sources were collected to answer the research questions, and multiple secondary sources of data facilitated the study and added depth to the descriptions. Participant lesson observations, reflective writing prompts and semi-structured interviews were the primary data sources.

Secondary sources of data were: a demographic survey, researcher's field journal, classroom observation protocol, screencasts of lessons, and audio recordings. The survey provided demographic data. The researcher's field journal maintained an audit trail. Information from the screencasts and audio recordings generated inquiry for the primary data sources of writing artifacts and semi-structured interviews. Each of the instruments is described in this section.

Role of Researcher

As the researcher, I was a principal instrument in this study (Creswell, 2007). I was involved in multiple functions: creating instruments; observing behavior; forming and asking questions; and gathering and analyzing data. Additionally, as a researcher, it was my duty to reflect and interpret the data collected. Multiple years in advanced graduate coursework in mathematics education, participation as a graduate assistant in qualitative research through the university, and execution of five educational action research projects rendered experience in research. Furthermore, 27 years of teaching experience supported my awareness to details in a classroom.

"To further de-emphasize a power relationship, we may collaborate directly with participants by having them review our research questions, or by having them collaborate with us during the data analysis and interpretation phases of research" (Creswell, 2007, p. 40). I understood my role as an observer, disengaging in discussions that might influence or persuade. Also, my questioning during semi-structured interviews was a tool to draw out participants' perceptions of their decisions, approaches, goals, and assessment in their pedagogical practice. Therefore, careful consideration and diplomacy were necessities in choosing writing prompts and interview questions.

Field Notes Journal

A field notes journal was kept as both a data source (Gay et al., 2012) and a tool for maintaining reliability (Yin, 2009). Two basic types of data recorded in field notes were descriptive information and reflective observations (Gay et al., 2012). Yin (2009) recommended establishing a “chain of evidence” (p. 122) that provided ground for a solid argument, which increased the quality of the study. To maintain an audit trail, I recorded every data collection piece from the study. Records included dates, times, utterances, communications, decisions, and reflections.

Survey

Demographic information was the primary goal for the initial data collection of an electronic survey, sent to the four participants (Appendix B). The survey served the purpose to introduce the research project, to collect demographic data, and set up communication conventions. The survey collected descriptive background of the participants and facilitated capturing contextual data to plan for the classroom observations.

Semi-Structured Interviews

Two phases of interviews were conducted: pre-interviews and post-interviews (see Appendix D). The pre-interview consisted of open-ended questions about teaching assignment, perceived uses of CAS in lessons, decisions in lesson planning, and curricular alternatives. The post-interview featured semi-structured questions and served the research in making clear particular attributes of each observed lesson. The questions probed into the decisions the participants made either before the lesson or upon a breakthrough. As well, the interview clarified any components of the lesson that lacked transparency from the artifacts collected. Finally, participants had the opportunity to reflect on the outcome of the lesson allowing them to share their perceptions of student reactions to the CAS utilization, that is, from the time of the lesson to the day of the interview. A listing of the dates of post-interviews can be found in Table 7.

Table 7

Pre and Post Interview Dates

Participant	Pre-Interview	Post-Interview 1	Post-Interview 2	Post-Interview 3
Springer	10-02-17	10-15-17	11-08-17	12-06-17
Shasta	10-02-17	10-04-17	11-06-17	12-22-17

Reflective Writing Artifacts

Individualized writing prompts (see Appendix E) were prepared after lesson observations. The purpose was to allow participants to reflect on CAS utilization regarding perceptions and attitudes, classroom dynamics, curriculum, and evaluation issues. Questions were adapted from Simonsen and Dick's (1997) interview protocol in which teachers' perceptions of calculators in mathematics classrooms were analyzed. The questions were modified, deleted, or added onto, after viewing the screencast for each lesson. Some questions were geared towards specific aspects of the lesson with the intent to draw out decisions and perceptions regarding teacher pedagogy. Table 8 shows a record of dates from the data collection and the number of prompts used in the analysis.

Table 8

Record of Written Reflections

Participant	Date Collected	Number of Prompts
Springer	10-13-17	5
Springer	11-04-17	6
Springer	11-30-17	9
Shasta	*10-04-17	6
Shasta	10-13-17	5
Shasta	12-20-17	6

Note. *This reflection was collected via a post-interview and is cited as an interview.

Lesson Artifacts

The primary lesson artifacts were screencasts, which captured a video of the computer screen and the participant's voice during the lesson. A backup audio recording was collected in some instances in the event that the computer microphone was of insufficient quality. During the On-site Visit, I gathered any papers, plans, or work products from the participants. Some of these artifacts were photographs of white board work, lesson plans, handouts, assessments, and screenshots of student computers that demonstrated innovative use of CAS tools.

Classroom Observation Protocol

The classroom observation protocol (see Appendix F) developed from the P-Map framework by Pierce and Stacy (2010) supported the collection of data related to the infrastructure of the lessons. Pierce and Stacy (2010) claimed that 10 pedagogical opportunities exist in the classroom, as documented in research and literature. The P-Map figure was part of the protocol, an expanded version was created from the literature providing descriptions and examples of pedagogical opportunities for quick reference. The P-Map served as a catalyst for examining the features of each lesson.

Procedures for Data Collection

The procedures of this study are described in the narrative that follows. The details are organized in the phases of Pre-visit, On-site Visit, and Post-visit. A research timeline is provided in Table 9. Permissions were granted from the school site, prior to the Institutional Review Board (IRB) submission.

Table 9

Research Timeline

Sent to IRB	IRB approval	Begin Pre-visit data collection	On site visit	Post-visit	Begin data analysis
Aug. 21	Sept. 28	Sept. 29	Oct. 2-6	Oct. 7 – Dec. 22	Nov. 1

Pre-visit: September 29 – October 1, 2017

This period extended three days from the time IRB approval protocol #18-2020 (Appendix G) was granted to the day the on-site visit commenced. Shasta, the lead teacher at the school site was contacted via email. Contact information for potential participants was requested. A video screencast was created for the purpose to introduce myself as a researcher, state the intent of the study, convey expectations for participants, and communicate the basic plan for follow-up. The video screencast was sent via email and included the IRB consent document and a link to the survey instrument. The first question on the survey required the respondent to authorize permission to participate in the study. As an auxiliary, I obtained physical signed consent forms during the on-site visit. The survey was completed during this phase or in the first day of the on-site visit. All four participants agreed to participate and provided consent. The survey served the purpose to introduce the research project, to collect demographic data, and set up communication conventions.

Shasta apprised me of a shift in his teaching assignment after the initial contact. An unexpected vacancy at the 6-8 school required him to modify the 9-12 teacher assignments and to absolve his responsibilities to fill that vacancy. Upon receiving this information, an addendum to the IRB was submitted, October 1, to the compliance office requesting approval to add the middle school as a second location. The compliance office granted the request on October 3. Shasta was interviewed at the high school prior to that date; however, no contact was made at the middle school until after permissions were granted.

On-site Visit: October 2 – October 6, 2017

The on-site visit allowed the researcher to conduct face-to-face classroom observations, collect lesson artifacts, administer lesson follow-up writing prompts, and conduct pre- and post-interviews. The classroom observation included several means of capturing the essence of teacher moves: video screencasts, an audio recording of the teacher, and a classroom observation protocol. Lesson plan worksheets, handouts, and blackboard work were collected with photographs to secure every aspect of the actual classroom experience. Follow up for each lesson came by means of a written reflection and a post-interview. One lesson cycle each for Springer and Shasta was conducted. In addition, a second lesson conducted by Shasta was observed.

The description that follows constituted one lesson cycle in terms of items and sequence of data collection: lesson artifacts, audio recording, screencast, observation protocol, reflective writing prompt, and post-interview. I collected three cycles for each of the participants for a total of six lesson cycles. The decision of which lesson to observe was given to the participant but with reminders of the goal of this project. Screencasts and audio recordings captured the lesson, in addition to a face-to-face observation. Notes were scribed on the classroom observation protocol and pertinent lesson artifacts collected. Questions selected for the writing prompt were decided post-observation. After reviewing the reflection, follow-up questions for a post-interview were organized.

Post-visit: October 7 – December 22, 2017

In the final phase, participants orchestrated additional lessons and transferred the data electronically to the researcher. In all, each teacher provided three lessons that utilized CAS technology. Digital transfer of video screencasts and any other lesson artifacts were accessed via cloud technology. Following each data transmission, participants were given a reflective writing prompt to both clarify and expand on thoughts from each lesson. Upon receipt of the written reflection the post-interview was conducted. The last phase of the study spanned 11 weeks. The lesson cycles and data collected related to each cycle are summarized in Table 10.

Table 10

Data Collected Related to Each Lesson Observation

Participant	Lesson Date	Written Reflection Date	Post-Interview Date	Vignettes Generated
Springer	10-06-17	10-13-17	10-15-17	1
Springer	10-20-17	11-04-17	11-08-17	2, 3
Springer	11-09-17	11-30-17	12-06-17	4, 5
Shasta	10-04-17	10-04-17	10-04-17	1, 2
Shasta	10-05-17	10-13-17	11-06-17	3
Shasta	12-04-17	12-20-17	12-22-17	4

Note. This data represents only data that was used in this study.

A list of all the lessons and the follow-up written reflections and interviews are provided in Table 10. Vignettes were generated as subsets from the lessons. The observed lesson descriptions aggregate the three data instruments for each lesson to tell the story.

Procedures for Data Analysis

The data analysis stage began when data were received in October and continued for eight months. Preparing and organizing the data for input to Atlas.ti (i.e., data analysis software) was ongoing as data were received. Written reflective artifacts were named with a pseudonym as described earlier and added to a database. Interviews were named, transcribed, and stored in the database. All other documents were date stamped, stored, and recorded in the field notes journal. Coding began in December and continued during the descriptive writing process. The next sections will provide more detail about the holistic analysis of the individual cases, the coding scheme, and the cross-case synthesis.

Holistic Analysis

Each case was analyzed separately accessing a holistic approach (Creswell, 2007). The lessons, written reflections, and interviews provided data for rich, thick descriptions of the participant's lessons, classified as vignettes. Each vignette was pattern matched (Yin, 2009) to the P-Map framework (Pierce & Stacey, 2010). Codes listed in Table 12 were used to identify instances of pedagogical opportunities. In the initial coding, I used a deductive process to assign hypothesis codes (Saldana, 2016) to selective pieces in the interviews and written artifacts. An evidence table was generated for each vignette to summarize pedagogy that matched the framework.

Next, the case for each participant was written, first Springer then Shasta. The five individual tables of evidence from Springer's vignettes were combined into a single table to create a perspective of the type of pedagogical opportunities that were in common and also those never identified. Consideration of each lesson was given once again to insure that no opportunities were omitted. The vignettes were imported into Atlas.ti for a second round of coding. Primarily the same P-Map codes for pattern coding were used; however, additional concept coding occurred simultaneously. Springer's case was completed first with the P-map themes aggregated. That phase involved looking for "converging lines of inquiry" (Yin, 2009, p. 115) that would reveal interesting ideals and form triangulation from the data sources. A nearly full analysis of Springer was completed prior to considering Shasta's data. The same procedures were followed for Shasta, except that only four lesson vignettes were produced.

Coding Scheme

The two primary coding schemes were hypothesis and concept coding. Initial coding accessed the P-Map codes from the theoretical framework, described by Saldana (2016) as hypothesis coding. P-map codes from Pierce & Stacey's (2010) framework were pre-determined prior to collecting data. The Classroom Observation Protocol (see Appendix F) guided the identification of P-Map codes prior to analysis. Thus, the framework informed decisions to select questions in the follow-up interviews and written reflections. Saldana names that coding method *hypothesis coding* from the basis of: "Application of a researcher-generated, predetermined list of codes . . . about what will be found in the data before they have been collected or analyzed" (p. 294). Saldana asserts that the hypothesis-coding scheme can explain the data. Simultaneously, while both holistic cases were analyzed by the P-Map codes, concept coding transpired. Saldana (2016) described concept coding as extracting *big picture* ideals. The codes will be explained in the sections that follow.

P-Map codes. Pierce and Stacey (2010) developed the taxonomy of various pedagogical opportunities that teachers take when utilizing MAS. The P-Map considered functional opportunities as the primary purpose of MAS, supporting users in the computation and manipulation of mathematical expressions and equations (Pierce & Stacey, 2010). The three levels of subject, classroom, and tasks categorize the opportunities. The codes given in A deductive analysis of the data occurred in the initial coding with the use of Table 11 codes.

Table 11 identify the P-Map level, the focused hypothesis codes abbreviation, and the description for each code that came directly from the P-Map (Pierce & Stacey, 2010).

The abbreviated codes represent the pedagogical opportunities from the P-Map and are identified in each lesson vignette summary. A deductive analysis of the data occurred in the initial coding with the use of Table 11 codes.

Table 11

Codes for Pedagogical Opportunities

Level	Code	Opportunity
Subject	S1	Exploit Contrast of Ideal and Machine Mathematics
	S2	Re-balance Emphasis on Skills, Concepts, and Applications
	S3	Build Metacognition and Overview
Class-room	C1	Change Classroom Social Dynamics
	C2	Change Classroom Didactic Contract
Tasks	T1	Learn Pen-and-paper Skills
	T2	Use Real Data
	T3	Explore Regularity and Variation
	T4	Simulate Real Situations
	T5	Link Representations

Note. Adapted from "Mapping pedagogical opportunities provided by MAS," by R. Pierce, & K. Stacey, 2010, *International Journal of Computers for Mathematical Learning, 15*, p. 6. Copyright 2010 by Springer International Publishing AG.

Codes for emergent themes. I documented my reflective thoughts in the field notes journal throughout the events of observing lessons, interviewing participants, transcribing lessons, writing the descriptions of lessons and coding all the data. Recurring thoughts became concept codes (Saldana, 2016) used in the analysis. The codes were phrases, gerunds, and concepts that either came directly from the data or via the literature review. Once produced, codes were migrated to Atlas.ti to become part of the coding scheme. Table 12 represents the concept codes showing the number of occurrences. Included in the table due to overlapping ideas from the P-Map framework, is the code *T5 Link representations*.

Table 12

Codes Grouped by Emergent Themes

Emerging Themes	Codes	Number of Events
Mathematical Consultant	Efficiency	28
	Connections	22
	Accuracy	19
	Student CAS use outside of classroom	12
	Empower students	6
	Ease of use	5
Verify Answers	Procedural skills	23
	Student trial and error	17
	Verify or check	17
Multiple Representations	*T5 Link representations	41
	Teacher valued multiple approaches	6
Regulate Access	Questions are different	35
	Creative (lesson design, questions, topics)	12
	Delay commands	6
Provide Guidance	CAS commands	49
	Syntax issues	41
	Student struggles	31
	Teach the tool	27
	Flexible with approach during lesson	18
	Work-around within CAS	9
Outsource Procedures	Outsource complex procedure	28
	Multiple examples quickly	23
	Reduce tedious calculations	16
Not Connected	Motivation for using CAS	61
	Enjoy	30
	Teacher new approach	24
	CAS platform choice	21

Note. *T5 was used in the P-Map code list also.

Cross-case Synthesis

Two developments materialized in the cross-case synthesis: aggregation of the P-Map findings and aggregation of the emergent themes. Summary tables from the holistic case analysis were merged into one with just the total of occurrences as extracted from the evidence tables for each lesson vignette. This new table was generated as a reflective tool to understand the relationship of the P-Map to both pedagogical affordances in a general sense and comparison of the two cases. A third round of coding the individual cases assisted in understanding the opportunities that participants were using to benefit their instruction. It was during this phase that a realization of very different classroom stories converged on pedagogical practices. The P-map similarities and differences were synthesized. Emergent themes arose out of that process. The individual case themes were explicated and the cross-case synthesis revealed six themes.

Ethical Considerations and Trustworthiness

There were concerns of an ethical nature that involved the school site, participants, and the body of evidence. Creswell (2007) noted that participants have little to gain from the research but still dedicate their time, energy, and emotions to the process. Furthermore, "unanticipated and unreviewed ethical issues can arise and need to be resolved on the spot" (Gay et al., 2012, p. 22). Alignment with an understanding of human subjects' treatment was imperative for the researcher. Gay et al. (2012) professed that participants may feel distressed about their activity and may desire knowledge of the researchers' written product. Also, the relationships that exist between researcher and participant can "create unintended influences on objectivity and data interpretation" (Gay et al., 2012, p. 22). Outlined in the following sections are the steps taken to ensure transparency, credibility, dependability, and transferability.

Transparency

It was imperative to take measures to maintain ethical standards in this research project. First, I conducted a proper review at the university through which this study commenced, submitting a proposal to the IRB. Second, I made every effort to minimize disruption and interference of the normal routines established by the teachers at the research school site. Third, no videos or photographs of individuals were secured: The video screencasts captured only the computer screen and audio in the room. Audio recordings filtered just the participants' voice and those participants signed proper consent forms before any data collection. Photographs that were taken only contained textual data from written work on the board or paper and screen shots of calculator devices or computer screens. Fourth, a dedicated journal of events documented the contact with participants, my perceptions of conversations, and any new questions necessary to explore. The journal was a trail of evidence, but also a reflective tool for me to listen to my voice as a way to maintain objectivity throughout the study.

Credibility

As a researcher, I had to consider only the elements of CAS utilization and not other classroom concerns such as classroom management, administrative disruptions or lack thereof, the pace of instruction, nor any other non-pedagogical issue. As a veteran teacher, I realized my own potential biases about what I observe in classroom culture and also in lesson design. However, the data collected were verifiable to ensure credible results. Multiple data sources were secured to corroborate the perspective. Triangulation from multiple data sources confirmed the process.

Dependability

Detailed descriptions were provided for each data source and through the methods of data collection. The researchers' field notes journal documented every spontaneous thought and action, providing a chain of evidence. Any quoted words from a participant were cataloged, dated, and certified. Concerning data analysis, qualitative data were analyzed using software tools that assisted in coding and quantifying information.

Transferability

This study was unique in its selection of site and participants that have access, resources, and minimal barriers to instruct with CAS technology. Since the study was context-bound, it was unlikely that a replication of this study at a different research site would match the findings. The study alluded that methodologies do exist that utilize CAS and advance the development of mathematical knowledge. Rich descriptions of lessons provided educational practitioners with lesson ideologies that have the potential for replication in other classes but not necessarily with similar outcomes. Teacher perceptions regarding decisions may be insightful for mathematics educators and may convey meaningful discussion.

Limitations

The limitations of this study foremost are not a result of the researcher's design, rather a result of participant selection, data collection procedures, member checks but, also, researcher epistemological bias. As one who has investigated this topic thoroughly, the perspective of viewing and identifying affordances may be gratuitous. However, ethical considerations always remained at the forefront of my mind in the data collection and analysis phases. The limitations are described in the following sections.

Participants

The original criterion for consideration in this study was teachers who utilized CAS as part of mathematics high school coursework. The researcher selected one participant, from the *Information Gathering Survey* results. This participant recruited three colleagues to participate in the study, with the knowledge of my criteria. Additional screening was not done; these four were the original participants. However, two of the participants were removed from the study during the post-visit phase. No additional data were received from those two participants and the on-site visit data did not appear to be CAS-rich. The two remaining participants provided ample evidence that was acceptable.

Data Collection

Limitations exist in the data collection phase that were beyond the researcher's control. However, these characteristics may have impacted the data either adversely or positively. First, participants selected lessons that represented a CAS-infused lesson. Second, one set of post-lesson data was collected the same day as the lesson. The conveyance of data may be limited due to lack of time to reflect. Alternatively, a different set of post-lesson data was collected one month after the lesson. Limitations may be due to ample time, thus, making recollection of pedagogy a challenge (see Table 10).

Screencasts may have limited the study in terms of the participant-selected lessons chosen for observation; three of the six lessons were observed via screencasts. The manner in which data were collected (i.e., screencasts) may have decreased the likelihood to observe particular kinds of classroom tasks, a limitation of the study. After analyzing the data through the lens of the P-Map, it was noticed that two types of tasks were not observed during this study: use of real data and simulation of real situations. As well, classroom level pedagogical opportunities, those that involve teacher-student and student-student interactions, could not be observed and, hence, were extrapolated from the screencasts. Teacher writing artifacts and interviews verified those researcher deductions.

Time Placement in the Course

The participants talked about their limitations of their CAS-oriented pedagogy during the first semester of their courses. Typically, students had little or no prior experience with CAS. As such, the participant indicated that he or she had to instruct how to use the CAS tool and also, ensure students had developed procedural fluency in the mathematics content first. The participants expressed that during the second semester students had greater knowledge both in the tool and in algebra skills, and, hence, classroom activities reflected more involved utilization of CAS. As a limitation, it is less about when during the course, rather the longevity of data collection. A broader length study may have disclosed additional findings.

Member Check

The researcher solely conducted transcriptions of the classroom lessons and interviews. Participants were asked to review their transcriptions for accuracy, verification, and clarification. However, both participants declined.

Delimitations

In general, utilization of CAS by secondary school teachers was rare; observing the avant-garde pedagogies was a unique opportunity. This study considered just some of the pedagogical affordances that high school teachers made; opportunities acknowledged were from an eighth grade algebra 1 and a twelfth grade calculus class. The prospect existed, in part, because access to CAS was not restricted at the school site. Furthermore, the scope only considered lessons that were offered to students one-fourth to one-half the way through the course.

Chapter Summary

This study focused on the real life phenomenon of innovative teacher practices as contemporary technologies surfaced in education culture. Therefore, a holistic qualitative multiple within-site case study was the chosen research method. Gay et al. (2012) classified particularistic studies, those that focus on one phenomenon, as a case study. Inspecting the two cases of teachers' administration of CAS in their classrooms enlightened aspects of how teachers integrate CAS-oriented instruction. Follow-up questions helped to understand the decisions teachers made regarding pedagogy.

Data collected were observations of lessons, interviews, and writing artifacts. Three lesson observations occurred on-site and three lessons were conducted via screen capture in the Post-visit. The data analysis phase included thick rich descriptions of nine lesson vignettes. Interview and writing artifact data were integrated into the stories of the lessons. Each lesson vignette was pattern matched to the P-Map framework using a deductive analysis. Following the two individual cases was a cross-case synthesis, which applied hypothesis coding to develop common themes. An emergent theme arrived from the analysis of both participants to answer the research question: How do secondary mathematics teachers orient their instructional practice to exploit CAS in the development of mathematical knowledge.

CHAPTER IV: FINDINGS

Introduction

Secondary teachers have been slow to act on the utilization of CAS as a tool to develop mathematical thinking (Heid et al., 2013; Zbiek & Hollebrands, 2008) due, in part, to multiple first-order and second-order barriers (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010). This study considered a school that had minimized those obstacles and habituated instruction with CAS as a cognitive tool. The integration of CAS into mathematics coursework begins with the teacher and his inventive pedagogy. This tool allows learners to participate in new ways of developing understandings of mathematics (Heid & Blume, 2008; Heid et al., 2013; Kutzler, 2003; Pierce & Stacey, 2010; Zbiek & Hollebrands, 2008).

The language in *Principles to Action* (NCTM, 2014) strongly encourages teachers to consider putting research into practice, with one of the eight principles highlighting tools and technology. This technology standard states, "An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (NCTM, 2014, p. 5). The data collected in this study potentially demonstrate that CAS technology is a tool that can be used to advance student reasoning and sense making through thoughtful presentation of mathematical content.

The purpose of this study was to understand (a) what pedagogical opportunities mathematics teachers exploit with the utilization of CAS, (b) how teachers align lessons to develop mathematical understandings, and (c) why these teachers wanted to orient their focus to exploit CAS in the development of mathematical knowledge. The following research question guided the study: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems in the development of mathematical knowledge?

The P-Map taxonomy of pedagogical opportunities described what secondary school teachers enacted with CAS technology. A multiple-case study of two participants was utilized. The data analysis involved pattern-matching logic (Yin, 2009) to ascertain features of lessons that bring about conceptual understanding in mathematics. This explanatory method captured a perspective from two teachers' points-of-view of the goals and intent of effectuating learning mathematics. This section begins with a brief description of the framework. Lesson vignettes from each participant will depict what occurred during instruction. After each vignette, I will clarify the identification of pedagogical affordances. Each participant's case reveals how the teacher oriented his or her instruction. Finally, a cross-case analysis will reveal similarities and differences between the two participants. Through the synthesis of the two participants the emergent themes were developed to provide more clarity of the findings.

Observed and Described Lessons

The lesson vignettes provided in the cases depict classroom practices that utilized CAS that I observed via face-to-face or a video screen capture (i.e., screencast). Participants completed interviews and responses to writing prompts immediately following each lesson to obtain clarity. In the vignettes, participants' spoken words from critical moments in the lesson are provided when available. Unfortunately, poor quality of audio recordings and conversations away from the microphone limited the possibility to detect precise words on occasion. However, follow-up interviews and written reflections captured the intent of the teacher in retrospection. Furthermore, questions asked post-lesson addressed pedagogical decisions. Throughout the vignettes, a time stamp is recorded to the left of phrases. Those times are formatted as minutes and seconds representing the amount of time into the recording of each lesson. In addition, I have described classroom activity during any breaks in time as parenthetical commentary. Generally speaking, the teacher and students had interactions that did not affect the instruction of the lesson. Rather those interactions may have been the teacher repeating instructions, clarifying points, waiting for students to engage, or answering off-topic questions that occurred as part of the culture of high school classroom activity. Transcriptions inserted represent the participants' spoken words addressing the class. The figures provided were teacher-generated or researcher re-created to display the technical aspects of CAS as a screenshot that the participants and their students generated.

The set of data is organized first by the participant, then by chronological lessons. Each lesson vignette is pattern matched to the P-Map framework (Pierce & Stacey, 2010) to identify parts of the lesson that demonstrate the pedagogical opportunities that the participant took in the presence of CAS technologies. "Such a logic compares an empirically based pattern with a predicted one. If the patterns coincide, the results can help a case study strengthen its internal validity" (Yin, 2009, p. 136). The discussion following the lesson vignette helps to describe what occurred to develop mathematical understanding.

The Case of Springer

The Grandview mathematics department coordinator for grades 9-12 was Ms. Springer. She had been at the school for nine years, the entirety of her teaching career. Her undergraduate degree was in mathematics education, and during her tenure at Grandview she earned a master's degree in educational technology. She had been utilizing CAS in her teaching practice for five years. During this study she taught high school courses: algebra 1, algebra 2, and calculus. It was the calculus classes that were featured in the lessons and conversations.

The sections below outline five distinct lessons that utilize CAS from the perspective of Springer. The narrative was created from lesson observations and was supported with the participant's reflection post-observation. After each vignette, components from her story that paired with pedagogical opportunities are identified, evidenced, and organized using the P-Map framework (Pierce & Stacey, 2010). The case analysis for Springer follows the lesson vignettes. Springer's case is first summarized as to the affordances from all five lessons using the P-Map and then emergent themes are tied to the pedagogical opportunities.

Springer Vignette 1: Finding Equations of Tangent Lines

A lesson using the *define* feature of the CAS involved students exploring the slope of the line tangent to the curve at a particular point defined on a function. Springer started the lesson by asking students to use the application, Desmos, on their personal computers and type the commands that she demonstrated via her computer-projected screen. The instruction was primarily teacher-centered, and Springer had students working through the technical procedures simultaneously as she modeled the mathematical syntax on the TI-Nspire™. At times when she asked questions, she paused to let students reflect and answer, but she did not call on individuals to respond. The lesson was presented in a straightforward manner, in that the teacher explained the content and asked questions of the students. Additionally, the mathematical content was typical for a calculus class—equations of tangent lines to the curve were written and checked.

Springer developed the idea of a limit to the slope of a line at the point of contact, in this case, $x = 1$, by calculating the slope of a secant line from the point of contact and another very close point, using numeric values. She then transitioned to finding the slope by using a difference quotient and taking the limit. This highlighted the definition of derivative by considering the limiting values of slope between two points with a horizontal distance of approximately zero. Instructions from Springer's lesson follow and are directly quoted. Student activity is noted in parentheses when significant but never quoted in the text.

1:17 Open up Desmos and go ahead and put in $y = x^2$ and put in the point (1, 1). We've done this problem before where we want to find the tangent line to $y = x^2$ at the point (1, 1). (Students prepared devices.)

1:47 We were making a table in Desmos (pause) and then we were finding values really close to one. And we were coming up with what we thought was the slope. Let's also review how we did that on the Nspire.

2:10 Let's Define $f(x) = x^2$. This is my favorite command.
(Students questioned the teacher what and how to input.)

2:38 If we wanted to find slopes around one, we could do $\left(f(0.99) - f(1)\right)/(0.99 - 1)$ and we get 1.99. Does everyone remember that this is what we have been doing? (Teacher talked about other course details unrelated to the lesson.)

3:52 The table method utilizes this idea of limits; that we're getting values really, really, really close to one. But never does the calculator ever actually give us two, or if it does give us two, it's a rounded two. It's not a definitive two. We were coming up with this estimating, guessing type of situation. What we are now going to get into today is actually, we can definitively come up with two. We practiced this a little bit last week. We are going to use the difference quotient. We are going to take the limit of the difference quotient, and it will definitively give us two. (Springer, Lesson, October 6, 2017)

Springer started with Desmos tools for graphical elements, but then shifted to the TI-Nspire™ for the CAS capabilities. Using CAS' *define* feature, she typed in the command "Define $f(x) = x^2$," projecting her CAS display on the wall while students keyed the same command into their personal devices. They all found the slope by setting up the following computation $(f(0.99) - f(1))/(0.99 - 1)$ and received an output of 1.99 shown in Figure 2. Springer connected this idea to a previous similar lesson, expounding that she and the students could choose other points really close to (1, 1) but never arrive at a definitive slope of two using this method. She described this method of finding the slope, a guessing or estimating type of situation.

Figure 2. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

Students were naturally pattern-seeking as the slope values got close to the number two. Springer reflected upon this lesson in a post-interview, stating the value of repetitive calculations.

> So we were evaluating *f*(0.9), *f*(0.99), *f*(0.999) real quickly. Generating those values and then we were calculating the slope, going back, and then grabbing those values and doing change in *y* over change in *x*. . . . We were able to come up with the values. And eventually the kids said, "You know, why do we have to do five of these? Why can't we just do *f*(0.999)?" I said, "Well, that's the whole idea of a limit. We are getting closer and closer and closer." (Springer, Interview, October 15, 2017)

Springer asked students to do these repetitive calculations with the hope that a meaningful shortcut would seem apparent to the students. In the interview, she shared how monotonous the calculations were for students, even when completed on a CAS. The recurrent task prepared the students for the definition of derivative.

The lesson then shifted to calculating the slope definitively using the limit of the difference quotient or the slope formula. Springer provided the specific difference quotient by using the function $f(x) = x^2$ and performed by-hand symbolic substitution with the general form for slope $(f(x+h) - f(x))/h$, having written $((x+h)^2 - x^2)/h$ with a marker on the whiteboard. She typed that expression into the CAS and got an output of $2x + h$. Next, Springer did two things on the CAS. First, she took the limit as h goes to zero. Second, she evaluated the value of x at the x-value for this instance, one, that matched her previous example. The computation involved using the CAS *such that* command and resulted in exactly the value of two as shown in Figure 3.

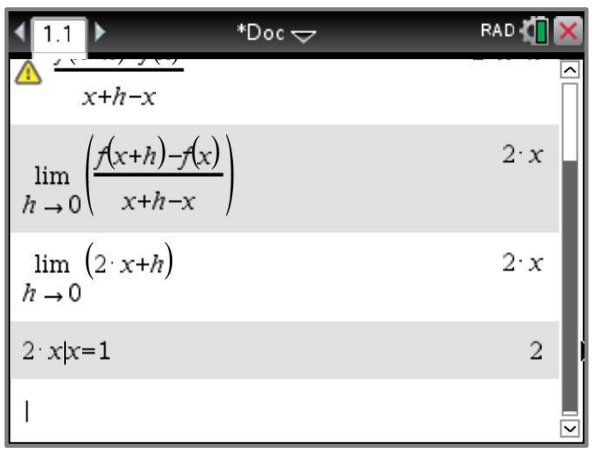

Figure 3. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

4:25 Let's go ahead and do our difference quotient of x^2. (Springer wrote this on the whiteboard simultaneously speaking.) So, for $f(x) = x^2$ we would have

$\frac{(x+h)^2 - x^2}{h}$.

So if your function is already defined to be x^2 . . . (Teacher explained and helped students put it into the calculator through circulating the classroom.)

5:00 So, now do $(f(x+h) - f(x))/h$ [on CAS] and you should get this, $2x + h$.

5:16 We're going to take our limit as h goes to zero and we're going to evaluate at the x value of whatever we want. If we take our limit as h goes to zero, what are we left with? (Students responded with $2x$. Other background discussion occurred.)

5:38 If I am trying to find my tangent line when x is one . . . We replace x with one. You get definitively two. That is the slope of the tangent line. (Teacher reiterated what happened on the CAS with the difference quotient and also how it was arrived with symbolic calculations by directing attention to the whiteboard.)

7:12 We are wanting the tangent line at this point (1, 1). So by replacing x as one, we are actually getting the slope definitively two. (Teacher directed students to put these ideas into their notes.)

7:51 The slope of any tangent line is just the difference quotient where we take our limit as h goes to zero. And then we plug in whatever x value it may be. If you want the tangent line at the point (3, 1), we are going to plug in x to be three. If we want it at the point (1, 0), x is going to be one. (Teacher reiterated the same ideas.)

8:53 Let's fill in what we have here. If we use point-slope for . . . I have this point (1, 1) and we figured out that two was our slope. I'm just going to put that into Desmos, $y - 1 = 2(x - 1)$. So there's our tangent line.

9:24 It's not that the table method doesn't work anymore. It's technically just a little more efficient because you can just directly do it. You are taking away this estimating, guessing situation. Which got a couple of us lost.

9:50 There is this piece of (pause) you are trying to approach something.

10:05 This is how we get the exact answer. (Springer directed students to repeat this exercise with other functions. Considerable time was provided for students to try a variety of examples. Springer walked around and assisted students.) (Springer, Lesson, October 6, 2017)

Finally, the goal was to write the equation of the tangent line, which was fulfilled. Springer used the definitive slope of the tangent line and the point of contact in the point-slope form to write the equation. Springer returned to the Desmos graph from the start of the lesson. She added in the equation of the tangent line, $y - 1 = 2(x - 1)$. This tactic provided a graphical representation that connected the given information of function and point with the computed equation of the tangent line. It confirmed the accuracy of the answer as shown in Figure 4.

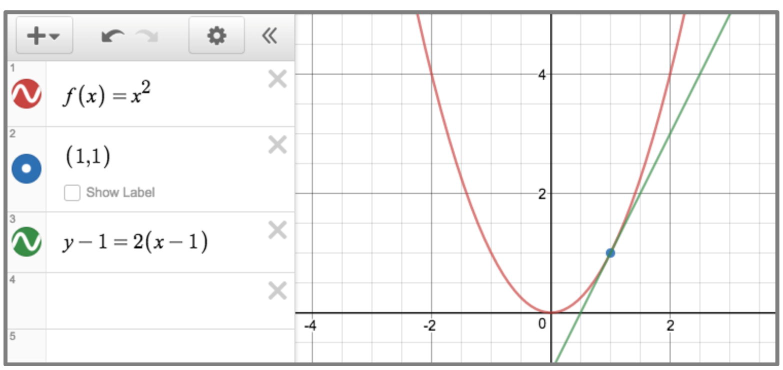

Figure 4. Springer's Desmos textual commands projected from computer to the classroom wall.

Springer Vignette 1: Pedagogical Opportunities

Springer's lesson on equations of tangent lines demonstrated utilization of CAS in the areas of building metacognition, exploring regularity and variation, learning pen-and-paper skills, and linking representations. The evidence summarized in

Table 13 uses pattern matching logic (Yin, 2009) with the data and the P-Map Framework (Pierce & Stacey, 2010). Descriptions from the vignettes facilitate how these characteristics were demonstrated. Evidence of connections to P-Map will be cited with time stamps from the lesson vignette as appropriate.

Table 13

Springer Lesson Vignette 1

P-Map	Evidence
S3	Use of the *define* tool to systematize slope calculations. Extended to abstraction of h as a very small value in a difference quotient and taking the limit as h approaches zero with symbolic representations.
T1	Springer uses the white board to perform by-hand algebraic simplification of the difference quotient.
T3	Regularity in the slope calculations to promote pattern recognition.
T5	Multiple representations of graphic, symbolic, and tabular forms.

Build metacognition and overview (S3). The development of a tangent line to the curve by finding the slope of a secant line was prevalent in the lesson with the goal of understanding derivatives by outsourcing complicated procedures to the CAS. Springer was able to systematize those calculations by using the *define* tool in CAS to calculate the slope of two very close values, those at $x = 0.99$ and $x = 1$ (Time stamp, 2:38). She connected this computation to the theoretical value of a very small difference between the selected points, namely the difference quotient $(f(x+h) - f(x))/h$. She used the CAS tools to demonstrate that the numerical computation is the same as for the algebraic computation. The CAS afforded the teacher and learners symbolic manipulation to simplify quickly and insert values to determine the slope. “This lesson we were more using CAS purely for the algebraic muscle of it to help us with the conceptual ideas of calculus" (Springer, Interview, October 15, 2017). Furthermore, this lesson provided the learner with the opportunity to conceptualize the limit as *h* approaches a value of zero.

Learn pen-and-paper skills (T1). There never seemed an intention to eliminate pen-and-paper skills; rather, Springer used an opportunity to review the simplification of the rational expression on the white board before keying the command into the CAS (4:25). She connected the procedural results to the output on the CAS, verifying her answer through the accuracy of the CAS.

Explore regularity and variation (T3). The slopes of the secant lines were calculated multiple times using progressively closer values to the point at which the line is tangent to the curve (3:52). A typical calculator with the ability of multiple line display could have accomplished the same demonstration of regularity in this lesson as Springer merely repeated numerical calculations. However, the ability to use the CAS command to *define* the function and display the values in function notation, as in Figure 2 above, supported the conceptual development of the definition of derivative.

Link representations (T5). Springer used multiple representations facilitated by the CAS tools to develop and verify the mathematical concepts. In this lesson, she utilized graphs, tables, and symbolic manipulations. Springer opened the lesson with a Desmos graph (1:17), building metacognition and an overview of the mathematical problem of finding the tangent line at a point. In previous lessons, Springer used a table in Desmos to gather two points close together with the purpose of finding the slope. Her discussion with the class pointed students to that recollection (1:47). The graphical tools in Desmos were used to link the representation of the algebraic function, a single point on the function, and the tangent line developed through CAS. She directed students to the TI-Nspire™ for symbolic manipulations of the algebraic function to find the slope (5:00). Once the slope was determined, the equation of the tangent line was written using the point-slope method using pen-and-paper tools. Students then returned to the graphical representation to verify the tangent line by keying the data into Desmos (8:53).

Springer Vignette 2: Development of the Concepts of Continuity and Differentiability

This lesson involved using the CAS *define* feature as in the previous lesson vignette but also required the *comDenom* command and a syntax input of the conjugate in the application of the formal definition of differentiation. The primary goal of this lesson was to develop students' conceptual understanding of continuity and differentiability at various domain values along the graph of the function. Springer had prepared several functions to explore: quadratic, rational, and radical. She also selected three points on each of those functions at critical places in the domain to lead students to reflect on the concept of differentiability. Finally, she used algebraic procedures with symbolic features of the CAS and graphical representations through the application of Desmos to advance student understanding. The format of the lesson was traditional in that several examples were explicated to facilitate the nuances of different function families. Those functions are displayed in Table 14 and provide Springer's class results that were discovered from this lesson vignette.

Table 14

Springer Examples Selected and Summarized for Continuity and Differentiability

Function	Continuous	Differentiable
$f(\text{x}) = x^2 + 4x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$f(\text{x}) = 1/x$	$(-\infty, 0), (0, \infty)$	$(-\infty, 0), (0, \infty)$
$f(\text{x}) = \sqrt{x+5}$	$[-5, \infty)$	$(-5, \infty)$

Quadratic function. Springer demonstrated procedures on the CAS projecting her computer screen on the wall, as students simultaneously mimicked those procedures on their personal devices. The first function she chose to explore was the quadratic function $f(x) = x^2 + 4x$. Her introduction to the lesson revealed Springer re-teaching the definition of derivative through the process of defining the function on the CAS, taking the difference quotient $(f(x+h) - f(x))/h$, and then finding the limit of that expression as h approaches zero. After calculating the derivative, she proceeded with questions regarding continuity and differentiability at three pre-selected points. Springer's directions follow.

0:08 We will explore evaluating derivatives at certain values. And then talk about where is the function continuous and where is the function differentiable. And we're going to write that in interval notation. Let's go ahead and get the derivative of this function.

0:45 Let's go ahead and define our function, $f(x) = x^2 + 4x$. And then what do we normally do? We are going to find the difference quotient. Why do we do $h = 0$ again? (Students answered incorrectly.) So it has to do with the definition of the derivative. What is the definition of the derivative? (Students discussed amongst one another.)

1:14 The definition of the derivative is $(f(x+h) - f(x))/h$ and we take the limit as h goes to zero, right? So that's why we evaluate h equal to zero. It's part of the definition of the derivative. So now we are going to evaluate it at $f'(-5)$, $f'(0)$, and $f'(5)$. (Students and teacher worked through the mechanics to generate the computation as seen in Figure 5.)

Define $f(x)=x^2+4\cdot x$	*Done*
$\frac{f(x+h)-f(x)}{h}$	$2\cdot x+h+4$
$2\cdot x+h+4\|h=0$	$2\cdot x+4$
$2\cdot x+4\|x=-5$	-6
$2\cdot x+4\|x=0$	4
$2\cdot x+4\|x=5$	14

Figure 5. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

2:34 Graph this on Desmos, $x^2 + 4x$. Go ahead and get a visual of it.

3:00 My next question is, where is this function continuous? Where is this function differentiable? So, just in case you weren't sure what the graph looked like. (Springer brings up the image of the graphed function on her compute. Students are discussing thoughts with one another.)

3:30 Differentiable just means, where can we take a derivative and evaluate it at let's say at a number. So you see, for example, I picked a positive number, a negative number, and zero. (Springer displays the Desmos graph and directs attention at the points previously evaluated.) Continuous from negative infinity to infinity . . . and you are differentiable from negative infinity to infinity. (Springer, Lesson, October 20, 2017)

Rational function. The second function explored, $f(x) = 1/x$, required students to work more complicated syntax on the CAS. This rational function differed from the last function because it allowed students to consider the break in continuity at x equal to zero. Springer keyed this example into her computer, directing students to enter on their devices. Some students were confused with the use of the command *comDenom* that was allowing the CAS to get a common denominator and combine the two fractions in a single command (see Figure 6). However, there were other discussions about syntax, regarding preferences of copying and pasting commands. Also, there was a surprising output of negative infinity when evaluating the function at zero as can be seen in Figure 6. Springer brought the discussion together through the use of a Desmos graph and the point of discontinuity.

4:08 Can you do your derivative for one over x? Define $f(x) = 1/x$, take the derivative, and evaluate at $f'(2)$, $f'(0)$, $f'(5)$. And you should get some different fractions. (Springer demonstrated on her device while students are keying this into their laptop computer.)

4:36 We've got to do our *comDenom*, right? (Springer noticed that the output from the difference quotient is a difference of two rational expressions. To find the limit as h approaches zero, the expression will need to be as just one rational expression.

The command *comDenom* on the TI-Nspire combines the two expressions as can be seen in Figure 6.

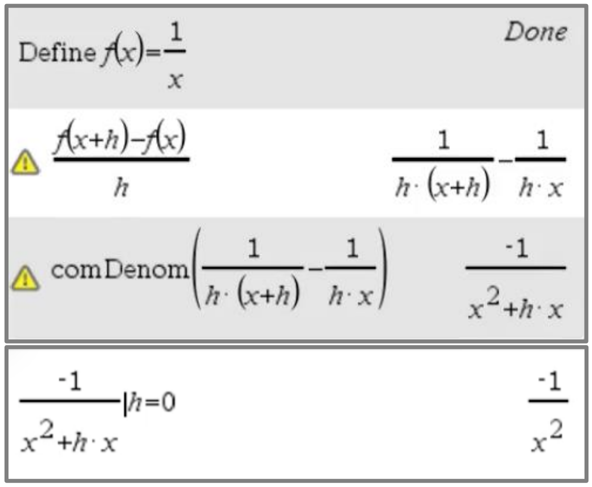

Figure 6. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

4:48 So, we've got our derivative. And then you are going to evaluate at *f'*(2), *f'*(0), *f'*(5). (Student talked to Springer- inaudible. Springer copied and pasted the previous input and changed the *x* values, rather than retyping the entire command.)

5:05 You just always retype it? (Further discussion by the student while Springer keyed commands into the CAS.)

5:12 That's kind of cool. Do you see how it came up with infinity? Yeah. What it's actually doing is, it's giving you the fact that there's an asymptote, right? (See Figure 7.)

Figure 7. Springer’s TI-Nspire™ textual commands projected from computer to the classroom wall.

5:26 Let’s go ahead and look at that (on a graph) (see Figure 8). I don’t think we have ever seen that before. Do you see, when we try to evaluate the function at zero, as we are approaching zero, (Springer is dragging points on the graph that animate from negative values of *x* to zero and then positive values of *x* to zero) we have an asymptote at zero. Notice we don’t actually get out a number.

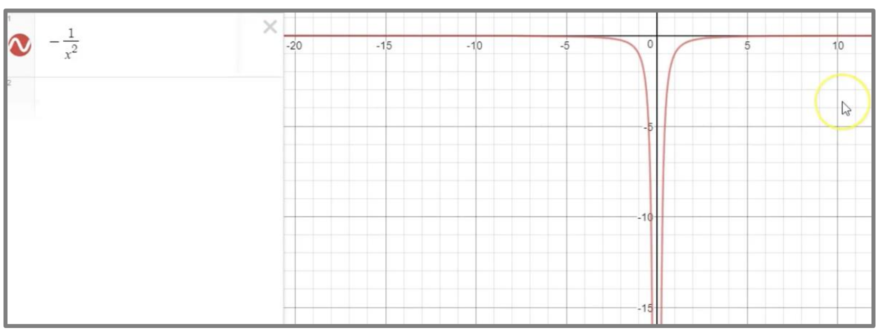

Figure 8. Springer’s Desmos textual commands projected from computer to the classroom wall.

6:10 And *f’*(0), because it’s not actually giving us a number, I am going to put *f’*(0) as undefined. Infinity is not actually a number. So now, where would you say this function is continuous and where would you say this function is differentiable? (Students are answering – inaudible. Springer was recording the answers (- ∞, 0), (0, ∞) for continuous and then the same answer for differentiable.)

6:40 Some people have been asking about this. What does it mean to be differentiable? You can differentiate. Differentiate means you can take a derivative at some point. If you're not actually getting out a value here, okay . . . This negative infinity is a little misleading. It's giving us something. It's not saying it's undefined. It's saying negative infinity just because there is an asymptote, but it's not actually getting out a real number. Since we are not getting a value at *f'*(0), it is not differentiable. (Springer, Lesson, October 20, 2017)

Radical function. The third and final function was a radical function that had a terminating point at one end of the function. The left-side domain value of the function showed continuity and non-differentiability at the terminating point. Springer displayed the graph in Desmos and used the command *Define* for the function $f(x) = \sqrt{x+5}$ on the TI-Nspire™. She found the difference quotient and evaluated the limit at h equal to zero. Springer directed students to evaluate at several points of the derivative, in particular at x equal to negative five. The procedures for finding the derivative were more complicated than previously. The resultant difference quotient required multiplying the conjugate on the numerator and denominator prior to evaluating at h equal to zero. At this point in the lesson students were confused at the syntax of the calculator inputs and outputs and also on the algebraic procedures. The transcription below indicates Springer's willingness to listen and backtrack to explain more thoroughly. At the end, Springer attempted to focus attention on the endpoint negative five to build her case for continuity and non-differentiability for radical functions. It could not be determined from this transcription if she was effective.

7:15 We're going to do one more like that. Go ahead and do the function $\sqrt{x+5}$. (Students had difficulty with the undefined value when evaluating this difference quotient at $h = 0$. Similar to the problem earlier, but this time undefined appeared in the output. See Figure 9.)

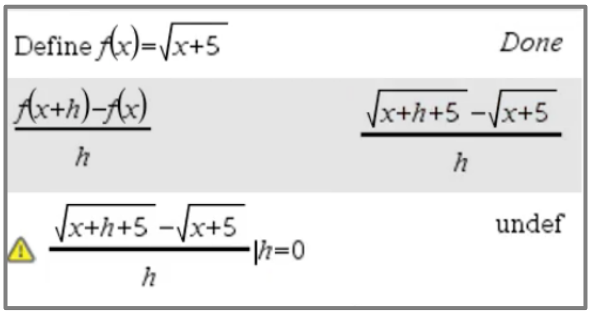

Figure 9. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

8:27 The issue is when you're setting the h equal to zero and it's undefined, it's because of this h in the denominator. So what do we need to multiply by? Yeah, we have to do the conjugate. So we are going to do the conjugate, and that's giving you your numerator. So that gives you your h, and then you have h times your conjugate in the denominator (as shown in Figure 10).

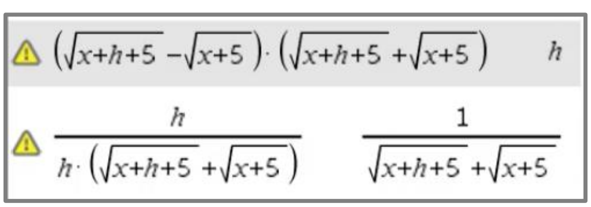

Figure 10. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

9:05 (Student was talking to the teacher about the complexities of entering this into the computer. He was confused and wanted Springer to slow down and re-explain.) It's not that bad. You can't do it all in one step. So that's maybe why you don't like it as much. Once you define your function and do your difference quotient because you can't evaluate with *h* equal to zero in the denominator. . . . That's undefined initially. All we need to do is one intermediary step where we multiply the numerator by its conjugate. That is how we would do it algebraically, right? You can kind of highlight and copy and paste it. So you take that times its conjugate. When you multiply by the conjugate, it's the numerator and denominator. (Other students were confused about a second issue that outputs undefined as in Figure 11. Yet, the output of undefined showed that the function was not differentiable. Another student was still confused about the previous issue that required *comDenom*.)

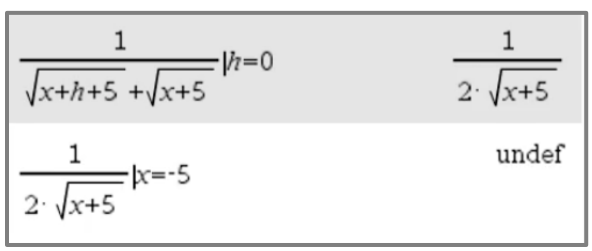

Figure 11. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

9:56 Yeah, you should get negative five is undefined (pause) because you would have zero in the denominator.

10:30 Alright, which *h*? Here? When I got that *h*? That was the numerator. In the denominator, there is still an *h* in the numerator. That is because I multiplied by the conjugate.

11:15 Let me ask you, at the moment, we only got one value that's undefined. We only tried three values. Let's try a couple more values. What if we try negative 6, what would come up? (Shown in Figure 12.)

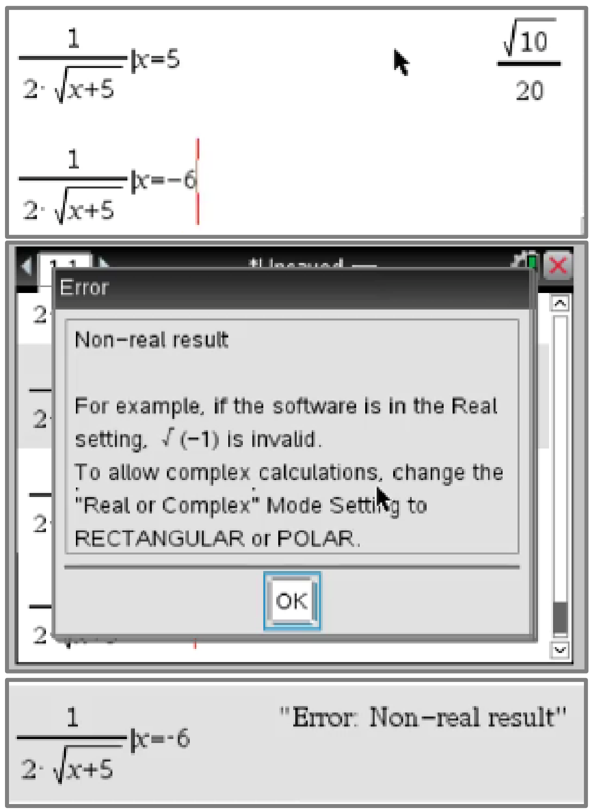

Figure 12. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

11:45 Negative five is undefined because we are dividing by zero. What would happen if I put negative six, negative seven, or negative eight?

11:57 We can't take the square root of a negative number. That would give you a non-zero result.

12:10 Where is this graph continuous? It's continuous including negative five to infinity. But differentiable is slightly different. (Springer displayed the graph as shown in Figure 13.)

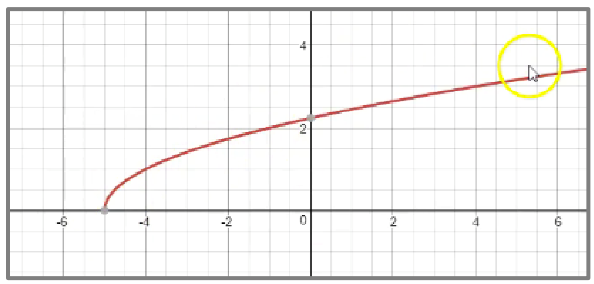

Figure 13. Springer's Desmos textual commands projected from computer to the classroom wall.

12:37 Is the function differentiable on this exact same interval? Go back here, notice at negative five, it was undefined. So it's not including negative five. So that would be a parenthesis [on the interval].

13:15 On the original graph, negative five is defined. So we were continuous there. So the idea is, you might be continuous and differentiable on the same interval. So in both the first two here, we were continuous and differentiable on the exact same interval. (Springer writes the answers on her screen of $[-5, \infty)$ and $(-5, \infty)$.)

13:33 That is a scenario. You might have a scenario where maybe, it's a very similar interval, but the difference is a bracket or a parenthesis. One other scenario could be that it is a totally different interval. Any questions on that? (Springer, Lesson, October 20, 2017)

The third example presented more challenging CAS utilization. It required the use of CAS to multiply the output by the conjugate and that produced a mathematical expression to take the derivative. The TI-Nspire™ had limitations of how to manage this. It did not permit the user to key in a fraction multiplied by another fraction without first automatically simplifying each fraction. As a work-around, Springer explained how to multiply the numerator and denominator separately on the CAS and then put it all back into one expression.

The third example also gave surprising outputs. In the instance of evaluating the derivative $1/\left(2\sqrt{x+5}\right)$ at x = -5, the output was *undefined*. However, in the second example when evaluating for zero in the derivative function, $f(x) = -1/x^2$, the output was -∞. Both inputs involved a zero in the denominator, but gave different outputs. In the third example a tangent line exists at x = -5 but it is a vertical tangent line, which has an undefined slope, hence making the derivative undefined. In example two there is no tangent line at x = 0 since $f(x) = 1/x$ is undefined at x = 0. Without looking at the graph, it was difficult to comprehend the difference between the endpoint (example 3) and the asymptotic behavior (example 2) as compared with the symbolic expressions that were output on the CAS.

The lesson closure involved examining the last function possessing the characteristic of being continuous but not differentiable at the endpoint. The first two examples illustrated functions where the intervals of continuity and differentiability were identical, whereas the third example highlighted the case where a function can be continuous to the right at a particular x-value, but not differentiable at that point. This last example of a rational function should have conjured curiosity when the interval for continuity and differentiability ended with different answers, unlike the other examples in which they were the same. Table 14 summarized the three examples and shows the intervals of continuity and differentiability. The graphs Springer showed during the lesson helped to clarify behavior at endpoints and asymptotes.

Springer Vignette 2: Pedagogical Opportunities

The lesson on developing concepts of differentiability and continuity that utilized CAS accompanied subject area consideration and specific tasks. The subject area pedagogy as identified in the P-Map afforded learning opportunities for the following: exploiting the contrast of ideal and machine mathematics; a re-balance of emphasis on skills and concepts; and building metacognition and overview. Tasks utilized pedagogical opportunities of exploring regularity and linking representations. The evidence summarized in Table 15 used pattern matching logic (Yin, 2009) with the data and the P-Map Framework (Pierce & Stacey, 2010). Descriptions from the vignettes facilitate how these characteristics were demonstrated.

Table 15

Springer Lesson Vignette 2

P-Map	Evidence
S1	1. The output from evaluating a value in the difference quotient was unexpected (infinity) 2. Solution output required extra commands on the device
S2	Less time on procedures, more time on development of concepts
S3	Emphasis shifted from procedural skills to consideration of both symbolic and graphical outputs that demonstrate continuity and differentiability to gain insight.
T3	Use of three different function families to compare and contrast
T5	Multiple representations of graphical and symbolic forms

Exploiting contrast of ideal and machine mathematics (S1). Pierce and Stacey (2010) described a contrast of ideal and machine mathematics as the difference between traditional paper-and-pencil expected answers versus the outputs the CAS provided. Often these issues are due to generalized mathematical conventions: a formatted difference, a rearrangement of terms, an error message, or an answer in a simplified form (Pierce & Stacey, 2010). In this lesson vignette, Springer managed to turn unexpected answers as an opportunity to teach the mathematical content of limits and the need for additional procedures. The first instance showed Springer evaluating the limit of a difference quotient for an indeterminate form, resulting in $f'(x) = -1/x^2$, for which the output gave the symbol for negative infinity (5:12). Syntax, in this case, provided the appropriate value, rather than simply "undefined." This surprised Springer and she reacted, "That's kind of cool. Do you see how it came up with infinity? Yeah. What it's actually doing is, it's giving you the fact that there's an asymptote, right?" (Springer, Lesson, October 20, 2017). She took the opportunity to display the graph of the function on Desmos, which allowed students to reflect (5:26).

The second instance Springer spontaneously adjusted was when the difference quotient provided the resultant output of the difference between two rational functions (4:36). Her response, "We've got to do our *comDenom*, right?" (Springer, Lesson, October 20, 2017). This command was used prior to this session, so it required students to recall this necessary mathematical procedure. "When I am creating these lessons, I'm thinking it through, but sometimes I have to be in the moment to come up with these extra commands. The Nspire has so many commands; so many little things." (Springer, Interview, November 8, 2017). Springer remained flexible to adjust in the moment as the need arose for additional instruction of CAS commands (9:05).

Re-balance emphasis on skills, concepts, and applications (S2). Mathematical skills and concepts were re-balanced because of the availability of the CAS. Springer was able to spend less time on skills by outsourcing the complicated procedure of difference quotients on the CAS, hence, reducing cognitive load and thereby freeing up working memory for concept building of continuity and differentiability. This outsourcing principle was apparent through the progression of challenging function examples (1:14, 4:48, and 7:15). When asked the question, "How does CAS facilitate student understanding of differentiability?" Springer wrote the reply,

> CAS facilitates student understanding of differentiability by the ease, speed, and accuracy of using the definition of the derivative to calculate derivatives and also evaluate derivatives at certain values. This is great because we can calculate $x = 1$, $x = 2$, $x = 3$ and see the calculator give us a value. But, then it will give us a result of "undefined" if the function is not differentiable at that value. (Springer, Written Reflection, November 4, 2017)

The functional opportunity of the CAS cleared the way to direct attention to the mathematical content. Springer justified this: "You need to know that the CAS gives you, like the accuracy and efficiency that allows you to just focus on the exploration and not like the work of it" (Springer, Interview, November 8, 2017). The appropriate tools for the computation enhanced student awareness to the central goal.

Build metacognition and overview (S3). Springer approached the topic of continuity and differentiability from a symbolic and graphic perspective, demonstrating connections facilitated through a CAS. She selected three types of functions (i.e., quadratic, rational, and radical) and looked at the graphs and the symbolic manipulation of each function to ascertain continuity and differentiability. She shifted away from the procedural skills and used CAS answers in the development of the mathematical concepts. It was both the reflection on the outputs and the look at graphical representations that prompted student consideration in assembling connections to the concepts of continuity and differentiability.

Explore regularity and variation (T3). The task considered the three types of functions mentioned as independent examples for comparing and contrasting the continuity and differentiability at particular points. CAS' efficiency supported the possibility of exploring those comparisons. Springer used the definition of derivative and evaluated each function separately at her pre-selected points (1:14; 4:48; 7:15). This quickly enabled her students to reflect on the outputs to consider differentiability.

Link representations (T5). CAS was employed on the TI-Nspire™ for symbolic manipulation and on Desmos for graphical representation. When the infinity symbol appeared as an output on the TI-Nspire™, Springer displayed the graph and toggled back to the symbolic output to facilitate connections (5:26). Furthermore, on the graph she was able to *drag* a point close to *x* equal to zero, illustrating the behavior of the infinite limit. Multiple representations of graph and symbolic forms heeded support in the development of the concepts.

Springer Vignette 3: Power Rule and Higher Derivatives

The goals for this lesson were to: (a) demonstrate procedures for finding higher derivatives from the definition of derivative (e.g., the second, third, fourth, tenth, and hundredth); (b) investigate the pattern in the resultant function of those derivatives; (c) generate a rule for determining each higher derivative algebraically based on those outputs (i.e., the power rule); and (d) check that rule using the definition of derivative with CAS. Springer stated her intentions for the lesson.

> The goal of the Calculus lesson was for students to explore calculating derivatives for polynomials and basic rational functions and then identify a pattern and determine the shortcut that arises when taking derivatives for these types of functions. Students were essentially deriving the differentiation technique known as the *power rule.* (Springer, Written Reflection, November 4, 2017)

This teacher-guided lesson was exploratory because the power rule was not provided first. Rather, the teacher posed questions to the students to facilitate algebraic and numeric connections to each function's higher derivatives.

On occasion, during the transcript, a student voice was noted, but the recording was primarily capturing the teachers' voice leaving student verbal responses off the record. However, it was apparent in the recording that the teacher often repeated student responses to her questions and those were documented in the transcription. An additional note is that both the teacher and students used an application on their laptop computers to take notes with a stylus and recorded their results from either the CAS outputs or from cognitive mathematical calculations. Several examples of those teacher annotations clarify Springer's discourse and are provided within this vignette.

Springer introduced the idea of taking higher derivatives: that is, after taking a first derivative, she demonstrated the procedure for finding a derivative from the answer to the first derivative and repeated for additional higher derivatives on the same function.

Springer began with an example of a rational function and later worked through an example of a fifth-degree polynomial. She ensured that students understood the power rule for a polynomial by practicing on two more polynomial functions and checking their by-hand calculations against the CAS. The fifth-degree polynomial was a less complicated rule because it only involved two algebraic manipulations per term and it dealt with positive exponents. She intentionally chose it to be the second example and then returned to the first example for a discussion on conversion of the rational function to negative exponents and the application of the power rule for the rational function. At the end of the lesson, she provided several more similar functions for students to work independently, verifying the algebraic rule against the definition of the derivative using CAS.

Rational function. In the exploration of the rational function, Springer intended for learners to develop three patterns: (a) the change in the numerator as some numerical value; (b) the recognition of alternating signs of positive and negative; and (c) the degree of the denominator as an increasing value. Springer began this exploration with defining the function $f(x) = 1/x$ on the CAS, and led students to find the higher derivatives $f'(x),\ f''(x),\ f'''(x)\ , f^4(x),\ f^{10}(x), and\ f^{100}(x)$. She guided students in the syntax of inputs and manipulations on the CAS. In addition, she modeled the documentation of outputs into notes that are taken on her digital notebook page as shown in Figure 14. This screenshot was captured towards the end the example but is a thorough representation of the development of the concept of higher derivatives of a rational function.

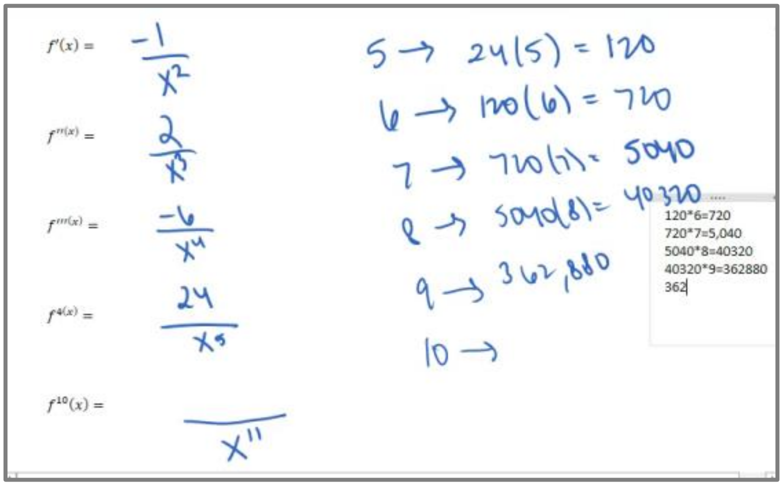

Figure 14. Springer's digital notes projected from computer to the classroom wall.

Springer led the class to find the first two higher derivatives and analyze results before computing the third derivative, $f'''(x)$. The prompts and questions for students to determine a pattern are noted in the following transcription of Springer.

1:51 Okay, so maybe what's happening is that it's alternating (pause) going positive, negative, positive.

1:55 So, maybe with the next one it will be negative, and then this coefficient was one for the first derivative. This coefficient was two for the second derivative. So what might it be for the third derivative? (Students are providing the answer.)

2:12 So, negative three over *x* to the fourth. I'm going to put a question mark, and now let's see if that's right. (Teacher typing into the CAS *define* $2/x^3$ and then taking the difference quotient and limit as *h* goes to zero.)

2:23 Sometimes we also might think we know the pattern and it's not quite what we think it is, so we gotta see. Alright, *comDenom* . . . So we actually get negative six over *x* to the fourth.

2:42 Okay so we were close. We knew that it was going to be negative. We knew the x to the fourth. (Springer, Lesson, October 20, 2017)

The first pattern recognition was the identification of alternating signs of positive and negative for each higher degree (1:51). The second point was the change in the numerator as some numerical value occurring (1:55). The students chose an incorrect value of three after determining the pattern was an increase in one for each increase in a higher derivative. Springer permitted the incorrect value to demonstrate the need for verification on the CAS. The third point of pattern recognition was determining the degree of x in the denominator to be four (2:12). Springer checked the cognitive guess of $-3/x^4$ by following these steps as shown in Figure 15. She keyed the commands to define the second derivative $f(x) = 2/x^3$ and allowed the CAS to compute the difference quotient $(f(x+h) – f(x))/h$. She used *comDenom* to combine the expression into one rational term. Finally, she found the limit as h goes to zero. The output did not match the guess. That led to student discussion and eventually the revelation that the constant in the numerator was the exponent multiplied by the coefficient.

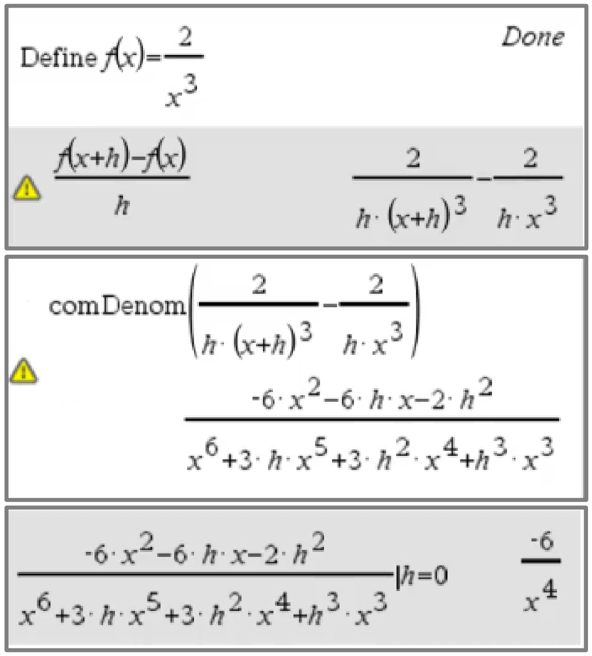

Figure 15. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

Rational function: Fourth derivative. Springer worked with the class on the second round of cognitive guess and CAS check for the fourth derivative. Both the mistakes and the accurate rule recognition were helpful in the students creating a cognitive guess for the fourth derivative. In the background noise, there was an abundance of murmuring of accurate guesses. Springer repeated those recommendations from students, although not seen in the transcription.

3:07 So then what do we think that would give us for the fourth derivative?

3:12 Maybe positive, cause it alternates, right? So, maybe positive 24 over *x* to the what? Over *x* to the fifth? So let's see if that's what happens. (Teacher keying into the CAS while talking.)

3:29 We're going to *define* and then we are going to do our difference quotient.

3:42 *Comdenom*, such that *h* equals zero, and there you go, 24 over *x* to the fifth. (Springer, Lesson, October 20, 2017)

Rational function: Tenth and hundredth derivatives. The next part of this example required learners to extend the rule onto much higher derivatives. Springer began an inquiry about building values from previous terms to develop subsequent terms as a method of recursion. She told students it was not necessary to write out all the derivatives, but then at the same time, she wrote out all the numerical calculations for the numerator up to the tenth term.

3:48 Can we extend that to get the tenth and hundredth derivative? So let's think about it for a minute. So to get the fourth derivative . . . (inaudible and paused, moving around the classroom, students working on the problem and teacher is talking with them)

4:20 I mean do we really have to keep going to 10?

4:25 What do we know about the denominator for sure? (Listening to student responses and repeating) *x* to the eleventh? And here's what we know about the even and odd. If it's an odd derivative, like one and three . . . (Springer is pointing to the handwritten pattern as seen in Figure 15 and a student says something to teacher that is inaudible.)

4:42 Actually no, that doesn't quite work out. Wait, so in . . . Oh yeah, okay wait, one and three, an odd derivative is going to be negative, right? And an even derivative is going to be positive. So, is this going to be positive or negative? (Student answers.) Okay, so you know it's going to be positive.

5:03 What would it be (the numerical value) in the numerator? (Students are sharing their thoughts with one another.)

5:35 Alright, so think about it. So part of the problem with this is . . . Have you guys heard the term recursive sequence? (Students are responding, yes.) What is a recursive sequence? It relies on the previous one, right?

5:50 So we have to kind of think about, what would it have been?

5:57 We don't have to write out all the derivatives. Cause we know most of the information. We just have to figure out the numerator. So [fifth derivative] would have been like a 24 times five?

6:18 So you don’t have to do every derivative because that would be a little time-consuming on the Nspire, right? All you have to do is go ahead and start multiplying a few numbers here. (Students are working on multiplying the numbers to get the numerator values up to ten.)

7:43 Okay so we should be getting 3,628,800 [for the tenth term]. Okay so now, let me ask you a question. We know that there's a pattern for the hundredth derivative, but do we really want to go through this pattern? (Students said no.) Okay, so there must be a shortcut. (Springer, Lesson, October 20, 2017)

Springer closed the example after finding the tenth derivative and left the hundredth derivative as an exercise to revisit after the second example function. By working through the numerator calculations, Springer demonstrated the monotony of finding the actual tenth derivative through patterns and recursion methods. She intended to pique their curiosity but needed an additional example to build the concept of the power rule.

Polynomial. The second function explored was the fifth degree polynomial $f(x) = 5x^5 - 2x^4 + 3x^3 + 9x^2 - 2x + 1$. Springer used her prepared presentation and asked students to find the first, second, third, fourth, tenth, and hundredth derivatives for this polynomial, just as she did in the previous example. She guided them through the first derivative, but then let students work independently through the remaining higher derivatives. After a period of time, she displayed those procedures on her computer projected on the classroom wall. Springer used a scaffold of questions and students answered correctly, in most cases. In the transcription, Springer repeated those student responses and brought clarity to several points of pattern seeking. One point was that the degree decreases for polynomials. The second point was that the constant term was eventually eliminated from the polynomial.

11:55 So the only commands I did was define it, difference quotient, and then once I got that answer, I did that h equals to 0. Alright, I got $25x^4 - 8x^3 + 9x^2 + 18x - 20$.

12:15 Alright so let's take a look for a minute, first of all, we had a fifth degree. Now, what do we have? One less, notice that's different from what we had above though, right? Here [referring to the work product of the rational function] we had like an x and it became x squared, and it became x cubed, it became x to the fourth. So a rational function the exponents seem to be increasing.

12:40 For a polynomial, what seems to be happening with the degree? It seems to be decreasing by one. What about, what are other things you notice? The $5x^5$ became a $25x^4$. Where might that 25 come from? So the five and the five, that coefficient and that exponent gave us the 25.

13:08 Does that work or is that a one-time thing? Because also five squared is 25. Two and a four, that's eight. What's three times three? Nine. What's nine times two? Eighteen.

13:26 What exponent is here? It's like a one and there's a two.

13:39 What happened? We kind of lost that one. If a $2x$ is $2x$ to the first, what is a one? A one would be like $1x$ to the 0. So if you multiplied a one by zero you get . . . What's 1*0? (Students answer zero.)

14:06 Zero and so it zeroed out. (Springer, Lesson, October 20, 2017)

Springer led students to consider the constant value that results in an elimination of the term from the function. Figure 16 reflects the written work product that Springer shared with the class of *x* to the first power and then *x* to the zero power. The explanation along with the product of the exponent and the coefficient facilitated student connection to the elimination of the term.

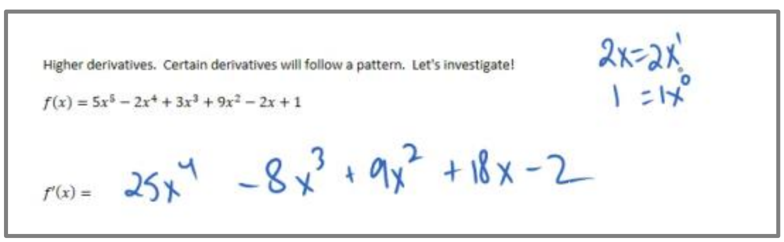

Figure 16. Springer’s digital notes projected from computer to the classroom wall.

Polynomial function: Second and third derivatives. The lesson continued with finding the second and third derivatives of the polynomial function on the CAS and documenting the results. Students worked independently and also talked with their nearby peers to assist with finding the results. Springer monitored their progress. An unexpected situation arose with syntax when finding the third derivative. The resultant function had a factored form, rather than the standard form (as seen in Figure 17). Springer proceeded to demonstrate on her computer the new command that was needed—

the *expand* feature. She explained why the command was necessary and how to facilitate its use in the transcription that follows.

16:45 Did you get this here (polynomial with a common numerical term factored out)? So all the Nspire does because it doesn't know what we're looking at, it's hard to see the pattern in that form . . .

16:53 If I type the word "expand," okay. Put a little double parenthesis and then bring this (the factored expression) into that parenthesis. It will give us, you know, all it did was distribute it. The advantage of that if you can't see the pattern if you factor out a GCF. Did everyone see how I did that *expand*? (Springer, Lesson, October 20, 2017)

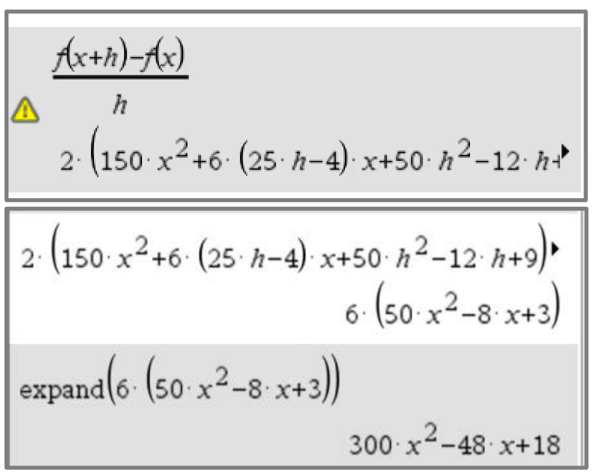

Figure 17. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

Polynomial function: Generalized higher derivatives. After students got the fourth, fifth, and sixth derivatives, Springer directed the students to consider the pattern, "Notice that the degree keeps decreasing, right? We were a degree five, then four, and then eventually a degree one. What would be the next after a degree one?" The goal at this point was to notice that the tenth and hundredth derivative would be zero (see Figure 18). She explained this to the class.

19:24 So basically, if you have a polynomial of degree five, you can get some unique answer up to the fifth derivative. Okay, when I say unique answer, meaning like you're going to get some polynomial of some answer. After it's exponent, meaning more than five, it just will zero out. That's unique to polynomials. Did that happen up here (referring to the rational function)? (Springer, Lesson, October 20, 2017)

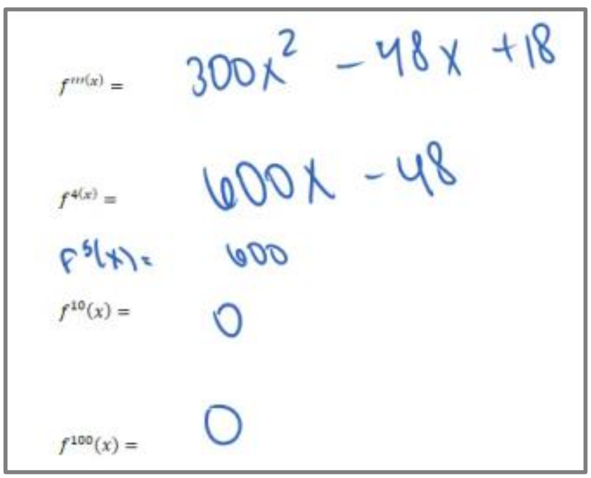

Figure 18. Springer’s digital notes projected from computer to the classroom wall.

The next five minutes, Springer helped the students to write a definitive rule for finding the derivative based on patterns. Students and teacher typed a description of the pattern into their notes, “The degree of the polynomial decreases by one. The exponent multiplies by the coefficient. Eventually, the derivatives zero out” (Springer, Lesson, October 20, 2017). Springer then created an additional polynomial example, $f(x) = 7x^{10} + 6x^2$, for students to first use the new rule and then check their answer against the CAS result. However, she ended up completing the example with the students, projected her work product on the wall, and talked through the solution.

Revisited the rational function. Springer returned to the first example of the rational function and showed how the derivatives could be found using the same method or rules discovered in the polynomial function. Although the class had explicitly stated the rule, Springer had not yet revealed the name for *the power rule*. Springer led students through the use of the power rule by rewriting the rational function using the definition of negative exponents. The transcription and digital notes by the teacher follow.

26:35 Is there a way to write one over *x* with a negative exponent? (Students respond.)

26:41 *x* to the negative 1 (Teacher repeated student answers.)

26:47 Okay, so let's see if this property that we just saw with polynomials can be extended now to a rational function. We said to get the first derivative you're going to decrease the exponent by one. You're going to subtract it. So negative one minus one would give us (pause) negative two. And then you'd multiply the exponent by the coefficient, which is negative one. Isn't that the same thing?

27:27 Right, so now let's go ahead and let's see what happens. If I, let's do it again. So *x* to the negative two . . . If I subtract one, what's negative two minus one? (Students responded.) Negative three. And what is negative one times negative two? (Students responded.) It’s two.

27:48 Isn't that the exact same thing? Isn’t that cool? (Students talking to teacher.)

27:57 Which way? You like that? Because it kind of looks like a polynomial? Let's keep going. Again. If I subtract one in this exponent, *x* to the negative three would become an x to the negative four. Two times negative three is negative six. (See notes projected on the wall in Figure 19.)

28:23 Okay, so let me ask you a question. Do you see now why, this is kind of cool? (pause) Do you see why it was an alternating; you know how it was like positive, negative, positive, . . . (Springer, Lesson, October 20, 2017)

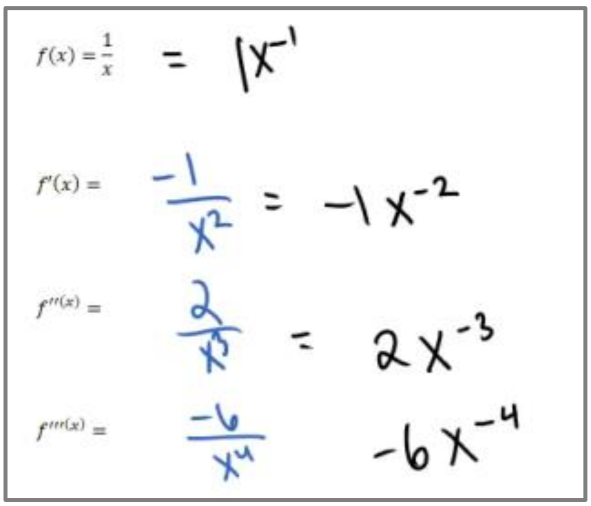

Figure 19. Springer's digital notes projected from computer to the classroom wall.

Springer brought the lesson to completion. After the development of the power rule in the polynomial example, she used the class-generated procedure on the rational function with negative exponents. However, she verified with not only the first derivative. She reinforced the idea of the power rule by computing the second and third derivatives to compare against the CAS.

Springer Vignette 3: Pedagogical Opportunities

This lesson reviewed students' understandings of derivatives and constructed knowledge of the power rule for derivatives through the utilization of the CAS. The evidence is paired using the P-Map taxonomy and shown in Table 16. Additionally, Springer posed questions about higher order derivatives. The function and its first three derivatives represented concepts of distance traveled, velocity, acceleration, and jerk. Springer had students explore derivatives beyond the first three. The patterns explored were merely a way to understand that derivatives of functions can *zero out*, meaning that at some point the higher derivatives do not exist. As seen in the case of the rational function, there was no end to finding a higher derivative. Springer's pedagogical decisions are explained.

Table 16

Springer Lesson Vignette 3

P-Map	Evidence
S1	Exploited the contrast of the unexpected outputs by teaching new commands to incur the intended outcome.
S2	Re-balanced the lesson by exploring higher derivatives and the patterns in those subsequent derivatives (new purpose)
S3	Delaying the CAS command of derivative until the definition of derivative has saturated the thought processes
C2	CAS became an external mathematical authority
T1	A cognitive guess was checked against the CAS output
T3	Multiple examples to explore patterns for both higher derivatives and a variety of types of functions.

Exploiting contrast of ideal and machine mathematics (S1). The CAS outputs compelled Springer to make pedagogical decisions during the lesson to elaborate on her mathematical goal of investigating patterns of the derivatives. Springer executed syntax for the definition of derivative as she had in lesson vignettes 1 and 2; however, in this example the output was an insufficient form to insert a value of $h = 0$. Springer introduced the *comDenom* command to combine two rational forms into one, thereby producing an output that $h = 0$ could be inserted to produce an expression of the derivative (2:23; 3:42). She remained flexible in her presentation, explaining the necessary commands.

> I come up with these things on the spot, that I remember. So when I am creating these lessons I'm thinking it through but sometimes I have to be in the moment to come up with these extra commands. Because there are so many, the Nspire has so many commands, so many little things. (Springer, Interview, November 8, 2018)

The syntax input of the CAS had to be manipulated to get a feasible output that would allow students to explore the pattern.

Re-balance emphasis on skills, concepts, and applications (S2). This lesson emphasized Springer's objectives in lesson design, the offloading of procedures to CAS to discover the power rule, and the initiation of a muse about higher derivatives. An explanation of how this was evident follows. Springer adjusted her teaching objectives for this calculus class. The goal of this lesson was originally to formulate the power rule through an exploration and pattern recognition. Springer talked about the decision to add this notion about higher derivatives.

> They were coming up with the power rule but also seeing that [they] would be getting multiple derivatives. And what would happen in, with the polynomials how it would zero out eventually, but with the rational function because it's a negative exponent it keeps increasing. There was some more to explore, I thought by sticking to one function. So like this was like a brand new idea that came to me. (Springer, Interview, November 8, 2018)

It was through the efficiency of CAS computations and reduction of tedious by-hand computations that made it possible to construct the power rule concept through pattern recognition.

> It's too tedious to do it by hand and you might get it wrong and then how could you explore something with a wrong answer. You need to know that the CAS gives you, like the accuracy and efficiency that allows you to just focus on the exploration and not like the work of it. We were able to have a discussion about it and not get bogged down with doing the algebra of it. (Springer, Interview, November 8, 2018).

The concession promoted an emphasis on the power rule for polynomial functions. But the lesson also allowed an extension to develop the conception of a function *zeroing out*.

> Well, I think with the polynomial, I wanted them to see that eventually it zeros out cause your exponents keep decreasing, decreasing, and decreasing. And so, basically, I am not going to use this term properly, but there is like a limit to how many derivatives you can take, right, but not limit in the calculus term, [rather] the normal term limit. . . . The pattern is different based on the family of functions. (Springer, Interview, November 8, 2018).

Springer developed this kind of hybrid lesson of CAS and pen-and-paper skills to conceptualize the term limits of higher derivatives. The lesson lent the opportunity for students to develop the power rule and access an idea regarding higher derivatives existence.

Build metacognition and overview (S3). Springer chose to prolong the use of the definition of derivative to build cognition. Springer retained the procedure for difference quotient for finding derivatives, although the CAS has its own direct command. She reflected on this decision.

> I decided to take a different approach this year, where we are still using the definition of derivative. I'm really trying to hone in on that and have them [students] really understand the difference quotient and the limit, so instead of using the command I was making them use the definition of derivative. (Springer, Interview, November 8, 2017)

Students had the ability to continue with the definition of derivative because the functional operations of CAS made it an effortless computation.

Change classroom didactic contract (C2). The reliance on the CAS gave credence to the pedagogical opportunity of a shift in the classroom didactic contract (Pierce & Stacey, 2010). It was through the authoritative knowledge of the CAS that students conceived the pattern in the higher derivatives. That allowed a shift from the teacher as sole source of knowledge. This shift was subtle as it was intertwined with the task.

Springer used the authority of the CAS to set the stage for students to construct mathematical understandings regarding the power rule. First, multiple derivatives were generated using CAS and analyzed for the pattern in each term of the function (1:51; 11:55). Then a cognitive guess of the pattern was discussed and agreed upon in written work (2:12; 2:23; 3:12; 12:40). Finally, by checking the guess against the CAS output, the hypothetical rule could be revised or accepted. CAS equipped the student to receive instant feedback, rather than relying on the teacher to substantiate their cognitive guess. This method of exploration demonstrated how CAS was used as an external authority.

Learn pen-and-paper skills (T1). Springer used CAS as a cognitive tool to generate multiple derivatives of a function. Student reflection on those derivatives prompted a targeted guess. Ultimately, a pen-and-paper skill with the use of the power rule for derivatives was imminent.

Explore regularity and variation (T3). The lesson was exploratory, using multiple derivatives for each of the functions (e.g., first-order, second-order) and also considered two different types of functions with the symbolic derivatives of each. Springer wrote about her decision for numerous calculations on the CAS.

> It was critical for me and the students to calculate multiple examples so they could identify a pattern with higher order derivatives. I hoped to accomplish that students were able to see the "shortcut/pattern" to taking derivatives. i.e.) The power rule for both positive and negative exponents. (Springer, Written Reflection, November 4, 2017)

Multiple derivatives taken quickly along with teacher prompts about those resultant derivatives provided the structure to allow students to make connections.

Springer Vignette 4: Derivatives of Trigonometric Functions via Graphic Exploration

The goal of this lesson was to introduce derivatives for trigonometric functions by exploring a function's derivative through a graphical representation. Given a trigonometric function, students were asked to create a graph and use the command $f'(x)$ in Desmos to produce a graph of the functions' derivative. Students then predicted the equation for the derivative based on the graph of $f'(x)$. Three functions were displayed in one Desmos graph: the original function, the derivative of that function, and the predicted function. The predicted function was compared with the graphical representation of $f'(x)$ for alignment. Springer thoughtfully chose this method to introduce the derivatives for sine, cosine, and tangent and described her rationale during an interview that followed the lesson.

> I thought that using Desmos would be a way for them to throw out a guess, and then actually be able to see if it was correct. It was kind of a quick, easy way and this is also visual. And I like how Desmos doesn't specifically give the derivative. It just shows the graph of the derivative, which is cool. (Springer, Interview, December 6, 2017)

In addition, she justified the visualization method to the class as the first look with the expectation that the formal proof of the derivative would follow.

> Here's the thing, to derive sine and cosine, okay, we would have to go back to the definition of the derivative and limits. Which I think I'm actually going to save for next class. For all the other ones— tan, cosecant, secant, cotangent— we actually just use the quotient rule. So we'll do that next class. Today I wanted to do it a little bit different on Desmos. (Springer, Lesson, November 9, 2017)

CAS was discernable in this vignette through Desmos. The computer application performed a derivation of the input function through the command $f'(x)$, although it did not display the symbolic representation of the function.

Quadratic function as introduction to Desmos command. Springer introduced the Desmos command of $f'(x)$ with her students by providing a basic example of a quadratic function and its derivative. She displayed her graph on the projector and explained to the class the procedures she was typing into Desmos as described in the transcription and as visible in Figure 20.

2:15 So, I don't know if I showed you this before, this really cool feature. If you graph $f(x) = x^2$, so go ahead and do that. What is the derivative of x squared? (Students answered $2x$.) Let me type in y equals $2x$ right there, okay?

2:38 And I'm going to change that just to be a dotted line, so we see it. That's what the derivative of that is (shown in the first graph of Figure 20.)

2:44 If I say to Desmos, graph f prime of x and prime is next to your enter key. Notice that it gives you that [the graph of $y = 2x$, the derivative] (the second graph in Figure 20). (Springer, Lesson, November 9, 2017)

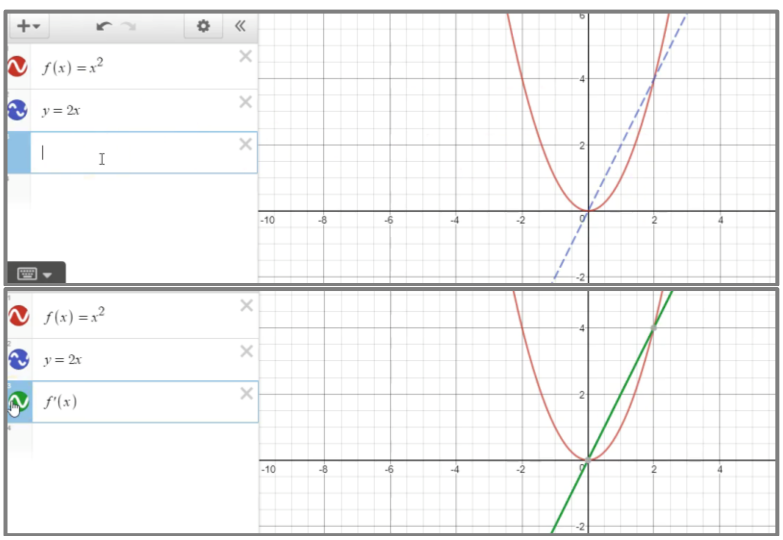

Figure 20. Springer's Desmos textual commands projected from computer to the classroom wall.

Several students asked why Ms. Springer had not shown that Desmos command to them previously. She replied, "I'm sorry. I have so many things to show you, I forget. I like thought of this yesterday" (2:58, Lesson, November 9, 2017). Furthermore, she described to the class the moment when she decided to use this approach, "So I was driving in my car. I was thinking about our lesson today, and I was remembering that we can do this on Desmos" (Springer, Lesson, November 9, 2017). The revelation of combining trigonometric functions with a graphing method resulted in this lesson methodology.

Sine. Springer introduced trigonometric derivatives by graphing the sine function with a solid line and its derivative with a dotted line. Unlike her introductory example, she used the dotted line for the Desmos $f'(x)$ command, and the solid line is used for the student-predicted guess for the function. The lesson employed the cognitive guess and CAS check similar to Springer Vignette 3. Her facilitation of the lesson follows and screenshots of Desmos graphs in Figure 21.

3:22 Okay, so we're going to see if we can figure out the derivative of sine and cosine by doing this. It gives you the picture but it doesn't tell you the answer. So here's sine of *x*.

3:34 And I'm going to type *f* prime of *x*. And I'm going to make it like a dotted line for right now, okay. So the question is, what is this dotted line? What is the derivative of sine? (Students are making some suggestions.)

3:54 So let's see, if I type in y = cosine *x*, that's a very perfect match. So yeah, so I'll put this on the board, the derivative of sine *x* is cosine *x*. (Springer, Lesson, November 9, 2017)

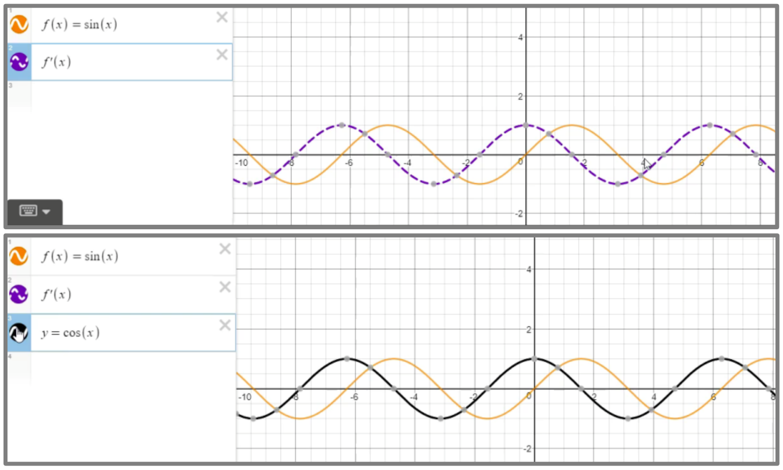

Figure 21. Springer's Desmos textual commands projected from computer to the classroom wall.

Cosine. The second trigonometric function explored was cosine *x*. Springer anticipated students suggesting sine of *x* and complied by inputting the function as the third line item. It clearly did not match the derivative of cosine as can be seen in Figure 22. To keep students advancing in their guesses, Springer suggested the other trigonometric functions of tangent and cosecant. The screen shots of those trial and error guesses are recorded in Figure 22. Meanwhile, several students were heard in the background suggesting a variation of cosine.

4:05 Okay, so now let's do the same thing, let's say I give you cosine *x*. Here's *f* prime of *x*.

4:14 What might be that derivative then? What do we think?

4:19 So here's sine of *x*. Not sine. Here's tangent. Cosecant?

4:36 Cosine, that's just the original. (A student said negative sine.)

4:43 That's it. So cosine is negative sine. (Springer, Lesson, November 9, 2017)

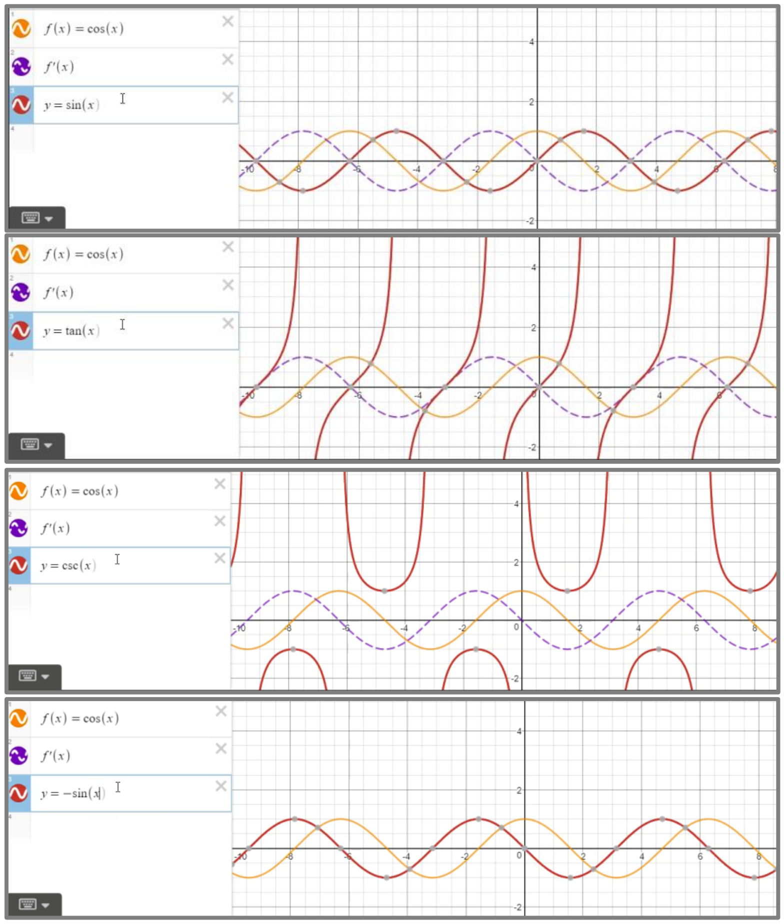

Figure 22. Springer's Desmos textual commands projected from computer to the classroom wall.

Springer reflected on the lesson in a post-interview. She engaged the class in this trial and error exploration of the derivative of sine. She conjectured that the sine function would be understandable, but that the derivative for a cosine function would be a confounding result because it involved a negative sign.

> So we did that for sine and then when it was time for cosine, then [the students] were all like, "Sine." It was interesting for them to see that it was not sine. They actually get that it was negative sine . . . I was impressed with that, that they were

able to gauge that. Then when we went on to tangent, you know they kind of went through the different trig derivatives. And it was kind of hard to come up with secant squared. (Springer, Interview, December 6, 2017)

Tangent. The lesson continued as Springer explored the derivative of tangent, a function much more difficult to guess-and-check. The derivative involved a squared function— either $\sec^2 x$ or $1/\cos^2 x$. Springer realized the challenge and quickly conducted a scaffold of guesses with a discussion about how secant x did not work but was closely related. Springer said to the class, "This one's not quite as intuitive" (Springer, Lesson, November 9, 2017). Figure 23 shows some of the cognitive guesses that students suggested and that Springer checked. Instead of waiting for students to guess accurately, she revealed the correct derivative $\sec^2 x$. The transcription from the lesson follows. The reader should keep in mind that Springer was typing these mathematical functions into Desmos.

5:10 Here's my derivative of tangent. It doesn't look like a sine or cosine graph, right? Cause sine kind of looks like that wave?

5:18 Cotangent looks actually very similar to tan but goes in a different direction.

5:25 Secant is really, really close but notice that the shape of it is slightly off. Right, like it's a little bit skinnier than secant x.

5:46 How about what? (student – Almost negative cotangent?)

5:55 You're saying the original graph, but we're trying to get the purple graph. Trying to get the purple graph because that's the derivative.

6:05 So this one's, this one's not quite as intuitive but it's secant x times secant x. Or also known as secant squared x. (Springer, Lesson, November 9, 2017).

Her reflection of the lesson conveyed the desire for additional planning. She realized that additional clues were needed for the tangent function to fully complete this lesson design.

> I think I would have to think through a little bit more about what kind of hints I could have given them— to maybe have them guess that. But at least they did get to see . . . it does look like a trig derivative. And I think they could tell it was like a cosecant or secant type graph. But the fact that it was secant squared . . . That was fun. So we did that on Desmos and then we jumped to some practice problems. (Springer, Interview, December 6, 2017)

This was a first-time presentation of this development of trigonometric derivatives and Springer was still conjuring her methodology of this approach. She summarized the three derivatives of sine, cosine, and tangent before shifting to student independent practice.

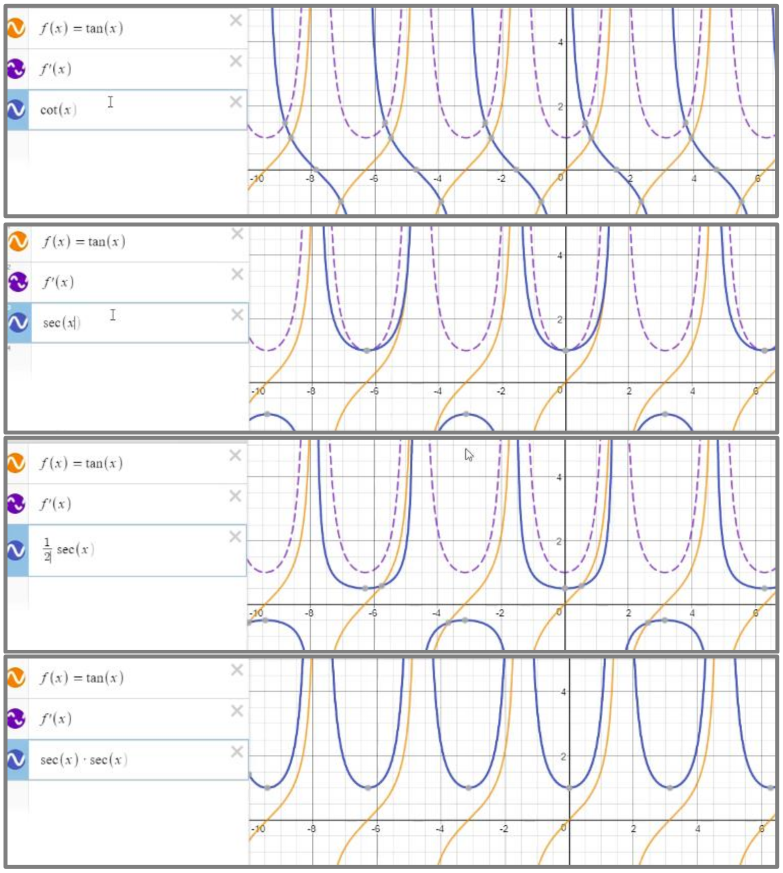

Figure 23. Springer's Desmos textual commands projected from computer to the classroom wall. The dashed line is the target for a cognitive guess.

Springer Vignette 4: Pedagogical Opportunities

Unlike Springer's other lesson vignettes, this one utilized Desmos and graphical representations singly aligning multiple elements in Pierce and Stacey's (2010) pedagogical taxonomy. Springer re-balanced emphasis to concept building, manifested an efficient perspective to develop an overview, continued to shift authority to the CAS, and appropriated a teaching task of representation linking graphical models to algebraic models as annotated in Table 17.

Table 17

Springer Lesson Vignette 4

P-Map	Evidence
S2	Re-sequencing - visual representation first; algebraic proof second
S3	Introduce derivation of trigonometric function from a visual perspective to develop deeper understanding
C2	Graph of $f'(x)$ created an authority shift
T5	Representation links graphical to algebraic models

Re-balance emphasis on skills, concepts, and applications (S2). A re-sequencing strategy altered the balance between conceptualization and skills development. Springer deliberately chose a visual representation of a derivative with a cognitive guess-and-check strategy to prove the derivations of trigonometric functions. "Here's the thing, to derive sine and cosine, okay, we would have to go back to the definition of the derivative and limits. Which I think I'm actually going to save for next class" (Springer, Lesson, November 9, 2017). This reorganization to graphical representation first is sometimes used; however, the approach with making a targeted guess and verifying against Desmos' $f'(x)$ command was a new approach for Springer (2:58).

Build metacognition and overview (S3). Springer chose a teaching approach based on the functional availability of Desmos graphical applications as a way for students to intuit the concept of trigonometric derivatives. "I decided to take the 'graphical' approach first to introduce trig derivatives because the derivatives of sine/cosine are somewhat intuitive – especially that *d/dx* of sine is cosine – and we could easily confirm that with the $f'(x)$ graph on Desmos" (Springer, Written Reflection, November 30, 2017). She exercised that approach with sine, cosine, and tangent functions. However, the tangent was more complex and less intuitive than the others because of the squared function (5:15).

Change classroom didactic contract (C2). Through the graphical approach, she covertly detached herself as the authority on the derivative. It was through the ease of technology and insightful prompts from Springer that students shifted their acceptance of knowledge to the CAS outputs. The teacher was not the final word on the precision of cognitive guesses. She allowed the tool with graphical representations to confirm or deny that precise answer (3:54; 4:43; 6:05).

Link representations (T5). Springer used the task of student trial and error to link representations from the graphical model of derivative to the algebraic equation. Springer described this in a written artifact, "[This] provides a multiple representation approach; i.e., the idea that we can represent a function numerically, graphically, algebraically, verbally" (Springer, Written Reflection, November 30, 2017). However, it was observed in this lesson that she merely linked the two representations, graphic and algebraic (3:54; 4:19; 5:15). The cosine and tangent examples captured incorrect algebraic guesses for graphical verification purposes (4:19; 5:15; 5:18; 5:25). This modeled for students how to revise those cognitive guesses. Additionally, she assembled a reverse approach; the graph was provided and recognition led to the algebraic model.

Springer Vignette 5: Applications that Involve Trigonometric Functions

This vignette followed the introduction to trigonometric functions derivatives in Vignette 4, but shifted to an application emphasis. Springer wanted students to practice using the derivative of trigonometric functions with calculus applications of finding horizontal tangent lines and finding equations of tangent lines on curves. The teaching approach was traditional in that the teacher was leading the students through examples and manipulating the technological tools and displaying those through a projector onto the wall. However, the manipulations on the TI-Nspire™ were quite complicated. At several points in the lesson Springer paused, re-explained, and answered specific questions about syntax issues particular to the device.

The next few paragraphs showcase Springer's utilization of some of CAS' most powerful characteristics, the *solve* command and the *such that* command, which restricted domains for functions. These commands were embedded within the calculus lesson. Springer described how CAS was executed in the lesson.

> We were using the Npsire CAS to solve various equations, looking for horizontal tangents, and we used the *solve* command, *such that*, x is between 0 and 2π, to just make our answers more friendlier. We were also using it to evaluate for certain values. (Springer, Interview, December 6, 2017)

The following transcriptions from critical moments in the lesson and screenshots from Springer's TI-Nspire™ and digital notebook assist the understanding of vignette components.

The first example Springer worked through in class portrayed minimal complexity of use, so the transcription begins at example two with the warning, "It's going to get a teeny bit harder. So we want to differentiate this function secant x over one plus tangent x. And for what values of x does the graph have a horizontal tangent?" (Springer, Lesson, November 9, 2017). Springer talked through the procedures that answer the question. First, the derivative was needed and required using the quotient rule that was learned in a previous' lesson. Second, the derivative was set equal to zero and solved, using a CAS command. She explained that those solutions were the places where the graph had horizontal tangents.

Springer directed students to follow her lead to perform by-hand calculations for the derivative and then to outsource solving the equation to the CAS utilizing the *solve* command. Springer expected students to use their knowledge of trigonometric derivatives from the previous lesson. The work product in Figure 24 shows the quotient rule being applied to the function. Springer wrote the actual TI-Nspire™ commands into her digital notebook prior to keying into the device.

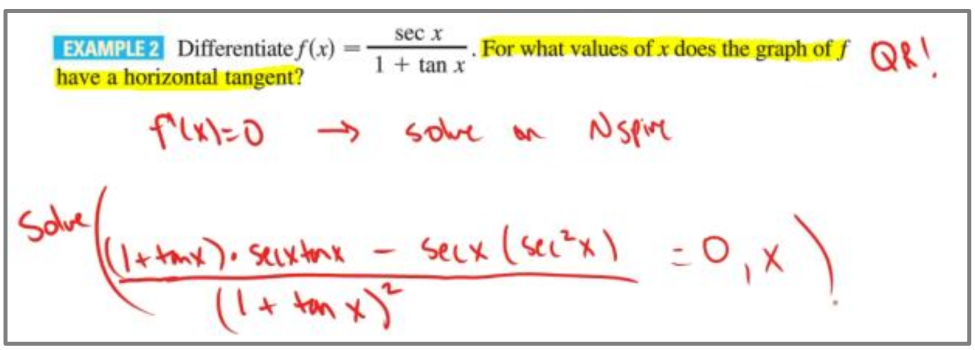

Figure 24. Springer's digital notes projected from computer to the classroom wall.

13:05 Now, so what we want to do is we want to solve this giant thing in our Nspire. We want to *solve* equal to zero comma *x*.

13:18 So now it's going to get a little hard to type this in because the Nspire doesn't like the this little two up here [the square on the sec *x*], so I'll show you how we're going to do that. You need a whole bunch of parentheses. (Springer set up the computer to key the TI-Nspire™ commands.)

13:59 Alright, so parentheses one plus tan *x*, secant *x*, times tangent *x*.

14:09 You have to put in that little dot when you're multiplying two functions on the Nspire. We have one plus tangent *x*, and wait the one plus tangent *x* has to be in its own parentheses, right, cause that's a quantity.

14:24 Minus and then the easiest way to do secant *x* times secant squared *x*, is going to be secant *x* times secant *x* times secant *x*. Just write it as a series of a product.

14:37 Isn't that secant squared *x* right there? (Student asks why she doesn't use a third power.)

14:43 You can. You're just going to have to do a giant parenthesis and a three [for the exponent].

14:48 Like it doesn't do the little one [$\sec^2 x$].

14:50 And then you have one plus tangent of *x* squared equals zero comma *x*. (Springer, Lesson, November 9, 2017)

The resultant output initiated new syntax issues, organized here in order of complexity. First, the output introduced the caution symbol in the front of the command as seen in the first line of Figure 25. Second, a new variable *n* appeared in the output expression due to the nature of a periodic function. This required Springer to stop and explain in further detail. She also showed students a work-around approach for the original command using the *such that* symbol (a vertical bar) in the CAS.

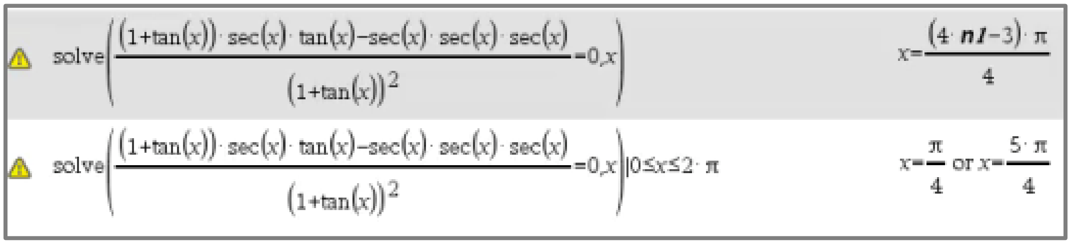

Figure 25. Springer's TI-Nspire™ textual commands projected from computer to the classroom wall.

Regarding the first syntax issue, the TI-Nspire™ always puts a triangle caution symbol next to the input expression with a message to remind the user of potential mathematical problems. In this case, since the input had a denominator of $(1 + \tan x)^2$ there was the potential that tan x could equal negative one. That would make the entire denominator zero, which in turn, caused the function to be undefined. A student in the class did not quite understand the reason for its presence, so Springer provided the explanation.

17:34 (Student asked, "Why is the caution symbol showing?")

17:37 It [triangle symbol] just has to do with the denominator and the domain. Just letting us know in terms of we can't divide by zero and there's going to be certain trig values that would make that denominator zero. (Springer, Lesson, November 9, 2017)

The other syntax error with the output given as a mathematical expression in terms of n involved an explanation of the output display: what it meant and how to resolve the issue on the CAS. Figure 25 showed the initial problem on the first line and the modified input on the second line. That change provided a simplified output with the values of the periodic function limited from zero to 2 *pi*.

15:21 We need to interpret that a little bit and we're going to fix our command. So do you remember from last year in precal when you were studying the unit circle? If you were taking, for example, the sin ($\pi/4$) you could go all the way around, and I'll be the same thing as let's say at sin ($9\pi/4$)? (Students talking to Springer.)

15:38 Right, so maybe sometimes your teacher just had you do like $2n\pi$.

15:44 Do you remember that? So the calculator is giving it with *n*'s. So in order for us to see, let's say, just the answers from 0 to 2π, here's what you're going to do.

15:57 Go ahead grab this back (copying the original command typed into the CAS) and we're going to do the vertical line [such that command] and what we're telling the calculator now is, 'solve this such that we're only between 0 and 2π.'

16:42 I want x to be wedged in between 0 and 2π. (Springer, Lesson, November 9, 2017)

Springer wrote the two solutions of $\pi/4$ and $5\pi/4$ for the horizontal tangents of the function in her digital notebook. She also took a screenshot of the CAS commands and pasted that into her notebook. In the background, students were still asking her the same questions about limiting the domain on the input for another three minutes. She added this explanation "The calculator is only as good as what we ask it, right? It has no idea if you want this general answer . . . We probably just think zero to two *pi*, but the calculator says there's infinite answers" (Springer, Lesson, November 9, 2017).

A little later in the lesson on a different problem, an interesting conversation between Springer and a student ensued. The student was confused between the *solve* and *such that* commands. Springer responded, "*Solve*, is solving for an x value. Like, think about factoring. You're solving for an x value. Evaluate means you are plugging in a value for the *x*" (Springer, Lesson, November 9, 2017). It was unclear if this answer satisfied the student, but it ended that exchange.

Pondering the exchange, it is noted that the procedure was different on the two examples and so the student may have memorized a procedure, void of cognition in problem solving. The mathematics problem asked for the equation of the tangent line to the curve at a particular *x*-coordinate. It was different from previous examples, in which the question inquired about *x*-coordinates at places on the curve that would have a horizontal tangent line. Students used the *solve* command to find those points, setting the slope equal to zero. The work-around of using the *such that* command restricted the domain and this eliminated the *n1* from the output. On this problem, however, students were finding the slope at a designated point. The derivative was computed first by-hand, followed by the *such that* command to evaluate the derivative at the point. Again, it was unclear if the answer Springer provided satisfied the student's inquiry.

Springer Vignette 5: Pedagogical Opportunities

The complex utilization of the CAS and demonstration of how CAS affected the content taught is outlined in Table 15. The syntax provided both a benefit and a constraint to learning applications of trigonometric functions in calculus. Springer took advantage of classroom level opportunities to enhance the interpersonal exchange with students as they used devices more fluently, even with the more complicated CAS procedures of this lesson. It was those symbolic algebra tasks that allowed Springer and her students to access the full power of CAS.

Table 18

Springer Lesson Vignette 5

P-Map	Evidence
S1	Caution symbol about the domain values for the function and output contained confusing syntax by introducing another variable for the periodic function
S2	Outsourced equation solving to the CAS
S3	Links concept of derivative, slope, and tangent line through symbolic manipulation of *solve* and *such that* commands
C1	Social contract teacher as part of the team in learning acquisition
C2	Allows the authority of the CAS to solve equations to provide accurate answers

Exploiting contrast of ideal and machine mathematics (S1). Springer exploited the contrast of the machine to human cognition in two distinct ways: a caution symbol alerted the user to a potential mathematical error, and the output of a periodic function introduced the variable *n1* for infinite solutions (see Figure 25). Students promptly recognized the alert with the caution symbol on the TI-Nspire™. Springer used the opportunity to explain potential mathematical domain errors (17:37).

The second issue appeared more problematic when the machine created the new function $x = ((4n1 - 3) \bullet \pi)/4$. Again, this output accommodated discussion about periodic functions and also initiated the opportunity to adjust syntax on the input to attain a favorable output (15:21). Springer described how to resolve this output in the post interview.

> For trig functions, we need to "restrict" the domain $[0, 2\pi)$ to come up with more "friendly" solutions that we can easily recognize and interpret. I actually like how it gives the $+ 2n\pi$ or $+ n\pi$ solutions to remind students that trig functions are

> periodic and infinite solution(s). But then I like that we can restrict the domain and see the "friendlier" answer too. (Springer, Written Reflection, November 30, 2017)

Students struggled more deeply with this issue, and it created confusion for a substantial length of time as observed in the lesson transcription.

Re-balance emphasis on skills, concepts, and applications (S2). The mathematics concepts were manageable by outsourcing the solving of complicated trigonometric functions to the CAS (13:05). "I didn't want to spend 20 minutes doing this algebraic computation because sometimes I think that I lose them" (Springer, Interview, December 6, 2017). However, the input of these equations had their own difficulties and Springer had to teach the tool in the midst of the example.

Build metacognition and overview (S3). Negotiating the CAS command *solve* allowed cognitive work to shift to the concept of the first derivative as the slope of the tangent line to the curve (13:05). "I'm just trying to get them to learn about these calculus concepts . . . I'm not having them master it" (Springer, Interview, December 6, 2017). Springer facilitated the connections to the application problems in an efficient manner by utilizing CAS.

Change classroom social dynamics (C1). Students advanced their connections to learning through the CAS as evidenced in their questions to Springer during the lesson and in student peer-to-peer interactions. Attempting to seek understanding, they questioned Springer's inputs and outputs. Springer fostered that classroom climate, "I try to create a dynamic where we are all working together; we are all a team. We're all studying this together" (Springer, Interview, December 6, 2017). She valued modeling the technical procedures of the device and how that transferred to student ability.

> I think it's important that I show them . . . I'm specifically showing them the commands on the screen. Like I'm typing it. So I think that is important, having that projected onto the screen and having me type it out and then also having them practice it on their own. (Springer, Interview, December 6, 2017)

Student observation of the teacher demonstration, student action on a personal device, and inquiry contributed to that climate of working together.

As well, peer interaction benefitted student learning according to Springer when she reflected on her lesson. "They're getting . . . a syntax error or it's not quite working out, their neighbor will look over and say, 'You're missing a comma or a parenthesis or you missed an exponent' or . . . whatever may be" (Springer, Interview, December 6, 2017). Springer described how students supported each other in the technical dimension of CAS.

Change classroom didactic contract (C2). The actuality of deploying equation solving to the CAS essentially gave confidence to the student in the capability of CAS. Springer talked about how students accessed other forms of CAS, yet, she also recognized that her students developed familiarty with the TI-Nspire™. As a result, they relied more on the TI-Nspire™ as a tool than other resources.

> I think there is a lot of kids using the Nspire in my class more than like Wolfram Alpha because . . . they get comfortable with it. Like it becomes [a resource] . . . They are just so used to the SOLVE command or so used to the vertical bar. Like

> it becomes so comfortable and natural and normal that they then don't go to these other sites. (Springer, Interview, December 6, 2017)

Springer still required students to solve equations for the application problems but relieved students of the expectation to perform a pen-and-paper skill. She assured that through appropriate commands, the solve feature portrayed precise answers. This authority shift Pierce and Stacey (2010) described as a change in the didactic contract.

Springer Case Analysis

The lessons, interviews, and written artifacts collected contained a wealth of information regarding Springer's pedagogical decisions and related instructional practice. The data that was specifically targeted was her usage and orientation of lessons that involved CAS. Many other elements were noticeable in her teaching practice; however, this study depicted just those elements of CAS utilization. The case analysis reported a generalization for her motivations to use CAS, an aggregated detail of elements that were identified using the P-Map framework (Pierce & Stacey, 2010), and several emergent themes about her instructional practices involving CAS.

Evidence showed that Springer implemented CAS in her pedagogy for several reasons: students enjoyed CAS utilization, CAS was a catalyst to get students engaged, and knowledge of such a tool was more realistic for life after formal education. Springer thoroughly enjoyed teaching with CAS and deriving new approaches to content knowledge through technology, in part because she felt her students enjoyed it. "I love that class. We can just have fun with the material, have fun with the technology, use it when it's appropriate" (Springer, Interview, December 6, 2017). Springer talked about a relaxed atmosphere because the students were non-AP and that reduced the time pressure in covering breadth of content. She reflected on her first experiences in teaching with CAS. She described her accelerated precalculus class from four years prior as a group of very energetic students. “They had never even seen it or heard about [CAS]. And I was able to show them how cool it was. They fell in love with it” (Springer, Interview, November 8, 2017). Her introspection may have been the beginning of a shift in belief that has continued to affect her teaching craft. Springer also sensed that mathematics could be too laborious for learners. “How do I get other kids to think that math is fun? Take out this tedious aspect” (Springer, Interview, October 15, 2017). CAS was acknowledged as a motivational tool for Springer.

Springer communicated that using technology had more real-life application. “In the real world [students know they are] going to have calculators and technology and so they enjoy using it to help them solve the problems” (Springer, Interview, October 15, 2017). One of her rationales for advocating CAS in teaching practice materialized in the last interview.

> [Students] can figure out other technology. So then you kind of have to embrace it and say, ‘Look, if they're going to use it, let me show them, you know how I would like them to use it,’ if that makes sense. . . . Why don't you kind of guide them, you know, to use it, how you'd want them to use it? . . . The more content you know, sometimes the more stuff you can do on CAS. (Springer, Interview, December 6, 2017)

Teaching to use the device to assist in learning was not due to real-world problems exactly. Springer was referring to the use of technology in the livelihood of business and careers.

Her strong mathematical background combined with an advanced degree in instructional technology contributed to her motivation of varied uses of CAS.

> I got a very strong math foundation and higher, upper level math courses at college. But also the background of education, philosophy, pedagogy, best practices . . . I wanted to do more that would help my classroom. I felt . . . that technology is the future. You know how can I learn different technologies? . . . Because it interests me, that was what I pursued [as a master's degree] and then I'm able to bring it to the classroom. . . . I'm kind of thinking strategically, how can the technology help just a little piece. (Springer, Interview, November 6, 2018)

This idea of strategically using technology to advance student learning was evident in her calculus class. Springer often conducted direct instruction with her computer projected on the screen and expected students to follow her lead in the lessons. "I can project my Nspire, you saw. The kids see exactly what I'm typing and we weren't able to do that before" (Springer, Interview, October 15, 2017). She paused at times during her lessons to give students the opportunity to ask questions, key commands into the CAS, or to ask a fellow student a clarifying question.

P-Map

The lesson vignettes spanned over eight weeks of time and captured selected instruction Springer qualified as lessons utilizing CAS. Writing artifacts and interviews support the decisions that Springer made in her pedagogy. Two different calculus classes provided the data of teacher instructional activity. The classes were not distinguished from one another in lesson descriptions. The five lesson vignettes were pattern matched

(Yin, 2009) in a deductive manner to the P-Map (Pierce & Stacey, 2010). Identification of pedagogical opportunities assisted in understanding the instruction that occurred in the participant's classroom as summarized in Table 19. Total number of occurrences for each segment of the P-map is provided to help understand frequencies of pedagogies observed. It is not the case that a higher number indicates a greater value of instruction. It merely helped to identify the more pronounced areas that Springer leveraged. The discussion that follows clusters the opportunities in the three levels of the subject, the classroom organization, and the student tasks.

Table 19

Springer Lesson Vignettes Summarized: The Occurrences of P-Map Opportunities that were Exploited During the Lesson Grouped by Subject, Classroom, and Tasks

P-Map	Vign 1	Vign 2	Vign 3	Vign 4	Vign 5	Total
S1		✓	✓		✓	3
S2		✓	✓	✓	✓	4
S3	✓	✓	✓	✓	✓	5
C1					✓	1
C2			✓	✓	✓	3
T1	✓		✓			2
T2						0
T3	✓	✓	✓			3
T4						0
T5	✓	✓		✓		4

Subject. Springer embraced CAS as a cognitive tool in her classroom as an essential resource to consider subject level opportunities in noteworthy ways. First, she exploited the contrast of ideal and machine mathematics to provoke student understandings of mathematical content and to elicit the need for understanding CAS commands that can provide feasible outputs. Second, she re-balanced and re-sequenced the presentation of mathematical content to reduce cognitive workload devoted to procedures and eased the capability for students to construct conceptual knowledge. Third, she used entry points on the CAS to augment students' metacognition and reflection on the content. These points are discussed in the next sections.

Exploiting contrast of ideal and machine mathematics (S1). The CAS machine produced unexpected outputs that needed consideration of connectedness to mathematical content. Several places in the lesson vignettes (Vignettes 2, 3, and 5) Springer exhibited flexibility in directing discussion regarding the output. She connected CAS error codes or caution symbols to bring understanding to the limitations of the output (e.g., potential domain errors, syntax errors, or unrecognizable answers). Also, CAS commands were recalled from a previous lesson or newly introduced to provide a suitable output as needed.

Vignette 5 had a moment in which the CAS gave an additional variable in the output (i.e., $(4 \cdot n1 - 3) \cdot \pi/4$). This was due to the nature of periodic functions and multiple solutions. Springer used the prospect to discuss numerous solutions for trigonometric functions by recalling the unit circle from students' precalculus class. The redirect connected prior mathematics knowledge to its relation with the new knowledge of derivatives that retain the periodic function. Typically, with trigonometric functions, students learn one answer and have to extrapolate the multiple solutions. In this situation the learners were reducing the multiple answers from the periodic form by restricting the domain. She accomplished this by using CAS commands that restricted the domain to $[0, 2\pi)$. Springer focused these unplanned moments as opportunities to build mathematical and technological connections.

Re-balance emphasis on skills, concepts, and applications (S2). Springer created lessons that involved a reversed sequence of events and relied on CAS to reduce by-hand procedures, thereby promoting concept development and mathematical application. Springer generated several lessons (Vignette 2, 3, and 4) with an approach that began with CAS to perform mathematical procedures, thereby allowing students to reflect on the results in the formation of mathematical definitions and conceptions (Vignettes 2, 3, and 4). In other situations (Vignettes 2, 3, 4, and 5), she outsourced mathematical procedures and forged through to teaching concepts or solving mathematical problems. Springer accessed the potential of CAS to compose inventive questions (Vignette 3) for students to hypothesize about mathematical patterns.

Build metacognition and overview (S3). Springer's pedagogy intermittently included CAS as a tool that provided structure for students to intuit and construct knowledge towards the calculus course goals (Vignettes 1, 2, 3, and 5). For example, the definition of derivative was a concept that was revisited in multiple lessons. In Vignette 1 Springer introduced the CAS command *define* to evaluate points close to another point, beginning the process for students to infer an idea about limits. CAS allowed her to use function notation with the definition of slope to illuminate the definition of derivative. In Vignette 2 the difference quotient was again used with the *define* command to explore concepts of continuity and differentiability. Even in Vignette 3, Springer was still using the *define* command with a difference quotient and limit for derivative. In a post-interview Springer explicitly stated that she was delaying the CAS *derivative* command until she saturated the thought processes of the difference quotient. Springer exploited the opportunity to continue with the definition because of the ease of the *define* command

and simple computation on the CAS. In this way, she provided her students the opportunity to develop a stronger connection to the definition of derivative.

Classroom Organization (C1 and C2). The largest distinction in the classroom level of the P-Map when CAS was utilized was the authority shift from Springer as the chief source of mathematical knowledge to CAS as an external mathematical consultant, observed in later Vignettes 3, 4, and 5. Springer relied on the CAS as the authority of symbolic manipulation to build student conceptions of the power rule for derivatives (Vignette 3). She directed students to use the outputs to develop a prototype and then verify a targeted guess using the CAS. This precedence created a gateway for students to use CAS for verification in other situations. It was observed in a similar way with Desmos in Vignette 4, as students predicted and checked their cognitive guess of trigonometric derivatives. Additionally, CAS power was accessed with the *solve* command in Vignette 5. Gradually, students accepted the shift in authority to the CAS as a tool in the learning processes and as a mathematical consultant that possesses great accuracy.

Less evidence was available about a change in classroom social dynamics (P-Map C1) due to CAS' presence. Springer discussed the social interactions in her interview, but it remained uncertain if student collaboration was improved due to CAS. During the first phase of the project Springer stated, "I would say at least in my Calculus class the kids are a lot more engaged when they're using CAS" (Springer, October 6, 2017). Springer responded to a question regarding differences in the presence of CAS in either student-to-student or student-to-teacher interaction.

> The kids are definitely working together, collaborating, you know somebody's like, "Why isn't this working? You missed a parenthesis. You spelled define wrong. . . . You missed a comma” . . . They kind of work together and are like saying, "Why isn't this working? Why am I not getting this?" We do a lot of stuff where on every problem I'll say, "Start working on this problem and look on your neighbor's computer (TI-Nspire™ was on the computer), make sure you guys are getting the same type of stuff" . . . They are definitely working together, but I would say that, whether it's paper and pencil or it's CAS. So that's just the environment of the class. (Springer, Interview, October 6, 2017)

This prompted the following question: “Would you say their questions then towards one another are really about, like, the syntax, which are just your input into the CAS that their questions are? Or are there questions that they have about content?” (Interview, October 6, 2017).

> I think it depends on what we're doing. If we're doing notes, where I could have presented something to them and I'm kind of walking through it, then it's a little bit more about syntax . . . When we start doing like related rates and optimization, we start doing more of these harder word problems. . . . Then they start trying to get creative and start talking a little bit more content. (Springer, Interview, October 6, 2017)

No lesson was ever observed regarding these types of word problems. Classroom social dynamics were not revealing of any uncommon opportunities in the presence of CAS. Yet, it was obvious that students relied on one another for learning the syntax of the tool.

Tasks. Springer accessed the functional opportunities of the CAS to help students learn pen-and-paper skills, to allow the exploration of regularity in algebraic functions and to link multiple representations for a deeper connection to the mathematics. She designed her lessons intentionally to incorporate technology. "I am always trying to find new, innovative ways to use technology in my math classroom– and I think CAS is the most helpful and efficient and powerful tool that the [students in our school] have easy access to" (Springer, Written Reflection, November, 30, 2017). It was routine for Springer to use the TI-Nspire™ for symbolic manipulations and to use the Desmos graphing calculator to connect either the initial prompt or some other resultant to a graphical representation.

Pen-and-paper skills (T1). Evidence of by-hand procedures reflected the value that Springer placed on students maintaining procedural fluency in mathematics. "I do feel that I'm doing [students in our school] a disservice if they don't know how to do certain problems by hand" (Springer, Interview, October 2, 2016). A problem in Vignette 1 required a simplification of a rational expression. Springer modeled the by-hand procedure prior to entering into the CAS. The CAS became a verification tool.

> I try to find this balance of us still doing procedural stuff. . . . And so I try to do a ton of that, like the first semester. And then by second semester we start picking up CAS a lot more because I feel like they have this good foundation of whatever it is I wanted them to have. (Springer, Interview, October 2, 2017)

On one occasion in Vignette 3 she used the CAS to first view the solution and then to work towards the answer through reasoning out a numeric pattern.

> Students were essentially deriving . . . the differentiation technique known as *The Power Rule*. I used a verbal exchange with me and the students [*sic*] as my formative assessment by asking the students to identify/explain the pattern/shortcut and then successfully calculate derivatives (without using the definition of the derivative and CAS). (Springer, Written Reflection, November 4, 2017)

After students had established the power rule they used pen-and-paper methods to work through an exercise set using procedural mental skills and then verified those answers with the CAS.

Explore regularity and variation (T3). Three different ways Springer was observed using CAS with regularity and variation are listed. First, she used multiple points and repetitive computation with the CAS to instill the conceptual idea of limit in Vignette 1. Second, she used three examples with various function characteristics to note places of continuity and differentiability in Vignette 2. Third, she evaluated each function at numerous input values to guide the learner to consider the functions behavior also in Vignette 2. The three situations drawn from vignettes, written reflections, and interviews provide the detail.

Overview of rationale. The lesson Power Rule and Higher Derivatives (Vignette 3) included an exploratory activity in which the output from the CAS was generated for the purpose of identifying patterns in numerical values. "Because of CAS, I am able to ask students to use the definition of the derivative to find MANY, MANY, MANY derivatives and it is not a tedious boring request anymore since they are not calculating them by hand" (Springer, Written Reflection, November 4, 2017). Springer valued the functional capabilities of the CAS for its efficiency.

Multiple points and repetition. Springer generated multiple points very close to a selected point to instill the idea of a *delta* value as was used in the definition of limit (Vignette 1). The CAS permitted her to calculate efficiently the slope of multiple secant lines as *delta* values approached small differences from the point. Springer reflected on a positive change in her teaching pedagogy as result of CAS utilization.

> I would be lazy and only do a couple points because like it was so tedious before. . . . You wanted to do it, but you couldn't really do it. . . . For a limit you want to see five, six, or ten points. You can do it so quickly on the Nspire. (Springer, Interview, October 15, 2017)

She used the points computed to calculate the slope of secant lines to help students notice the limiting value of the slope (i.e., the derivative). It was the repetition of evaluation of the derivative at particular points that guided students to think about creating a *shortcut* rule. Springer talked about students' recognition for the ability to bypass the multiple calculations.

> When we were doing tangent lines by *defining* the function and then doing like $f(0.9)$, $f(0.99)$, $f(0.999)$, you know we were having those conversations with, why are we doing that? Talking about the limit. Talk about approaching it. And then the kids were like, "What? Can't we just jump to this 0.999, cause that's going to give us the closer value?" (Springer, Interview, October 6, 2017)

Through repetition students had the opportunity to develop their mathematical conceptualization of limiting values.

Multiple examples. Springer was quickly able to calculate derivatives for three separate functions and also able to evaluate at multiple points on those functions' derivatives through the functional capability of the CAS. This enabled her students to view the similarities and differences between the functions and their derivatives. As Springer compared the three functions side-by-side, students were equipped to make connections to the type of function and the places at which the function was continuous, and likewise, where it was differentiable.

Generated multiple outputs. In Vignette 3 the goal required focused attention on the outputs from the CAS. Those expressions formed the basis for the exploratory task. "It was critical for me and the students to calculate multiple examples so they could identify a pattern with higher order derivatives" (Springer, Written Reflection, November 4, 2017). Those multiple examples were simply the first, second, third, etc. derivatives of the one function, a simple CAS computation. She explained that, in the past, she used multiple different functions and computed just the first derivative using the CAS *derivative* command; but she imagined the lesson unfolding in a new way for the current year.

> I stuck to one polynomial and one rational function for them to see the pattern. . . . I decided to take a different approach this year, where we are still using the definition of derivative. I'm really trying to hone in on that. And have them really understand with the difference quotient and the limit, so instead of using the command [CAS *derivative*] I was making them use the definition of derivative. (Springer, Interview, November 8, 2017)

She maintained the use of the difference quotient and the limit. The higher derivatives allowed variation to explore the patterns that would soon be constructed into the *power rule* for derivatives.

Representations (T5). Springer often linked various representations of functions together. She talked about how much she loved using Desmos, so much that she would always have a tab open for it. "I definitely think that in general [multiple representations] can help you have a deeper understanding. So like [the students are] seeing the algebraic approach and then they're seeing graphically" (Springer, Interview, October 15, 2017). She demonstrated this use of multiple representations in Vignettes 1, 2, and 4. The first two vignettes illustrated how Springer performed the symbolic computation first and then compared the output to the graph. The fourth vignette was interesting because she showed the graphical representation of the function and its derivative first, and then students produced a cognitive guess strategy to arrive at the symbolic representation to verify against the graphical representation. "The Desmos graph of $f'(x)$ (hopefully) facilitated student understanding that the derivative of a certain trig function will be another trig function (without me explicitly telling them that information)" (Springer, Written Reflection, November 30, 2017).

Summary of P-Map. CAS was leveraged in Springer's pedagogy by allowing her to adjust areas at the subject level of the P-Map, giving opportunity to students to access an external mathematical authority, and enriching her instruction with tasks that utilize technology. Springer found satisfaction in her CAS-infused methodology, in part, because of her belief that students enjoyed learning mathematics with CAS.

Emergent Themes from Springer's Data

The P-Map illuminated components of teacher practice for instruction that was oriented when CAS was exploited in the development of mathematical knowledge. Specific themes emerged through the analysis of Springer's case that explained how Springer oriented her pedagogy. The insight prescribed support to instruction that utilized CAS. Springer's emergent themes can be catalogued as outsourcing procedures, providing guidance, verifying answers, regulating access, and viewing CAS as a mathematical consultant. Each theme that was palpable from the lesson was listed with the P-Map description following the vignette in Table 20. The detail for each theme and its evidence validates the selection. In some lesson vignettes, the evidence was identified within open coding of the lesson and referenced as *lesson* in the table rather than a P-Map code. The order in which they are presented has no relevance to a ranking of importance or strength of case. In principal, themes were points that overlapped and formed interlocking pieces of pedagogy and are conveyed in Figure 26.

Table 20

Emergent Themes Evidence: Springer

	Outsource Procedures	Provide Guidance	Verify Answers	Mathematical Consultant	Regulate Access
Vign 1	S3		T1, T5		Lesson
Vign 2	S2, S3	S1			
Vign 3	Lesson	S1		C2	
Vign 4		Lesson	Lesson	C2	
Vign 5	S2, S3			C2	Lesson

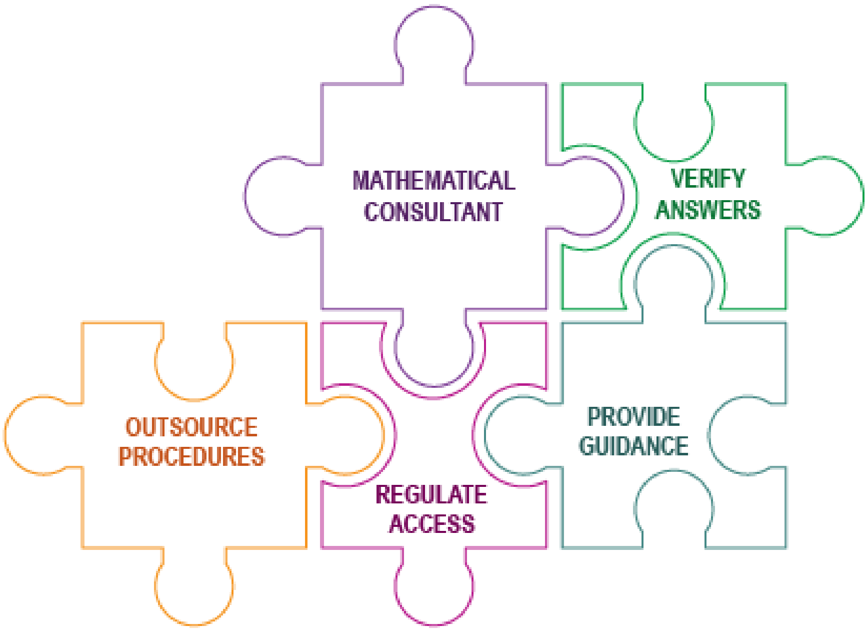

Figure 26. Emergent Themes Schema: Springer.

Outsource procedures. Oftentimes Springer allowed the CAS to do the work of complex computations (e.g., difference quotients for the derivative, adding rational expressions, restricting domains), referred to as *outsourcing procedures* since students' cognitive function only required knowledge of what and how to input the command. At times, Springer directed students to focus on particular mathematical concepts by outsourcing a procedure, hoping to reduce cognitive demand on working memory. "I see different uses that I try to use CAS and like I said, sometimes it's to help us with the algebra, so we can understand the concepts of calculus. And this next lesson we will use it to discover derivative rules with polynomials" (Springer, Interview, October 15, 2017). She was referring to Vignette 3 and the exploration of number patterns in higher derivatives. Springer used the power of CAS to compute many derivatives quickly and accurately for the purpose of critical analysis.

Multiple times Springer allowed CAS to perform symbolic calculations in lieu of traditional pen-and-paper computations (Vignettes 1, 2, 3 and 5), to reinforce procedural skills of by-hand computations (Vignettes 1 and 3), or to target other aspects of the mathematics (Vignette 5). The following examples provided additional detail. The final section discussed potential benefits of outsourcing procedures.

In lieu of traditional pen-and-paper computations. One of her favorite commands was *define* because she let the device retain a function in its memory to be called upon to evaluate the function at both numeric and symbolic points. When she performed these algebraic manipulations for a difference quotient (Vignettes 1, 2, 3, and 5), CAS did all the procedures whether or not they were complicated. This act permitted students to attend to the content rather than the mathematical procedure, departing from traditional pen-and-paper procedures.

Reinforce procedural skills. Other times she reinforced correct procedures through the use of CAS. Springer wrote out the procedure for derivative on the whiteboard (Vignette 1) and then confirmed accuracy on the CAS. After that example, she outsourced all future derivatives to the CAS in that lesson. Similarly, in Vignette 3 when students were learning the by-hand procedures of the power rule, she checked by-hand procedures against the CAS thus outsourcing procedures. The actions fortified student procedural knowledge of the by-hand skill of students either taking a limit of a difference quotient or finding a derivative by means of the power rule.

Target other aspects of mathematics. Springer targeted mathematical problem solving by outsourcing mathematical procedures to the CAS using the *solve* command in Vignette 5. The application problems involved trigonometric function derivatives that required the product, quotient, and chain rules. Springer instructed her students to let the device do all the work internally.

> It's so much algebra computation. . . . As much as I love doing the definition of the derivative and doing it on the Nspire, it was going to be like 20 minutes and then I felt like they were going to be exhausted and we wouldn't have even started practicing problems. I wanted to just entice them, interest them, and do something quick and easy. (Springer, Interview, December 6, 2017)

The lesson was directed towards applications and not the learning of the procedures.

Benefits to outsourcing procedures. This idea of using the calculator for symbolic manipulation and computation referred to as *outsourcing procedures* to the CAS may have its advantages. Springer talked about the benefits of functional opportunities that are possible because of CAS.

> The advantages are you can do [complicated], messy problems and it can be accurate. It's sometimes more efficient depending on what type of problems [we] are working on. So it helps you. Shasta refers to it as like, it does the muscle work for you. So you can more focus on the big ideas and let the CAS do that muscle work. (Springer, Interview, October 6, 2017)

CAS functional opportunities provided students with accuracy and efficiency for any type of problem, even very challenging procedures.

Provide guidance. With proper guidance students had the potential for success with CAS. Springer gave consideration to teaching the tool, creating an awareness of CAS capabilities, offering flexibility during instruction, and thoughtful preparation. The components are explained in the next sections.

Teaching the tool. Springer often adjusted her instruction to teach the CAS tool or to propose new CAS commands. She was careful to introduce new commands at the stage that it was necessary. "Sometimes I have to be in the moment to come up with these extra commands" (Springer, Interview, November 8, 2018). Her classroom climate supported students assisting one another with the device, particularly in terms of syntax. When confounding results displayed on the screen, Springer paused her instruction to talk about how to interpret those outputs. Flexibility during the lesson was imperative to Springer to support student challenges with CAS. She always assessed the situation to improve student knowledge of the tool in addition to honing in on content-specific questions.

> Just using CAS in your classroom isn't great. You have to, kids need some guidance, they need some help and they can't be stressed out about it. And like they need to want to use CAS and to learn from the CAS. The teacher has to be more on-board and more showing them how to use the CAS. (Springer, Interview, November 8, 2017)

Her outlook was to teach the tool by easing in new commands and also to accept the class time involved in creating a smooth transition for students.

CAS capabilities. In interviews and informal conversations, it was evident that Springer was concerned about raising awareness of the CAS capabilities to her students. It was also observed that her personal experience with technological tools had depth of knowledge, manipulating the device with ease and speed. She clearly wanted her students to reach a more advanced level on the use of CAS without becoming too frustrated.

> I say to them, you are speaking the computer language. You have to know when to do a parenthesis and when to do a comma and I taught them the vertical bar, such that, like they had never even see that. I think they find that part of it cool. . . . They have all these different commands that they can ask the calculator to walk us through the steps. (Springer, Interview, October 15, 2017)

Even when students got error messages on their output, she used those opportunities to help students persist by looking at the details of their input. "They know they're getting this error and they know that they need to correct the error. They know that there's something that is needed to fix their input" (Springer, Interview, October 15, 2017). Her patience and guidance back to the CAS input progressed students' technical abilities.

Flexibility. Springer demonstrated flexibility to teach the tool at various times in her lessons (Vignettes 2, 3, and 4). Twice in Vignette 2 she adopted positive reactions when the output was unexpected: first, with an output of infinity for an indeterminate form; and second, when the command *comDenom* was required to get the derivative. "That's kind of cool. Do you see how it came up with infinity? Yeah. What it's actually doing is, it's giving you the fact that there's an asymptote, right?" (5:12). She provided guidance with the infinity symbol by comparing the output to the graph. When *comDenom* was needed to get a feasible output, she simply stated, "We've got to do our *comDenom*, right?" (4:36). She modeled how students can use the output to interpret meaning and build mathematical connections.

Springer also showed flexibility during instruction in Vignette 4, promoting students to think about the derivative of the tangent function. In this instance, Springer provided scaffolding of cognitive guesses due to student struggles with content. The trigonometric derivative was not obvious. She provided suggestions to students that were variations of other trigonometric functions with the hope that student guesses would be on target.

Preparation. When questioned about designing lessons that involve CAS, Springer responded, "I think about the commands that we have available and [then] how could that help us to do whatever we're doing" (Springer, Interview, October 2, 2017). The commands were shared just at that moment when they were needed. She coined this phrase, "I definitely let [teaching commands] happen organically as things come up" (Springer, Interview, December 6, 2017). She followed up with her rationale, "If it's not natural like then it's not going to stick with them." It was evident that she coordinated providing guidance on CAS with teaching mathematical content.

Verify answers. CAS was used as a tool repeatedly to corroborate answers: by-hand work products versus the CAS (Vignettes 1 and 3) or setting one representation against another (Vignettes 1 and 4). Springer modeled usage of CAS to her students in an effort to cultivate knowledge acquisition. I observed her purposively inputting incorrect answers to develop an idea of self-correcting (Vignette 4). She guided students to consider incorrect outputs and reflect on necessary changes to deliver accuracy. "It's important for students to enjoy the learning process and have success with learning the material to gain confidence with their math abilities and be confident in their abilities to learn future concepts" (Springer, Written Reflection, October 14, 2017). Springer believed that verification led to confidence.

Mathematical consultant. The authority shift from the teacher to the device was subtle. Springer demonstrated how CAS could provide instant feedback to the accuracy of pen-and-paper skills as students were learning the power rule in Vignette 3. Similarly, in Vignette 4 students were making cognitive guesses of the derivative of trigonometric functions and matching those to the graph. The teacher was not needed as the authority when the CAS was used as a resource. A slightly different sense of reliability was found in Vignette 5 when students used the *solve* command within an application problem. To utilize that feature, the individual had to recognize the accuracy and precision of the CAS tool. Springer was conveniently relying on its functional opportunity, and, hence, the students had to trust its capability as well.

Regulate access. Students accessed the CAS commands when Springer displayed them on her computer-projected screen. She realized that students might have accessed other TI-Nspire™ commands or other CAS platforms. However, her obligation was to teach how to use specific tools to advance mathematical understanding. She accomplished this through (a) direction of the command to utilize; (b) sequence of the order in which commands were accessed; (c) degree of difficulty of commands; and (d) permissions for using CAS in assignments and assessments. These are described in the sections that follow.

Direction. Repeatedly Springer directed students to use particular commands and to key them in while simultaneously observing her projections on the classroom wall. The class seemed surprised when a new command became available. This occurred in Vignette 1 when she first performed a difference quotient with the symbolic expression rather than numerical values and again in Vignette 4 when she showed the Desmos feature of graphing $f'(x)$.

Sequence. Springer set the order and placement of commands. When producing an equation for the tangent line to a curve in Vignette 1 she withheld a more efficient command.

> They don't know about the command for tangent line. Like there is a tangent button that will do it. So I make them do like all these steps, you know, so that they understand and see all the steps that go into it. Eventually they may figure that out. . . . I try not to show them that because I want to use the CAS but it's like I want them to use it as . . . like you're telling the CAS what to do and then it does it type [of] thing. (Springer, Interview, October 15, 2017)

Springer also delayed the derivative command (d/dx) opting to stay with the use of a difference quotient throughout all the vignettes. Although the use of d/dx was never observed, Springer talked about staying with the difference quotient after instructing in Vignette 3. "I'm really trying to hone in . . . and have [students] really understand the difference quotient and the limit, so instead of using the [derivative] command I was making them use the definition of derivative" (Springer, Interview, November 8, 2017). Springer postponed the derivative command until conceptualization was formed.

Degree of difficulty. There was varying degree of difficulty with CAS commands and she continued to motivate students to adapt to more challenging use of those commands. This was particularly apparent in Vignette 5 with the trigonometric application problems. The CAS line items were lengthier and resulted in complex outputs, those that had additional variables for a periodic function.

Permissions on assessments. Assessing student learning required consideration or alteration by the teacher, although this was not observed in the lesson vignettes.

Springer had different ways that she managed CAS work in assessments and assignments: She restricted access to CAS and changed question format. “There's definitely a combination of sometimes no CAS and CAS assessments” (Springer, Interview, October 2, 2017). Her expectations of work products had slight adjustments. She explained how she might word a question on a quiz.

> Set up the problem, you know, show what it is, and then go ahead and type it into the CAS. And then make me a little note that just says, used CAS, so I know where that came from. But there's typically . . . I don't want them just to give me an answer. (Springer, Interview, October 2, 2017)

However, it was evident that the type of questions in the presence of CAS was a challenge to develop.

> I do have to get a little bit more creative with the types of questions that we’re studying. . . . If you're going to be able to use the CAS on homework and assessments, how do I come up with other questions that are not, like plug and chug? (Springer, Interview, October 2, 2017)

Springer’s pedagogical content knowledge made it possible for her inventive questioning strategies. She thought deeply about what and how to instruct and assess using CAS through regulation of the features.

Summary of Springer

CAS was an essential tool for learning calculus in Springer’s classroom. Springer was inventive in her lesson design, altering the presentation sequence to bring about conceptualization of calculus concepts. She regulated access to CAS commands, revealing only those that she wanted students to access. Yet, she valued student exploration of TI-Nspire™ tools. She was aware that students might choose other web-based CAS tools, so she endorsed utilization in productive ways.

Springer remained flexible during instruction to call upon CAS commands that were needed to complete a computation, as those commands varied depending on the output. As well, she intermittently had to retrace procedures, as students may have gotten lost. Multiple examples presented explored patterns of regularity and differences. Oftentimes she outsourced procedures to the CAS. When those computations on the CAS got complicated students leaned on one another to manage the syntax of inputs. “We're doing notes, where I could have presented something to them and I'm kind of walking through it. Then [student collaboration is] a little bit more about syntax” (Springer, October 6, 2017).

Springer purposefully incorporated technology in her lessons. Her decisions were rooted in the functional opportunities of the CAS and belief that it made learning more enjoyable. “How can we study really hard concepts like calculus and topics in calculus that is a college level course and use the technology to help us understand it and create it? You know, make it enjoyable” (Springer, Interview, December 6, 2017). She began with content and thought of ways to use technology to more easily advance learning.

Themes that emerged from the analysis of data were outsourcing procedures, providing guidance, verifying answers, regulating access, and CAS as a mathematical consultant. They were presented in no particular order as they are interlocking pieces in their presence of Springer’s pedagogy.

The Case of Shasta

Shasta served as the Grandview mathematics department chair for Preschool-12 and secondary mathematics teacher. He had been at the school for five years, teaching secondary mathematics for 28 years. Shasta's educational background included the following degrees: Bachelor degrees in mathematics and philosophy; Master's degrees in education and mathematics; and Doctorate of Education. Personally, he had been using CAS and other technologies from his first days of teaching and continued usage during his nearly 30-year teaching career. Generally, he had taught high school precalculus, statistics, and calculus courses, but during the year of this study, he shifted to middle school to fill a vacant mathematics teacher role. Classes described in Shasta's lesson vignettes were eighth grade algebra one. He had previous experience teaching eighth-grade mathematics.

The second participant's case follows a similar layout as Springer's analysis. First is a description of each lesson vignette using transcriptions, images from notes, and screenshots of the details on CAS. The narrative was created from lesson observations and was supported with the participant's reflection post-observation. After each vignette is a calibration to the P-Map framework (Pierce & Stacey, 2010) to isolate particular aspects of Shasta's pedagogical practices. When all lesson vignettes are thoroughly explained, the individual case is aggregated through the use of the P-Map. Finally, Shasta's instructional methods revealed emerging themes, and in closing, they will be explained.

Shasta Vignette 1: Distributive Property and Combining Like Terms

Prior to the lesson, students had been making mistakes in the algebraic procedures of the distributive property and combining like terms. Shasta recognized the problem of different skill levels: some students had success and others continued to err. He saw the opportunity to use CAS to promote students independent checking of their work amidst the practice of similar math problems. Shasta provided a rationale to the class about why he chose to introduce the CAS on this particular lesson.

> 2:56 Everybody needs kind of a different level of understanding. So what I'm going to introduce you to, is the side of the TI-Nspire that lets you do the amount of practice that you need to do. The side of the TI-Nspire that I'm going to introduce to you is actually the next class of software that is called computer algebra systems, so CAS. (Shasta, Lesson, October 4, 2017)

Later in the lesson, he supported students learning through CAS as a tool to help each person progress at their own pace. "Think about this machine as your best mathematical non-judgmental friend that you will ever have. It doesn't care how much practice you need. It will keep practicing . . . with you until you just run out of time" (Shasta, Lesson, October 4, 2017).

The lesson was structured first to consider the distributive property and disclose how to check work and second, to combine like terms and confirm answers. Students verified the accuracy of their procedures by entering the original mathematical expression given, inputting an equal sign, and then entering their pen-and-paper answer. When the output was correct, the CAS output *true*. In the case of an incorrect answer, the CAS output the same input, possibly with a rearrangement of the terms. The students revised their answers and made another attempt as needed. Shasta selectively instructed the commands of CAS to his students. Parts of those instructions were shared along with screenshots that represent the account.

10:44 We're all going to type the top line of this. So negative three. (Pause as students keyed in the commands.)

10:55 Now as soon as I type in negative, I get two options (see Figure 27).

11:00 Am I going to minus something or am I negating something? I'm negating— I'm getting the negative of three. So choose negate and then type a three.

Figure 27. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

11:10 Do you know what to type next? Open parentheses, the left parenthesis. Something really cool happens. What did it automatically put in? (Shown in the first line of Figure 28. Student answered. Shasta repeated.) The other end.

11:22 Now type $x - 10$. When you press x, something really exciting happens.

11:29 It put a dot in-between the negative three and the parentheses because what's always assumed when you put anything side-by-side? Multiplication. (Shown in the second line of Figure 28.)

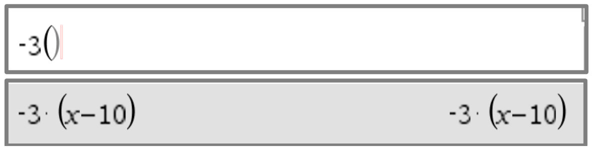

Figure 28. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

12:07 Have you ever typed something into a calculator and made a mistake, and you had to go back and retype the whole formula? That happens pretty frequently. (Inaudible)

12:18 If you hit up arrow, what happens? (Student: It selects it. It highlights it.) While it is highlighted, you could control C, copy it, and then move down and paste. There is actually an easier way to do it on the Nspire, press enter. (Student: Oh, cool.) Once it's highlighted you just hit enter and it will copy the whole line. (Shown in Figure 29.)

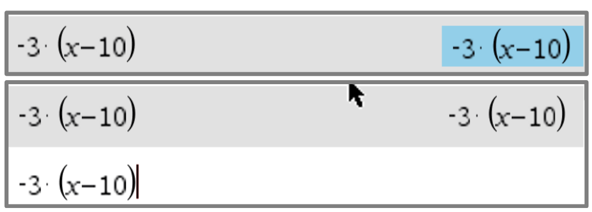

Figure 29. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

12:50 Imagine this was like a long computation, and you made one mistake. You just go grab the whole computation and drop it down and see what it turns into.

13:00 Now type equals. What was the first bad mistake that we made on this? (Shasta referring to the examples and work product the class has created earlier on the white board. Student answered -3x – 30.) Alright, I want you to type -3x – 30. Press enter. Anything exciting? No? (Shown in Figure 30, line 2.)

13:22 Alright, let's do the really wacky one [mistake]. Oh, how do I get that whole line copied? Up arrow, highlight, so now I'm just going to delete off that line [the end parts] and put x – 13.

13:37 Same thing happen? What did your CAS do? It just reprinted what you just typed? (Shown in Figure 30 line 3.)

13:50 I now want you to copy down one more time, delete off the x – 13. And now type the correct answer, -3x + 30 and press enter. (Students said true.)

14:04 So, what just happened? (Student: It discerned that it was correct. Shown in Figure 30, line 4.)

$-3\cdot(x-10)$	$-3\cdot(x-10)$
$-3\cdot(x-10)=-3\cdot x-30$	$-3\cdot(x-10)=-3\cdot x-30$
$-3\cdot(x-10)=-3\cdot x-13$	$-3\cdot(x-10)=-3\cdot x-13$
$-3\cdot(x-10)=-3\cdot x+30$	true

Figure 30. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

14:14 How do you know when you've actually distributed correctly? (Students answered.) It says true.

14:23 Is there ever a reason for anybody to walk back in here again and not know that their distribution problem is correct?

14:35 Could this be a really quick, easy check for you every time? (Shasta, Lesson, October 4, 2017)

Shasta presented Boolean logic CAS commands for the first time to this class. He talked systematically through each command, ensuring that students had the opportunity to type the commands, to view the output, and to interpret the results with analysis. He informed the students of his expectations to create and check individual work before returning to class with questions and also for learning by-hand procedures.

The second part of the lesson revolved around combining like terms with similar commands. However, this time the CAS performed the operations in spite of incorrect solutions. Shasta knew the potential problem and warned the students, "I'm about to do something dangerous on the Nspire – dangerous for your own warning" (Shasta, Lesson, October 4, 2017). Shasta was speaking in reference to the CAS combining like terms automatically, even when there was not a command keyed with the input.

14:44 Here is another problem that was a little funky for you guys. $x + 10 – 2x + 5$. We had surprisingly large number of people having issues combining like terms. (Shasta performed by-hand procedures; having written the problem, solution, and mistakes on the whiteboard.)

15:50 So how can we check to see whether this was actually correct? What do I type? (Students were giving Shasta suggestions to input in the CAS.) $x + 10 – 2x + 5$

16:08 Type out the problem, equals, what do we think it was? $-x + 15$ (Shasta pressed the enter key and reflected on the output as shown in Figure 31.)

16:15 We can combine like terms.

Figure 31. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

17:22 What would happen if, up here, instead of + 15, what if it was + 14? What is the machine going to tell me? (Student said false.) Did it tell me false up above when I made the mistake? What did it give me back? The equation? [Is it] alright if I try that? It just did something funky. (Student said it combined the one side as is shown in Figure 32.) It combined the left side.

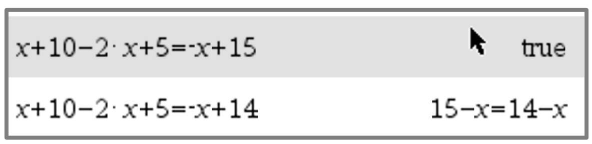

Figure 32. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

17:56 Is 15 – *x* what I have? What was my answer? - *x* + 15? What's it saying on the left side? 15 – *x*?

18:18 When you have addition, can you add the thing out of order? What's that called? What's it called when you can swap the order? (Students responded the commutative property.) (Shasta, Lesson, October 4, 2017)

Shasta reinforced the input and output on the CAS, solidifying the CAS procedure of re-ordering the terms. He honed in on the fact that CAS combined the like terms, even when computed with an incorrect by-hand computation. Finally, he was able to draw out students' vocabulary of commutative property and he showed an example of how that knowledge was applied.

Shasta was clear that he wanted to teach his students how to use CAS as a tool for checking answers primarily to improve accuracy through cognitive processing. He was aware of students' mistakes but also keen on the idea that learners do not recognize when they make a mistake. In addition to teaching students to learn pen-and-paper skills, he used student written work as a heuristic to engage students in the process of mathematical understanding.

> I make them commit in writing. And that is like a major piece of what I do before I'll talk about solutions to a class— before I'll do anything else. I make them commit to an answer. Now they can go back and erase it. I have no control over that piece, but what I'm trying to do is to get them to recognize in those moments, "Is there something going on that my written work doesn't align with what is coming out of the technology?" And it is just forcing them to slow down a minute so that they can actually give the cognitive side of their brain a chance to recognize it. (Shasta, Interview, October 4, 2017)

Shasta used CAS as a tool to heighten awareness of each step and he evoked learners to think, reflect, and understand. As part of the assignment, when a student made a mistake in their cognitive guess, he or she was directed to take a screenshot of the CAS display and paste into their notes for discussion at a later time.

Shasta Vignette 1: Pedagogical Opportunities

Shasta's lesson on the distributive property and combining like terms demonstrated utilization of CAS in the following areas: exploiting the contrast of ideal and machine mathematics; re-balancing skills and concepts; building metacognition;

adjusting the classroom didactic contract; exploring regularity and variation; learning pen-and-paper skills; and linking representations. The evidence summarized in Table 21 used pattern matching logic (Yin, 2009) with the data and the P-Map Framework (Pierce & Stacey, 2010). Descriptions from the vignette facilitated how these characteristics were demonstrated. Evidence of connections to P-Map will be cited with time stamps from the lesson vignette as appropriate.

Table 21

Shasta Lesson Vignette 1

P-Map	Evidence
S1	Contrasted expected with unexpected outputs with critical analysis of mistakes
S2	Structured re-teaching skills based on availability of CAS
S3	1. Foregoing the CAS command of combining like terms 2. Sequenced lesson to avoid revealing automaticity of combining like terms on the CAS until students learn the logic arguments
C2	CAS became an external mathematical consultant
T1	A cognitive guess was checked against the CAS output
T3	Student generated multiple examples to explore for accuracy

Exploiting contrast of ideal and machine mathematics (S1). The lesson was centered on the contrast of by-hand versus CAS computations. “All of the students walked out with an awareness that they have an ability to check their work” (Shasta, Interview, October 4, 2017). Shasta expressed the value of students’ knowledge of verification, but Shasta also valued the critical analysis of mistakes. “What this does, is it highlights a particular piece– makes you slow down and copy something and pay attention to where the mistake happened” (Shasta, Interview, October 4, 2017). The Boolean logic of the CAS was helpful in developing a connection of the individual mistake that a learner made by reflecting on the machine outputs (Time stamps 11:10, 11:29, 13:37, 13:50, 16:15, 17:22). In one instance, knowledge of the commutative property was required to interpret the output.

Re-balance emphasis on skills, concepts, and applications (S2). This lesson was re-teaching skills that students had not yet mastered. Shasta carefully presented the skill with a new action that was possible because of the availability of the CAS. Students worked to develop strong pen-and-paper procedures and were presented the opportunity to refine their conceptual understanding of algebraic structure. Students had different skill levels as they entered the classroom. The creativity of unrestricted examples in the lesson provided differentiated instruction.

Build metacognition and overview (S3). Shasta used Boolean logic of true or not true to aid students in finding their errors (14:14). The CAS distribution property command (i.e., *expand*) was withheld from the scope of student knowledge intentionally. Shasta shared, "One kid who just asked, 'Is there a way that I could just get it to do it for me?' . . . There is an *expand* command. I didn't give it to them" (Shasta, Interview, October 4, 2017). Shasta purposely delayed access to the command. Instead, Shasta set students to the task of making a decisive answer from their own cognition. Students then reflected on the verification of their answer choice, the output, to complete their understanding. "I am hoping the kid is metacognitively aware enough or becomes aware enough, that they can parse it apart and see the number part was good, maybe and my variable coefficient was off or whatever else it was" (Shasta, Interview, October 4, 2017). Shasta bridged student thought processes with CAS outputs through questioning tactics.

Shasta carefully sequenced the lesson to present the distributive property first (14:23) and then combining like terms (16:15). On the CAS, inputs were automatically simplified and rearranged to include combining like terms.

> The reason I started with the distribution and not the combine like terms is this particular CAS doesn't automatically distribute. So it slowed the problem down, and they could hit enter and they wouldn't and, hopefully I was getting them into the mindset of, "I have to write before and after." So I was deliberately trying to keep them away from an Nspire CAS feature. (Shasta, Interview, October 4, 2017)

Shasta knew that to reveal the CAS functionality too soon, students might have missed the point of the lesson, that is, to check answers against the CAS. Since the distributive property would not reveal that side of CAS, he sequenced that part of the lesson first.

Change classroom didactic contract (C2). CAS evolved into an authoritative tool when Shasta instructed students to check assigned problems on the CAS prior to coming to class. "The big change is like this shift in authority. Some kids are starting to for the first time, to not just be told that their answer is wrong and it should have been this" (Shasta, Interview, October 4, 2017). Shasta modeled and promoted that shift through classroom practice.

> Every single individual student solved those six problems, and they did extras in his or her own way and was able to individually confirm without me looking at anybody's screen whether they got it and whether they needed to do more work. And they could do so at their own pace. They didn't need me controlling the pace of the classroom. (Shasta, Interview, October 4, 2017)

He monitored student activity and offered guidance, prompts, and inquiry to keep students engaged in the task. He began the process of allowing CAS to be an external mathematical consultant to students for learning.

Learn pen-and-paper skills (T1). Shasta generated the lesson as a result of student failure to perform algebraic procedures correctly on a non-CAS assessment. Shasta's purpose was to re-teach and re-direct students to learn pen-and-paper skills. When asked about learning by-hand calculations Shasta stated, "The initial expectations were, I need to get their brains to engage" (Shasta, Interview, October 4, 2017). He believed that students had absolute confidence that their answer was correct, even when it was not, so CAS leveraged the immediate feedback.

> Recognition for the students that when they write down their algebraic equivalent from the distribution and combining like terms, every one of them is 100% convinced that they've got the right answer. So the check step with an infallible machine, just to have that added safety check. (Shasta, Interview, October 4, 2017)

The ultimate goal was to learn pen-and-paper skills (2:56). CAS was the efficient tool for the learner to ensure that the mathematical skills were processed accurately.

Explore regularity and variation (T3). Shasta first provided discoveries and examples, and later left the student to create their own inquiries. "I could not have done this level of individualized problem solving without technology" (Shasta, Interview, October 4, 2017). CAS allowed students variation of problems specific to each persons need.

Shasta Vignette 2: Solving Equations

The goal of the lesson was to learn how to solve basic linear equations error-free. The student entered the entire equation as provided. Whatever operation the student chose to isolate the variable was typed into the machine. The CAS operated accurately to both sides of the equation. This method assisted the student by providing a visual representation of when that operation helped to isolate the variable in the equation. It also reduced any potential computational errors (e.g., incorrect addition or subtraction).

Shasta chose this instructional method as a follow-up to the previous lesson on the distributive property and combining like terms. It was given on the same day but is explained here as a stand-alone lesson. Shasta shared only one example with variations in his step-by-step instruction of this CAS procedure.

31:28 If I look at this [equation on the whiteboard], what would be the first thing I should do? So, $20x - 17 = 5$. (Students were answering. Shasta was repeating.) Add 17.

31:43 Now what do you think a classmate might do? What's a common mistake? Subtract 17. (Shasta justified why he was trying an incorrect operation to solve on the CAS.)

31:57 What if I errantly thought I was supposed to subtract 17. I'm going to type minus17, and this time I'm going to do *Ans* – 17 (see Figure 33).

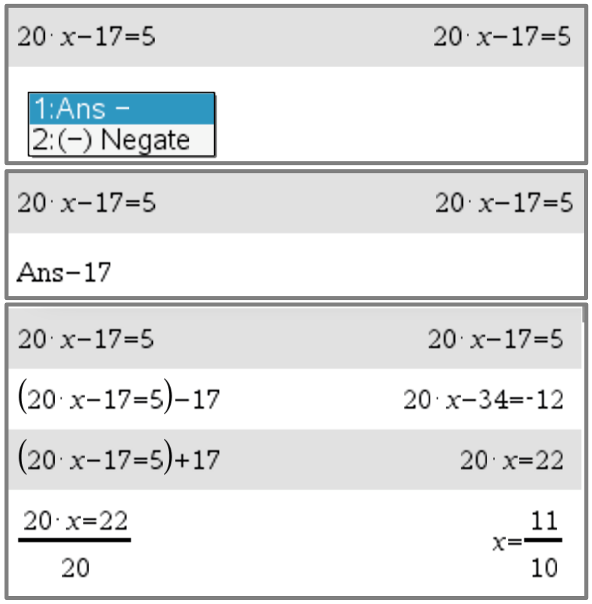

Figure 33. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall. Sequential steps that demonstrated solving an equation.

32:37 What happens to my equation? Is it prettier or uglier? But you're supposed to be in a way, like simplifying this thing down. (Result shown in Figure 33, line 2.)

32:51 At this point would you recognize that subtracting 17 wasn't the best way to go?

32:59 So how can I correct it? What's the easiest way to fix it on the Nspire? How can I copy down my previous line? Up arrow until it's highlighted?

33:11 And I really wasn't supposed to be minus 17, what was I supposed to do? So let's change the minus to a plus, and now did I simplify the problem? (Result shown in Figure 33, line 3.)

33:27 Can you tell really quickly that you've done a silly mistake?

33:31 So, I was supposed to be + 17 + 17. I get $20x = 22$. What would be my next algebraic step? (Student: Divide 20.) So how is this going to attach to the x? So how are you going to do it? Divide by, whatever it was? x is 22 over 20? And is that a perfectly fine number? (Students: Simplify that.) So, what would I do to both sides up here?

34:36 What buttons do I press? Slash, divide by 20 and watch what it does. Not only, it wasn't like 22 over 20, but it went ahead and simplified the fraction for you. (Result shown in Figure 33, line 4; Shasta, Lesson, October 4, 2017)

Shasta modeled solving an equation by providing step-by-step instructions to key in the commands for the laptop version of the TI-Nspire. Line items on the CAS do not display in a similar manner to pen-and-paper work products. Shasta's inquiries intended to connect pen-and-paper work products to CAS syntax. Shasta summarized the goal, "I don't expect anyone to walk back in here again without having checked your work. You'll still have some of these that you don't understand, but you should know absolutely whether you got it right" (Shasta, Lesson, October 4, 2017). His mission was to teach the tool well enough that students could independently utilize CAS.

Shasta Vignette 2: Pedagogical Opportunities

Solving equations by scaffolding steps on the CAS was a technique Shasta used to advance student cognition to procedural fluency in solving mathematical equations. The lesson facets were summarized in Table 22 as exploiting the contrast in machine mathematics, building metacognition and overview, adjusting the classroom didactic contract, and learning pen-and-paper skills.

Table 22

Shasta Lesson Vignette 2

P-Map	Evidence
S1	Equation solving appears different on CAS from typical by-hand work products
S3	1. CAS alerted student to a mistake when solving equations 2. Foregoing the solve command
C2	CAS became an external mathematical consultant
T1	CAS kept student work error-free through step-by-step checks facilitating the learning of pen-and-paper skills

Exploiting contrast of ideal and machine mathematics (S1). The TI-Nspire™ displayed the solving of equations in an uncommon manner (32:37). The CAS embraced the entire equation in parentheses and displayed just one operation performed to the whole equation as in Figure 34. Shasta exploited those differences. Mathematical operations typically are depicted as performed on both sides of an equation. Commonplace by-hand procedures do not show just one-sided operations such as the CAS did when solving an equation.

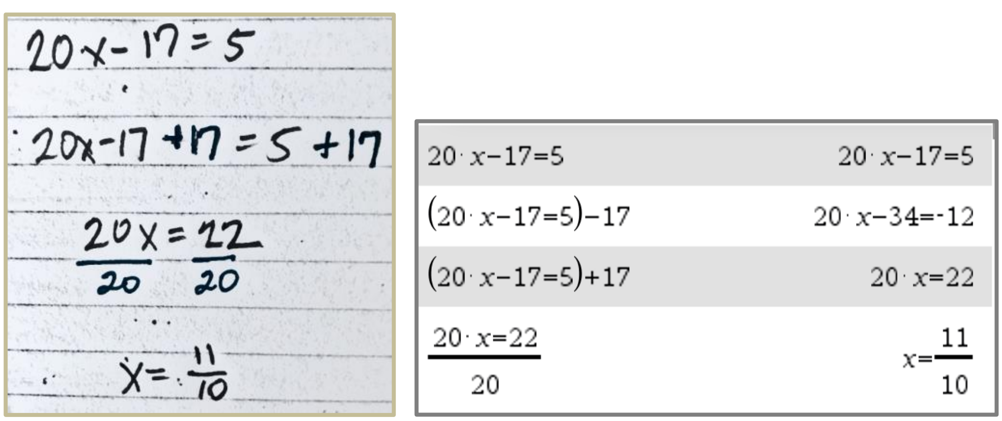

Figure 34. Typical by-hand procedures contrasted with CAS.

Build metacognition and overview (S3). Shasta purposively typed a mistake into the CAS to provide a visual representation of how the CAS would display the aberration (31:57). He described the output as *pretty or ugly* and added that it did not help to isolate the variable (32:37). This modeled critical analysis of CAS outputs when you made a mistake and also provided value to students' reasoning when each independently made a similar mistake.

CAS had a command that avoided the step-by-step procedures. Shasta talked about why he did not reveal that aspect of the tool. "Note that there is absolutely a *solve* command; I am not giving it to them yet. They will get it. They aren't getting it yet. They need to learn the fundamental vocabulary first" (Shasta, Interview, October 4, 2017). Shasta was careful to use CAS to build procedural skills and he anticipated the timing of introducing CAS commands.

Change classroom didactic contract (C2). A primary goal was to help students become more independent learners by checking each algebraic step of an equation solve with the CAS. "You should know absolutely whether you got it right" (Shasta, Lesson, October 4, 2017). Shasta shifted the responsibility to the student with a CAS outside the classroom boundaries.

Learn pen-and-paper skills (T1). The task was designed to help students become proficient in solving equations with complete accuracy. Through mistakes that were displayed on the CAS (32:51), students recognized the error in the midst of solving the equation. The CAS provided the opportunity to self-correct.

Shasta Vignette 3: Introducing Linear Functions

This lesson vignette spotlighted an introductory approach to linear functions through numeric patterns and arithmetic sequences. There had not been any discussion about slope, graphs, or *y*-intercepts previously. Shasta provided a list of numbers that formed an arithmetic sequence that generated a linear pattern. He instructed students to explore the values as data points by finding several additional points and identifying patterns. He then asked students to enter the data into a spreadsheet on the TI-Nspire™, create a scatterplot, drop in a moveable line, and adjust it to fit the data points. The TI-Nspire™ attached an equation to that moveable line. Shasta questioned students about the equation to correlate to the original number pattern. In a post-interview, Shasta explained the lesson design and why he chose this approach.

> I connected back to find the next number in the sequence stuff that the kids were heavily used to from their lower school, their elementary school. And basically, we knew if the first term was this, the second term was this, and the third term was this, they see what you are adding or subtracting every time. So they are easily able to make predictions. We turn those into ordered pairs. Those can then go onto a graphics screen. There were lots of choices, but the TI-Nspire™ is the only one that I knew of where the kids could actually take their mouse or because they have touch screens, literally lay their fingers on the line and maneuver the line so that it fits the graph. . . . I wanted the tactile laying the hands on the line. They then had their equation that sort of pops out. . . . So basically they develop $y = mx + b$, but they do it as the *n*th term of an arithmetic sequence rather than as slope plus *y*-intercept. So basically, I tried to make it connect to something they

had experience with, rather than forcing them into the abstract as their first exposure to lines. (Shasta, Interview, November 6, 2017)

Shasta set the lesson up in the class digital notebook for students to explore with table partners before giving the explicit directions on the CAS. He walked around the room assisting students with computer skills and directions. After 30 minutes he began the class discussion. Images from the computer screen shown in Figure 35 represent the sequence of steps in data entry.

1:50 If I was to give you those first three, could you recreate this entire list? What's the next number? (Students: 37, 39, 41) Okay, as soon as you got enough there– the first term and the second term. I happen to have three, watch what happens. If you don't know this trick already I need eyes on screen.

2:13 Hover your mouse over the lower right hand corner. What happens just as I get to the right hand corner? Turns into a plus. (The image projected on the wall in Figure 35 showed the columns highlighted but not the plus sign that would be at the corner of the blue highlight.)

2:21 Click and hold, drag it to the bottom of the list, it already, you've already defined the pattern, when I release it just fills them in for you. (Students were amazed and trying it their device.) (Shasta, Lesson, October 5, 2017)

	A x	B y	C
=			
1	0	31	
2	1	33	
3	2	35	
4			

	A x	B y	C
=			
3	2	35	
4	3	37	
5	4	39	
6	5	41	
7	6	43	

Figure 35. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall.

Shasta had given small groups of students a drag-down tip to automatically fill the columns when he was monitoring students at the beginning of class. He provided a rationale for revealing this command on the spreadsheet.

> What I figured out was the kids really just don't have spreadsheet experience, so I am slowly building that in over the year. . . . It enabled me to get the kids to really quickly get all of their information in. (Shasta, Interview, November 6, 2017)

He wanted to teach the tool and expedite the procedures with accuracy. After the data were entered, he questioned students to extract their knowledge of *data and statistics* applications on the TI-Nspire™.

2:33 Now just like we did with the river, just like we've done with other things, when I have data and I want a picture? (Students responded.) Yeah that works, alright?

2:50 So insert, Data and Statistics. And my *x* value was? Yes, very, very exciting terminology there. The *x* value was *x-value*. And *y* value was *y-value* and whoa that's way too pretty. (Shasta, Lesson, October 5, 2017)

Some of Shasta's students had already determined this method during the first part of the class. Shasta accommodated all the students and ensured that the linear equation was determined both with the CAS and by pen-and-paper. The directions continued to affirm how the CAS revealed the formula for the sequence of numbers. The images of the sequences of events are shown in Figure 36.

3:09 So what did you do next? . . . Moveable line? I was really impressed with everybody. You guys figured out how to do this without me even giving you a single piece of instruction.

3:26 Did you grab the ends? (Student talking to teacher about directions on CAS with regards to a grid command to move up/down and curly arrows to swivel the line.)

3:42 Yeah, but you guys like we're in, is that pretty close? Pretty close to right there in the center? Let's see if I can get that top line a little bit better, okay.

4:03 What is that number look like it's really close to? Two and this one looks like it’s close to? 31? So let me show you one more nice trick. $2x + 31$? Yes.

4:17 Alright, click on the line, I'm going to make it go away. $2x + 31$, watch this, analyze [CAS command], plot function. What did we say this was equal to? $2x + 31$ (Shasta typed the precise function into the CAS and it lined up perfectly with the data points.)

4:40 Do you think we might have the pattern? (Shasta, Lesson, October 5, 2017)

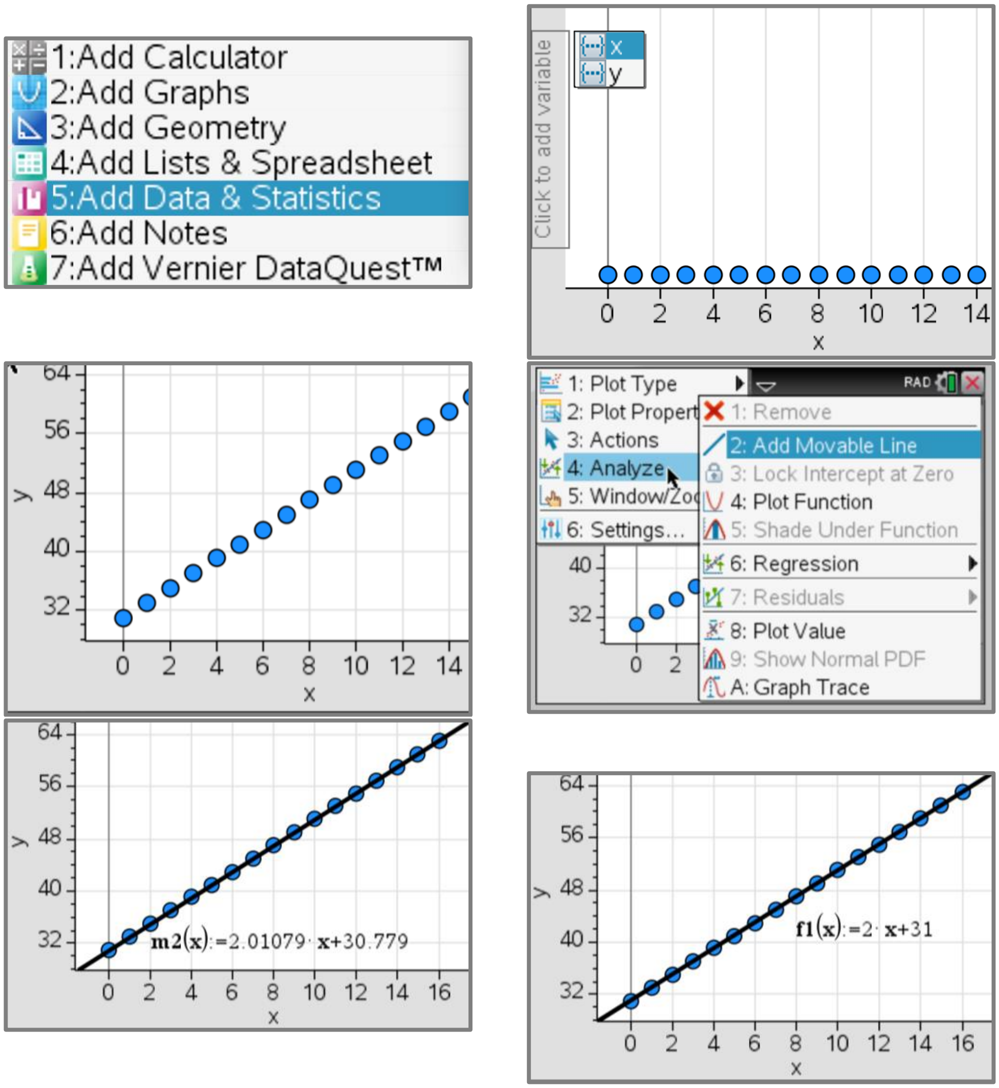

Figure 36. Shasta's TI-Nspire™ textual commands projected from computer to the classroom wall. Sequential steps that demonstrated data manipulation from a spreadsheet application to a graph plot.

After the class discussion resulted in a definite formula, Shasta directed his student's attention back to the whiteboard and the original list of values. The discussion that followed illuminated student understanding of the connection to mathematics; the parts of the equation that related to those initial values.

5:25 What kind of formula did we come up with? (Students all worked together to get $y = 2x + 31$)

5:41 This formula has something to do with this list. (Students screaming: It's the number that you start with.)

5:53 It's not the first number. (Students: It's the zeroth number!)

6:05 We are calling it the zeroth number so that we don't confuse it with the first.

6:13 What's the two got to do [with it]? (Student: It's how much it goes up by. It's our sequence and pattern.)

6:22 So, if that's true, if this really is the formula, could you tell me what the ninth term in this list is, without looking?

Students and teacher continued working several more examples both by-hand and with this same technique. Shasta continued with inquiries from all different perspectives. He summarized the lesson, "So you have a [graphic], an algebraic, a numeric, and a verbal description of the line. Are we good? Given any one, can you re-create the other three?" (Lesson 18:54, October 5, 2017). Students were given an assignment in which three geometric figures were drawn as a growing pattern and they had to determine the number of blocks in the *n*th figure. This was Shasta's closure to the lesson.

Shasta's post-interview was obtained several weeks after the lesson. He was asked, "What has changed in terms of student understanding about linear relations?" Since the approach was new to him, he was very reflective about student learning. He described events that took place after the observed lesson.

> I started giving them sequences where the terms, they were no longer adjacent to the other. So I gave them like the second term and the seventh term. They had to

figure out the common difference, and they had to figure out the zeroth term. They pretty quickly, like it was just intuitive, from the second term to the seventh term it's five steps, so whatever change my numbers had done, I just divide by five, it's the least common differences. . . . And then, I took the two ordered pairs, drew a vertical line down the board. So I am writing directly beside what they just solved. I do slope between the two ordered pairs. And the screams coming from the room, "No, you didn't. You tricked us." (Shasta, Interview, November 6, 2017)

Introducing linear equations by providing an arithmetic sequence first was genuinely avant-garde pedagogy for Shasta.

> I had never taught arithmetic sequences as anything other than a linear function. I never taught it as separate formulas. But this is actually a little bit of serendipity. This was the very first moment that I ever actually 100% developed for a kid in their first year algebra course . . . developed this [approach]. It was just sort of gut instinct from lots of experiences. All my other stuff didn't work perfectly; let me try something else. I'm thinking this is going to work out well. (Shasta, Interview, November 6, 2017)

He chose this methodology because he knew that students had familiarity with number patterns. What he realized in retrospection was that he was able to circumvent the abstraction of linear relationships to build conceptual understanding with students. "I tried to make it connect to something they had experience with, rather than forcing them into the abstract as their first exposure to lines" (Shasta, Interview, November 6, 2017). Finally, Shasta did not anticipate student development of the slope concept so concretely that the knowledge became intuitive for his students. They recognized slope as a difference between terms in a sequence of values.

Shasta Vignette 3: Pedagogical Opportunities

Shasta's innovative approach revealed multiple aspects of pedagogy that were effectuated as a result of CAS. Some codes numbered in the evidence column of Table 23 were due to features of the lesson that warranted in-depth discussion. A thorough explanation follows.

Table 23

Shasta Lesson Vignette 3

P-Map	Evidence
S1	Moveable line gave an approximate linear equation. Collectively, the class arrived at the exact equation.
S2	Tangible lesson to increase student conceptualization based on prior mathematical knowledge
S3	1. Use of various representations: spreadsheet, graphs, algebraic, numeric 2. Reinforce syntax procedures through repetitive motions
C1	1. Exploration activity lent to social exchange 2. Social interaction enhanced mathematical connections from graphic to algebraic perspective
T1	The instant answers provided feedback to the student
T3	Patterns were explored through the extension in the spreadsheet and in the placement of the moveable line
T5	Multiple representations: sequence, ordered pairs, scatter plots, linear functions, and geometric patterns

Exploiting contrast of ideal and machine mathematics (S1). CAS inserted two types of lines: the moveable line (3:09) and the plot function feature (4:17). Shasta used his approximation of the moveable line to determine the precise equation. He verified that it was the actual formula by plotting that function on the scatter plot using the CAS command *plot function* to display a visual representation that showed a perfect fit.

Re-balance emphasis on skills, concepts, and applications (S2). Shasta chose to approach this lesson from a tangible activity of number patterns. He knew that CAS would allow students to connect the concept of increasing values by addition for subsequent terms by graphing a moveable line and then looked at the numerical value as the coefficient of x.

> I tapped the moveable line feature in the data and statistics window as another essentially . . . I now consider it a CAS feature, this ability to like have a function, have a line, and actually manipulating on an object geometrically, rather than manipulating it algebraically. (Shasta, Interview, November 6, 2017)

Shasta chose a conceptual delivery of linear functions that was grounded in students' prior knowledge of numerical patterns. This allowed him to delay the abstraction of linear functions and at the same time cultivate a connection to the increasing values at a constant rate of change. Additionally, he constructed a gateway to a visual representation of the graphs for these number patterns. He talked about the pedagogy to construct the mathematical knowledge.

> Algebraic representations are extremely non-intuitive to early algebra students. By holding off the algebraic manipulations as long as possible, my students are able to discover the slope/common difference and y-intercept/0-term relationships that define the $y = mx + b$ form. They discover the fundamentals for themselves; they don't memorize what I lecture. It's a much deeper, organic, and long-lasting effect. (Shasta, Written Reflection, October 14, 2017)

This lesson demonstrated access to linear functions through a re-balancing of procedural skill and conceptual development.

Build metacognition and overview (S3). This lesson included multiple representations with the purpose of developing a richer understanding of linear functions. Shasta purposely started with numeric data and had students translate into graphical representations. He shared how those various representations build mathematical understanding.

> I've 'preached' multiple representations and how an answer or aspect of a problem that isn't obvious in one form can 'appear' when you translate between forms. The human brain is quick to see numeric patterns in arithmetic sequences, which is why recursive formulas for sequences are simpler for students. (Shasta, Written Reflection, October 14, 2017)

He approached multiple representations as a foundational aspect of teaching any mathematics concept. Shasta regarded the cognitive functioning from reflection on those different forms a complex brain activity and he would do everything possible in his instructional practice to ease students into understanding more deeply the connections.

Shasta managed the availability of technical features. In this lesson he let students tire over tedious data entry. When students had entered a substantial number of points in the spreadsheet, he instructed the use of dragging to populate the table based on number values already entered as the first part of a sequence. Shasta shared his rationale.

> The dragging technique is an immensely powerful tool. If given at the beginning, students are less likely to remember it. By feeling mundane data entry before experiencing the wonder of having it automatically generated, the students can (1)

> verify the automatic results, and (2) clearly understand how much time the approach saves. Both "dramatic" experiences increase the likelihood of long-term memory encoding. (Shasta, Written Reflection, October 14, 2017)

He divulged these shortcuts or features of the device during classroom cooperative learning times. When he was monitoring student progress he would show the drop-down feature to small groups of students. It was only later that he demonstrated this drag-down action to the whole class on the computer-projected screen (2:13).

Change classroom social dynamics (C1). Students were encouraged to work in partners or small groups to discuss both the process of completing the task and the results obtained. Shasta viewed student discussion as a critical component to externalize the theories that a student is conjecturing.

> Student conversations and discovery are central to all of this. I'm asking my student not to memorize, but to define patterns they see, explain how these appear in the equation that appears as the output of the moveable line, and ultimately to hypothesize results for sequences they create on their own. (Shasta, Written Reflection, October 14, 2017)

In this particular lesson, the moveable line approximations necessitated sharing to build strong connections to a precise formula. Students got a variety of equations based on their own manipulation of the line.

> Student interaction is also critical here because the moveable line fits don't all create exactly the same equations— very small pixel variations create differences in coefficients. By looking at all of the equations a group creates, they can more confidently hypothesize something of an average equation that will tend to be

> closer to the true equation— something like the central limit theorem of discovering linear equations! (Shasta, Written Reflection, October 14, 2017)

Shasta hoped students were not relying on their own approximation but looked at classmate's equation for the same data. In this way, collaboration provided each student greater assurance in his or her conjecture.

In addition, students were communicating mathematics, causing students to adjust their perspective to more clearly direct information to their peers.

> I've got a point person and a sequence person sitting at the same table working on a problem together. I'm watching the kids like change their language and change their interpretation so the other person can understand what they are saying. (Shasta, Interview, November 6, 2017)

This tapped into multiple approaches to solutions and extracting connections.

Learn pen-and-paper skills (T1). Student graphical representation on the CAS instantly revealed whether the numeric calculations the student performed to find subsequent terms were correct or not. Students noticed that sometimes a point would not line up with the other seven points. The output helped the student reflect on their pen-and-paper skills and self-correct.

Explore regularity and variation (T3). Students moved fluidly from a spreadsheet of data to a graphical form. The CAS allowed students to insert a moveable line that could be manipulated (3:09). This variation permitted learner flexibility in exploring the function; thereby, making connections to the values of slope as related to the common differences between terms.

Link representations (T5). The activity associated a sequence to ordered pairs, scatter plots, and linear functions. "By shifting to the graphical representation, the linear relationship in the data jumps in your face" (Shasta, Written Reflection, October 14, 2017). As an extension to the lesson, Shasta linked a geometrical pattern to the sequence. This lesson blended the representations fluidly so that students potentially could have multiple entry points.

Shasta Vignette 4: Quadratic Factorization

The goal of this lesson was to understand the relationship between binomial linear factors and the product as a quadratic expression. Shasta taught students the by-hand distributive calculations using a box method and a rainbow arc method. He shifted to the CAS to generate multiple examples expeditiously in order to analyze number patterns from the results. The pedagogical move by Shasta intended to promote students to think deeply about numeric relationships. "My goal was to use the CAS to avoid memorized patterns in boxes in most textbooks, expand what they had learned without technology, and return to the CAS to expedite and reinforce what they had learned" (Shasta, Written Reflection, December 20, 2017).

Shasta was slow to move to CAS in this lesson. He taught the by-hand procedure first before cautiously revealing the CAS commands to students. He introduced the *factor* and *expand* CAS commands to the class. "I'm going to show you a couple new commands for your computer algebra system. They are really nice commands. We're going to do a little exploration and try to speed up this whole process" (Shasta, Lesson, December 4, 2017). At the same time, he warned his students, "While these commands can dramatically speed up your homework and your practice time, if you don't know how to do this, you're going to be completely hopeless when you're facing a quiz” (Shasta, Lesson, December 4, 2017). Shasta expressed this conflicted feeling about using CAS to help develop conceptual understanding, knowing that it could adversely affect student learning. He projected the warning on the wall (see Figure 37) to be sure the concern was acknowledged.

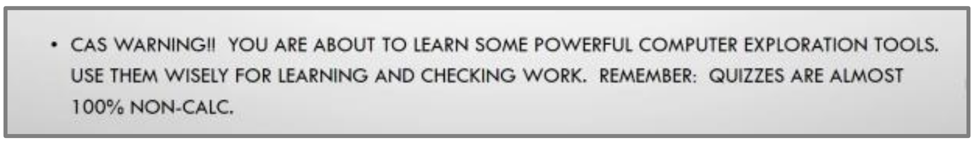

Figure 37. Shasta’s presentation notes projected from computer to the classroom wall.

Shasta began with the command *factor* on the CAS and used the exact factoring problems that were just solved with by-hand calculations. Shasta instructed students to type in the CAS. He waited to key the commands until after students had completed typing. He explained the TI-Nspire™ recognition of an internal command. When a command was typed rather than accessed through the CAS’ menu, the display font was showing differently for text and internal commands as shown in Figure 38.

25:32 These are two problems that we just did. [$x^2 + 12x + 36$ and $x^2 + 6x + 9$] The CAS command is factor. How can you tell . . . (Students typed into their device, but Shasta is not yet keying this.)

25:55 So as soon as you type the *r* [in the word factor] and it straightens up and says what?

26:09 That’s the signal that [CAS] knows what you’re talking about (Shasta, Lesson, December 4, 2017).

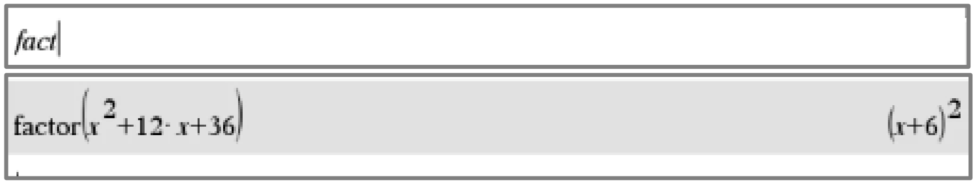

Figure 38. Shasta's TI-Nspire™ projected from computer to the classroom wall. Note the font prior to typing the *r* in factor.

Next, Shasta allowed students to interpret the result of the factorization of $x^2 + 12x + 36$ since the result displayed $(x + 6)^2$ rather than the two binomials as a product. It was assumed from Shasta's questions that the by-hand calculations were not combined into a single binomial squared.

26:22 Alright, so I'm going to type factor [on the computer display] and it was $x^2 + 12x + 36$.

27:19 What did we get when we factored this on the wall? (Students answer *x* plus six times *x* plus six).

27:31 Is it the same [on the CAS]? Why? (A lot of discussion among students.)

27:51 It's not exactly what we expected, but isn't it (pause) exactly what we got on the wall? (Shasta, Lesson, December 4, 2017)

Shasta relayed the connection between the symbolic algebra and the work product from a visual box method solution on the wall.

After a few more examples that use *factor*, Shasta introduced the *expand* command. Shasta projected instructions on his wall as shown in Figure 39 and said, "The computer command for distribution is *expand*" (Shasta, Lesson, December 4, 2017). He provided time for students to enter commands in CAS according to the presentation directions.

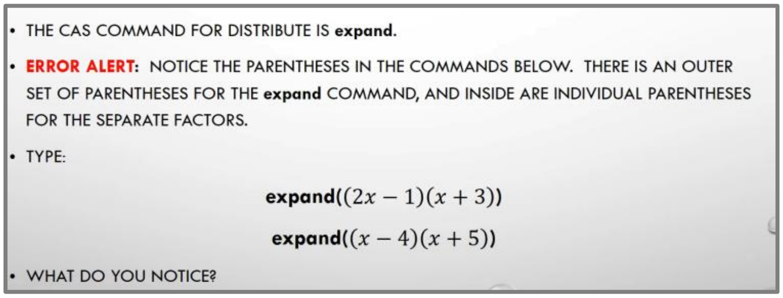

Figure 39. Shasta's presentation notes projected from computer to the classroom wall.

Shasta anticipated student error in keying the commands. He modeled the correct key sequence and explained the potential mistakes to students. The reader should note the warning in the transcript and the actual key commands that Shasta performed in Figure 40.

30:03 Remember, I gave a warning. The reason was up there on the slide.

30:11 The parentheses are really, really important.

30:17 Expand, there is automatically a big set of parentheses and inside that you have to put everything you want to expand.

30:27 Now, that expression on the inside, if it has parentheses, you have to get them all in there. Notice here, when in looking . . .

30:35 If you want a set of parentheses on the outside then I have all of these parentheses going on the inside of these [the big parentheses].

30:58 (Shasta showed the correct syntax for input while talking.) Expand. Now I have x minus 4 in parentheses.

31:05 So I have to open up another set of parentheses, so there's x minus four.

31:10 Open up another parentheses for $x + 5$.

31:14 And when I have all the parentheses of the original problem contained within one extra set of parentheses, then I can expand.

31:34 Do you have ways that you can confirm for distribute and undistribute [factor]? (Shasta, Lesson, December 4, 2017)

$\text{expand}((2\cdot x-1)\cdot(x+3))$	$2\cdot x^2+5\cdot x-3$
$\text{expand}((x-4)\cdot(x+5))$	x^2+x-20

Figure 40. Shasta’s TI-Nspire™ projected from computer to the classroom wall.

Shasta was prepared for some students to make the mistake with parentheses before further explaining the correct syntax to the class. After working through the correct procedure, he questioned students’ ability to verify their own work through the CAS.

Class time shifted to student-centered work for a few minutes. Students performed multiple factoring computations on the CAS and explored the relationships between the numbers in the original quadratic expressions and the factored expression as displayed in Figure 41. Shasta then brought the class to a group discussion to talk about the numerical patterns.

37:22 Is there a relationship between the expanded and the distributed form?

37:36 There is a relationship between the numbers in the factored form and the expanded form.

37:51 Look here, the numbers 13 and 36 are somehow connected to 4 and 9 (Shasta, Lesson, December 4, 2017).

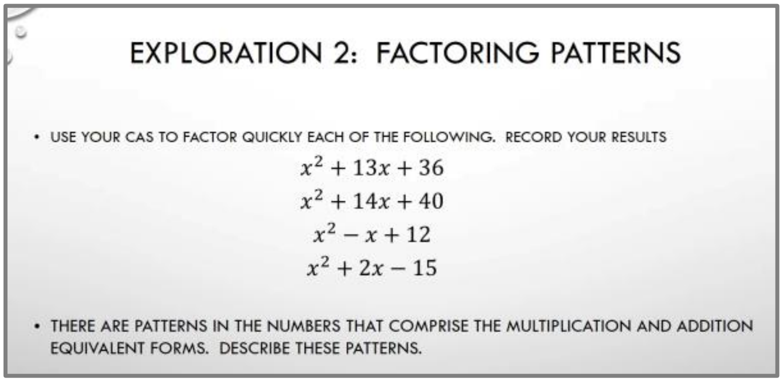

Figure 41. Shasta's presentation notes projected from computer to the classroom wall.

Students discussed these relationships amongst one another and also with Shasta. Once it was established that the last number was the product of the two linear factors and the middle coefficient of *x* was the sum of the two linear factors, Shasta changed the quadratic trinomial to an example that would not follow that pattern. He projected the example $2x^2 + 27x + 36$ on his wall as shown in Figure 42. He asked students to factor and to consider the reason why the newly discovered procedure did not work.

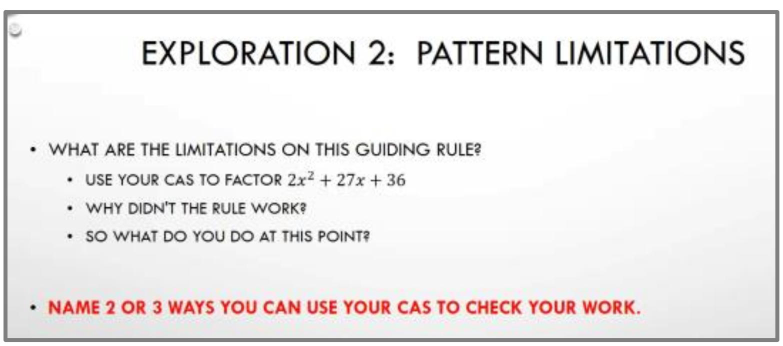

Figure 42. Shasta's presentation notes projected from computer to the classroom wall.

43:41 Factor this $2x^2 + 27x + 36$. What should happen here? (Students answering the pattern discovered and Shasta repeated.) It seems like it should add to 27, multiply to 36? Right? Let's find out. (Shasta typing factor into the CAS followed by the trinomial. The output was $(x + 12)(2x + 3)$.)

44:51 So those two numbers multiply to 36 but do they add to 27? (Students saying no.)

44:58 Why? What is different? (Student says that it is two x squared.)

45:01 It's $2x^2$. This is really important everybody. What is common in all the first examples?

45:09 All these rules (pause) the fast rules only work for what form? (Shasta, Lesson, December 4, 2017)

Shasta briefly explained how to factor when the trinomial starts with something other than one by referring students back to the box method. He mentioned that this type requires a little more thought to develop the mathematical patterns to factor and that a different day will be devoted to that. Shasta closed the lesson by asking students to complete an exit ticket prompting students to factor a few problems with the sum and product procedure was developed by analysis of CAS outputs.

Shasta Vignette 4: Pedagogical Opportunities

The lesson widely involved pen-and-paper skills with reasoning of numeric patterns. CAS provided instructional opportunities similar to those in Shasta's Vignettes 1 and 2. However, in this instance CAS performed all computation instantly. Shasta was compelled to ask questions to draw out the connections for students to ensure that the CAS was truly a consultant to student learning as opposed to an outsourcing of procedures. The evidence of the cultivation of trinomial factorization is summarized in Table 24.

Table 24

Shasta Lesson Vignette 4

P-Map	Evidence
S1	Explored the contrast of perfect square trinomials with CAS and by-hand calculations
S3	Used CAS commands to provide an overview of factoring quadratic functions, delaying the menu options
C2	Factor and expand commands allowed an authority shift
T1	Learn how to factor without CAS; with mental math
T3	Regularity in the problems that do factor; variation in problems that factor with a different pattern
T5	Visual models banded with symbolic representation

Exploiting contrast of ideal and machine mathematics (S1). The by-hand calculations of the trinomial $x^2 + 12x + 36$ resulted in $(x + 6)\ (x + 6)$, but the CAS syntax was $(x + 6)^2$ (Time stamp 26:22). "Even though your answer looks different doesn't mean it is" (Shasta, Interview, December 22, 2017). Shasta questioned students about the accuracy of the answers allowing connections to be discovered by the students individually.

Build metacognition and overview (S3). Shasta prescribed access to the CAS commands to prevent gratuitous use of CAS. He did not teach the *factor* command via menus on the CAS. Instead, he directed students to type the word out and to observe the letters changing from an italicized font to a regular bold font, indicating an internal CAS command (25:55). He shared his rationale in the reflection.

> The CAS is an extremely powerful tool. I haven't pulled any commands from the menus yet. Partly, that is to keep students unaware of some additional commands. Also, I'm trying to get students to be "intuitive" about commands. I ask them to

> think about what they are trying to do and to ask the CAS to do that. They focus on "words" becoming un-italicized to recognize when they have hit upon a command the CAS knows. (Shasta, Written Reflection, December 20, 2017)

Shasta hoped this purposeful tactic would enable students to be eager in their future encounters with the CAS. He also wanted to restrain the specific features to preserve learning of some foundational mathematical skills.

Change classroom didactic contract (C2). He used the CAS to reveal other ways to write factorizations, instead of telling them. The example of one binomial squared was discussed above. Another case was the sequence of factors.

Learn pen-and-paper skills (T1). Shasta made clear the goal was to develop a method to complete factorization of trinomials with the mental ability of number pattern recognition. He used CAS as a path for students to explore the number patterns of these relationships.

Explore regularity and variation (T3). This lesson was set up as an exploratory activity. "I wanted them to have some practice with the 'mechanics' of factoring and had set the stage for them to 'discover' the sum & product features of the coefficients" (Shasta, Written Reflection, December 20, 2017). Shasta carefully selected mathematical expressions that would present both affirming and problematic situations to students.

Students saw supporting evidence in factorization when the output of the CAS matched their expected result from by-hand computation. Later, students saw variation when an unexpected output produced of one expression that combined the factors. The output $(x + 6)^2$ supported learning to reveal equivalent expressions (27:19).

Shasta intentionally presented an example that was problematic to students. When students tried to factor a quadratic function that had a leading coefficient of two, the output did not align with their newly developed rule. When asked about the placement of this example Shasta responded, "I was drawing attention to this feature *after* they had some practice so they could understand *why* the coefficients combined this way and to shine a spotlight on features some had already started to recognize intuitively" (Shasta, Written Reflection, December 20, 2017). Shasta used the variation to advance student learning through consideration of how that two affected the terms in the factorization.

Link representations (T5). The by-hand methods at the beginning of the lesson were reinforced by the CAS results. Shasta facilitated the connections of visual and algebraic representations.

> With the mechanics already in hand (to various degrees across the class), some were frustrated with having only the box (visual approach) to factor. By explicitly naming the algebraic/numeric relationship, students again had multiple ways (visual—and now algebraic) to solve their problems. (Shasta, Written Reflection, December 20, 2017)

He continued to compel students to explicitly state the patterns and relationships between the numbers, solidifying the concept.

Shasta Case Analysis

Shasta taught secondary school while exploiting CAS in mathematics pedagogy since the beginning of his teaching career, 30 years prior. He served as the lead supervisor of mathematics in the school system and carried the burden of fulfilled all duties related to mathematics at his school. At the time of the study, he was reassigned from teaching high school precalculus courses to eighth grade algebra one and geometry classes, filling an unexpected vacancy. Shasta's stories revealed instructions from a basic foundational algebra perspective that emanated from three eighth-grade classes.

P-Map

The first three lesson vignettes spanned one week; the fourth vignette occurred two months later. Shasta's pedagogical opportunities, summarized in Table 25, indicated that subject-level prospects occurred most frequently in these algebra classes. The total pieces of evidence comprised nearly half of the pedagogical opportunities observed in Shasta's lessons. The classroom didactic contract was clearly affected by the promise of CAS as a cognitive tool (i.e., students feel empowered to take over their own learning). Only three of the five task opportunities from the P-Map were observed. Of these, CAS' support of pen-and-paper skills understandings stood out as a primary task. The total number of occurrences aid in understanding pedagogies observed more frequently. It is not the case that a higher number indicated superior instruction. The discussion that follows clustered the opportunities in the three levels of the subject, the classroom organization, and tasks.

Table 25

Shasta Lesson Vignettes Summarized: The Occurrences of P-Map Opportunities that were Exploited During the Lesson Grouped by Subject, Classroom, and Tasks

P-Map	Vign 1	Vign 2	Vign 3	Vign 4	Total
S1	✓	✓	✓	✓	4
S2	✓		✓		2
S3	✓	✓	✓	✓	4
C1			✓		1
C2	✓	✓		✓	3
T1	✓	✓	✓	✓	4
T2					0
T3	✓		✓	✓	3
T4					0
T5			✓	✓	2

Subject (S1, S2, and S3). Shasta exploited the differences in the machine and by-hand procedures in all of the lesson vignettes to advance student learning and also to recognize equivalency when the syntax produced an unexpected result. The contrast of the ideal was used as a verification tool for procedural accuracy. Shasta's lessons (Vignettes 1, 2, and 4), presented as the distributive property, combining like terms, multiplying and factoring quadratic equations, necessitated exactly one solution. CAS returned true, the identity of input, or the result of a procedure. In some cases when the machine provided an unexpected result, the output was actually correct and equivalent but with a different form. When that occurred learning shifted to knowledge of mathematical properties, arrangement of terms, and computer syntax. The output of the device created a situation in which the teacher facilitated student interpretation of results. Otherwise, students may have misread the output, disrupting the purpose of verification.

Shasta used dynamic features as a catalyst to construct mathematical knowledge. The changeable properties of a moveable line were accessed in Vignette 3 to compile characteristics of linear functions. The ability for students to have a tactile interaction with the device to generate a line that best fit the data provided insight about the steady rate of increase. The discussion that followed brought out two concepts: (a) slope as the difference between terms; and (b) *y*-intercept as the term that preceded the first term in the sequence. The activity augmented student conjecturing and justifying conditions about the characteristics of linear functions.

Classroom organization (C1 and C2). The four lesson vignettes portrayed Shasta instructing students in the use of CAS tools as an external mathematical authority. First, he had the expectation that students would not only collaborate to interpret results but also view CAS as an independent unbiased tool. The functional capabilities of the CAS in terms of ease, efficiency, and accuracy entitled students to monitor their own learning. It was left to the student to integrate the tool into their personal practice. Shasta modeled in Vignettes 1 and 2 how to use CAS as an external mathematical authority during class. Second, he provided several problems that he wanted the students to work on independently in class. Finally, students were directed to develop their own problems to conduct extra practice. Shasta described this shift to students managing their own work, Shasta described as giving agency to the students for their learning.

Tasks. Shasta was observed teaching activities that included learning pen-and-paper skills, exploring regularity and variation in algebraic structure, and making connections amongst multiple representations. He directed instruction for part of the class; other times, students worked in small groups or completed independent assessments. A large portion of class time was devoted to teaching the tool. Yet, the tasks were part of a mathematics curriculum and ideology that encouraged students to extend beyond the norms of traditional instruction. Shasta deliberately promoted the development of habits with the CAS that would allow students to be creative in their approach to mathematical knowledge. The next sections will describe evidence of Shasta's task-level opportunities in the P-Map.

Learning pen-and-paper skills while exploring regularity and variation in algebraic structure (T1 and T3). A strong emphasis was on the development of algebraic procedures and structure through the utilization of CAS as a tool that assisted in the acquisition of pen-and-paper skills (Vignettes 1, 3, and 4). Shasta expected that students would learn the algebraic manipulations to be performed without the assistance of technological tools. However, CAS provided the exact tool to develop those skills through its ability to verify with accuracy and precision. Classroom tasks supported teaching students how to use the tool to draw out understanding.

Shasta selected examples that purposively would pique students' curiosity with the irregular and unanticipated outputs from CAS. He used non-equivalent forms in Vignettes 1, 2, and 4 to expose the CAS outputs of varying outputs of differences. In the lesson on combining like terms (Vignette 1) students compared their cognitive guesses with the CAS output. In that analysis the order of the terms was reversed. Shasta drew attention to these outputs and offered inquiry regarding algebraic properties. He facilitated students making connections to the structure of algebra. Similarly, in Vignette 4 this occurred when CAS output a factorization of a squared binomial rather than the product of two linear expressions. CAS was proficient to make adjustments in algebraic structure; thus, Shasta helped raise students' awareness of those modifications.

Representations (T5). Shasta talked about a strong belief in the value of multiple representations, yet I only found evidence in Vignettes 3 and 4. Shasta used a variety of representation to develop student conceptions.

Shasta's Vignette 3 began with numerical data, shifted to tabular arrangements, and ultimately accessed graphical representations to form the basis for a linear function. All representations with the exception of the numerical data were performed with different CAS functions. Shasta provided his rationale for the task as it connected to student learning.

> I've "preached" multiple representations and how an answer or aspect of a problem that isn't obvious in one form can "appear" when you translate between forms. The human brain is quick to see numeric patterns in arithmetic sequences. . . . By shifting to the graphical representation, the linear relationship in the data jumps in your face. (Shasta, Written Reflection, October 14, 2017)

He asked students to find 10 points in the sequence because he wanted the linear representation to be obvious. Later in the lesson, he directed students to consider all the representations as potential access points to equivalent forms of a linear relationship. "So you have a [graphic], an algebraic, a numeric, and a verbal description of the line. . . . Given any one, can you re-create the other three?" (Shasta, Lesson, October 5, 2017). Shasta valued students' developing cognitive abilities that allowed for flexibility in mathematical representation. Shasta closed the lesson with a connection to geometric patterns after making reference to a past problem. He connected the past problem solution to an arithmetic sequence.

Vignette 4 presented another example of representation. Shasta used visual by-hand representations and connected those to CAS computations in the mathematical procedures of multiplying and factoring quadratic functions. This differed from the Vignette 3 example: a drawn-out area model for multiplication connection was made to CAS symbolic manipulations. In addition, Shasta connected a second hand drawing to associate the other two models. In this instance, Shasta used an arc method of multiplication of binomials. He focused student attention to the numerical coefficients of the terms in the trinomial, drawing out understanding about the number patterns from the CAS symbolic representation.

Summary of P-Map. Shasta exploited the CAS in his instruction to facilitate student reasoning and sense making of mathematical knowledge. He approached teaching the CAS tool with warning and caution to his students first prior to having them perform action on the device. The affordances of CAS in Shasta's vignettes primarily included subject-level opportunities. Through interviews and written artifacts Shasta's creativity of lesson design was thoughtful and often revealed a re-balancing of skills, concepts, and applications. Shasta conveyed a message that CAS can operate as an external mathematical guide for students inside and outside of the classroom environment. The expectation from Shasta was that students must access CAS. Tasks were designed to build procedural understandings and to develop connections from multiple representations.

Emergent Themes from Shasta's Data

Shasta demonstrated and shared perspectives of his philosophy of CAS utilization in secondary education. Through lengthy discussions, several ideas materialized. A section of an interview following the first lesson encapsulated some of these ideas: precision in language and syntax, verification in mathematics, and agency to the student though CAS as an external consultant. Following this excerpt will be a more comprehensive list of the emergent themes.

The post-lesson interview described the value of utilizing CAS to support student understanding. Shasta explained his actions while students practiced problems during class. The description demonstrated CAS capability to differentiate instruction for varying student abilities. Shasta was asked, "Can you talk to me about when you say a *non-judgmental mathematical friend*?"

> When I get into the algebraic solving, and so I would say mathematics as a language, the hardest thing that students face is the very tight and mercilessly precise language of mathematical writing. And they have to get it right. So if they combine variables in the wrong way, distributed the wrong way, solved for something in the wrong way, they are not going to get the solution that they need. . . . What they need really is varying levels of practice. . . . I was sort of circling around. . . . Some kids I was giving additional challenges to. Some kids I was helping them to decode their responses to figure out where the mistakes were. So teaching them to, how to decode– Some, teaching them how to create their own problem. But what's beautiful about the CAS and about like writing down their answers before checking against what the machine is going to say, is the machine

> truly has no emotions. And so having . . . someone or some place that [students] can go to and they can never ever feel stupid because they can always throw away the paper, or erase the file, they can get all the practice done that they want, as much as they are willing to do. This machine will keep giving them feedback and keep giving them practice until they have just had enough for the day or until they learned their topic. So for me, that is like, the pitch. [CAS] is a great *mathematical friend* [emphasis added]. It will work as hard, it will do the nastiest math problems you ever give it to solve, and it just doesn't care. It is non-judgmental, and it is your friend. It will help you if you are willing to engage. (Shasta, Interview, October 4, 2017)

Shasta pointed out that syntax in mathematics problems had to be accurate. CAS was a tool that allowed for verification in accuracy. Finally, CAS was non-judgmental in its ability to check precision. In the activity, CAS was the agency for the student to increase their mathematical conceptualization.

Themes that rose out of the entirety of Shasta's data were the verification of answers, the need for providing guidance, the idea of multiple representation, the teacher regulating access to CAS, and CAS as a mathematical consultant. These themes presented in no particular order are outlined in Table 26 with notable identification in the lesson vignettes. In some lesson vignettes, the evidence was identified within open coding of the lesson and referenced as *lesson* in the table rather than a P-Map code. Figure 43 is offered to conceptualize the five themes. They do not overlap; however, they interlock showing that a relationship exists between adjacent components. For example, *Mathematical Consultant* impacts *Verifying Answers* in that the pair of components link but do not overlap. The pedagogy of verification of answers requires exploitation of CAS as a mathematical consultant. Each of the components of the emergent themes will be explained in the following sections.

Table 26

Emergent Themes Evidence: Shasta

	Verify Answers	Provide Guidance	Multiple Representation	Mathematical Consultant	Regulate Access
Vign 1		Lesson		C2	S3
Vign 2	C2			T3	S3
Vign 3	T1, T5	Lesson	T5		S3
Vign 4		Lesson	T5		S3

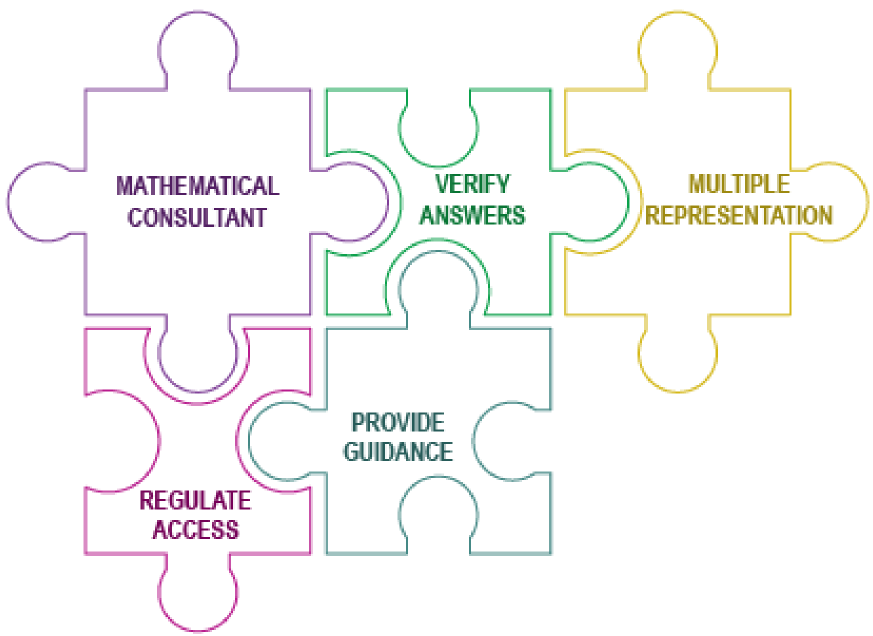

Figure 43. Emergent Themes Schema: Shasta.

Verify answers. The introductory excerpt from Shasta embodied the value of verifying answers through CAS. Shasta illustrated how helpful the tool was given the correct guidance. First, it released the need for a solution manual or teacher to affirm correct answers. Second, CAS was available for other questions that students might need answered. Third, it delivered results without judgment and it did not retain a memory of incorrect answers. Finally, CAS was expeditious.

Multiple times in this algebra class students employed CAS to check and verify answers. The value of performing this routinely was to develop an awareness of the type of symbolic and algebraic patterns that occurred. Shasta instructed students on CAS' symbolic manipulation features; meanwhile, students made mistakes that he did not anticipate. Shasta called this "an innocent round of symbolic manipulation . . . The kids never would have raised this issue if they hadn't seen the CAS not giving back the response that they were expecting" (Shasta, Interview, October 4, 2017). The presence of the CAS empowered students to take another look at their work product. Shasta said in his pre-interview, "You can't keep that instant feedback" (October 2, 2017), meaning that when a student received CAS' feedback, he felt compelled to revise and retry.

Shasta was asked the question, "What changed in terms of content knowledge about solving equations or distributive property or combining like terms in the presence of CAS, if at all?" He quickly responded.

> The biggest piece is all of the students walked out with an awareness that they have an ability to check their work. And they don't require an authority figure to do it for them. There was a very deliberate offsetting of power and authority within the classroom. (Shasta, Interview, October 4, 2017)

Shasta set a standard for students to routinely check their assignments against CAS. He felt it was imperative in student learning that students were consistent with verifications.

Provide guidance. Shasta provided direction to his students with very specific instructions on the use of the CAS. Shasta's awareness of students first encounter with CAS compelled him to groom detailed syntax guidelines. He was observed walking the room and individually assisting students on the interpretation of outputs. He provided guidance with syntax, interpretation of outputs, and orchestrated opportunities for students to intuit concepts. These are explained in the sections that follow.

Syntax. Shasta carefully and thoroughly provided instructions on keying in commands to the CAS. The computer screen projected CAS manipulations on the wall as he provided directions. Students mirrored his procedures on personal devices. This excerpt from Vignette 1 exhibited how specific Shasta was with his instruction, as well as, revealing mental cues with precise mathematical language. "Now as soon as I type in *negative* [emphasis added], I get two options. Am I going to minus something or am I negating something? I'm negating– I'm getting the negative of three. So choose *negate* [emphasis added], and then type a three" (Shasta, Lesson, October 4, 2017). These were very specific decisions that the student would need to proceed. Furthermore, Shasta demonstrated multiple options for manipulating the device. "While it is highlighted, you could control *C*, copy it, and then move down and paste. There is actually an easier way to do it on the Nspire, press enter" (12:18, Lesson, October 4, 2017). Students learned how to be fluent with CAS.

Interpreting output. Just as Shasta helped students to understand the syntax on input to CAS, he facilitated discussions about how to decipher the outputs. "I was helping them to decode their responses to figure out where the mistakes were. So teaching them to, how to decode, some teaching them how to create their own problems" (Shasta, Interview, October 4, 2017).

Intuit through repetition. Shasta had this notion about the benefit of repetition. There were two sides to this: repetitions in pen-and-paper procedures indicated that CAS power could be helpful, and generating repetitive arguments in the CAS builds connections to mathematical concepts. Shasta shared this in the first interview. "Being repetitive means my machine is waiting for me to ask the right question, but I don't know what that question is yet" (Shasta, Interview, October 2, 2017). This matched evidence in Vignette 3 when students found multiple points for a sequence and entered data into the CAS.

> Let me tell you the big thing is . . . so I was having them enter in 10 data points. In retrospect I think I could have probably done with four or five. I wanted more than just two. I wanted it to be really, really clear that this was a line. (Shasta, Interview, November 1, 2017)

After entering all the data points, the learner questioned whether the points would always fall on a line. I then asked Shasta how many different sequence problems that he had prepared for students to explore in this lesson.

> I had them do three. Not too many as to become mind-numbing and repetitive, but enough to give them a data set because I didn't tell them . . . remember that I told them nothing at all about what the equation was. Just drop a moveable line and then compare the results of the equation after you get a decent fit. (Shasta, Interview, November 1, 2017).

In this second situation he was using multiple examples to hone the conception about linear functions. However, he limited repetition to just three to retain engagement.

Multiple representations. Shasta's lessons revealed occasional use of representation. However, in both cases (Vignettes 3 and 4) Shasta delved into several distinct models. The third vignette representations went from numerical data to tabular data, followed by graphical representation and then to symbolic equations. Vignette 4 examples began with two pen-and-paper representations and then went to CAS. "I want them playing and shifting between multiple representations of math ideas" (Shasta, October 4, 2017). The purpose was to establish mathematical connections. He recounted this allegory.

> I tell them the algebra is always trying to whisper something to them– if they are willing to listen to the story. Can they look at that equation and look at the picture and figure out what the equation was trying to whisper back to them? (Shasta, Interview, October 2, 2017)

All of Shasta's interviews heeded his value on various models. Three significant points Shasta explained: the need to change form, the value in naming a relationship, and the potential for a CAS representative form. These are explicated here.

Change form. These two quotes from separate interviews reflect Shasta's philosophy: "If I could translate this into a different way of thinking I can probably find my answer . . . No math problem was ever solved without . . . manipulating an algebraic expression/equation . . . Doing nothing more than changing between forms" (Shasta, Interview, October 2, 2017). The second quote was, "I've 'preached' multiple representations and how an answer or aspect of a problem that isn't obvious in one form can ‘appear’ when you translate between forms” (Shasta, Written Reflection, October 14, 2017). Vignette 3 was the perfect demonstration of how the linear relationship was exposed when the data points were placed on a graph.

Naming the relationship. “By explicitly naming the algebraic/numeric relationship, students again had multiple ways (visual—and now algebraic) to solve their problems” (Shasta, Written Reflection, December 20, 2017). This referenced Vignette 4 when Shasta coined the phrases rainbow arcs and box method. Both were by-hand sketched models.

A new representation. Shasta hoped that CAS formed a new approach to students thinking about mathematics. He believed that CAS was its own type of representation.

> For me there's basically numbers, algebra, pictures, and words, however you want to work those. I am now convinced that CAS and it's . . . a fifth representation. That by being able to translate your idea into a form that the computer can work on [the idea], changes your understanding of what the problem is. (Shasta, Interview, October 4, 2017)

Likewise, Shasta shared during a later interview the idea of the use of a tool as additional representation. He explained that representation goes beyond the CAS.

> I've said that rule of four in math: algebra, numerical, graphical, and verbal. I'm starting to think that there's a fifth rule now. It is interacting with technology or interacting with other tools. And when you have a tool that is there, knowing how to use it can get you an answer. I would argue compass and straightedge, knowing how to construct a perpendicular allows you to do something that you can't conveniently do algebraically, numerically or anything else. It's learning

> how to ask or use a tool in a way that's helpful in solving the problem in front of you. It is not intuitive how you construct a perpendicular with a compass and a straightedge when you first get it, though. Is not intuitive to know what kinds of commands to ask a CAS when you're first exploring and experimenting. (Shasta, Interview, December 22, 2017)

This elaboration on the idea of a fifth representation required a technological device for students to generate those models. The capability of CAS must be fluid for a student to have the ability to create multiple models. Shasta reflected on a more personal experience of his capabilities as a CAS consumer and as a practitioner with a wealth of CAS utilization background.

Mathematical consultant. The lesson observances revealed an emphasis on CAS as a mathematical consultant for students. Shasta advocated for students to use CAS to verify and check work, even outside the classroom. He subtly shifted a portion a mathematical expertise of knowledge to the CAS.

> I try to keep giving students agency. How does it make sense to you? Make sure you learn, even if it's not your way of thinking. Learn how to listen to somebody else. Learn how to listen to how somebody else is solving it. You can do it your own way, when it's on your time. You need to be able to read and give feedback to a colleague. (Shasta, Interview, December 22, 2017)

He knew that at this juncture of student learning it was more about students developing a strategy to gain access to understanding. "There's like this whole sort of self-driven side of learning, if they're sharing and motivated enough to figure it out" (Shasta, Interview, October 2, 2017). Shasta prompted learners to critically consider the information set before them. "The kids started asking questions because of, sort of the information that they saw in front of them" (Shasta, Interview, October 4, 2017). When students asked him advanced questions, he would give those individuals feedback that would encourage them to explore. Furthermore, Shasta acknowledged that students were less likely to ask him to verify answers. Rather, students would use the CAS as an external resource. "The big change is like this shift in authority" (Shasta, October 4, 2017).

Regulate access. CAS had the potential for many procedures and as a result Shasta purposively refrained from using some of the power in his instruction. "I was deliberately trying to keep them away from an Nspire CAS feature" (Shasta, Interview, October 4, 2017). Shasta directed each CAS command. He chose how and when to reveal CAS commands in every lesson. He gave warnings to students about outsourcing procedures and potentially missing the opportunity to learn through the use of CAS. He often limited CAS permissions on assessments. However, Shasta had flexibility for students with greater desire to explore more thoroughly on the CAS. The next sections will discuss how Shasta gave direct commands, withheld access to the CAS menu, sequenced commands, and managed permissions on assessments.

Direct commands to access CAS. When distributing terms, the CAS command *expand* completed computations fully. Shasta preferred for students to type the input of the problem with the command *equals* and their answer to let CAS verify internally as observed in Vignette 1. A second example of withholding a CAS command was in the solving of equations. Shasta never revealed the CAS command *solve*, but rather had students scaffold the steps on the CAS to arrive at the answer.

CAS menu. Shasta directed students to type CAS commands rather than proceed to the CAS menu options (Vignette 4). His rationale was to train a student to think about the procedure they would like to adopt and search for it on the device.

> The CAS is an extremely powerful tool. I haven't pulled any commands from the menus yet. Partly, that is to keep students unaware of some additional commands. Also, I'm trying to get students to be "intuitive" about commands. I ask them to think about what they are trying to do and to ask the CAS to do that. They focus on "words" becoming un-italicized to recognize when they have hit upon a command the CAS knows. (Shasta, Written Reflection, December 20, 2017)

Shasta felt the menu interfered with students' cognition of mathematical operations. His desire was for students to learn mathematical operations and ask the machine to perform it, as opposed to look on the device for an operation and observe what it did.

Sequence. Shasta carefully chose to go with the distributive property first and then combining like terms, as described in Vignette 1. On the CAS, inputs are automatically simplified and rearranged to include combining like terms. By choosing the distributive property first, this feature was concealed.

Permissions on assessments. Shasta relied on CAS' functionality for students to develop procedural fluency. He gave warnings in class (e.g., the *factor* and *expand* commands in Vignette 4). "If you don't know how to do this, you're going to be completely hopeless when you're facing a quiz" (Shasta, Lesson, December 4, 2017). He discussed how he assessed learning without the CAS until a point in the year when he believed all students had command of the objective.

> Right now I'm trying to maintain a few reigns on a really, really powerful tool. By the end of the year they're going to be "no holds barred" in trying to explore and use. For right now, I almost think of it like a learner's permit. They can get behind the wheel; they just can't go everywhere they want to on their own yet. (Shasta, Interview, December 22, 2017)

He realized his personal responsibility to teach the tool and also to regulate access to the many features of CAS.

Flexibility. There was also a hint of various student capabilities with the CAS. “Anytime a student asks about a command or asks to do something and they don't know how, I will always introduce the command for them” (Shasta, Interview, December 22, 2017). He was the gatekeeper to procedures on the CAS and would gladly provide access to individual students when they inquired.

Summary of Shasta

The lesson vignettes illustrated a teacher-centered instructional approach with a focus on showing students how to access features of CAS to learn mathematics. Shasta regulated access, empowered student learning, and shifted authority to the CAS as a mathematics consultant for students. The tasks that Shasta engaged in spotlighted discussions on algebraic structure and multiple representations. It was clear that algebra one classes were exploring regularity and variation as a way to build procedural fluency.

> Because you need . . . if you have this intuitive sense that the machine can do something but you don't know what the command is, I think it's [the teacher’s] responsibility to teach [students] how to independently discover what that thing is for themselves. (Shasta, Interview, December 22, 2017)

There was an investment of time to learn the tool: commands, syntax, output interpretations, and strategies for using CAS. Shasta stated his purpose for using CAS, "You can learn on a CAS without knowing the rules and the kids are deeply aware that they can use CAS and technology in their learning" (Shasta, Interview, October 4, 2017). His role was to manage pedagogy to develop meaningful mathematical content.

Shasta stressed how CAS enabled a discovery approach to learning concepts and why that was important. "They discover the fundamentals for themselves; they don't memorize what I lecture. It's a much deeper, organic, and long-lasting effect" (Shasta, Interview, October 13, 2017). He crafted his lessons to lead students to the edge of discovering mathematical ideas.

Cross-Case Synthesis

The Cases of Springer and Shasta were thoroughly examined and explained prior to considering the cross-case synthesis. First, pedagogical opportunities were analyzed and compared using the P-Map codes with a concept coding methodology (Saldana, 2016). Second, themes emerged through a comparison of the individual cases' P-Map findings and the application of concept coding. These themes and new codes were cross-referenced within each case. The following sections will discuss those findings from the P-Map and emergent themes.

P-Map

Initially, themes drawn from the individual cases appeared to have oriented their pedagogy quite different from one another. However, further analysis involving the creation of Table 27 and a comparison of hypothesis codes to the number of occurrences from the pattern matching analysis, revealed both participants with similar results. The taxonomies are arranged vertically. Three categories of occurrences (i.e., none, moderate, and strong) clarify the evidence of each code by participant. The data aligned very similarly; approximately the same category matched the P-Map code for both participants. Three categories did not match but came within one strength level (e.g., S2 had strong for Springer and moderate for Shasta). Observations made from the comparison of the two tables do not acknowledge the different stories of the two participants. Each level (e.g., subject, classroom, and task) from the P-Map was compared thus revealing similarities and differences about the two participants.

Table 27

Participants' Pedagogical Opportunities Compared

P-Map	Springer	Shasta
S1	Strong	Strong
S2	Strong	Mod
S3	Strong	Strong
C1	Mod	Mod
C2	Strong	Strong
T1	Mod	Strong
T2	No	No
T3	Strong	Strong
T4	No	No
T5	Strong	Mod

Note. Totals from evidence within five lessons vignettes of Springer and four vignettes of Shasta
No: No evidence
Mod: Moderate evidence (1 or 2 pieces of evidence)
Strong: Strong evidence (> 2 pieces of evidence)

Subject. Both participants exploited differences in the contrast between the ideal and machine mathematics (S1). They each crafted questions from the output on the CAS to provide opportunities to advance student understanding of mathematics content. As well, they discussed feasible outputs (e.g., differences in algebraic or numeric forms) and used the results for verification of the mathematics problem. However, Shasta took an additional step by urging students to critically analyze the outputs to seek recognition of equivalency of the expected answer to the output. "Even though your answer looks different doesn't mean it is" (Shasta, Interview, December 22, 2017).

Both participants rebalanced skills and concepts (S2) in the coursework to develop focused mathematical connections. Springer often reduced cognitive workload with the CAS by outsourcing procedures (Springer, Vignettes 2, 3, 4, and 5). This was not observed in Shasta's classes. Shasta re-sequenced the order of presentation in Vignette 3. His lesson began with a given sequence. Students converted those values as data points, plotted them on a graph, created a line graph, and developed the equation (three actions using CAS). The activity promoted students to make connections. Shasta shared these thoughts regarding that lesson.

> I needed the sequence and I needed those points accurately and I needed them to line up. And the sooner I can get kids on to that, then they're shifting their focus from, "I have an arithmetic sequence," to "Oh my goodness, they always make a line on a graph." (Shasta, Interview, November 1, 2017)

Shasta led students to an analysis of the values within the equation as each related to the sequence's numerical patterns. He utilized dynamic features as a catalyst in constructing knowledge.

Finally, participants chose different entry points to lessons exploiting CAS to build metacognition and overview (S3). Springer consistently used the definition of derivative with the *define* command to take the limit of a difference quotient for a function, rather than using short-cut methods for derivative or CAS command *d/dx* (Vignettes 1, 2, 3, and 5). This delay to introduction of other methods was intentional to build conceptual knowledge of the definition of derivative by shifting student focus to structure and intuition of calculus derivations. In contrast, Shasta instilled students with analysis of CAS outputs to verify procedures. He provided step-by-step guidance for students to notice syntax on both inputs and outputs. Shasta's instruction was directed at building students' metacognitive habits.

Classroom (C1 and C2). The participants used CAS as an external authority to change the classroom didactic contract. Evidence suggested that teachers taught students how to use CAS as a mathematical authority and, in turn, developed those expectations for their students. In both participants' cases, the classroom social dynamics may or may not have changed due to CAS' presence. There were no pre- or post-observations to record those changes.

Tasks. Table 27 revealed that the exact three task opportunities (T1, T3, and T5) afforded as opportunities to adjust pedagogy, but also two task opportunities (T2 and T4) were absent from both participants' lesson observations. In the sections that follow each of the five opportunities will be compared between the two cases.

Pen-and-paper skills (T1). Springer oriented instruction around CAS to develop rules for pen-and-paper skills that she referred to as *shortcuts*. Her pedagogy reflected value of students maintaining procedural fluency (Springer Vignettes 1 and 3). In contrast, Shasta oriented instruction with pen-and-paper and exploration tasks to emphasize algebraic structure (Shasta Vignettes 1 – 4). He claimed that the efficiency, accuracy, and precision of CAS was the perfect tool to assist learners with mathematical skill development. Shasta also used CAS as a verification tool to insure that students understood algebraic properties with complete precision.

Explore regularity and variation (T3). Springer's case provided more robust examples of the explore regularity and variation tasks from the P-Map. As an example, points selected from a function to find the slope of a secant line were generated rapidly with the CAS (Vignette 1). Springer led the class in finding points with input values (i.e., 0.9, 0.99, 0.999) that were approaching the value of one. This variation of input points provided the opportunity for students to understand the concept of limit. Similarly, Springer's pedagogy in Vignette 3 reflected repeated derivatives with the intent for student to recognize the patterns through variation.

Exploration was observed in a different manner in Shasta's lesson (Vignette 3). Students plotted points with the CAS and inserted a *moveable line* that provided the opportunity for students to manipulate the line to approximate the pattern in the data. The dynamic feature of the CAS permitted the student to explore the position of the line as it related to the algebraic equations that CAS was providing. Students then compared answers with one another to seek a consensus on the pattern for the data. This process was repeated for three sets of data.

As a third situation, both participants selected multiple examples for input to the CAS to explore the variation of outputs (Shasta Vignettes 1, 3, and 4; Springer Vignette 2). The selection of multiple examples was not novel for teachers; however, the participants planning with CAS outputs into consideration was important. Shasta focused on the outputs and the variation in algebraic form based on the algebraic expression.

Link representations (T5). Both participants took advantage of linking representations to mathematical concepts. Shasta took the opportunity to access multiple representations (Vignettes 3 and 4) with visual by-hand models and graphic, numeric, tabular, and symbolic forms. He shared how the unique forms could connect cognition to mathematical ideas in mysterious ways. The identification and naming of the different representations likely supported learners in seeing relationships. Furthermore, his extensive experience utilizing CAS brought him to a position to theorize CAS as its own representation.

Springer primarily used the different forms for checking and verifying work (Vignettes 1, 2, and 4). Repeatedly Springer used a symbolic form and compared the result to a graphical form, or vice versa. She also specifically chose to introduce a lesson from a graphical representation or a symbolic expression to achieve her content goals. In another instance, she talked about numeric tables (Vignette 1). The way she used the table was described to students but not observed. However, she recalled a time during a lesson that she used tabular points to find slope of secant lines. Springer was then observed finding numeric values of slope from ordinal points that were evaluated using the *define* feature. She connected the numeric representation of slope to an algebraic difference quotient representation exploiting CAS' symbolic feature of the *define* command.

Use real data and simulate real situations (T2 & T4). Absent from both cases were the pedagogical opportunities of real data and real situations. The objectives for these two tasks imply application of mathematics to real-world contexts. Neither case produced evidence of pedagogy in these task opportunities.

Emergent Themes

Springer and Shasta had similar themes in orienting their mathematics instruction: providing guidance, verifying answers, regulating access, and viewing CAS as a mathematical consultant (see Figure 44). Each participant was recognized as having an additional theme. Springer's lessons revealed outsourcing procedures as a fifth major theme within her pedagogy. Furthermore, Shasta's lessons portrayed the feature of multiple representations in the utilization of CAS. Within each theme there were some variations on specifics of pedagogy. The detail of the emergent themes will be described in the sections that follow.

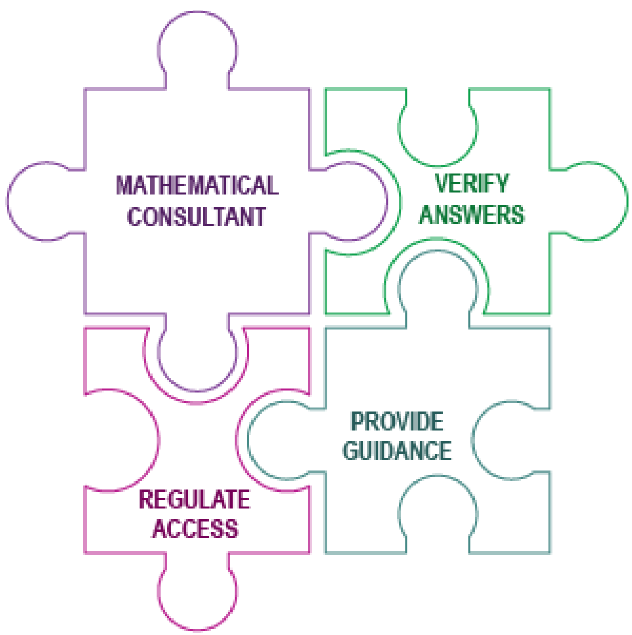

Figure 44. Emergent Themes: Springer and Shasta.

Verify answers. The element of CAS' ability to check accuracy and precision of algebraic solutions was helpful for students to gain agency in their learning. Both Springer and Shasta afforded those opportunities. Shasta clarified the benefits.

> I did the example of the distribution on top and then underneath in red, I wrote all of the different wrong things kids can do. Sometimes, rather than burying it, saying this is what you should do, pulling it out and naming where the mistakes are . . . Naming the things that kids are doing, makes them aware of it. (Shasta, Interview, October 4, 2017)

Learners needed awareness to understand their mistakes so that they could self-correct. The efficiency of CAS provided students feedback at the moment it was needed (Shasta Vignettes 1, 2, and 4).

> "Even though your answer looks different doesn't mean it is. . . . Do you now know how to go back and look at the command and make sure you asked the right question? Or can you take that and tease apart— here's the part of the question that I got right, and here's the part that I got wrong. Can you go back and figure out your own error and where it occurred?" (Shasta, Interview, December 22, 2017)

Shasta communicated about the need to consider both inputs and outputs to build understanding. In addition to learning from mistakes by looking at CAS outputs, Shasta would say CAS verified without judgment, another element that benefits the student. Shasta also recognized that CAS permitted students varying levels of practice. The CAS did not distinguish challenging, complicated procedures from simple ones. CAS could handle both types of problems with the same ease.

Springer viewed CAS as a tool for verification and to build students' confidence. Joy came from student successes of performing operations accurately. "It's important for students to enjoy the learning process and have success with learning the material to gain confidence with their math abilities and be confident in their abilities to learn future concepts" (Springer, Written Reflection, October 13, 2017). Observation revealed that like Shasta, she purposively keyed mistakes into the CAS to give confidence to learners that aberrations could be helpful. It was through analysis of mistakes that learners not only developed methods to avoid the miscue but also gained depth of understanding.

Provide guidance. A substantial amount of guidance assisted the students in adapting to the technology and utilizing it for framing access to mathematical knowledge. The two participants varied on management style of students' syntax issues. Springer offered flexibility in her instruction to provide help in the moment. Shasta managed syntax issues more on the frontend by providing warnings and very specific directions. He was systematic and provided step-by-step instructions. "I don't want to frighten students off because they see me just whipping through something really quickly," (Shasta, Interview, December 22, 2017). Shasta methodically worked through examples with his students.

However, Springer was inclined to rely on students to assist classmates with small syntax issues. When parenthesis or multiplication dots were missing the CAS would output an error. At times she purposely let those errors be revealed, which were then used as discussion points to connect either to syntax issues or mathematical ideals. In contrast, Shasta would be more inclined to circle the room and give more individualized feedback to students regarding those syntax problems.

An important aspect in both participants' instruction was that they modeled the exact commands and procedures on the computer as it was projected on the wall.

Springer had talked about the time when that was not possible. Current technology supported students watching the teacher and mimicking identical keystrokes.

Mathematical consultant. There was evidence of a slight shift in mathematical authority from the teacher to the device for both participants. CAS provided specific, instant, and accurate feedback. Students were empowered to check all problems, as well as invent their own inquires. Students then potentially achieved competency in their skills. Shasta encouraged CAS as a tool that granted students a strategy to gain access to individualized learning. He transferred agency to each student to develop knowledge at his or her preferred pace.

By comparison, Springer entrusted CAS as a reliable source for mathematical procedures to help with solving problems. Springer granted permissions to outsource procedural problems to CAS only after skills had been mastered. Her concern was rooted in two potential areas: tedious by-hand skills that could frustrate students and the possibility of lost focus due to significant procedures in the midst of learning mathematical conceptions. Exploiting the CAS as an external authority brought organization to learning.

Regulate access. The participants permitted students regular access to CAS. Student laptop computers were pre-loaded with the TI-Nspire™. However, Springer and Shasta regulated student's use of CAS. Springer's case revealed the following management practices: direction of the command to utilize; sequence of the order in which commands were accessed; regulation to the degree of difficulty of commands and permission for using CAS in assignments and assessments. Shasta also managed students with these three practices: giving directions, sequencing the commands accessed, and restricting permissions on assessments. The four different management practices will be explained in the following paragraphs.

Springer and Shasta withheld particular commands at times, only to later release them for student use. As an example, Springer chose to delay the derivative *d*/*dx* command throughout all the lesson vignettes. In doing so, she hoped that students would develop a richer understanding of a limit of a difference quotient. "I'm really trying to hone in . . . and have [students] really understand the difference quotient and the limit, so instead of using the [derivative] command I was making them use the definition of derivative" (Springer, Interview, November 8, 2017). Shasta also delayed commands frequently. He cautiously proposed each new command with words of warning. This was rooted in his fear that a student would inadvertently outsource procedures to the CAS in lieu of advancing their learning potential. Shasta's shared his perception.

> I don't want [students] looking in the menus yet. Some of them will. One or two of them already have, but for the most part I want them using the tool rather . . . using the tool for what we're doing, rather than sort of like investigating the fastest way out. Nobody has figured out *solve* yet. So again evidence that . . . like I have gone half a year in this class and nobody knows *solve* yet and I'm making huge use of the CAS in class. (Shasta, Interview, December 22, 2017)

Shasta directed students to aspects of the CAS that he chose, rather than releasing control.

When Springer moved to application problems students struggled to keep up with CAS commands. Although these were not new commands, they were used in a new manner creating a challenge. In one situation (Vignette 5), Springer differentiated a rational function containing trigonometric functions (see Figure 25). The derivative output an additional variable (i.e., *n1*) due the fact that the derivative was also a trigonometric function producing a periodic function as the output. The CAS output revealed multiple solutions to the problem, forcing the user to consider restricting answers to a particular domain. The restriction of the domain required additional commands that Springer had to teach creating the extra challenge with syntax, despite students having performed similar procedures. Springer allocated CAS commands to gradually incorporate their functionality.

Both participants regulated access particularly when assessing students. Shasta was insistent on assessing student performance in the absence of CAS. Shasta talked of a future day when he would be less concerned about the distinction of non-CAS assessments, but this was not observed. “There's a difference between assessing a student’s ability to do mathematical manipulations and assessing a student's ability to solve problems mathematically” (Shasta, Interview, December 22, 2018). He was observed withholding access to CAS in a post-lesson quiz.

Springer, who was not observed assessing in the presence of CAS, managed assessment a little differently. “If you're going to be able to use the CAS on homework and assessments, how do I come up with other questions that are not, like plug and chug” (Springer, Interview, October 2, 2017). Rather than remove CAS completely, Springer chose to alter the types of questions on assessments.

Outsource procedures. Springer repeatedly directed students to let CAS perform procedures in order to focus on other mathematical concepts. She felt that redistributing computations gave students an advantage in their ability to attend to new mathematical knowledge. The following excerpt conveyed Springer's impression of how outsourcing procedures benefitted students.

> There are so many algebra steps for [students] to make mistakes. And it's like they could understand the calculus. They could understand what to do, but then don't know how to *expand* something. Or don't know how to go through the process. I think they enjoy this idea of, from what they've told me, they enjoy, they understand what it is they have to do and the calculator is kind of . . . helping them along the way, accomplish what it is they have to do. (Springer, Interview, October 15, 2017)

Springer took opportunities to outsource procedures in these ways: produce results of algebraic procedures (e.g., solve, expand, simplify rational expressions), reinforce procedural skills by verifying by-hand skills with CAS, and target other areas of mathematics. In contrast, this was never observed in Shasta's algebra class, although he hinted at the potential of that occurring later in the year.

Multiple representations. Both participants provided opportunities for students to learn multiple representations, but Shasta's case stood out as a greater necessity for his pedagogical practice. For example, in Vignette 3 he asked the students in his class if they would be able to rewrite the linear function in any of the forms: numeric, graphic, verbal, and symbolic representation. "By shifting to the graphical representation, the linear relationship in the data jumps in your face" (Shasta, Written Reflection, October 14, 2017). Shasta aimed to develop students' capabilities in working with multiple representations to access any given math problem from all representations and to change it to another form. In contrast, Springer valued representations as a way to verify answers and also to access different characteristics of functions. Springer showed how the symbolic output of a derivative presented negative infinity ($-\infty$) and compared to the functions' graph (see Figure 8, Vignette 2). The different representations provided an opportunity for students to connect symbolic output to the graphs asymptotic behavior.

Summary of Cross-Cases

The pedagogical opportunities around which both participants oriented their instruction around were primarily in the subject area level of the P-Map. They also had similar ties to a change in the classroom didactic contract and the tasks that they chose to employ. Much of the participants' utilization of technology was grounded in CAS' functional capabilities. Emerging from the data analysis were six themes: four that were common for both participants, two distinct themes that were uncommon. Those in common were: CAS as a verification tool, the need for teachers to provide guidance and regulate access to the CAS, and the benefit of CAS as an external mathematical consultant. The participants oriented their pedagogy with these themes in mind. One participant also esteemed multiple representations to the point of elevating CAS models as potentially its own form of representation. The other participant commonly outsourced procedures to the CAS to alleviate some of the tedium of mathematical procedures and also to enhance lessons by easing the cognitive workload. All six themes are combined and will be presented as the Schema for CAS-Oriented Instruction in the next chapter.

Chapter Summary

Lesson vignettes depicted teacher CAS-infused lessons that were observed in this study to reveal teacher pedagogical opportunities. Each lesson vignette used pattern-matching logic with hypothesis coding (Saldana, 2016) of the P-Map framework to illuminate, clarify, and define elements of the lesson in which the teacher afforded the opportunity to exploit CAS to develop mathematical understandings. Ms. Springer's lessons were analyzed first; followed by Shasta's lessons. The next step involved a cross-case synthesis to compare the participants P-Map cases. The evidence revealed very similar affordances despite the lessons seeming to have different pedagogies. Through a retrospective analysis and concept-coding (Saldana, 2016) of the cases and cross-case six emergent themes materialized. The themes were then applied to the individual cases. Four themes were in common for both participants and one was added for each participant that were unique to the other four. The next chapter will present the emergent theme and connect the scheme to related literature.

CHAPTER V: SUMMARY AND DISCUSSION

Introduction

Teacher pedagogy in a CAS-rich milieu was the focus of this study. NCTM (2014) acknowledged mathematical tools and technology as essential resources integrated into classrooms to benefit learners as communicators, mathematical problem solvers, and reasoning and sense making citizens. Integration of technological devices presents challenges to teachers to imagine and develop methodology for integration of such tools (Blume & Heid, 2008). Australian educators have been implementing CAS for nearly 20 years and shared a perspective of pedagogical opportunities that teachers have afforded in CAS-rich classrooms (Garner, 2004; Garner & Pierce, 2016; Kendal et al., 2005; Pierce & Stacey, 2002, 2004, 2008, 2010, 2013). The theoretical framework generated by Pierce and Stacey (2010) provided the lens to describe the essence of pedagogical practices of teachers. In particular, it was specifically those lessons in which CAS was exploited to develop mathematical understandings.

This qualitative study examined the pedagogy exhibited by secondary mathematics classroom teachers as they utilized CAS technology. Pierce and Stacey's (2010) P-Map illuminated affordances that two teachers actualized in their classroom lessons and shared in both written reflections and interviews. The opportunities that teachers achieved exploited CAS in the development of mathematical knowledge.

In this chapter, a restatement of the research problem, a review of methodology, and a summary of results are presented. The discussion of results from the study includes an interpretation of research findings and connections to prior research. The chapter closes with implications for practice and potential areas for future research developments.

The Research Problem

Research surrounding the development of rich pedagogies when it pertains to using CAS technologies is limited (Heid & Blume, 2008; Heid et al., 2013; Pierce & Stacey, 2010; Schultz, 2003; Usiskin, 2006; Zbiek & Hollebrands, 2008). The mathematics education community is overdue for a new immersion into technological tools with the teacher as the change agent (NCTM, 2014; Zbiek & Hollebrands, 2008). NCTM supported technological development of mathematical concepts through *Principles and Standards for School Mathematics* (2000) stating "The computational capacity of technological tools extends the range of problems accessible to students and also enables them to execute routine procedures quickly and accurately, thus allowing more time for conceptualizing and modeling" (NCTM, 2000, p. 25). Through the functional opportunities of CAS, teacher pedagogy can include CAS tools that motivate and promote students' grasp of mathematical knowledge (Heid et. al., 2013; NCTM, 2014; Pierce & Stacey, 2010). Far reaching effects of CAS can encompass many areas in mathematics education such as curriculum, assessment, accessibility, teacher beliefs and attitudes, and more. This study specifically focused on teacher pedagogy.

Obstacles to teaching with technology are numerous (Ertmer, 1999; Ertmer & Ottenbreit-Leftwich, 2010; Hicks, 2010; Kaput, 1992). Ertmer (1999) referred to obstacles as first- and second-order barriers to change. First-order barriers involve issues that are external to the teacher, such as lack of equipment. Teacher beliefs about teaching and learning with technology fall in the latter, as a second-order barrier to change. Reasonably, one can conclude that by reducing barriers, the issue of limited research of teacher pedagogy that exploits CAS technology can be centralized and examined. This study investigated teacher pedagogies in a milieu of minimal barriers to change so as to answer the research question: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems (CAS) in the development of mathematical knowledge?

Review of Methodology

A qualitative holistic multiple-case design (Yin, 2009) was deliberately used to garner the views of two distinct examples on the real-life phenomenon of innovative teacher practices as contemporary technologies of CAS surface in education culture. Gay et al. (2012) classified particularistic studies, those that focus on one phenomenon, as a case study. Insufficient pedagogical opportunities that utilized CAS produced a chasm between current practices and potential instruction. Since the teacher drives the pedagogical decisions, the examination of teachers' administration of CAS acquaints educators with CAS-enriched pedagogies that aid in the development of mathematical knowledge.

This multiple case design employed a within-site scheme. By keeping the study limited to one school, the cultural aspects remained fixed; two teachers with numerous lessons were varied. Data from two participants provided more robust results due to intentional replication of conditions (e.g., similar students and school culture) (Yin, 2009). However, differences between teachers availed the opportunity for deeper analysis of the theoretical framework according to Yin (2009).

After data were collected, detailed descriptions of classroom lessons were written. The first round of coding involved a deductive analysis. Teacher activities were mapped to the pedagogical opportunity taxonomy (Pierce & Stacey, 2010) revealing key features of the participants' decisions. The individual cases went through a second cycle of open coding to reveal emergent themes. The cross-case analysis revealed similarities and differences between the individual cases. A *naturalistic generalization* (Creswell, 2007) from the comparative analysis of the two cases augmented the Schema for CAS-Oriented Instruction.

Review of Results

Two secondary teachers supplied data for nine lesson vignettes that captured the essence of CAS-infused instruction. Each narrative was aligned with Pierce and Stacey's P-Map (2010) to highlight teacher pedagogical affordances. The individual cases of the participants (i.e., Springer and Shasta) revealed notable aspects in decisions to exploit CAS to develop mathematical understandings. The retrospective analysis of the data revealed several emergent themes. Participants oriented their instruction in the development of mathematical knowledge through the practices of providing guidance, verifying answers, regulating access, viewing CAS as a mathematical consultant, outsourcing selected procedures, and accessing multiple representations. Four of these themes were consistent with both participants; one additional theme for each participant completed the schema. Nevertheless, themes interlocked to form the Schema for CAS-Oriented Instruction with the six emerging themes illustrated in Figure 45. The center four pieces represent the common themes for the two participants. The themes interlocked forming a more complete illustration of teacher pedagogy that gives direction to other teachers in the presence of CAS. Each component is explicated in the following sections.

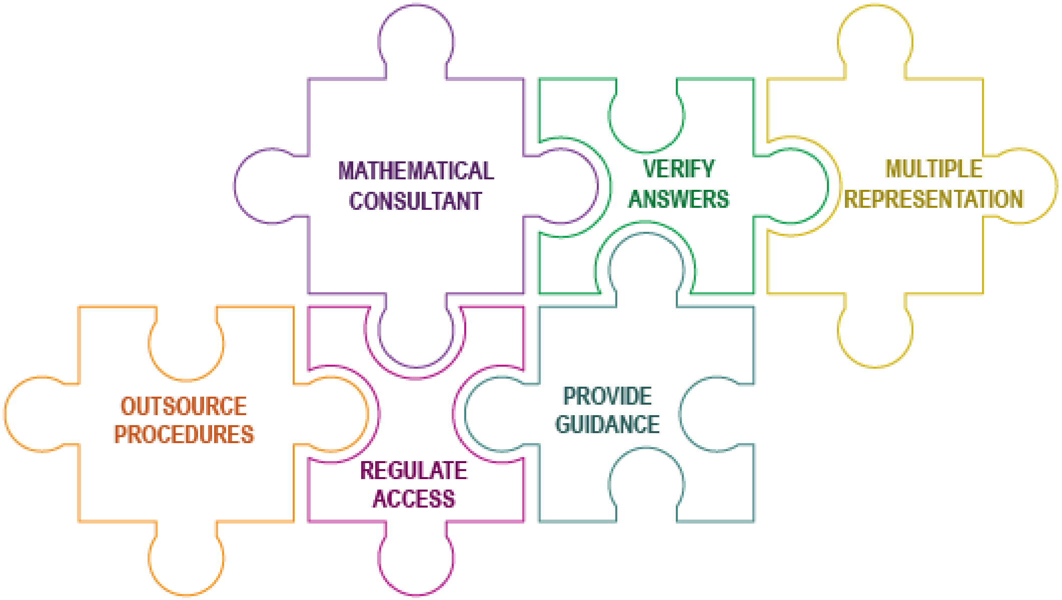

Figure 45. Schema for CAS-Oriented Instruction.

Mathematical Consultant

The phrase *mathematical consultant* regards the student or teacher's ability to access CAS as an external mathematical authority. The word *consultant* was used to recognize that CAS might be called upon as an additional resource, thus honoring the absolute mathematical authority (e.g., teacher, etc.). Shasta preferred to give students agency to their learning; CAS enabled students to practice the type and amount of mathematics problems as needed to achieve competence. Shasta described CAS in the context of a *non-judgmental mathematical friend* with which students consult. Teaching students how to perform the operations to achieve this type of usefulness overlapped into the themes *provide guidance* and *verify answers*. Alternatively, Springer preferred students to achieve accuracy in their work for success in a larger contextual problem. At times, Springer directed students' focus on conceptual development of broader mathematics and CAS assisted as an external authority for mathematical procedures branching into the theme *outsource procedures*.

Verify Answers

The functionality of CAS accommodated learners checking answers. The value of CAS as a verification tool was embedded in its accuracy and precision. Springer's students and Shasta's students verified their pen-and-paper skills against the CAS. Syntax issues had to be resolved in order to input into the device and to understand the outputs. The theme *provide guidance* overreached into the *verify answers* schema. Furthermore, Shasta regarded CAS as a tool that reserved no judgment. He demonstrated the usefulness of CAS and how it allowed the student, after making a mistake, an opportunity to immediately determine the error and self-correct. The expeditious capability of CAS proved useful.

Multiple Representations

The term *multiple representations* referred to all forms or models that a mathematical idea can behold: numeric, tabular, graphic, symbolic, or written. The first four models are readily represented on a CAS. Shasta worked through all the forms in Vignette 3. He generated multiple representations within CAS more earnestly than Springer. Springer accessed graphic features of Desmos to use in comparison with symbolic outputs on the TI-Nspire™. She utilized multiple representations primarily as a checking tool, similar in nature to the theme *verify answers*. In contrast, Shasta utilized multiple representations as a way to conjure additional mathematical understandings.

Regulate Access

Both Springer and Shasta regulated student knowledge of CAS commands. Each participant selected certain commands for students to access during specific lessons. Shasta worked with students in an introductory algebra class, and they had less familiarity with the tools. He regulated access through sequencing commands, directing usage of commands, or withholding commands. On one occasion in Vignette 1, Shasta directed students to key the command rather than access the command through menu features. Part of his rationale was that he wanted students to make decisions regarding a mathematical command; he did not want students looking at the menu to select choices. Shasta deliberately assessed students without the CAS on occasions. Similarly, Springer explained that she opted for a similar assessment at times, but she prompted students to set up problems and write out the words, "I used CAS to solve."

Springer's outlook on accessibility to CAS differed slightly from Shasta's view. She felt that students would use CAS tools on assignments when she was not monitoring their usage. Her perception was that by teaching students how to use the tools to deepen their mathematical knowledge, she would, in effect, benefit her students. They would then have the knowledge of using CAS in a productive manner. *Regulating access* interlocks with *providing guidance*.

Provide Guidance

Substantial time was allocated to teaching how to use CAS; the teacher *provided guidance* to students throughout the utilization of CAS. Each participant managed instruction with CAS with a distinctive manner. Springer began with CAS projections and had students mirroring her commands. She forged ahead with her lesson plan, pausing at times not only to give students the opportunity to ask one another syntax questions and to key in commands but also to walk around the room and monitor student activity. Shasta generally started a CAS lesson very methodically, providing instruction in advance of keying commands into the CAS. The theme *provide guidance* intersected with all aspects in the CAS-Oriented Instruction schema. Without guidance, students will not develop the technical knowledge to utilize CAS to effectuate learning.

Outsource Procedures

Springer exploited CAS to outsource procedures with the potential for students to reduce cognitive struggle surrounding procedures, thereby targeting other mathematical conceptions. CAS was integrated to do the procedural work (e.g., finding derivatives, solving equations, or other computations) in the midst of broader mathematical problems. Springer also used outsourcing procedures in conjunction with developing procedural fluency, balancing pen-and-paper skills acquisition with developing connections to the algorithms. Although I never observed Shasta outsourcing procedures, he shared this philosophy as the mathematics department chair, Springer's superior in command. He offered the following insights:

> We never ever said that you shouldn't learn how to do that computation. What I am saying is that in the midst of an application, in the midst of extending your knowledge into a new realm– that is not the time to be making computation mistakes. It is not the time to be making data entry mistakes. (Shasta, Interview, November 6, 2017)

Shasta's intent to the phrase *that computation* was a very general purpose of algebraic or symbolic computation. His philosophical statement reflected Springer's actions on outsourcing procedures. As Springer's supervisor and mentor, Shasta's views may be the initial perspectives adopted by Springer.

Summary of Results Overview

This investigation of teachers orienting their pedagogy in the utilization of CAS technology revealed six interlocking elements. Four of these elements were action oriented: verifying answers, regulating access, providing guidance, and outsourcing procedures. These actions occurred as teachers performed instruction seamlessly. The other two elements (i.e., mathematical consultant and multiple representations) were conceptions or products of CAS. These two formed an overarching philosophy about what CAS contributes to pedagogical practice. Springer and Shasta yielded sufficient evidence to support these elements as individual pieces. The junction of the six elements formed a more complete illustration in answer to the research question: How do secondary mathematics teachers orient their instructional practices to exploit computer algebra systems (CAS) in the development of mathematical knowledge?

Discussion of Results

CAS-oriented lessons were rich with stories that revealed the complexities of teacher pedagogy. Multiple actions occurred simultaneously. As teachers reflected on their decisions, they shared deep-held beliefs about their own teaching and learning practice. The communications from those interviews and written reflections sustained the lesson vignettes and helped to formulate the Schema for CAS-Oriented Instruction. In the following sections, the emergent themes are connected to the literature reviewed in this study.

Connections to the Prior Research

The pedagogy participants implemented with CAS connected to several pieces of literature regarding both CAS and broader educational ideas. The two different approaches to the data analysis (i.e., deductive analysis using the P-Map and emergent themes from the data) are addressed separately in this section. First, the effectiveness of P-Map as a researcher's tool is disclosed. Second, themes are described within the context of the literature review.

P-Map deductive analysis. Pierce and Stacey's (2010) P-Map addressed all six components from the schema and formed the basis of the analysis. Of the 10 pedagogical affordances, eight were identified in the combined observations as clear evidence in the individual cases. However, a significant result from this study was the absence of two tasks and a minimal connection to the pedagogical opportunity of a change in classroom social dynamics. Addressing the change in dynamic first, the data collected did not consider a time prior to CAS as a comparative analysis. In fact, it was unlikely that one can make a true parallel to the change in the classroom didactic contract. Both of these opportunities attend to the classroom level on the P-Map, noted as *change* in the description of the taxonomy. The research of instruction through CAS was not a transformative process of comparing a change in authority; however, the class culture emitted a presence of CAS as a mathematical authority. CAS did become an authoritative resource for the students and for the teacher, and, thus, gave light to the pedagogical opportunity *change classroom didactic contract*.

Unfortunately, two types of tasks were not observed during this study: use of real data and simulation of real situations. The noticeable absence of these tasks does not lessen the value of the study. Limitations of the number of lessons observed and the manner in which data were collected (i.e., as a screencast in a classroom) may have decreased the likelihood to observe such tasks.

Mathematical consultant. Pierce and Stacey (2010) labeled one of the classroom level pedagogical opportunities a *change in classroom didactic contract*. The discussion of the change comes about through an authority shift when CAS was accessed in classrooms because "students may gain a new sense of personal authority" (Pierce & Stacey, 2010, p. 9). Given that the P-Map was a tool to pattern-match data in this study, a sense of an external authority came from the device that subsumed mathematical knowledge. This definition is supported by Langer-Osuna (2017) who stated, "The most relevant type of authority is that of the expert who possesses mathematical knowledge that is taken as true" (p. 238). This classification was evident in teacher pedagogy and classrooms. The term *consultant*, instead of authority, was the chosen description for the Schema to recognize the teacher as a more significant authority of mathematical knowledge in the classroom. Springer and Shasta encouraged students to verify procedures through the tool both in and outside of class. Springer endorsed CAS as a tool to outsource procedures, thereby assigning mathematical authority to CAS.

Verify answers. Shasta and Springer used CAS extensively to verify answers. They accomplished verifying answers by pitting by-hand procedures against the CAS or comparing multiple representations one against the other. "[Students] are focused on making sense of mathematics, comparing varied approaches to solving problems, and defending, confirming, verifying, or rejecting possible solutions" (NCTM, 2014, p. 109). Shasta verified answers to refine and produce procedural fluency. He found that students developed a good ability to reason their procedures when they made mistakes. Zbiek and Hollebrands (2008) claimed, "Feedback offered by different computer resources may allow students access to information that allows them to correct their own errors" (p. 312). This was precisely how Shasta promoted CAS.

Multiple representations. NCTM (2014) recognized the functionality of CAS. "Graphing applications can allow students to examine multiple representations of functions and data by generating graphs, tables, and symbolic expressions that are dynamically linked" (p. 78). Shasta showed that those representations provided for student's opportunity to learn. His accounts reflected instruction of creating the different forms with by-hand methods and through a CAS device. However, Shasta created multiple representations to assist learners in the development of connections to the problem. As such, he regarded three points as significant to representation: the need to change form, the value in naming a relationship, and the confirmation that each form provided. Shasta's view was reminiscent of Pea's (1985) claim "that a primary role for computers is changing the tasks we do by reorganizing our mental functioning, not amplifying it" (p. 168). The points Shasta made take Pea's idea into a pedagogical realm by naming and claiming the different representations.

Shasta held strong beliefs regarding multiple representations of functions. "I've 'preached' multiple representations and how an answer or aspect of a problem that isn't obvious in one form can 'appear' when you translate between forms" (Shasta, Written Reflection, October 14, 2017). Likewise, Fonger (2012) made the case that the representations create more fluent learners. They possess the potential for greater success in problem solving regardless of the form presented. A rationale for multiple representation benefitted students beyond their ability to create the representation.

Regulate access. Students were permitted regular access to CAS devices; however, the teacher limited exposure to certain commands. Participants regulated access three particular ways: (a) delayed or withheld commands, (b) determined access on assessments, and (c) managed the release of commands. First, both participants delayed or withheld commands until the need arose. Kastberg and Leatham (2005) presented the finding from a meta-analysis that the teacher mediated calculator access and decided how and when to utilize CAS. This study confirmed the notion of teachers controlling student interactions with CAS.

Second, the determination of whether to permit CAS on assessments was an issue to which participants tended. If CAS were permitted, then question adjustment necessitated planning. Weigand (2014) had supposed that meaningful assessment questions were particularly challenging in a digital technology environment. Both participants agreed that the goal of the assessment had to be considered first if CAS were permitted.

Third, consideration of regulating access as part of the schema was only logical. The participants controlled access; however, approach to the usage of CAS required the participants' management to draw out student understandings. Students often lacked knowledge of the device and its commands; hence, instruction facilitated student designation for use. Access was considered a second-order barrier in previous studies (Wachira & Keengwe, 2011; Ertmer, 1999; Ertmer & Leftwich-Ottenbreit, 2010; Ivy &

Franz, 2016). Springer and Shasta carefully sequenced the release of knowledge of the device as the mathematical content necessitated it.

Provide guidance. The theory of instrumental genesis (Artigue & Diderot, 2002) emanating from the work of Verillon and Rabardel (1995) applies to learning the CAS as a cognitive tool in the development of mathematical knowledge. Instrumental genesis "attributes a major role to artefacts [*sic*] that mediate the human's activity for carrying out the task" (Drijvers et. al., 2013, p. 27). Considering this theory in light of the CAS tool, one cannot assume an automatic assist. The tool has interplay with human interaction or acts as an extension to brain activity. The intermediary of the participant seized the opportunity to develop this relationship to the tool. Springer and Shasta provided guidance for students in the syntax, representation, and interpretation of outputs on the device. Springer described how she introduced new commands organically as the need for a command arose in the lesson. This theme aligned with the theory of instrumental genesis in that the object (i.e., CAS) became a tool (i.e., cognitive aspects) for learning mathematics.

Academics issued concern due to the complexities of learning a technological device (Artigue & Diderot, 2002; Blume & Heid, 2008; Kieran & Saldanha, 2008). Technical and conceptual knowledge has the potential to confound learners as they attend to both simultaneously (Blume & Heid, 2008). Kieran and Saldanha (2008) designed an exploratory lesson with the principle "aim at supporting the development of conceptual knowledge within technical activity" (p. 399). This secondary classroom lesson reflected a similar approach in Springer and Shasta's lesson vignettes. The participants guided the technical and mathematical knowledge every step through the lesson.

Shasta and Springer placed importance on providing specific individualized directions to students on the syntax of CAS. Springer shared her view that she was intentional to avoid frustrating students with the CAS. Jakucyn and Kerr (2002), secondary teachers that also utilized CAS, had similar feelings. "The syntax sensitivity of a CAS can frustrate and discourage students. Providing clear instructions on using a CAS and keeping the introduction of new commands to a minimum, were therefore important" (p. 629). In addition, Shasta and Springer both used computer projectors to support student progress. Doerr and Zangor (2000) recognized that a projection device visibly displaying procedures on CAS assisted the teacher to develop classroom discussions. Springer's lessons revealed student interaction, with all primarily regarding syntax. Ivy and Franz (2016) also claimed that students assisted one another with syntax issues.

Outsource procedures. Concern regarding the use of technology to perform computations in lieu of by-hand procedures pervades education culture (Cedillo & Kieran, 2003; Drijvers, 2000; NCTM, 2014; Ozgun-Koca, 2009). The concern stems from an apprehension that teachers will neglect building procedural fluency, and students will develop a reliance on technology relinquishing attainment of procedural fluency. NCTM (2014) defined procedural fluency as "the meaningful and flexible use of procedures to solve problems" (p. 7). CAS' functionality provides for production of procedures in absence of skill.

The term *black box* technology, which originated with Buchberger (1990), helps to understand the dilemma. The black box concept refers to a technology that is disconnected to knowledge about the functional operation of the command. The user only seeks the output without knowledge of the inner workings. For example, inputting $3x + 4x$ provided an output of $7x$. If the user were unfamiliar with addition of variables, the symbol for addition (i.e., +) may be meaningless. However, a CAS step-by-step approach that involved a factorization or expansion of terms would not be considered a black box technology. Springer utilized CAS with the black box approach in some instances. She used the *solve* command within application problems by outsourcing steps to retrieve answers to equations. In a different lesson, she generated rounds of higher derivatives to analyze the outputs, again utilizing CAS as a block box tool. These situations did not deter conceptualization of mathematics because the focus was not on the procedures.

In a similar manner to Springer's application problems, Drijvers (2000) pointed out a concern that students lacked conceptual understanding of derivative while solving optimization problems. Drijvers explained the black box approach here.

> The concept of the derivative as a 'rate of change' has been taught to the students, but they do not yet know how to apply the rules for differentiation. They are forced to leave the derivation of the functions that model the optimization problems to the symbolic calculator. Computer algebra thus serves as a 'black box' that may motivate the students to learn the rules after the experiment is finished. (Drijvers, 2000, p. 197)

In contrast to the black box technology, Springer used an additional model to insure that students developed understanding of the derivative of a function.

Springer's model was comparable to the *white box* technology. Particularly, she accessed a white box perspective with derivatives in calculus. Instead of utilizing d/dx

commands, she performed calculations using the limit of a difference quotient. She continued to instruct with this white box approach as she introduced application problems. The argument for Springer's approach was that she maintained a connection to derivative with her methodology.

Summary regarding connections to literature. Functional opportunities of CAS are the basis for pedagogy as represented in the emergent Schema for CAS-Oriented Instruction (Figure 45). The purpose of CAS-rich instruction was to present opportunities to advance student mathematical understandings. NCTM (2014) technology recommendations acclaimed mathematical action technologies as essential elements for instruction.

> Given the accelerating ease with which technology can be used to carry out nearly any mathematical procedure that students might be asked to perform, mathematics educators may need to raise questions about the balance of procedural and conceptual knowledge required for mathematical proficiency. (NCTM, 2014, p. 88)

Beyond the scope of this study are questions about the balance of knowledge; yet, the intent of the participants was to produce opportunities for understanding. As well, procedural knowledge was valued. The emergent themes from the schema intended to capture the essence of participants' eliciting an advancement of student knowledge. The connection of the schema was to widespread literature.

The P-Map framework (Pierce & Stacey, 2010) used in the deductive analysis proved a useful tool to organize, categorize, and illuminate pedagogical opportunities. Limited examples were produced in the classroom level and in realistic tasks.

Conventions in pedagogical practice conveyed characteristics related to prior research and literature. Mathematical authority is recognized in related and unrelated mathematical literature (Langer-Osuna, 2017). In this context, mathematical consultant was presented as an additional resource that may provide agency to student learning, calling upon Langer-Osuna's definition (2017) "of the expert who possesses mathematical knowledge that is taken as true" (p. 238). As such, CAS had the capability to be a checking tool.

Verification of mathematics problems affords learners with sense-making opportunities through instant feedback (Zbiek & Hollebrands, 2008). This verification can extend to multiple representations of mathematical expressions, by pitting access to one form against another. Through representations learners adapt information to mathematical problems with flexibility and earnestness (Fonger, 2012). A concern and obstacle to CAS-enriched instruction (Wachira & Keengwe, 2011; Kastberg & Leatham, 2005) regarding access to technological devices was validated. Yet, through appropriate fidelity to regulation, participants in this study facilitated designated uses of CAS to students for the purpose of drawing out mathematical understandings.

As a theme, regulating access is an appropriate criterion. The teacher must be a guide to assist learners in developing both technical and conceptual knowledge (Blume & Heid, 2008). An important aspect for teachers of technological devices is to understand the tool as a device to which students gradually adapt. Theory of instrumental genesis (Artigue & Diderot, 2002; Verillon & Rabardel, 1995) recognizes the transition of the CAS tool to learners' frame of reference. As such, the utility of technology becomes an extension to the learner's ability to reorganize cognitive functioning (Artigue & Diderot, 2002; Pea, 1995). The result endows the learner with opportunity to engage reflectively on the mathematical knowledge. Participants in this study provided necessary guidance.

The final theme, outsource procedures connects the *black box/white box* (Buchberger, 1990) model to this study. Heid (1988) in her seminal study utilized a black box approach in teaching calculus to college students. She outsourced procedures to the CAS for the purpose of devoting instruction towards conceptions. Springer utilized a black box approach for students to efficiently and accurately produce answers in the midst of solving problems and focusing on other key concepts. CAS utilization afforded the participants opportunities to develop sufficient practices that benefitted their teaching craft for the purpose of enriching student knowledge.

Implications for Practice

CAS pedagogy has provided evidence that it can be exploited with the potential to develop mathematical understandings. The nine lesson vignettes highlighted methodologies for accomplishing that instruction. Implications can be extended to pre-service teacher educators, professional development designers, and secondary mathematics teachers.

The Schema for CAS-Oriented Instruction provides education leaders with three discussion points in pedagogical areas for teachers to deliberate. First, when practitioners are presented with the idea of CAS as a mathematical consultant to benefit student learning, teachers may begin to form an opinion about CAS as a tool that fills a void as an external mathematical authority. That belief may generate interest in teachers pursuing methodologies that promote student independence. Since instructional practice revealed several ways to utilize CAS, educators have the option to utilize it as a mathematical consultant for the students as proven by Springer. She shared knowledge about students who accessed other types of CAS devices through online and free sources.

Second, as teachers consider the theme *regulating access*, a host of managerial tasks may come to mind. However, presentation of techniques that Springer and Shasta implemented could ease the concern. That is, participants sequenced the release of commands, delayed the revealing of commands, and manipulated the device via keying the commands rather than accessing the menu. In addition, they managed permissions on assessments based on the intended goal of the lesson. However, all of these ideas were cradled by the significant guidance from the participants. Students were taught how to use the tools to advance their learning not replace it. Furthermore, students were given the opportunity to outsource procedures after the learning was assessed. That notion lends an incentive to students, thereby, helping practitioners to understand some guidelines for teaching with a CAS.

Finally, the lesson vignettes illuminated lesson designs that may be applicable to teacher curriculum and lesson design. Each vignette supplied a rationale from the participant, a step-by-step method of instruction, actual implementation, and images to understand the exact commands accessed. Lesson vignettes produced were Algebra I and calculus classes; yet, the overlapping of topics in Algebra II and precalculus could be applied to extensions on these lessons. At the very minimum, the lessons provided a model for instruction via the CAS.

Contribution to Literature

The P-Map proved an impressive tool for identification of pedagogical opportunities. As the data were coded, three levels within the P-Map seemed to parallel with the purpose of this research, which considered these inquiries: (a) what pedagogical opportunities mathematics teachers exploited with the presence of CAS, (b) how teachers aligned lessons to develop mathematical understandings, and (c) why these teachers wanted to orient their focus to exploit CAS in the development of mathematical knowledge. Figure 46 displays an image of how I viewed the purposes of the study as they aligned to Pierce and Stacey's (2010) three levels.

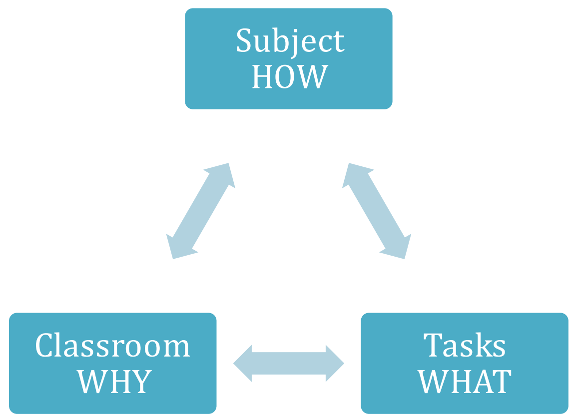

Figure 46. Alignment to Pierce & Stacey's (2010) three levels of pedagogical opportunities.

What. Pierce and Stacey (2010) described the task level opportunities as those "representing five different ways in which MAS affords opportunities for improved teaching and learning tasks" (p. 6). The tasks in which students engaged as described in this studies lesson vignettes, suggested the answer to the first point about what occurred. However, only three of Pierce and Stacey's (2010) five tasks reconciled with this study. Limited data do not adequately support this assumption of the how, why, and what parallelism.

Why. The classroom level issues were more abstract in terms of classroom social dynamics. The charge was that as students become more engaged in a CAS-enriched environment, social dynamics between teacher-student and student-student become enriched with questions, agency of exploration, and reliance on the CAS as another mathematical resource (Pierce & Stacey, 2010). The characteristics of classroom level and social dynamics may be generative of ideals that strengthen teaching and learning in mathematics. Pierce and Stacey (2010) conceived that "opportunities arise from the improved way in which mathematical working and results can be displayed and shared in the classroom" (p. 8). Pierce and Stacey's (2010) phrase *mathematical authority* arose out of changes that occur at the classroom level. The inference is that classroom level pedagogy may imply reasons why the teacher may want to orient their focus to exploit CAS in the development of mathematical knowledge.

How. Finally, the subject level reconciled the question of how teachers aligned lessons to develop mathematical knowledge. Pierce and Stacey's (2010) subject level categorized "opportunities for technology to support new or changed goals or teaching methods for a mathematics course as a whole and new understandings of mathematics as a field of human endeavor" (p. 9). This study's emergent theme components can be interpreted as instructional practices that teachers adopt as they exploit CAS while adjusting his or her subject level goals to develop students' mathematical understandings.

This theory of the answers to questions (i.e., what, why, and how) as it parallels Pierce and Stacey's (2010) pedagogical opportunities framework (i.e., tasks, classroom, and subject) alignment is purely inconclusive. I mention it so that developers and users of the P-Map may ruminate on this information. There may be practicality and connectedness to this paralleled theory. I used the P-Map as a researcher's tool to understand teacher pedagogy. In the process, I was able to imagine the P-Map levels through a different lens.

Recommendations for Future Research

Four areas that are directly connected to this research are accessible for future research. First, I presented a Schema for CAS-Oriented Instruction developed through an introspective analysis, and this schema can be scrutinized further. Second, Pierce and Stacey's (2010) P-Map was used as a researcher's tool to investigate pedagogy. There may be additional depth and breadth to the framework that one can tease out related to its categories. Third, the participants in this study were chosen based on their experience using CAS. Questions regarding the transition of teachers from a non-CAS user to one that becomes an active, effective teacher have potential in the realm of research. Finally, a student perspective as a learner in the milieu of CAS may be examined. These four areas are elaborated in the following sections.

Emergent schema. The Schema for CAS-Oriented Instruction could benefit from implementation as a protocol and, hence, refinement or extension. The scheme was developed based on the results of two teachers and six observed lessons with follow-up interviews and writing artifacts. Additional data will support the components and possibly add to the richness of this discussion.

Components from the schema each provide a place for further investigation. In particular, CAS as an external mathematical consultant provides the learner with a host of opportunities. This study exhibited CAS as a supplement to the teacher. New venues for teaching and learning secondary mathematics have taken the stage. Perhaps in light of the Information Age and rising online coursework, CAS can support more profound utilization in the absence of the teacher.

It was purposeful to leave the Schema for CAS-Oriented Instruction with open puzzle pieces. The schema may display only part of a complete picture of pedagogical components for CAS-rich instruction. Additional components may be revealed through similar studies.

P-Map. The P-Map framework impacted this study to advance an understanding about teachers' pedagogy. The P-Map did not elaborate on assessment or curriculum; two areas affected by the functional opportunities of MAS according to Pierce and Stacey (2010). Through this examination, I had proposed to uncover some factors from the participants' pedagogy that may have interacted with curriculum and assessment. Several writing prompts (Appendix E, *Curriculum and Evaluation Issues*) were directed to the participants on these ideals. These question prompts did not produce constructive developments. Future research may produce knowledge pertaining to curriculum and assessment.

Teacher exemplars. Shasta and Springer were identified as exemplars of CAS utility prior to the study due to their educational background. Shasta proclaimed his utilization of CAS for over 20 years, when he was first introduced to the TI-81 by his-then mathematics department chairperson. Shasta was active in many CAS-platforms: TI-92, TI-89, Wolfram-alpha (e.g., web-based tool), and the TI-Nspire™ CAS. Finally, he was an active member with the MEECAS-USACAS groups in the years from 2000 up to the date of this study. In comparison, Springer's interest in technology and education led her to complete a master's degree in instructional technology. She explained her philosophy about technology integration in mathematics curriculum.

> How can I use technology in my math classroom to help [students] guide and explore and help them better understand the process or investigate the process and even if it's five or ten minutes. Just to create that opportunity for growth and development, you know, using the technology and coming up with it on their own. (Springer, Interview December 6, 2017).

I wondered about the motivations of the two participants that exhibited CAS utility and what compelled them to transition their pedagogy. Each retained technological capability characteristics and formulated CAS-observed lessons with what appeared natural and relative ease.

Zbiek and Hollebrands (2008) considered a perspective that elaborated on Beaudin and Bowers (1997) PURIA model (Play, Use, Recommend, Incorporate, Assess); a framework of modalities that identify teacher transitional phases in the utilization of technologies.

> We found a perspective that allowed for explicit consideration of teachers' needs to learn the technology, to learn to do mathematics with technology, to use the technology with students, and to attend to student learning as a guide for innovation. (Zbiek & Hollebrands, 2008, p. 294)

Further research may consider the characteristics of teachers like Shasta and Springer, as exemplars in the utilization of CAS. As well, longitudinal studies could reveal transitional phases from a novice user to a consumer at advanced stages of CAS integration.

Student perspective. A premise of this study was CAS as a tool for developing mathematical knowledge. The investigation considered teacher activities that utilized CAS to provide opportunities for student learning to deepen and broaden his or her knowledge base. The student as a learner on the receiving side of instruction is a viable next step to bring light to learning in a CAS-rich milieu.

Researcher's Reflection

Springer and Shasta expressed joy both in their presentation of CAS-rich lessons and in their students' acceptance of CAS-enriched pedagogy. In my relations with CAS enthusiasts and researchers alike, I have found this emotion plentiful. "We are sure that the use of technology will increase the joy and interest of the students and they will experience the learning of mathematics in a more meaningful way because we can offer them a more meaningful mathematics" (Heugl, 2005, p. 11). Springer's case reflected enjoyment for herself and her students by utilizing CAS technology. "[Students] had never even seen it or heard about [CAS]. And I was able to show them how cool it was. They fell in love with it" (Springer, Interview, November 8, 2017). The joy was an expected outcome that was in common with Austrian didactics expert Helmut Heugl. Likewise, Shasta shared such excitement multiple times and in this interview.

> [Students] eyes lit up around the room as they were typing *factor*. And when they typed the last *r* and it went un-italics. The kids were like, "Oh my goodness, it can do this," like just the recognition that we've stumbled upon a hidden tool, was kind of fun for them. (Shasta, Interview, December 22, 2017)

It is my own joy to share this research and contribute to the ever-changing technology landscape that influences educational practice.

Chapter Summary

CAS-oriented pedagogy was regarded as underdeveloped and this holistic qualitative study, therefore, adds value to the literature base. After analysis of two teachers in the reduction of potential barriers an emergent theme was discovered. Data were collected over the course of three months obtaining six lessons, which equated to nine lesson vignettes. Thick descriptions provided the framework for analysis. A deductive analysis was the first step in coding. Each vignette was explicated through the use Pierce and Stacey's (2010) P-Map. These lessons revealed the emergent themes in orienting the participants' mathematics instruction: viewing CAS as a mathematical consultant, verifying answers, applying multiple representations, regulating access, providing guidance, and outsourcing procedures. The components interlock with one another to form a cohesive depiction of pedagogical decisions in the presence of CAS-rich classrooms.

NCTM's (2014) guiding principle for tools and technology stated, "An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (p.78). CAS has been regarded as an essential tool in some circles (Roschelle & Leinwand, 2011; Usiskin, 2006; Waits & Demana, 1998, 2000). As well, CAS is a tool that learners can conceive new ways of developing an understanding of mathematics (Heid & Blume, 2008; Heid et al., 2013; Kutzler, 2003; Pierce & Stacey, 2010; Zbiek & Hollebrands, 2008). Participants Shasta and Springer supplied the stories of CAS integration in their program and represented CAS as an essential tool. Their examples serve to exhibit methods that educators can use to create opportunities for student reasoning and sense making in mathematical knowledge.

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APPENDICES

APPENDIX A: Information Gathering Survey

Sent electronically, November 2015 – January 2016, via Google Forms

Hello, my name is Candace Terry, a doctoral student at Middle Tennessee State University researching the topic of computer algebra systems (CAS). You were selected as a participant in that you were a technology conference attendee and/or a known technology user. I am asking for 10 minutes of your time to participate in this information gathering survey. Your participation is completely voluntary and anonymous.

Consent

By proceeding with this questionnaire, I am agreeing to participate in an information gathering survey. I have read the consent form approved by the IRB office and understand the purpose, benefits, and risks.

What is CAS technology?

Computer algebra systems (CAS) can be viewed as any accessible tool that has features of standard scientific calculators and may also feature one or more of the following: graphing 2D, graphing 3D, dynamic geometry, tables and spreadsheets, numerical calculations, symbolic calculations, and symbolic manipulations. Multiple platforms contain these capabilities: handheld calculators, computer software programs, and tablet applications.

Are you utilizing CAS technology in all or part of your instructional practice? Yes or No

Are you familiar with Computer Algebra System (CAS) technology? Yes, Somewhat, or No

What platform(s) of CAS do teachers use? (check all that apply)

Handheld
Tablet
Computer

For those who use handheld technology, which device do you specifically use?

I don't use handheld technology for math.
TI-Nspire CAS
TI-Nspire (non-CAS)
HP Prime
TI-84 or TI-83
Other (specify)

For those who use tablets, which application program do you use?

I don't use a tablet for math.
Geogebra
TI-Nspire CAS
TI-Nspire (non-CAS)
Desmos
Other (specify)

For those who use desktop or laptop computers, which program do you use?
I don't use computers for math.
Geogebra
TI-Nspire CAS or (non-CAS)
TI-84 or TI-83
Desmos
Wolfram alpha
Mathematica
Other (specify)

In what ways do you as the teacher utilize CAS technology?
(rate: not at all, have tried once or twice, sometimes, frequently)
Graphing
Dynamic Geometry
Investigative with algebra
Regression and Modeling
Data & Statistics
Numerical Solve of Equations
Symbolic Algebra Manipulations
Systems of Equations
TI developed activities

Lesson Scenarios

We want to explore how a geometric sequence relationship can be viewed in its' graphical representation, numerical table, algebraic model, and any other representation. Describe what features you would utilize with CAS technology.

The lesson involves solving a system of equations with multiple variables. Describe what features you would utilize with technology.

The class is curious about the factorization for different polynomial functions. Does the polynomial factor? Under what circumstances? What would the factors look like? Explain how you would investigate this using technology?

Open-Ended
What is another of your favorite tools and/or lessons that utilize CAS technology?
Beliefs regarding teaching and learning with CAS technology

Concerns that you have regarding CAS technology... Rate from low (1) to high (3) or 0 (does not apply)

Students will not learn mathematical concepts without learning the procedures first.

Time to teach utilizing CAS technology and paper/pencil technology can be overwhelming and not address the real needs of learning mathematics.

A different type of mathematical question can be asked when utilizing CAS technology.

Teaching Assignment

What level do you teach?

Middle School Mathematics
High School Mathematics
Community College or Two-Year College
University (4 year)
Other

What courses do you teach?

Basic Algebra
Geometry
Algebra 2
Pre-Calculus
Honor's Courses
Calculus AB or Calc 1
Calculus BC or Calc 2
Statistics
Integrated Mathematics
Other

Thank you

I appreciate your time thoughtfully answering questions about CAS technology.
If you are willing to discuss in a little more detail about how you are utilizing CAS technology, please provide an email contact.

APPENDIX B: Follow-up Interview Protocol

Do you mind me tape-recording this call so that I can focus on the conversation rather than taking notes?

1. Just so that I understand your frame of reference, can you give me a brief description of your teaching assignment?
2. Explain your lesson scenario from the survey regarding [choice depending on survey answer- copy and paste here for reference]
    a. Geometric sequence
    b. System of equations
    c. Polynomial functions and factoring
3. This is a two-part question. Do you use any of the symbolic features of CAS? As in the solve, factor, expand, and such with algebraic expressions… In what ways do you use these tools?
4. Talk to me a little bit about the difference in your instruction as you first began to use CAS and later as you personally became more adept at using it as a tool?
5. What kinds of things do you think about when developing lessons involving CAS technology?
6. How does teaching using CAS technology affect your curricular choices?
7. What beliefs do you hold about teaching utilizing CAS technology? On your survey, you responded that… [choice depending on survey answer-copy and paste here for reference]
    a. Students will not learn mathematical concepts without learning the procedures first.
    b. Time to teach utilizing both CAS technology and paper/pencil technology can be overwhelming and does not address the real needs of learning mathematics.
    c. A different type of mathematical question can be asked when utilizing CAS technology.
8. What problem, if any, do you foresee with teaching using CAS technology

APPENDIX C: Survey

NOTE: This survey has been created in Google forms. A portable document format (pdf) of the form is attached. In sending this link to participants, the IRB consent form will be attached to the email.

Introduction CAS Survey - Demographic Data

Hello, my name is Candace Terry, a high school mathematics teacher and a doctoral student at Middle Tennessee State University researching the topic of computer algebra systems (CAS). In particular, I am interested in the ways that teachers teach using CAS.

I want to thank you for your participation in this study. At any time if you have questions or concerns about the study, please contact me at (931) 247-7220 or candace.terry@mtsu.edu.

The purpose of this survey is to introduce myself and begin data collection. First, I will ask for your consent to participate in the study this fall. Second, I have a few background data questions. Third, I need your school schedule to assist in planning a campus visit. And finally, I will request your contact information.

This survey should take about 10 minutes to complete.

Email Address: This form is collecting email addresses.

Protocol ID: _________ **Consent Form**

1. By proceeding with this questionnaire, I am agreeing to participate in an information gathering survey. I have read the attached consent form approved by the IRB office and understand the purpose, benefits, and risks.

☐ I agree.

☐ I have a few questions before consenting. Please contact me.

Question number 1 employs skip logic that depending on the answer choice skips to the next logical question. The first choice continues to the next question; the second choice skips to contact information #9.

Demographic Data

2. Please provide your gender.

☐ Male

☐ Female

3. Please provide your age.

(short answer response)

4. Please provide your ethnic background.

☐ Caucasian

☐ Hispanic or Latino

☐ Black or African American

☐ Native American or American Indian

☐ Middle Eastern

☐ Asian or Pacific Islander

☐ Other: (fill in)

5. What is your educational background? (Select all that apply.)

☐ Bachelors degree mathematics major

☐ Bachelors degree mathematics minor

☐ Bachelors degree with a mathematics related degree, but not mathematics

☐ Bachelors degree in an area other than mathematics

☐ Masters degree mathematics

☐ Masters degree in education

☐ Masters degree in an area other than mathematics or education

☐ Advanced degree beyond masters

☐ Other: <u>(fill in)</u>

6. How many years have you been at this school, including this year?

☐ 0-1

☐ 2-4

☐ 5-6

☐ 7-8

☐ 9-10

☐ beyond 10

7. How many years have you been teaching high school?

☐ 0-1

☐ 2-4

☐ 5-6

☐ 7-8

☐ 9-10

☐ 11-12

☐ 13-15

☐ 16-20

☐ beyond 20

8. Please give a brief background of your education experience. For example, any additional discussion about degrees earned, certificates, or previous experience.

(short answer response)

Contact Information

9. Please provide the email address you will use for communication purposes with me.

If you will be using the one previously supplied, please skip this question.

(short answer response)

10. Do you have a SKYPE account that I can use for interviews?

☐ Yes

☐ No, but I can create one. Link: https://www.skype.com/en/

☐ No, I prefer a different video teleconference tool.

Question number 9 employs skip logic that depending on the answer choice skips to the next logical question. The first two continue number 11; the third choice skips to #12.

SKYPE

11. What is your SKYPE user name?

(short answer response)

Question number 11 employs skip logic. This answer will skip to the end.

Other Teleconference Tool

12. The video teleconference application program I like to use is...

(short answer response)

Thank you!
I appreciate your time thoughtfully answering questions. I look forward to talking with you.
Candace Terry

APPENDIX D: Interviews

Pre-Interview

Hello, my name is Candace Terry, a doctoral student at Middle Tennessee State University researching the topic of computer algebra systems (CAS).
Do you mind me tape-recording this interview so that I can focus on the conversation rather than taking notes?

1. Please provide a brief description of your teaching assignment.
2. Talk to me a little bit about the difference in your instruction as you first began to use CAS and later as you personally became more adept at using it as a tool.
3. What kinds of things do you think about when developing lessons involving CAS technology?
4. How does teaching using CAS technology affect your curricular choices, if at all?

Post-Interviews

Do you mind me tape-recording this interview so that I can focus on the conversation rather than taking notes?

1. Please summarize the lesson for me.
2. What prompted you to approach the topic in this way?
3. Were there previous outcomes that helped you form an opinion about this method? Can you explain?
4. I wasn't clear about ____. Please help me understand by clarifying that for me.

APPENDIX E: Reflective Writing Prompts

NOTE: These questions may be modified, deleted, or added on, after viewing screencasts for each lesson. Only a selection of questions will be asked on each cycle. These questions will be sent and received via participant electronically.

Introduction on the email: Thank you for your inspiring views in our interview the other day. I have several questions here for you to preview. Please read them and after a time of reflection, draft answers to the questions and reply to this email. It should take about 15 minutes to type out responses. If you are more comfortable writing in a separate document and attaching, that is fine too. Please plan to complete this within five days of receiving this.

Teacher Perceptions and Attitude

1) What do you see as advantages of using CAS for teaching mathematics?
   a) Why do you consider these advantages?
   b) Why are these advantages important, if at all?
2) What do you see as disadvantages of using CAS for teaching mathematics?
   a) Why do you consider these disadvantages?
   b) Why are these disadvantages important, if at all?
3) Was there a time in your teaching career when you didn't teach with CAS?
   a) What were the different expectations for your students, if any?
   b) Why do you think you decided to teach using CAS?

Classroom Dynamics

4) Has the presence of CAS technology in your classroom changed your teaching in any way? Specifically, what difference, if any, has it made in your presentation of the material? In your students' participation? In your role as teacher?
5) Discuss the impact that CAS has had on the interaction between students in the classroom, if any.
6) Discuss the impact that CAS has had on the interaction between you and your students, if any.

Curriculum and Evaluation Issues

7) What effect has CAS had on the goals and content of the mathematics course you teach, if any?
8) What effect has CAS had on your evaluation of the students you teach, if any?
9) What had changed it terms of preparation, if any?

Content Specific Questions

10) How is CAS helpful in understanding (insert here the mathematical concept)?
11) Are there extensions to the lesson on (insert here the mathematical concept) that were possible with the CAS, that were nearly impossible without?
12) What had changed it terms of content knowledge about (insert here the mathematical concept), if any?

*Adapted from Simonsen and Dick (1997) Interview Protocol

APPENDIX F: Classroom Observation Protocol

Teacher/Participant:	Date/Time:
Mathematics Course:	Mathematical Topic:

In what ways does the teacher utilize technology? Check all that apply

☐ Teacher Demonstration
☐ Student Demonstration
☐ Scientific calculator
☐ Graphing calculator
☐ Dynamic Geometry tools
☐ Investigative with CAS
☐ Investigative with geometry
☐ Regression and Modeling
☐ Spreadsheets
☐ Function Tables
☐ Statistical Calculation
☐ Symbolic Algebraic
☐ Numerical Solve of Equations
☐ Systems of Equations
☐ TI developed activities
☐ Scientific probe ware
☐ Other________________

Comments: Description of how the technology was used.

Detailed Description of Pedagogical Opportunities

Type	Opportunity	Description	Example
Subject	Exploit Contrast of Ideal and Machine Mathematics	Teachers deliberately use 'unexpected' error messages, format of expressions, graphical displays as catalyst for rich mathematical discussion	Syntax in the device provides an unexpected output, different from pen-and-paper solutions.
	Re-balance Emphasis on Skills, Concepts, and Applications	Teacher adjusts goals: spend less time on routine skills; more time on concepts and applications. Increase on mathematical thinking.	Heid's seminal research on re-sequencing of concepts and skills in a calculus course Dynamic geometry can shift from memorization of facts to conjecturing and proving through visual arguments
	Build Metacognition and Overview	Teachers give overview as introduction or summation: link concepts through manipulation of symbolic expressions and use of multiple representations	Promote curiosity or instill a question, questioning strategies for reflection on the mathematical concept(s)
Class-room	Change Classroom Social Dynamics	Teachers facilitate rather than dictate. Encourage group work. Encourage students to initiate discussion and share their learning with the class.	Linking action with mathematical reflection Constructivist approach to instruction
	Change Classroom Didactic Contract	Teachers allow technology to become a new authority. Change what is expected of students/teachers. Permit or constrain explosion of available methods.	Role changes for both teacher and student, possibly teacher as facilitator and student as consultant.

Tasks	Learn Pen-and-paper Skills	Use instant 'answers' as feedback when learning processes.	Solve equations one step at a time. Use of a symbolic math guide (tutorial program within the device)
	Use Real Data	Work on real problems involving calculations that, done by hand, are error prone and time consuming.	Collect real data through the device, such as the height of a ball or the temperature of a cup of water.
	Explore Regularity and Variation	Strategically vary computations. Search for patterns. Observe effects of parameters. Use general forms.	Use of sliders to dynamically change the graph of a function. Alter a geometric shape with drag features. Expand or factor algebraic expressions and make observations.
	Simulate Real Situations	Use dynamic diagrams, drag, and collect data for analysis. Use technology generated statistical data sets.	Random function generator repeated times to create a histogram for 1000 tosses of two dice.
	Link Representations	Move fluidly between geometric, numeric, graphic, and symbolic representations.	Equation of a circle in symbolic sense, input numerical values, graphed, and drawn with geometry tools

Note. Adapted from "Mapping pedagogical opportunities provided by MAS," by R. Pierce, & K. Stacey, 2010, *International Journal of Computers for Mathematical Learning*, *15*, p. 6. Copyright 2010 by Springer International Publishing AG.

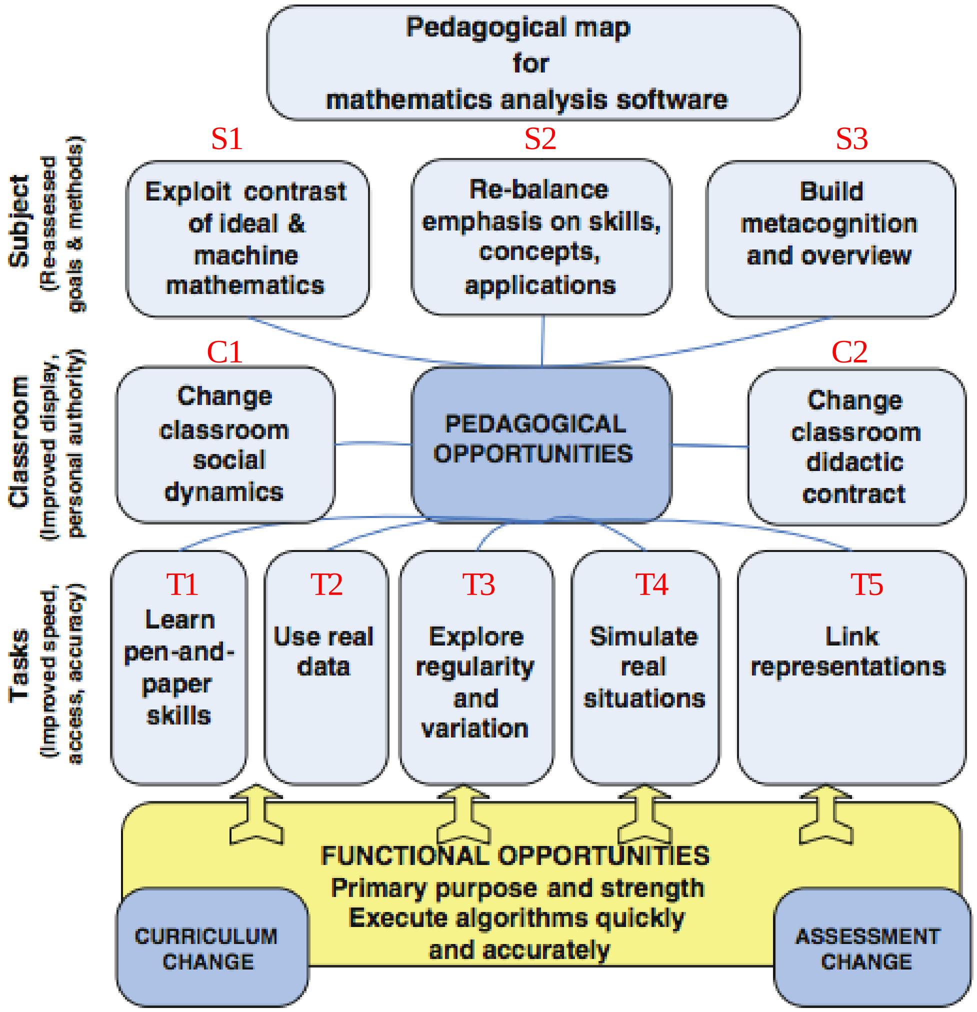

Figure 47. Pedagogical Map (P-Map). Adapted from "Mapping Pedagogical Opportunities Provided by Mathematics Analysis Software," by R. Pierce and K. Stacey, 2010, *International Journal of Computers for Mathematical Learning*, *15*, p. 6. Copyright 2010 by Springer International Publishing AG.

APPENDIX G: IRB Approval

IRB
INSTITUTIONAL REVIEW BOARD
Office of Research Compliance,
010A Sam Ingram Building,
2269 Middle Tennessee Blvd
Murfreesboro, TN 37129

MIDDLE TENNESSEE STATE UNIVERSITY

IRBN001 - EXPEDITED PROTOCOL APPROVAL NOTICE

Thursday, September 28, 2017

Principal Investigator	Candace Terry (Student)
Faculty Advisor	Michaele Chappell
Co-Investigators	NONE
Investigator Email(s)	candace.terry@mtsu.edu; michaele.chappell@mtsu.edu
Department	Mathematics
Protocol Title	Secondary Mathematics Teachers' Pedagogy Through the Tool of Computer Algebra Systems
Protocol ID	**18-2020**

Dear Investigator(s),

The above identified research proposal has been reviewed by the MTSU Institutional Review Board (IRB) through the **EXPEDITED** mechanism under 45 CFR 46.110 and 21 CFR 56.110 within the category *(7) Research on individual or group characteristics or behavior* A summary of the IRB action and other particulars in regard to this protocol application is tabulated as shown below:

IRB Action	APPROVED for one year from the date of this notification
Date of expiration	**9/30/2018**
Participant Size	5 [FIVE]
Participant Pool	Adult Private School Teachers
Exceptions	1. Video and audio recording permitted for transcription purposes 2. Name and email address collection allowed
Restrictions	**1. Informed consent must be obtained** **2. Participants must be adults age 18 or over** **3. Vidoe and audio recordings to be destroyed once analyzed** **4. Identifiable information to be destroyed on data are analyzed**
Comments	NONE

This protocol can be continued for up to THREE years (**9/30/2020**) by obtaining a continuation approval prior to **9/30/2018**. Refer to the following schedule to plan your annual project reports and be aware that you may not receive a separate reminder to complete your continuing reviews. Failure in obtaining an approval for continuation will automatically result in cancellation of this protocol. Moreover, the completion of this study MUST be notified to the Office of Compliance by filing a final report in order to close-out the protocol.

Continuing Review Schedule:

Reporting Period	Requisition Deadline	IRB Comments
First year report	9/20/2018	TO BE COMPLETED
Second year report	9/30/2019	TO BE COMPLETED
Final report	9/30/2020	TO BE COMPLETED

Post-approval Protocol Amendments:

Date	Amendment(s)	IRB Comments
NONE	NONE	NONE

The investigator(s) indicated in this notification should read and abide by all of the post-approval conditions imposed with this approval. Refer to the post-approval guidelines posted in the MTSU IRB's website. Any unanticipated harms to participants or adverse events must be reported to the Office of Compliance at (615) 494-8918 within 48 hours of the incident. Amendments to this protocol must be approved by the IRB. Inclusion of new researchers must also be approved by the Office of Compliance before they begin to work on the project.

All of the research-related records, which include signed consent forms, investigator information and other documents related to the study, must be retained by the PI or the faculty advisor (if the PI is a student) at the secure location mentioned in the protocol application. The data storage must be maintained for at least three (3) years after study completion. Subsequently, the researcher may destroy the data in a manner that maintains confidentiality and anonymity. IRB reserves the right to modify, change or cancel the terms of this letter without prior notice. Be advised that IRB also reserves the right to inspect or audit your records if needed.

Sincerely,

Institutional Review Board
Middle Tennessee State University

Quick Links:

Click here for a detailed list of the post-approval responsibilities.
More information on expedited procedures can be found here.

IRB

INSTITUTIONAL REVIEW BOARD
Office of Research Compliance,
010A Sam Ingram Building,
2269 Middle Tennessee Blvd
Murfreesboro, TN 37129

IRBN001 - EXPEDITED PROTOCOL APPROVAL NOTICE

Tuesday, October 03, 2017

Principal Investigator	Candace Terry (Student)
Faculty Advisor	Michaele Chappell
Co-Investigators	NONE
Investigator Email(s)	candace.terry@mtsu.edu; michaele.chappell@mtsu.edu
Department	Mathematics
Protocol Title	Secondary Mathematics Teachers' Pedagogy Through the Tool of Computer Algebra Systems
Protocol ID	**18-2020**

Dear Investigator(s),

The above identified research proposal has been reviewed by the MTSU Institutional Review Board (IRB) through the **EXPEDITED** mechanism under 45 CFR 46.110 and 21 CFR 56.110 within the category *(7) Research on individual or group characteristics or behavior* A summary of the IRB action and other particulars in regard to this protocol application is tabulated as shown below:

IRB Action	APPROVED for one year
Date of expiration	9/30/2018
Participant Size	5 [FIVE]
Participant Pool	Adult Private School Teachers
Exceptions	1. Video and audio recording permitted for transcription purposes 2. Name and email address collection allowed
Restrictions	**1. Informed consent must be obtained** **2. Participants must be adults age 18 or over** **3. Vidoe and audio recordings to be destroyed once analyzed** **4. Identifiable information to be destroyed on data are analyzed**
Comments	NONE

This protocol can be continued for up to THREE years (9/30/2020) by obtaining a continuation approval prior to 9/30/2018. Refer to the following schedule to plan your annual project reports and be aware that you may not receive a separate reminder to complete your continuing reviews. Failure in obtaining an approval for continuation will automatically result in cancellation of this protocol. Moreover, the completion of this study MUST be notified to the Office of Compliance by filing a final report in order to close-out the protocol.

Continuing Review Schedule:

Reporting Period	Requisition Deadline	IRB Comments
First year report	9/20/2018	TO BE COMPLETED
Second year report	9/30/2019	TO BE COMPLETED
Final report	9/30/2020	TO BE COMPLETED

Post-approval Protocol Amendments:

Date	Amendment(s)	IRB Comments
10.03.2017	Permitted to recruit participants from Hawken School (Lyndhurst, OH 44124-2595 - www.hawken.edu - (440) 423-4446).	IRB review

The investigator(s) indicated in this notification should read and abide by all of the post-approval conditions imposed with this approval. Refer to the post-approval guidelines posted in the MTSU IRB's website. Any unanticipated harms to participants or adverse events must be reported to the Office of Compliance at (615) 494-8918 within 48 hours of the incident. Amendments to this protocol must be approved by the IRB. Inclusion of new researchers must also be approved by the Office of Compliance before they begin to work on the project.

All of the research-related records, which include signed consent forms, investigator information and other documents related to the study, must be retained by the PI or the faculty advisor (if the PI is a student) at the secure location mentioned in the protocol application. The data storage must be maintained for at least three (3) years after study completion. Subsequently, the researcher may destroy the data in a manner that maintains confidentiality and anonymity. IRB reserves the right to modify, change or cancel the terms of this letter without prior notice. Be advised that IRB also reserves the right to inspect or audit your records if needed.

Sincerely,

Institutional Review Board
Middle Tennessee State University

Quick Links:

Click here for a detailed list of the post-approval responsibilities.
More information on expedited procedures can be found here.

APPENDIX H: P-Map Permission Letter

Robyn Pierce r.pierce@unimelb.edu.au via mtmailmtsu.onmicrosoft.com 4/3/16
to Candace, Kaye, r.p

Dear Candace
Thank you for your interest in our work. We are very pleased for you to make use of our pedagogical opportunities map for your research and would be interested to read any resulting publications.

I have attached a published article:
Pierce, R. & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*. 15(1) 1- 20 DOI : 10.1007/s10758-010-9158-6.

Our P-Map did evolve over some time in response to feedback from our international colleagues – the Australian Senior Mathematics Journal paper was an early version. The one in the article attached is our final version - or at least we have not made changes since then! As technology changes so do specific opportunities but in generic terms we still think that version of the map works.
You ask:
Is there any significance to the four arrows going from the functional opportunities to the model?
No there was no deep meaning in there being four arrows. (They don't specifically link to particular Tasks) We are trying to convey that it is the functional opportunities that support the pedagogical opportunities. The easily accessible functional capacity of technology like CAS allows us to think about different pedagogy.

And do the blue lines connecting all 10 opportunities to the center important in any other way?

Again these lines just represent the interconnectedness of the opportunities eg Exploring regularity and variation may be the trigger for a change in classroom didactic contract OR link representations may support an Overview of a topic.

From the sound of your proposed project you may also be interested in the second article I have attached:
Pierce, R. & Stacey, K. (2013). Teaching with new technology: four 'early majority' teachers. *Journal of Mathematics Teacher Education, 16* (5), 323-347.

Very best wishes for your research
Robyn

Robyn Pierce | Associate Professor
Melbourne Graduate School of Education

Level 3, 234 Queensberry St
The University of Melbourne, Victoria 3010 Australia
T: +61 3 8344 8519 M: 0417 110 902 E: r.pierce@unimelb.edu.au
W: unimelb.edu.au

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